References

1. Andreotti, A., Mayer, A.L.: On period relations for abelian integrals on algebraic curves. Ann. Sc. Norm. Sup. Pisa **21**, 189–238 (1967)
2. Barth, W.: Verallgemeinerung des Bertinischen Theorems in Abelschen Mannigfaltigkeiten. ibid. **23**, 317–330 (1969)
3. Barth, W.: Some properties of stable rank-2 vector bundles on $\mathbb{P}_n$. Math. Ann. **226**, 125–150 (1977)
4. Beauville, A.: Prym varieties and the Schottky problem. Invent. Math. **41**, 149–196 (1977)
5. Goresky, M., MacPherson, R.: Intersection homology theory. Topology **19**, 135–162 (1980)
6. Matsusaka, T.: On a characterization of a Jacobian variety. Mem. Coll. Sci. Kyoto Ser. A. **23**, 1–19 (1959)
7. Mumford, D.: Curves and their Jacobians. Ann. Arbor: U. of Michigan Press, 1975
8. Ran, Z.: On subvarieties of abelian varieties. Invent. Math. **62**, 459–479 (1981)
9. Weil, A.: Zum Beweis des Torellischen Satzes. Nachr. Akad. Wiss. Göttingen, 33–53 (1957)
10. Zariski, O.: Algebraic Surfaces. 2d ed. Berlin-Heidelberg-New York: Springer 1971

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Inventiones mathematicae

The Enright Functor on the Bernstein-Gelfand-Gelfand Category $\mathcal{O}$

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1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $R \subset \mathfrak{h}^*$ the set of non-zero roots, $R^+ \subset R$ a system of positive roots, $B \subset R^+$ a set of simple roots, $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ the triangular decomposition of $\mathfrak{g}$ corresponding to B and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$.

On a certain category of $\mathfrak{g}$ modules, Enright ([8], Sect. 3) defined for each $\alpha \in B$ a functor C_α. This is constructed by first considering a $\mathfrak{g}$ module M as a module for the $sl(2)$ subalgebra defined by α and then defining C_α on each indecomposable summand (which must be one of a very limited number of easily described forms). Remarkably one can find "matching conditions" which ensure that the resulting $sl(2)$ module $C_\alpha M$ can be given the structure of a $\mathfrak{g}$ module. Furthermore the functor C_α has a number of attractive properties: it is covariant, left exact, takes Verma modules to Verma modules, commutes with the translation functor and of most interest admits the lifting of a contravariant form from M to $C_\alpha M$.

Each of the above results was established by long calculations involving in each case a detailed verification of the appropriate matching condition. In this some simplification was introduced Deodhar [4] who recognized that C_α was just a subfunctor of a certain localization functor. In particular he was thus able to prove (a conjecture of Enright) that restricted to $\mathfrak{n}^-$ free modules (the only modules considered by either Enright or Deodhar) the C_α: $\alpha \in B$ generate a "singular Hecke algebra". Yet he was not able (or did not attempt) to establish the possibility of lifting a contravariant form, which is certainly the most interesting and difficult property.

In this paper, we establish a definition of the Enright functor in the so-called $\mathcal{O}$ category for which all of the above stated properties become easy and natural verifications. A main tool is an equivalence of categories theorem ([2], Sects. 5, 6; [9], 1.16) which is itself a consequence of the classical Kostant multiplicity theorem (cf. [5], 8.3.8) for the $\mathfrak{g}$ module structure of the space of harmonic elements of $U(\mathfrak{g})$.

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The main point of the paper was to take up a remark of Gabber who suggested that because the contravariant form on M can be lifted to $C_\alpha M$ it should be possible to use the Enright functor to settle the Jantzen conjecture ([11], 5.18). The importance of doing this is that the latter is known to imply ([10], Sect. 4) the Kazdhan-Lusztig conjecture, and even a significant refinement of it which for example is needed to compute the Duflo set ([15], Sect. 4).

What actually turns out is that this construction leads only to an equivalent orthogonality conjecture which has nevertheless the advantage of only involving a single Verma module, whereas Jantzen's conjecture involved a pair. At the same time a number of further important properties of Enright's functor were discovered.

Now the discussion of the Jantzen conjecture requires the replacement of the base field by a certain commutative ring. It was therefore thought better to reserve this paper for the more elementary case of a field, where in any case less strong restrictions on the modules can be imposed.

As by-products the following additional results are obtained. We are able to precisely compute $C_\alpha L$ when L is a simple object in $\mathcal{O}$. From this we can discuss how the Hecke algebra property fails on the whole of $\mathcal{O}$. Again the lifting of the contravariant form leads to a purely algebraic construction of the Kunze-Stein intertwining operators for the corresponding complex Lie group. This in turn gives the Duflo-Zelobenko ([6], V, 1.10) four-term exact sequence for principal series modules. Moreover assuming that the Kazhdan-Lusztig is true (it nearly is! [1, 3]) we are able to compute the restriction of the Kunze-Stein intertwining operators to certain submodules which occur in Kostant's problem ([12], Sects. 1, 2).

Finally we remark that in the above we need not take α to be an absolutely simple root; indeed C_α is defined for any root α. Furthermore the equivalence of categories theorem allows one to replace a relatively simple root by an absolutely simple one. This is important for example in studying Jantzen's conjecture.

2. The Enright Functor

The notations and conventions of this paper are those of [9] refered to as G.J. Occasionally less standard terminology is redefined and of course some new definitions will be needed. We fix throughout this paper $\lambda \in \mathfrak{h}^*$ dominant and regular. Set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\forall \alpha \in R$.

2.1. Let $\mathcal{O}$ (notation G.J, 1.4) denote the Bernstein-Gelfand-Gelfand category of $U(\mathfrak{g})$ modules. Set $\Lambda = \lambda + P(R)$ (notation G.J, 1.12) and let Λ^+ denote the set of dominant elements of Λ.

Let $\mathcal{O}_\Lambda$ denote the full subcategory of $\mathcal{O}$ consisting of all those modules M satisfying $\Omega(M) \subset \Lambda$ (notation G.J, 1.10).

For each $\alpha \in B_\lambda$ (notation G.J, 1.2) we define the Enright functor C_α on $\mathcal{O}_\Lambda$ through

$$C_\alpha M = L(M(s_\alpha \lambda), M) \otimes_{U(\mathfrak{g})} M(\lambda) \quad \text{(notation G.J, 1.5).}$$

It is obviously covariant. Although C_α could be defined for any $\alpha \in R$ it is the case of $\alpha \in B_\lambda$ (i.e. relatively simple roots) which is important and has the most interesting properties. To avoid confusion we restrict ourselves to this case.

2.2. Let $\mathscr{H}$ denote the category of Harish-Chandra modules for the pair $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{k})$ (notation G.J, 1.3, 1.7) and $\mathscr{H}_{\hat{\lambda}}$ the full subcategory of $\mathscr{H}$ consisting of those modules on which $1 \otimes Z(\mathfrak{g})$ acts through $1 \otimes z \to \chi_\lambda(\check{z})$ (notation G.J, 1.16). Then the functor $T: \mathcal{O}_\Lambda \to \mathscr{H}_{\hat{\lambda}}$ defined by $T(M) = L(M(\lambda), M)$ is exact (G.J, 1.12).

Furthermore (G.J, 1.16) T has left adjoint T' the functor $T'(V) = V \otimes_{U(\mathfrak{g})} M(\lambda)$ and the maps $\mathscr{I}d_{\mathscr{H}_{\hat{\lambda}}} \to TT'$, $T'T \to \mathscr{I}d_{\mathcal{O}_\Lambda}$ are isomorphisms. In particular T' is exact and so C_α is left exact. Again since $\mathscr{H}_{\hat{\lambda}}$ depends only the orbit $\hat{\lambda}$ of λ under the Weyl group W, we can replace λ by any dominant element of $\hat{\lambda}$. This means that C_α: $\alpha \in B_\lambda$ is equivalent to a $C_{\alpha'}$ for some $\alpha' \in B$ (the required combinatorial property of W being just say ([7], I, Lemme 2)),

We may also regard C_α as being defined through the U module isomorphism (notation G.J, 1.5).

$$L(M(\lambda), C_\alpha M) \cong L(M(s_\alpha \lambda), M). \tag{$*$}$$

From this using translation principles (cf. G.J, 1.13) it follows that C_α does not depend (to within $P(R)$ translates) on the choice of λ dominant and regular. (This does *not* necessarily imply the independence described previously.)

2.3. Let $\mathscr{E}$ denote the category of all finite dimensional $U(\mathfrak{g})$ modules.

Lemma. *For each $M \in \operatorname{Ob} \mathcal{O}_\Lambda$, $E \in \mathscr{E}$, we have $E \otimes M \in \operatorname{Ob} \mathcal{O}_\Lambda$ and $C_\alpha(E \otimes M) \cong E \otimes C_\alpha(M)$.*

The first part is well-known. For the second part observe that

$$\begin{aligned} L(M(\lambda), E \otimes C_\alpha M) &\cong \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, E) \otimes L(M(\lambda), C_\alpha M) \\ &\cong \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, E) \otimes L(M(s_\alpha \lambda), M), \quad \text{by } (*) \\ &\cong L(M(s_\alpha \lambda), E \otimes M)), \end{aligned}$$

and apply T'.

2.4. The embedding $M(s_\alpha \lambda) \hookrightarrow M(\lambda)$ gives a natural U module homomorphism $L(M(\lambda), M) \to L(M(s_\alpha \lambda), M)$ and hence a natural $U(\mathfrak{g})$ module homomorphism $M \to C_\alpha M$. The latter is injective if and only if $L(M(\lambda)/M(s_\alpha \lambda), M) = 0$. Recall that each $M \in \operatorname{Ob} \mathcal{O}_\Lambda$ has finite length with simple subquotients amongst the $L(\mu)$: $\mu \in \Lambda$ (notation G.J, 1.2). Call $L(\mu)$ α-finite if $(\alpha^\vee, \mu) \in \mathbb{N}^+$ and call $M \in \operatorname{Ob} \mathcal{O}_\Lambda$ α-finite if all its simple subquotients are α-finite. When α is absolutely simple (i.e. when $\alpha \in B$) M is α-finite if and only if $X_{-\alpha}$ (notation G.J, 1.1) acts locally finitely on M. Equivalently when M is a direct sum of finite dimensional $\mathfrak{r}_\alpha := \mathfrak{h} \oplus \mathbb{C} X_\alpha \oplus \mathbb{C} X_{-\alpha}$ modules.

Lemma. $\operatorname{Ker}(M \to C_\alpha M)$ *is the largest α-finite submodule of M.*

Suppose M is α-finite. We show that $C_\alpha M = 0$, equivalently that $L(M(s_\alpha \lambda), M) = 0$. It suffices to establish the latter when $M = L(w\mu)$: $w \in W_\lambda$, $\mu \in \Lambda^+$.

By the hypothesis of α-finiteness $L(w\mu)$ is a quotient of $M(w\mu)/M(s_\alpha w\mu)$. In the proof of (G.J, 3.2) (taking $w_1=s_\alpha$, $w_2=w$ and interchanging λ,μ) we showed that for any $E\in\mathscr{E}$, $L(s_\alpha\lambda)$ is not a subquotient of $E\otimes M(w\mu)/M(s_\alpha w\mu)$ so a fortiori it is not a subquotient of $E\otimes L(w\mu)$. Hence $L(M(s_\alpha\lambda), L(w\mu))=0$, as required.

Let N be a submodule of M. By functoriality and the left exactness of C_α we have a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_\alpha N \arrow[r] & C_\alpha M \arrow[r] & C_\alpha(M/N) \arrow[r] & {} \end{tikzcd}$$

Let K be the largest α-finite submodule of M. By our first observation $C_\alpha K=0$ and so taking $K=N$ in the diagram we obtain $K\subset\operatorname{Ker}(M\to C_\alpha M)$. Again for the opposite inclusion it suffices to show that the map $M/K\to C_\alpha(M/K)$ is injective. As noted above this holds if and only if $L(M(\lambda)/M(s_\alpha\lambda), M/K)=0$.

Observing that M/K has no α-finite submodules the latter assertion obtains exactly as in the proof of (G.J, 3.6(i)).

2.5. Lemma. *For all $\mu\in\Lambda^+$, $w\in W_\lambda$, $\alpha\in B_\lambda$ one has*

$$C_\alpha M(w\mu)\cong\begin{cases} M(s_\alpha w\mu): & (\alpha, w\mu)<0,\\ M(w\mu): & \textit{otherwise}.\end{cases}$$

This follows from 2.2. (*) and taking $w_1\in\{1,s_\alpha\}$, $w_2\in\{w, s_\alpha w\}$ in (G.J, 3.4).

2.6. Let $a\to\check{a}$ (resp. $a\to{}^t a$) denote the principal (resp. Chevalley) antiautomorphism of $U(\mathfrak{g})$ (notation G.J, 1.3). Let M be a $U(\mathfrak{g})$ module. Its dual M^* is given a $U(\mathfrak{g})$ module structure through

$$(af, m)=(f,\check{a}m),\quad a\in U(\mathfrak{g}),\quad f\in M^*,\quad m\in M.$$

Again observe that $\tau\colon a\to{}^t\check{a}$ is an automorphism of $U(\mathfrak{g})$. We define M^τ to be the $U(\mathfrak{g})$ module with the same underlying vector space as M; but with the action $(a,m)\to\tau(a)m$. Through the definition (G.J, 1.3, 1.5) of the action of U on $\operatorname{Hom}_{\mathbb{C}}(M,N)$ we have the isomorphism $\operatorname{Hom}_{\mathbb{C}}(M,N)\cong N^\tau\otimes M^*$ of U modules. For $E,F\in\mathscr{E}$ there results the U module isomorphism $\operatorname{Hom}_{\mathbb{C}}(M,N)\otimes(F^\tau\otimes E^*)\cong\operatorname{Hom}(M\otimes E, N\otimes F)$. Similar definitions apply when $\mathfrak{g}$ is replaced by $\mathfrak{g}\times\mathfrak{g}$ and so in particular τ is defined on U modules.

2.7. Given $M\in\operatorname{Ob}\mathcal{O}_\Lambda$, let $\delta(M)$ denote the submodule of $(M^*)^\tau$ of all $\mathfrak{h}$ finite elements. One easily checks that $\delta(M)\in\operatorname{Ob}\mathcal{O}_\Lambda$ and that $\dim\delta(M)_\mu=\dim M_\mu$ (notation G.J, 1.2) for all $\mu\in\Lambda$. This last relation shows that the contravariant functor δ on $\mathcal{O}_\Lambda$ is exact and involutory. One has $\delta(L(\mu))\cong L(\mu)$, $\mu\in\Lambda$ and in particular $E^*\cong E^\tau$, $\forall E\in\mathscr{E}$.

2.8. Define an isomorphism η on U through $\eta(X,Y)=(Y,X)$, $X,Y\in\mathfrak{g}$. Let $M,N\in\operatorname{Ob}\mathcal{O}_\Lambda$. It is immediate that $(L(M\otimes N)^*)^\eta\cong L(N\otimes M)^*$ (notation G.J, 1.5). By (G.J, 1.9) this gives a U module isomorphism.

$$L(M,N)\cong L(\delta N,\delta M)^\eta. \tag{$**$}$$

Given $V\in\operatorname{Ob}\mathscr{H}_{\hat{\lambda}}$ we let $\delta(V)$ denote the largest $\mathfrak{k}$ locally finite submodule of $V^{*\tau}$.

One checks that $\delta(V)\in\operatorname{Ob}\mathscr{H}_{\hat{\lambda}}$ and that $[V:E]=[\delta(V):E]$, $E\in\mathscr{E}$ (notation G.J, 1.11). It follows that the contravariant functor δ on $\mathscr{H}_{\hat{\lambda}}$ is exact and involutory.

Proposition. *For each $\mu\in\Lambda$, $N\in\operatorname{Ob}\mathcal{O}_\Lambda$ we have an homomorphism ζ_μ^N: $L(M(\mu),\delta N)\to\delta(L(M(\mu),N))$ of U modules. It is an isomorphism if μ is dominant.*

Let $\mathbb{C}$ denote the trivial one-dimensional $\mathfrak{k}$-module. By Frobenius reciprocity one has $[L(-\mu,-\mu):\mathbb{C}]=1$ (G.J, 1.5, 1.12). We let P denote the projection of $L(-\mu,-\mu)$ onto $\mathbb{C}$ defined by the direct sum decomposition of $L(-\mu,-\mu)$ into $\mathfrak{k}$-isotypical components. Clearly P is a $\mathfrak{k}$-module map and so $P(j(X)v)=0$ (notation G.J, 1.3) for all $X\in\mathfrak{g}$, $v\in L(-\mu,-\mu)$.

Through (G.J, 1.9) we may identify $L(-\mu,-\mu)$ with $L(M(\mu),\delta M(\mu))$. With respect to the natural $U(\mathfrak{g})-U(\mathfrak{g})$ bimodule structure on $L(M(\mu),\delta M(\mu))$ we have $P(av)=P(va)$, $a\in U(\mathfrak{g})$, $v\in L(M(\mu),\delta M(\mu))$.

We prove the existence of ζ_μ^N by constructing a contravariant pairing $L(M(\mu),N)\times L(M(\mu),\delta N)\to\mathbb{C}$ which we also denote by ζ_μ^N. To do this recall that by ($**$) we have an isomorphism

$$\varphi\colon L(M(\mu),N)\xrightarrow{\sim} L(\delta N,\delta M(\mu))^\eta.$$

Yet $L(\delta N,\delta M(\mu))\,L(M(\mu),\delta N)\subset L(M(\mu),\delta M(\mu))$ and so it makes sense to define $\zeta_\mu^N(v,w):=P(\varphi(v)w)$, $v\in L(M(\mu),N)$, $w\in L(M(\mu),\delta N)$.

For all $a,b\in U(\mathfrak{g})$ we have

$$\begin{aligned}\zeta_\mu^N(((a\otimes b)\cdot v),w)&=P(\varphi((a\otimes b)\cdot v)\,w)\\ &=P(((b\otimes a)\cdot\varphi(v))\,w)\\ &=P({}^t\check{b}\,\varphi(v)\,\check{a}\,w)\\ &=P(\varphi(v)\,\check{a}\,w\,{}^t\check{b})\\ &=P(\varphi(v)\,(({}^t a\otimes{}^t b)\cdot w))\\ &=\zeta_\mu^N(v,({}^t a\otimes{}^t b)\,w)\end{aligned}$$

which establishes contravariance and so the first part of the proposition.

Now suppose $\mu\in\Lambda^+$ and consider $L(M(\mu),\delta N)$. By ([13], 4.3) every submodule of $L(M(\mu),\delta N)$ takes the form $L(M(\mu),\delta Q)$ for some quotient Q of N. Since $\delta(M(\mu))$ is an injective module, the natural map $L(\delta N,\delta M(\mu))\to L(\delta Q,\delta M(\mu))$ is surjective and so the restriction of ζ_μ^N to $L(M(\mu),\delta Q)$ coincides with ζ_μ^Q. Consequently it is enough to establish the injectivity of ζ_μ^N when N is a simple module. In this we can obviously assume that $L(M(\mu),\delta N)\neq 0$. It then remains to shows that $P(\varphi(v)\,L(M(\mu),\delta N))=0$: $v\in L(M(\mu),N)$ implies $v=0$. Set $V=U(\mathfrak{g})\,\varphi(v)\,L(M(\mu),N)$. Through the $\mathfrak{k}$-invariance of P it is enough to show that $v\neq 0\Rightarrow P(V)\neq 0$.

Now δN is simple and $L(M(\mu), \delta N) \neq 0$ and so $\varphi(v) L(M(\mu), \delta N) M(\mu) = \varphi(v) \delta N \neq 0$. Consequently V is a non-zero U submodule of $L(-\mu, -\mu)$. Since $L(\mu) \cong \delta(L(\mu))$ is an essential and simple submodule of $\delta(M(\mu))$ it follows from ([13], 4.3) that $L(M(\mu), L(\mu))$ is an essential and simple submodule of $L(M(\mu), \delta M(\mu)) \cong L(-\mu, -\mu)$. Obviously $[L(M(\mu), L(\mu)) : \mathbb{C}] = 1$ (in fact this U module naturally coincides with $U(\mathfrak{g})/\operatorname{Ann} L(\mu)$) and so $P(V) \neq 0$.

To show that ζ_μ^N is surjective it is now enough to show that $L(M(\mu), \delta N)$ and $\delta(L(M(\mu), N))$ are isomorphic as k-modules. Take $E \in \mathscr{E}$ simple. Then

$$\begin{aligned}[\delta(L(M(\mu), N)) : E] &= [L(M(\mu), N) : E] \\ &= \dim \operatorname{Hom}_{\mathfrak{g}}(M(\mu) \otimes E, N), \quad \text{by (G.J, 1.6).}\end{aligned}$$

Since μ is dominant, $M(\mu)$ and hence $M(\mu) \otimes E$ are projective in $\mathcal{O}_\Lambda$. Thus the last expression above depends only the multiplicity of the simple subquotients of N and so by (G.J, 1.8) equals $\dim_{\mathfrak{g}}(M(\mu) \otimes E, \delta N)$ which in turn equals $[L(M(\mu), \delta N) : E]$. This proves the required isomorphism and hence the surjectivity of ζ_μ^N.

2.9. Lemma. *For all $N \in \operatorname{Ob} \mathcal{O}_\Lambda$ one has $L(M(\lambda), N) M(\lambda) = N$.*

Set $N' = L(M(\lambda), N) M(\lambda)$. It is clear that N' is a submodule of N and that the natural embedding $L(M(\lambda), N') \hookrightarrow L(M(\lambda), N)$ is an isomorphism. Since λ is dominant, $M(\lambda)$ must be projective and so the latter gives $L(M(\lambda), N/N') = 0$. Since λ is regular, T is faithfull (G.J, 1.16, (iii)) and so $N/N' = 0$, that is $N' = N$.

2.10. Fix $M \in \operatorname{Ob} \mathcal{O}_\Lambda$, $\alpha \in B_\lambda$. If the natural homomorphism $M \to C_\alpha M$ is an embedding we call M α-free.

By 2.4 this is equivalent to saying that every simple submodule $M(\mu)$ of M satisfies $(\alpha^\vee, \mu) \notin \mathbb{N}^+$. We say that M is B_λ-free if M is α-free for each $\alpha \in B_\lambda$. Equivalently every simple submodule of M has the form $M(\mu)$: $-\mu \in \Lambda^+$. Hence M is B_λ-free if and only if M is $U(\mathfrak{n}^-)$ free in the sense of ([8], Sect. 3).

Lemma. *Suppose M is α-free. Then $\operatorname{Ann} M = \operatorname{Ann} C_\alpha M$.*

We have $\operatorname{Ann} M \supset \operatorname{Ann} C_\alpha M$ as a trivial consequence of the hypothesis. For the opposite inclusion set $V = L(M(\lambda), C_\alpha M)$ and let $l(V)$ denote its annihilator as a left $U(\mathfrak{g})$ module. Taking $N = C_\alpha M$ in 2.9 gives $l(V) = \operatorname{Ann} C_\alpha M$, whereas by 2.2(*) we have $l(V) \supset \operatorname{Ann} M$ which proves the assertion.

2.11. Fix $N \in \operatorname{Ob} \mathcal{O}_\Lambda$, $\alpha \in B_\lambda$. Let $D_\alpha^+ N$ be the smallest submodule of N such that $N/D_\alpha^+ N$ is α-finite. Again set $D_\alpha^- N = N/\operatorname{Ker}(N \to C_\alpha N)$ and $D_\alpha N := D_\alpha^+(D_\alpha^- N) \cong D_\alpha^-(D_\alpha^+ N)$. A map $f: N \to M$ defines by restriction a map $f^+: D_\alpha^+ N \to D_\alpha^+ M$ and by functoriality a map $\operatorname{Ker}(N \to C_\alpha N) \to \operatorname{Ker}(M \to C_\alpha M)$, hence a map f^-: $D_\alpha^- N \in D_\alpha^- M$ (so that the appropriate diagrams commute). It follows that D_α, D_α^+, D_α^- are covariant functors on $\mathcal{O}_\Lambda$. If f is injective (resp. surjective) then so are $f^\pm$; but none of these functors is either left or right exact. All are idempotent and D_α, D_α^+ coincide on α-free modules.

Lemma. *For all $\mu \in \Lambda^+$, $w \in W_\lambda$, $\alpha \in B_\lambda$ one has*

$$D_\alpha M(w\mu) \cong \begin{cases} M(w\mu): & (\alpha, w\mu) \leq 0, \\ M(s_\alpha w\mu): & \text{otherwise.} \end{cases}$$

Recall first that any Verma module is $U(\mathfrak{n}^-)$ free and hence α-free. Suppose $(\alpha, w\mu) > 0$. Then $(\alpha^\vee, w\mu) \in \mathbb{N}^+$ and so $M(s_\alpha w\mu)$ is a submodule of $M(w\mu)$ and the quotient is α-finite. On the other hand $M(s_\alpha w\mu)$ admits a unique simple quotient namely $L(s_\alpha w\mu)$ and this is α-free. These observations prove the lemma.

2.12. Assume $\alpha \in B_\lambda \cap B$ and let $\tilde{C}_\alpha$ denote the Enright functor as defined by Deodhar ([4], Sect. 2).

We show that C_α and $\tilde{C}_\alpha$ coincide on α-free objects in $\mathcal{O}_\Lambda$. Given $M, N \in \operatorname{Ob} \mathcal{O}_\Lambda$ which are α-free we have $L(\tilde{C}_\alpha M, \tilde{C}_\alpha N) \xrightarrow{\sim} L(D_\alpha M, D_\alpha N)$ defined by restriction (see [14], 5.5 noting difference of notation). Taking $M = M(\lambda)$ we obtain from ([14], 5.3) and 2.11 that

$$L(M(\lambda), \tilde{C}_\alpha N) \xrightarrow{\sim} L(M(s_\alpha \lambda), D_\alpha N) \xrightarrow{\sim} L(M(s_\alpha \lambda), N)$$

where for the last isomorphism we have used that $L(M(s_\alpha \lambda), Q) = 0$ if Q is α-finite. By 2.2(*) and applying T' we obtain $\tilde{C}_\alpha N \cong C_\alpha N$ as required.

3. Coherent Continuation

In this section we shall determine the precise extent to which C_α fails to be right exact. This allows us to obtain a number of additional properties of the C_α. The analysis relates C_α to the functor θ_α (notation G.J, 2.1) of coherent continuation across the α-wall. Combined with 2.8 it leads to a natural way of lifting contravariant forms.

3.1. For each $\mu \in \Lambda^+$, let T_λ^μ, T_μ^λ denote the Jantzen translation functors on $\mathcal{O}_\Lambda$ (notation G.J, 1.13).

For each $\alpha \in B_\lambda$, fix $\lambda_\alpha \in \Lambda^+$ as in (G.J, 2.1) on the α-wall and set $\psi_\alpha = T_\lambda^{\lambda_\alpha}$, $\varphi_\alpha = T_{\lambda_\alpha}^\lambda$. By (G.J, 1.7) φ_α is both a left and a right adjoint to ψ_α and we set $\theta_\alpha = \varphi_\alpha \psi_\alpha$. All these functors are exact. Again we have exact functors $\psi_\alpha^l := R_\lambda^{\lambda_\alpha}$, $\psi_\alpha^r := S_\lambda^{\lambda_\alpha}$ (notation G.J, 1.13) on $\mathscr{H}$ (defined also by tensoring with finite dimensional representations and taking primary decomposition) such that $\psi_\alpha^l L(M, N) \cong L(\psi_\alpha M, N)$, $\psi_\alpha^r L(M, N) \cong L(M, \psi_\alpha N)$ for all $M, N \in \operatorname{Ob} \mathcal{O}$. We take a similar definition for φ_α^l, φ_α^r and set $\theta_\alpha^l = \varphi_\alpha^l \psi_\alpha^l$, $\theta_\alpha^r = \varphi_\alpha^r \psi_\alpha^r$. Given $M \in \operatorname{Ob} \mathcal{O}$ (resp. $V \in \operatorname{Ob} \mathscr{H}$) the identity map on $\psi_\alpha M$ (resp. $\psi_\alpha^l V$) induces maps

$$\begin{aligned} &I'_M : M \to \theta_\alpha M, \quad I''_M : \theta_\alpha M \to M \\ (\text{resp. } &I'_V : V \to \theta_\alpha^l V, \quad I''_V : \theta_\alpha^l V \to V). \end{aligned}$$

Lemma. *For all $M, N \in \operatorname{Ob} \mathcal{O}$ the diagram*

$$\begin{tikzcd} L(\theta_\alpha M, N) \arrow[r, "I'_M"] & L(M, N) \\ \theta_\alpha^l(L(M, N)) \arrow[u, "\iota"] \arrow[ur, "I''_{L(M,N)}"'] & \end{tikzcd}$$

commutes.

Let $E \in \mathscr{E}$ be the representation occuring in the definition of ψ_α. One has for all $M \in \operatorname{Ob} \mathcal{O}$

$$\psi_\alpha M = p'(E \otimes p(M)), \qquad \varphi_\alpha M = p(E^* \otimes p'(M))$$

where p, p' are projections into appropriate primary components. In particular $\theta_\alpha M$ is a direct summand of $E^* \otimes E \otimes M$ and one easily checks that I''_M: $\theta_\alpha M \to M$ is the restriction of the map $\xi \otimes e \otimes m \mapsto (\xi, e) m$ of $E^* \otimes E \otimes M \to M$.

Let $\{e_i\}$ be a basis for E and $\{\xi_i\}$ be a dual basis for E^*. One easily checks that I'_M: $M \to \theta_\alpha M$ is given by $m \mapsto \sum p(\xi_i \otimes p'(e_i \otimes p(m)))$. Similar formulae hold for θ^l_α replaced by θ_α, $\mathcal{O}$ by $\mathscr{H}$ and $M \in \operatorname{Ob} \mathcal{O}$ by $V \in \operatorname{Ob} \mathscr{H}$. Here we interpret p, p' as functors on $\mathscr{H}$ with respect to the primary decomposition defined by $1 \otimes Z(\mathfrak{g})$.

Take $w \in \theta^l_\alpha(L(M, N))$. Then $p(w) = w$ and we can write $w = \sum v_{ij} \otimes (1 \otimes \eta_i) \otimes (1 \otimes f_j)$ for some $\eta_i \in E^*$, $f_j \in E$, $v_{ij} \in L(M, N)$ satisfying $p(v_{ij}) = v_{ij}$, $p(\sum (v_{ij} \otimes (1 \otimes \eta_i))) = \sum (v_{ij} \otimes (1 \otimes \eta_i))$. It follows from the above that

$$I''_{L(M,N)}(w) = \sum v_{ij}(\eta_i, f_j).$$

On the other hand setting $w' := I'_M(w)$, it follows that w' is the map $m \mapsto w(\sum p(\xi_i \otimes p'(e_i \otimes p(m))))$ of M to N.

Yet

$$\begin{aligned} w(\sum p(\xi_k \otimes p'(e_k \otimes p(m)))) &= \sum (\xi_k, f_j)(\eta_i, e_k)\, v_{ij}(m) \\ &= \sum (\eta_i, f_j)\, v_{ij}(m). \end{aligned}$$

Combined with our previous formula this establishes the required commutativity.

3.2. Proposition. *For all $\alpha \in B_\lambda$, $M \in \operatorname{Ob} \mathcal{O}_\Lambda$, there is an exact sequence*

$$0 \to L(M(\lambda), C_\alpha M) \to L(\theta_\alpha M(\lambda), M) \to L(M(\lambda), D^+_\alpha M) \to 0.$$

From the exact sequence ([10], 3.13)

$$0 \to M(\lambda) \xrightarrow{I'_{M(\lambda)}} \theta_\alpha M(\lambda) \to M(s_\alpha \lambda) \to 0,$$

we obtain the long exact sequence

$$0 \to L(M(s_\alpha \lambda), M) \to L(\theta_\alpha M(\lambda), M) \xrightarrow{I'_{M(\lambda)}} L(M(\lambda), M) \to$$

Set $V = L(M(\lambda), M)$. By 3.1 we can write the above sequence as

$$0 \to L(M(s_\alpha \lambda), M) \to \theta^l_\alpha V \xrightarrow{I''_V} V \to \operatorname{Coker} I''_V \to 0.$$

By ([10], 3.12) $\operatorname{Coker} I''_V$ is the largest quotient Q of V such that $\theta^l_\alpha Q = 0$. Now every simple submodule of V takes the form $V(-w\mu, -\lambda)$ (notation G.J, 1.12) with $\mu \in \Lambda^+$, $w \in W_\lambda$. Furthermore $\theta^l_\alpha V(-w\mu, -\lambda) = 0$ if and only if $(\alpha, w\mu) > 0$. (This well-known result can be proved as follows. First by translation principles we can reduce to the case when μ is regular. Applying the map η (2.8) the first condition is equivalent to $\theta^r_\alpha V(-\lambda, -w\mu) = 0$ and so by ([6], I, 4.1) to $\theta^r_\alpha V(-w^{-1}\lambda, -\mu) = 0$. Since ([13], 4.7) $V(-w^{-1}\lambda, -\mu) \cong L(M(\mu), L(w^{-1}\lambda))$ the condition is equivalent to $\theta_\alpha L(w^{-1}\lambda) = 0$ and so by ([11], 2.11) equivalent to $w^{-1}\alpha > 0$, that is to $(\alpha, w\mu) > 0$.) Applying T it follows from the definition of D^+_α that $Q \cong L(M(\lambda), M/D^+_\alpha M)$. Since $M(\lambda)$ is projective in $\mathcal{O}_\Lambda$ it follows that $\operatorname{Im} I''_V \cong L(M(\lambda), D^+_\alpha M)$. Recalling 2.2(*) then gives the proposition.

3.3 Corollary. *For all $\alpha \in B_\lambda$ one has $C_\alpha D^+_\alpha = C_\alpha$.*

It is enough to show that the map $L(M(\lambda), C_\alpha D^+_\alpha M) \to L(M(\lambda), C_\alpha M)$ (induced by $D^+_\alpha M \to M$) is an isomorphism. Since $D^{+\,2}_\alpha = D^+_\alpha$ it follows from 3.2 and functoriality that we only have to show that the map $L(\theta_\alpha M(\lambda), D^+_\alpha M) \to L(\theta_\alpha M(\lambda), M)$ is an isomorphism. This will result if we can show that $\theta^l_\alpha L(M(\lambda), M/D^+_\alpha M) = 0$. Now every simple subquotient $L(w\mu)$: $\mu \in \Lambda^+$, $w \in W_\lambda$ of $M/D^+_\alpha M$ is α-finite and hence satisfies $(\alpha, w\mu) > 0$. Thus $\theta^l_\alpha(L(M(\lambda), L(w\mu))) = \theta^l_\alpha V(-w\mu, -\lambda) = 0$ (as shown in 3.2) and the assertion follows from the exactness of θ^l_α.

3.4. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence $\mathfrak{M}$ in $\mathcal{O}_\Lambda$ and fix $\alpha \in B_\lambda$. Then we have a complex

$$0 \to D^+_\alpha M_1 \to D^+_\alpha M_2 \to D^+_\alpha M_3 \to 0.$$

Let $H(D^+_\alpha \mathfrak{M})$ denote its cohomology. That is

$$H(D^+ \mathfrak{M}) = \operatorname{Ker}(D^+_\alpha M_2 \to D^+_\alpha M_3)/D^+_\alpha M_1.$$

Lemma. *There is an exact sequence*

$$0 \to C_\alpha M_1 \to C_\alpha M_2 \to C_\alpha M_3 \to H(D^+_\alpha \mathfrak{M}) \to 0.$$

By 3.2, the left exactness of C_α, the exactness of θ_α and the projectivity of $M(\lambda)$, we have the commuting diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L(M(\lambda), C_\alpha M_1) \arrow[r] \arrow[d] & L(\theta_\alpha M(\lambda), M_1) \arrow[r] \arrow[d] & L(M(\lambda), D^+_\alpha M_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L(M(\lambda), C_\alpha M_2) \arrow[r] \arrow[d, "\sigma"'] & L(\theta_\alpha M(\lambda), M_2) \arrow[r] \arrow[d] & L(M(\lambda), D^+_\alpha M_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L(M(\lambda), C_\alpha M_3) \arrow[r] & L(\theta_\alpha M(\lambda), M_3) \arrow[r] \arrow[d] & L(M(\lambda), D^+_\alpha M_3) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where all rows and columns are exact, with the exception of the last column whose cohomology is hence Coker σ. Applying T' gives the assertion of the lemma.

3.5. Let $L \in \operatorname{Ob} \mathcal{O}_\Lambda$ be simple. We can write $L = L(\mu) := M(\mu)/\overline{M(\mu)}$ (notation G.J, 1.2) for some $\mu \in \Lambda$. Take $\alpha \in B_\lambda$. Then $C_\alpha L = 0$ exactly when $(\alpha, \mu) > 0$.

Corollary. *Suppose $(\alpha, \mu) \leqq 0$. Then there is an exact sequence*

$$0 \to C_\alpha \overline{M(\mu)} \to M(s_\alpha \mu) \to C_\alpha L(\mu) \to \overline{M(\mu)}/D_\alpha^+ \overline{M(\mu)} \to 0.$$

Apply 3.4 using 2.5, 2.11.

3.6. Lemma. *For all $\alpha \in B_\lambda$ one has*

(i) $D_\alpha^- C_\alpha = C_\alpha$.
(ii) $D_\alpha^+ C_\alpha = D_\alpha^+ C_\alpha D_\alpha^-$.
(iii) $C_\alpha^2 D_\alpha^- = C_\alpha^2$.
(iv) $D_\alpha^+ C_\alpha = D_\alpha$.
(v) $C_\alpha^2 = C_\alpha D_\alpha^-$.

(i) We must show that $\operatorname{Hom}_\mathfrak{g}(L(\mu), C_\alpha M) = 0$, whenever $\mu \in \Lambda$ satisfies $(\mu, \alpha) > 0$. Set $V = L(M(\lambda), L(\mu)) \cong V(-\mu, -\lambda)$. Then $\operatorname{Hom}_U(V, L(\theta_\alpha M(\lambda), M)) \cong \operatorname{Hom}_U(\theta_\alpha^t V, L(M(\lambda), M)) = 0$, since $\theta_\alpha^t V = 0$ (as noted in 3.2). Then by 3.2 we obtain $\operatorname{Hom}_\mathfrak{g}(L(\mu), C_\alpha M) \cong \operatorname{Hom}_U(V, T(C_\alpha M)) = 0$.

(ii), (iii) From the surjection $M \to D_\alpha^- M$ which has an α-finite kernel and 3.4 we obtain the exact sequence

$$0 \to C_\alpha M \to C_\alpha D_\alpha^- M \to \operatorname{Ker}(D_\alpha^+ M \to D_\alpha M) \to 0. \qquad (*)$$

Now $\operatorname{Ker}(D_\alpha^+ M \to D_\alpha M)$ is α-finite. Thus applying D_α^+ to $(*)$ gives (ii) and applying C_α gives (iii).

(iv). (v) By 3.3 and (i), (iii) we can assume that $D_\alpha^+ M \xrightarrow{\sim} M$ and $M \xrightarrow{\sim} D_\alpha^- M$.

Apply the exact functor θ_α to the exact sequence in 3.2. Since $\theta_\alpha^2 = \theta_\alpha \oplus \theta_\alpha$ ([10], 3.6, (vi)) and $D_\alpha^+ M \xrightarrow{\sim} M$ we obtain the exact sequence

$$0 \to L(\theta_\alpha M(\lambda), C_\alpha M) \to (L(\theta_\alpha M(\lambda), M) \oplus L(\theta_\alpha M(\lambda), M)) \to L(\theta_\alpha M(\lambda), M) \to 0.$$

Yet $M \xrightarrow{\sim} D_\alpha^- M$ and so we have a natural embedding σ_2:

$$L(\theta_\alpha M(\lambda), M) \hookrightarrow L(\theta_\alpha M(\lambda), C_\alpha M).$$

Yet

$$\begin{aligned} L(\theta_\alpha M(\lambda), C_\alpha M) &\cong \theta_\alpha^t L(M(\lambda), C_\alpha M) \cong \theta_\alpha^t L(M(s_\alpha \lambda), M) \cong L(\theta_\alpha M(s_\alpha \lambda), M) \\ &\cong L(\theta_\alpha M(\lambda), M) \qquad \text{(by [11], 2.10a)).} \end{aligned}$$

Hence σ_2 is an isomorphism. From 3.2 (applied to both $M, C_\alpha M$) we obtain the commuting diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & L(M(\lambda), C_\alpha M) \arrow[r] \arrow[d, "\sigma_1"] & L(\theta_\alpha M(\lambda), M) \arrow[r] \arrow[d, "\sigma_2"] & L(M(\lambda), D_\alpha^+ M) \arrow[r] \arrow[d, "\sigma_3"] & 0 \\ 0 \arrow[r] & L(M(\lambda), C_\alpha^2 M) \arrow[r] & L(\theta_\alpha M(\lambda), C_\alpha M) \arrow[r] & L(M(\lambda), D_\alpha^+ C_\alpha M) \arrow[r] & 0 \end{tikzcd}$$

where σ_i: $i = 1, 2, 3$ are the natural maps. By (i) and 2.4, σ_1 is an embedding. By the hypothesis $M \xrightarrow{\sim} D_\alpha^- M$ and 2.4 the map $M \to C_\alpha M$ is an embedding and hence (cf. 2.11) so is σ_3. Since σ_2 is an isomorphism it follows that so are σ_1, σ_3. Applying T' gives (iv), (v).

3.7. Corollary. *For all $\alpha \in B_\lambda$, $M \in \operatorname{Ob} \mathcal{O}_\Lambda$ one has*

$$L(M(s_\alpha \lambda), M)\, M(s_\alpha \lambda) = D_\alpha^+ M.$$

Set $M' = L(M(s_\alpha \lambda), M)\, M(s_\alpha \lambda)$. For any α-finite module N one has $L(M(s_\alpha \lambda), N) = 0$ (G.J, 3.2) and so M' is a submodule of $D_\alpha^+ M$. Yet the definition of M' gives $L(M(s_\alpha \lambda), M') \xrightarrow{\sim} L(M(s_\alpha \lambda), M)$ which by 2.2(*) gives $C_\alpha M' \xrightarrow{\sim} C_\alpha M$. Then by 3.6, (iv) $D_\alpha M = D_\alpha^+ C_\alpha M = D_\alpha^+ C_\alpha M' = D_\alpha M'$ which together with our first observation proves the lemma.

3.8. Lemma. *Take $\alpha \in B_\lambda$ and $M, N \in \operatorname{Ob} \mathcal{O}_\Lambda$. Let M' be a submodule of $C_\alpha M$ containing* $\operatorname{Im}(M \to C_\alpha M)$ (so we have a map $M \to M'$). *Then the natural maps*

(i) $L(M', C_\alpha^2 N) \to L(M, C_\alpha^2 N)$,
(ii) $L(D_\alpha^+ M, D_\alpha^- N) \to L(D_\alpha^+ M, C_\alpha^2 N)$

are isomorphisms.

(i) Set $M'' = \operatorname{Im}(M \to C_\alpha M) \cong D_\alpha^- M$.
By 3.6(iv), $C_\alpha M/M''$ and hence $K := M'/M''$ is α-finite.
By 3.6(i), $C_\alpha^2 N$ is α-free and so $L(K, C_\alpha^2 N) = 0$ (notation G.J, 3.6, (ii)) which proves injectivity.

From $C_\alpha M \cong L(M(s_\alpha \lambda), M) \otimes_{U(\mathfrak{g})} M(\lambda)$, it follows that $L(M, C_\alpha^2 N)\, C_\alpha M$ is defined and contained in

$$L(M(s_\alpha \lambda), C_\alpha^2 N) \otimes_{U(\mathfrak{g})} M(\lambda) \cong C_\alpha^3 N \cong C_\alpha^2 N \qquad (3.6(\text{v})).$$

This proves surjectivity.

(ii) By definition, $D_\alpha^- N$ is α-free so by 2.4 the map $D_\alpha^- N \to C_\alpha^2 D_\alpha^- N \cong C_\alpha^2 N$ is an embedding. This proves injectivity in (ii). For surjectivity it is enough to show that $L(D_\alpha^+ M, C_\alpha^2 N/D_\alpha^- N) = 0$. Every simple quotient of $D_\alpha^+ M$ is α-free, whereas by 3.6(iv) $C_\alpha^2 N/D_\alpha^- N$ is α-finite and so the assertion follows from (G.J, 3.6(ii)).

3.9. Corollary. *Take $N \in \operatorname{Ob} \mathcal{O}_\Lambda$. For all $\alpha \in B_\lambda$, $w \in W_\lambda$ satisfying $s_\alpha w > w$ one has* $L(M(w\lambda), C_\alpha^2 N) \cong L(M(s_\alpha w \lambda), D_\alpha^- N)$.

If N is α-free then $L(M(w\lambda), C_\alpha N) \cong L(M(s_\alpha w \lambda), N)$.

Apply 3.8 taking account of 2.5, 2.11 and 3.6(v).

3.10. Lemma. *Take $M \in \operatorname{Ob} \mathcal{O}_\Lambda$, $\alpha \in B_\lambda$. Then* $\operatorname{Im}(M \to C_\alpha M)$ *is an essential submodule of $C_\alpha M$.*

Set $N = \mathrm{Im}(M \to C_\alpha M)$. By 3.6(i) $C_\alpha M$ and hence N is α-free. By 3.6(iv) $D_\alpha^+(C_\alpha M) = D_\alpha M = D_\alpha^+ N$ which is hence a submodule of N. Thus $(C_\alpha M)/N$ is α-finite. It follows that every non-zero submodule of $C_\alpha M$ intersects N non-trivially.

3.11. For each $w \in W_\lambda$ define the functor C_w on $\mathcal{O}_\Lambda$ through $C_w M := L(M(w^{-1}\lambda), M) \otimes_{U(\mathfrak{g})} M(\lambda)$. Set $S_\lambda = \{s_\alpha : \alpha \in B_\lambda\}$. We view the pair (W_λ, S_λ) as a Coxeter group and let l_λ denote its length function. Let $\mathcal{O}_{\Lambda,f}$ denote the full subcategory of $U(\mathfrak{n}^-)$ free objects in $\mathcal{O}_\Lambda$. By 3.10, $\mathcal{O}_{\Lambda,f}$ is stable under each C_α: $\alpha \in B_\lambda$.

Lemma. *$\mathcal{O}_{\Lambda,f}$ is stable under the C_w: $w \in W_\lambda$ and restricted to this subcategory they satisfy*

(i) $C_s^2 = C_s, \quad s \in S_\lambda$.
(ii) $C_w C_{w'} = C_{ww'}$ *if* $l_\lambda(w) + l_\lambda(w') = l_\lambda(ww')$.

Apply 3.6(v) and 3.8.

Remarks. When $B_\lambda = B$ this result is due to Deodhar, though his proof involves some unpleasant computations. In this special case the result could already have been read off from ([14], 5.5) taking account of 2.12.

3.12. Lemma. *Take $M, N \in \mathrm{Ob}\,\mathcal{O}_\Lambda$. If N is α-finite then for all $i \in \mathbb{N}$ one has*

(i) $\mathrm{Ext}^i(C_\alpha M, N) \xrightarrow{\sim} \mathrm{Ext}^{i+1}(D_\alpha^+ M, N)$.
(ii) $\mathrm{Ext}^i(N, D_\alpha^+ M) \xrightarrow{\sim} \mathrm{Ext}^{i+1}(N, C_\alpha M)$.

Set $V = L(M(\lambda), N)$. As we saw in 3.2 the hypothesis that N is α-finite gives $\theta_\alpha^t V = 0$.

Since θ_α^t is self-adjoint this gives $\mathrm{Ext}^i(L(\theta_\alpha M(\lambda), M), V) = 0$, $\forall i$. Hence (i) results from 3.2 and the equivalence of categories (G.J, 1.16). (ii) is similar.

3.13. Corollary. *Take $M, N \in \mathrm{Ob}\,\mathcal{O}_\Lambda$ with N α-finite.*

(i) $\mathrm{Ext}^i(N, C_\alpha^2 M) = 0$.
(ii) $C_\alpha^2 M = C_\alpha M$ *if and only if $D_\alpha^+ M$ is α-free.*

Taking $i = 0$ in 3.12(ii) gives the commuting square

$$\begin{array}{ccc} \mathrm{Hom}(N, D_\alpha^+ M) & \xrightarrow{\sim} & \mathrm{Ext}^1(N, C_\alpha M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(N, D_\alpha^+ C_\alpha M) & \xrightarrow{\sim} & \mathrm{Ext}^1(N, C_\alpha^2 M). \end{array}$$

By 3.6(i), $C_\alpha M$ is α-free and so $\mathrm{Hom}(N, D_\alpha^+ C_\alpha M) = 0$. This gives (i). If $D_\alpha^+ M$ is not α-free then we can choose some N α-finite such that $\mathrm{Hom}(N, D_\alpha^+ M) \neq 0$. Then $\mathrm{Ext}^1(N, C_\alpha M) \neq 0$ and so $C_\alpha M \neq C_\alpha^2 M$ by (i). Conversely if $D_\alpha^+ M$ is α-free, then by 3.3, 3.6 we obtain $C_\alpha M = C_\alpha D_\alpha^+ M = C_\alpha D_\alpha^- D_\alpha^+ M = C_\alpha^2 D_\alpha^+ M = C_\alpha^2 M$.

Example 1. Take $\mathfrak{g} \cong sl(2)$ and λ integral. Then $C_\alpha(\delta(M(\lambda))) \cong M(s_\alpha \lambda)$, whereas $C_\alpha^2(\delta(M(\lambda))) \cong M(\lambda)$.

Example 2. We see from 3.12 (ii) taking $M = M(\lambda)$ that $\mathrm{Ext}^1(L(w\lambda), M(\lambda)) = 0$ unless $w = w_\lambda$. Thus the projective module $M(\lambda)$ is also "almost" injective.

Example 3. Take $\mathfrak{g} \cong sl(3)$ and λ integral. Set $B = \{\alpha, \beta\}$. Then $\mathrm{Ext}^1(M(s_\beta s_\alpha \lambda), L(s_\beta \lambda)) \neq 0$; even though $L(s_\beta \lambda)$ is α-finite and $M(s_\beta s_\alpha \lambda) \cong C_\alpha M(s_\beta s_\alpha \lambda)$.

3.14. As shown by Example 1 above, 3.11(i) fails on the whole of $\mathcal{O}_\Lambda$. Yet there is good evidence that 3.11(ii) always holds. Indeed just as in ([4], 4.2) it is enough to verify this for certain special "pair relations" that is when B': $= \mathrm{supp}\, w w'$ satisfies card $B' = 2$. Choose $w \in W$, $B'' \subset B$: card $B'' = 2$ such that $wB' \subset \mathbb{N} B''$. Through equivalence of categories (2.2) there is no less in generality in assuming $w = 1$. Set

$$\mathfrak{r} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \mathbb{Z}B'' \cap R} \mathbb{C} X_\alpha \Big).$$

Each $M \in \mathrm{Ob}\,\mathcal{O}_\Lambda$ is an infinite direct sum of $\mathfrak{r}$ modules defined in the corresponding $\mathcal{O}_\Lambda$ category relative to $\mathfrak{r}$. It can be shown that C_α: $\alpha \in B'$ restricted to each direct summand is just the corresponding functor defined relative to $\mathfrak{r}$ modules. Consequently it is enough to verify 3.11(ii) for rank 2. Using the known multiplicity formulae for $\theta_\alpha L(w\lambda)$, 3.2 and 3.12(ii) one easily verifies that 3.11(ii) holds (in rank 2) for all simple modules.

3.15. Since Enright did not define his functor $\tilde{C}_\alpha$ on modules which are not α-free it makes no sense to ask if $\tilde{C}_\alpha = C_\alpha$ on the whole of $\mathcal{O}_\Lambda$. However Deodhar's definition (again given only in the α-free case) suggests that we might take $\tilde{C}_\alpha M$ to be the largest X_α locally finite submodule of the localized module $M_{X_{-\alpha}}$.

Then trivially $\tilde{C}_\alpha D_\alpha^- = \tilde{C}_\alpha$ and so $\tilde{C}_\alpha = C_\alpha D_\alpha^-$ by 2.12. Yet (3.6,(v)) $C_\alpha D_\alpha^- = C_\alpha^2 \neq C_\alpha$. What is particularly surprising is that although $\tilde{C}_\alpha$ satisfies 3.11(i) it fails to satisfy 3.11(ii). Indeed in the notation of Example 3 above one finds that $\tilde{C}_\alpha \tilde{C}_\beta \tilde{C}_\alpha L(s_\alpha s_\beta \lambda) = \tilde{C}_\alpha L(s_\alpha s_\beta \lambda)$ whereas $\tilde{C}_\beta L(s_\alpha s_\beta \lambda) = 0$.

3.16. Lemma. *Suppose $P \in \mathrm{Ob}\,\mathcal{O}_\Lambda$ is projective. Then $P \xrightarrow{\sim} C_\alpha P$, $\alpha \in B_\lambda$.*

It is well-known that every projective module in $\mathcal{O}_\Lambda$ is a direct summand of $E \otimes M(\lambda)$ for an appropriate $E \in \mathscr{E}$. Since C_α commutes with direct sums it is enough by 2.3 to establish the assertion for $M(\lambda)$. Since λ is dominant, this follows from 2.5.

Remark. The referee showed that the converse of 3.16 fails if card $B_\lambda > 1$, by the argument below. Let P be the projective cover of $M(w_\lambda \lambda)$ and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & M(w_\lambda \lambda) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_\alpha M & \longrightarrow & C_\alpha P & \xrightarrow{\varphi} & C_\alpha M(w_\lambda \lambda) & & \end{array}$$

where the rows are exact and the vertical maps injective. Since $M(w_\lambda \lambda)$ is simple $M \subsetneqq C_\alpha M$ implies $C_\alpha M = P$ and so $\varphi = 0$, contradicting that

$P \to M(w_\lambda \lambda) \hookrightarrow C_\alpha M(w_\lambda \lambda)$ is $\neq 0$. By BGG duality M has a filtration by all the $M(w\lambda)$, $w \neq w_\lambda$ each occurring once. As $M(\lambda)$ occurs only once M cannot be a decomposable projective, as all the $M(s_\alpha w_\lambda \lambda)$: $\alpha \in B_\lambda$ occur and card $B_\lambda > 1$, it cannot be an indecomposable one either.

4. Lifting of Contravariant Forms

Throughout this section a form on $M \in \mathrm{Ob}\,\mathcal{O}_\Lambda$ (or $V \in \mathrm{Ob}\,\mathscr{H}$) means a contravariant form ζ. That is a symmetric bilinear map $\zeta\colon M \times M \to \mathbb{C}$ satisfying $\zeta(m, a\,n) = \zeta({}^t a\,m, n)$, $\forall a \in U(\mathfrak{g})$, $m, n \in M$ (or $a \in U(\mathfrak{g} \times \mathfrak{g})$, $m, n \in V$). The space of forms on M (or V) identifies canonically with $\mathrm{Hom}_{\mathfrak{g}}(M, \delta M)$ (or $\mathrm{Hom}_U(V, \delta V)$ through $m \mapsto (n \mapsto \zeta(m, n), \forall n \in M)$ and we shall use $\zeta(m)$ to denote the image of $m \in M$ under this map.

4.1. We use 2.8, 3.2 to lift a form from M to $C_\alpha M$. By the above remarks and the isomorphism of 2.8 we have only to describe how to lift a form from $L(M(\lambda), M)$ to $L(M(\lambda), C_\alpha M)$. Now there is a standard way (pioneered by Janzten) to define a form on $\theta_\alpha M$ given one on M. First recall that $\psi_\alpha M$ is a primary component of the tensor product $M \otimes E$ for a suitable simple $E \in \mathscr{E}$. This tensor product admits a natural form, namely the product of the form on M with the unique (up to scalars) non-degenerate form on E. By contravariance primary decomposition is an orthogonal direct sum for this form and so restriction gives a form on $\psi_\alpha M$ or equivalently a map $\psi_\alpha M \to \delta(\psi_\alpha M)$. Moreover it is clear that the above construction defines an isomorphism $\psi_\alpha(\delta M) \xrightarrow{\sim} \delta(\psi_\alpha M)$, that is to say δ, ψ_α commute. Similar considerations apply with ψ_α replaced by ψ_α^t or by φ_α, φ_α^t and hence for θ_α, θ_α^t.

Now suppose we have a map $\xi\colon M \to \delta M$. Then we have the following diagram

$$\begin{tikzcd}
\delta(L(M(\lambda), C_\alpha M)) & \delta(L(M(s_\alpha \lambda), M)) \arrow[l, "\sim"'] & \theta_\alpha^t(L(M(\lambda), M))) \arrow[l, two heads] \\
L(M(\lambda), \delta(C_\alpha M)) \arrow[u, "\zeta_\lambda^{C_\alpha M}"] & L(M(s_\alpha \lambda), \delta M) \arrow[u, "\zeta_{s_\alpha \lambda}^M"] \arrow[r, hook] & \theta_\alpha^t(L(M(\lambda), \delta M)) \arrow[u, "\zeta_\lambda^M"] \\
L(M(\lambda), C_\alpha M) \arrow[u, "F_\alpha^M(\xi)"] \arrow[r, "\sim"] & L(M(s_\alpha \lambda), M) \arrow[u, "\xi"] \arrow[r, hook] & \theta_\alpha^t(L(M(\lambda), M)) \arrow[u, "\xi"]
\end{tikzcd}$$

where $F_\alpha^M(\xi)$ is defined by requiring the periphery of the diagram to commute. That is we set

$$F_\alpha^M(\xi) = (\zeta_\lambda^{C_\alpha M})^{-1} (\zeta_\lambda^M \xi)|_{L(M(s_\alpha \lambda), M)}.$$

This formula describes how to lift a form from M to $C_\alpha M$.

4.2. Fix $M \in \mathrm{Ob}\,\mathcal{O}_\Lambda$, $E \in \mathscr{E}$. Through the isomorphism $E \xrightarrow{\sim} \delta E$ we obtain an isomorphism $E \otimes \delta M \xrightarrow{\sim} \delta(E \otimes M)$. Yet by 2.3 we have an isomorphism $E \otimes C_\alpha M \xrightarrow{\sim} C_\alpha(E \otimes M)$. It is an immediate consequence of the definition of F_α that we have the

Lemma. *Fix $\xi \in \mathrm{Hom}_{\mathfrak{g}}(M, \delta M)$. The diagram*

$$\begin{tikzcd}
E \otimes C_\alpha(M) \arrow[r, "F_\alpha^M(\xi)"] \arrow[d, "\iota"] & E \otimes \delta(C_\alpha(M)) \arrow[d, "\iota"] \\
C_\alpha(E \otimes M) \arrow[r, "F_\alpha^{E \otimes M}(\xi)"] & \delta(C_\alpha(E \otimes M))
\end{tikzcd}$$

commutes.

Remark. This is also a property of Enright's construction and which emerges in his work as a consequence of some quite elaborate computations ([8], Sect. 8).

4.3. The diagram in 4.1 naturally contains a simplified formula for $F_\alpha^M(\xi)$. This can be expressed by the following

Proposition. *The diagram of 4.1 commutes. In particular*

$$\zeta_\lambda^{C_\alpha M} F_\alpha^M(\xi) = \zeta_{s_\alpha \lambda}^M \xi.$$

The bottom right hand square commutes by functoriality. Consider the top right hand square. Given $V \in \mathrm{Ob}\,\mathscr{H}$, let V_0 denote its trivial $\mathfrak{k}$-isotypical component and $P\colon V \to V_0$ the projection defined by the isotypical decomposition of V. In particular if $V = \mathrm{Hom}(E, \delta E)$: $E \in \mathscr{E}$, then $V_0 = \mathrm{Hom}_{\mathfrak{g}}(E, \delta E)$.

From the construction of θ_α there is defined $E \in \mathscr{E}$ and surjections $E \otimes M(\lambda) \twoheadrightarrow \theta_\alpha M(\lambda) \twoheadrightarrow M(s_\alpha \lambda)$. Now choose $u \in L(M(s_\alpha \lambda), \delta M)$, $v \in L(M(s_\alpha \lambda), M)$. In the conventions and notations of 2.8 we have

$$\zeta_{s_\alpha \lambda}^M(u, v) = P(\varphi(v)\,u).$$

On the other hand with respect to the embeddings

$$L(M(s_\alpha \lambda), N) \hookrightarrow \theta_\alpha^t(L(M(\lambda), N)) \hookrightarrow \mathrm{Hom}(E, \mathbb{C}) \otimes L(M(\lambda), N)$$

where we take either $N = M$ or δM we may write $u = \sum u_i \otimes e_i$, $v = \sum v_j \otimes f_j$ for some $u_i \in L(M(\lambda), \delta M)$, $v_j \in L(M(\lambda), M)$, $e_i, f_j \in \mathrm{Hom}(E, \mathbb{C})$ and then

$$\begin{aligned}
\zeta_\lambda^M(u, v) &= \sum \zeta_\lambda^M(u_i, v_j)\, P(\varphi(f_j)\, e_i) \\
&= \sum P(\varphi(v_j)\, u_i)\, P(\varphi(f_j)\, e_i) \\
&= (P \otimes P)\left(\sum (\varphi(v_j)\, u_i \otimes \varphi(f_j)\, e_i)\right).
\end{aligned}$$

From the natural embedding

$$\sigma\colon L(M(s_\alpha \lambda), \delta M(s_\alpha \lambda)) \hookrightarrow L(E \otimes M(\lambda), \delta(E \otimes M(\lambda))$$

we have a diagram of $\mathfrak{k}$ maps

$$\begin{tikzcd}
L(-s_\alpha \lambda, -s_\alpha \lambda) \arrow[r, "\sigma"] \arrow[d, "P"] & L(E \otimes M(\lambda), \delta(E \otimes M(\lambda)) \arrow[r, "\sim"] \arrow[d, "P"] & L(-\lambda, -\lambda) \otimes \mathrm{Hom}_{\mathbb{C}}(E, \delta E) \arrow[d, "P \oplus P"] \\
L(-s_\alpha \lambda, -s_\alpha \lambda)_0 \arrow[r, "\sigma"] & L(E \otimes M(\lambda), \delta(E \otimes M(\lambda)))_0 \arrow[r, "\varepsilon"] & L(-\lambda, -\lambda)_0 \otimes \mathrm{Hom}_{\mathfrak{g}}(E, \delta E)
\end{tikzcd}$$

which commutes for an appropriate choice of ε. That is $(P\otimes P)\sigma=(\varepsilon\sigma)P$. Substituting in the above and recalling that $\varepsilon\sigma\in\operatorname{End}\mathbb{C}$ we have now only to check that $\varepsilon\sigma\neq 0$. Should $\varepsilon\sigma=0$, then $F_\alpha^M(\xi)=0$ for any $M\in\operatorname{Ob}\mathcal{O}_\Lambda$ and any $\xi\in\operatorname{Hom}_{\mathfrak{g}}(M,\delta M)$. Yet take $M=M(w_\lambda\lambda)$ and give the latter its non-degenerate form. Then the product form on $L(\theta_\alpha M(\lambda), M(w_\lambda\lambda))$ is non-degenerate and from 3.2 we have an exact sequence

$$0\to L(M(\lambda), M(s_\alpha w_\lambda\lambda))\to L(\theta_\alpha M(\lambda), M(w_\lambda\lambda))\to L(M(\lambda), M(w_\lambda\lambda))\to 0.$$

It follows that the restriction of the form to $L(M(\lambda), M(s_\alpha w_\lambda\lambda))$ is either non-degenerate or has kernel isomorphic to the simple and strict U submodule $L(M(\lambda), M(w_\lambda\lambda))$. In either case the form is non-zero and so the proposition results. (Since a non-zero form on $M(s_\alpha w_\lambda\lambda)$ has kernel $M(w_\lambda\lambda)$ the latter situation holds.)

4.4. We may complete the diagram of 4.1 to obtain a map

$$(\zeta_\lambda^{C_\alpha M})^{-1}\zeta_{s_\alpha\lambda}^M\colon\ L(M(s_\alpha\lambda),\delta M)\to L(M(\lambda),\delta(C_\alpha M)).$$

Applying T' and 2.2(*) we obtain a map $\kappa_\alpha^M\colon C_\alpha(\delta M)\to\delta(C_\alpha M)$. This result summarizes succintly the lifting of a form from M to $C_\alpha M$. It also leads to an algebraic version of the Kunze-Stein intertwining operators (for the corresponding complex group) and the Duflo-Zelobenko four term exact sequence. This is discussed in the following sections. We start with the

Lemma. $\quad \operatorname{Ker}\kappa_\alpha^M=\operatorname{Im}(\delta M\to C_\alpha(\delta M))=D_\alpha^-(\delta M)$.

The last equality is just 2.4. The inclusion $\operatorname{Ker}\kappa_\alpha^M\supset\operatorname{Im}(\delta M\to C_\alpha(\delta M))$ will result if we can show that $\zeta_{s_\alpha\lambda}^M(\operatorname{Im}(L(M(\lambda),\delta M)\to L(M(s_\alpha\lambda),\delta M))=0$. This holds because $L(\delta M,\delta M(s_\alpha\lambda))\,L(M(\lambda),\delta M)\subset L(M(\lambda),\delta(M(s_\alpha\lambda)))\cong L(-s_\alpha\lambda,-\lambda)$ and by Frobenius reciprocity the principal series module $L(-s_\alpha\lambda,-\lambda)$ has no trivial $\mathfrak{k}$-type. To obtain equality consider the commuting diagram of exact sequences

$$\begin{tikzcd}
0 & \delta(L(M(s_\alpha\lambda),M)) \arrow[l] & \delta(L(\theta_\alpha M(\lambda),M)) \arrow[l] & \delta(L(M(\lambda),D_\alpha^+ M)) \arrow[l] & 0 \arrow[l] \\
0 \arrow[r] & L(M(s_\alpha\lambda),\delta M) \arrow[r] \arrow[u, "\zeta_{s_\alpha\lambda}^M"] & L(\theta_\alpha M(\lambda),\delta M) \arrow[u, "\zeta_\lambda^M"'] & &
\end{tikzcd}$$

which results from 3.2 applied to M and to δM. This gives

$$\operatorname{Ker}\zeta_{s_\alpha\lambda}^M\subset(\zeta_\lambda^M)^{-1}\,\delta(L(M(\lambda),D_\alpha^+ M))=L(M(\lambda),\delta(D_\alpha^+ M)).$$

Since $\delta(D_\alpha^+ M)\cong D_\alpha^-(\delta M)$ applying T' shows that $\operatorname{Ker}\kappa_\alpha^M$ is isomorphic to a submodule of $D_\alpha^-(\delta M)$. Finiteness of length of objects in $\mathcal{O}_\Lambda$ then implies the required equality.

4.5. Fix $\xi\in\operatorname{Hom}_{\mathfrak{g}}(M,\delta M)$ and set $\xi'=F_\alpha^M(\xi)$.

Corollary. (i) $\operatorname{Ker}\xi'\supset D_\alpha^-(M)+C_\alpha(\operatorname{Ker}\xi)$.
(ii) *If* $\operatorname{Ker}\xi=0$, *then equality holds in* (i).

(i) We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\xi \arrow[r] \arrow[d] & M \arrow[r, "\xi"] \arrow[d, "\sigma"] & \delta M \arrow[d, "\sigma'"] & \\
0 \arrow[r] & C_\alpha(\operatorname{Ker}\xi) \arrow[r] & C_\alpha M \arrow[r, "C_\alpha(\xi)"] & C_\alpha(\delta M) \arrow[r, "\kappa_\alpha^M"] & \delta(C_\alpha M)
\end{tikzcd}$$

with $\xi'=\kappa_\alpha^M C_\alpha(\xi)$. Then

$$\xi'(\sigma(M))=\kappa_\alpha^M(C_\alpha(\xi)\,\sigma(M))=\kappa_\alpha^M(\sigma'(\xi(M)))\subset\kappa_\alpha^M(\sigma'(\delta M))=0,\quad\text{by 4.4.}$$

So $\operatorname{Ker}\xi'\supset\operatorname{Im}(M\to C_\alpha M)=D_\alpha^-(M)$ (by 2.4). Clearly $\operatorname{Ker}\xi'\supset\operatorname{Ker}C_\alpha(\xi)=C_\alpha(\operatorname{Ker}\xi)$ and combined with the previous inclusion this gives (i).

(ii) The hypothesis $\operatorname{Ker}\xi=0$ implies that the form induced on $L(\theta_\alpha M(\lambda),M)$ is non-degenerate. Recall that ξ' obtains by restriction of this non-degenerate form and so by 3.2, applying T' we obtain an isomorphism $\operatorname{Ker}\xi'\cong\delta(D_\alpha^+ M)$. Yet $\delta(D_\alpha^+ M)\cong D_\alpha^-(\delta M)$ and $M\cong\delta M$ (because $\operatorname{Ker}\xi=0$). Hence (ii).

4.6. Choose $\alpha\in B_\lambda$, $w\in W_\lambda$ such that $s_\alpha w>w$. Choose $\mu\in\Lambda^+$ such that $s_\alpha w\mu<w\mu$ (equivalently that $(\alpha,w\mu)\neq 0$). Then (in a unique fashion) $M(s_\alpha w\mu)$ identifies with a strict submodule of $M(w\mu)$ and $\delta M(s_\alpha w\mu)$ with an image of $\delta M(w\mu)$. Recalling 2.5 we observe that $C_\alpha M(s_\alpha w\mu)=M(w\mu)$.

Lemma. *The sequence*

$$0\longrightarrow\delta M(s_\alpha w\mu)\longrightarrow C_\alpha(\delta M(s_\alpha w\mu))\xrightarrow{\kappa_\alpha^{M(s_\alpha w\mu)}}\delta M(w\mu)\longrightarrow\delta M(s_\alpha w\mu)\longrightarrow 0$$

is exact.

The unique simple submodule $L(s_\alpha w\mu)$ of $\delta M(s_\alpha w\mu)$ is α-free and so $D_\alpha^-(\delta M(s_\alpha w\mu))=\delta M(s_\alpha w\mu)=\operatorname{Ker}\kappa_\alpha^{M(s_\alpha w\mu)}$ (4.4). By 3.6(iv) and 4.4, $\operatorname{Im}\kappa_\alpha^{M(s_\alpha w\mu)}$ is α-finite and so $\operatorname{Coker}\kappa_\alpha^{M(s_\alpha w\mu)}\supset\delta M(s_\alpha w\mu)$ since the unique simple submodule of $\delta M(s_\alpha w\mu)$ is α-free.

Apply T (which is exact) to the above conclusions taking account of 3.8 and (G.J, 2.9). This gives the exact sequence of principal series modules

$$0\to L(-s_\alpha w\mu,-\lambda)\to L(-s_\alpha w\mu,-s_\alpha\lambda)\to L(-w\mu,-\lambda)\to L(-s_\alpha w\mu,-\lambda).$$

By Frobenius reciprocity the centre terms are isomorphic as $\mathfrak{k}$-modules. It follows that the last map is surjective and applying T' gives the assertion of the lemma.

4.7. Choose $\mu_1,\mu_2\in\Lambda$ such that $s_\alpha\mu_i<\mu_i$: $i=1,2$. Let $\sigma_\alpha^{\mu_i}$ be the unique (up to scalars) embedding $M(s_\alpha\mu_i)\hookrightarrow M(\mu_i)$.

Corollary. *The sequence of principal series modules*

$$0\longrightarrow L(-s_\alpha\mu_1,-\mu_2)\xrightarrow{\sigma_\alpha^{\mu_2}}L(-s_\alpha\mu_1,-s_\alpha\mu_2)\xrightarrow{\kappa_\alpha^{M(s_\alpha\mu_1)}}L(-\mu_1,-\mu_2)$$
$$\xrightarrow{\sigma_\alpha^{\mu_1}}L(-s_\alpha\mu_1,-\mu_2)\longrightarrow 0$$

is exact.

By 4.6 with $\mu_1 = w\mu$ we have the exact sequence

$$0 \to \delta M(s_\alpha \mu_1) \to C_\alpha(\delta M(s_\alpha \mu_1)) \to \delta(M(\mu_1)/M(s_\alpha \mu_1)) \to 0.$$

Apply the functor $L(M(\mu_2), -)$. Since $\operatorname{Ext}^1(E \otimes M(\mu_2), \delta M) = 0$, for any $E \in \mathscr{E}$ and any $\mathfrak{n}^-$-free module M (G.J, 4.2) we obtain using 2.5 and 3.8 the exact sequence

$$0 \to L(-s_\alpha \mu_1, -\mu_2) \to L(-s_\alpha \mu_1, -s_\alpha \mu_2) \to L(M(\mu_2), \delta(M(\mu_1)/M(s_\alpha \mu_1))) \to 0,$$

and hence the exact sequence

$$0 \to L(-s_\alpha \mu_1, -\mu_2) \to L(-s_\alpha \mu_1, -s_\alpha \mu_2) \to L(-\mu_1, -\mu_2) \to L(-s_\alpha \mu_1, -\mu_2).$$

The centre terms are isomorphic as $\mathfrak{k}$-modules (Frobenius reciprocity) and so the last map is surjective as required.

Remark. For α absolutely simple a similar four step exact sequence was given by Duflo ([6], V, 1.10) following the work of Zelobenko. In their work the maps are the Kunze-Stein intertwining operators. It can no doubt be proved that these maps concide with ours (as is obvious in certain special cases); but as this does not seems of any special importance we leave the proof to the energetic reader.

4.8. Let B' be a subset of B_λ and suppose that $M \in \operatorname{Ob} \mathcal{O}_\Lambda$ is B'-free. Given a form ξ on M we can lift it to a form on any $C_w M$: $w \in W_{B'}$ by taking a reduced decomposition $w = s_{\alpha_1} \cdot s_{\alpha_2} \dots s_{\alpha_l}$ for w and successively lifting the form from $C_{w_i}(M)$ to $C_{\alpha_i}(C_{w_i}(M))$ where $w_i = s_{\alpha_{i+1}} \dots s_{\alpha_l}$. By an argument similar to that given in 4.3 the form obtained is exactly that obtained by directly applying the map $\zeta^M_{w^{-1}\lambda}$.

That is we have the formula

$$\zeta^{C_w(M)}_\lambda F_1(F_2 \dots (F_l(\xi)) \dots) = \zeta^M_{w^{-1}\lambda}\xi,$$

where $F_i = F^{C_{w_i}(M)}_{\alpha_i}$. This result should be of interest in studying Enright's lattice functor. We expect it to also hold without the restriction of B'-freeness (cf. 3.13).

5. The Jantzen Conjecture and Kostant's Problem

5.1. In view of 4.5(ii) the optimist would probably conclude that equality always holds in 4.5(i).

Unfortunately this is not so and the fact that it does hold in certain special cases is a deep question intimately related to the Kazhdan-Lusztig and Jantzen conjectures. For example, take $\alpha \in B_\lambda$, $\mu \in \Lambda$ with $\mu > s_\alpha \mu$ and $M = M(s_\alpha \mu)$. Take ξ in 4.5 to be the unique up to scalars non-zero form on $M(s_\alpha \mu)$. Then $\operatorname{Ker} \xi = \overline{M(s_\alpha \mu)}$ (notation G.J, 1.2). Recall (2.5) that $C_\alpha M(s_\alpha \mu) = M(\mu)$ so if $\xi' \neq 0$ one has $\operatorname{Ker} \xi' = \overline{M(\mu)}$. Now $M(s_\alpha \mu)$ is α-free so if equality holds in 4.5(i) (for these particular choices) we obtain

$$\overline{M(\mu)} = M(s_\alpha \mu) + C_\alpha(\overline{M(s_\alpha \mu)}). \qquad (*)$$

Now consider the exact sequence

$$0 \to C_\alpha(\overline{M(s_\alpha \mu)}) \to M(\mu) \xrightarrow{\sigma} C_\alpha(L(s_\alpha \mu))$$

which results from the left exactness of C_α. Set $Q(\mu) = \operatorname{Im} \sigma$. Clearly $Q(\mu)$ admits a unique simple quotient and this is isomorphic to $L(\mu)$. Again by 3.9, $L(s_\alpha \mu)$ is an essential submodule of $C_\alpha(L(s_\alpha \mu))$ and hence an essential submodule of $Q(\mu)$. Since $\overline{M(s_\alpha \mu)}$ is a submodule of $C_\alpha(\overline{M(s_\alpha \mu)})$ (by 2.4) and the quotient is α-finite (3.6) it follows that $(M(s_\alpha \mu) + C_\alpha(\overline{M(s_\alpha \mu)}))/C_\alpha(\overline{M(s_\alpha \mu)}) \cong L(s_\alpha \mu)$. We therefore see that $(*)$ is equivalent to the assertion that $Q(\mu)$ is a non-trivial extension of $L(s_\alpha \mu)$ by $L(\mu)$. Now by ([9], 3.16)

$$\dim \operatorname{Ext}^1(L(\mu), L(s_\alpha \mu)) = \dim \operatorname{Hom}(\overline{M(\mu)}, L(s_\alpha \mu)) \leqq [M(\mu) : L(s_\alpha \mu)] = 1.$$

(The last equality which is well-known follows for example from the trivial identity $[M(s_\alpha \mu) : L(s_\alpha \mu)] = 1$ and the fact that $C_\alpha(M(s_\alpha \mu))/M(s_\alpha \mu) \cong M(\mu)/M(s_\alpha \mu)$ is α-finite whereas $L(s_\alpha \mu)$ is α-free.) Thus if $\operatorname{Ext}^1(L(\mu), L(s_\alpha \mu)) \neq 0$, then $L(s_\alpha \mu)$ is a quotient of $\overline{M(\mu)}$ and any quotient of $M(\mu)$ which admits $L(s_\alpha \mu)$ as an essential submodule is isomorphic to the unique non-trivial extension of $L(s_\alpha \mu)$ by $L(\mu)$. We may summarize our conclusions by the

Lemma. *Take $\alpha \in B_\Lambda$, $\mu \in \Lambda$ with $\mu > s_\alpha \mu$. The following three assertions are equivalent.*

(i) $\overline{M(\mu)} = M(s_\alpha \mu) + C_\alpha(\overline{M(s_\alpha \mu)})$.
(ii) $\dim \operatorname{Ext}^1(L(\mu), L(s_\alpha \mu)) = 1$.
(iii) *$Q(\mu)$ is isomorphic to a non-trivial extension of $L(s_\alpha \mu)$ by $L(\mu)$.*

5.2. It is known that 5.1(ii) is a consequence of the Kazhdan-Lusztig conjecture. To see this we may assume (by translation principles) that $\mu = s_\alpha w \lambda$ for some $w \in W_\lambda$: $s_\alpha w < w$. We have a complex

$$0 \to L(w^{-1}\lambda) \to \theta_\alpha L(w^{-1}\lambda) \to L(w^{-1}\lambda) \to 0$$

and we let $U_\alpha(L(w^{-1}\lambda))$ denote its cohomology. As pointed out by Vogan ([16], Sect. 3) the Kazhdan-Lusztig conjecture implies that $U_\alpha(L(w^{-1}\lambda))$ is semisimple. By 3.2 and equivalence of categories ([9], 1.16) this is equivalent to the semisimplicity of $C_\alpha(L(w\lambda))/L(w\lambda)$ (and moreover $[C_\alpha(L(w\lambda))/L(w\lambda) : L(w'\lambda)] = [U_\alpha(L(w^{-1}\lambda)) : L(w'^{-1}\lambda)]$, $\forall w' \in W_\lambda$, (by [6], 1, 4.1)). Since $[C_\alpha(L(w\lambda)) : L(s_\alpha w\lambda)] = 1$ and $L(w\lambda)$ is an essential submodule of $C_\alpha L(w\lambda)$ (3.9) it follows that $\operatorname{Ext}^1(L(s_\alpha w\lambda), L(w\lambda)) \neq 0$ which in turn gives 5.1(ii).

5.3. Take α, μ, M as in 5.1 and recall (4.6) that we have an exact sequence

$$0 \to \delta M(s_\alpha \mu) \to C_\alpha(\delta M(s_\alpha \mu)) \xrightarrow{\kappa} \delta M(\mu).$$

Recall that $\delta(L(s_\alpha \mu)) \cong L(s_\alpha \mu)$ and set $N = C_\alpha(L(s_\alpha \mu))$. The embedding $L(s_\alpha \mu) \hookrightarrow \delta(M(s_\alpha \mu))$ defines via 2.4 an embedding $N \hookrightarrow C_\alpha(\delta(M(s_\alpha \mu)))$. Let κ_0 denote the restriction of κ to N.

Lemma. *The following three assertions are equivalent.*

(i) $\overline{M(\mu)} = M(s_\alpha \mu) + C_\alpha(\overline{M(s_\alpha \mu)})$.

(ii) $\operatorname{Im} \kappa_0 = L(\mu)$.

(iii) $N/N \cap \delta(M(s_\alpha \mu)) = L(\mu)$.

(ii)$\Leftrightarrow$(iii) is immediate. Set $K = (N \cap \delta(\overline{M(s_\alpha \mu)}))/L(s_\alpha \mu)$. This is a submodule of $\delta(\overline{M(s_\alpha \mu)})$ and by 3.6 it is α-finite. Thus δK is a quotient of $\overline{M(s_\alpha \mu)}/D_\alpha^+(\overline{M(s_\alpha \mu)})$.

Comparison with 3.5 then shows that (iii) implies 5.1(iii) and hence (i). That (i)$\Rightarrow$(ii) obtains as a special case of the more general result given below.

5.4. Take α, μ as 5.1 and choose $v \in \Lambda$ such that $s_\alpha v < v$. Recall (4.7) that we have an exact sequence

$$0 \to L(M(\mu), \delta M(s_\alpha v)) \to L(M(s_\alpha \mu), \delta(M(s_\alpha v))) \xrightarrow{\kappa'} L(M(\mu), \delta M(v)).$$

Let κ_0' denote the restriction of κ' to the submodule $L(L(s_\alpha \mu), \delta M(s_\alpha v))$ of $L(M(s_\alpha \mu), \delta M(s_\alpha v))$ (defined by the map $M(s_\alpha \mu) \twoheadrightarrow L(s_\alpha \mu)$). Let $Q(\mu)$ denote the quotient of $M(\mu)$ defined in 5.1.

Proposition. *Suppose* $Q(\mu)/L(s_\alpha \mu) = L(\mu)$.

Then $\operatorname{Im} \kappa_0'$ *is a submodule of* $L(L(\mu), \delta(M(v)/M(s_\alpha v)))$ *(the latter being identified with a submodule of* $L(M(\mu), \delta M(v))$*).*

By 4.7 we have the commuting diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & L(M(\mu), \delta M(s_\alpha v)) \arrow[r] & L(M(\mu), C_\alpha(\delta M(s_\alpha v))) \arrow[r, "\kappa'"] & L(M(\mu), \delta(M(v)/M(s_\alpha v))) \\
0 \arrow[r] & L(Q(\mu), \delta M(s_\alpha v)) \arrow[r] \arrow[u] & L(Q(\mu), C_\alpha(\delta M(s_\alpha v))) \arrow[r, "\kappa_0''"] \arrow[u] & L(Q(\mu), \delta(M(v)/M(s_\alpha v))) \arrow[u] \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

where κ_0'' is defined by restriction.

In 3.8 take $N = \delta(M(s_\alpha v))$, $M = L(s_\alpha \mu)$ (which are both α-free) and $M' = Q(\mu)$ which by its definition is a submodule of $C_\alpha(L(s_\alpha \mu))$. By 3.8 (i), (ii) we obtain $L(Q(\mu), C_\alpha(\delta M(s_\alpha v))) \cong L(L(s_\alpha \mu), \delta M(s_\alpha v))$. Again $\delta(M(v)/M(s_\alpha v))$ is α-finite whereas $L(s_\alpha \mu)$ is α-free. Given that $Q(\mu)/L(s_\alpha \mu) = L(\mu)$ it follows that the natural embedding $L(L(\mu), \delta(M(v)/M(s_\alpha v))) \hookrightarrow L(Q(\mu), \delta(M(v)/M(s_\alpha v)))$ is an isomorphism. This proves the proposition.

Remark. Take $v \in \Lambda$ dominant and regular. The conclusion of 5.3 (ii) is equivalent to the existence of an exact sequence

$$0 \to L(M(v), \operatorname{Ker} \kappa_0) \to L(M(v), C_\alpha(\delta L(s_\alpha \mu))) \xrightarrow{\kappa_0} L(M(v), \delta L(\mu)) \to 0. \qquad (**)$$

In 3.8 taking $M = M(s_\alpha v)$, $M' = C_\alpha(M(s_\alpha v))$, $N = \delta(L(s_\alpha \mu)) \cong L(s_\alpha \mu)$ gives $L(M(v), C_\alpha(\delta L(s_\alpha \mu))) \cong L(M(s_\alpha v), \delta L(s_\alpha \mu))$.

Through η (2.8) and (G.J, 1.9), (**) is equivalent to the exact sequence

$$0 \to L(\delta(\operatorname{Ker} \kappa_0), \delta M(v)) \to L(L(s_\alpha \mu), \delta M(s_\alpha v)) \xrightarrow{\kappa_0'} L(L(\mu), \delta M(v)) \to 0.$$

We conclude that the conclusion of 5.3 (ii) is just the special case of the conclusion of 5.4 when v is dominant and regular.

Hence (i)$\Rightarrow$(ii) in 5.3.

5.5. Assume the Kazhdan-Lusztig conjecture to hold (cf. [1, 3]).

Corollary. *Take* $\alpha \in B_\lambda$, $\mu \in \Lambda$ *with* $\mu > s_\alpha \mu$.

Then the map κ': $L(M(s_\alpha \mu), \delta M(s_\alpha \mu)) \to L(M(\mu), \delta M(\mu))$ *restricts to a map* $L(L(s_\alpha \mu), L(s_\alpha \mu))$ *into* $L(L(\mu), L(\mu))$.

Apply 5.4 with $\mu = v$ and use η of 2.8.

5.6. The above result is a comparison theorem for Kostant's problem. The above analysis shows that its proof requires an essential use of a deep consequence (namely 5.1 (*)) of the Kazhdan-Lusztig conjecture. Let M be a $U(\mathfrak{g})$ module. The action of $U(\mathfrak{g})$ on M defines a homomorphism of $U(\mathfrak{g})$ into $L(M, M)$ with kernel $\operatorname{Ann} M$.

For M simple, Kostant has asked if this map is always surjective. This question has a very far from positive answer even assuming $M \in Ob\,\mathcal{O}$ (see [14], Sects. 1, 2 and references therein). Through 5.5 we have a map κ_0': $L(L(s_\alpha \mu), L(s_\alpha \mu)) \to L(L(\mu), L(\mu))$ which defines by restriction a map κ_{00}': $U(\mathfrak{g})/J(s_\alpha \mu) \to U(\mathfrak{g})/J(\mu)$ (notation G.J, 1.2). Recalling that κ_0' is the restriction of a map κ' whose kernel $L(M(\mu), \delta(M(s_\alpha \mu)))$ does not contain the trivial $\mathfrak{k}$ module, it follows that κ_{00}' is surjective. Consequently $J(s_\alpha \mu) \subset J(\mu)$ with equality if and only if $\operatorname{Ker} \kappa_{00}' = 0$. (The above inclusion is an old result of Duflo and was based on a rather similar argument; but using product formulae for the Kunze-Stein intertwining operators.)

Suppose $J(s_\alpha \mu) = J(\mu)$. In general this does *not* imply $\operatorname{Ker} \kappa_0' = 0$. Yet in this case $\operatorname{Ker} \kappa_0'$ is a submodule of $L(L(s_\alpha \mu), L(s_\alpha \mu))$ distinct from $U(\mathfrak{g})/J(s_\alpha \mu)$ and can be non-zero only if the embedding $\operatorname{Fract}(U(\mathfrak{g})/J(s_\alpha \mu)) \hookrightarrow \operatorname{Fract} L(L(s_\alpha \mu), L(s_\alpha \mu))$ is strict (cf. [14], Sect. 2). In view of ([12], 9.1) this gives the

Lemma. *Assume that* R_λ *(notation G.J, 1.2) has only type* A_n *factors. Take* $\alpha \in B_\lambda$, $\mu \in \Lambda$: $s_\alpha \mu < \mu$. *If* $J(s_\alpha \mu) = J(\mu)$, *then* κ_0': $L(L(s_\alpha \mu), L(s_\alpha \mu)) \to L(L(\mu), L(\mu))$ *is an embedding.*

5.7. Unfortunately 5.6 does not lead to any new cases for which Kostant's problem has a positive answer. What we need is a corresponding control on $\operatorname{Coker} \kappa_0'$. Assume α, μ as in 5.6 with $\mu \in W_\lambda \lambda$. The most optimistic conjecture is that either $\operatorname{Coker} \kappa_0' = 0$ or $\operatorname{Soc}_U(\operatorname{Coker} \kappa_0')$ is a simple U module of the same Gelfand-Kirillov dimension as $U(\mathfrak{g})/J(\mu)$.

Admitting the Kazhdan-Lusztig conjecture to hold we have by ([15], 4.13) that $L(L(\mu), L(\mu))/\operatorname{Soc}_U L(L(\mu), L(\mu))$ has strictly lower Gelfand-Kirillov dimension then $U(\mathfrak{g})/J(\mu)$. We would then conclude that $\operatorname{Soc}_U(\operatorname{Coker} \kappa_0')$ is a simple submodule of $L(L(\mu), L(\mu))$ and furthermore by the argument of ([7], II, 2) isomorphic to some $V(-\sigma\lambda, -\lambda)$ with $\sigma \in W_\lambda$ an involution. By say ([12], 4.1) this involution must be outside the "Duflo set" ([15], 3.5) and we expect every such involution to occur. We remark that $\mathrm{Soc}_U(\mathrm{Ker}\,\kappa_0')$ need not always be simple (as in type C_3). Also the U module homomorphism κ_0': $L(L(s_\alpha\mu), L(s_\alpha\mu)) \to L(L(\mu), L(\mu))$ is not in general an algebra morphism (for example in type G_2).

5.8. Take $\mu \in \Lambda$ and let $\{M^i(\mu)\}_{i\in\mathbb{N}}$ be a Jantzen filtration ([10], 4.1) for the Verma module $M(\mu)$. Now assume $\alpha \in B_\lambda$ satisfies $\mu > s_\alpha\mu$. The considerations of 5.1 lead us to the following which we state as a

Conjecture. $M^i(\mu) = M^{i-1}(s_\alpha\mu) + C_\alpha(M^i(s_\alpha\mu)), \forall i.$

When $i=0$, it is just 2.5 and for $i=1$ it is just 5.1(*). It may be regarded as a refinement of the Jantzen conjecture (cf. [10], 4.2). It therefore implies the Kazhdan-Lusztig conjecture and in fact an important refinement of it ([10], 4.9). Conversely this refinement can be shown to imply the above conjecture. In a subsequent paper we shall show that the conjecture is equivalent to a certain orthogonality conjecture for the canonical form ([10], 1.3.7) on a Verma module defined over $C[t]$ localized at its augmentation ideal. In specialization (i.e. to zero order in t) this conjecture is equivalent to 5.1 (*) and is therefore implied by the Kazhdan-Lusztig conjecture.

Index of Notation

Symbols used frequently are listed below in order of appearance.

1.	$\mathfrak{g}, \mathfrak{h}, R, R^+, B, \mathfrak{n}^+, \mathfrak{n}^-, U(\mathfrak{g})$.
2.	$\lambda, \alpha^\vee$.
2.1.	$\mathcal{O}, \Lambda, P(R), \Lambda^+, \mathcal{O}_\Lambda, \Omega, B_\lambda, C_\alpha, L(,)$.
2.2.	$\mathcal{H}, \mathfrak{k}, \mathcal{H}_{\hat{\lambda}}, T, M(\lambda), T', \hat{\lambda}, W, U$.
2.3.	ε.
2.4.	$L(\mu), X_{-\alpha}$.
2.6.	$\check{a}, {}^t a, \tau$.
2.7.	δ.
2.8	$\eta, \zeta_\mu^N, L(-\mu, -\nu), P$.
2.11.	$D_\alpha^+, D_\alpha^-, D_\alpha$.
2.12.	$\tilde{C}_\alpha$.
3.1.	$\psi_\alpha, \varphi_\alpha, \theta_\alpha, \psi_\alpha^l, \varphi_\alpha^l, \theta_\alpha^l$.
3.10.	$C_w, S_\lambda, l_\lambda, \mathcal{O}_{\Lambda,\mathcal{S}}$.
4.1.	F_α^M.
4.4.	κ_α^M.
5.1.	$\overline{M(\mu)}, Q(\mu)$.

References

1. Beilinson, A.A., Bernstein, I.N.: Localisation of $\mathfrak{g}$ modules. Comptes Rendus **292** A, 15–18 (1981)
2. Bernstein, I.N., Gelfand, S.I.: Tensor products of finite and infinite dimensional representations of semisimple Lie algebras. Compos. Math. **41**, 245–285 (1980)
3. Brylinski, J.L., Kashiwara, M.: Kazhdan-Lusztig conjecture and holonomic systems. Invent. Math. **64**, 387–410 (1981)
4. Deodhar, V.V.: On a construction of representations and a problem of Enright. Invent. Math. **57**, 101–118 (1980)
5. Dixmier, J.: Algèbres envoloppantes. Cahiers Scientifiques, XXXVII, Gauthier-Villars, Paris, 1974
6. Duflo, M.: Représentations irréductibles des groupes semi-simples complexes. In: Analyse Harmonique sur les Groupes de Lie. pp. 26–88, Lecture Notes in Mathematics, vol. 497. Berlin-Heidelberg-New York: Springer 1975
7. Duflo, M.: Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple. Ann. of Math. **105**, 107–130 (1977)
8. Enright, T.J.: On the fundamental series of a real semisimple Lie algebra: their irreducibility, resolutions and multiplicity formulae. Ann. Math. **110**, 1–82 (1979)
9. Gabber, O., Joseph, A.: The Bernstein-Gelfand-Gelfand resolution and the Duflo sum formula. Compos. Math. **43**, 107–131 (1981)
10. Gabber, O., Joseph, A.: Towards the Kazhdan-Lusztig conjecture. Ann. Ec. Norm. Sup. **14**, 261–302 (1981)
11. Jantzen, J.C.: Moduln mit einen höchsten Gewicht. Lecture Notes in Math., vol. 750. Berlin-Heidelberg-New York: Springer 1980
12. Joseph, A.: Kostant's problem, Goldie rank, and the Gelfand-Kirillov conjecture. Invent. Math. **56**, 193–204 (1980)
13. Joseph, A.: Dixmier's problem for Verma and principal series submodules. J. London Math. Soc. **20**, 193–204 (1979)
14. Joseph, A.: Kostant's problem and Goldie rank. In: Lecture Notes in Math., vol. 880. Berlin-Heidelberg-New York: Springer 1981
15. Joseph, A.: Goldie rank in the enveloping algebra of a semisimple Lie algebra. III. J. Alg. **73**, 295–326 (1981)
16. Vogan, D.: Irreducible characters of semisimple Lie groups II. The Kazhdan-Lusztig conjectures. Duke Math. J. **46**, 805–859 (1979)

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