that $\omega' = F^M(K_1, \ldots, K_{m+1}) \wedge x$, and $\omega = F^m(K_1, \ldots, K_{m+1})$. Consequently, ω is reducible.

Finally we prove Theorem 6′: Let R be a Noetherian OP ring and P be a projective module of odd rank. Then $P = R^{2n} \oplus I$, and $\operatorname{rg} I = 1$.

Proof. Consider a projective R-module $L = P \oplus (\det P)^{-1}$, $\operatorname{rg} L = 2n + 2$. Then let $L \oplus X = R^p$. This is possible because L is projective. Thus L is closed in R^p and projective. According to the reduction theorem for Noetherian rings, there exists $R^e \subset X$ which is closed. Thus $P \oplus \det P^{-1} \oplus R^e \subset RP$ is closed and $\operatorname{rg}(P + \det P^{-1} \oplus R^e) = p - 1$. Obviously $\det(P + \det P^{-1} \oplus R^e) = R$ and, hence, by the Kleiner Theorem, $P \oplus \det P^{-1} \oplus R^e = R^{p-1}$. Also, stable free modules are free because R is an OP-ring and $P \oplus \det P^{-1} = R^{2n+2}$. Using the Bass Theorem [3], we get that $P = R^{2n} \oplus \det P$.

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Goldie Rank in the Enveloping Algebra of a Semisimple Lie Algebra, I

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Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$, and Prim $U(\mathfrak{g})$ the set of primitive ideals of $U(\mathfrak{g})$. In this two-part series the Goldie ranks of the primitive quotients $\{U(\mathfrak{g})/J : J \in \operatorname{Prim} U(\mathfrak{g})\}$ are computed. Let $\mathfrak{h}$ be a Cartan subalgebra for $\mathfrak{g}$ and $\mathfrak{h}^*$ the dual of $\mathfrak{h}$. After Duflo [4] there exists a surjection $\lambda \mapsto J(\lambda)$ of $\mathfrak{h}^*$ onto Prim $U(\mathfrak{g})$. In the first part it is shown that $\mathfrak{h}^*$ can be written as a disjoint union of infinite subsets with the following property. On each subset Λ there exists a polynomial p_Λ such that for all $\lambda \in \Lambda$, $p_\Lambda(\lambda)$ is the Goldie rank of $U(\mathfrak{g})/J(\lambda)$. In Part II the computation of the p_Λ is reduced to a knowledge of the multiplicities of the simple factors of the Verma modules.

1. Introduction

Unless otherwise specified all vector spaces are assumed over the complex field $\mathbf{C}$. In general a field is *not* taken to be commutative. Hom denotes $\operatorname{Hom}_{\mathbf{C}}$.

1.1. Let A be a ring. Call A Noetherian (resp. Goldie) if it is both left and right Noetherian (resp. Goldie). A multiplicative subset T of regular elements of A is said to be Ore if it is both left and right Ore. Then $T^{-1}A$ denotes the localization of A at T and for each left (resp. right) A module M we write $T^{-1}M$ (resp. MT^{-1}) for $T^{-1}A \otimes_A M$ (resp. $M_A \otimes T^{-1}A$). Given A prime Noetherian, let rk A denote the maximum number of direct summands of left (or right) ideals of A. It is called the *Goldie rank* of A. If $n = \operatorname{rk} A$, then the ring of fractions Fract A of A is isomorphic to an $n \times n$ matrix ring over a field, which we call the *Goldie field* of A.

1.2. For each ring A, let Prim A (resp. Max A) denote the set of primitive (resp. maximal) ideals of A. For each vector space V, let $S(V)$ denote the symmetric algebra over V and V^* the dual of V. For each $m \in \mathbb{N}$, let $S_m(V)$ denote the subspace of $S(V)$ of homogeneous polynomials of degree m. For each Lie algebra $\mathfrak{a}$, let $U(\mathfrak{a})$ denote the enveloping algebra of $\mathfrak{a}$ and $Z(\mathfrak{a})$ the center of

$U(\mathfrak{a})$. For each finite group G let $\hat{G}$ denote the set of equivalence classes of irreducible representations of G.

1.3. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra for $\mathfrak{g}$, R the set of nonzero roots, $R^+ \subset R$ a system of positive roots, $B \subset R^+$ the set of simple roots, s_α the reflection corresponding to the root α, W the group generated by the s_α: $\alpha \in B$, and ρ the half sum of the positive roots. Fix a Chevalley basis for $\mathfrak{g}$ and let X_α denote the element of this basis of weight $\alpha \in R$. Set

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathbb{C}X_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in R^+} \mathbb{C}X_{-\alpha}, \qquad \mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}.$$

For each $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}e_\lambda$ denote the one-dimensional $U(\mathfrak{b})$ module defined by $He_\lambda = (H, \lambda)\, e_\lambda$: $H \in \mathfrak{h}$, $Xe_\lambda = 0$: $X \in \mathfrak{n}^+$, and set $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}e_{\lambda-\rho}$. Let $L(\lambda)$ denote the unique simple quotient of $M(\lambda)$ and set $J(\lambda) = \operatorname{Ann} L(\lambda)$.

1.4. For each $\lambda \in \mathfrak{h}^*$, set $R_\lambda = \{\alpha \in R: 2(\alpha, \lambda)/(\alpha, \alpha) \in \mathbb{Z}\}$, $R_\lambda^+ = R_\lambda \cap R^+$, $B_\lambda \subset R_\lambda^+$ the subset of simple roots for the root system R_λ and W_λ the subgroup of W generated by the s_α: $\alpha \in B_\lambda$.

1.5. After Duflo [4, Theorem 1] the map $\lambda \mapsto J(\lambda)$ of $\mathfrak{h}^*$ into Prim $U(\mathfrak{g})$ is surjective. Set $\hat{\lambda} = W\lambda$ and $\mathscr{X}_{\hat{\lambda}} = \{J(\mu): \mu \in \hat{\lambda}\}$ and let π be the projection of Prim $U(\mathfrak{g})$ onto Max $Z(\mathfrak{g})$ defined by $\pi(I) = I \cap Z(\mathfrak{g})$. Then $\hat{\lambda}$ identifies with an element of Max $Z(\mathfrak{g})$ such that $\mathscr{X}_{\hat{\lambda}} = \pi^{-1}(\hat{\lambda})$.

1.6. Let $P(R)$ denote the lattice of integral weights. For each $\lambda \in \mathfrak{h}^*$ we write $\Lambda = \lambda + P(R)$. Define a function p_Λ on Λ through

$$p_\Lambda(\mu) = \operatorname{rk} U(\mathfrak{g})/J(\mu): \mu \in \Lambda.$$

The aim of this paper is to determine these functions. Our main result (Corollary 5.12) is that each Λ can be written as a finite union of disjoint infinite subsets $\{\Lambda_i\}_{i=1}^n$: $n = \operatorname{card} W_\lambda$, such that the restriction p_{Λ_i} of p_Λ to Λ_i is a polynomial. This result generalizes the special csae [10, 10.4] when $\mathfrak{g}$ has only type A_n factors (Cartan notation), though the method of proof is rather different. The main technical advance achieved here is to obliviate the need to consider induction from a parabolic subalgebra. As by-products we are able to show that a question of Kostant has a negative answer even for Harish-Chandra modules (see 4.7) and that the Goldie field of $U(\mathfrak{g})/J(\mu)$ depends only on the subset Λ_i to which μ belongs (Theorem 4.8). A further corollary is that the lower bound on card $\mathscr{X}_{\hat{\lambda}}$ given in [9, 5.2] holds without the restriction on B_λ previously imposed (Theorem 6.2). The aim of Part II is to determine the p_{Λ_i}. In particular it is shown that $\{1, 2, \ldots, n\}$ can be written as a disjoint union of subsets with the following property. For each subset there exists $m \in \mathbb{N}$ such that the corresponding functions span a simple W_λ submodule of $S_m(\mathfrak{h})$. Furthermore this submodule is not isomorphic to any submodule of $S_n(\mathfrak{h})$: $n < m$.

2. Preliminaries

2.1. Let $u \mapsto {}^t u$ (resp. $u \mapsto u^\vee$) denote the antiautomorphism of $U(\mathfrak{g})$ defined through ${}^t X_\alpha = X_{-\alpha}$: $\alpha \in R$, ${}^t H = H$: $H \in \mathfrak{h}$ (resp. $X^\vee = -X$: $X \in \mathfrak{g}$). Identify $U := U(\mathfrak{g}) \otimes U(\mathfrak{g})$ canonically with $U(\mathfrak{g} \times \mathfrak{g})$. Define a map j: $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ through $j(X) = (X, -{}^t X)$, set $\mathfrak{k} = j(\mathfrak{g})$ and identify $j(\mathfrak{h})^*$ with $\mathfrak{h}^*$. Here j is defined for any semisimple Lie algebra and so in particular for $\mathfrak{g} \times \mathfrak{g}$.

2.2. After [2, 7.8.15] we call a $U(\mathfrak{g})$ module *regular* if it is generated by a finite-dimensional $U(\mathfrak{b})$ module in which $\mathfrak{h}$ acts by semisimple endomorphisms and let $\mathscr{R}$ (resp. $\mathscr{R}_{\hat{\lambda}}$: $\lambda \in \mathfrak{h}^*$) denote the Abelian category of regular $U(\mathfrak{g})$ modules (resp. with infinitesimal character $\hat{\lambda}$). Any finite-dimensional $U(\mathfrak{g})$ module belongs to $\mathscr{R}$ and $\mathscr{R}$ is closed under tensoring with a finite-dimensional $U(\mathfrak{g})$ module. Given $U(\mathfrak{g})$ modules M, N consider $\operatorname{Hom}(M, N)$ as a U module through $(a \otimes b)$. $x = {}^t a^\vee x b^\vee$ for all $a, b \in U(\mathfrak{g})$, $x \in \operatorname{Hom}(M, N)$. Let $L(M, N)$ denote the subspace of all $\mathfrak{k}$-finite elements of $\operatorname{Hom}(M, N)$ (which is again a U module). Given $M, N \in \mathscr{R}$, then by [8, 4.3; 2, 7.8.15] $L(M, N)$ has finite length as a U module and hence is finitely generated as a left or as a right $U(\mathfrak{g})$ module. In particular any U subring of $L(M, M)$ is Noetherian.

2.3. Given L a finitely generated left $U(\mathfrak{g})$ module define the left Gelfand–Kirillov dimension $d(L)$ (resp. multiplicity $e(L)$) of L as in [8, 2.1]. That is, we take the normal filtration of L induced by the canonical filtration of $U(\mathfrak{g})$ and let $(e(L)/d(L)!)\, k^{d(L)}$: $k \in \mathbb{N}$, denote the leading term of the associated Hilbert–Samuel polynomial. If L is also a finitely generated right $U(\mathfrak{g})$ module, we denote its right Gelfand–Kirillov dimension (resp. multiplicity) by $d'(L)$ (resp. $e'(L)$). Recall [8, 2.3] that if L is a locally $\mathfrak{k}$ finite U module then $d(L) = d'(L)$, $e(L) = e'(L)$. Call L *smooth* (resp. *quasi-simple*) if $d(L) = d(L_0)$ (*resp.* $d(L) > d(L/L_0)$) for every nonzero submodule L_0 of L. A normal series $L = L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_{t+1} = 0$ for L is called a *quasi-composition* series if each factor L_i/L_{i+1} is quasi-simple and satisfies $d(L_i/L_{i+1}) = d(L)$. Clearly any smooth module of finite length admits a quasi-composition series. For each U module L set $l(L) = \{a \in U(\mathfrak{g}): aL = 0\}$, $r(L) = \{a \in U(\mathfrak{g}): La = 0\}$.

2.4. For each finite-dimensional $U(\mathfrak{g})$ module E, consider E^* as a $U(\mathfrak{g})$ module through transposition. Given $U(\mathfrak{g})$ modules M, N we have (cf. [11, 3.3]) the canonical identifications

$$\operatorname{Hom}_{\mathfrak{k}}(E, \operatorname{Hom}(M, N)) = \operatorname{Hom}_{\mathfrak{k}}(M \otimes E, N) = \operatorname{Hom}_{\mathfrak{k}}(M, N \otimes E^*).$$

Let M be a nonzero submodule of $N \otimes E^*$. By the second isomorphism a quotient of $M \otimes E$ is isomorphic to a nonzero submodule of N. Thus if N is smooth we have $d(M) = d(M \otimes E) \geqslant d(N) = d(N \otimes E^*)$, so $N \otimes E^*$ is smooth.

Hence the set $\mathscr{R}^0$ of smooth regular $U(\mathfrak{g})$ modules is closed under tensoring by finite-dimensional modules.

2.5. Let M, N, E, F be $U(\mathfrak{g})$ modules with E, F finite dimensional. Define a linear map Ψ: $\mathrm{Hom}(M, N) \otimes \mathrm{Hom}(E, F)$ (considered as a U module for the diagonal action of $\mathfrak{g} \times \mathfrak{g}$) into $\mathrm{Hom}(M \otimes E, N \otimes F)$ through $(\Psi(x \otimes y))(m \otimes e) = xm \otimes ye$. It is elementary that Ψ is a U module isomorphism and that the restriction of Ψ to $L(M, N) \otimes \mathrm{Hom}(E, F)$ has image $L(M \otimes E, N \otimes F)$. This was the following two important consequences. First we identify $(E \otimes \mathbb{C})$ as a U module with $\mathrm{Hom}(\mathbb{C}, E)$ and consider $L(M, N) \otimes (E \otimes \mathbb{C})$ (resp. $N \otimes E$) as a U module (resp. $U(\mathfrak{g})$) module for the diagonal action of $\mathfrak{g} \times \mathfrak{g}$ (resp. $\mathfrak{g}$). Define the map Ψ': $L(M, N) \otimes (E \otimes \mathbb{C}) \to L(M, N \otimes E)$ through $\Psi'(x \otimes e)m = xm \otimes e$.

Lemma. (i) *Ψ' is an isomorphism. Suppose $M, E \in \mathscr{R}$ and simple. Then*

(ii) *$L(M, M), L(M \otimes E, M \otimes E)$ are prime, Noetherian rings, and*

(iii) *$\mathrm{rk}\, L(M \otimes E, M \otimes E) = \mathrm{rk}\, L(M, M) \cdot \dim E$.*

2.6. Call $\lambda \in \mathfrak{h}^*$ *dominant* (resp. *regular*) if $(\alpha, \lambda) \geqslant 0$ (resp. $(\alpha, \lambda) \neq 0$) for all $\alpha \in R_\lambda^+$. For each $\lambda \in \mathfrak{h}^*$, set $R_\lambda^0 = \{\alpha \in R_\lambda : (\lambda, \alpha) = 0\}$ and let W_λ^0 denote the subgroup of W_λ generated by the s_α: $\alpha \in R_\lambda^0$. Let w^λ denote the unique element of W_λ such that $w^\lambda \lambda$ is dominant and that $w^\lambda(R_\lambda^0 \cap R^+) \subset R^-$. Set $\Lambda = \lambda + P(R)$, $\bar{F}_\lambda = \{\mu \in \Lambda : R_\mu^0 \supset R_\gamma^0,\ w^\lambda \mu \text{ dominant}\}$, $\hat{F}_\lambda = \{\mu \in \bar{F}_\lambda : w^\lambda(R_\mu^0 \cap R^+) \subset R^-\}$, $F_\lambda = \{\mu \in \bar{F}_\lambda : R_\mu^0 = R_\lambda^0\}$. Recall that Λ is a disjoint union of the $\hat{F}_{w\lambda}$: $w \in W_\lambda$. (For all this see [6, 1.3, 2.7]).

2.7 For each $\lambda, \mu \in \mathfrak{h}^*$ consider $(M(-\lambda) \otimes M(-\mu))^*$ as a U module by transposition and let $L(\lambda, \mu)$ denote the subspace of all $\mathfrak{k}$-finite elements (which is again a U module and is of finite length [3, IV, 2]). If $\lambda - \mu = \nu$ for some $\nu \in P(R)$ then the simple $\mathfrak{k}$ module with extreme weight ν occurs with multiplicity one in $L(\lambda, \mu)$ and we denote by $V(\lambda, \mu)$ the unique simple subquotient of $L(\lambda, \mu)$ containing this $\mathfrak{k}$-submodule.

Theorem [11, 4.7]. *Fix $\lambda \in \mathfrak{h}^*$ dominant and regular. Then for all $\mu \in \Lambda$ one has*

$$L(M(\lambda), L(\mu)) = V(-\mu, -\lambda),$$

up to isomorphism.

Remarks. Since λ is assumed dominant, it is trivial that $M(\lambda)$ is projective in $\mathscr{R}$ and so [11, 3.5] for each $M \in \mathscr{R}$ the simple factors of $L(M(\lambda), M)$ are just the $L(M(\lambda), L)$ as L runs over the simple factors of M. Again $L(M(\lambda), L(\mu)) = L(M(\lambda), L(\mu'))$ if and only if $L(\mu) = L(\mu')$, up to isomorphisms of U (resp. $U(\mathfrak{g})$) modules.

3. A Commutant Theorem

3.1. Throughout this section we fix $\lambda \in \mathfrak{h}^*$ dominant and regular and $\mu \in \Lambda$. Set $V = L(M(\lambda), L(\mu))$, which by 2.7 is a simple U module.

Lemma. *For each finite-dimensional $U(\mathfrak{g})$ module E one has*

$$\dim \mathrm{Hom}_{j(\mathfrak{g}\times\mathfrak{g})}(V \otimes (E \otimes \mathbb{C}), V) = \dim \mathrm{Hom}_{j(\mathfrak{g})}(L(\mu) \otimes E, L(\mu)).$$

Indeed

$$\begin{aligned} &\dim \mathrm{Hom}_{j(\mathfrak{g}\times\mathfrak{g})}(V \otimes (E \otimes \mathbb{C}), V) \\ &\quad = \dim \mathrm{Hom}_{j(\mathfrak{g}\times\mathfrak{g})}(L(M(\lambda), L(\mu) \otimes E), V), \qquad \text{by 2.5(i)}, \\ &\quad = \dim \mathrm{Hom}_{j(\mathfrak{g})}(L(\mu) \otimes E, L(\mu)), \qquad \text{by 2.7 and Remarks.} \end{aligned}$$

3.2. Let M be a maximal submodule of $M(\lambda)$ such that $VM = 0$. Then $l(V) = J(\mu)$ and $r(V) = \mathrm{Ann}\, M(\lambda)/M$ which by [11, 4.12] is a primitive ideal. Set $A_2 = U(\mathfrak{g})/r(V)$, which acts injectively in V by right multiplication and $A_1' = L(L(\mu), L(\mu))$ which acts injectively in V by left multiplication. In particular we have an embedding $A_1' \hookrightarrow \mathrm{End}_{A_2} V$.

Theorem. $\mathrm{End}_{A_2} V = A_1'$.

Fix $\theta \in \mathrm{End}_{A_2} V$. For each $X \in \mathfrak{g}$, define $\theta_X \in \mathrm{End}_{A_2} V$ through

$$\theta_X(x) := X\theta(x) - \theta(Xx), \qquad \text{for all } x \in V.$$

Obviously

$$\theta_X(x) = [X, \theta(x)] - \theta([X, x]). \tag{*}$$

Recalling that V is simple and locally $\mathfrak{k}$ finite, choose a finite-dimensional generating subspace V_0 which is $\mathfrak{k}$ stable. Then $V_0 U(\mathfrak{g}) = V$ and so $\theta(V_0) = 0$ implies $\theta = 0$. Yet by (*) we have

$$\theta_X(V_0) \subset [X, \theta(V_0)] + \theta(V_0)$$

from which it easily follows that $\theta \in L(V, V)$. Now let E_1 be the finite-dimensional $(\mathfrak{g} \times \mathfrak{g})$ module generated by θ. Since $\theta \in \mathrm{End}_{A_2} V$, it follows that E_1 takes the form $(E \otimes \mathbb{C})$, where E is a finite-dimensional $U(\mathfrak{g})$ module. Then applying 3.1 to E gives the assertion of the theorem.

3.3. Let Σ_λ denote the set of involutions of W_λ. Fix $\mu \in \bar{F}_\lambda$ (so μ is dominant) and $J \in \mathscr{X}_\Lambda$. By [11, 4.3(i)] $J = \mathrm{Ann}(M(\mu)/JM(\mu))$.

LEMMA. (i) *There exists a unique submodule* $M_J(\mu)$ *of* $M(\mu)$ *containing* $JM(\mu)$ *and maximal with respect to* $J = \mathrm{Ann}(M(\mu)/M_J(\mu))$.

(ii) *The map* $J \mapsto M(\mu)/M_J(\mu)$ *is a bijection of* $\mathscr{X}_\Lambda$ *onto the set of quasi-simple quotients of* $M(\mu)$.

(iii) *The unique simple submodule of* $M(\mu)/M_J(\mu)$ *takes the form* $L(\sigma\mu)$: $\sigma \in \Sigma_\lambda$, $\sigma\mu \in \hat{F}_{\sigma\lambda}$.

Since J is prime, $U(\mathfrak{g})/J$ is quasi-simple as a U module and we let V denote its unique simple submodule. We may write $V = V(-\sigma\mu, -\mu)$, for some $\sigma \in W_\lambda$ (see [4, Sect. II]). By [11, 4.6; 3, I, 4.1] we can assume without loss of generality that $\sigma^{-1}(R^0_{\sigma\mu} \cap R^+) \subset R^-$ (equivalently that $\sigma\mu \in \hat{F}_{\sigma\lambda}$) and that $\sigma^{-1}(R_\mu{}^0 \cap R^+) \subset R^-$. Moreover these conditions determine σ uniquely. By [4, II, 2] $\sigma^{-1} \in W_\mu{}^0 \sigma W_\mu{}^0$ and so by [11, 4.6] $\sigma^{-1} = \sigma$, that is $\sigma \in \Sigma_\lambda$. We may write $V = J'/J$ for some uniquely determined ideal J' of $U(\mathfrak{g})$ and we set $M = JM(\mu)$, $M' = J'M(\mu)$. Let M'' be a maximal submodule of M' containing M. By [11, 3.6, 4.3] $J' = L(M(\mu), M')$, $J = L(M(\mu), M)$, up to U module isomorphisms. Hence

$$V = J'/J = L(M(\mu), M')/L(M(\mu), M''), \qquad \text{by the simplicity of } V,$$
$$= L(M(\mu), M'/M''), \qquad \text{by [11, 3.5]}.$$

Yet

$$V = L(M(\mu), L(\sigma\mu)), \qquad \text{by [11, 4.7], and the choice of } \sigma.$$

Since M'/M'' is simple, it follows again by [11, 4.7] that $M'/M'' = L(\sigma\mu)$. Conversely by [11, 3.5, 4.7] it is clear that $[M'/M : L(\sigma\mu)] = [J'/J : V(-\sigma\mu, -\mu)] = 1$ and so M'/M admits a unique simple quotient. Let $M_J(\mu)$ denote the unique maximal submodule of M' containing M. Since by [7, 2.8; 11, 4.3] $2d(M(\mu)/M') = d(U(\mathfrak{g})/J') < d(U(\mathfrak{g})/J) = 2d(M(\mu)/M)$, the assertions of the lemma follow easily.

Remark. If μ is regular, then $M_J(\mu) = JM(\mu)$ and assertion (ii) of the lemma is just [11, 5.2].

3.4. Recall that λ is a *fixed* dominant and regular element of $\mathfrak{h}^*$. Let Σ_λ^0 denote the set of all $\sigma \in W_\lambda$ such that $L(\sigma\lambda)$ is the unique simple submodule of a quasi-simple quotient of $M(\lambda)$. By 3.3, $\Sigma_\lambda{}^0 \subset \Sigma_\lambda$ and card $X_{\hat{\lambda}} =$ card $\Sigma_\lambda{}^0 \leqslant$ card Σ_λ. This is an equality if and only if R_λ has only simple factors of type A_n [10, 4.4, Remark; 15, Theorem 6.5].

THEOREM. *For all* $\mu \in \bar{F}_\lambda$ *one has*

(i) $\mathscr{X}_\Lambda = \{J(\sigma\mu) : \sigma \in \Sigma_\lambda{}^0, \sigma\mu \in \hat{F}_{\sigma\lambda}\}$.

(ii) Fract $U(\mathfrak{g})/J(\sigma\mu) =$ Fract $L(L(\sigma\mu), L(\sigma\mu))$ *for all* $\sigma \in \Sigma_\lambda{}^0$ *satisfying* $\sigma\mu \in \hat{F}_{\sigma\lambda}$.

For each $\mu \in \bar{F}_\lambda$, let $T_\lambda{}^\mu : \mathscr{R}_{\hat{\lambda}} \to \mathscr{R}_\Lambda$ be the Jantzen translation functor [6, 2.10]. Suppose Q is a quasi-simple quotient of $M(\lambda)$ and let $L(\sigma\lambda)$: $\sigma \in \Sigma_\lambda{}^0$ denote its unique simple submodule. Given $\sigma\mu \in \hat{F}_{\sigma\lambda}$, then by [6, 2.11] we have $T_\lambda{}^\mu L(\sigma\lambda) = L(\sigma\mu)$. Then [6, 3.4, 2.18] adjointness and exactness of $T_\lambda{}^\mu$ imply that $T_\lambda{}^\mu Q$ is quasi-simple with $L(\sigma\mu)$ its unique simple submodule. Then (i) obtains from 3.3 and [1, 2.12].

For (ii) suppose that N is a submodule of $M(\mu)$ such that $Q := M(\mu)/N$ is quasi-simple. Set $J = \mathrm{Ann}\, Q$. By [11, 3.6, 4.3(i)], $U(\mathfrak{g})/J$ identifies with $L(M(\mu), M(\mu))/L(M(\mu), N)$, which by [11, 3.5] further identifies with $L(M(\mu), Q)$. Yet $L(Q, Q) = \{a \in L(M(\mu), Q) : aN = 0\} = \{a \in U(\mathfrak{g})/J : aN \subset N\} = U(\mathfrak{g})/J$. Now let L be the unique simple submodule of Q. By [10, 6.2] there exists an embedding of $L(Q, Q)$ into $L(L, L)$ and furthermore Fract $L(Q, Q) =$ Fract $L(L, L)$. Hence (ii).

4. LOCALIZATION

4.1. Throughout this section V denotes a smooth locally $\mathfrak{k}$-finite U module of finite length. The argument of [12, 3.3] gives the

LEMMA. *Let* V_0 *be a nonzero left* $U(\mathfrak{g})$ *submodule of* V. *Then* $d(V_0) = d(V)$.

4.2. Set $P_1 = l(V)$, $P_2 = r(V)$, $A_i = U(\mathfrak{g})/P_i : i = 1, 2$. Let S_i denote the set of regular elements of $A_i : i = 1, 2$.

LEMMA. (i) *If* $sv = 0$, $s \in S_1$, $v \in V$, *then* $v = 0$.

(ii) *If* $vs = 0$, $s \in S_2$, $v \in V$, *then* $v = 0$.

Consider (i). Set $V_0 = A_1 v$. Then V_0 is isomorphic to a quotient of $A_1/A_1 s$. Hence $d(V_0) \leqslant d(A_1/A_1 s) < d(A_1)$, by [12, 2.5(i)]. Thus (i) obtains from 4.1. Assertion (ii) follows similarly.

4.3. In general A_1, A_2 will *not* be prime rings. In this subsection we assume that they are subrings of prime rings A_1', A_2' such that each A_i' is locally $\mathfrak{k}$ finite, finitely generated as a U module (so in particular finitely generated as either a left or as a right $U(\mathfrak{g})$ module) and such that the identity of A_i is also the identity of A_i'. Under these conditions [12, Sect. 3] S_i is an Ore set for A_i and furthermore [12, 2.8] holds.

PROPOSITION. $S_1^{-1}V = VS_2^{-1}$.

Fix $v \in V$, $s \in S_2$; set $K = \{t \in A_1 : tv = 0\}$, $L = \{t \in A_1 : tv \in Vs\}$. Obviously $L \supset K$ and $Lv = A_1 v \cap Vs$. Through the embedding $A_1 v/(A_1 v \cap Vs) \hookrightarrow V/Vs$ and 4.2(i) we obtain $d(A_1 v/(A_1 v \cap Vs)) \leqslant d(V/Vs) < d(V) = d(A_1)$, where the last equality obtains from [7, Sect. 3]. Yet $A_1 v/(A_1 v \cap Vs) = A_1 v/Lv =$

$(A_1/K)/(L/K) = A_1/L$ and so $d(A_1/L) < d(L)$. Thus $L \cap S_1 \neq \varnothing$ by [12, 2.8]. This gives $VS_1^{-1} \subset S_2^{-1}V$. The reverse inclusion follows similarly.

4.4. *Fix* $\lambda \in \mathfrak{h}^*$ dominant and regular and $\mu \in \Lambda$. In the remainder of this section we take $V = L(M(\lambda), L(\mu))$ which by 2.7 is a simple and hence smooth U module. Adopt the notation of 4.2 and recall [11, 4.12] that P_1, P_2 are prime ideals. Set $A_1' = L(L(\mu), L(\mu))$ and recall [12, 3.7(ii)] that S_1 is an Ore set for A_1' and that $S_1^{-1}A_1' = \operatorname{Fract} A_1'$. Set $\mathscr{A}_2 = S_2^{-1}A_2$, $\mathscr{A}_1' = S_1^{-1}A_1'$, $\mathscr{V} = S_1^{-1}V = VS_2^{-1}$ (by 4.3). Then $\mathscr{V}$ is a left $\mathscr{A}_1'$ module and a right $\mathscr{A}_2$ module.

THEOREM. $\operatorname{End}_{\mathscr{A}_2}\mathscr{V} = \mathscr{A}_1'$.

Fix a finite-dimensional generating subspace V_0 of V which is $\mathfrak{k}$ stable and note that $V_0A_2 = V$. Fix $\theta \in \operatorname{End}_{\mathscr{A}_2}\mathscr{V}$. Then $\theta(V_0)$ is a finite-dimensional subspace of $\mathscr{V}$ and so there exists $s \in S_1$ such that $s\theta(V_0) \subset V$. Define $\theta' \in \operatorname{End}_{\mathscr{A}_2}\mathscr{V}$ through $\theta'(x) = s\theta(x)$ for all $x \in \mathscr{V}$. Then $\theta'(V_0) \subset V$ and so $\theta'(V) \subset V$. Hence by 3.2, there exists $a \in A_1'$ such that $\theta'(x) = ax$, for all $x \in V$. Thus $\theta(x) = (s^{-1}a)x$, for all $x \in V$ and hence by 4.3 for all $x \in \mathscr{V}$. Yet $s^{-1}a \in \mathscr{A}_1'$ which is just the assertion of the theorem.

4.5. Retain the notation and hypothesis of 4.4.

THEOREM. *There exists a field extension* K *of* $\mathbb{C}$ *and finite-dimensional* K-*vector spaces* E_1, E_2 *over* K *such that*

(i) $\mathscr{A}_1' = \operatorname{End}_K E_1$, *as a ring isomorphism,*

(ii) $\mathscr{A}_2 = \operatorname{End}_K E_2$, *as a ring isomorphism,*

(iii) $\mathscr{V} = \operatorname{Hom}_K(E_2, E_1)$, *as a left* $\mathscr{A}_1'$ *and a right* $\mathscr{A}_2$ *module isomorphism.*

Recall that $\mathscr{A}_2$ is a simple Artinian ring and fix K, E_2 to satisfy (ii). Recall that V is finitely generated as a right $U(\mathfrak{g})$ module and hence $\mathscr{V}$ is finitely generated as a right $\mathscr{A}_2$ module. Let n be the rank of $\mathscr{V}$ over $\mathscr{A}_2$. Then $\mathscr{V}$ is the direct sum of n copies of the unique simple module over $\mathscr{A}_2$. Since the endomorphism ring of that simple module is K it follows that $\operatorname{End}_{\mathscr{A}_2}\mathscr{V}$ identifies with $\operatorname{End}_K E_1$ for some K-vector space E_1 of dimension n and $\mathscr{V}$ identifies with $\operatorname{Hom}_K(E_2, E_1)$. This gives (iii) and (i) follows on taking 4.4 into account.

4.6. COROLLARY. $\operatorname{End}_{\mathscr{A}_1'}\mathscr{V} = \mathscr{A}_2$.

4.7. The action of U in V defines an embedding of $U/\operatorname{Ann} V$ into $L(V, V)$.

PROPOSITION. *Suppose* $U/\operatorname{Ann} V = L(V, V)$. *Then* $\operatorname{Fract} L(L(\mu), L(\mu)) = \operatorname{Fract} U(\mathfrak{g})/J(\mu)$.

Just as in 3.2 it follows that $\operatorname{End}_{A_1} V \subset L(V, V)$. Since $\operatorname{Ann} V = l(V)^{\vee} \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes (r(V))^{\vee}$ by say [11, 4.12] the hypothesis implies that $A_2 = \operatorname{End}_{A_1} V$. Just as in 4.4, it follows that $\mathscr{A}_2 = \operatorname{End}_{\mathscr{A}_1}\mathscr{V}$, and just as in 4.6 this gives $\mathscr{A}_1 = \operatorname{End}_{\mathscr{A}_1'}\mathscr{V}$. Hence by 4.4, $\mathscr{A}_1 = \mathscr{A}_1'$ as required.

Remarks. If R_λ has only simple factors of type A_n the conclusion of the proposition is never violated. This follows as in [10, Sect. 4] taking [15, Theorem 6.5] into account. Yet if B_λ has, say, two roots which define a subsystem of type B_2 the conclusion can be violated [10, 9.3]. This shows that a question of Kostant (cf. [10, Sect. 9]) has a negative answer even for the simple Harish-Chandra modules.

4.8. Recall that $\lambda \in \mathfrak{h}^*$ is dominant and regular.

THEOREM. *Fix* $w \in W_\lambda$. *Up to isomorphisms*

(i) *The Goldie field of* $L(L(\mu), L(\mu))$ *is independent of the choice of* $\mu \in \hat{F}_{w\lambda}$.

(ii) *The Goldie field of* $U(\mathfrak{g})/J(\mu)$ *is independent of the choice of* $\mu \in \hat{F}_{w\lambda}$.

(i) Let $V = V(-\mu, -\lambda)$. By 4.5 it is enough to show that $r(V)$ is independent of the choice of $\mu \in \hat{F}_{w\lambda}$. This follows from [11, 4.12]. Assertion (ii) follows from (i) and 3.4(ii).

Remark. It is to be expected that in both cases the Goldie field is isomorphic to a Weyl field of index $d(L(\mu))$ over $\mathbb{C}$. For R_ν of type A_n this holds whenever $J(\mu)$ is almost induced—that is, if $J(\mu)$ is the minimal prime over an induced ideal. Direct computation also verifies this conjecture if all the simple factors of $\mathfrak{g}$ have rank $\leqslant 2$.

5. THE ADDITIVITY PRINCIPLE AND GOLDIE RANK

5.1. Fix $\lambda \in \mathfrak{h}^*$ dominant and regular, $\mu \in \Lambda$, and set $V_0 = L(M(\lambda), L(\mu))$. Fix a finite-dimensional simple $U(\mathfrak{g})$ module E and set $V = V_0 \otimes (E \otimes \mathbb{C})$. Consider $L(\mu) \otimes E$ as a $U(\mathfrak{g})$ module for the diagonal action. By 2.5(i) we have $V = L(M(\lambda), L(\mu) \otimes E)$, up to a U module isomorphism. Set $M = L(\mu) \otimes E$. If $M = M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_{t+1} = 0$ is a composition series for M then $V = V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{t+1} = 0$, with $V_i = L(M(\lambda), M_i)$ is a composition series for V. Furthermore $d(V_i/V_{i+1}) = 2d(M_i/M_{i+1})$ (see [7, Sect. 2.3]) so in particular V is a smooth module. Again if the former is a quasi-composition series then so is the latter. Now set $A_1 = U(\mathfrak{g})/l(V)$, $A_2 = U(\mathfrak{g})/r(V)$, $A_1' = L(M, M)$ and let S_i denote the set of regular elements of A_i. It is clear that $r(V) = r(V_0)$ and so A_2 is a prime ring. In general, $l(V)$ is not a prime ideal; yet A_1 is a U subring of the finitely generated and locally $\mathfrak{k}$-finite prime ring A_1' which has the same identity as A_1. Thus the conditions of 4.3 are satisfied, so S_i is Ore in A_i and we have $\mathscr{V} := S_1^{-1}V = VS_2^{-1}$. Again by [12, 3.7] each $s \in S_1$ is regular in A_1', S_1 is an Ore subset for A_1' and $\mathscr{A}_1' := S_1^{-1}A_1' = \operatorname{Fract} A_1'$. Set $\mathscr{A}_i = S_i^{-1}A_i : i = 1, 2$.

5.2. Recall that V is finitely generated as a left A_1 module and that $d(V) = d(A_1)$.

LEMMA. *Let $V' \supset V''$ be A_1 submodules of V. Then $S_1^{-1}V' = S_1^{-1}V''$ if and only if $d(V'/V'') < d(V)$.*

Necessity follows as in [8, 5.2]. Suppose $d(V'/V'') < d(V)$. Set $I = l(V'/V'')$ (computed in A_1). Then by [7, Sect. 3] $d(A_1/I) = d(V'/V'') < d(V) = d(A_1)$, and so by [12, 2.8] we have $I \cap S_1 \neq \varnothing$ as required.

5.3. Consider V as a U module and $\mathscr{V}$ as an $\mathscr{A}_1$–$\mathscr{A}_2$ bimodule.

COROLLARY. *Let $V = V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{t+1} = 0$, be a quasi-composition series for V. Then $S_1^{-1}V_i = V_iS_2^{-1}$, $\mathscr{V} = S_1^{-1}V_1 \supsetneq S_1^{-1}V_2 \supsetneq \cdots \supsetneq S_1^{-1}V_{t+1} = 0$ is a composition series for $\mathscr{V}$ and the map $\{V_i\} \mapsto \{S_1^{-1}V_i\}$ sets up a bijection from the set of quasi-composition series for V to the set of composition series for $\mathscr{V}$.*

5.4. Define E_1, E_2, K as in 4.5 with respect to V_0 and recall that $r(V_0) = r(V)$, so then $\mathscr{A}_2 = \operatorname{End}_K E_2$. By elementary computation we have $\mathscr{V} = VS_2^{-1} = (V_0 \otimes (E \otimes \mathbb{C}))S_2^{-1} = V_0S_2^{-1} \otimes (E \otimes \mathbb{C}) = \operatorname{Hom}_K(E_2, E_1) \otimes (E \otimes \mathbb{C}) = \operatorname{Hom}_K(E_2, E_1 \otimes E)$ up to isomorphisms. Again by 2.5 and 4.5(i), we have $\mathscr{A}_1' = \operatorname{Fract} A_1' = \operatorname{Fract}(L(L(\mu), L(\mu)) \otimes \operatorname{End} E) = \operatorname{Fract} L(L(\mu), L(\mu)) \otimes \operatorname{End} E = \operatorname{End}_K E_1 \otimes \operatorname{End} E = \operatorname{End}_K(E_1 \otimes E)$, up to isomorphisms of U modules. Moreover the left action of $\mathscr{A}_1'$ in $\mathscr{V}$ just becomes the natural action of $\operatorname{End}_K(E_1 \otimes E)$ in $\operatorname{Hom}_K(E_2, E_1 \otimes E)$. This gives the following.

PROPOSITION. $\operatorname{End}_{\mathscr{A}_2}\mathscr{V} = \mathscr{A}_1'$.

5.5. Let $\mathscr{V} = \mathscr{V}_1 \supsetneq \mathscr{V}_2 \supsetneq \cdots \supsetneq \mathscr{V}_{t+1} = 0$ be a composition series for $\mathscr{V}$ considered as an $\mathscr{A}_1$–$\mathscr{A}_2$ bimodule. By the complete reducibility of $\mathscr{V}$ as a right $\mathscr{A}_2$ module we have the following.

LEMMA. *For each $i = 1, 2,\ldots, t$, $\operatorname{End}_{\mathscr{A}_2}(\mathscr{V}_i/\mathscr{V}_{i+1})$ is a simple Artinian ring and*

$$\sum_{i=1}^{t} \operatorname{rk}(\operatorname{End}_{\mathscr{A}_2}(\mathscr{V}_i/\mathscr{V}_{i+1})) = \operatorname{rk}(\operatorname{End}_{\mathscr{A}_2}\mathscr{V}).$$

5.6. Here and in the next three subsections we take an arbitrary element $M \in \mathscr{R}^0$ with $M = M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_{t+1} = 0$ a quasi-composition series for M. Set $L_i = M_i/M_{i+1}$. By [10, 6.2] each $L(L_i, L_i)$ is a prime, Noetherian ring.

THEOREM. *Assume that $L(M, M)$ is a prime ring. Then there exist $\mathscr{I} \subset \{1, 2,\ldots, t\}$ and $y_i \in \mathbb{N}^+ : i \in \mathscr{I}$ such that*

$$\operatorname{rk} L(M, M) = \sum_{i \in \mathscr{I}} y_i^{-1} \operatorname{rk} L(L_i, L_i).$$

This is proved in several stages given below. We remark that one *can* have $\mathscr{I} \subsetneq \{1, 2,\ldots, t\}$ (see [10, 8.10]); but we have no examples where $y_i > 1$.

5.7. Fix M satisfying the hypotheses of 5.6 and set $D = L(M, M)$, which is prime and Noetherian. Let M' be the maximal $U(\mathfrak{g})$ submodule of M such that $D_1 := \{a \in D: aM' = 0\} \neq 0$. Then M/M' is smooth by [10, 6.1(i)], the smoothness of M and the maximality of M'. Again $D_1M' = 0$ and $M' = \{m \in M: D_1m = 0\}$. Set $D_2 = \{a \in D: D_1a = 0\}$. Then $D_1D_2 = 0$, which gives $D_2M \subset M'$ and since M is a faithful D module one has $D_2 = \{a \in D: aM \subset M'\}$. Set $L = M/M'$ and let P denote the annihilator of M/M' computed in $U(\mathfrak{g})$. Then $D_1/(D_1 \cap D_2)$ identifies with a U subring of $L(L, L)$ which is non-zero. Furthermore from the maximality of M', it easily follows that both $U(\mathfrak{g})/P$ and $D_1/(D_1 \cap D_2)$ are prime rings. Set $D_3 = \{a \in D: D_1a \subset D_1\}$. Then $D_1D_3M' \subset D_1M' = 0$ and so $D_3M' \subset M'$ by choice of M'. Thus D_3/D_1 identifies with a U-subring of $L(M', M')$ containing $U(\mathfrak{g})/\operatorname{Ann} M'$. Also $D_3 = \{a \in D: aM' \subset M'\}$.

Set $A = U(\mathfrak{g})/\operatorname{Ann} M$ and let S denote the set of regular elements of A. Since D is prime the conclusions of [12] apply. In particular S is contained in the set of regular elements of D, it is Ore in both A and D with $S^{-1}D = \operatorname{Fract} D$. We have the following.

LEMMA. (i) *$D_1/(D_1 \cap D_2)$, D_3/D_1 are prime rings,*

(ii) *$\operatorname{rk} D = \operatorname{rk} D_1/(D_1 \cap D_2) + rk\, D_3/D_1$,*

(iii) *$d(M/M') = d(M)$.*

Consider $C := D \cap S^{-1}D_1$. Then C is a U submodule of D containing D_1 and so in particular is finitely generated as a left $U(\mathfrak{g})$ module. For each $b \in C$, there exists $s \in S$ such that $sb \in D_1$ and so $d(C/D_1) < d(A)$ (as in 4.3). Set $Q = l(C/D_1)$. Then by [8, 2.4(i)] we have $d(U(\mathfrak{g})/Q) = d(C/D_1) < d(A) = 2d(M)$. Set $N = CM'$. Then $QN = QCM' \subset D_1M' = 0$ and so $Q \subset \operatorname{Ann} N$ which gives $d(U(\mathfrak{g})/Q) \geqslant d(U(\mathfrak{g})/\operatorname{Ann} N) = 2d(N)$ by [7, 3.8]. Thus $d(N) < d(M)$ and so by smoothness $N = 0$, which gives $C = D_1$. Using the equality of left and right Gelfand–Kirillov dimension (recall 2.3) a similar argument gives $D \cap D_1S^{-1} = D_1$.

Given $s \in S$, $a \in D_1$ we can choose $t \in S$, $b \in D$ such that $at = sb$. Since $S \subset A$, we have $atM' \subset aM' = 0$ and so $b \in S^{-1}D_1 \cap D = D_1$. Hence $S^{-1}D_1 \subset D_1S^{-1}$. A similar argument gives the opposite inequality and so $S^{-1}D_1 = D_1S^{-1}$. The remaining steps in the proof of (i), (ii) follow exactly as [9, 4.2]. Finally $2d(M/M') = d(U(\mathfrak{g})/P) = d(A) = 2d(M)$ by [7, 2.8; 12, 3.9(i)]. Hence (iii).

5.8. Choose $M \in \mathscr{R}$ such that

(a) *M is simple as an $L(M, M)$ module,*

(b) *$\{a \in L(M, M): aM' = 0\} = 0$, for every nonzero $U(\mathfrak{g})$ submodule M' of M.*

LEMMA. (i) *M is smooth,*

(ii) $\operatorname{rk} L(M_0, M_0) = \operatorname{rk} L(M, M)$, *for every nonzero simple* $U(\mathfrak{g})$ *submodule* M_0 *of* M.

Set $D = L(M, M)$, $D_0 = L(M_0, M_0)$, $I = L(M, M_0)$. Then I is a right ideal of D. If $\overline{M} \subsetneq M$ is maximal, then $I = 0 \Rightarrow L(M/\overline{M}, M_0) = 0 \Rightarrow L(M_0, M/\overline{M}) = 0 \Rightarrow L(M, M)\, M_0 \subset \overline{M}$ contradicting (a). Thus $IM = M_0$ and by (b), I is essential. Since D is prime (and also Noetherian), I contains a regular element of D. Considered as a ring it easily follows that I does not admit an infinite direct sum of left ideals. Since as a subring of D it inherits the ascending chain condition on left annihilators, it follows that I is a left Goldie ring. Again

$$L(M/M_0, M_0) \subset L(M/M_0, M) = \{a \in L(M, M): aM_0 = 0\} = 0,$$

by (b). It follows that the U module homomorphism $\varphi: I \mapsto D_0$ defined by restriction is injective. Hence $\varphi(I)$ is a nonzero left ideal of the prime, Noetherian ring D_0. Since M_0 is a simple $U(\mathfrak{g})$ module and I is a U module it follows that $\varphi(I)$ is essential in D_0. Hence as a ring, I is right Goldie and so Goldie. As a subring of D_0 it satisfies Fract I = Fract D_0 and as a subring of D it satisfies Fract I = Fract D. Hence (ii). Again by [7, 2.8; 9, 2.1] and the essentiality of I we have

$$2d(M) = d(L(M, M)) = d(L(M, M_0)) = d(L(M_0, M_0)) = 2d(M_0).$$

Hence (i).

5.9. Return to the proof of 5.6. Let D' be any prime U-subring of $D := L(M, M)$ and $L_i' : i = 1, 2, \ldots, t$, the unique simple submodule of L_i. (Note that the L_i' are exactly the simple subquotients of M satisfying $d(L_i') = d(M)$.) By [10, 6.2 (iii)] we have $\operatorname{rk} L(L_i, L_i) = \operatorname{rk} L(L_i', L_i')$. With S as in 5.7 it follows from [12, 3.7] that S is Ore in both D and D' and that $S^{-1}D$ = Fract D, $S^{-1}D'$ = Fract D'. By [12, 3.8] $\operatorname{rk} D/\operatorname{rk} D' \in \mathbb{N}^+$. Hence the assertion of the theorem is equivalent to showing that there exists $\mathscr{I}' \subset \{1, 2, \ldots, t\}$ and $y_i' \in \mathbb{N}^+$ such that

$$\operatorname{rk} D' = \sum_{i \in \mathscr{I}'} y_i'^{-1} \operatorname{rk} L(L_i', L_i'),$$

for each D'. Then by 5.7 it is enough to prove this assertion in the case when $\{a \in D': aM' = 0\} = 0$ for every nonzero submodule M' of M. Moreover there is then no loss of generality in assuming that M is simple as a D' module and a fortiori simple as an $L(M, M)$ module. We are thus eventually reduced to the case when 5.8 applies and from its conclusion we obtain 5.6.

5.10. Now fix M as in 5.1 and define M_i, L_i as in 5.6. Set $V_i = L(M(\lambda), M_i)$. Then $V_i/V_{i+1} = L(M(\lambda), M_i/M_{i+1})$, by [11, 3.5]. Set $\mathscr{V}_i = V_i S_2^{-1}$ (notation 5.1). Then by 5.1 and 5.3, $\mathscr{V} = \mathscr{V}_1 \supsetneq \mathscr{V}_2 \supsetneq \cdots \supsetneq \mathscr{V}_{t+1} = 0$ is a composition series for $\mathscr{V}$. Since V_i/V_{i+1} is a faithful left $L(L_i, L_i)$ module, so is $\mathscr{V}_i/\mathscr{V}_{i+1} = (V_i/V_{i+1})\, S_2^{-1}$ (flatness of localization). This gives an embedding $L(L_i, L_i) \hookrightarrow \operatorname{End}_{\mathscr{A}_2}(\mathscr{V}_i/\mathscr{V}_{i+1})$ (notation 5.1) and hence (recalling [10, 6.2 (iii)] that $L(L_i, L_i)$ is prime) the

LEMMA. $\operatorname{rk} L(L_i, L_i) \leqslant \operatorname{rk} \operatorname{End}_{\mathscr{A}_2}(\mathscr{V}_i/\mathscr{V}_{i+1})$.

5.11. THEOREM. *For each $\mu \in \mathfrak{h}^*$ and each finite-dimensional simple $U(\mathfrak{g})$ module E one has*

$$\operatorname{rk} L(L(\mu), L(\mu)) \cdot \dim E = \sum_{i=1}^{t} \operatorname{rk} L(L_i', L_i'),$$

where the L_i' run over the simple subquotients of $L(\mu) \otimes E$ satisfying $d(L_i') = d(L(\mu))$.

This follows from 2.5, 5.4–5.6, 5.10 and [10, 6.2 (iii)].

5.12. Fix $\lambda \in \mathfrak{h}^*$ dominant and regular. For each $w \in W_\lambda$ we define functions q_w, p_w on $\hat{F}_{w\lambda}$ through $q_w(\mu) = \operatorname{rk} L(L(\mu), L(\mu))$, $p_w(\mu) = \operatorname{rk}(U(\mathfrak{g})/J(\mu))$.

COROLLARY. *For each $w \in W_\lambda$*

(i) *q_w extends to a polynomial on $\mathfrak{h}^*$,*

(ii) *p_w extends to a polynomial on $\mathfrak{h}^*$,*

(iii) *$q_w/p_w \in \mathbb{N}^+$.*

Assertion (i) follows from 5.11 exactly as in the proof of [14, Theorem 1.1]. Assertion (ii) follows from (i) and 3.4(ii). By [12, 3.8], we have $q_w(\mu)/p_w(\mu) \in \mathbb{N}^+$ and by [9, 2.3, 3.3] there exists $c \in \mathbb{N}^+$ depending only on $\mathfrak{g}$ such that $q_w(\mu)/p_w(\mu) \leqslant c$, for all $\mu \in \hat{F}_{w\lambda}$. Hence there exists a Zariski dense subset $\Omega \subset \hat{F}_{w\lambda}$ on which this quotient is a fixed positive integer. Then (iii) follows from (i), (ii).

Remark. See [10, 9.4] for an example when $q_w/p_w > 1$.

6. A LOWER BOUND ON card X_λ

6.1. Fix $\lambda \in \mathfrak{h}^*$ dominant and regular. For each $w \in W_\lambda$ we define a function r_w on $F_\lambda \times F_{w\lambda}$ through $r_w(\nu, w\mu) := e(V(-w\mu, -\nu))$: $\mu, \nu \in F_\lambda$. After Vogan [14, Theorem 1.1], r_w is a polynomial. Recall the definition of p_w: $w \in W_\lambda$ given in 5.12.

THEOREM. *$r_w(\nu, w\mu)/p_w(w\mu)\, p_{w^{-1}}(w^{-1}\nu)$ is a rational number independent of the choice of $\mu, \nu \in F_\lambda$.*

With $V = V(-w\mu, -\nu)$, define $\mathscr{V}$, $\mathscr{A}_1'$, $\mathscr{A}_2$ as in 4.5. Recall that $e(V) = e'(V)$ by 2.3. By 4.5. $\mathscr{V}$ is a direct sum of $q_w(w\mu)$ isomorphic simple right $\mathscr{A}_2$ modules. By [11, 4.12] $\mathscr{A}_2$ depends only on ν and so taking [8, 5.2] into account it follows that $r_w(\nu, w\mu)/q_w(w\mu)$ is independent of $\mu \in F_\lambda$. Again $V(-w\mu, -\nu) = V(-\mu, -w^{-1}\nu)$, up to isomorphism [3, I, 4.1] and so $r_w(\nu, w\mu) := e(V(-w\mu, -\nu)) = e(V(-\mu, -w^{-1}\nu)) = e(V(-w^{-1}\nu, -\mu)) =: r_{w^{-1}}(\mu, w^{-1}\nu)$. Hence by the first part, $r_w(\nu, w\mu)/q_{w^{-1}}(w^{-1}\nu)$ is independent of ν. Since r_w, q_w, $q_{w^{-1}}$ are all polynomials the assertion of the theorem follows on taking 5.12(iii) into account.

6.2. For each $B' \subset B_\lambda$, define $p_{B'} \in S(\mathfrak{h})$ through

$$p_{B'} := \prod_{\alpha \in \mathbb{N}B' \cap R^+} \alpha.$$

After MacDonald [13] the W_λ module generated by $p_{B'}$ is simple and we denote by $\Omega_{R,\lambda}$ the subset of $\hat{W}_\lambda$ of all such irreducible representations.

THEOREM. *For all $\lambda \in \mathfrak{h}^*$ dominant and regular, one has*

$$\operatorname{card} \mathscr{X}_{\hat{\lambda}} \geqslant \sum_{\sigma \in \Omega_{R,\lambda}} \dim \sigma.$$

This follows from 6.1 and [5, Lemmas 5, 8].

Remarks. In the special case for which there exists $w \in W$ such that $wB_\lambda \subset B$, the assertion is just [9, 5.2]. In general the above inequality is strict (see [9, 5.4]). However, if R_λ has only type A_n factors then $\Omega_{R,\lambda} = \hat{W}_\lambda$ and so equality results from Duflo's upper bound [4, Proposition 9], namely, $\operatorname{card} \mathscr{X}_{\hat{\lambda}} \leqslant \operatorname{card} \Sigma_\lambda$. This last equality for general B_λ was first obtained independently by Jantzen and by Vogan [15, Theorem 6.5]. The present method of proof is quite different and goes back to the ideas of [8]. It completes—presumably to Duflo's satisfaction—the last remark of [5].

APPENDIX: Index of Notation

Symbols appearing frequently throughout the text are given below in order of appearance. In particular, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$.

1.1. $T^{-1}A$, rk A
1.2. Prim A, $S(V)$, V^*, $U(\mathfrak{a})$, $Z(\mathfrak{a})$, $\hat{G}$
1.3. $\mathfrak{g}$, $\mathfrak{h}$, R, R^+, B, s_α, W, ρ, X_α, $\mathfrak{n}^+$, $\mathfrak{n}^-$, $\mathfrak{b}$, e_λ, $M(\lambda)$, $L(\lambda)$, $J(\lambda)$
1.4. R_λ, R_λ^+, B_λ, W_λ
1.5. $\hat{\lambda}$, $\mathscr{X}_{\hat{\lambda}}$
1.6. $P(R)$, Λ
2.1. ${}^t u$, $\check{u}$, U, j, $\mathfrak{k}$
2.2. $\mathscr{R}$, $\mathscr{R}_{\hat{\lambda}}$, $L(M, N)$
2.3. $d(L)$, $e(L)$, $l(L)$, $r(L)$
2.4. $\mathscr{R}^0$
2.6. R_λ^0, W_λ^0, w^λ, $\bar{F}_\lambda$, $\hat{F}_\lambda$, F_λ
2.7. $L(\lambda, \mu)$, $V(\lambda, \mu)$
3.3. Σ_λ
5.12. q_w, p_w
6.1. r_w
6.2. $p_{B'}$, $\Omega_{R,\lambda}$

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