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Stable base change $\mathbb{C}/\mathbb{R}$ of certain derived functor modules

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0. Introduction

Suppose $G(\mathbb{C})$ denotes the complex points of a connected semisimple linear algebraic group defined over $\mathbb{Q}$. Let σ_G be the action on $G(\mathbb{C})$ of the nontrivial element of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$. Let $G(\mathbb{R})$, or, for short, G, denote the σ_G-fixed points of $G(\mathbb{C})$. In [3], Arthur considered the problem of formulating some of the principles of functoriality with respect to L-groups in such a way as to be valid for non-tempered representations, making a number of conjectures. This paper proves a special case of these principles (although base change was not mentioned in [3]).

In [7], Clozel investigated stable base change $\mathbb{C}/\mathbb{R}$ for tempered representations of G. The irreducible tempered representations of G are partitioned into L-packets (see (3.3)), also called tempered. These tempered L-packets are classified by Langlands parameters, which are (equivalence classes of) admissible homomorphisms $\phi: W_{\mathbb{R}} \to {}^LG$, where $W_{\mathbb{R}}$ denotes the Weil group of $\mathbb{R}$ and ${}^LG$ is a (non-connected) complex group. The L-packet parametrized by ϕ will be denoted Π_ϕ, in spite of the confusion that arises. Every Π_ϕ gives rise to an irreducible tempered representation of $G(\mathbb{C})$ by means of an operation called base change lifting. (The L-packets of a complex group are sets with a single element.) Call this representation Π. The operation can be characterized in two independent ways, which are proven to be equivalent. The first is the base change lift of parameters: given ϕ, one may regard $W_{\mathbb{R}} = \mathbb{C}^\times$ (the Weil group of $\mathbb{C}$) as a subgroup of $W_{\mathbb{R}}$; the L-groups of $G(\mathbb{C})$ is the connected component of the identity of ${}^LG$, denoted ${}^LG^0$. Hence $\phi|_{W_{\mathbb{C}}}$ is a Langlands parameter for $G(\mathbb{C})$, and parametrizes the (L-packet of the) representation Π.

On the other hand, suppose given Π a representation of $G(\mathbb{C})$. We say Π is σ stable if, defining Π^σ to be the representation of $G(\mathbb{C})$ on the same space as Π but given by $\Pi^\sigma(g) = \Pi(\sigma_G(g))$ for $g \in G(\mathbb{C})$, we have that Π is equivalent to Π^σ. Let A_σ denote an intertwining operator yielding this equivalence. If Π is σ-stable and

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irreducible (or at least if Schur's lemma holds) then we can require that $(A_\sigma)^2 = 1$. Then we can define the twisted character of Π as follows. One can extend Π to a representation Π^* of $G(\mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by letting σ act by A_σ. Clozel (see also Bouaziz [5]) has shown that the character of Π^* is a locally L^1 function. When restricted to $G(\mathbb{C}) \times \{\sigma\}$ this character can be regarded as a function on $G(\mathbb{C})$, is denoted $\tilde{\chi}_\Pi$, and is called the twisted character of Π. (Note that there is an ambiguity of sign in this definition.)

If Π is the base change lift of Π_ϕ then Π is σ-stable. But it is not the case that every σ-stable irreducible tempered representation has a parameter coming from $G(\mathbb{R})$ as described above. Those that do are called stably σ-stable. In this paper, as in [7], we treat all parameters coming from $G(\mathbb{R})$. When the parameter Π does come from the parameter ϕ for $G(\mathbb{R})$, the twisted character of Π is related to the usual character χ_ϕ of the representation of $G(\mathbb{R})$,

$$\sum_{\pi \in \Pi_\phi} \pi$$

as follows. Assume for simplicity that $G(\mathbb{C})$ is simply connected. The intersection of a regular conjugacy class of $G(\mathbb{C})$ with $G(\mathbb{R})$ is called a stable conjugacy class. There is a norm "map" N from suitably nice elements of $G(\mathbb{C})$ to stable conjugacy classes of $G(\mathbb{R}) - N(g)$ can be taken to be any element in the stable conjugacy class which is the intersection of the conjugacy class of $g\sigma_G(g)$ in $G(\mathbb{C})$ with $G(\mathbb{R})$. Then under the foregoing hypotheses, Clozel has established the (stable) base change character identity,

$$\tilde{\chi}_\Pi(g) = \pm\, \chi_\phi(N(g)). \tag{0.1}$$

This is an example of L-functoriality, which (for local fields) asserts the existence of (specific) character identities between representations whose Langlands parameters are connected by suitable homomorphisms between L-groups (the homomorphism is described in Sect. 3).

For non-tempered representations this precise formulation cannot be correct, for reasons discussed by Arthur in [3]. For example, the ambiguity in the definition of $N(g)$ requires that χ_ϕ take the same value on the different elements in the stable conjugacy class of $N(g)$. If ϕ (i.e. Π_ϕ) is non-tempered, this may fail. Thus the formulation of the stable base change character identity should use Arthur's (conjectured) enlarged packets, see [3]. These are packets $\hat{\Pi}_\psi$, parametrized by Arthurian parameters, no longer disjoint. Arthur's conjectures, in the case of a local field, include the following (let's assume the field is archimedean). Under a simple condition on $\psi, \hat{\Pi}_\psi$ contains only unitary representations. There exist integers c_i, specified in a simple way by ψ, such that

$$\sum_{\pi_i \in \hat{\Pi}_\psi} c_i \chi_{\pi_i}$$

(here χ_π denotes the usual character of a representation π) is a stable distribution, i.e. is well-defined on stable conjugacy classes. Analogous results should hold for the enlarged packets to the results which Shelstad [14–16] has established for tempered L-packets, i.e. the cases of L-functoriality applied to endoscopic groups.

In [2], Adams and this author defined enlarged packets for a certain class of derived functor modules, and verified some of the above conjectures (unitarity is due to Vogan [14] and, independently, Wallach [26]). This is the same class that is studied in this paper, so let us review here the definition of this class. In [22], Kumaresan and [25], Vogan–Zuckerman classified all irreducible unitary modules affording non-zero $(\mathfrak{g}_0, K)$-cohomology (here K is a maximal compact subgroup of $G, \mathfrak{g}_0$ is the Lie algebra of G, and $\mathfrak{g}$ is the Lie algebra of $G(\mathbb{C})$). These modules are called $A_{\mathfrak{q}}(\lambda)$'s, for various parabolic subalgebras $\mathfrak{q} \subseteq \mathfrak{g}$ and unitary characters λ of $\mathfrak{q}$, and are constructed by a derived functor construction. This is the class for which [2] verifies Arthur's conjectures.

The aim of the present paper is to formulate and prove an analogue to (0.1) for these enlarged packets (when the infinitesimal character of such a representation $A_{\tilde{\mathfrak{q}}}(\tilde{\lambda})$ of $G(\mathbb{C})$ is the same as that of $F \otimes F, F$ an irreducible finite-dimensional representation of $\mathfrak{g}_0$, it will be shown that $A_{\tilde{\mathfrak{q}}}(\tilde{\lambda})$ is stably σ-stable). The method has some similarities to the method of [2]. The enlarged packets have character formulas expressing the stable character associated to the packet in terms of pseudo-L-packets. Then there are two questions.

The first one is similar to that of [2]: (note that Clozel has shown that an analogue of (0.1) holds for pseudo-L-packets). It is; do these character formulas behave well under base change? The answer is as follows. On identifying a representation with its character, we have from [9] that

$$A_{\tilde{\mathfrak{q}}}(\tilde{\lambda}) = \sum_{w \in W(\tilde{\mathfrak{l}})} (-1)^{\ell(w)} X_w \tag{0.2}$$

where $\tilde{\mathfrak{l}}$ is the Levi factor of $\tilde{\mathfrak{q}}$, $W(\tilde{\mathfrak{l}})$ is the Weyl group of $\tilde{\mathfrak{l}}, \ell(w)$ is the length of w, and X_w is a pseudo-L-packet (again, a singleton) associated to w of representations of $G(\mathbb{C})$. Then there is $\mathfrak{q}$, a parabolic subalgebra of $\mathrm{Lie}(G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}$, and λ such that, putting $\hat{\Pi}_\psi$ as the enlarged packet defined by (note, *not* "containing" since this wouldn't be unique) $(A_{\mathfrak{q}}(\lambda), \mathfrak{q}, \lambda)$, we have, firstly,

$$\sum_{\pi \in \hat{\Pi}_\psi} (-1)^{\ell(w)} \chi_\pi = \sum_{\eta \in T} (-1)^{\ell(\eta)} X_\eta \tag{0.3}$$

where T is a certain set of parameters of pseudo-L-packets (analogous to $W(\tilde{\mathfrak{l}})$), $\ell(\eta)$ means the length of η, and X_η is the (character of the) pseudo-L-packet associated to η, and secondly, the base change lifting of parameters matches bijectively the set T with $\{w \in W(\tilde{\mathfrak{l}}): X_w \text{ is } \sigma\text{-stable}\}$.

The second one is not similar. It is, is there an analogue to (0.2) for twisted characters? Of course all the preceding is useless without this. In particular, Schur's lemma does not hold for the right hand side of (0.2), so the ambiguity in the definition of twisted character is much worse than that of a single sign. In this paper it is shown, firstly, that the intertwining operators can be chosen such that the twisted character of the left hand side of (0.2) is equal to the twisted character of the right hand side of (0.2). Secondly, it is shown that with this choice, one has also, for each $w \in W(\tilde{\mathfrak{l}})$ such that X_w is σ-stable, for nice $g \in G(\mathbb{C})$,

$$\tilde{\chi}_{X_w}(g) = \pm \chi_\eta(N(g)) \cdot (-1)^{\ell(w) + \ell(\eta)} \tag{0.4}$$

for a unique $\eta \in T$, with the sign independent of w and η. It is then clear that combining (0.4, 0.3), and the typographically horrible twisted analogue of (0.2), we have the appropriate analogue of (0.1).

The outline of the body of the paper is as follows. In Sect. 1 we introduce some of the notation in use throughout. In Sect. 2 we recall the definitions and results of Clozel needed to formulate the (stable) base change character identity. In Sect. 3 we recall the definitions of the pseudo-L-packets that play a role, the so-called standard modules (which occur in the Langlands classification), and reformulate the Langlands parameter formulation of base change in terms of Vogan's $\mathbb{Z}/2\mathbb{Z}$ character data.

In Sect. 4 we show that the intertwining operators can be chosen consistently for our purposes. We are forced to make a different choice of signs than Clozel's, in order to obtain a twisted character formula. To do this, we need to treat of the trivial representation in great detail. We rely heavily on an analogue for Harish–Chandra modules of the BGG resolution of a finite dimensional module, established in [10], and the related character formulas for $A_{\mathfrak{q}}(\lambda)$'s.

In Sect. 5 we recall the stable base change character identity proved by Clozel for tempered L-packets, and in fact, for pseudo-L-packets, including the ones intervening in Sect. 4 and the rest of the paper. We reformulate all this in terms of $\mathbb{Z}/2\mathbb{Z}$ character data, which is more convenient for the proofs of this paper. We show that the base change lift of a $\mathbb{Z}/2\mathbb{Z}$ character data parameter depends only on a positive root system together with an involution, in the sense of (10.9) of [21]. Finally, in Sect. 6 we simply combine all these results and deal with the question which was similar to that of [2].

1. Some notation

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{R}$. Then $G(\mathbb{C})$ will denote its complex points, and σ_G, the action of σ, the nontrivial element of the Galois group of $\mathbb{C}$ over $\mathbb{R}$, on $G(\mathbb{C})$. Let $G(\mathbb{R})$ be the σ_G-fixed points of $G(\mathbb{C})$. We will sometimes denote $G(\mathbb{R})$ by G. Every such group has a quasisplit inner form, and it will be obvious that the results of this paper behave well under inner twistings. For simplicity, then, we will assume G is quasisplit. Let $\tilde{\mathbf{G}} = \mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathbf{G}$ (notation of [19 p. 12]; cf [18 p. 399]) be the algebraic group obtained by considering $\mathbf{G}$ as defined over $\mathbb{C}$ and then restricting scalars to $\mathbb{R}$. We may take $\tilde{\mathbf{G}}$ concretely as follows: $\tilde{\mathbf{G}}(\mathbb{C}) = G(\mathbb{C}) \times G(\mathbb{C})$. Now $\tilde{\mathbf{G}}$ is defined over $\mathbb{R}$, and the action of σ on $\tilde{\mathbf{G}}(\mathbb{C})$ is denoted $\sigma_{\tilde{G}}$ and is given by $\sigma_{\tilde{G}}(x, y) = (\sigma_G y, \sigma_G x)$, $x, y \in G(\mathbb{C})$. Then $\tilde{G}(\mathbb{R}) = \{(x, \sigma_G x) : x \in G(\mathbb{C})\}$ and $G(\mathbb{C}) \xrightarrow{\sim} \tilde{G}(\mathbb{R}) : x \mapsto (x, \sigma_G x)$.

Let $\mathfrak{g}_0$ be the (real) Lie algebra of G. Let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, and similarly for other groups, including $\tilde{G}(\mathbb{R})$. Then the isomorphism of $G(\mathbb{C})$ with $\tilde{G}(\mathbb{R})$ yields an isomorphism (of real Lie algebras) of the Lie algebra of $G(\mathbb{C})$, identified with $\mathfrak{g}$, with the Lie algebra of $\tilde{G}(\mathbb{R})$, which is $\tilde{\mathfrak{g}}_0$. Then $\tilde{\mathfrak{g}} \cong \mathfrak{g} \times \mathfrak{g}$ by the usual map which satisfies two properties: firstly, giving $\tilde{\mathfrak{g}}$ the complex structure from $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \otimes_{\mathbb{R}} \mathbb{C}$, and giving $\mathfrak{g} \times \mathfrak{g}$ the product of the complex structures of $\mathfrak{g}$ and $\mathfrak{g}$ (from $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, respectively), this map is an isomorphism of complex Lie algebras. Secondly, identifying $\mathfrak{g}$ with $\tilde{\mathfrak{g}}_0 \to \tilde{\mathfrak{g}}$ via the preceding, the map sends $x \in \mathfrak{g}$ to $(x, \sigma_G x)$: note this is fixed under $\sigma_{\tilde{G}}$. Here σ_G acts on $\mathfrak{g}$ in the way induced by its action on

$G(\mathbb{C})$, and similarly $\sigma_{\tilde{G}}$. Now the map σ_G transferred to $\tilde{G}(\mathbb{R})$ via this isomorphism is in agreement with an algebraic automorphism α of $\tilde{G}(\mathbb{C})$ restricted to $\tilde{G}(\mathbb{R})$, $\alpha(x, y) = (y, x)$.

We will always let H denote a fixed maximally split Cartan subgroup of G. For any reductive group, similar notation to the above holds, thus $\sigma_H = \sigma_G|_{H(\mathbb{C})}$, σ_H acts on $\mathfrak{h}$, etc. Then $\tilde{\mathfrak{h}} \cong \mathfrak{h} \times \mathfrak{h}$ by the above isomorphism.

Given two choices Δ_i^+, $i = 1, 2$, of positive systems for the roots of $\mathfrak{h}$ in $\mathfrak{g}$, denoted $\Delta(\mathfrak{g}, \mathfrak{h})$, we let (Δ_1^+, Δ_2^+) be the positive system for $\Delta(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ defined by $\{(\alpha, 0): \alpha \in \Delta_1^+\} \cup \{(0, \alpha): \alpha \in \Delta_2^+\}$ where $(\alpha, 0) \in \mathfrak{h}^* \times \mathfrak{h}^* \cong \tilde{\mathfrak{h}}^*$ has the obvious meaning. Similarly for other Cartan subgroups H' of G, $\tilde{\mathfrak{h}}' = \mathfrak{h}' \times \mathfrak{h}'$, etc. If $\mathfrak{u}$ is any subspace of $\mathfrak{g}$ normalized by $\mathfrak{h}'$, let $\Delta(\mathfrak{u}, \mathfrak{h}')$ denote the set of roots of $\mathfrak{h}'$ in $\mathfrak{u}$, and $\rho(\mathfrak{u})$ or $\rho(\Delta(\mathfrak{u}, \mathfrak{h}'))$ denote

$$(1/2) \sum_{\alpha \in \Delta(\mathfrak{u}, \mathfrak{h}')} \alpha;$$

similarly, $\rho(\Delta_i^+)$. Let

$$\bar{\mathfrak{u}} = \bigoplus_{-\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}^\alpha$$

if $\mathfrak{u}$ as above is a subalgebra of $\mathfrak{g}$. Then

$$\Delta(\bar{\mathfrak{u}}) = -\Delta(\mathfrak{u}).$$

Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$. That is, let $W_{\mathbb{R}} = \mathbb{C}^\times \cup \tau\mathbb{C}^\times$ where $\tau^2 = -1$ and $\tau z \tau^{-1} = \bar{z}$, the complex conjugate of z. Let $W_{\mathbb{C}} = \mathbb{C}^\times \subseteq W_{\mathbb{R}}$ in the obvious way. Let $\Gamma = \{1, \sigma\} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then $W_{\mathbb{R}}$ is equipped with a projection homomorphism $W_{\mathbb{R}} \to \Gamma$ sending τ to σ and making

$$1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma \to 1$$

an exact sequence. We write $\tau = 1 \times \sigma$.

For unexplained notation, see [4, especially p. 29f.]: ${}^LG^0$ is the connected complex group defined there; the inner equivalence class of G defines an action of Γ and hence of $W_{\mathbb{R}}$ on ${}^LG^0$ by what is notated μ_G, we denote the action of σ on ${}^LG^0$ by ${}^L\sigma_G$ or simply ${}^L\sigma$, this is algebraic, and ${}^LG = {}^LG^0 \rtimes W_{\mathbb{R}}$ via this action. (If G is split, this action is trivial.) If $\mathbf{G}$ is considered as a group over $\mathbb{C}$, then we denote its L-group by ${}^LG_{\mathbb{C}}$ to avoid confusion, by definition ${}^LG_{\mathbb{C}} = {}^LG^0 \times W_{\mathbb{C}} = {}^LG^0 \times \mathbb{C}^\times$. If necessary to avoid confusion, we will notate ${}^LG$ as ${}^LG_{\mathbb{R}}$. We have ${}^LG_{\mathbb{C}} \subseteq {}^LG_{\mathbb{R}}$.

We let K be a maximal compact subgroup of G such that if θ is the corresponding Cartan involution of G and $\mathfrak{g}$, then H (and $\mathfrak{h}$) is θ-stable. Similarly for $\tilde{K}, \tilde{\theta}$, and $\tilde{\mathfrak{h}}$.

2. Base change character identity

We also denote by α the dual automorphism of ${}^L\tilde{G} \cong ({}^LG^0 \times {}^LG^0) \times W_{\mathbb{R}}$ which sends $(x, y) \times w$ to $(y, x) \times w$, $x, y \in {}^LG^0$, $w \in W_{\mathbb{R}}$. Here τ, or σ, acts on ${}^LG^0 \times {}^LG^0$ via ${}^L\sigma_{\tilde{G}}$ (or ${}^L\tilde{\sigma}$): $(x, y) \mapsto ({}^L\sigma(y), {}^L\sigma(x))$. Recall that a Langlands parameter for $G(\mathbb{R})$ is the equivalence class under conjugation by elements of ${}^LG^0$ of an admissible homomorphism $\phi: W_{\mathbb{R}} \to {}^LG$, denoted $\{\phi\}$. (Later in the paper we ignore the brackets.)

That is, $\phi(w) = g \times w$ if $w \in W_{\mathbb{R}}$, for some $g \in {}^LG^0$. Similarly, taking $\tilde{G}$ for G. A Langlands parameter for $G(\mathbb{C})$ as a group over $\mathbb{C}$ is an (equivalence class of an) admissible homomorphism from $W_{\mathbb{C}}$ into ${}^LG_{\mathbb{C}}$. These parametrize the L-packets of irreducible admissible Harish–Chandra modules of their respective groups. (For $G(\mathbb{C})$ and $\tilde{G}(\mathbb{R})$ these packets are singletons. Since $G(\mathbb{C}) \cong \tilde{G}(\mathbb{R})$ as Lie groups, we have two ways of parametrizing the same representations.) Suppose $\{\phi\}$ is a parameter for $G(\mathbb{R})$. We say that $\{\phi\}$ is α-stable if $\{\alpha \circ \phi\} = \{\phi\}$. We say that, if $\{\phi'\}$ is a parameter for $G(\mathbb{C})$, that $\{\phi'\}$ is σ_G-stable, or σ-stable, if, putting $\phi'^{\sigma}(z) = {}^L\sigma_G \phi'(\bar{z})$ we have $\{\phi'\} = \{\phi'^{\sigma}\}$.

Let us agree to call $\{\phi\}$ stably α-stable if there exists a $\psi \in \{\phi\}$ such that $\psi = \alpha \circ \psi$. if $\{\phi\}$ is stably α-stable then $\{\phi\}$ is "from $G(\mathbb{R})$" in the following sense: with ψ as above we have that ψ factors through the "diagonal embedding" $\imath : {}^LG \to {}^L\tilde{G}$ given by $\imath(g \times w) = (g, g) \times w$ for $g \in {}^LG^0$ and $w \in W_{\mathbb{R}}$. In general, not every α-stable parameter is stably α-stable, but Shelstad [18] has shown that every α-stable parameter factors through an α-twisted endoscopic group. In this paper we treat all derived functor modules in our class whose parameters are from $G(\mathbb{R})$.

Given a parameter $\{\phi\}$ for $G(\mathbb{R})$ we define its base change lift to be $\{\phi|_{W_{\mathbb{C}}}\}$ a parameter for $G(\mathbb{C})$, or equally, $\{\phi \circ \imath\}$, a parameter for $\tilde{G}(\mathbb{R})$, and denote (either) by $\mathrm{Lift}_{\mathbb{R}}^{\mathbb{C}} \phi$ or simply Lift ϕ. The precise form which the Langlands classification takes when restricted to the Harish–Chandra modules (with a fixed infinitesimal character) which occur in the resolution of a finite dimensional irreducible module will be recalled in Sect. 3. However, we here wish to recall some of the form. Via the Langlands classification, Lift ϕ determines a standard representation (see Sect. 3, under our assumptions, this means induced from a discrete series representation of a cuspidal parabolic in such a way that the Langlands subquotient is the unique irreducible submodule) of $G(\mathbb{C})$ or $\tilde{G}(\mathbb{R})$, denoted X (Lift ϕ). Let $\psi : W_{\mathbb{C}} \to {}^LG_{\mathbb{C}}$ yield a parameter for $G(\mathbb{C})$. Then $\{\psi\}$ is σ_G-stable precisely when the parameter $\tilde{\phi} : W_{\mathbb{R}} \to {}^L\tilde{G}$ which parametrizes the same standard representation as $X(\psi)$ is α-stable, and $\{\psi\} = \{\phi|_{W_{\mathbb{C}}}\}$ for some ϕ precisely when ϕ is stably α-stable.

Suppose Π is an irreducible representation of $G(\mathbb{C})$ (or $\tilde{G}(\mathbb{R})$). Then we call Π σ-stable (or α-stable) when, defining $\Pi^{\sigma}(g) = \Pi(\sigma_G(g))$ (or $\Pi^{\alpha}((g, h)) = \Pi((h, g))$) we have $\Pi \cong \Pi^{\sigma}$ (or $\Pi \cong \Pi^{\alpha}$) via an intertwining operator A_{σ}. We normalize A_{σ} by requiring $A_{\sigma}^2 = 1$, as we may by Schur's lemma. Repka and Clozel have shown that if Π is tempered then it is σ-stable (or α-stable) if and only if its parameter is. Then A_{σ} is determined up to a sign.

On the other hand, suppose $X = \mathrm{Ind}_{(MA)N}^{G}(\pi_{\delta} \otimes 1)$ is a standard module with infinitesimal character the same as some finite dimensional module of $G(\mathbb{C})$, with π_{δ} a representation of MA in the discrete series. Since X has a unique submodule $\bar{X}, \bar{X}$ is σ_G-stable if X is. Suppose the parameter $\{\phi\}$ for X is from $G(\mathbb{R})$. Then Clozel and Repka have shown the following. The representation π_{δ} is σ_{MA}-stable, so there exists a (normalized) intertwining operator for $\pi_{\delta}, A_{\sigma_{MA}}$. But this induces an operator on the space of X, called $\mathrm{Ind}(A_{\sigma_{MA}})$. Let A be the operator on the space of X (realized as functions on G) such that $(Af)(g) = f(\sigma_G g)$. Then $A_{\sigma} = A \circ \mathrm{Ind}\, A_{\sigma_{MA}}$ is an intertwining operator for X and X^{σ} and so X is σ_G-stable. Now if X is not σ_G-stable, then π_{δ} is not, so let $\pi_{\delta'} = \pi_{\delta}^{\sigma_{MA}}$. Then $X^{\sigma} = \mathrm{Ind}_{MAN}(\pi_{\delta'} \otimes 1)$ is not equivalent to X and so $\bar{X}^{\sigma} \neq \bar{X}$.

Note that since the standard modules possess unique irreducible submodules, A_σ is again determined up to a sign simply by the conditions of being a normalized intertwining operator.

If π is a representation then χ_π denotes the locally L^1-function which is its character (under the hypothesis that π is an admissible Harish–Chandra module of finite length). Sometimes, to avoid typography, we will identify a representation with its character, and, indeed, an L-packet (or pseudo-L-packet) with the sum of the characters of the elements of the packet (or even enlarged packet). If Π is a σ-stable representation of $G(\mathbb{C})$ and A_σ is a choice of normalized intertwining operator then, under mild assumptions on Π (admissibility, finite length which conditions we suppress, once and for all), Clozel has shown that the distribution on $G(\mathbb{C})$ defined by $f \mapsto \mathrm{Trace}(A_\sigma \circ \pi(f))$, for $f \in C_c^\infty(G(\mathbb{C}))$ and

$$\pi(f) = \int_{g \in G(\mathbb{C})} \pi(g) f(g) dg$$

in the usual way, can be represented by a locally L^1 function which we denote by $\tilde{\chi}_\pi$. This depends on the choice of A_σ. If Π is reducible then, for example, different choices of A_σ could make $\tilde{\chi}_\pi$ sometimes zero, identically, or sometimes a multiple of the twisted trace of an irreducible. When Π is either irreducible or standard we will always use a specified (in Sect. 4) choice of A_σ, and hence $\tilde{\chi}_\pi$, which differs from Clozel's by a sign which will be discussed in detail later. When Π is one of the direct sums of standard modules occurring in, for example, the BGG resolution, we will need to choose A_σ carefully, see Sect. 4.

We will show later that an α-stable standard module's parameter is stably α-stable provided that the infinitesimal character is one of the type which we consider, namely, that of an α-stable finite dimensional module of $\tilde{G}(\mathbb{R})$.

We will need later the exact relationship between $\tilde{\phi}$ and ϕ when ϕ is a Langlands parameter for $G(\mathbb{C})$, $X(\phi)$ denoting the standard module parameterized by ϕ, similarly for $\tilde{\phi}$ a parameter for $\tilde{G}$ and $X(\tilde{\phi})$, when $X(\phi) = X(\tilde{\phi})$ and $\tilde{\phi}$ factors through the diagonal ι. This is, as is easy to see, that ϕ can be taken to be (after conjugation) Lift ψ where $\tilde{\phi} = \iota . \psi$.

With ψ now any parameter subject to our assumptions on infinitesimal character for $G(\mathbb{R})$, we denote by $X(\psi)$ the pseudo-L-packet (4.2 of [2]) parametrized by ψ: recall that this means the set of standard modules whose (unique, by assumption) Langlands submodules make up the usual L-packet parameterized by ϕ. As remarked above, we will also use $X(\psi)$ to denote both

$$\chi_\psi = \sum_{\pi \in X(\psi)} \chi_\pi$$

and

$$\bigoplus_{\pi \in X(\psi)} \pi.$$

Let $G^* = G(\mathbb{C}) \rtimes \Gamma$ where σ acts on $G(\mathbb{C})$ as σ_G (this group is sometimes called the base change group). A representation Π of $G(\mathbb{C})$ extends to a representation of G^* if and only if Π is σ-stable and there exists a choice of normalized intertwining operator A_σ. Then Π can be extended by taking any such choice of A_σ and defining $\Pi(1 \times \sigma) = A_\sigma$. Then

$$\tilde{\chi}_{\Pi}(g) = \chi_{\Pi}(g \times \sigma)$$

(cf. Bouaziz, [5]). This point of view will be used at one point.

Shelstad [17, 2.5] has defined a norm map in general for our G. Given $x \in G(\mathbb{C})$ such that $(\sigma_G x) \cdot x$ is regular semisimple (in $G(\mathbb{C})$) (such x are called σ-regular), $N(x) \in G(\mathbb{R})$ is defined up to stable conjugacy in $G(\mathbb{R})$. We omit the definition of N in general, but recall it in two cases: if G is semisimple and simply-connected then the conjugacy class in $G(\mathbb{C})$ of $\sigma_G(x) \cdot x$ is defined over $\mathbb{R}$, so, by a theorem of Steinberg, meets $G(\mathbb{R})$, by definition this is a stable conjugacy class of $G(\mathbb{R})$ any element of which may be called $N(x)$; if G is a torus then $N(x) = \sigma_G(x) \cdot x \in G(\mathbb{R})$ is well-defined; the general case is a combination of these.

It is proved in [7, 2.4.3] that every Cartan subgroup T of G satisfies: every regular element in the connected component of the identity of T is a norm of some σ-regular, σ-semisimple x.

Clozel has proved the following character identity:

$$\tilde{\chi}_{\Pi}(g) = \varepsilon_M \chi_{\phi}(N(g)) \tag{2.1}$$

for $g \in G(\mathbb{C})$ σ-regular, $\phi: W_{\mathbb{R}} \to {}^L G$ a tempered Langlands parameter, $\Pi = X(\mathrm{Lift}_{\mathbb{R}}^{\mathbb{C}} \phi)$ and $\varepsilon_M = \pm 1$ is determined when A_σ is chosen in a specific fashion which we do not recall since it differs from ours (to be specified later) by a sign. (Note that since ϕ is tempered, the notions of pseudo-L-packet and L-packet yield the same representation.)

Furthermore, suppose that F is an irreducible finite dimensional representation of $G(\mathbb{R})$ and that ϕ is such that

$$0 \to F \to X(\phi)$$

(in particular ϕ is not tempered). Then there is an irreducible finite dimensional representation $\mathrm{Lift}(F)$ of $G(\mathbb{C})$ with

$$0 \to \mathrm{Lift}(F) \to X(\mathrm{Lift}\, \phi)$$

and we have

$$\tilde{\chi}_{\mathrm{Lift}(F)}(g) = \chi_F(N(g)). \tag{2.2}$$

In fact, identifying $\tilde{\mathfrak{g}}$ with $\mathfrak{g} \times \mathfrak{g}$ and $\tilde{G}(\mathbb{R})$ with $G(\mathbb{C})$, Lift $F = F \otimes F$, with $A_\sigma(x \otimes y) = (y \otimes x)$. Clozel does not deduce this from (2.1), it is an easy calculation.

In fact, Clozel has proved that (2.1) holds for ϕ arbitrary, since we have used the symbol denoting pseudo-L-packets in the formulation of (2.1). But the form of (2.1) does not allow one to even formulate a base change character identify for Π an irreducible non-tempered representation, as remarked in the introduction to this paper. We will formulate and prove an analogue for the class of irreducible unitary modules affording non-zero $(\mathfrak{g}, K)$-cohomology.

3. Base change of parameters

We recall the definition of standard modules, but impose for our own purposes an extra condition, "a)" below, not present in the standard definition (and allowing us to reformulate conditions b) and c)).

(3.1) **Definition.** Any module X which satisfies the following three conditions is called a standard module.

a) The infinitesimal character of X is the same as F, some finite dimensional representation of G.

b) There exist a cuspidal parabolic subgroup P of G with Langlands decomposition $P = MAN$, a discrete series representation π_δ of M, and a character ν of A, such that

$$X = \mathrm{Ind}_{MAN}^{G}(\pi_\delta \otimes \nu \otimes 1).$$

c) The module X has a unique irreducible submodule, denoted $\bar{X}$, which contains the lowest K-type of X (and hence is the Langlands subquotient of X).

As in [22, p. 99] and [20, p. 387f], we can parametrize the distinct standard modules with the same infinitesimal character as F by $\mathbb{Z}/2\mathbb{Z}$ character data.

(3.2) **Definition.** A set of $\mathbb{Z}/2Z$ character data is (with an implicit choice of F) the G-conjugacy class of a triple (H_1, Δ^+, χ) such that H_1 is a θ-stable Cartan subgroup, $\Delta^+ \subseteqq \Delta(\mathfrak{g}, \mathfrak{h}_1)$ is a choice of positive system, and $\chi: H_1/H_1^0 \to \mathbb{Z}/2\mathbb{Z}$ is a character. The standard module parametrized by (H_1, Δ^+, χ) is denoted $X(H_1, \Delta^+, \chi)$ and its unique submodule, by $\bar{X}(H_1, \Delta^+, \chi)$.

Then the Langlands classification of irreducible admissible Harish–Chandra modules with infinitesimal character the same as F says that every such module is isomorphic to some $\bar{X}(H_1, \Delta^+, \chi)$ with the conjugacy class of (H_1, Δ^+, χ) uniquely determined.

(3.3) **Definition.** An L-packet of representations with infinitesimal character the same as F is a set of the form $\{\bar{X}: X = \mathrm{Ind}_{M_0A_0N_0}^{G}(\pi_\delta \otimes \nu_0 \otimes 1)$ and π_δ has the same central character and infinitesimal character as $\pi_{\delta_0}\}$ where $M_0A_0N_0, \pi_{\delta_0}$, and ν_0 are chosen fixed such that π_{δ_0} is a discrete series representation of M_0 such that

$$X_0 = \mathrm{Ind}_{M_0A_0N_0}^{G}(\pi_{\delta_0} \otimes \nu_0 \otimes 1)$$

is a standard module with the same infinitesimal character as F.

(3.4) **Definition.** A pseudo-L-packet of representations with infinitesimal character the same as F is a set of the form

$$\{X \text{ a standard module: } \bar{X} \in \Pi\}$$

where Π is an L-packet as above.

The pseudo-L-packets are disjoint, and so the unique pseudo-L-packet containing X is denoted $\Pi(X)$.

We need to use the connection between (P, δ, ν) and (H_2, Δ^+, χ), $\delta \in \mathfrak{h}_1^*$, given by

$$\Pi(\mathrm{Ind}_{MAN}^{G}(\pi_\delta \otimes \nu \otimes 1) = \Pi(X(H_2, \Delta^+, \chi)), \tag{3.5}$$

at least when $\chi = 1$ and $F = \mathbb{C}$. (Here $\mathfrak{h}_1$ is the fundamental Cartan of MA and $\delta \otimes \nu$ is the Harish–Chandra parameter of $\pi_\delta \otimes \nu$.) It is:

(3.6) a) H_2 is the fundamental Cartan in MA, so $H_2 = TA$ where T is a compact Cartan subgroup of M;

b) $\rho(\Delta^+)|_{\mathfrak{t}} = \delta$;

c) $\rho(\Delta^+)|_{\mathfrak{a}} = d\nu$,

where $d\nu$ is the differential of ν, also written ν;

d) the central character of π_δ is chosen so that the central character of

$$\mathrm{Ind}_{MAN}^G(\pi_\delta \otimes \nu \otimes 1)$$

is the same as that of $\mathbb{C}$.

Note that, as we will prove later (5.3), we can ignore condition d if we are only interested in the values of the characters on the image of the norm map.

We will also need (part of) the connection between Langlands parameters and $\mathbb{Z}/2\mathbb{Z}$ character data. Given (H_1, Δ^+, χ) define (P, δ, ν) as above. Then $\Pi(\pi_\delta \otimes \nu)$ is a discrete series L-packet and thus determines an admissible homomorphism $\phi\colon W_{\mathbb{R}} \to {}^L(MA)$. By [4], we have that for $z \in \mathbb{C}^\times \subset W_{\mathbb{R}}$,

$$\phi(z) = z^{\mu} \bar{z}^{\nu} \times z \in {}^L H_1^0 \times \{z\} \tag{3.7}$$

with $\nu = {}^L\sigma_M w \mu$, $\mu = \rho(\Delta^+)$, and $w \in W(\mathfrak{g}, \mathfrak{h}_1)$ chosen so that ${}^L\sigma_M \circ w$ is the automorphism of $\mathfrak{h}_1^*$ which is -1 on the span of the roots of M in $\mathfrak{h}_1^*$, and 1 on the center of $\mathfrak{m} \oplus \mathfrak{a}$. Here ${}^L\sigma_M$ is the action of τ on ${}^L(MA)^0$ defining the L-group of MA, restricted to ${}^L H^0$ and transferred to $\mathfrak{h}_1^*$ the Lie algebra of ${}^L H_1^0$. Since μ is in the span of these imaginary roots, then ${}^L\sigma_M w \mu = \sigma_{H_1} \mu$ where σ_{H_1} is the Galois action of σ_G on $H_1(\mathbb{C})$ inherited by

$$\mathfrak{h}_1 = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}_1)} \mathbb{R} \cdot H_{1,\alpha} \otimes_{\mathbb{R}} \mathbb{C} \tag{3.8}$$

and transferred to $\mathfrak{h}_1^*$ via the map which sends $H_{1,\alpha}$, the α-root-vector, to the root $\alpha \in \mathfrak{h}_1^*$. We will not need to know $\phi(\tau)$.

All the preceding concepts are defined with $\tilde{G}$ in place of G. In this case, $\Delta^+ = (\Delta_1^+, \Delta_2^+)$, and so $X(\tilde{H}, (\Delta_1^+, \Delta_2^+), 1)$ corresponds to an admissible homomorphism

$$\tilde{\phi}\colon W_{\mathbb{R}} \to {}^L\tilde{G} = ({}^L G^0 \times {}^L G^0) \rtimes W_{\mathbb{R}} \tag{3.9}$$

such that, putting a superscript zero on a parameter to denote the projection onto ${}^L\tilde{G}^0$ or ${}^L G^0$,

$$\tilde{\phi}^0(z) = z^{\rho((\Delta_1^+, \Delta_2^+))} \bar{z}^{\sigma_{\tilde{H}}(\rho(\Delta_1^+, \Delta_2^+))} \in {}^L\tilde{G}^0 \tag{3.10}$$

by the above, for here $\tilde{H} = MA$ is abelian, and so ${}^L\sigma_M = \sigma_{\tilde{H}}$, and $w = 1$.

Identifying $\mathfrak{h}^* \times \mathfrak{h}^*$ with $\tilde{\mathfrak{h}}^*$ in the obvious way, we can rewrite $\tilde{\phi}^0(z)$ as

$$(z^{\rho(\Delta_1^+)} \bar{z}^{\sigma_H \rho(\Delta_2^+)}, z^{\rho(\Delta_2^+)} \bar{z}^{\sigma_H \rho(\Delta_1^+)}) \in {}^L G^0 \times {}^L G^0, \tag{3.11}$$

using the fact that $\sigma_{\tilde{H}}(h_1, h_2) = (\sigma_H h_2, \sigma_H h_1)$. We may also assume $\tilde{\phi}(W_{\mathbb{R}}) \subseteqq {}^L\tilde{H}$ with ${}^L\tilde{H}$ embedded in ${}^L\tilde{G}$ as a Levi factor.

We wish to determine when $\tilde{\phi}$ is α-stable. It is clear that $\tilde{\phi}$ is equivalent to $\tilde{\phi}^\alpha$ if and only if the characters of $\tilde{H}$ determined by the factorization of $\tilde{\phi}$ and $\tilde{\phi}^\alpha$ through ${}^L\tilde{H}$ are such as, when induced from $\tilde{B}_1^*$ (respectively $\tilde{B}_2^*$) to $\tilde{G}$, yield the same representation (standard) of $\tilde{G}$ (here $\tilde{B}_i^*$ is a Borel subgroup of $\tilde{G}$ containing $\tilde{H}$ and defined over $\mathbb{R}$, which was notated P in (3.5)). For this to be true, it must be the case that $\tilde{\phi}^0$ and $(\tilde{\phi}^\alpha)^0$ are conjugate under the real Weyl group of $\tilde{H}$, and furthermore this suffices. Thus, to determine whether or not $\tilde{\phi}$ and $\tilde{\phi}^\alpha$ are conjugate, it suffices to check their values on $\mathbb{C}^\times$, since the fact that $\tilde{H}$ is connected implies that the value of $\tilde{\phi}(\tau)$ doesn't affect the characters we are considering (see [15, 4.1]).

Now $\tilde{\phi}^\alpha(z) = \alpha \circ \phi(z)$, and this is, by the above, the parameter associated to $X(\tilde{H}, (\Delta_2^+, \Delta_1^+), 1)$. Then $\tilde{\phi}^\alpha$ is conjugate to $\tilde{\phi}$, i.e. $X(\tilde{H}, (\Delta_2^+, \Delta_1^+), 1)$ is equivalent to $X(\tilde{H}, (\Delta_1^+, \Delta_2^+), 1)$, if and only if (Δ_1^+, Δ_2^+) is conjugate under the real Weyl group of $\tilde{H}$, denoted $W(\tilde{G}, \tilde{H})$, to (Δ_2^+, Δ_1^+). If G is split, this is equivalent to saying that (Δ_1^+, Δ_2^+) are in the same relative position as (Δ_2^+, Δ_1^+), or, writing $\Delta_2^+ = w\Delta_1^+$, with $w \in W(\mathfrak{g}, \mathfrak{h})$, that $\Delta_1^+ = w\Delta_2^+$, i.e. $w^2 = 1$. If G is not inner-to-split, an element of the real Weyl group does not act diagonally, but with a twist. Suppose $v \in W(\mathfrak{g}, \mathfrak{h})$ is represented by $g \in G(\mathbb{C})$. The image of g in $\tilde{G}(\mathbb{R})$ is $(g, \sigma_G g)$ so this conjugates (Δ_1^+, Δ_1^+) to $(v\Delta_1^+, \sigma_G(v\sigma_G\Delta_2^+))$. Then $(\Delta_1^+, w\Delta_1^+)$'s being conjugate to $(w\Delta_1^+, \Delta_1^+)$ is equivalent to having

$$\sigma_G w \sigma_G w \Delta_1^+ = \Delta_1^+, \tag{3.12}$$

which in fact implies

$$(\sigma_G w)^2 \Delta_j^+ = \Delta_j^+ \tag{3.13}$$

for Δ_j^+ any positive system. We abbreviate the relation 3.13 by writing

$$(\sigma_G w)^2 \equiv 1. \tag{3.14}$$

Given a (conjugacy class of a) triple $\gamma = (H_1, \Delta^+, \chi)$ as above, we recall from [22, 8.1.4], that the definition of length of the standard module is defined to be

(3.15) **Definition.**

$$\ell(\gamma) = (\tfrac{1}{2})(\dim(\mathfrak{h}_1 \cap \mathfrak{p}) - \dim A^c) + \tfrac{1}{2}\#\{\alpha \in \Delta_1^+ : \theta\alpha \notin \Delta_1^+\}.$$

This is an integer. Here H^c is the fundamental Cartan subgroup of G and if H_i is any θ-stable Cartan subgroup of G we write $H_i = T_i A_i$ where $T_i = H_i \cap K$ and $A_i = H_i \cap \exp(\mathfrak{p}_0)$ is a vector group, with $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$ the Cartan decomposition. See [11] for the fact that length is constant on pseudo-L-packets (and hence on L-packets, if we put $\ell(\bar{X}(\gamma)) = \ell(\gamma)$). Length is not necessarily constant on Arthur's enlarged packets.

4. Compatible B.G.G. resolution

If Y is an admissible module of finite length, let JH(Y) denote its Jordan–Holder series (as a set with multiplicities), and for η a set of $\mathbb{Z}/2\mathbb{Z}$ character data, let $m(\eta, Y)$ denote the multiplicity of $\bar{X}(\eta)$ in J.H.(Y). In the special case of a complex group $\tilde{G}$, we have, if $\eta = (\tilde{H}, (\Delta^+, w\Delta^+), 1)$ that $\ell(\eta) = n - \ell(w)$ where n is the length of the longest element of $W = W(\mathfrak{g}, \mathfrak{h})$. Suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is the Levi decomposition of a θ-stable parabolic subalgebra of $\mathfrak{g}$ (in the sense of [22, 5.2.1]). Let $\Delta^+(\mathfrak{l})$ be a choice of positive system for $\Delta(\mathfrak{l}, \mathfrak{h}_1)$ for $\mathfrak{h}_1$ a θ-stable Cartan subgroup of $\mathfrak{l}$. Suppose there is a fixed choice of isomorphism (defined over $\mathbb{C}$) of $\mathfrak{h}_1$ with $\mathfrak{h}$, sending $\Delta(\mathfrak{g}, h_1)$ to $\Delta(\mathfrak{g}, \mathfrak{h})$. Let S be the image of $\Delta(\mathfrak{u})$ under this isomorphism, and $\Delta_{\mathfrak{l}}^+$ be the image of $\Delta^+(\mathfrak{l})$. We define ${}^L\ell(\eta)$ to be $\ell(w)$ if $\eta = (\tilde{H}, (\Delta_{\mathfrak{l}}^+ \cup S, w\Delta_{\mathfrak{l}}^+ \cup (-S), 1)$ for some $w \in W(\mathfrak{l}) \subseteq W(\mathfrak{g}, \mathfrak{h})$, and be undefined if no such w exists. Let $\max \mathrm{JH}(Y) = \{\mathbb{Z}/2\mathbb{Z}$ character data $\eta: \ell(\eta)$ is maximal in $\{\ell(\mu): \bar{X}(\mu) \in \mathrm{JH}(Y)\}$ and $\bar{X}(\eta) \in \mathrm{JH}(Y)\}$.

Define $\ell(Y) = \ell(\eta)$ for any $\eta \in \operatorname{Max} \mathrm{JH}(Y)$.

In [9], an analogue of the BGG resolution of a finite dimensional module, $\mathbb{C}$, say, was constructed. In this paper, we will refer to this as the BGG resolution. (In fact, for $\tilde{G}(\mathbb{R})$, it is the dual of the usual BGG resolution.) In [9], this was constructed by showing that $X(\eta)$ has a universal mapping property relative to $\bar{X}(\eta)$, in the full subcategory, denoted $\mathscr{C}(\ell(\eta))$, of admissible Harish-Chandra modules Y of finite length with infinitesimal character the same as that of $\mathbb{C}$, and satisfying $\ell(Y) \leqq \ell(\eta)$. We recall this:

(4.1) **Theorem.** *Let $Y \in \mathscr{C}(\ell(\eta))$. Suppose*

$$0 \to \bar{X}(\eta) \xrightarrow{f} Y$$

is an injection. Then there exists a map h making the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & \bar{X}(\eta) \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, "h"] \\ & X(\eta) & \end{tikzcd}$$

where g is the inclusion of $\bar{X}(\eta)$ into $X(\eta)$ as unique irreducible submodule. Furthermore,

$$\dim_{\mathbb{C}} \operatorname{Hom}_{(\mathfrak{g},K)}(Y, X(\eta)) = m(\eta, Y).$$

We recall two of the most important properties of the standard modules [22]:

$$\ell(X(\eta)) = \ell(\eta); \tag{4.2}$$

$$\text{if } \ell(\eta_1) \geqq \ell(\eta) \quad \text{then} \quad m(\eta_1, X(\eta)) = 0 \tag{4.3}$$

unless $\eta_1 = \eta$, in which case $m(\eta, X(\eta)) = 1$.

Now write

$$X^i = \bigoplus_{\substack{\gamma = (H', \Delta^+, 1) \\ \ell(\gamma) = n - i}} X(\gamma).$$

Then for any real group G (including $\tilde{G}$) it is shown in [10] that there exist maps f^i such that

$$0 \to \mathbb{C} \xrightarrow{f^0} X(\gamma_0) \xrightarrow{f^1} X^{\ell(\gamma_0)-1} \to \cdots \to X^c \to 0 \tag{4.4}$$

is an exact sequence, where c is the length of any $\mathbb{Z}/2\mathbb{Z}$ character data of minimal length, and γ_0 is the (unique) $\mathbb{Z}/2\mathbb{Z}$-character data such that $\gamma_0 = (H, \Delta^+, 1)$ and $\ell(\gamma_0)$ is maximal among the lengths of $\mathbb{Z}/2\mathbb{Z}$ character data. (Equivalently, $\mathbb{C} = \bar{X}(\gamma_0)$.) It was shown that each X^i is in fact a sum of pseudo-L-packets [11]:

$$X^i = \bigoplus_{\eta \in I'(i)} \Pi(X(\eta))$$

for $I'(i)$ some set of $\mathbb{Z}/2\mathbb{Z}$ character data.

If we unstitch the exact sequence (4.4) with $\tilde{G}$ for G, we obtain

$$0 \to K_i \xrightarrow{\tilde{f}^i} \bigoplus_{\eta \in I(i)} X(\eta) \xrightarrow{f^{i+1}} K_{i+1} \to 0 \tag{4.5}$$

where $\tilde{f}^i$ is obtained from f^i by factoring the latter through the cokernel of f^{i-1}, etc., and taking $I(i)$ to be

$$\{\gamma : \chi = 1 \text{ and } \ell(\gamma) = n - i\}$$

so that, of course,

$$X^i = \bigoplus_{\eta \in I(i)} X(\eta).$$

It was proved in [10] that $K_i \in \mathscr{C}(i)$ and that, writing

$$\tilde{f}^i = \prod_{\eta \in I(i)} \tilde{f}^\eta$$

where $\tilde{f}^\eta : K_i \to X(\eta)$ (and similarly, $f^i = \prod f^\eta$), each $\tilde{f}^\eta$ could be arbitrarily varied by a scalar but was determined up to this scalar by the universal mapping property (4.1) since it was also shown that $m(\eta, K_i) = 1$. (Varying $\tilde{f}^\eta$ by a scalar of course changes K_{i+1} and hence (4.5) taking $i+1$ for i.)

Write $I(i) = A(i) \cup B(i) \cup C(i)$ where

$$A(i) = \{\eta \in I(i) : \bar{X}(\eta) \text{ is } \sigma\text{-stable}\}$$

$$\eta \in B(i) \Leftrightarrow \bar{X}(\eta)^\sigma = \bar{X}(\eta') \quad \text{for some} \quad \eta' \in C(i)$$

and the union is disjoint. Let

$$XA(i) = \bigoplus_{\eta \in A(i)} X(\eta)$$

and if $A_{\sigma,\eta}$ is any choice of normalized intertwining operator for all $\eta \in A(i)$, as in Sect. 3, set

$$A_i = \bigoplus_{\eta \in A(i)} A_{\sigma,\eta}.$$

For $\eta \in B(i)$ set

$$B_{\sigma,\eta} : X(\eta) \oplus (X(\eta))^\sigma \to X(\eta) \oplus (X(\eta))^\sigma$$

to be $A_\sigma|_{X(\eta)} \oplus A_\sigma|_{X(\eta')}$ composed with interchange of the two factors where $A_\sigma : X(\gamma_0) \to X(\gamma_0)$ is the canonical intertwining operator of $X(\gamma_0)$ with $X(\gamma_0)^\sigma$ (i.e., when $X(\gamma_0)$ is realized as functions on $G(\mathbb{C})$ which are invariant under a Borel subgroup, defined over $\mathbb{R}$, then $A_\sigma f = f \circ \sigma_G$), $X(\eta)$ and $X(\eta')$ are regarded as quotients of $X(\gamma_0)$ (by passing to the usual BGG resolution), and $X(\eta') = A_\sigma(X(\eta)) \simeq X(\eta)^\sigma$. Then let

$$A^i = \left(\bigoplus_{\eta \in B(i)} B_{\sigma,\eta} \right) \oplus A_i.$$

act on X^i in the obvious way.

Now regard X^i as G^*-modules (more precisely, as the analogues of Harish–Chandra modules for G^*).

(4.6) **Lemma.** *The* BGG resolution of $\mathbb{C}$ (*i.e.*, *4.4*) *can be chosen so that the maps* f^i *commute with the operators* A^i, *provided that for each* i *and each* $\eta \in A(i)$,

$$A_{\sigma,\eta} = A_\sigma|_{X(\eta)}.$$

Such a BGG resolution is called in this paper a compatible BGG resolution. More loosely expressed, one could say "is a resolution of $\mathbb{C}$ as a G^*-module."

(4.7) **Corollary.**

$$\tilde{\chi}_{\mathbb{C}} = \sum_i (-1)^{i+1} \tilde{\chi}_{XA(i)}.$$

Proof of Corollary. Clearly the twisted character of $\mathbb{C}$ is the alternating sum of the twisted characters of the X^i in a compatible BGG resolution. But the twisted character of $X(\eta) \oplus X(\eta)^\sigma$ for $(\eta \in B(i))$ is zero. QED

Proof of Lemma. Clearly, if the analogous statement is true for each exact sequence (4.5), where (4.5) is viewed with the inherited choices of $\tilde{f}^i$, f^i, and intertwining operators, then we are done. Now if $i = 0$ the analogous statement is clear, by definition of inherited: i.e.

$$0 \to \mathbb{C} \xrightarrow{f^0} X(\gamma_0) \to K_1 \to 0 \tag{4.8}$$

it is clear, since $\mathbb{C}$ has the inherited intertwining operator from $X(\gamma_0)$, that $f^0 = \tilde{f}^0$ commutes with this intertwining operator; by definition, K_1 inherits the quotient G^*-module structure (and f^0 can still be varied by a complex scalar).

Now assume by induction that the analogous statement holds for all sequences (4.5) with $i < j$, and hence that K_j has inherited the intertwining operator from X^{j-1}. Consider the exact sequence

$$0 \to K_j \xrightarrow{\tilde{f}^j} X^j \to K_{j+1} \to 0. \tag{4.9}$$

It is not true that the intertwining operator on K_j is inherited from X^j, unless we choose $\tilde{f}^j$ to make this true. For this, it suffices to define f^η and hence $\tilde{f}^\eta$ for $\eta \in I(j)$ so that each map of

$$K_j \xrightarrow[\eta \in I(j)]{\prod \tilde{f}^\eta} \bigoplus_{\eta \in I(j)} X(\eta) \xrightarrow{f^{j+1}} K_{j+1} \to 0 \tag{4.10}$$

commutes with the intertwining operators. There are two cases.

If $\eta \in A(j)$, then as is easily seen, both $\tilde{f}^\eta$ and $\tilde{f}^\eta_\sigma = A_\sigma \circ \tilde{f}^\eta \circ A^{j-1}$ are $(\tilde{\mathfrak{g}}, \tilde{K})$-module homomorphisms from K_j to $X(\eta)$ and so differ by a scalar (here A_σ is regarded as acting on $X(\eta)$ as follows: Schur's Lemma holds for $X(\eta)$ and so we only have to choose the sign of A_σ; this sign is determined by the action of A_σ on the highest weight space of $X(\eta)$; denote by $M(\eta)$ the Verma module with the same highest weight as $X(\eta)$, similarly $M(\gamma_0)$, then $M(\eta)$ embeds into $M(\gamma_0)$, on which A_σ acts naturally, and on $M(\eta)$, inheritedly, and so on the highest weight space of $M(\eta)$ with a definite sign, the one we choose for $X(\eta)$).

This scalar can be determined by considering the action of $\tilde{f}^{\eta}$ and $\tilde{f}^{\eta}_{\sigma}$ on a highest weight vector, v, of $\bar{X}(\eta)$: both maps take v to a highest weight vector of $X(\eta)$. Now A_σ preserves the space of highest weight vectors of $X(\eta)$ and acts as a sign $\varepsilon = \pm 1$. But since K_j is a quotient of X^{j-1} (in fact a factor sub-$\tilde{\mathfrak{h}}$-module), the sign $\varepsilon' = \pm 1$ with which A_σ acts on v is the same as the sign for the action of A^{j-1} on the image $s(v) \in X^{j-1}$ where s is any $\tilde{\mathfrak{h}}$-splitting section $s: K_j \to X^{j-1}$. But then ε' is the same as the sign of the action of A^i restricted to any non-zero component v' of $s(v)$, $v' \in X(\mu)$. We may take v' to be killed by $\mathfrak{n}$, and so $v' = v$ up to a scalar. Any such sign is again inherited from $X(\gamma_0)$ in the same way as before, and so $\varepsilon' = \varepsilon$. But then

$$\tilde{f}^{\eta}_{\sigma}(v) = A_\sigma(\tilde{f}^{\eta}(A^{j-1}v)) = \varepsilon'\varepsilon\tilde{f}^{\eta}(v) = \tilde{f}^{\eta}(v)$$

which takes care of the first case.

If $\eta \in B(j)$ then we may choose $\tilde{f}^{\eta}$ arbitrarily and put, if $X(\eta)^\sigma = X(\eta')$,

$$\tilde{f}^{\eta'} = A_\sigma \circ \tilde{f}^{\eta} \circ A^{j-1}. \quad \text{QED}$$

It requires an argument to know whether, if

$$\bar{X}(\gamma) = \sum_{w \in W} m(w,\gamma) X(\tilde{H}, (\Delta^+, w\Delta^+), 1)$$

(only as $(\tilde{\mathfrak{g}}, \tilde{K})$-modules), then

$$\tilde{\chi}_{\bar{X}(\gamma)} = \sum_{\substack{w \in W \\ (\sigma w)^2 \equiv 1}} m(w,\gamma) \tilde{\chi}_{X(\tilde{H}, (\Delta^+, w\Delta^+), 1)},$$

assuming $\bar{X}(\gamma)$ is σ-stable. Certainly this is false unless the correct choice of intertwining operators is made. The previous Lemma showed which choice of intertwining operators made this equation true for $\bar{X}(\gamma) = \mathbb{C}$. In order to reach the point where a completely similar argument (which will be omitted, of course), for exactly the same choice of intertwining operators, will work for $\bar{X}(\gamma)$ a unitary representation affording non-zero $(\tilde{\mathfrak{g}}, \tilde{K})$-cohomology, we will need to first recall the construction of [10] of their resolutions by direct sums of standard modules and then show that the maps constructed there satisfy the same universal mapping properties as above (this was not treated in [10]).

In [25], Vogan and Zuckerman classify all irreducible unitary modules π such that $H^*(\mathfrak{g}, K; \pi \otimes F^*) \neq 0$ for F an irreducible finite dimensional representation of $\mathfrak{g}$. Firstly, π must have the same infinitesimal character as F, secondly the π come in coherent families so we may reduce to the case $F = \mathbb{C}$, and finally, in this case (due to Kumaresan, [12]), each π can be written $\pi = A_{\mathfrak{q}}(\mathbb{C})$ where $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is the Levi decomposition of a θ-stable parabolic subalgebra of $\mathfrak{g}$; this means π is the module obtained by applying the cohomological parabolic induction functor (in a certain degree) from $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules to the trivial representation of L. Furthermore, the Langlands parameters of $A_{\mathfrak{q}}(\mathbb{C})$ are calculated. In the notation of $\mathbb{Z}/2\mathbb{Z}$ character data this amounts to

$$A_{\mathfrak{q}}(\mathbb{C}) = \bar{X}(H_0, \Delta_0^+, 1)$$

where H_0 is the maximally split Cartan subgroup of L, and $(H_0, \Delta_0^+, 1)$ is of maximal length among the set of all $\mathbb{Z}/2\mathbb{Z}$ character data $(H', \Delta^{+\prime}, 1)$ such that $H' = H_0$ (even $H' \subseteq L$) and $\Delta^{+\prime} \supseteq \Delta(\bar{\mathfrak{u}})$. One may take $\Delta_0^+ = \Delta_0^+(\mathfrak{l}) \cup \Delta(\bar{\mathfrak{u}})$ where $\bar{X}(H_0, \Delta_0^+(\mathfrak{l}), 1)$ is the trivial representation of L: i.e. $\theta\Delta_0^+(\mathfrak{l}) = -\Delta_0^+(\mathfrak{l})$ if L is quasisplit. From now on we switch notation and write $A_{\mathfrak{q}}(\mathbb{C})$ for $A_{\bar{\mathfrak{q}}}(\mathbb{C})$ (or else write $\mathfrak{q} = \mathfrak{l} + \bar{\mathfrak{u}}$).

In [10] it was shown that any $A_{\mathfrak{q}}(\mathbb{C})$ has a resolution by direct sums of standard modules

$$(4.11) \qquad 0 \to \bar{X}(H_0, \Delta_0^+, 1) \to X(H_0, \Delta_0^+, 1) \to \bigoplus_{\gamma \in Q(m-1)} X(\gamma) \to \cdots \to \bigoplus_{\gamma \in Q(r)} X(\gamma) \to 0,$$

where $Q(i) = \{\mathbb{Z}/2\mathbb{Z}$ character data $\gamma = (H, \Delta^+, 1)$: $H \subseteq L, \Delta^+ \supseteq \Delta(\mathfrak{u})$, and $\ell(\gamma) = i\}$, $m = \ell(H_0, \Delta_0^+, 1)$, and $r =$ minimal length of all $\mathbb{Z}/2\mathbb{Z}$ character data for G. It is constructed by taking the BGG resolution for $\mathbb{C}$ as a representation of L, and applying the cohomological parabolic functor (in a certain degree) which is, as it happens, an exact functor on the modules in question.

We now wish to rewrite this in the special case of a complex group, i.e. taking $\tilde{G}$ for G. We will determine a criterion for a derived functor module with infinitesimal character the same as that of a finite dimensional representation, F, of $\tilde{G}$ to be stably σ-stable (i.e. stably α-stable). Assume first that $F = \mathbb{C}$.

(4.12). **Proposition.** *Suppose that* $\tilde{\mathfrak{q}} = \tilde{\mathfrak{l}} + \bar{\mathfrak{u}}$ *is the Levi decomposition of a* $\tilde{\theta}$*-stable parabolic. Then* $A_{\tilde{\mathfrak{q}}}(\mathbb{C})$ *is* α*-stable.*

Proof. It clearly suffices to show that $X(\tilde{H}, \tilde{\Delta}_0^+, 1)$ is. We wish to show, then, that, writing $\tilde{\Delta}_0^+ = (\Delta_1^+, w\Delta_1^+)$ with $w \in W(\mathfrak{g}, \mathfrak{h})$, we have $(\sigma_G w)^2 \equiv 1$ since by 3.14 this implies α-stability.

Since $\tilde{\mathfrak{q}}$ is $\tilde{\theta}$-stable, (by Definition 5.2.1 of [22]) there exists $\tilde{\lambda} \in i\tilde{\mathfrak{t}}_0^*$ such that $\Delta(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) = \{\alpha \in \Delta(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) : \langle \tilde{\lambda}, \alpha \rangle = 0\}$ and $\Delta(\tilde{\mathfrak{u}}, \tilde{\mathfrak{h}}) = \{\alpha \in \Delta(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) : \langle \alpha, \tilde{\lambda} \rangle > 0\}$ (since $\tilde{\lambda} \in i\tilde{\mathfrak{t}}_0^*$, $\langle \alpha, \tilde{\lambda} \rangle \in \mathbb{R}$). Identify $\tilde{\mathfrak{t}}^*$ with $\tilde{\mathfrak{t}}$ via the Killing form. Then $\tilde{\lambda} = (\lambda, -\sigma_G \lambda)$ for some $\lambda \in \mathfrak{h}$, and since $\tilde{\lambda} \in i\tilde{\mathfrak{t}}_0$, we have $\lambda \in \mathfrak{h}_{\mathbb{R}}$, the real span of the roots. Suppose $\alpha = (\alpha_1, 0) \in \Delta(\tilde{\mathfrak{u}})$. Then $\langle \alpha, \tilde{\lambda} \rangle = \langle \alpha_1, \lambda \rangle > 0$. But if $\alpha = (\alpha_1, 0) \in \Delta(\tilde{\mathfrak{l}})$, $\langle \alpha, \lambda \rangle = 0$. Therefore, putting $S \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$, the projection on the first factor of $\Delta(\tilde{\mathfrak{u}}, \tilde{\mathfrak{h}})$, we have that

$$\sum_{\alpha \in \Delta(\mathfrak{l}) \cup S} \mathfrak{g}^\alpha$$

is a parabolic subalgebra whose Levi factor is $\mathfrak{l}$, where $\Delta(\mathfrak{l})$ is defined as $\{\beta \in \Delta(\mathfrak{g}, \mathfrak{h}) : (\beta, 0) \in \Delta(\tilde{\mathfrak{l}})\}$, whence the notation: $\tilde{L}$ is the complexification of L a Levi subgroup of G. Now as was shown in 2.5 of [2], we may vary L among the inner forms in G of L, and thus may assume that L is quasisplit, since G is.

Now let c_1 be a Cayley transform of H to H^c, the fundamental Cartan subgroup of G. If

$$c_1^{-1} H_\beta c_1 = H_{\beta'}, \quad \beta' \in \Delta(\mathfrak{g}, \mathfrak{h}), \quad \beta \in \Delta(\mathfrak{g}, \mathfrak{h}^c),$$

then define $\lambda' \in \mathfrak{h}^c$ by $\langle \lambda', \beta \rangle = \langle \lambda, \beta' \rangle$. In fact, since $\tilde{\lambda} \in i\tilde{\mathfrak{t}}_0$ we have $\lambda' \in i\mathfrak{t}_0^c$. Thus λ' defines a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, whose Levi factor may as well be identified with $\mathfrak{l}$, and with $\Delta(\mathfrak{u}) = \mathrm{Ad}\, c_1(S)$ if $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Then

$$\Delta(\bar{\mathfrak{u}}) = (\Delta(\mathfrak{u}), -\sigma_G \Delta(\mathfrak{u})) = (\Delta(\mathfrak{u}), \sigma_G \sigma_L \Delta(\mathfrak{u})).$$

Then the $\mathbb{Z}/2\mathbb{Z}$ character data of $A_{\mathfrak{q}}(\mathbb{C})$ is $(H', \Delta^+(\mathfrak{l}) \cup c_2 c_1 S, 1)$ where H' is a maximally split Cartan subgroup of L (θ-stable of course), $\Delta^+(\mathfrak{l})$ is any σ_G and σ_L stable choice of positive system for $\Delta(\mathfrak{l}, \mathfrak{h}')$—for example, $c_2 c_1 \Delta^+(\mathfrak{l})$ in the usual notation, the ambiguities in the choices of c_2 and c_1 make no material difference—, c_2 is a Cayley transform within $L(\mathbb{C})$ of H^c to H' and hence preserves $\mathfrak{u}$, and $c_2 c_1 S = \Delta(\mathfrak{u}, \mathfrak{h}')$.

There exists $w \in W(\mathfrak{g}, \mathfrak{h}')$ such that

(4.13) $$W(\Delta^+(\mathfrak{l}) \cup c_2 c_1 S) = \Delta^+(\mathfrak{l}) \cup \sigma_G(c_2 c_1(-S)) = \sigma_G \sigma_{H'}(\Delta^+(\mathfrak{l}) \cup c_2 c_1 S).$$

We may take w to be, see Lemma 5.14, $c_2^{-1} \bar{c}_2$ if G is split, and $c_2^{-1} \bar{c}_2$ followed by some element of $W(\mathfrak{l})$ if not. Then it is clear that $(\sigma_G w)^2 \equiv 1$ since $w(\Delta(\mathfrak{l})) = \Delta(\mathfrak{l})$ and so $w\Delta(\mathfrak{u}) = -\Delta(\mathfrak{u})$. But, transferring w to $\Delta(\mathfrak{g}, \mathfrak{h})$ via c_2 and c_1, we find that

$$w(\Delta^+(\mathfrak{l}) \cup S) = \sigma_G(\Delta^+(\mathfrak{l}) \cup -S).$$

Now we are reduced to the following lemma:

(4.14). **Lemma.** *The $\mathbb{Z}/2\mathbb{Z}$ character data of $A_{\tilde{\mathfrak{q}}}(\mathbb{C})$ is*

(4.15) $$(\tilde{H}, (\Delta^+(\mathfrak{l}) \cup S, \sigma_G(\Delta^+(\mathfrak{l}) \cup -S)), 1) = (\tilde{H}, (\Delta^+(\mathfrak{l}) \cup S, \sigma_G \sigma_{H'}(\Delta^+(\mathfrak{l}) \cup S)), 1).$$

Proof. Since $\tilde{L} = \mathbf{R}_{\mathbb{C}/\mathbb{R}} L$, then

$$\mathbb{C} = \bar{X}(\tilde{H}', (\Delta^+(\mathfrak{l}), \sigma_L \Delta^+(\mathfrak{l}), 1)$$

for any choice of positive system for $\Delta(\mathfrak{l}, \mathfrak{h}')$. Then by [10], the $\mathbb{Z}/2\mathbb{Z}$ character data for $A_{\tilde{\mathfrak{q}}}(\mathbb{C})$ is

$$(\tilde{H}', (\Delta^+(\mathfrak{l}), \sigma_L \Delta^+(\mathfrak{l})) \cup \Delta(\tilde{\mathfrak{u}}, \tilde{\mathfrak{h}}'), 1).$$

Of course we may arrange that $\sigma_L \Delta^+(\mathfrak{l}) = \Delta^+(\mathfrak{l})$. Now

$$\Delta(\tilde{\mathfrak{u}}, \tilde{\mathfrak{h}}') = \mathrm{Ad}(c_2 c_1, \bar{c}_2 \bar{c}_1) \Delta(\tilde{\mathfrak{u}}, \tilde{\mathfrak{h}}) = \mathrm{Ad}(c_2 c_1, \bar{c}_2 \bar{c}_1)(S, \sigma_{H'} \sigma_H S).$$

If we conjugate

$$(\tilde{H}', (\Delta^+(\mathfrak{l}) \cup \mathrm{Ad}(c_2 c_1) S, \Delta^+(\mathfrak{l}) \cup \mathrm{Ad}(\bar{c}_2 \bar{c}_1)(\sigma_{H'} \sigma_H S)), 1)$$

to $\tilde{H}$ via $\mathrm{Ad}(c_1^{-1} c_2^{-1}, \bar{c}_1^{-1} \bar{c}_2^{-1})$ we obtain

$$\begin{aligned} &\mathrm{Ad}(c_1^{-1}, \bar{c}_1^{-1}) \mathrm{Ad}(c_2^{-1}, \bar{c}_2^{-1})(\tilde{H}', (\Delta^+(\mathfrak{l}) \cup \mathrm{Ad}\, c_2 \mathrm{Ad}\, c_1 S, \Delta^+(\mathfrak{l}) \cup \mathrm{Ad}(\bar{c}_2) \mathrm{Ad}(\bar{c}_1) \sigma_{H'} \sigma_H S), 1) \\ &\quad = (\tilde{H}, (\mathrm{Ad}(c_1^{-1} c_2^{-1}) \Delta^+(\mathfrak{l}) \cup S, \mathrm{Ad}(\bar{c}_1^{-1} \bar{c}_2^{-1}) \Delta^+(\mathfrak{l}) \cup \sigma_{H'} \sigma_H S), 1). \end{aligned}$$

Now if we identify $\Delta^+(\mathfrak{l})$ with $\mathrm{Ad}(c_1^{-1} c_2^{-1}) \Delta^+(\mathfrak{l})$ and transfer $\sigma_{H'}$ to $H(\mathbb{C})$ via this identification, we may rewrite this as

$$(\tilde{H}, (\Delta^+(\mathfrak{l}) \cup S, \mathrm{Ad}(\bar{c}_1^{-1} \bar{c}_2^{-1} c_2 c_1) \Delta^+(\mathfrak{l}) \sigma_{H'} \sigma_H S), 1).$$

On applying Lemma 5.14 twice, and noting that $\sigma_{H'}$ and σ_H commute, and that $\sigma_H|_{\mathfrak{h}_\mathbb{R}} = \sigma_G$ by definition, we obtain that the $\mathbb{Z}/2\mathbb{Z}$ character data of $A_{\tilde{\mathfrak{q}}}(\mathbb{C})$ is

$$(\tilde{H}, (\Delta^+(\mathfrak{l}) \cup S, \sigma_H \sigma_{H'} \Delta^+(\mathfrak{l}) \cup \sigma_{H'} \sigma_H S), 1) = (\tilde{H}, (\Delta^+(\mathfrak{l}) \cup S, \sigma_G(\Delta^+(\mathfrak{l}) \cup -S)), 1)$$

since $\sigma_{H'} S = -S$ and $\sigma_{H'} \Delta^+(\mathfrak{l}) = \Delta^+(\mathfrak{l})$. QED

We now wish to rewrite (4.11) in the special case of a complex group, and then prove analogues of (4.6) and (4.7). Taking $\tilde{G}$ for G in (4.11) and using Lemma 4.14, we obtain

(4.16)

$$0 \to A_{\tilde{\mathfrak{q}}}(\mathbb{C}) \xrightarrow{f_0} X(\tilde{H}, (\Delta^+(\mathfrak{l}) \cup S, \Delta^+(\mathfrak{l}) \cup (-S)), 1) \xrightarrow{f_1} \bigoplus_{\substack{w \in W(\mathfrak{l}) \\ \ell(w)=1}} X(\tilde{H}, (\Delta^+(\mathfrak{l})$$
$$\cup S, w\Delta^+(\mathfrak{l}) \cup (-S)), 1) \xrightarrow{f_2} \cdots \xrightarrow{f_{\ell(w_0)}} X(\tilde{H}, (\Delta^+(\mathfrak{l}) \cup S, -\Delta^+(\mathfrak{l}) \cup (-S)), 1) \to 0.$$

We need to establish that each map f_n of (4.16) is a direct product of maps which are provided by the universal mapping property of the standard modules. Let $\ell^L(\eta)$ be $\ell(w)$ if $\eta = \eta(w)$ which we define by

$$\eta(w) = (\tilde{H}, (\Delta^+ \cup S, w\Delta^+ \cup (-S)), 1)$$

with $\Delta^+ = \Delta^+(\mathfrak{l})$, $w \in W(\mathfrak{l})$, and let $\ell^L(\eta)$ be undefined if no such w exists. Let Q_n be the cokernel of f_n and $Q_{-1} = A_{\tilde{\mathfrak{q}}}(\mathbb{C})$. Write

$$f_n = \prod_{\substack{w \in W(\mathfrak{l}) \\ \ell(w)=n}} f_w,$$

where

$$f_w\colon \bigoplus_{\substack{v \in W(\mathfrak{l}) \\ \ell(v)=n-1}} X(\eta(v)) \to X(\eta(w)),$$

and similarly

$$\tilde{f}_n = \prod_{\substack{w \in W(\mathfrak{l}) \\ \ell(w)=n}} \tilde{f}_w .$$

with $\tilde{f}_w$ the map induced by f_w on Q_{n-1} (n is now a variable, and not fixed as before).

(4.17). **Lemma.** a) *If* $\eta \in \max(J.H.(Q_n))$ *then* $\ell^L(\eta) = n+1$.
b) *If* $\ell^L(\eta) = n+1$ *then* $m(\eta, Q_n) = 1$.
c) *The map* $\tilde{f}_w$ *is obtained by the universal mapping property of standard modules if* $\ell(w) = n+1$.

Proof. These are true for $n = -1$ by inspection. We now proceed by induction on n, assuming the results true for all $Q_i, i \leqq n-1$. Consider the following short exact sequences, obtained by unstitching (4.16):

$$(4.18) \qquad 0 \to Q_{n-1} \xrightarrow{\tilde{f}_n} \bigoplus_{\substack{w \in W(\mathfrak{l}) \\ \ell(w)=n}} X(\eta(w)) \xrightarrow{f_{n+1}} Q_n \to 0$$

$$(4.19) \qquad 0 \to Q_n \xrightarrow{\tilde{f}_{n+1}} \bigoplus_{\substack{\ell(w)=n+1 \\ w \in W(\mathfrak{l})}} X(\eta(w)) \xrightarrow{f_{n+2}} Q_{n+1} \to 0.$$

Suppose that $\ell^L(\mu) = n+1$. The $\ell(\mu) = \ell(X(\eta(w)))$ for any $w \in W(\mathfrak{l})$ such that $\ell(w) = n+1$, and so $\bar{X}(\mu)$ cannot occur in the Jordan–Holder series of any $X(\eta(w))$ with $\ell(w) = n+2$, and hence cannot occur in JH (im (f_{n+2})). (Note that $\ell(X(\eta(v))) =$

$-\ell(v)+c$ for some constant c.) But $\bar{X}(\mu)$ does occur in

$$\mathrm{JH}\left(\bigoplus_{\substack{\ell(w)=n+1\\ w\in W(\mathfrak{l})}} X(\eta(w))\right)$$

precisely once (by a standard property of standard modules) and hence $m(\mu, Q_n)=1$, which proves b.

Now suppose $\mu\in\max \mathrm{JH}(Q_n)$. If μ' is such that $\ell(\mu')>\ell(X(\eta(w)))$ for $\ell(w)=n+1$ then $m(\mu', \mathrm{im}(\tilde{f}_{n+1}))=0$ and hence $m(\mu, Q_n)=0$. It follows that $\ell(Q_n)\leqq\ell(\eta(w))$ for $\ell(w)=n+1$. But since Q_n is a submodule of

$$\bigoplus_{\substack{w\in W(\mathfrak{l})\\ \ell(w)=n+1}} X(\eta(w)),$$

it must contain one of the unique submodules $\bar{X}(\eta(w))$ in its composition series (and none of these more than once), say $\bar{X}(\eta(v))$. So $m(\eta(v), Q_n)=1$, so $\ell(Q_n)=\ell(\eta(v))$, and hence $\eta(v)\in\max\mathrm{JH}(Q_n)$. Then $\ell(\eta(v))=\ell(\mu)$. But then, by 4.3, $\bar{X}(\mu)$ does not occur in $\mathrm{JH}(X(\eta(v)))$ unless $\mu=\eta(w)$. But $\bar{X}(\mu)$ must so occur, for some $w\in W(\ell)$ with $\ell(w)=n+1$. Hence $\mu=\eta(w)$ for some w as above. Then $\ell^L(\mu)$ is defined and equal to $n+1$, which proves a. Now c follows at once a and b, since Q_n is evidently in $\mathscr{C}(c-n-1)$. QED

(4.20) **Lemma.** *The $\{f_w\}$ can be varied by multiplication by scalars, in the obvious sense.*

Proof. Either the same as in [9] for the BGG resolution, or notice that the f_w are obtained by applying an exact functor to any BGG resolution of $\mathbb{C}$ as an $\tilde{L}$-module. QED

Note that this lemma is in sufficient for the application we have in mind, c of Lemma 4.17 is needed.

(4.21) **Theorem.** *With the same choices of intertwining operators as in* (4.6), *we have*

$$\tilde{\chi}_{A_{\bar{\mathfrak{q}}}(\mathbb{C})}=\sum_{\substack{w\in W(\mathfrak{l})\\ (\sigma_G w)^2\equiv 1}}(-1)^{\ell(w)+1}\tilde{\chi}_{X(\eta(w))}. \tag{4.22}$$

Proof. Repeating the same arguments as in the proofs of (4.6) and (4.7), we obtain a resolution of $A_{\bar{\mathfrak{q}}}(\mathbb{C})$ compatible with the intertwining operator. QED

5. Determination of a sign

In this section we determine the sign which appears in 0.4 when the intertwining operators are chosen as in (4.6). This sign differs from ε_M defined by Clozel. We first indicate how to reduce the general case of Theorem 6.6 to the case where all modules in it have infinitesimal character the same as $\mathbb{C}$.

In [7, 8.6], Clozel has shown that the infinitesimal character of $\mathrm{Lift}_{\mathbb{R}}^{\mathbb{C}}(\pi)$ depends only on that of π (at least for π-tempered, but it follows immediately for π a pseudo-L-packet and hence for stable combination of $A_{\mathfrak{q}}(\mathbb{C})$'s). But by (2.2), if π has infinitesimal character the same as F a finite dimensional irreducible module of $G(\mathbb{R})$, then Lift (π) has infinitesimal character the same as Lift $(F) = F \otimes F$. So if $A_{\bar{q}}(\mathbb{C}) \subseteq X(\tilde{\gamma}_0)$ is such that $X(\tilde{\gamma}_0)$ is stably stable and has infinitesimal character the same as a finite dimensional representation of $\tilde{G}$, then there exists F a finite dimensional representation of G such that $A_{\bar{q}}(\mathbb{C})$ has infinitesimal character the same as $F \otimes F$.

As is remarked in 4-2-9 Bouaziz, [6], it is easy to see from this that the coherent continuation (without crossing any walls) by F followed by base change lift is the same as base change lift followed by coherent continuation by $F \otimes F$. So we may as well assume $F = \mathbb{C}$.

We next recall the twisted character formula for induced representations of Repka [13] and Clozel [7, 8.1], analogous to Hirai's character formula for principal series representations. Suppose π^M is a stable combination of (characters of) irreducible representations of M where MN is the Levi decomposition of a cuspidal parabolic P of G. Suppose further that Π^M is a representation of $M(\mathbb{C})$ (or $\tilde{M}(\mathbb{R})$) which is Galois stable, and that Π^M is either irreducible or at any rate that the only $M(\mathbb{C})$ self-intertwining operators of Π^M are scalars (this is well known for standard modules). Suppose further that A_M is an intertwining operator from Π^M to $(\Pi^M)^{\sigma_M}$ (where σ_M is the Galois action of $M(\mathbb{C})$), normalized so that $(A_M)^2 = 1$. Suppose further that both Π^M and A_M are such that, for some $\varepsilon = +1$ or -1,

$$\varepsilon \tilde{\chi}_{\Pi^M}(g) = \chi_{\pi^M}(N(g)) \tag{5.1}$$

for suitable $g \in M(\mathbb{C})$. Writing $A_G = A \circ \operatorname{Ind}(A_M)$ as defined previously (A_σ, Sect. 2) (cf. Clozel, [7], 8.1 and *not* 8.7) we have, as remarked before, that A_G is an intertwining operator for

$$\Pi^G = \operatorname{Ind}_{M(\mathbb{C})N(\mathbb{C})}^{G(\mathbb{C})}(\Pi^M \otimes 1)$$

and $(A_G)^2 = 1$. Let

$$\pi^G = \operatorname{Ind}_{M(\mathbb{R})N(\mathbb{R})}^{G(\mathbb{R})}(\pi^M \otimes 1).$$

Then 8.1 of [7] says

$$\varepsilon \tilde{\chi}_{\Pi^G}(g) = \chi_{\pi^G}(N(g)) \tag{5.2}$$

for suitable $g \in G(\mathbb{C})$.

(5.3) **Lemma.** *The functions $\chi_{\pi^G}(g)$ are linearly independent functions (on the set of regular elements in a suitable connected neighborhood of the identity of $G(\mathbb{R})$) when the modules π^G run over the set B of pseudo-L-packets occurring in the BGG resolution of $\mathbb{C}$.*

(5.4) **Corollary.** *The left hand sides of* (5.2) *are linearly independent functions, on the set of σ-regular elements of $G(\mathbb{C})$ in a suitable neighbourhood of the identity of $G(\mathbb{C})$, when the modules Π^G run over any set C of the same cardinality as B and such that every element of C satisfies* (5.2) *for some element of B. Furthermore, C may be taken to be the set $\bigcup_i A(i)$, notation as in the discussion following* (4.5).

(5.5) **Corollary.** Lift $(\Pi(X(\gamma_1))) \neq$ Lift $\Pi(X(\gamma_2))$, $\gamma_i \in B$, $\gamma_1 \neq \gamma_2$.

Proof of Lemma 5.3. It is well known that the characters of standard modules form a basis (of some vector space). It is easy to see from this that the characters of pseudo-L-packets also form a linearly independent set. In spite of this, suppose the contrary, and that there is a non-trivial dependence. Since all but those of $\mathbb{Z}/2\mathbb{Z}$ character data involving the fundamental Cartan subgroup H^c are induced from a proper subgroup, hence vanish on H^c, one must have a dependency relation between the $\Pi(X(H^c,\Delta_i^+,\chi_i))|_{H^c_{\mathrm{reg}}}$. But there is always a simple explicit formula for any $\Pi(X(H_i,\Delta_i^+,\chi_i))|_{H_{i,\mathrm{reg}}}$ which follows from Osborne's conjecture [8], see also Vogan, [23], and 4.7 of [20]. (Or, alternatively, Shelstad's formula for the stable sum of discrete series on any Cartan of M_i, and Hirai's induced character formula, 5.2 of [14].) But this implies that this restricted dependence is trivial. So no $\Pi(X(H^c,\Delta_i^+,\chi_i))$ occur in the original dependence. Now arguing by induction on the dimension of the split component of H_i, we are done.

The next claim is that the characters of $\Pi(X(H_1,\Delta_1^+,\chi_1))$ and $\Pi(X(H_1,\Delta_1^+,\chi_2))$ agree on the connected component of the identity in $H_1(\mathbb{R})$, denoted $H_1(\mathbb{R})^0$. This follows from Osborne's conjecture and 4.7 of [20] as well (or, again alternatively, Hirari and Shelstad): on $H_1(\mathbb{R})^0$ it is immediate. But, in fact the exponents of the characters on other Cartan subgroups are determined by those on $H_1(\mathbb{R})^0$ by the Harish–Chandra matching conditions since on every Cartan not conjugate to H_1, the leading exponents are zero, by 4.7 of [20]. And the restrictions of the exponents to $H_i(\mathbb{R})^0$ are determined by the restrictions of the matching exponents to $(\mathrm{Ad}(c)(H^i))(\mathbb{R})^0$ (where c is the appropriate Cayley transform). So, in fact this holds for the connected component of the identity in any Cartan subgroup.

Now it follows from this, using the same sort of induction as before, that if we take $\chi = 1$ always, but restrict to regular elements each of which is in the connected component of the identity of some Cartan subgroup, we still have linear independence. But this is precisely the set B. QED

Proof of Corollary 5.4. More precisely, the neighbourhood of (5.3) can be chosen to be a neighbourhood of the identity; every regular element of which is a norm: it is well known that for any $H_1, H_1/H_1^0$ is a two-group, so squares of regular elements lie in $H_1^0(\mathbb{R})$. But by Corollary 2.11 of Clozel, [7], every norm is in the stable conjugacy class of a regular square. Corollary 5.4 now follows immediately. QED

We now give a sketch of a new proof of Clozel's Theorem 2.1, which we include because we will need later some of the observations.

We know from [7, Chap. 4] that the twisted character is represented by a locally L^1 function. So we are justified in regarding $\tilde{\chi}_{X(\tilde{H},(\Delta_1^+,\Delta_2^+),1)}$ as a function. We also know that (5.2) holds whenever (5.1) holds. We know (5.1) holds for finite dimensionals. By induction on the dimension of G we may assume that (5.1) holds, for, say, discrete series of M whenever MN is a proper parabolic. Then (5.2) holds for all standard modules except discrete series (so we may as well assume G is equal rank). So we are reduced to showing that (5.1) holds for discrete series when $M = G$.

Now we have already established the existence of a compatible BGG resolution of $\mathbb{C}$—we assume now that the signs in the ambiguity of signs in the choices of A_M and hence A_G are chosen as specified in Sect. 4. Hence the twisted character of $\mathbb{C}$ is a linear combination of twisted characters of standard modules. These are linearly independent, expect possibly for the one standard module of minimal length. But at least the twisted character of $\mathbb{C}$ is linearly independent of the twisted characters of all standard modules except that of minimal length, since these last are not supported on $\{g \in G(\mathbb{C})\sigma\text{-regular}: N(g) \text{ is elliptic}\}$. Hence, writing i_0 for the minimum length,

$$\tilde{\chi}_{XA(i_0)}(g) = \sum_i (-i)^i \tilde{\chi}_{XA(i)}(g) + \tilde{\chi}_{\mathbb{C}}(g) \tag{5.7}$$

shows that the left hand side depends only on $N(g)$. Assume now Lemma 5.16. Then the right hand side can be rewritten as

$$\sum_i (-1)^i \varepsilon_i \chi_{\Pi(X(\gamma_i))}(N(g)) + \chi_{\mathbb{C}}(N(g)) \tag{5.8}$$

by the inductive hypothesis, (5.15) and (5.5). It remains only to show

$$(-1)^{\ell(\gamma_i)} \varepsilon_i = (-1)^i. \tag{5.9}$$

We omit the proof of this.

In order to prepare for the proof of (5.9) which will be included in this paper, we next calculate Lift $\Pi(X(X_1, \Delta^+, 1))$, in the sense of base-change lifting of parameters. Note that, by the agreement on connected components of Cartans, Lift $\Pi(X(H_1, \Delta^+, \chi))$ is independent of χ. Because of this, we need not keep track of images of τ in what follows.

Consider $\phi: W_{\mathbb{R}} \to {}^L G$ associated to $(H_1, \Delta^+, 1)$ as in Sect. 3. It follows that, writing $\tilde{\phi} = \text{Lift}\, \phi$ we have

$$\tilde{\phi}(z) = (z^{\rho(\Delta^+)} \bar{z}^{\sigma_{H_1} \rho(\Delta^+)}, z^{\rho(\Delta^+)} \bar{z}^{\sigma_{H_1} \rho(\Delta^+)}) = z^{(\rho(\Delta^+), \rho(\Delta^+))} \bar{z}^{(\sigma_{H_1} \rho(\Delta^+), \sigma_{H_1} \rho(\Delta^+))}. \tag{5.10}$$

By Sect. 3, this corresponds to $X(\tilde{H}_1, (\Delta^+, \Delta^+), 1)$. However, the $\mathbb{Z}/2\mathbb{Z}$ data of standard modules for $\tilde{G}$ in the compatible BGG resolution of $\mathbb{C}$ are all more conveniently written with respect to our fixed Cartan $\tilde{H}$. We wish, then, to conjugate the data $(\tilde{H}_1, (\Delta^+, \Delta^+), 1)$ to data, the first element of which is $\tilde{H}$.

If s is a Cayley transform such that $sH_1(\mathbb{C})s^{-1} = H(\mathbb{C})$ then $(s, \bar{s})\tilde{H}_1(s^{-1}, \bar{s}^{-1}) = \tilde{H}$ over $\mathbb{C}$ and over $\mathbb{R}$, respectively. Hence

$$X(\tilde{H}_1, (\Delta^+, \Delta^+), 1) = X(\tilde{H}, (\text{Ad}(s)\Delta^+, \text{Ad}(\bar{s})\Delta^+), 1).$$

Using Ad(s) as a sort of pseudo-diagonalization (Shelstad [16]) identify Δ^+ with $\text{Ad(s)}\Delta^+$. Then

$$X(\tilde{H}_1, (\Delta^+, \Delta^+), 1) = X(\tilde{H}, (\Delta^+, \text{Ad}(\bar{s}s^{-1})\Delta^+), 1). \tag{5.11}$$

Now, by Lemma 5.14, $\text{Ad}(\bar{s}s^{-1})$ acts on $\rho(\Delta^+) \in \mathfrak{h}_{\mathbb{R}}$ the same way as does the transfer of σ_{H_1} to $H(\mathbb{C})$ (via this pseudo-diagonalization), and this is represented by an element of the Weyl group $w = \bar{s}s^{-1}$ (with $w^2 = 1$) if G is split. Hence, if G is split,

$$X(\tilde{H}_1, (\Delta^+, \Delta^+), 1) = X(\tilde{H}, (\Delta^+, w\Delta^+), 1) \tag{5.12}$$

or

$$\text{(5.13)} \qquad \operatorname{Lift} X(H_1, \Delta^+, 1) = X(\tilde{H}, (\Delta^+, \sigma_{H_1}\Delta^+), 1).$$

For general quasisplit G, we prove

(5.14) **Lemma.** $\operatorname{Ad}(\bar{s}s^{-1})\rho(\Delta^+) = \sigma_H\sigma_{H_1}\rho(\Delta^+)$.

Proof. We may assume s is a standard Cayley transform through a set of strongly orthogonal real roots of H. (Recall H is maximally split in G, quasisplit.) Then s is a product of commuting Cayley transforms through one root at a time. So we may assume s is a standard Cayley transform through one real root α of H, and that $sH_1(\mathbb{C})s^{-1} = H(\mathbb{C})$. The following considerations reduce this to the case of $SL_2(\mathbb{R})$, which is clear: now $\mathfrak{h}_{\mathbb{R}}$ breaks up into an orthogonal direct sum of three spaces: V_1, the subspace of $\mathfrak{a}\cap\mathfrak{h}_{\mathbb{R}}$ orthogonal to α; V_2, that spanned by α; and $V_3 = \mathfrak{t}\cap\mathfrak{h}_{\mathbb{R}}$. But $\sigma_H\sigma_{H_1}$ is 1 on V_1, 1 on V_3, and -1 on V_2. It is clear that $\operatorname{Ad}(s\bar{s}^{-1})$ is 1 on V_1 and V_3, and the calculation of it on V_2 is the case of $SL_2(\mathbb{R})$. QED

So in general, $(\sigma_G w)^2 \equiv 1$, which is compatible with Sect. 3, and we have

$$\text{(5.15)} \qquad \operatorname{Lift} X(H, \Delta^+, 1) = X(\tilde{H}, (\Delta^+, \sigma_G\sigma_{H_1}\Delta^+), 1).$$

By the stable base change character identity, it follows that, with our choice of intertwining operators,

$$\tilde{\chi}_{X(\tilde{H},(\Delta^+,\sigma_G\sigma_{H_1}\Delta^+),1)}(g) = \varepsilon(w)\chi_{X(H_1,\Delta^+,1)}(N(g))$$

where $w\in W(\mathfrak{g},\mathfrak{h})$ is chosen so that $w\Delta^+ = {}^L\sigma\sigma_{H_1}\Delta^+$, and $\varepsilon(w) = \pm 1$ (and was called ε or ε_i in (5.2) and (5.9)). We can rewrite (5.9) to obtain $\pm\varepsilon(w) = (-1)^{\ell(w)}(-1)^{\ell(H_1,\Delta^+,1)}$ where the sign is dependent only on G, and we will prove this using the following lemma.

It was shown in Sect. 3 that every α-stable standard module is of the form $X(\tilde{H}, (\Delta^+, w\Delta^+), 1)$ for w such that $(\sigma_G w)^2 = 1$.

(5.16) **Lemma.** *Every α-stable standard module is stably α-stable.*

Proof. It suffices to show that for every $w\in W(\mathfrak{g},\mathfrak{h})$ such that $(\sigma_G w)^2 \equiv 1$, there exists a θ-stable Cartan subgroup H_1 such that for some positive system $\Delta^+ \subseteq \Delta(\mathfrak{g},\mathfrak{h})$, $w\rho(\Delta^+) = \sigma_G\sigma_{H_1}\rho(\Delta^+)$ upon transferring σ_{H_1} to $H(\mathbb{C})$. Recall that such transference is via a pseudo-diagonalization of H_1, i.e. an identification of the abstract Cartan in the sense of [21] with the concrete Cartan $\mathfrak{h}_1$ and a choice of positive system.

Consider the twist of σ_G by w. This means let θ be the involution of $\Delta(\mathfrak{g},\mathfrak{h})$ such that $\theta = \sigma_G\circ w$. Using Lemma 10.9 of [21] let $\sigma_{G'}$ be the involution of $G(\mathbb{C})$ agreeing with θ on $\mathfrak{h}$. This defines an inner form of G, called G': in fact G' is an inner form of G since if $g\in G(\mathbb{C})$ represents w, then $\sigma_{G'}\circ\sigma_G\circ\operatorname{Ad}g^{-1}$ is an automorphism of $\mathfrak{g}$ which is 1 on $\mathfrak{h}$, and so cannot (by 2.14 of [19]) be an outer automorphism. Since w preserves H, $\sigma_{G'}$ acts on $H(\mathbb{C})$, call the resulting group defined over $\mathbb{R}$, H'. Take $\Delta_1^+ \subseteq \Delta(\mathfrak{g}',\mathfrak{h}')$ to be the same set as the given $\Delta^+ \subseteq \Delta(\mathfrak{g},\mathfrak{h})$ used in the definition of ${}^L\sigma$, i.e. it is stable under σ_G. Then

$${}^L\sigma\sigma_{G'}\Delta_1^+ = {}^L\sigma\sigma_G\circ\operatorname{Ad}(g)\Delta^+ = {}^L\sigma\sigma_G w\Delta^+ = w\Delta^+$$

since ${}^L\sigma\sigma_G$ is 1 on $\mathfrak{h}_{\mathbb{R}}\cap[\mathfrak{g},\mathfrak{g}]$. Since G' is an inner form of G, the Cartan subgroup H' is isomorphic over $\mathbb{R}$ via an inner automorphism of $G(\mathbb{C})$ with some Cartan subgroup H_1 of G. Since the isomorphism is defined over $\mathbb{R}$, it respects the actions of $\sigma_{G'}|_{H'(\mathbb{C})}$ and $\sigma_{H_1}=\sigma_G|_{H_1(\mathbb{C})}$. Replacing Δ_1^+ by its image contained in $\Delta(\mathfrak{g},\mathfrak{h}_1)$ we obtain the desired H_1^+ and Δ_1^+. QED

Now by the remarks preceding (5.9), note that there we assume the truth of (2.1), we have already shown that the twisted character of $\mathbb{C}$ can be expressed by either of two expressions:

$$\pm\sum_{\gamma_i\in K}(-1)^{\ell(\gamma_i)}\tilde{\chi}_{X(\gamma_i)}(g) \tag{5.17}$$

where K is the set of $\mathbb{Z}/2\mathbb{Z}$ character data of Galois stable standard modules, and the sign is $+$ if the number of positive roots is even, $-$, otherwise; and, since $\tilde{\chi}_{\mathbb{C}}(g)=\chi_{\mathbb{C}}(N(g))$

$$\pm\sum_{\eta_i\in B}(-1)^{\ell(\eta_i)}\Pi(X(\eta_i)) \tag{5.18}$$

with the sign $+$ if the maximal length of $\mathbb{Z}/2\mathbb{Z}$ character data is even, -, otherwise. Hence, in particular, these signs depend only on G.

Furthermore, in (5.18) each term is linearly independent of the others. But each $\gamma_i\in K$ satisfies

$$\tilde{\chi}_{X(\gamma_i)}(g)=\varepsilon_i X(\eta(\gamma_i))(N(g)) \tag{5.19}$$

for some $\eta(\gamma_i)\in B$, by Clozel's Theorem (2.1) and (5.16). It is clear from (5.19) and linear independence (5.4 and 5.5) that $\gamma_i\mapsto\Pi(X(\eta(\gamma_i)))$ is a well defined function on K, and is a left inverse of $\mathrm{Lift}_{\mathbb{R}}^{\mathbb{C}}\colon B\to K$. So it is a bijection. So, using (5.4) again (applied to equating (5.18) with (5.19)) we conclude

(5.20) **Proposition.** $\pm\varepsilon_i=(-1)^{\ell(\gamma_i)-\ell(\eta(\gamma_i))}$ with the sign depending only on G.

We will need this in the next section.

6. Conclusion

The calculation of the base change lift of an enlarged packet of $A_{\mathfrak{q}}(\mathbb{C})$'s is easy, now that (5.20) and (4.21) are available. We will proceed as follows. The stable linear combination of the characters in the enlarged packet will be written down in (6.1). Using Theorem 4.21 the twisted character of $A_{\bar{\mathfrak{q}}}(\mathbb{C})$ will be written as a linear combination of twisted characters of standard modules, each of which in turn can be rewritten, using (5.20) and (5.2), as a (stable) pseudo-L-packet for $G(\mathbb{R})$ composed with the norm map. This will be compared with (6.1) and the desired theorem, (6.6), will be established.

Let H^c be a fundamental Cartan subgroup of G, as before, θ-stable. Given a θ-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$, we define, for $w\in W(G(\mathbb{C}),H^c(\mathbb{C}))^\theta$ (the elements of the Weyl group which commute with θ)

$$\mathfrak{q}^w=\left(\bigoplus_{\alpha\in\Delta(\mathfrak{q})}\mathfrak{g}^{w\alpha}\right)\oplus\mathfrak{h}^c=\mathfrak{l}^w\oplus\mathfrak{u}^w$$

and L_w is then the normalizer in $G(\mathbb{R})$ of $\mathfrak{q}^w$. Let $\gamma(w) = (\frac{1}{2}) \dim (L_w/L_w \cap K)$. (When $G(\mathbb{R})$ is connected this theorem is due to Zuckerman). It was proved in [2] that the following is a stable combination of $A_{\mathfrak{q}}(\mathbb{C})$'s:

$$\sum_{w \in S} (-1)^{\gamma(w)} A_{\mathfrak{q}^w}(\mathbb{C}) = \varepsilon \sum_{\eta \in \Gamma} (-1)^{\ell(\eta)} X(\eta) \tag{6.1}$$

where ε, which depends only on $\{\mathfrak{q}^w : w \in W\}$ is ± 1,

$$S = W(G, H^c) \backslash W(G(\mathbb{C}), H^c(\mathbb{C}))^\theta / W(L(\mathbb{C}), H^c(\mathbb{C})),$$

and $\Gamma = \bigcup_{w \in S} \Gamma_w$ where Γ_w is the set of $\mathbb{Z}/2\mathbb{Z}$ character data $\eta = (H, \Delta^+, 1)$ with $H \subseteq L_w$ and $\Delta^+ \supseteq \Delta(\mathfrak{u}^w)$.

In fact the right hand side of (6.1) is a linear combination of pseudo-L-packets, each coefficient being ± 1. Furthermore, it follows that, putting E to be a set of representatives in Γ for the pseudo-L-packets in the right hand of (6.1), we may arrange that $E \subseteq B$ (notation as in (5.18)).

(This formula conceals a trivial subtlety. In [2] it was only proved that the left hand side of (6.1) was stable, and in [10] it was shown that $A_{\mathfrak{q}^w}(\mathbb{C})$ was an alternating sum over Γ_w but with $\ell(\eta)$ defined differently: as the length, here we will notate it $\ell_w(\eta)$, of $(H, \Delta^+ \cap \Delta(\mathfrak{l}), 1)$ as a parameter for L_w, if $\eta \in \Gamma_w$. But one can easily see that $\ell_w(\eta) + \gamma(w)$ is independent of w, and that the parity modulo 2 of $\ell_w(\eta) + \gamma(w) - \ell(\eta)$ is independent of η.)

By redefining length to shifted by a constant, we can arrange that $\varepsilon = 1$.

On the other hand, we have, rewriting (4.21)

$$\tilde{\chi}_{A_{\tilde{\mathfrak{q}}}(\mathbb{C})} = \sum_{\substack{w \in W(\mathfrak{l}) \\ (\sigma_L w)^2 \Delta_1^+(\mathfrak{l}) = \Delta_1^+(\mathfrak{l})}} \tilde{\chi}_{X(\tilde{H}, (\Delta_1^+, \sigma_G \sigma_L w \Delta_1^+), 1)} \tag{6.2}$$

where Δ_1^+ is a fixed choice of positive system such that Δ_1^+ contains the projection on the first factor of $\tilde{\mathfrak{h}}^*$ of $\Delta(\tilde{\mathfrak{u}}, \tilde{\mathfrak{h}})$, and $\Delta_1^+ \cap \Delta(\mathfrak{l})$ is stable under σ_L. (The only point to consider about this rewriting is that $\sigma_G \sigma_L w \Delta_1^+$ should run over all positive systems Δ_2^+ such that both Δ_2^+ contains the projection on the second factor of $\Delta(\tilde{\mathfrak{u}}, \tilde{\mathfrak{h}})$, i.e. $-S$ (in the notation of Sect. 4) and $(\tilde{H}, (\Delta_1^+, \Delta_2^+), 1)$ is α-stable.)

Now suppose $(\tilde{H}, (\Delta_1^+, \Delta_2^+), 1)$ is one of the representatives of $\mathbb{Z}/2\mathbb{Z}$ character data occurring in (6.2). Then, by (5.15), we know that it is the stable base change lift of $X(H_1, \Delta_1^+, 1)$ for some Cartan subgroup H_1 of G satisfying $\sigma_G \sigma_{H_1} \Delta_1^+ = \Delta_2^+$ (making appropriate identifications via some implicit pseudo-diagonalization, which depends on Δ_2^+). Then $\sigma_G \sigma_{H_1} \Delta_1^+ \supseteq \sigma_G \sigma_L \Delta(\mathfrak{u}) = \sigma_G \Delta(\bar{\mathfrak{u}})$ by hypothesis on Δ_2^+. Since L is defined over $\mathbb{R}$, σ_G and σ_{H_1} preserve $\Delta(\mathfrak{l})$ (same for any L_w): hence

$$\sigma_{H_1} \Delta(\mathfrak{u}) = \Delta(\bar{\mathfrak{u}}). \tag{6.3}$$

Now it was shown in [11] that the pseudo-L-packet containing $X(H_1, \Delta_1^+, 1)$ is

$$\Pi(X(H_1, \Delta_1^+, 1)) = \{X(H_1, w\Delta_1^+, 1) : w \in W(\mathfrak{m}_1, \mathfrak{h}_1)\}$$

where $H_1 = T_1 A_1$ as usual and $M_1 = \mathrm{Cent}_G(A_1)$. By Sect. 10 of [2], we may represent $w \in S$ by $w \in W(\mathfrak{m}^c, \mathfrak{h}^c)$. Furthermore, it was shown in [2] that every L_w occurring in (6.1) is an inner form of a fixed $L \in \{L_w\}$ which is quasisplit. So every Cartan subgroup occurring in the $\eta \in \Gamma$ is actually isomorphic over $\mathbb{R}$ to a Cartan subgroup of L.

We wish to show

(6.4) **Lemma.** *Under the above hypothesis H_1 can be taken to be a Cartan subgroup of L.*

Proof. By the above remarks, it suffices to show that $H_1(\mathbb{C})$ can be taken to be a complex Cartan subgroup of $L(\mathbb{C})$. (It is necessary to make this observation, since in what follows, the implicit pseudo-diagonalizations obviously cannot preserve the property of being a non-compact imaginary root.)

We have $\sigma_{H_1}\Delta_1^+ = \sigma_L w \Delta_1^+$. Let H^c be the fundamental Cartan subgroup of L (and hence of G) and be θ-stable. Then H_1 is a Cayley transform of H^c through some set of strongly orthogonal non-compact imaginary roots. It suffices to show that all these roots belong to L. Let α be one. Then $\sigma_{H_1}\alpha = \alpha$. If $\alpha \in \Delta(\mathfrak{u})$ then (6.3) shows that $\sigma_{H_1}\alpha \in \Delta(\bar{\mathfrak{u}})$ which is a contradiction. Similarly $\alpha \notin \Delta(\bar{\mathfrak{u}})$, so we conclude $\alpha \in \Delta(\mathfrak{l})$. QED

(6.5) **Lemma.** *Suppose $H_1 \subseteqq L$ is a Cartan subgroup and $(H_1, \Delta_1^+, 1)$ is an arbitrary set of $\mathbb{Z}/2\mathbb{Z}$ character data subject to this condition on H_1 and the condition that $\Delta_1^+ \supseteqq \Delta(\mathfrak{u})$. Then* Lift $\Pi(X(H_1, \Delta_1^+, 1))$ *occurs in the right hand side of (6.1).*

Proof. It suffices to show that $\Delta_2^+ = \sigma_G \sigma_{H_1} \Delta_1^+$ occurs in (6.2), provided we take Δ_1^+ there to be the same as Δ_1^+ here, by choosing an appropriate pseudo-diagonalization. So it suffices to show that $\Delta_2^+ \supseteqq \sigma_G \Delta(\bar{\mathfrak{u}})$, i.e. $\sigma_G \sigma_{H_1} \Delta_1^+ \supseteqq \sigma_G \Delta(\bar{\mathfrak{u}})$. Since $H_1 \subseteqq L, \sigma_{H_1} = \sigma_L$, but since $\Delta(\mathfrak{u})$ is defined by the condition that $\alpha \in \Delta(\mathfrak{u})$ whenever $\mathrm{Re}\langle \alpha, \lambda \rangle > 0$ for a fixed $\lambda \in i\mathfrak{t}_0^{c*}$, we have $\sigma_L \lambda = {}^L\sigma_L \lambda = -\lambda \in \mathfrak{h}_1$ (since λ is in the (image of the) center of $\mathfrak{l}$ (under the Killing form), λ is in any Cartan subalgebra of $\mathfrak{l}$) so $\sigma_{H_1}\Delta(\mathfrak{u}) = \Delta(\bar{\mathfrak{u}})$. QED

Now for every $\eta \in \Gamma$ there exists $\eta' \in \Gamma_1$ such that $\Pi(X(\eta)) = \Pi(X(\eta'))$. Hence the right hand side of (6.1) may be rewritten, after relabelling if necessary,

$$\varepsilon \sum_{\eta \in E_1} (-1)^{r(\eta)} \Pi(X(\eta))$$

with E_1 a set of representatives in Γ_1 for the pseudo-L-packets. (Even though it is not immediately apparent from (6.4) that $\eta = (H_1', \Delta_1^{+\prime}, 1)$ is $G(\mathbb{R})$-conjugate to some $(H_1, \Delta_1^+, 1)$ with $H_1 \subseteqq L$ and $\Delta_1^+ \supseteqq \Delta(\mathfrak{u})$, it is clear that η is $G(\mathbb{C})$-conjugate to some such, and moreover, so, via a conjugacy which respects the Galois actions on H_1' and H_1. But then Lift η and Lift $(H_1, \Delta_1^+, 1)$ agree by (5.15). Hence, finally, by (5.5), we must have $\Pi(X(\eta)) = \Pi(X(H_1, \Delta_1^+, 1))$, and so we may as well have assumed $\eta = (H_1, \Delta_1^+, 1)$.)

But we have already shown, in the proof of (5.20), that the correspondence between B and K given by *Lift* is bijective: and in (6.1) and (6.2) we have subsets of these, so (6.4) and (6.5) together imply that *Lift* gives a bijection between E_1 and $\{w \in W(\mathfrak{l}) : (\sigma_L w)^2 \Delta_1^+(\mathfrak{l}) = \Delta_1^+(\mathfrak{l})\}$. But (5.20) shows that the signs in the base change character identities for these pseudo-L-packets are consistent with the signs in (6.1) and (6.2). So we have proved

(6.6) **Theorem.** $\tilde{\chi}_{A_{\bar{\mathfrak{q}}}(\mathbb{C})}(g) = \pm \sum_{w \in S} (-1)^{\gamma(w)} A_{\mathfrak{q}^w}(\mathbb{C})(N(g))$.

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