Unipotent Representations and Quantum Induction

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Abstract: In this paper, we construct unipotent representations for the real orthogonal groups and the metaplectic groups in the sense of Vogan.

Introduction

The existence of certain irreducible unitary representations, often called unipotent representations, was conjectured by Arthur, Barbasch and Vogan. Precise predictions about the representation-theoretic invariants associated with these representations was given in [Ar83], [BV85], [VO89] and [Ar]. The goal of this article is to verify the existence of unipotent representations for $O(p, q)$ and $Mp_{2n}(\mathbb{R})$ in the sense of Vogan. The construction of unipotent representations for other classical groups of type I can be carried out in the same spirit.

Unipotent representations play a crucial role in the representation theory of real reductive groups and in the theory of automorphic forms. Constructing unipotent representations is one core part of Vogan's program to classify the unitary duals of real reductive groups ([VO86]). For general linear groups over Archmidean fields, unipotent representations can be produced by parabolic induction and the classification of unitary duals was carried out by Vogan himself in [Vo86]. For complex semisimple groups, unipotent representations were constructed and studied by Barbasch-Vogan ([BV85]). Later, Barbasch proved the unitarity of these representations which led to his classification of the unitary duals of complex classical groups ([B89]). For other types of real reductive groups, Adams, Barbasch and Vogan gave certain descriptions about special unipotent representations in Arthur packets ([ABV] and [Ar]). The unitarity and construction of these representations remain open problems.

In this paper, we propose a different way of obtaining unipotent representations that will automatically be unitary. The motivation came from the work of Segal-Shale-Weil on the oscillator representations and the work of Howe, Li, Przebinda and many others on theta correspondences. In [Heq], we defined the concept of quantum induction as composition of theta correspondences (see [Ho89]) within a certain range. Moreover we proved that quantum induction, if nonvanishing, preserves unitarity. In this paper, we further study quantum induction in the framework of the orbit method ([VO86], [VO94]). Using quantum induction, we attach to certain real rigid nilpotent orbits $\mathcal{O}$ a packet of irreducible **unitary** representations, $\mathcal{N}(\mathcal{O})$. Each irreducible representation satisfies the characterizations given in [VO89].

Let me illustrate our construction by an example. Let $\mathcal{O}_{\mathbf{D}}$ be a real nilpotent coadjoint orbit of $Sp_{30}(\mathbb{R})$, with $\mathbf{D} =$

$$\begin{array}{llllll}
\boxed{-} & \boxed{+} & \boxed{-} & \boxed{+} & \boxed{-} & \boxed{+} \\
\boxed{-} & \boxed{+} & \boxed{-} & \boxed{+} & \boxed{-} & \\
\boxed{+} & \boxed{-} & \boxed{+} & \boxed{-} & \boxed{+} & \\
\boxed{-} & \boxed{+} & \boxed{-} & \boxed{+} & & \\
\boxed{+} & \boxed{-} & \boxed{+} & \boxed{-} & & \\
\boxed{+} & \boxed{-} & & & & \\
\boxed{+} & \boxed{-} & & & & \\
\boxed{-} & & & & & \\
\boxed{+} & & & & &
\end{array}$$

Notice that $\mathcal{O}_{\mathbf{D}}$ is not special in the sense of Lusztig ([Lus], [CM]). Let j be a positive integer. Define $\mathbf{D} - \mathbf{j}$ to be the Young diagram obtained by deleting the first j columns from the left. Let p_j be the number of $+$ in $\mathbf{D}-\mathbf{j}$ and q_j be the number of $-$ in $\mathbf{D}-\mathbf{j}$. Construct the sequence

$$[(p_0, q_0), (p_1, q_1), (p_2, q_2), \ldots],$$

obtaining

$$[(15, 15), (10, 11), (7, 7), (5, 4), (2, 2), (1, 0)].$$

Let χ be a unitary character of $O(1)$. Define

$$\begin{aligned}
&Q(30; 10, 11; 14; 5, 4; 4; 1)(\chi) \\
=&\theta_s(O(10, 11), Mp_{30}(\mathbb{R}))\theta_s(Mp_{14}(\mathbb{R}), O(10, 11))\theta_s(O(5, 4), Mp_{14}(\mathbb{R})) \\
&\theta_s(Mp_4(\mathbb{R}), O(5, 4))\theta_s(O(1, 0), Mp_4(\mathbb{R}))(\chi).
\end{aligned} \tag{1}$$

where θ_s is the theta correspondence in the semistable range (see Definitions 17 and 22). As proved in [Heq], this is a genuine irreducible unitary representation of $Mp_{30}(\mathbb{R})$ if it does not vanish (see Theorem 3.6 and Theorem 3.7). In Chapter 4, we prove that $Q(*)(\chi) \neq 0$. Thus $\mathcal{Q}(*)(\chi)$ is an irreducible unitary representation of $Mp_{30}(\mathbb{R})$. Then in Chapter 5 and 6, we compute the infinitesimal character and the associated variety of $Q(*)(\chi)$ based on the results obtained by Przebinda ([PR96], [PR93]). We further prove that the associated variety of $Q(*)(\chi)$ is indeed the closure of the complexified orbit $\mathcal{O}_{\mathbb{C}}$.

Here is one of the highlights in this paper.

Theorem 0.1 *Let $G = Sp_{2n}(\mathbb{R})$ or $G = O(p, q)$ with $p + q$ even. Let $\mathcal{O}_{\mathbf{d}}$ be a special rigid nilpotent adjoint orbit of $G_{\mathbb{C}}$ parametrized by the partition $\mathbf{d}$ (see Ch 6.3,7.3 [CM]). Let $\mathcal{O}$ be a real adjoint orbit of G in $\mathcal{O}_{\mathbf{d}}$. Then there exists a nonempty set $\mathcal{N}(\mathcal{O})$ of irreducible unitary representations of G such that for every $\pi \in \mathcal{N}(\mathcal{O})$ the associated variety of π, $\mathcal{V}(Ann\,\pi) = cl(\mathcal{O}_{\mathbf{d}})$. Let $\mathbf{d}^t = (m_1 > m_2 > m_3 > \ldots > \ldots)$ be the transpose of $\mathbf{d}$ (considered as a Young diagram). Then the infinitesimal character $\mathcal{I}(\pi)$ only depends on $\mathbf{d}$.*

- *If $G = Sp_{2n}(\mathbb{R})$, $\mathcal{I}(\pi) = (\rho(\mathfrak{sp}_{m_1}(\mathbb{C})), \rho(\mathfrak{o}(m_2, \mathbb{C})), \rho(\mathfrak{sp}_{m_3}(\mathbb{C})), \rho(\mathfrak{o}(m_4, \mathbb{C})), \ldots)$;*
- *If $G = O(p, q)$, $\mathcal{I}(\pi) = (\rho(\mathfrak{o}(m_1, \mathbb{C})), \rho(\mathfrak{sp}_{m_2}(\mathbb{C})), \rho(\mathfrak{o}(m_3, \mathbb{C})), \rho(\mathfrak{sp}_{m_4}(\mathbb{C})), \ldots)$.*

Here $\rho(\mathfrak{g})$ is the half sum of the positive roots of $\mathfrak{g}$.

Let me make a few remarks here.

Remark 0.1 1. Assuming $\mathcal{O}_{\mathbf{d}}$ to be special and rigid implies that all m_i must be even. Thus all m_i in Theorem 0.1 are assumed to be even. On the one hand, a rigid orbit is defined to be an orbit that is not induced. Roughly, $\mathcal{O}_{\mathbf{d}}$ is rigid if $\mathbf{d}^t$ is multiplicity-free. The classification of rigid orbits for classical Lie algebras is due to Kempken and Spaltenstein, and can be found in [CM]. Special nilpotent orbits, on the other hand, are defined by Lusztig in [Lus]. For Lie algebras of type C and D, $\mathcal{O}_{\mathbf{d}}$ is special if any odd number in $\mathbf{d}^t$ occurs with even multiplicities (see [Lus] also Proposition 6.3.7 [CM]). Therefore, $\mathcal{O}_{\mathbf{d}}$ being rigid and special forces m_i to be even for all i.

2. It is generally understood that representations attached to special nilpotent orbits should be representations of the linear group G. Theorem 0.1 certainly reinforces this understanding. According to Barbasch and Vogan, the representations in Theorem 0.1 should really be called special unipotent. These representations are attached to special nilpotent orbits in the sense of Lusztig. They are unitary representations of *the linear group G*. Needless to say, there are unipotent representations of the covering of G that are not special.

3. As illustrated by the example for $Mp_{30}(\mathbb{R})$, our construction of $\mathcal{N}(\mathcal{O})$ goes beyond special rigid orbits. It produces non-special unipotent representations as well. The most well-known examples of unipotent representations that are not special, are the two irreducible constituents of the oscillator representation, also called the Segal-Shale-Weil representation. The representations constructed in this paper can be regarded as derivations of the oscillator representations.

4. Some of the representations in this paper have been known in one way or another. For example, minimal representations, the representations attached to minimal orbits, are studied by Brylinski-Kostant in a series of papers ([BK]) and by Binegar-Zierau for $O(p, q)$ ([BZ]) from a different angle. Beyond that, representations attached to a perhaps wider class of small rigid orbits have also been studied by Kashiwara-Vergne, Howe, Li, Sahi, Tan, Huang, Zhu and others (see [KV], [Ho84], [Sahi], [HT], [ZH], [HL] and the references within them). Our approach is a generalization of the latter.

5. The intrinsic connection between coadjoint orbits and the unitary dual of a Lie group was first explored by Kirillow and Kostant. It is now known as the orbit method. One core part of the orbit method for real reductive Lie groups is to construct unipotent representations attached to rigid nilpotent orbits. Once we know how to attach representations to rigid nilpotent orbits, there are various ways to attach unitary representations to induced orbits (see [VO94], [VO87]). What we have accomplished in this paper is the construction of some unipotent representations attached to special rigid orbits and some other nonspecial nilpotent orbits. It is not clear whether our list of $\mathcal{N}(\mathcal{O})$ exhausts all special unipotent representations. At least for some special rigid orbits, it does. I tend to believe that $\mathcal{N}(\mathcal{O})$ is exhaustive for special rigid orbits.

Also included in this paper is a theorem concerning the relationship between quantum induction and parabolic induction. Based on a theorem of Kudla-Rallis [KR] and of Lee-Zhu [LZ], we decompose certain parabolic induced representation $I^{\alpha}(\pi)$ of $Mp_{2n}(\mathbb{R})$ into direct sum of quantum induced representations $Q(p,q)(\pi)$ with $p+q=n+1$ and a fixed parity on p (see Definition 24, Theorem 4.5 and Theorem 4.8). Each $Q(p,q)(\pi)$ is either irreducible or vanishes. This is the limit case for which quantum induction can be constructed from parabolic induction. Correspondingly, there is a decomposition theorem for the wave front set of $I^{\alpha}(\pi)$, namely

$$WF(I^{\alpha}(\pi)) = \cup_{p+q=n+1, p \ even} WF(Q(p,q)(\pi)).$$

On the one hand, $WF(I^{\alpha}(\pi))$ is computable and consists of finite number of irreducible components. On the other hand, the wave front set of each $Q(p,q)(\pi)$ possesses certain distinctive characteristic that can be derived from [PR93] and [PAN]. Under certain hypotheses on π, we sort out the occurence of $WF(Q(p,q)(\pi))$ in $WF(I^{\alpha}(\pi))$ completely. This provides us with one nonvanishing theorem for $\theta_s(p,q)(\pi)$ and for $Q(p,q)(\pi)$.

Let me also point out one advantage of our construction related to the theory of automorphic forms. Theta correspondence originated from the studies of theta series (see [Si], [WE65], [We65], [Ho79]). As pointed out to me by Jian-Shu Li, theta correspondence should map automorphic representations to automorphic representations (see [Ho79], [Ra87], [Li94]). Thus representations in $\mathcal{N}(\mathcal{O})$ should all be automorphic. In the final part of this paper, along the lines of [VO86], we make some conjectures regarding the automorphic dual in the sense of Burger-Li-Sarnak [BLS].

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Contents

Chapter 1

Invariants

In mathematics, classification problems are often approached by constructing invariants. In this chapter, we will attach invariants to the equivalence classes of irreducible Hilbert representations of a reductive group G. One hopes that these invariants could shed some light on the classification and construction of irreducible unitary representations. Our purpose here is not to give a historical account of these invariants, but rather, to review some basic facts we need concerning these invariants. The main references are [KN] and [Wallach].

1.1 Notation

Reductive groups and semisimple groups in this paper are assumed to have at most a finite number of components. Let G be a Lie group. We adopt the following notation:

1. G_0—the identity component of G;
2. $\mathfrak{g}$—the real Lie algebra of G_0;
3. $\mathfrak{g}_{\mathbb{C}}$—the complex Lie algebra of G_0;
4. $U(\mathfrak{g})$—the complex universal enveloping algebra;
5. $Z(\mathfrak{g})$—the center of $U(\mathfrak{g})$;

Let G is a reductive real group. We adopt the following notation:

1. K—a maximal compact subgroup of G;
2. $\Pi(G)$—the set of equivalence classes of irreducible $(\mathfrak{g}, K)$-modules, or equivalently, the set of infinitesimal equivalence classes of irreducible Hilbert representations;
3. $\Pi_u(G)$—the set of equivalence classes of unitarizable $(\mathfrak{g}, K)$- modules, or equivalently, the set of equivalence classes of irreducible unitary representations of G.
4. $(,)$—an invariant quadratic form on $\mathfrak{g}$ such that $(,)|_{\mathfrak{k}}$ is positive definite;
5. $\mathfrak{p}$—the orthogonal complement of $\mathfrak{k}_0$ with respect to $(,)$;

6. $\mathfrak{a}$—a maximal Abelian Lie subalgebra of $\mathfrak{p}$;

7. A—the connected Abelian group generated infinitesimally by $\mathfrak{a}$.

8. KAK—a Cartan decomposition.

Let V be a finite dimensional space over $\mathbb{F}$. We use V^* to denote $Hom_{\mathbb{F}}(V, \mathbb{F})$. Let G be a real reductive group.

Notation 1 *Let $(\pi, \mathcal{H})$ be a Hilbert representation of G. Fix a maximal compact subgroup K. Let $V(\pi)$ be the space of K-finite vectors in $\mathcal{H}$.*

Then $V(\pi)$ is a $(\mathfrak{g}, K)$-module. π is said to be admissible if each K-type occurs in $V(\pi)$ with finite multiplicity. If, in addition, $V(\pi)$ is finitely generated as $U(\mathfrak{g})$-module, $V(\pi)$ is often called a Harish-Chandra module.

An irreducible Hilbert representation of a reductive Lie group is always admissible. If π is unitary and irreducible, then the Hilbert norm is unique up to a scalar multiplication. So, equivalence classes of irreducible unitary representations are in one-to-one correspondence with irreducible unitarizable Harish-Chandra modules.

All unitary representations in this paper, unless otherwise stated, are taken as suitable unitarized Harish-Chandra modules.

This convention is consistent with the way we construct unitary representations. First, we will construct a $(\mathfrak{g}, K)$-module (π, V). Then we will show that V is an irreducible $(\mathfrak{g}, K)$-module. Hence V is a Harish-Chandra module. Finally, we will prove the existence of an invariant inner product on V. By a Theorem of Harish-Chandra, π is an irreducible unitary representation of G. For simplicity, we use π to denote the group action, the Lie algebra action and sometimes the $(\mathfrak{g}, K)$-module. We define three involutions in the category of Harish-Chandra modules:

1. π^*, the contragredient representation;

2. π^c, the representation π equipped with the conjugate complex linear structure;

3. π^h, the Hermitian dual representation of π.

We have $(\pi^*)^c = \pi^h$. If π is unitary, then $\pi^h \cong \pi$.

Notation 2 *Suppose $a, b \in \mathbb{R}^n$. Write $a \preceq b$ if and only if for every $1 \leq k \leq n$,*

$$\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j;$$

define $a \prec b$ if and only if

$$\sum_{j=1}^{k} a_j < \sum_{j=1}^{k} b_j.$$

The ordering $\preceq$ is a partial ordering.

1.2 Infinitesimal Character and Harish-Chandra Homomorphism

Let $\mathfrak{g}$ be a complex reductive Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $W(\mathfrak{g},\mathfrak{h})$ be the Weyl group generated by the root system $\Sigma(\mathfrak{g},\mathfrak{h})$. Let $U(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})}$ be the space of $W(\mathfrak{g},\mathfrak{h})$-invariant vectors in $U(\mathfrak{h})$. Then the Harish-Chandra homomorphism

$$\mathcal{H}C : Z(\mathfrak{g}) \to U(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})}$$

is an algebra isomorphism (see [KN], Chapter VIII. 5 or [Wallach]). Identify $U(\mathfrak{h})$ with the symmetric algebra of $\mathfrak{h}$. For each vector Λ in the complex dual space of $\mathfrak{h}$, define a character χ_Λ of $Z(\mathfrak{g})$ by

$$\chi_\Lambda(x) = \Lambda(\mathcal{H}C(x)) = \mathcal{H}C(x)(\Lambda) \qquad (x \in Z(\mathfrak{g}))$$

It is well-known that every character of $Z(\mathfrak{g})$ can be obtained this way and that Λ is unique up to the action of $W(\mathfrak{g},\mathfrak{h})$. In short, $Spec(Z(\mathfrak{g})) \cong \mathfrak{h}^*//W(fg,\mathfrak{h})$. Here the categorical quotient $h^*//W(\mathfrak{g},\mathfrak{h})$ coincides with the geometric quotient since $W(\mathfrak{g},\mathfrak{h})$ is finite.

Let G be a connected real reductive Lie group with Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G. Let $(\pi,\mathcal{H})$ be an irreducible Hilbert representation of G. Let $V(\pi)$ be the Harish-Chandra module of $(\pi,\mathcal{H})$, consisting of all the K-finite vectors in the Hilbert space $\mathcal{H}$. We retain π for the infinitesimal action of $U(\mathfrak{g})$ on the smooth vectors of $(\pi,\mathcal{H})$. Since π is irreducible, $V(\pi)$ is an irreducible $(U(\mathfrak{g}),K)$-module. Since G is connected, $Z(\mathfrak{g}) = U(\mathfrak{g})^G$. By Schur's lemma, $Z(\mathfrak{g})$ acts on $V(\pi)$ by a character

$$\chi : Z(\mathfrak{g}) \to \mathbb{C}.$$

Thus there exists a Λ such that $Z(g)$ acts on $V(\pi)$ by χ_Λ. For simplicity, we call Λ a **infinitesimal character** of π.

Let G be a real reductive group with a finite number of components. Let $\mathfrak{h}$ be a complex cartan subalgebra in $\mathfrak{g}_\mathbb{C}$. Then $U(\mathfrak{g})^{G_0} = Z(\mathfrak{g})$. Let $U(\mathfrak{g})^G$ be the G-invariant vectors in $U(\mathfrak{g})$. Consider the adjoint action of G/G_0 on $Z(\mathfrak{g})$. Each element in G/G_0 acts on $Z(\mathfrak{g})$ as an algebra automorphism. Furthermore, $U(\mathfrak{g})^G$ is precisely the subalgebra of $Z(\mathfrak{g})$ invariant under the action of G/G_0. By the Harish-Chandra homomorphism, G/G_0 acts on

$$U(\mathfrak{h})^{W(\mathfrak{g}_\mathbb{C},\mathfrak{h})}$$

as algebra automorphisms. This action induces an action of G/G_0 on

$$Spec(U(\mathfrak{h})^{W(\mathfrak{g}_\mathbb{C},\mathfrak{h})}).$$

Since G/G_0 is finite, by invariant theory

$$Spec(\mathcal{H}C(U(\mathfrak{g})^G))$$

is precisely in one-to-one correspondence with the G/G_0-orbits in

$$Spec(U(\mathfrak{h})^{W(\mathfrak{g}_\mathbb{C},\mathfrak{h})}).$$

Let π be an irreducible $(\mathfrak{g},K)$-module. By Schur's lemma, $U(\mathfrak{g})^G$ must act by a character ξ.

Definition 1 *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We say that $\Lambda \in \mathfrak{h}^*$ is an infinitesimal character of π if $\chi_\Lambda|_{U(\mathfrak{g})^G} = \xi$.*

Λ is unique up to the action of G/G_0 and $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. For semisimple Lie group G, the action of G/G_0 on $\mathfrak{h}^*//W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ is fairly easy to understand. Let ι be the action of G/G_0 on $\mathfrak{h}^*//W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. Then ι induces an action of G/G_0 on any closed Weyl chamber, consequently on the positive root system. Hence $\iota(G/G_0)$ can be regarded as automorphisms of the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$. The automorphisms of the Dynkin diagram are easy to classify.

The infinitesimal character is one of the main tools used in the literature to study irreducible admissible representations. In fact, there is only a finite number of infinitesimal equivalence classes of irreducible representations with a fixed infinitesimal character.

Notation 3 *Let $\Pi_\Lambda(G)$ be the set of infinitesimal equivalence classes of irreducible representations of G with infinitesimal character Λ.*

1.3 Leading Exponents of Irreducible Representations

Let G be a real reductive Lie group. Fix a maximal compact subgroup K. Unless otherwise stated, matrix coefficients in this paper are assumed to be K-finite. Fix a nondegenerate real invariant bilinear form $(,)$ and a maximal Abelian Lie subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ as in (1.1). Let r be the real dimension of $\mathfrak{a}$. We call r the real rank of G. Let $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system.

Fix a positive root system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Let $\rho(G)$ be the half sum of all positive roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$. Let M be the centralizer of $\mathfrak{a}$ in K. Let $W(G, \mathfrak{a})$ be the normalizer of $\mathfrak{a}$ in K modulo M. We call $W(G, \mathfrak{a})$ the real Weyl group. Let $W(\mathfrak{g}, \mathfrak{a})$ be the Weyl group generated by the root system. Clearly, $W(\mathfrak{g}, \mathfrak{a}) \subseteq W(G, \mathfrak{a})$.

For $Sp_{2n}(\mathbb{R})$, $\mathfrak{a}$ is isomorphic to $\mathbb{R}^n$. The real Weyl group $W(Sp_{2n}(\mathbb{R}), \mathfrak{a})$ is generated by permutations and sign changes on n variables. For $O(p,q)$, $\mathfrak{a}$ is isomorphic to $\mathbb{R}^{\min\{p,q\}}$. The real Weyl group $W(O(p,q), \mathfrak{a})$ is also generated by permutations and sign changes on $\min\{p,q\}$ variables.

Attached to each irreducible admissible representation π of a connected G of real rank r, is a finite number of r-dimensional complex vectors, called leading exponents. Leading exponents are vectors in the complex dual of $\mathfrak{a}$. They are the main data used to produce the Langlands classification (see [LA], [KN], [Wallach]). I shall emphasize that leading coefficients depend on the choice of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. For a disconnected group G, we define the leading exponents of $\pi \in \Pi(G)$ to be the leading exponents of π with respect to G_0.

Leading exponents are closely related to the infinitesimal character. For an irreducible finite dimensional representation π of a real reductive group G with real rank equal to complex rank, leading exponents are just the highest weights of π with respect to a maximally split Cartan subgroup. In this situation, finite dimensional representation theory says that a highest weight

v has the property that

$$v + \rho = w\Lambda$$

for some w in $W(G, \mathfrak{h})$. A similar statement holds for leading exponents for any irreducible admissible representation of a real reductive group. By Theorem 8.33 from [KN], we have

Theorem 1.1 *Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{m}$. Take*

$$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}.$$

Suppose π is an irreducible admissible representation of a real reductive group G. Let v be a leading exponent v of π. Then there exists an infinitesimal character Λ of π in $\mathfrak{h}^*_{\mathbb{C}}$ such that

$$v + \rho(G) = \Lambda|_{\mathfrak{a}}.$$

Notice that for non-split groups, v is in $\mathfrak{a}^*_{\mathbb{C}}$ and Λ is in $(\mathfrak{a} \oplus \mathfrak{b})^*_{\mathbb{C}}$. By Theorem 1.1, for any $\pi \in \Pi_\Lambda$, the set of leading exponents is finite.

Notation 4 *Let v be a complex vector. Denote the real part of v by $\Re(v)$.*

Leading exponents are extracted from a certain expansion of the matrix coefficients at ∞. Therefore, they control the growth of matrix coefficients. We cite the following estimate from [KN].

Theorem 1.2 *Let π be an irreducible admissible representation of a real reductive group G. Let v_0 be in the real dual of $\mathfrak{a}$. Let $a(g) \in A^+$ be the middle term of the KAK decomposition of g. If every leading exponent v of π satisfies*

$$(v_0 - \Re(v))(H) \geq 0 \qquad (\forall \;\; H \in \mathfrak{a}^+), \tag{1.1}$$

then there is an integer $q \geq 0$ such that each K-finite matrix coefficient of π is dominated on by a multiple of

$$\exp(v_0 \log a(g))(1 + (\log a(g), \log a(g))^{\frac{1}{2}})^q.$$

The converse also holds.

Theorem 1.2 provides a uniform bound for matrix coefficients for all $\pi \in \Pi_\Lambda(G)$. Let π be a unitary representation in $\Pi_\Lambda(G)$. If Λ is "small ", we can easily find a v_0 to dominate the set

$$\{w\Lambda|_{\mathfrak{a}} - \rho(G) \mid w \in W(G, \mathfrak{h}_{\mathbb{C}})\}.$$

By Theorem 1.2, we gain fairly good control over the growth of the matrix coefficients for all $\pi \in \Pi_\Lambda$. If Λ is large, then the convex cone spanned by the set

$$\{w\Lambda|_{\mathfrak{a}} - \rho(G) \mid w \in W(G, \mathfrak{h}_{\mathbb{C}})\}$$

is widely spread. The uniform bound from Theorem 1.2 does not yield any useful information. In fact, for unitary representations in $\Pi(G)$, the matrix coefficients are all bounded by constant functions. Theorem 1.2 then implies

Corollary 1.1 *Let G be a real reductive group. If $\pi \in \Pi_u(G)$, then every leading exponent v of π satisfies*
$$\Re(v)(H) \leq 0 (\forall\ H \in \mathfrak{a}^+).$$

In this paper, we will mostly deal with small Λ. To give a broader picture of the importance of $\Pi_u(G)$ with small infinitesimal character, we recall one definition from [VO00] and [SV].

Definition 2 *A unitary representatation $\pi \in \Pi_\Lambda(G)$ is called "unitarily small" if Λ is in the convex hull spanned by*
$$\{w(\rho(\mathfrak{g}_{\mathbb{C}})) \mid w \in W(\mathfrak{g}, \mathfrak{h})\}.$$

Salamanca-Vogan conjectured that any irreducible unitary representation of G not small may be constructed by parabolic or cohomological induction from an irreducible unitarily small representation of a reductive subgroup of G.

It is precisely for "unitarily small" representations that Theorem 1.2 will produce useful information about matrix coefficients beyond what is stated in the corollary. One goal of this paper is to study some unitarily small representations far away from the tempered unitary representations.

1.3.1 Example I: The groups $Mp_{2n}(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$

Let G be either $Mp_{2n}(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. Fix

$$\mathfrak{a} = \{\operatorname{diag}(a_1, a_2, \ldots a_n, -a_1, -a_2, \ldots - a_n)\}.$$

Then the Weyl group $W(G, \mathfrak{a})$ is generated by permutations and sign changes on $\{a_i\}_{i=1}^n$. Fix positive roots $\Sigma^+ = \{e_i \pm e_j \mid i \leq j; i, j \in [1, n]\}$. Then

$$\rho(G) = (n, n-1, \ldots, 1)$$

and

$$\mathfrak{a}^+ = \{a_1 \geq a_2 \geq \ldots \geq a_n \geq 0\}.$$

Clearly,

$$A^+ = \{\operatorname{diag}(A_1, A_2, \ldots A_n, A_1^{-1}, A_2^{-1}, \ldots A_n^{-1}) \mid A_1 \geq A_2 \geq A_n \geq 1\}.$$

Finally, (1.1) is equivalent to

$$v_0 \preceq \Re(v)$$

(see Notation 2 and 4).

1.3.2 Example II: The Groups $O(p, q)$

Let $G = O(p, q)$ with $q \geq p$. Here $O(p, q)$ is the isometry group of

$$(x, y) = \sum_{i=1}^{p} x_i y_{p+i} + x_{p+i} y_i + \sum_{j=1}^{q-p} x_{2p+i} y_{2p+i}.$$

Fix
$$\mathfrak{a} = \{ \operatorname{diag}(a_1, a_2, \ldots a_p, -a_1, -a_2, \ldots - a_p, 0, 0, \ldots 0) \mid a_i \in \mathbb{R}\}.$$
Then the Weyl group $W(G, \mathfrak{a})$ is again generated by permutations and sign changes on $\{a_i\}_{i=1}^p$. Notice for $p = q$, $W(G, \mathfrak{a})$ is bigger than $W(\mathfrak{g}, \mathfrak{a})$.

Fix restricted positive roots
$$\Sigma^+ = \{e_i \pm e_j \mid i < j; i, j \in [1, p]\} \cup \{e_i \mid i \in [1, p]\} \quad \text{if } p < q$$
and
$$\Sigma^+ = \{e_i \pm e_j \mid i < j; i, j \in [1, p]\} \quad \text{if } p = q.$$
Then
$$\rho(G) = (\overbrace{\frac{p+q-2}{2}, \frac{p+q-4}{2}, \ldots \frac{q-p}{2}}^{p}).$$
Fix a Weyl chamber
$$\mathfrak{a}^+ = \{a_1 \geq a_2 \geq \ldots a_n \geq 1\}.$$
(1.1) is again equivalent to $v_0 \preceq \Re(v)$. Notice that for $O(p, p)$, the Weyl chamber is not uniquely determined by Σ^+ due to the action of $O(p, p)/SO_0(p, p)$.

1.4 Global Characters

The global character is also known as the Harish-Chandra character. For each irreducible Hilbert representation π of G and a compactly supported smooth function $f(g)$ on G, define $\Theta(\pi)(f)$ to be the "trace" of
$$\pi(f) = \int f(g)\pi(g)dg.$$
$\Theta(\pi)$ is a distribution on G. A theorem of Harish-Chandra states that $\Theta(\pi)$ can be identified with a locally integrable function (still denoted by $\Theta(\pi)$) and $\Theta(\pi)$ is real analytic on the set of regular semisimple elements of G(see [HC]).

For π unitary, Miličić defined a notion of the rate of growth of $\Theta(\pi)$ which we will not recall here. We will state one theorem relating γ to the matrix coefficients of π (see [MI]).

Theorem 1.3 (Miličić) *Let $\pi \in \Pi_u(G)$. Let $\Xi(g)$ be Harish-Chandra's Ξ function. Then the following are equivalent:*

1. *$\Theta(\pi)$ has the rate of growth $\gamma \in \mathbb{R}$;*
2. *the K-finite matrix coefficients of π are bounded by*
$$C\Xi^{1-\gamma}(g)(1 + \|\log(a(g))\|)^s \qquad (C, s \geq 0, a(g) \in A^+);$$

3. every leading exponent v of π satisfies

$$\Re(v)(H) \leq \ (\gamma-1)\rho(G)(H) \qquad (\forall\ H \in \mathfrak{a}^+).$$

So $\gamma = 0$ if π is tempered and $\gamma = 1$ if π is trivial. For $G = Mp_{2n}(\mathbb{R})$ or $G = O(p,q)$, the last statement is equivalent to

$$\Re(v) \preceq (\gamma-1)\rho(G).$$

1.5 Associated Variety, Asymptotic Cycle and Wave Front Set

Recall that $U(\mathfrak{g})$ has a natural filtration

$$\mathbb{C} = U_0(\mathfrak{g}) \subseteq U_1(\mathfrak{g}) \subseteq U_2(\mathfrak{g}) \subseteq \oplus U_3(\mathfrak{g}) \subseteq \ldots \subseteq U_n(\mathfrak{g}) \subseteq \ldots.$$

1.5.1 Annihilator and Associated Variety

Let π be an admissible irreducible representation of a reductive group G. Let $V(\pi)$ be the Harish-Chandra module of π. Then $V(\pi)$ is a $U(\mathfrak{g})$ module. Consider the annihilator of π,

$$Ann(V(\pi)) = \{D \in U(\mathfrak{g}) \mid D \cdot V = 0\}.$$

$Ann(V(\pi))$ is an ideal of $U(\mathfrak{g})$ and inherits a filtration from the standard filtration of $U(\mathfrak{g})$. It follows that the induced graded algebra $gr(Ann(V(\pi)))$ is an ideal of $gr(U(g)) = S(g)$ and necessarily commutative.

Notation 5 *Let $\mathcal{V}(Ann\ \pi)$ be the associated variety of $gr(Ann(V))$.*

Theorem 1.4 (Borho-Brylinski) *Suppose G is a connected semisimple group and π is an irreducible representation of G. Let $\mathfrak{g}^*$ be the linear dual space of $\mathfrak{g}$. Then $\mathcal{V}(Ann\ \pi)$ is the closure of a single nilpotent orbit in $\mathfrak{g}^*$.*

Notation 6 *From now on, we will identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using a fixed invariant bilinear form on $\mathfrak{g}$.*

Thus $\mathcal{V}(Ann\ \pi)$ will be the closure of one single nilpotent adjoint orbit.

For G having a finite number of components, $\mathcal{V}(Ann\ \pi)$ will be the closure of a finite number of nilpotent adjoint orbits of equal dimension. In fact, G/G_0 acts on the set of nilpotent adjoint orbits. The associated variety $\mathcal{V}(Ann\ \pi)$ will be the closure of the union of one nilpotent orbit with its translations under $Ad(G/G_0)$. For a detailed account of associated varieties of Harish-Chandra modules, see [VO89].

1.5.2 Asymptotic Cycle and Wave Front Set

Let G be a semisimple Lie group with finite many components. Let us consider the global character $\Theta(\pi)$. One can lift $\Theta(\pi)$ to an invariant distribution D on $\mathfrak{g}_0$. There exists a Taylor expansion of D near 0,

$$D(f(tx)) \cong \sum_{i=-r}^{\infty} t^i D_i(f(x)).$$

In [BV80], Barbasch-Vogan proved that the Fourier transform $\widehat{D_i}$ is supported on the nilcone of $\mathfrak{g}$.

Definition 3 *Define the asymptotic cycle $AS(\pi)$ as the closure of the union of supports of $\widehat{D_i}$.*

Theorem 1.5 (Barbasch-Vogan) *Suppose G is a connected semisimple Lie group. Then the real dimension of $AS(\pi)$ is equal to the complex dimension of $\mathcal{V}(Ann\ \pi)$. $AS(\pi)$ is a union of nilpotent orbits contained in $\mathfrak{g}_0 \cap \mathcal{V}(Ann\ \pi)$. Let $r = \frac{\dim(AS(\pi))}{2}$. Then D_{-r} is the lowest nonzero term of the Taylor expansion of D and supp$(\widehat{D_{-r}})$ is of maximal dimension in $AS(\pi)$.*

This theorem holds for G with a finite number of components. See [BV80].

Another notion similar to $AS(\pi)$ is the wave front set of π defined by Howe ([Ho81]). Originally, $WF(\pi)$ was defined as a closed subset of the cotangent bundle T^*G. Because of the G-action, $WF(\pi)$ can be regarded as a closed subset of $\mathfrak{g}^*$. Howe then showed that for π irreducible $WF(\pi)$ is in the characteristic variety of $\mathfrak{g}^*$. For G semisimple, this is to say that $WF(\pi)$ is in the nilpotent cone of $\mathfrak{g}^*$. Howe further studied the behavior of wave front sets $WF(\pi)$ under restrictions to certain subgroups (see [Ho81]).

Suppose that π is irreducible and unitary and G is reductive. Rossmann proved that $WF(\pi)$ and $AS(\pi)$ are identical (see Theorem C, [**?**]). In what follows, we will not distinguish between $WF(\pi)$ and $AS(\pi)$ as long as $\pi \in \Pi_u(G)$. There are two basic facts the reader should keep in mind. The first fact is that $WF(\pi)$ lies in the real Lie algebra $\mathfrak{g}$. The second fact is that the algebraic closure of $WF(\pi)$ is exactly the associated variety $\mathcal{V}(Ann\ \pi)$.

Chapter 2

Nilpotent Orbits

An element x in a Lie algebra $\mathfrak{g}$ is called nilpotent if $ad(x)$ is nilpotent. Let G be a Lie group with Lie algebra $\mathfrak{g}$. If x is nilpotent and $g \in G$, then $Ad(g)x$ is also nilpotent. It follows that under the adjoint action of G, nilpotent elements are grouped into G-orbits. Each G-orbit is called a nilpotent adjoint orbit. For semisimple Lie groups, there are finitely many nilpotent orbits and the classification of nilpotent adjoint orbits is completely known. We cite Collingwood-McGovern's book [CM] as the main reference for this chapter. Unlike in [CM], nilpotent orbits in this paper depend on the Lie groups, not just on the Lie algebra. In particular, for the orthogonal groups, a nilpotent orbit may not be connected. For our convenience, we state the Springer-Steinberg theorem slightly differently than in [CM]. The reason for doing this is given in Theorem 2.6.

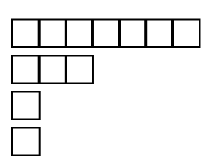
2.1 Young Diagrams and Complex Nilpotent Orbits

A sequence of positive integers

$$\mathbf{d} = (d_1 \geq d_2 \geq \ldots \geq d_{r-1} \geq d_r > 0 = d_{r+1})$$

is said to be a partition of n if $n = \sum_{j=1}^{r} d_j$. Write $|\mathbf{d}| = n$.

Notation 7 *In this paper, a partition of n will be represented by a Young diagram of n boxes, arranged as follows:*



with the $i-th$ row of length d_i. The transpose of $\mathbf{d}$ is denoted by $\mathbf{d}^t$.

In our notation, the Young diagram above can be written as $\mathbf{d} = (7, 3, 1, 1)$ and its transpose $\mathbf{d}^t = (4, 2, 2, 1, 1, 1, 1)$.

Definition 4 *If d_j is odd for every $j \leq r$, we say $\mathbf{d}$ is very odd. If d_j is even for every $j \leq r$, we say $\mathbf{d}$ is very even. If the nonzero d_j are all distinct, we say $\mathbf{d}$ is multiplicity free. A row of even length is called an even row. A row of odd length is called an odd row.*

2.1.1 Complex Nilpotent Orbits of $Sp_{2n}(\mathbb{C})$

Definition 5 *A Young diagram $\mathbf{d}$ is said to be a symplectic Young diagram of size $2n$ if odd rows occur with even multiplicity and $\|\mathbf{d}\| = 2n$. We use $\mathcal{YD}_-(2n)$ to denote the set of symplectic Young diagrams of size $2n$.*

A symplectic group is the linear group that preserves a nondegenerate skew-symmetric form.

Theorem 2.1 *Nilpotent coadjoint orbits of $Sp_{2n}(\mathbb{C})$ are in one to one correspondence with $\mathcal{YD}_-(2n)$.*

We denote the nilpotent coadjoint orbit corresponding to $\mathbf{d}$ by $\mathcal{O}_{\mathbf{d}}(-)$. For our convenience, we denote the symplectic group $Sp_{2n}(\mathbb{C})$ by $G(\mathcal{O}_{\mathbf{d}}(-))$. The group $Sp_{2n}(\mathbb{C})$ is thus attached to the orbit implicitly. The subscript $-$ is used to indicate that the group G is defined with respect to a *skew*-symmetric form.

2.1.2 Complex Nilpotent Orbits of $O(n, \mathbb{C})$

Definition 6 *A Young diagram $\mathbf{d}$ is said to be an orthogonal Young diagram of size n if even rows occur with even multiplicity and $\|\mathbf{d}\| = n$. We use $\mathcal{YD}_+(n)$ to denote the set of all orthogonal Young diagrams of size n.*

An orthogonal group is a linear group preserving a nondegenerate symmetric form.

Theorem 2.2 *Nilpotent coadjoint orbits of $O(n, \mathbb{C})$ are in one to one correspondence with $\mathcal{YD}_+(n)$.*

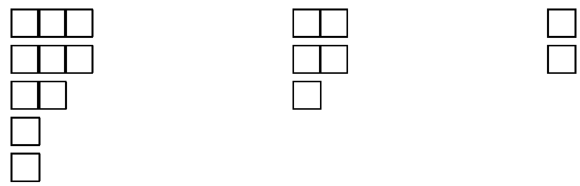
We denote the nilpotent coadjoint orbit corresponding to $\mathbf{d}$ by $\mathcal{O}_{\mathbf{d}}(+)$. If the orbit $\mathcal{O}_{\mathbf{d}}(+)$ is known to be orthogonal, we will simply denote it by $\mathcal{O}_{\mathbf{d}}$. In this context, the orthogonal group $O(n, \mathbb{C})$ is denoted by $G(\mathcal{O}_{\mathbf{d}}(+))$ or $G(\mathcal{O}_{\mathbf{d}})$. The script $+$ is used to indicate that the group G preserves a *symmetric* form.

Depending on the context, $\mathcal{O}_{\mathbf{d}}$ can refer to either $\mathcal{O}_{\mathbf{d}}(+)$ or $\mathcal{O}_{\mathbf{d}}(-)$.

2.1.3 Algorithm -1

Definition 7 *Let $\mathbf{d}$ be a Young diagram of size n. Define $\mathbf{d} - \mathbf{1}$ to be the new Young diagram obtained by deleting the first column of $\mathbf{d}$. Define $\mathbf{d} - \mathbf{i}$ to be the new Young diagram obtained by deleting the first i columns of $\mathbf{d}$.*

Consider $\mathbf{d} = [3^2, 2^1, 1^2]$. The Young diagrams $\mathbf{d}$, $\mathbf{d} - \mathbf{1}$ and $\mathbf{d} - \mathbf{2}$ are listed as follows



Notice that $\mathbf{d}$ is a symplectic Young diagram, $\mathbf{d}-\mathbf{1}$ is an orthogonal Young diagram, and $\mathbf{d}-\mathbf{2}$ is a symplectic Young diagram.

Theorem 2.3 *If* $\mathbf{d}$ *is a symplectic Young diagram, then* $\mathbf{d}-\mathbf{1}$ *is an orthogonal Young diagram. If* $\mathbf{d}$ *is an orthogonal Young diagram, then* $\mathbf{d}-\mathbf{1}$ *is a symplectic Young diagram.*

2.2 Signed Young Diagrams and Real Nilpotent Orbits

Type I classical groups are subgroups of the general linear groups that preserve certain sesquilinear forms (see [LI89] for the definition). Let G be a real classical group of type I. Real nilpotent orbits of G are simply nilpotent G-orbits in the real Lie algebra $\mathfrak{g}$. For type I classical groups, the real nilpotent orbits for G are parameterized by equivalence classes of signed Young diagrams.

Definition 8 ((see [CM] 9.3)) *Signed Young diagrams are Young diagrams with $+$ or $-$ labeling the boxes in such a way that signs alternate across rows . Two signed Young diagrams are considered to be equivalent if one signed diagram can be obtained from the other by interchanging the rows of same lengths.*

Let $\mathbf{D}$ be a signed Young diagram. We will use D^+ to denote the number of positive boxes in $\mathbf{D}$ and D^- to denote the number of negative boxes in $\mathbf{D}$. We call (D^+, D^-) the signature of $\mathbf{D}$.

In this chapter, we are only interested in the nilpotent orbits of $Sp_{2n}(\mathbb{R})$ and $O(p,q)$.

Notation 8 *For the sake of our discussion, we fix a matrix realization for each G. So the group $O(p,q)$ and $Sp_{2n}(\mathbb{R})$ in this paper, will come with a fixed sesquilinear form.*

The correspondence between signed Young diagrams and real nilpotent orbits depends on the sesquilinear form. I would like to thank the referee for pointing this to me.

2.2.1 Real Orbits of $Sp_{2n}(\mathbb{R})$

Theorem 2.4 (Springer and Steinberg) *Nilpotent adjoint orbits of $Sp_{2n}(\mathbb{R})$ are parametrized by the equivalence classes of signed Young diagrams with the following properties:*

- *the signature of* $\mathbf{D}$ *is* (n,n)*;*
- *for every s, rows of length $2s+1$ must occur with even multiplicity and must have their leftmost boxes labeled*

$$-,+,-,+,\ldots,-,+$$

  from the highest row to the lowest row.

Clearly, the signed Young diagrams that satisfy our second condition must have signature (n, n). So, the first condition is redundant. In the second condition, the choice of the sign pattern of rows of odd lengths is artificial. In fact, it suffices that there are same number of boxes labeled $+$ and $-$ in the rows of length $2s+1$. Nevertheless, I have included this in Theorem 2.4 for two purposes. The first is to maximize the analogy with Theorem 2.5. The second is to make the signed young diagrams easy to manipulate.

We denote the set of signed Young diagrams in Theorem 2.4 by $\mathcal{Y}D_{-}(n, n)$. The following signed Young diagrams are in $\mathcal{Y}D_{-}(n, n)$:

$$\begin{array}{l}
\boxed{-}\boxed{+}\boxed{-}\boxed{+}\boxed{-}\boxed{+}\boxed{-} \\
\boxed{+}\boxed{-}\boxed{+}\boxed{-}\boxed{+}\boxed{-}\boxed{+} \\
\boxed{+}\boxed{-}\boxed{+}\boxed{-} \\
\boxed{+}\boxed{-} \\
\boxed{-} \\
\boxed{+}
\end{array}
\qquad\qquad
\begin{array}{l}
\boxed{-}\boxed{+}\boxed{-}\boxed{+}\boxed{-}\boxed{+} \\
\boxed{-}\boxed{+}\boxed{-}\boxed{+}\boxed{-}\boxed{+} \\
\boxed{-}\boxed{+}\boxed{-}\boxed{+}\boxed{-} \\
\boxed{+}\boxed{-}\boxed{+}\boxed{-}\boxed{+} \\
\boxed{-}\boxed{+}\boxed{-}\boxed{+}\boxed{-} \\
\boxed{+}\boxed{-}\boxed{+}\boxed{-}\boxed{+}
\end{array}$$

Notice that $Mp_{2n}(\mathbb{R})$-adjoint orbits coincide with $Sp_{2n}(\mathbb{R})$- adjoint orbits.

Notation 9 *We denote the nilpotent orbit corresponding to* $\mathbf{D}$ *by* $\mathcal{O}_{\mathbf{D}}(-)$ *or simply* $\mathcal{O}_{\mathbf{D}}$ *if* $\mathbf{D}$ *is specified to be symplectic. We denote the group* $Mp_{2n}(\mathbb{R})$ *by* $G(\mathcal{O}_{\mathbf{D}})$.

Definition 9 *Define*

$$\tau(x) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} x \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad (x \in \mathfrak{s}p_{2n}(\mathbb{R})).$$

Notice that

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = - \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and changing the symplectic form $<,>$ to $-<,>$ results in the same group $Sp_{2n}(\mathbb{R})$. Therefore τ defines an involution on $\mathfrak{sp}_{2n}(\mathbb{R})$ and on $Sp_{2n}(\mathbb{R})$. By sorting out the signs in the proof of Lemma 9.3.1 in [CM], we obtain

Lemma 2.1 *1. τ defines an involution on the real Lie algebra $\mathfrak{s}p_{2n}(\mathbb{R})$.*

2. τ induces an involution on the set of nilpotent orbits, namely

$$\tau(\mathcal{O}_{\mathbf{D}}) = \mathcal{O}_{\tau(\mathbf{D})}.$$

Here $\tau(\mathbf{D})$ is the signed Young diagram obtained by switching the signs $(+ \leftrightarrow -)$ for the even rows of $\mathbf{D}$.

3. $\mathcal{O}_{\mathbf{D}}$ is geometrically identical to $\mathcal{O}_{\tau(\mathbf{D})}$.

The proof is left to the reader.

Notice that the odd rows of $\tau(\mathbf{D})$ remain the same as those of $\mathbf{D}$. For example, τ:

$$\begin{array}{lllllll} - & + & - & + & - & + & - \\ + & - & + & - & + & - & + \\ + & - & + & - & & & \\ + & - & & & & & \\ - & & & & & & \\ + & & & & & & \end{array} \quad \longrightarrow \quad \begin{array}{lllllll} - & + & - & + & - & + & - \\ + & - & + & - & + & - & + \\ - & + & - & + & & & \\ - & + & & & & & \\ - & & & & & & \\ + & & & & & & \end{array}$$

One can further explore the involution τ on the Lie group level.

Definition 10 *1. Define τ on $Sp_{2n}(\mathbb{R})$ by*

$$\tau(g) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Clearly τ defines an involution on $Sp_{2n}(\mathbb{R})$. τ is a topological homeomorphism and an isometry with respect to certain left invariant Riemannian metric.

2. Lift τ to an involution on $Mp_{2n}(\mathbb{R})$. The lift τ exists and is unique. By abusing notation, denote this involution by τ.

3. Let $\pi \in \Pi(Mp_{2n}(\mathbb{R}))$. Define a new representation $(\pi^\tau, \mathcal{H}_\pi)$ by

$$\pi^\tau(g) = \pi(\tau(g)).$$

Lemma 2.2 *$\pi \to \pi^\tau$ defines an involution on $\Pi(Mp_{2n}(\mathbb{R}))$ and on $\Pi_u(Mp_{2n}(\mathbb{R}))$.*

2.2.2 Real Orbits of $O(p, q)$

Theorem 2.5 (Springer & Steinberg) *Nilpotent coadjoint orbits of $O(p, q)$ are parametrized by equivalence classes of signed Young diagrams with the following properties*

- *the signature of $\mathbf{D}$ is (p, q);*
- *for every s, rows of even length $2s$ must occur with even multiplicity and must have their leftmost boxes labeled*

  $$+, -, +, -, \ldots, +, -$$

  from the highest row to the lowest row.

Again, the choice of sign patterns for rows of even length is artificial. We denote the set of signed Young diagrams in Theorem 2.5 by $\mathcal{Y}D_+(p, q)$. The following signed Young diagrams are in $\mathcal{Y}D_+(7, 9)$:

$$\begin{array}{llllll} + & - & + & - & + & - \\ - & + & - & + & - & + \\ - & + & - & & & \\ - & & & & & \end{array} \qquad \begin{array}{lllll} - & + & - & + & - \\ + & - & + & - & \\ - & + & - & + & \\ - & + & - & & \end{array}$$

Notation 10 *We use $\mathcal{O}_{\mathbf{D}}$ to denote the nilpotent orbit corresponding to $\mathbf{D} \in \mathcal{Y}D_+(p,q)$. We use $G(\mathcal{O}_{\mathbf{D}})$ to denote the group $O(p,q)$. Any $\mathbf{D}$ in this paper is interpreted as a signed Young diagram in a previously chosen set $\mathcal{Y}D_-$ or $\mathcal{Y}D_+$. The orbit $\mathcal{O}_{\mathbf{D}}$ refers to either $\mathcal{O}_{\mathbf{D}}(+)$ or $\mathcal{O}_{\mathbf{D}}(-)$ with the understanding that $+$ or $-$ is implicitly known once $\mathbf{D}$ is given.*

Of course, if $\mathbf{D}$ is specified to be in $\mathcal{Y}D_+$ or $\mathcal{Y}D_-$, the notation $\mathcal{O}_{\mathbf{D}}$ causes no confusion.

2.3 Nilpotent Orbits of Class $\mathcal{U}$

Notation 11 *Let $\mathcal{O}_{\mathbf{D}}$ be a real nilpotent orbit. We use $\mathbf{d}$ to denote the Young diagram obtained from $\mathbf{D}$ by removing the signs.*

Every real nilpotent orbit $\mathcal{O}$ induces a complex nilpotent orbit $\mathcal{O}_{\mathbb{C}}$ by considering the complex group G_{ad} acting on $\mathcal{O}$ in $\mathfrak{g}$. This map is simply

$$\mathcal{O}_{\mathbf{D}} \to \mathcal{O}_{\mathbf{d}}$$

for orthogonal groups and symplectic groups.

Definition 11 *Let $\mathbf{D}$ be a signed Young diagram. Define $\mathbf{D}-\mathbf{1}$ to be the signed Young diagram obtained from $\mathbf{D}$ by deleting the first column.*

Theorem 2.6 $-\mathbf{1}$ *defines an operation from $\mathcal{Y}D_+(p,q)$ to the disjoint union*

$$\cup_{n \leq \min(p,q)} \mathcal{Y}D_-(n,n)$$

$-\mathbf{1}$ *also defines an operation from $\mathcal{Y}D_-(n,n)$ to the disjoint union*

$$\cup_{\max(p,q) \leq n} \mathcal{Y}D_+(p,q)$$

By Theorem 2.4 and Theorem 2.5, $-\mathbf{1}$ induces an orbital correspondence from real nilpotent orbits of symplectic groups to real nilpotent orbits of orthogonal groups, and conversely.

Definition 12 *We say that $\mathcal{O}_{\mathbf{D}}$ (or $\mathcal{O}_{\mathbf{d}}$) is pre-rigid if $\mathbf{d}^t$ is multiplicity free.*

This amounts to saying that every integer between 1 and d_1 appears in the partition $\mathbf{d}$. In other words, the partition $\mathbf{d}$ is of the following form

$$([d_1]^{m_1}, [d_1-1]^{m_2}, [d_1-2]^{m_3}, \ldots, [1]^{m_{d_1}})$$

with each multiplicity $m_i \geq 1$.

Definition 13 (Nilpotent orbits of Class $\mathcal{U}$)

1. *$\mathcal{U}$ consists of a class of real nilpotent orbits $\mathcal{O}_{\mathbf{D}}$, yet to be defined. First, for technical reasons, we* **exclude** *from $\mathcal{U}$ those $\mathbf{D}$ whose last 2 columns are of the same length and are of the following forms*

$$\begin{array}{cc} - & + \\ - & + \\ - & + \\ - & + \\ - & + \end{array} \qquad\qquad \begin{array}{cc} + & - \\ + & - \\ + & - \\ + & - \\ + & - \end{array}$$

2. *Let $\mathbf{D} \in \mathcal{Y}D_-(n,n)$ and $\mathbf{d}^t = (m_1 \geq m_2 \geq \ldots \geq m_{d_1})$. The nilpotent orbit $\mathcal{O}_{\mathbf{D}}$ of $Sp_{2n}(\mathbb{R})$ is said to be in $\mathcal{U}(Sp_{2n}(\mathbb{R}))$ if $\mathbf{d}^t$ satisfies the following three conditions:*
   - *(a) $\mathbf{d}^t$ is either very even or very odd;*
   - *(b) For every i, $m_{2i} > m_{2i+1}$ and $m_{2i+1} \geq m_{2i+2}$;*
   - *(c) (1).*
3. *Let $\mathbf{D} \in \mathcal{Y}D_-(p,q)$ and*
$$\mathbf{d}^t = (m_1 \geq m_2 \geq \ldots \geq m_{d_1}).$$
The nilpotent orbit $\mathcal{O}_{\mathbf{D}}$ is said to be in $\mathcal{U}(O(p,q))$ if $\mathbf{d}^t$ satisfies the following three conditions:
   - *(a) $\mathbf{d}^t$ is either very even or very odd;*
   - *(b) For every i, $m_{2i} \geq m_{2i+1}$ and $m_{2i+1} > m_{2i+2}$;*
   - *(c) (1).*

Here are two $\mathbf{D}$, one in $\mathcal{U}(Sp_{30}(\mathbb{R}))$, the other in $\mathcal{U}(O(15,22))$.

$$\begin{array}{llllll} - & + & - & + & - & + \\ - & + & - & + & - & \\ + & - & + & - & + & \\ - & + & - & + & & \\ + & - & + & - & & \\ + & - & & & & \\ + & - & & & & \\ - & & & & & \\ + & & & & & \end{array} \qquad\qquad \begin{array}{llllll} + & - & + & - & + & - \\ - & + & - & + & - & + \\ - & + & - & + & - & \\ - & + & - & + & - & \\ - & + & - & + & - & \\ - & + & - & + & - & \\ - & + & - & & & \\ - & & & & & \\ - & & & & & \end{array}$$

The condition (b) amounts to saying that all multiplicities of $\mathbf{d}$ except perhaps the first are even. The condition (c) is weaker than saying $\mathbf{d}^t$ is multiplicity free. Recall

Lemma 2.3 (Theorem 7.3.5 and Proposition 6.3.7, [CM]) *If $\mathcal{O}_{\mathbf{d}}$ is rigid, then $\mathcal{O}_{\mathbf{d}}$ is pre-rigid. If $\mathcal{O}_{\mathbf{d}}$ is special and rigid, then $\mathbf{d}^t$ must be either very even or very odd and must be multiplicity free.*

Corollary 2.1 *Special rigid orbits of $O(p,q)$ and $Sp_{2n}(\mathbb{R})$ are contained in $\mathcal{U}$.*

By Theorem 2.6, we have

Corollary 2.2 *If $\mathcal{O}_{\mathbf{D}}$ is in $\mathcal{U}(Mp)$, then $\mathcal{O}_{\mathbf{D-1}}$ is in $\mathcal{U}(O)$. If $\mathcal{O}_{\mathbf{D}}$ is in $\mathcal{U}(O)$, then $\mathcal{O}_{\mathbf{D-1}}$ is in $\mathcal{U}(Mp)$.*

2.4 Induced Orbits

2.4.1 Complex Induced Orbits

Let $\mathfrak{g}$ be either $\mathfrak{o}(n,\mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be a parabolic subalgebra. Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in $\mathfrak{l}$.

Definition 14 *Define $Ind_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}}$ to be the nilpotent G-orbit that intersects $\mathcal{O}_{\mathfrak{l}}+\mathfrak{n}$ in an open dense set.*

Our definition differs slightly from in Ch. 7.1 from [CM] as we require $Ind_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}}$ be a G-orbit and G may not be connected. In any case, $\mathcal{O}_{\mathfrak{g}}$ is the G-orbit of the "generic" elements in $\mathcal{O}_{\mathfrak{l}}+\mathfrak{n}$. In addition, $Ind_{\mathfrak{l}}^{\mathfrak{g}}O_{\mathfrak{l}}$ only depends on $\mathfrak{l}$, not on the choice of $\mathfrak{p}$ (see Ch. 7.1 [CM]).

Lemma 2.4 (Proposition 7.1.4 [CM]) *Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ be two Levi subalgebras of $\mathfrak{g}$ and $\mathfrak{l}_1 \subset \mathfrak{l}_2$. Then*

$$Ind_{\mathfrak{l}_2}^{\mathfrak{g}}(Ind_{\mathfrak{l}_1}^{\mathfrak{l}_2}\mathcal{O}_{\mathfrak{l}_1}) = Ind_{\mathfrak{l}_1}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}_1}.$$

Thus orbital induction is "associative".

Nilpotent orbits of $SL(n,\mathbb{C})$ are parametrized by partitions of n in terms of the Jordan form.

Lemma 2.5 *Let $n = n_1 + n_2$. Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$. Let $\mathfrak{l}$ be the block-diagonal matrices of size (n_1, n_2) in $\mathfrak{g}$. Let $\mathcal{O}_{\mathbf{s}}$ and $\mathcal{O}_{\mathbf{t}}$ be nilpotent orbits in $\mathfrak{sl}(n_1,\mathbb{C})$ and $\mathfrak{sl}(n_2,\mathbb{C})$ respectively. Then*

$$Ind_{\mathfrak{g}}^{\mathfrak{l}}\mathcal{O}_{\mathbf{s}} \times \mathcal{O}_{\mathbf{t}} = \mathcal{O}_{\mathbf{d}}$$

with

$$d_j = s_j + t_j \qquad (\forall\ j).$$

We call $\mathbf{d}$ the merging of $\mathbf{s}$ and $\mathbf{t}$. For $G = O(n,\mathbb{C})$ or $G = Sp_{2n}(\mathbb{C})$, computation of induced orbits is slightly more complicated that simply a "merging". One needs the concept of "collapse" (see Ch. 6 [CM]). For the sake of simplicity, we will not introduce the concept of "collapse". We only state one special result concerning $Sp_{2n}(\mathbb{C})$. The general result can be found in [CM] (Lemma 6.3.3) and is due to Gerstenhaber.

Lemma 2.6 *Let $G = Sp_{2n}(\mathbb{C})$ and $L = Sp_{2m}(\mathbb{C}) \times GL(n-m,\mathbb{C})$. Let $\mathcal{O}_{\mathbf{s}}$ be a nilpotent orbit of $Sp_{2m}(\mathbb{C})$ such that the number of rows in $\mathbf{s}$ is less than or equal to $n-m$. Then*

$$Ind_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathbf{s}} \times \{0\} = \mathcal{O}_{\mathbf{d}}$$

with

$$d_j = s_j + 2 \qquad (\forall\, 1 \leq j \leq n-m).$$

So under the assumption that the number of the rows in $\mathbf{s}$ is less or equal to $n-m$, $\mathbf{d}$ is the merging of $\mathbf{s}$ with two copies of 1^m. $\mathbf{d}$ remains symplectic since $\mathbf{s}$ is symplectic.

2.4.2 Induced Real Orbits

Let G be a reductive Lie group and L be a Levi subgroup. Let $P = LN$ be a parabolic subgroup of G. Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in $\mathfrak{l}$.

Definition 15 *Define $Ind_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}}$ to be the union of the nilpotent G-orbits $\mathcal{O}_{\mathfrak{g}}^{(i)}$ that contain an open subset of $\mathcal{O}_{\mathfrak{l}} + \mathfrak{n}$.*

Implicit in the notation is the fact that $Ind_{\mathfrak{l}}^{\mathfrak{g}}$ does not depend on the choice of $\mathfrak{n}$. The proof of this fact is essentially the same as the proof for the complex orbits (see Theorem 7.1.3 of [CM]). $Ind_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}}$ for a classical group of type I may no longer be a single nilpotent orbit. Nevertheless, it is contained in a single complex nilpotent orbit.

Lemma 2.7 *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{l}} & \xrightarrow{complex\ orbit} & \mathcal{O}_{\mathfrak{l}_{\mathbb{C}}} \\ \Big\downarrow Ind & & \Big\downarrow Ind \\ Ind_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}} & \xrightarrow{complex\ orbit} & Ind_{\mathfrak{l}_{\mathbb{C}}}^{\mathfrak{g}_{\mathbb{C}}}\mathcal{O}_{\mathfrak{l}_{\mathbb{C}}} \end{array} \tag{2.1}$$

Let $\mathcal{O}_{\mathbf{S}}$ be a nilpotent orbit in $\mathfrak{sp}_{2\mathfrak{m}}(\mathbb{R})$. By the lemma above,

$$Ind_{\mathfrak{gl}(n-m,\mathbb{R})\oplus\mathfrak{sp}_{2m}(\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})}\{0\} \times \mathcal{O}_{\mathbf{S}}$$

must be contained in the complex induced orbit

$$Ind_{\mathfrak{gl}(n-m,\mathbb{C})\oplus\mathfrak{sp}_{2m}(\mathbb{C})}^{\mathfrak{sp}_{2n}(\mathbb{C})}\{0\} \times \mathcal{O}_{\mathbf{s}}.$$

Under the assumption in Lemma 2.6, this complex induced orbit is parametrized by $\mathbf{d}$ with

$$d_j = s_j + 2 \quad (\forall\, 1 \leq j \leq n-m).$$

It is an linear algebra exercise to show that

$$Ind_{\mathfrak{gl}(n-m,\mathbb{R})\oplus\mathfrak{sp}_{2m}(\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})}\{0\} \times \mathcal{O}_{\mathbf{S}}$$

is the intersection of $\mathcal{O}_{\mathbf{d}}$ with $\mathfrak{sp}_{2n}(\mathbb{R})$. Thus we obtain

Theorem 2.7 *Let $\mathbf{s} = (s_1 \geq s_2 \geq \ldots \geq s_r > 0)$ be in $\mathcal{Y}D_-(m,m)$. Suppose that $n - m \geq r$. Then*

$$Ind_{\mathfrak{gl}(n-m,\mathbb{R})\oplus\mathfrak{sp}_{2m}(\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})}\{0\} \times \mathcal{O}_{\mathbf{S}} = \cup_{j=0}^{n-m-r}\mathcal{O}_{\mathbf{D}^{(j)}}.$$

Each $\mathbf{D}^{(j)} \in \mathcal{Y}D_-(n,n)$ is uniquely defined as follows:

1. *merge one column of length $n-m$ to $\mathbf{S}$ from the left;*
2. *merge one column of length $n-m$ to $\mathbf{S}$ from the right;*
3. *extend the signs of $\mathbf{S}$ for rows of even lengths;*

4. fill in the $n-m-r$ *rows of length* 2 *in* $\mathbf{D}^{(j)}$ *with* j $\boxed{-}\boxed{+}$ *s and* $n-m-r-j$ $\boxed{+}\boxed{-}$ *s.*

The signs of $\mathbf{D}^{(j)}$ for rows of odd length are uniquely dictated by the rules we set for $\mathcal{Y}D_{-}(n,n)$, thus not subject to change.

Example Given $\mathbf{t}$, $\mathbf{S}$, $\mathbf{t}$,

$$\begin{array}{l} \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} \qquad \begin{array}{l} - + - + - \\ + - + - + \\ - + - + \\ - + \\ - + \end{array} \qquad \begin{array}{l} \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array}$$

one obtains three symplectic nilpotent orbits corresponding to

$$\begin{array}{l} - + - + - + - \\ + - + - + - + \\ + - + - + - \\ + - + - \\ + - + - \\ + - \\ + - \end{array} \qquad \begin{array}{l} - + - + - + - \\ + - + - + - + \\ + - + - + - \\ + - + - \\ + - + - \\ + - \\ - + \end{array} \qquad \begin{array}{l} - + - + - + - \\ + - + - + - + \\ + - + - + - \\ + - + - \\ + - + - \\ - + \\ - + \end{array}$$

respectively. These three signed Young diagrams are $\mathbf{D}^{(0)}$, $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$.

Corollary 2.3 *Let* $\mathbf{t} = [1^n]$. *Then*

$$Ind_{\mathfrak{gl}(n,\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})} \mathcal{O}_{\mathbf{t}}$$

is the union of $n+1$ *orbits consisting of all* $\mathcal{O}_{\mathbf{D}^{(j)}}$ *with* $\mathbf{d}^{(j)} = [2^n]$.

Theorem 2.8 *Let* $\mathcal{O}_{\mathbf{S}}$ *be a nilpotent orbit of* $Sp_{2m}(\mathbb{R})$. *Then*

$$Ind_{\mathfrak{sp}_{2m}(\mathbb{R}) \oplus \mathfrak{gl}(n-m,\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})} \tau(\mathcal{O}_{\mathbf{S}}) \times \{0\}$$

is a union of nilpotent orbits $\mathcal{O}_{\mathbf{D}^{(j)}}$ *and each* $\mathbf{D}^{(j)}$ *can be obtained by*

- *merging two copies of* $[1^{n-m}]$ *to* $\mathbf{S}$ *from the left;*
- *extending the signs of* $\mathbf{S}$*;*
- *filling the* $n-m-r$ *rows of length* 2 *in* $\mathbf{D}^{(j)}$ *with* j $\boxed{-}\boxed{+}$*and* $n-m-r-j$ $\boxed{+}\boxed{-}$.

The proof is omitted.

Chapter 3

Theta Correspondences and Quantum Induction

Fix a dual pair $(O(p,q), Sp_{2n}(\mathbb{R}))$ in $Sp_{2n(p+q)}(\mathbb{R})$ (see [Ho79]). Let $Mp_{2n(p+q)}(\mathbb{R})$ be the metaplectic covering of $Sp_{2n(p+q)}(\mathbb{R})$. Let $\{1, \epsilon\}$ be the preimage of $1 \in Sp_{2n(p+q)}(\mathbb{R})$. For any subgroup G of $Sp_{2n(p+q)}(\mathbb{R})$, let MG be the preimage of G under the metaplectic covering. Fix a maximal compact subgroup K for $Mp_{2n(p+q)}(\mathbb{R})$ such that $K \cap MO(p,q)$ is a maximal compact subgroup of $MO(p,q)$ and $K \cap MSp_{2n}(\mathbb{R})$ is a maximal compact subgroup of $MSp_{2n}(\mathbb{R})$. Let $\omega(p,q;2n)$ be the oscillator representation of $Mp_{2n(p+q)}(\mathbb{R})$ equipped with a fixed dual pair $(O(p,q), Sp_{2n}(\mathbb{R}))$. If the parameters $(p,q;2n)$ are apparent, we will write ω. Let $V(\omega)$ be the Harish-Chandra module. Unless otherwise stated, any representation π of MG in this paper satisfies $\pi(\epsilon) = -1$. Very often, this assumption reduces our discussion to the representation theory of the linear group.

Notation 12 *Let (G_1, G_2) be a dual pair in Sp. Let $\mathcal{R}(MG, \omega)$ be the equivalence classes of irreducible Harish-Chandra modules of MG that occur as quotients of $V(\omega)$.*

Theorem 3.1 (Howe, [Ho89]) *Let (G_1, G_2) be a dual pair in Sp. Then $\mathcal{R}(MG_1G_2, \omega)$ yields a one-to-one correspondence between $\mathcal{R}(MG_1, \omega)$ and $\mathcal{R}(MG_2, \omega)$.*

This correspodence is often called (local) theta correspondence, Howe's correspondence or duality correspondence. Let $\theta(p,q;2n)$ be the theta correspondence from $\mathcal{R}(MO(p,q), \omega)$ to $\mathcal{R}(MSp_{2n}(\mathbb{R}), \omega)$. Let $\theta(2n;p,q)$ be its inverse. If the parameters $(p,q;2n)$ are apparent, we will just write θ. The description of the sets $\mathcal{R}(MO(p,q), \omega)$ and $\mathcal{R}(MSp_{2n}(\mathbb{R}), \omega)$ is not known in general.

For $p+q$ odd, $MSp_{2n}(\mathbb{R})$ is the metaplectic group $Mp_{2n}(\mathbb{R})$ and $MO(p,q)$ splits; for $p+q$ even, $MSp_{2n}(\mathbb{R}) \cong Sp_{2n}(\mathbb{R}) \times \{\pm 1\}$ and $MO(p,q)$ splits. In both cases, θ can be regarded as a one-to-one correspondence between a certain subset of $\Pi(Mp_{2n}(\mathbb{R}))$ and a certain subset of $\Pi(O(p,q))$. Since this is the viewpoint in some literature (for example [KR]), we will also take this viewpoint if necessary.

3.1 Theta Correspondence in Semistable Range

We adopt the notation from (1.3.1) and (1.3.2). Let $\mathbf{n}$ be the constant vector $(n, n, n, \ldots, n)$ of a fixed dimension.

Definition 16 *An irreducible representation π of $MO(p,q)$ is said to be in the semistable range of $\theta(p,q;2n)$ if and only if every leading exponent v of π satisfies*

$$\Re(v) - \mathbf{n} + 2\rho(O(p,q)) \prec 0.$$

An irreducible representation π of $MSp_{2n}(\mathbb{R})$ is said to be in the semistable range of $\theta(2n;p,q)$ if and only if every leading exponent v of π satisfies

$$\Re(v) - \frac{\mathbf{p}+\mathbf{q}}{\mathbf{2}} + 2\rho(Sp_{2n}(\mathbb{R})) \prec 0.$$

We denote the semistable ranges by $\mathcal{R}_s(MO(p,q),\omega)$ and $\mathcal{R}_s(MSp_{2n}(\mathbb{R}),\omega)$ respectively.

Definition 17 ([LI89], [He00]) *Consider $(G_1, G_2) = (O(p,q), Sp_{2n}(\mathbb{R}))$ or $(G_1, G_2) = (Sp_{2n}(\mathbb{R}), O(p,q))$. Let $\pi \in \mathcal{R}_s(MG_1,\omega)$. Define a bilinear form $(,)_\pi$ on $V(\omega) \otimes V(\pi^c)$*

$$(\phi \otimes u, \psi \otimes v)_\pi = \int_{MG_1} (\omega(g)\phi, \psi)(v, \pi(g)u) dg \qquad (\phi, \psi \in V(\omega), u, v \in V(\pi)).$$

Let $\mathcal{R}_\pi$ be the radical of $(,)_\pi$. Define

$$\theta_s(MG_1, MG_2)(\pi) = V(\omega(MG_1, MG_2)) \otimes V(\pi^c)/\mathcal{R}_\pi.$$

$\theta_s(MG_1, MG_2)(\pi)$ inherits an infinitesimal MG_2 action from $\omega(MG_1, MG_2)$. It is a $(\mathfrak{g}, K)$-module of MG_2.

Theorem 3.2 ([He00]) *Suppose π is a unitarized Harish-Chandra module in the semistable range of $\theta(MG_1, MG_2)$. Then $(,)_\pi$ is well-defined. If $(,)_\pi \neq 0$, then $\theta_s(MG_1, MG_2)(\pi)$ is an irreducible Hermitian Harish-Chandra module of MG_2 and $\theta_s(MG_1, MG_2)(\pi)$ is equivalent to $\theta(MG_1, MG_2)(\pi)$.*

This theorem basically says that if $(,)_\pi$ is well-defined and nonvanishing, then θ_s is Howe's correspondence ([Ho89]) on the Harish-Chandra module level. By Howe's Theorem, $\theta_s(MG_1, MG_2)(\pi)$ is an irreducible Harish-Chandra module of MG_2. The notation $(,)_\pi$ is essentially due to Jian-Shu Li ([LI89]).

In [He01], we proved the following nonvanishing theorem.

Theorem 3.3 ([He01]) *Suppose $p + q \leq 2n + 1$. Suppose $\pi \in \mathcal{R}_s(MO(p,q),\omega)$. Let $\det$ be the lift of the determinant of $O(p,q)$ to $MO(p,q)$, i.e.,*

$$\ker(\det) = MSO(p,q).$$

Then either $\theta_s(p,q;2n)(\pi) \neq 0$ or $\theta_s(p,q;2n)(\pi \otimes \det) \neq 0$.

3.2 Unitarity and Strongly Semistable Range

In [Heu], we proved the unitarity of $\theta_s(\pi)$ for π in the strongly semistable range.

Definition 18 *An irreducible admissible representation π is in $\mathcal{R}_{ss}(MO(p,q),\omega)$ if every leading exponent of π satisfies*

$$\Re(v) - (\mathbf{n} - \frac{\mathbf{p}+\mathbf{q}}{\mathbf{2}}) + \rho(O(p,q)) \preceq 0. \tag{3.1}$$

An irreducible admissible representation π of $MSp_{2n}(\mathbb{R})$ is in $\mathcal{R}_{ss}(MSp_{2n}(\mathbb{R}),\omega)$ if every leading exponent of π satisfies

$$\Re(v) - (\frac{\mathbf{p}+\mathbf{q}}{\mathbf{2}} - \mathbf{n} - \mathbf{1}) + \rho(Sp_{2n}(\mathbb{R})) \preceq 0.$$

We call $\mathcal{R}_{ss}$ the strongly semistable range.

Notice that $\mathcal{R}_s(MSp_{2n}(\mathbb{R}),\omega)$ and $\mathcal{R}_{ss}(MO(p,q),\omega)$ only depend on $p+q$, not on a particular pair (p,q). Since

$$\Re(v) - \mathbf{n} + 2\rho(O(p,q)) \prec \Re(v) - (\mathbf{n} - \frac{\mathbf{p}+\mathbf{q}}{\mathbf{2}}) + \rho(O(p,q))$$

and

$$\Re(v) - \frac{\mathbf{p}+\mathbf{q}}{\mathbf{2}} + 2\rho(Sp_{2n}(\mathbb{R})) \prec \Re(v) - (\frac{\mathbf{p}+\mathbf{q}}{\mathbf{2}} - \mathbf{n} - \mathbf{1}) + \rho(Sp_{2n}(\mathbb{R})),$$

by the definition of semistable range, $R_{ss} \subseteq R_s$. Thus θ_s in the strongly semistable range is the same as the original θ.

Theorem 3.4 ([Heu]) *Suppose*

- $p+q \leq 2n+1$;
- $\pi \in \mathcal{R}_{ss}(MO(p,q),\omega)$;
- π *is unitary;*

Then $(,)_\pi$ is positive semidefinite. If $\theta_s(p,q;2n)(\pi) \neq 0$, then $\theta_s(p,q;2n)(\pi)$ is an irreducible unitary representation of $MSp_{2n}(\mathbb{R})$.

Theorem 3.5 ([Heu]) *Suppose*

- $n < p \leq q$
- $\pi \in \mathcal{R}_{ss}(MSp_{2n}(\mathbb{R}),\omega)$;
- π *is unitary;*

Then $(,)_\pi$ is positive semidefinite. If $\theta_s(2n;p,q)(\pi) \neq 0$, then $\theta_s(2n;p,q)(\pi)$ is an irreducible unitary representation of $MO(p,q)$.

Strictly speaking, $\theta_s(*)(\pi)$ is an irreducible unitarizable Harish-Chandra module. Notice that $(,)_\pi$ can be regarded as an inner product on $\theta_s(*)(\pi)$. We can simply complete $(,)_\pi$ to obtain the Hilbert space of $\theta_s(*)(\pi)$. In general, invariant inner product on an irreducible Harish-Chandra module is unique up to the multiplication of a constant.

Theorem 3.4 and Theorem 3.5 provide us the basis to construct unitary representation through the theta correspondence in the strongly semistable range.

3.3 Quantum Induction $\mathcal{Q}$

Let $\mathcal{E}(p'+q, q'+p; 2n)$ be the representation of $O(p'+q, q'+p)$ studied by Zhu-Huang in [ZH]. It is essentially $\theta(2n; p'+q, q'+p)(\text{trivial})$ restricted to the $O(p'+q, q'+p)$ component in $MO(p'+q, q'+p)$. Notice here that $MO(p'+q, q'+p)$ splits into $O(p'+q, q'+p)$ and $O(p'+q, q'+p)\epsilon$.

Definition 19 *Suppose*

$$p'+q \geq 2n+1, \qquad p+q' \geq 2n+1, \qquad p+q \equiv p'+q' \pmod 2.$$

Let $\pi \in \Pi(O(p,q))$ be such that

$$(u_1 \otimes v_1, u_2 \otimes v_2) = \int_{O(p,q)} (\mathcal{E}(p'+q, q'+p; 2n)(g)u_1, u_2)(\pi(g)v_1, v_2) dg$$

converges absolutely for every $u_1, u_2 \in V(\mathcal{E}(p'+q, q'+p; 2n))$ and every $v_1, v_2 \in V(\pi)$. Let $\mathcal{R}$ be the radical of $(,)$ as a Hermitian form on $V(\mathcal{E}(p'+q, q'+p; 2n)) \otimes V(\pi)$. Then

$$(V(\mathcal{E}(p'+q, q'+p; 2n)) \otimes V(\pi))/\mathcal{R}$$

inherits an infinitesimal $O(p', q')$-action. Define

$$\mathcal{Q}(p, q; 2n; p', q')(\pi) = V(\mathcal{E}(p'+q, q'+p; 2n)) \otimes V(\pi)/\mathcal{R}.$$

$\mathcal{Q}(p, q; 2n; p', q')(\pi)$ is a $(\mathfrak{g}, K)$-module of $O(p', q')$. The representation $\mathcal{E}(p'+q, q'+p; 2n)$ is denoted by π_n in [ZH].

Conjecture 1 *If $\mathcal{Q}(p, q; 2n; p', q')(\pi) \neq 0$, then $\mathcal{Q}(p, q; 2n; p', q')(\pi)$ is an irreducible admissible representation of $O(p', q')$. If π is unitary, then $\mathcal{Q}(p, q; 2n; p', q')(\pi)$ is also unitary.*

For $p+q \leq n+n'+1$, put $\mathcal{E}(2n+2n'; p, q) = \theta(p, q; 2n+2n')(\text{trivial})$. If $p+q \leq n+n'$, $\mathcal{E}(2n+2n'; p, q)$ is unitary according to Howe-Li's theory on stable range dual pairs ([Ho84], [LI89]). If $p+q = n+n'+1$, $\mathcal{E}(2n+2n'; p, q)$ is unitary according to Przebinda's results on almost stable range dual pairs (see Lemma 8.6 [PR93]). $\mathcal{E}(2n+2n'; p, q)$ is a genuine representation of $Mp_{2n+2n'}(\mathbb{R})$ if $p+q$ is odd.

Definition 20 *Suppose $n+n'+1 \geq p+q$. Let π be an irreducible representation of $Mp_{2n}(\mathbb{R})$ such that the following Hermitian form $(,)$ on $V(\mathcal{E}(2n+2n'; p, q)) \otimes V(\pi)$ converges:*

$$(\varphi \otimes u, \varsigma \otimes v) = \int_{MSp_{2n}(\mathbb{R})} (\mathcal{E}(2n+2n'; p, q)(g)\varphi, \varsigma)(\pi(g)u, v) dg$$

$$\forall \ \ \varphi, \varsigma \in V(\mathcal{E}(2n+2n'; p, q)); u, v \in V(\pi).$$

Define $\mathcal{Q}(2n; p, q; 2n')(\pi)$ to be $V(\mathcal{E}(2n; p, q; 2n')) \otimes V(\pi)$ modulo the radical of $(,)$. $\mathcal{Q}(2n; p, q; 2n')(\pi)$ inherits an infinitesimal $MSp_{2n'}(\mathbb{R})$-action from $\mathcal{E}(2n+2n'; p, q)$. It is a $(\mathfrak{g}, K)$-module of $MSp_{2n'}(\mathbb{R})$.

Conjecture 2 *If $\mathcal{Q}(2n; p, q; 2n')(\pi) \neq 0$, then $\mathcal{Q}(2n; p, q; 2n')(\pi)$ is an irreducible admissible representation of $Mp_{2n'}(\mathbb{R})$. If π is unitary, then $\mathcal{Q}(2n; p, q; 2n')(\pi)$ is also unitary.*

We call $\mathcal{Q}(*)$ quantum induction.

3.4 Unitary Quantum Induction Q

In [Heq], we study the leading exponents of $\theta_s(*)(\pi)$. Under certain assumptions, one can compose θ_s with another θ_s. Surprisingly, the strongly semistable range plays a crucial role in the composability of θ_s. By identifying $\mathcal{Q}$ with a certain composition of θ_s, we established the unitarity and irreducibility of $\mathcal{Q}(\pi)$ for π in the strongly semistable range. Unitary quantum induction then enables us to construct certain irreducible unitary representations whose existence has been conjectured by Arthur and Barbasch-Vogan.

Theorem 3.6 ([Heq]) *Let π be an irreducible unitary representation in*

$$\mathcal{R}_{ss}(MO(p, q), \omega(p, q; 2n)).$$

Suppose

- $q' \geq p' > n$;
- $p' + q' - 2n \geq 2n - (p+q) + 2 \geq 1$;
- $p + q = p' + q' \pmod 2$.

Then

1. $\theta_s(p, q; 2n)(\pi) \in \mathcal{R}_{ss}(MSp_{2n}(\mathbb{R}), \omega(p', q'; 2n))$;
2. $\mathcal{Q}(p, q; 2n; p', q')(\pi) \cong \theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi)$;
3. *If $\mathcal{Q}(p, q; 2n; p', q')(\pi) \neq 0$, then $\mathcal{Q}(p, q; 2n; p', q')(\pi)$ is unitary.*

Let me say a few words about the proof of this theorem. (1) is proved in [Heq] through an estimate on the matrix coefficients of $\theta_s(p, q; 2n)(\pi)$. By (1), $\theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi)$ is well-defined. In fact, by Definition 17, the Harish-Chandra module of $\theta_s(p, q; 2n)(\pi)$ consists of "distributions" of the following form

$$\int_{MO(p,q)} \omega(p, q; 2n)(g_1)\phi \otimes \pi^c(g_1) u dg_1 \qquad (\phi \in V(\omega(p, q; 2n)), u \in V(\pi)).$$

For the same reason, the Harish-Chandra module of $\theta_s(2n;p',q')\theta_s(p,q;2n)(\pi)$ consists of "distributions" of the following form

$$\int_{MSp_{2n}(\mathbb{R})} \omega(p',q';2n)(g_2)\psi\otimes\theta_s(p,q;2n)(\pi)^c(g_2)v dg_2 \qquad (\psi\in V(\omega(p',q';2n)), v\in V(\theta_s(p,q;2n)(\pi))).$$

Combining these two statements, $V(\theta_s(2n;p',q')\theta_s(p,q;2n)(\pi))$ consists of "distributions" of the form

$$\int_{g_2\in MSp_{2n}(\mathbb{R})}\int_{g_1\in MO(p,q)} \omega(p',q';2n)(g_2)\psi\otimes\omega(p,q;2n)^c(g_2g_1)\phi\otimes\pi(g_1)u dg_1 dg_2$$

$$(\psi\in V(\omega(p',q';2n)), \phi\in V(\omega(p,q;2n)), u\in V(\pi)).$$

Notice that $\omega(p,q;2n)^c\cong\omega(q,p;2n)$ and

$$\omega(p',q';2n)\otimes\omega(q,p;2n)\cong\omega(p'+q,q'+p;2n).$$

Furthermore, by the theorems of Howe-Li,

$$\{\int_{g_2\in MSp_{2n}(\mathbb{R})} \omega(p'+q,q'+p;2n)(g_2)(\psi\otimes\phi)dg_2 \mid \psi\in V(\omega(p',q';2n)), \phi\in V(\omega(q,p;2n))\}$$

can be identified with the Harish-Chandra module of $\mathcal{E}(p'+q,q'+p;2n)$. Therefore,

$$V(\theta_s(2n;p',q')\theta_s(p,q;2n)(\pi))$$

is equivalent to

$$\{\int_{MO(p,q)} \mathcal{E}(p'+q,q'+p;2n)(g_1)\eta\otimes\pi(g_1)u dg_1 \mid \eta\in V(\mathcal{E}(p'+q,q'+p;2n)), u\in V(\pi)\}.$$

We have assumed that $\pi(\epsilon)=-1$ from the beginning. The integral over $MO(p,q)$ is just twice the integral over $O(p,q)$. It follows that $V(\theta_s(2n;p',q')\theta_s(p,q;2n)(\pi))$ is equivalent to

$$\{\int_{O(p,q)} \mathcal{E}(p'+q,q'+p;2n)(g_1)\eta\otimes\pi(g_1)u dg_1 \mid \eta\in V(\mathcal{E}(p'+q,q'+p;2n)), u\in V(\pi)\}$$

which is exactly $\mathcal{Q}(p,q;2n;p',q')(\pi)$. The absolute convergences proved in [Heq] allow us to change the order of the integrals. (2) is proved. (3) follows easily by Theorems 3.4 and 3.5.

Definition 21 *Under the hypotheses in Theorem 3.6, define*

$$Q(p,q;2n;p',q')(\pi)=\theta_s(2n;p',q')\theta_s(p,q;2n)(\pi).$$

We call Q unitary quantum induction. By Theorem 3.6, $Q(p,q;2n;p',q')(\pi)$ is well-defined and equivalent to $\mathcal{Q}(p,q;2n;p',q')(\pi)$. A similar statement holds for $\mathcal{Q}(2n;p,q;2n')(\pi)$.

Theorem 3.7 ([Heq]) *Let π be a unitary representation in $\mathcal{R}_{ss}(MSp_{2n}(\mathbb{R}),\omega(p,q;2n))$. Suppose*

- $2n' - p - q \geq p + q - 2n - 2$;
- $n < p \leq q$.

Then

1. $\theta_s(2n; p, q)(\pi) \in \mathcal{R}_{ss}(MO(p, q), \omega(p, q; 2n'))$;
2. $\mathcal{Q}(2n; p, q; 2n')(\pi) \cong \theta_s(p, q; 2n')\theta_s(2n; p, q)(\pi)$;
3. *if* $\mathcal{Q}(2n; p, q; 2n')(\pi) \neq 0$, *then* $\mathcal{Q}(2n; p, q; 2n')(\pi)$ *is unitary.*

Definition 22 *Under the hypotheses of Theorem 3.7, define*

$$Q(2n; p, q; 2n')(\pi) = \theta_s(p, q; 2n')\theta_s(2n; p, q)(\pi).$$

By Theorem 3.7, $Q(2n; p, q; 2n')(\pi)$ is well-defined and equivalent to $\mathcal{Q}(2n; p, q; 2n')(\pi)$.

In summary, $\mathcal{Q}$ is a generalization of $Q = \theta_s \circ \theta_s$. The advantage of introducing $\mathcal{Q}$ will be manifested when we begin to relate unitary quantum induction to parabolic induction. Based on the studies on parabolic induction, we will establish the nonvanishing of $\theta_s(2n; p, q)(\pi)$ under some restrictions. The nonvanishing of $\theta_s(2n; p, q)(\pi)$ is otherwise very hard to establish if p or q is less than $2n$.

3.5 Moment Map and $\Theta(G_1, G_2)$

From now on, let $G_1 = O(p, q)$ and $G_2 = Sp_{2n}(\mathbb{R})$. Write

$$G_{1\mathbb{C}} = O(p + q, \mathbb{C}), \qquad G_{2\mathbb{C}} = Sp_{2n}(\mathbb{C}).$$

Let $Mat(p + q; 2n)$ be the set of $p + q$ by $2n$ real matrices. Let

$$W = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Definition 23 *Define the moment maps*

$$m_1 : x \in Mat(p + q, 2n) \to I_{p,q} x W_n x^t \in \mathfrak{o}(p, q),$$

$$m_2 : x \in Mat(p + q, 2n) \to W_n x^t I_{p,q} x \in \mathfrak{sp}_{2n}(\mathbb{R}).$$

We say that x is nilpotent if $m_1(x)$ is nilpotent.

Observe that

Lemma 3.1 *$m_1(x)$ is nilpotent if and only if $m_2(x)$ is nilpotent.*

Thus we obtain a set of nilpotent $Sp_{2n}(\mathbb{R}) \times O(p,q)$-orbits in $Mat(p+q, 2n)$. Clearly, m_i induces a map from the nilpotent orbits in $Mat(p+q, 2n)$ to nilpotent orbits in $\mathfrak{g}_i$.

Similarly, we define the complex moment maps, still denoted by m_i. This should not cause any notational problem. For example, the usage of $\mathbf{D}$ usually points to the complex moment map and the usage of $\mathbf{d}$ usually points to the real moment map.

For a complex nilpotent orbit $\mathcal{O}_1$ in $\mathfrak{g}_\mathbf{1}$, consider the closure of

$$m_2(m_1^{-1}(\text{Closure of } \mathcal{O}_1)).$$

By a Theorem of Daszkiewicz-Kraskiewicz-Przebinda, the closure of

$$m_2(m_1^{-1}(\text{Closure of } \mathcal{O}_1))$$

is the closure of a unique nilpotent orbit in $\mathfrak{g}_\mathbf{2}$ (see [DKP]). This yields a correspondence between certain nilpotent orbits in $\mathfrak{g}_1$ and certain nilpotent orbits in $\mathfrak{g}_2$. We denote this correspondence by $\Theta(G_{1\mathbb{C}}, G_{2\mathbb{C}})$. Similarly, we define $\Theta(G_{2\mathbb{C}}, G_{1\mathbb{C}})$. The reader must be warned that $\Theta(G_{1\mathbb{C}}, G_{2\mathbb{C}})$ is NOT the inverse of $\Theta(G_{2\mathbb{C}}, G_{1\mathbb{C}})$. For the nilpotent orbits we are concerned with, $\Theta(G_{1\mathbb{C}}, G_{2\mathbb{C}})$ and $\Theta(G_{2\mathbb{C}}, G_{1\mathbb{C}})$ are quite easy to describe ([DKP]).

Lemma 3.2 ([DKP]) *Let $\mathcal{O}_\mathbf{D} \in \mathcal{U}$ be a nilpotent orbit of either $O(p,q)$ or $Sp_{2n}(\mathbb{R})$. Construct an alternating sequence of complex orthogonal orbits and complex symplectic orbits*

$$\mathcal{O}_\mathbf{d}, \mathcal{O}_\mathbf{d-1}, \mathcal{O}_\mathbf{d-2}, \ldots \mathcal{O}_{\mathbf{d}-d_1+1}$$

and a corresponding alternating sequence of complex orthogonal groups and symplectic groups:

$$G(\mathcal{O}_\mathbf{d}) = G(d_1)_\mathbb{C}, G(d_1-1)_\mathbb{C}, \ldots, G(1)_\mathbb{C}.$$

Then $\forall$ j,

$$\Theta(G(d_1-j)_\mathbb{C}, G(d_1-j+1)_\mathbb{C})(cl(\mathcal{O}_{\mathbf{d_1-j}})) = cl(\mathcal{O}_{\mathbf{d_1-j+1}}).$$

We define $\Theta(G_2, G_1)$ and $\Theta(G_1, G_2)$ for real nilpotent orbits in the same fashion. Pan showed that $\Theta(G_2, G_1)(\mathcal{O}_\mathbf{D})$ is the closure of at most two real nilpotent orbits (Theorem 8.11 [PAN]). We give two lemmas that can be easily deduced from the descriptions of $\Theta(G_2, G_1)$ and $\Theta(G_1, G_2)$ in [PAN].

Lemma 3.3 *Suppose $p+q \leq 2n$. Then the real nilpotent orbit $\mathcal{O}_\mathbf{D}$ occurs in the image of m_2 if and only if $G(\mathcal{O}_{\mathbf{D}-1}) = O(r,s)$ with $r \leq p$ and $s \leq q$.*

Lemma 3.4 *Let $\mathcal{O}_\mathbf{D} \in \mathcal{U}$ be a real nilpotent orbit of either $O(p,q)$ or $Sp_{2n}(\mathbb{R})$. Construct an alternating sequence of real nilpotent orbits of symplectic groups and orthogonal groups*

$$\mathcal{O}_\mathbf{D}, \mathcal{O}_\mathbf{D-1}, \mathcal{O}_\mathbf{D-2}, \ldots \mathcal{O}_\mathbf{D-d_1+1}$$

and a corresponding sequence of real groups

$$G(d_1), G(d_1-1), \ldots, G(1).$$

Then $\forall$ j,

$$\Theta(G(d_1-j), G(d_1-j+1))(cl(\mathcal{O}_\mathbf{D-j})) = cl(\mathcal{O}_\mathbf{D-j+1}).$$

3.6 Przebinda's Results on $\mathcal{V}(Ann\ \theta(\pi))$

In [PR93], Przebinda proved that under some conditions, the following diagram commutes:

$$\begin{array}{ccc} \pi & \xrightarrow{\theta} & \theta(MG, MG')(\pi) \\ \Big\downarrow{\scriptstyle Ass} & & \Big\downarrow{\scriptstyle Ass} \\ \mathcal{V}(Ann\ \pi) & \xrightarrow{\Theta(G,G')} & \mathcal{V}(Ann\ \theta(MG, MG')(\pi)) \end{array} \tag{3.2}$$

We state Przebinda's theorem for $O(p,q)$ and $Mp_{2n}(\mathbb{R})$. Recall from Theorem III.1 [MI] that π has the rate of growth γ if and only if for every leading exponent v of π,

$$\Re(v) \preceq (\gamma - 1)\rho(G).$$

Theorem 3.8 ([PR93]) *Let π be an irreducible unitary representation of $O(p,q)$. Suppose*

1. $2n + 1 \geq p + q$;
2. $\Theta(\pi)$ *has rate of growth* γ *with*

$$\gamma + 1 < \frac{2n}{p + q - 2}$$

[Equivalently, every leading exponent v of π satisfies

$$\Re(v) \prec (\frac{2n}{p + q - 2} - 2)\rho(O(p,q))$$

(see Theorem III.1 of Miličić, Theorem 8.48 of [KN] and Theorem 4.5 of [PR93])];

3. $(,)_\pi$ *converges and does not vanish;*
4. $(,)_\pi$ *is positive semidefinite;*
5. *there exists a full rank element $x \in Mat(p+q, 2n)$ such that the $O(p,q)$-orbit generated by $m_1(x)$ is of maximal dimension in $AS(\Theta(\pi))$.*

Then $\theta(p,q;2n)(\pi)$ is unitary and its associated variety is the complex orbit

$$\Theta(O(p+q,\mathbb{C}), Sp_{2n}(\mathbb{C}))(\mathcal{V}(Ann\ \pi)).$$

Let me make two remarks here. Firstly, the assumption (1) does not appear in Przebinda's original theorem. Notice that for a unitary representation π, $\gamma \geq 0$. Thus (2) implies $2n > p + q - 2$. We add (1) as an assumption for the sake of clarity. The reader should also notice that condition (1) guarantees the nonvanishing of $(,)_\pi$. Secondly, condition (5) is satisfied in the setting of Lemma 3.4. More precisely, let $\mathcal{O}_{\mathbf{D}}$ be a nilpotent orbit of $Mp_{2n}(\mathbb{R})$ such that $G(\mathcal{O}_{\mathbf{D}-1}) = O(p,q)$. Then there exists a full rank element $x \in Mat(p+q, 2n)$ such that the $O(p,q)$-orbit generated by $m_1(x)$ is $\mathcal{O}_{\mathbf{D}-1}$. The argument goes as follows. According to Lemma

3.3, $\mathcal{O}_{\mathbf{D}}$ occurs in the image of m_2. According to Lemma 3.4, there exists x in $Mat(p+q,2n)$ such that $m_2(x)\in\mathcal{O}_{\mathbf{D}}$ and $m_1(x)\in cl(\mathcal{O}_{\mathbf{D}-1})$. Then

$$rank(W_n x^t I_{p,q} x) = rank(m_2(x)) = p+q.$$

But $x\in Mat(p+q,2n)$. So x must be of full rank. For x of full rank, observe that

$$[m_2(x)]^r = W_n x^t [m_1(x)]^{r-1} I_{p,q} x.$$

By writing out the defining equations for $m_2(x)\in\mathcal{O}_{\mathbf{D}}$, we obtain $m_1(x)\in\mathcal{O}_{\mathbf{d-1}}$. But $m_1(x)\in cl(\mathcal{O}_{\mathbf{D}-1})$. It follows that $m_1(x)\in\mathcal{O}_{\mathbf{D-1}}$.

Lemma 3.5 *If $\mathcal{O}_{\mathbf{D}-1}$ is of maximal dimension in $AS(\Theta(\pi))$ and*

$$O(p,q)=G(\mathcal{O}_{\mathbf{D}-1}), \qquad Mp_{2n}(\mathbb{R})=G(\mathcal{O}_{\mathbf{D}}),$$

then (5) *in Theorem 3.8 holds.*

Similary, Przebinda's theorem for $Mp_{2n}(\mathbb{R})$ can be formulated as follows.

Theorem 3.9 ([PR93]) *Let π be an irreducible unitary representation of $Mp_{2n}(\mathbb{R})$. Suppose*

1. *$p+q>2n$;*
2. *$\Theta(\pi)$ has the rate of growth γ with*
$$\gamma+1<\frac{p+q}{2n};$$
Equivalently, every leading exponent v of π satisfies
$$\Re(v)\prec(\frac{p+q}{2n}-2)\rho(Mp_{2n}(\mathbb{R}));$$
3. *$(,)_\pi$ converges and does not vanish;*
4. *$(,)_\pi$ is positive semidefinite;*
5. *there exists a full rank element $x\in Mat(p+q,2n)$ such that the $Sp_{2n}(\mathbb{R})$-orbit generated from $m_2(x)$ is of maximal dimension in $AS(\Theta(\pi))$.*

Then $\theta(2n;p,q)(\pi)$ is unitary and its associated variety is the complex nilpotent orbit

$$\Theta(Sp_{2n}(\mathbb{C}),O(p+q,\mathbb{C}))(\mathcal{V}(Ann\ \pi)).$$

Let $\mathcal{O}_{\mathbf{D}}$ be a nilpotent orbit of $O(p,q)$ such that $G(\mathcal{O}_{\mathbf{D}-1})=Mp_{2n}(\mathbb{R})$. Then according to Lemma 3.4 and the definition of $\Theta(Sp_{2n}(\mathbb{R}),O(p,q))$, $\mathcal{O}_{\mathbf{D}}$ occurs in the image of m_1 and $m_1m_2^{-1}(cl(\mathcal{O}_{\mathbf{D}-1}))=cl(\mathcal{O}_{\mathbf{D}})$. Let x be an element in $Mat(p+q,2n)$ such that $m_1(x)\in\mathcal{O}_{\mathbf{D}}$ and $m_2(x)\in cl(\mathcal{O}_{\mathbf{D}-1})$. Then x is necessarily of full rank since $rank(I_{p,q}xW_nx^t)=2n$. By writing out the defining equations for $cl(\mathcal{O}_{\mathbf{D}})$, one can easily show that $m_2(x)\in\mathcal{O}_{\mathbf{D}-1}$.

Lemma 3.6 *If $\mathcal{O}_{\mathbf{D}-1}$ is of maximal dimension in $AS(\Theta(\pi))$ and*

$$O(p,q) = G(\mathcal{O}_{\mathbf{D}}), \qquad Mp_{2n}(\mathbb{R}) = G(\mathcal{O}_{\mathbf{D-1}}),$$

then (5) *in Theorem 3.9 holds.*

Last let us recall the following theorem of Przebinda.

Theorem 3.10 *(Corollary 2.8, [PR93]) Consider the dual pair $(O(p,q), Sp_{2n}(\mathbb{R}))$.*

- *If $\pi \in \mathcal{R}(MO(p,q), \omega)$, then $WF(\pi)$ is a subset of $m_1(Mat(p+q; 2n))$.*
- *If $\pi \in \mathcal{R}(MSp_{2n}(\mathbb{R}), \omega)$, then $WF(\pi)$ is a subset of $m_2(Mat(p+q; 2n))$.*

Chapter 4

A Nonvanishing Theorem

In [He01], we established a nonvanishing theorem for $\theta_s(p,q;2n)$ with $p+q \leq 2n+1$. This theorem is cited as Theorem 3.3 in the last Chapter. In this chapter, we will prove a nonvanishing theorem for $\theta_s(2n;p,q)$. These two nonvanishing theorems will then be used to construct a packet of irreducible unitary representations $\mathcal{N}(\mathcal{O})$ attached to a real special rigid orbit $\mathcal{O}$ and more generally any $\mathcal{O} \in \mathcal{U}$ (see Definition 13). We start with a trivial lemma.

Lemma 4.1 *Suppose that*

$$\theta_s(p,q;2n_2)\theta_s(2n_1;p,q)(\pi) = \mathcal{Q}(2n_1;p,q;2n_2)(\pi).$$

If $\mathcal{Q}(2n_1;p,q;2n_2)(\pi) \neq 0$*, then* $\theta_s(2n_1;p,q)(\pi) \neq 0$.

Combining with Theorem 3.7, we obtain

Theorem 4.1 *Let* π *be a unitary representation in* $\mathcal{R}_{ss}(MSp_{2n_1}(\mathbb{R}), \omega(p,q;2n_1))$*. Suppose*

1. $2n_2 - p - q \geq p + q - 2n_1 - 2$;
2. $n_1 < p \leq q$.

If $\mathcal{Q}(2n_1;p,q;2n_2)(\pi) \neq 0$*, then* $\theta_s(2n_1;p,q)(\pi) \neq 0$.

At first glance, $\mathcal{Q}(2n_1;p,q;2n_2)(\pi)$ seems to be more difficult to treat than $\theta_s(2n_1;p,q)$. This is true with one exception. As predicted in [Heq], quantum induction should produce some parabolic induced module when (1) is an equality. In this chapter, we will examine $\mathcal{Q}(2n_1;p,q;2n_2)(\pi)$ for $n_1+n_2+1 = p+q$. We will prove that precisely for $n_1+n_2+1 = p+q$ and $n_1 \leq n_2$, $\mathcal{Q}(2n_1;p,q;2n_2)(\pi)$ can be obtained from parabolically induced representations.

First of all, a theorem of Kudla and Rallis says that, for some α, the parabolic induced representation

$$I^\alpha = Ind_{MP_{n_1+n_2}}^{Mp_{2n_1+2n_2}(\mathbb{R})}\chi^\alpha$$

decomposes into a direct sum of irreducible representations, namely $\mathcal{E}(2n_1+2n_2;p,q)$ for some p and q. Here $P_{n_1+n_2}$ is the Siegel parabolic subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$. Regard I^α as a representation of $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$. For certain $\pi \in \Pi(Mp_{2n_1}(\mathbb{R}))$, I^α induces a representation of $Mp_{2n_2}(\mathbb{R})$, $I^{\alpha}(\pi)$ (see Definition 24). Using the theorem of Kudla and Rallis, we show that $I^{\alpha}(\pi)$ decomposes into a direct sum of $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ for some $p + q = n_1 + n_2 + 1$.

Next, we study the restriction of I^{α} to $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$. Notice that the Harish-Chandra module $V(I^{\alpha})$ consists of sections of a homogeneous line bundle over the Lagrangian Grassmannian of $\mathbb{R}^{2n_1+2n_2}$. By analyzing the action of $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$ on the Grassmanian, we prove that

$$I^{\alpha}(\pi) \cong Ind_{Mp_{2n_1}(\mathbb{R})MGL(2n_2-2n_1)N}^{Mp_{2n_2}(\mathbb{R})} \pi^{\tau} \otimes \chi^{\alpha}.$$

The details are given in 4.2 and 4.3.

Now, we know that $I^{\alpha}(\pi)$ is nonvanishing. The remaining question is which $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ is nonvanishing. This turns out to be a hard question. We don't have any answer in general. But for some π, we can detect the nonvanishing of $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ precisely, thanks to the notion of wave front sets. Notice that the decomposition of $I^{\alpha}(\pi)$ into $\mathcal{Q}(*)(\pi)$ induces a decomposition on the wave front level. On the one hand, we know that the wave front set of the parabolic induced representation is the induced orbit. As we have pointed out earlier, the induced orbit consists of a few irreducible components. Theorems 2.7 and 2.8 provide us the precise information about the wave front set of $I^{\alpha}(\pi)$ for the π we are concerned with. On the other hand, Theorem 3.10 of Przebinda gives some control on which nilpotent orbit occurs in the wave front set of $\theta(p, q; 2n_2)(\pi)$ ([PR93]). Lemma 3.3 of Pan gives the parametrization of these nilpotent orbits([PAN]). We have now two sets of nilpotent orbits, one from the induced orbit, the other from the moment map m_2 with respect to various $(O(p, q), Sp_{2n_2}(\mathbb{R}))$. By matching these two sets of wave front sets, we obtain a nonvanishing theorem for $\mathcal{Q}(2n_1; p, q; 2n_2)$ and for $\theta_s(2n_1; p, q)$. Some of the argument here will appear in Chapter 6. In this Chapter, we will prove a generic nonvanishing theorem.

Theorem 4.2 *Consider the group $Mp_{2n_1+2n_2}(\mathbb{R})$ with $n_1 \leq n_2$. Let π be a unitary representation in*

$$\mathcal{R}_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1 + n_2 + 1, 0; 2n_1)).$$

Let $\mathcal{O}_{\mathbf{D}}$ be a nilpotent orbit of maximal dimension in

$$Ind_{\mathfrak{sp}_{2n_1}(\mathbb{R}) \oplus \mathfrak{gl}(n_2-n_1,\mathbb{R})}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} WF(\pi).$$

If $\mathcal{O}_{\mathbf{D}}$ occurs only in the image of the moment map m_2 associated with $(O(p, q), Sp_{2n_2}(\mathbb{R}))$ for a finite set

$$S \subseteq \{(p, q) \mid p + q = n_1 + n_2 + 1\},$$

then there exists a pair $(p, q) \in S$ such that $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi) \neq 0$. For such a pair, $\theta_s(2n_1; p, q)(\pi) \neq 0$.

Notice

$$\mathcal{R}_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1 + n_2 + 1, 0; 2n_1)) = \mathcal{R}_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(p, q; 2n_1))$$

for every $p + q = n_1 + n_2 + 1$. In this chapter, we take θ and θ_s as a correspondence between representations of $Mp_{2n}(\mathbb{R})$ and representations of $O(p, q)$.

Throughout this Chapter, we assume $n_2 \geq n_1$.

4.1 Results of Kudla-Rallis and Lee-Zhu

Consider the Siegel parabolic subgroup P_n of $Sp_{2n}(\mathbb{R})$. Let $P_n = MAN$ be the Langlands decomposition with $MA = GL(n, \mathbb{R})$. Let $Mp_{2n}(\mathbb{R})$ be the metaplectic covering of $Sp_{2n}(\mathbb{R})$. The group $MGL(n, \mathbb{R})$ can be expressed as

$$\{(\xi, x) \mid x \in GL(n, \mathbb{R}), \xi^2 = \det x\}.$$

Put $\chi(\xi, x) = \frac{\xi}{|\xi|}$. Then χ is a character of $MGL(n, \mathbb{R})$ of order 4. We extend χ trivially on N. χ becomes a character of MP_n.

Theorem 4.3 (Kudla-Rallis, [KR]) *Suppose $p + q = n + 1$ and $\alpha \equiv p - q \pmod 4$. Let $I^\alpha = Ind_{MP_n}^{Mp_{2n}(\mathbb{R})} \chi^\alpha$ be the unitarily induced representation of $Mp_{2n}(\mathbb{R})$. Then $\theta(p, q; 2n)(\mathrm{trivial})$ is a subrepresentation of I^α.*

By studying the K-types in $\theta(p, q; 2n)(\mathrm{trivial})$ for various $p + q = n + 1$, one has

Theorem 4.4 ([KR], [LZ])

$$I^\alpha = \oplus_{p+q=n+1, \alpha \equiv p-q \pmod 4} \theta(p, q; 2n)(\mathrm{trivial}).$$

More precisely, for n odd,

$$I^0 = Ind_{P_n}^{Sp_{2n}(\mathbb{R})} \mathrm{trivial} = \oplus_{p+q=n+1, p \equiv q \pmod 4} \theta(p, q; 2n)(\mathrm{trivial}),$$

$$I^2 = Ind_{P_n}^{Sp_{2n}(\mathbb{R})} \mathrm{sgn}(\det) = \oplus_{p+q=n+1, p-q \equiv 2 \pmod 4} \theta(p, q; 2n)(\mathrm{trivial}).$$

For n even,

$$I^1 = \oplus_{p+q=n+1, 1 \equiv p-q \pmod 4} \theta(p, q; 2n)(\mathrm{trivial}),$$

$$I^3 = \oplus_{p+q=n+1, 3 \equiv p-q \pmod 4} \theta(p, q; 2n)(\mathrm{trivial}).$$

This decomposition theorem can be derived directly from the results in [KR]. It was explicitly stated in [LZ].

Notation 13 *From now on, let $P_{n_1+n_2}$ be the Siegel parabolic subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$. Let $GL(n_1 + n_2)N$ be the Langlands decomposition of $P_{n_1+n_2}$. Write*

$$MP_{n_1+n_2} = \{(\xi, gn) \mid g \in GL(n_1 + n_2), n \in N, \det g = \xi^2\}.$$

Let $\chi(\xi, gn) = \frac{\xi}{|\xi|}$. Write

$$I^\alpha = Ind_{MP_{n_1+n_2}}^{Mp_{2n_1+2n_2}(\mathbb{R})} \chi^\alpha.$$

Definition 24 *Let $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ be embedded diagonally into $Sp_{2n_1+2n_2}(\mathbb{R})$. The embedding $Sp_{2n_1}(\mathbb{R}) \to Sp_{2n_1+2n_2}(\mathbb{R})$ lifts to $Mp_{2n_1}(\mathbb{R}) \to Mp_{2n_1+2n_2}(\mathbb{R})$ (see for example [Heu]). Consider I^α. Let $\pi \in \Pi(Mp_{2n_1}(\mathbb{R}))$ such that the form $(,)$*

$$(u_1 \otimes \phi_1, u_2 \otimes \phi_2) = \int_{Mp_{2n_1}(\mathbb{R})} (\pi(g)u_1, u_2)(I^\alpha(g)\phi_1, \phi_2) dg$$

converges for every $u_1, u_2 \in V(\pi), \phi_1, \phi_2 \in V(I^\alpha)$. Define $I^\alpha(\pi)$ to be $V(\pi) \otimes V(I^\alpha)$ modulo the radical of $(,)$. $I^\alpha(\pi)$ is a $(\mathfrak{g}, K)$-module for $Mp_{2n_2}(\mathbb{R})$.

From the definition of quantum induction $\mathcal{Q}$ and Theorem 4.4, we obtain

Theorem 4.5 *Suppose $\alpha \equiv n_1 + n_2 + 1 \pmod 2$. Then as $(\mathfrak{g}, K)$-modules*

$$I^\alpha(\pi) = \oplus_{p+q=n_1+n_2+1, p-q \equiv \alpha \pmod 4} \mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$$

whenever one side is well-defined.

Let me remind the reader that $\theta(p, q; 2n)$(trivial) is denoted by $\mathcal{E}(2n; p, q)$ in the definition of $\mathcal{Q}$.

It is known, at least under the assumptions of Theorem 3.7, that $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ is a unitarized Harish-Chandra module of $Mp_{2n_2}(\mathbb{R})$. Theorem 4.5 then implies that $I^\alpha(\pi)$ is also a unitarized Harish-Chandra module in this case. Later, in 4.3, we will show directly that $I^\alpha(\pi)$ is a unitaried Harish-Chandra module of $Mp_{2n_2}(\mathbb{R})$ in a much more general setting.

4.2 Lagrangian Grassmannian X

We are interested in the $Mp_{2n_2}(\mathbb{R}) \times Mp_{2n_1}(\mathbb{R})$-action on

$$I^\alpha = Ind_{MP_{n_1+n_2}}^{Mp_{2n_1+2n_2}(\mathbb{R})} \chi^\alpha.$$

Recall that representation I^α consists of sections of a certain homogeneneous line bundle over

$$Sp_{2n_1+2n_2}(\mathbb{R})/P_{n_1+n_2},$$

the Lagrangian Grassmannian. Denote this Lagrangian Grassmannian by X. X parametrizes Lagrangian subspaces of a symplectic space of dimension $2n_1 + 2n_2$. We are thus led to the problem of analyzing the $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$-action on X.

The existence of an open dense $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$-orbit in X can be proved using algebraic group theory. But this is not sufficient for our purpose. Not only do we need to know the existence of an open dense orbit, but also the detailed structure of this orbit. For this reason, we give two elementary lemmas and one theorem in this section. We shall use them in the next section to identify the restriction of I^α to $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$.

Suppose from now on $n_2 > n_1 > 0$. The case $n_1 = n_2$ has been studied in [He0], [He99]. Let me begin by recalling the following result.

Theorem 4.6 ([He0]) *Let $L(4n)$ be the Lagrangian Grassmannian of $(\mathbb{R}^{4n}, \Omega)$. Then $Sp_{2n}(\mathbb{R}) \times Sp_{2n}(\mathbb{R})$ acts on $L(4n)$ with $2n+1$ orbits. Furthermore, there is an open dense orbit in $L(4n)$ which can be identified with $Sp_{2n}(\mathbb{R})$.*

4.2.1 Notations

Notation 14 *Let $(\mathbb{R}^{2n_2+2n_1}, \Omega)$ be a symplectic space. Let X be the Lagrangian Grassmannian of $(\mathbb{R}^{2n_2+2n_1}, \Omega)$. Fix a standard real basis*

$$\{e_1, e_2, \ldots, e_{n_2+n_1}, f_1, f_2, \ldots f_{n_2+n_1}\}$$

on $\mathbb{R}^{2n_1+2n_2}$ such that

$$\Omega(e_i, e_j) = 0, \qquad \Omega(f_i, f_j) = 0$$
$$\Omega(f_j, e_i) = \delta_i^j,$$

where δ_i^j is the Kronecker symbol. Write

$$\mathbb{R}^{2n_1} = span\{e_1, \ldots, e_{n_1}, f_1, \ldots, f_{n_1}\}$$

as $\mathbb{R}^{2n_1}$ and

$$\mathbb{R}^{2n_2} = span\{e_{n_1+1}, \ldots, e_{n_1+n_2}, f_{n_1+1}, \ldots, f_{n_1+n_2}\}.$$

Let $Sp_{2n_1}(\mathbb{R})$ be the subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$ fixing e_j, f_j for every $j > n_1$. Let $Sp_{2n_2}(\mathbb{R})$ be the subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$ fixing e_j, f_j for every $j \leq n_1$. The group $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ is then diagonally embedded into $Sp_{2n_1+2n_2}(\mathbb{R})$.

Let

$$V_1 = span\{e_1 - e_{n_2+1}, e_2 - e_{n_2+2}, \ldots e_{n_1} - e_{n_2+n_1}\}$$
$$V_2 = span\{f_{n_2+1} + f_1, f_{n_2+2} + f_2, \ldots, f_{n_1+n_2} + f_{n_1}\}$$
$$V_0 = span\{e_{n_1+1}, \ldots, e_{n_2}\}$$
$$V_0' = span\{f_{n_1+1}, \ldots f_{n_2}\}.$$

4.2.2 $Sp_{2n_2}(\mathbb{R})$-action and the Generic Orbit X_0

Notice that $V_1 \oplus V_2 \oplus V_0$ is a Lagrangian subspace in $\mathbb{R}^{2n_1+2n_2}$. Put

$$V = V_1 \oplus V_2 \oplus V_0.$$

Consider the action of $Sp_{2n_2}(\mathbb{R})$ on V. Let X_0 be the $Sp_{2n_2}(\mathbb{R})$-orbit of V. X_0 is a subset of X.

Lemma 4.2 *Let $P_{n_2-n_1}$ be the maximal parabolic subgroup in $Sp_{2n_2}(\mathbb{R})$ stabilizing V_0. Let $Sp_{2n_1}(\mathbb{R})GL(n_2-n_1)$ be the Levi subgroup stabilizing both V_0 and V_0'. Let $Sp_{2n_1}(\mathbb{R})GL(n_2-n_1)N$ be the Langlands decomposition of $P_{n_2-n_1}$. Then $(Sp_{2n_2}(\mathbb{R}))_V = GL(n_2-n_1)N$ and*

$$X_0 \cong Sp_{2n_2}(\mathbb{R})/GL(n_2-n_1)N.$$

Furthermore, X_0 is open in X.

Proof: Let $g \in Sp_{2n_2}(\mathbb{R})$. Then g fixes e_i, f_i for every $i \leq n$. Suppose $gV = V$.

1. Since g stabilizes V and $\mathbb{R}^{2n_2}$, g stabilizes $V_0 = V \cap \mathbb{R}^{2n_2}$. So $g \in P_{n_2-n_1}$.

2. Let $j \leq n_1$. Since $ge_j = e_j$ and $g\mathbb{R}^{2n_2} = \mathbb{R}^{2n_2}$,
$$g(\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) = \mathbb{R}e_j \oplus \mathbb{R}^{2n_2}.$$
It follows that
$$g[(\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) \cap V] = [(\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) \cap V].$$
Notice that
$$(\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) \cap V = (\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) \cap (V_1 \oplus V_2 \oplus V_0) = \mathbb{R}(e_j - e_{n_2+j}) \oplus V_0.$$
So
$$g(\mathbb{R}(e_j - e_{n_2+j}) \oplus V_0) = \mathbb{R}(e_j - e_{n_2+j}) \oplus V_0.$$
Since $ge_j = e_j$,
$$g(e_j - e_{n_2+j}) \in (e_j - e_{n_2+j}) + V_0.$$
Hence for every $j \leq n_1$, $ge_{n_2+j} \in e_{n_2+j} + V_0$.

3. Similarly, since $gf_j = f_j$ for $j \leq n_1$,
$$gf_{n_2+j} \in f_{n_2+j} + V_0, \qquad (j \leq n_1).$$

4. Thus, for every
$$v \in span\{e_{n_2+1}, \ldots, e_{n_2+n_1}, f_{n_2+1}, \ldots f_{n_2+n_1}\},$$
$g(v) = v \oplus V_0$. This shows that $g \in GL(n_2 - n_1)N$.

5. Conversely, if $g \in GL(n_2-n_1)N$, then g fixes e_{n_2+j}, f_{n_2+j} for every $j \leq n_1$. So g preserves $V_1 \oplus V_2$. In addition, $g \in P_{n_2-n_1}$. So g stabilizes V_0. It follows that $gV = V$.

Therefore, $(Sp_{2n_2}(\mathbb{R}))_V = GL(n_2 - n_1)N$ and $X_0 \cong Sp_{2n_2}(\mathbb{R})/GL(n_2 - n_1)N$. To show that X_0 is open in X, we compute the dimensions of X_0 and X:

$$\begin{aligned}
\dim X_0 =& \dim(Sp_{2n_1}(\mathbb{R})) + \dim(Sp_{2n_2}(\mathbb{R})/Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)N) \\
=& \dim(Sp_{2n_1}(\mathbb{R})) + \frac{1}{2}\dim(Sp_{2n_2}(\mathbb{R})/Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)) \\
=& 2n_1^2 + n_1 + \frac{2n_2^2 + n_2 - 2n_1^2 - n_1 - (n_2 - n_1)^2}{2} \\
=& \frac{4n_1^2 + 2n_1 + 2n_2^2 + n_2 - 2n_1^2 - n_1 - n_1^2 + 2n_1n_2 - n_2^2}{2} \\
=& \frac{n_1^2 + n_2^2 + 2n_1n_2 + (n_1 + n_2)}{2} \\
=& \frac{2(n_1 + n_2)^2 + (n_1 + n_2) - (n_1 + n_2)^2}{2} \\
=& \frac{1}{2}\dim(Sp_{2n_1+2n_2}(\mathbb{R})/GL(n_1 + n_2, \mathbb{R})) \\
=& \dim(Sp_{2n_1+2n_2}(\mathbb{R})/P_{n_1+n_2}) \\
=& \dim X.
\end{aligned} \tag{4.1}$$

$\square$.

For $n_2 = n_1$, $P_{n_2-n_1}$ will be trivial. Our Lemma says that X_0 can be identified with $Sp_{2n_2}(\mathbb{R})$. This case is already treated in [He0] (see Theorem 4.6).

4.2.3 $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_1}(\mathbb{R})$-Action

Next, consider the $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ action on X. Again, our symplectic space is $\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}$. The basis for $\mathbb{R}^{2n_1}$ is

$$\{e_1, e_2, \ldots e_{n_1}, f_1, f_2, \ldots, f_{n_1}\}.$$

Let

$$U = span\{e_{n_2+1}, e_{n_2+2}, \ldots, e_{n_2+n_1}, f_{n_2+1}, f_{n_2+2}, \ldots, f_{n_2+n_1}\}.$$

Identify $\mathbb{R}^{2n_1}$ with U by mapping

$$e_j \to e_{n_2+j}, f_j \to f_{n_2+j}, \qquad (j \in [1, n_1]).$$

For each $g \in Sp_{2n_1}(\mathbb{R})$, define $i(g) \in Sp_{2n_2}(\mathbb{R})$ such that

$$i(g)|_{V_0 \oplus V_0'} = identity; \qquad i(g)|_U = g.$$

In other words, $Sp_{2n_1}(\mathbb{R})$ is embeded into $Sp_{2n_2}(\mathbb{R})$ as the first factor of the Levi subgroup $Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)$. Recall for every $g \in Sp_{2n_1}(\mathbb{R})$, $\tau(g)$ is defined to be

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{n_1} \end{pmatrix} g \begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{n_1} \end{pmatrix}.$$

Lemma 4.3 *Let $g \in Sp_{2n_1}(\mathbb{R})$. Let $i(g) \in Sp_{2n_2}(\mathbb{R})$ such that $i(g)|_{V_0 \oplus V_0'} = 1$. Regarding $(g, \tau(i(g)))$ as an element in $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$, we have*

$$(g, \tau(i(g)))V = V.$$

Proof: Let $g \in Sp_{2n_1}(\mathbb{R})$. We borrow some ideas from [He0]. Define an isomorphism $\phi : \mathbb{R}^{2n_1} \to U$ by

$$\phi(e_j) = -e_{n_2+j} \qquad (j = 1, 2, \ldots, n_1)$$
$$\phi(f_j) = f_{n_2+j} \qquad (j = 1, 2, \ldots, n_1).$$

Then $V_1 \oplus V_2$ is the graph of ϕ and $\tau(i(g))\phi = \phi g$. Notice that

$$\begin{aligned}(g, \tau(i(g)))(V_1 \oplus V_2) =& \{(gu, \tau(i(g))\phi(u)) \mid u \in \mathbb{R}^{2n_1}\} \\ =& \{(u, \tau(i(g))\phi(g^{-1}u)) \mid u \in \mathbb{R}^{2n_1}\} \\ =& \{(u, \phi u) \mid u \in \mathbb{R}^{2n_1}\} \\ =& V_1 \oplus V_2.\end{aligned} \tag{4.2}$$

Thus $V_1 \oplus V_2$ is preserved by the actions of $(g, \tau(i(g)))$. Clearly, $(g, \tau(i(g)))$ also preserves vectors in V_0. It follows that $(g, \tau(i(g)))V = V$. $\square$

Combining with Lemma 4.2, we obtain

Theorem 4.7 *Let $P_{n_2-n_1}$ be the maximal parabolic subgroup of $Sp_{2n_2}(\mathbb{R})$ which preserves V_0. Let $Sp_{2n_1}(\mathbb{R})GL(n_2-n_1)N$ be the Langlands decomposition of $P_{n_2-n_1}$ as in Lemma 4.2. Then the isotropy group*

$$(Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R}))_V = \{(\tau(g), gh) \mid g \in Sp_{2n_1}(\mathbb{R}), h \in GL(n_2-n_1)N\}.$$

Moreover,

$$X_0 \cong (Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R}))/(Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R}))_V$$

and X_0 is open and dense in X.

Here $Sp_{2n_1}(\mathbb{R})$ and $Sp_{2n_2}(\mathbb{R})$ are of the standard matrix form. The $Sp_{2n_1}(\mathbb{R})$ in the Levi factor $Sp_{2n_1}(\mathbb{R})GL(n_2-n_1)$ of $Sp_{2n_2}(\mathbb{R})$ is identified with the standard $Sp_{2n_1}(\mathbb{R})$. We avoid using i again.

Proof: We only need to prove that X_0 is dense in X. This follows from an argument similar to that in [He0]. Briefly, for every Lagrangian subspace W, define

$$Ind(W) = (\dim(W \cap \mathbb{R}^{2n_1}), \dim(W \cap \mathbb{R}^{2n_2})).$$

Notice that $\dim(V \cap \mathbb{R}^{2n_1}) = 0$ and $\dim(V \cap \mathbb{R}^{2n_2}) = \dim(V_0) = n_2 - n_1$. In general, for every $W \in X$, we have

$$\dim(W \cap \mathbb{R}^{2n_1}) \geq 0, \qquad \dim(W \cap \mathbb{R}^{2n_2}) \geq n_2 - n_1.$$

We say a Lagrangian subspace W is generic if $Ind(W) = (0, n_2 - n_1)$. Observe that the group action $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ preserves Ind. Therefore, every element in X_0 is generic.

Conversely, every generic Lagrangian subspace W must be an element in X_0. Let W be generic. Since $W \cap \mathbb{R}^{2n_1} = \{0\}$, the projection

$$p_2 : \mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2} \to \mathbb{R}^{2n_2}$$

restricted to W is injective. So $\dim(p_2(W)) = n_1 + n_2$. Similarly, since $\dim(W \cap \mathbb{R}^{2n_2}) = n_2 - n_1 = \dim(W) - \dim(\mathbb{R}^{2n_1})$, the projection

$$p_1 : \mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2} \to \mathbb{R}^{2n_1}$$

restricted to W must be surjective. It can then be shown that W is uniquely determined by a symplectic map from $p_2(W)$ to $\mathbb{R}^{2n_1}$. The kernel of this symplectic map is exactly $W \cap \mathbb{R}^{2n_2}$. By basic linear group theory, the group $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ acts on the set of generic elements transitively. So the orbit X_0 consists of all generic Lagrangian subspaces. Therefore X_0 is dense in X. $\square$

4.3 Parabolic Induction and $I^\alpha(\pi)$

Recall that $I^\alpha(\pi)$ is induced from $\pi \in \Pi(Mp_{2n_1}(\mathbb{R}))$ as follows. If

$$(u_1 \otimes \phi_1, u_2 \otimes \phi_2)_1 = \int_{Mp_{2n_1}(\mathbb{R})} (\pi(g)u_1, u_2)(I^\alpha(g)\phi_1, \phi_2) dg$$

converges for every $u_1, u_2 \in V(\pi), \phi_1, \phi_2 \in V(I^\alpha)$, then $I^\alpha(\pi)$ is defined to be $V(\pi) \otimes V(I^\alpha)$ modulo the radical of $(,)_1$. $I^\alpha(\pi)$ is only a $(\mathfrak{g}, K)$-module for $Mp_{2n_2}(\mathbb{R})$. Let τ and π^τ be as in Definition 9.

Theorem 4.8 *Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$ be such that $I^0(\pi)$ is well-defined. Suppose $n_1 \leq n_2$ and $\pi(\epsilon)\chi^\alpha(\epsilon) = 1$. Let $P_{n_2-n_1} = Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1, \mathbb{R})N$ be as in Lemma 4.2. Realizing $MP_{n_2-n_1}$ as the quotient group*

$$Mp_{2n_1}(\mathbb{R}) \times MGL(n_2 - n_1)N/\{(1,1), (\epsilon, \epsilon)\}.$$

Then $I^\alpha(\pi)$ coincides with the Harish-Chandra module of $Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$. Furthermore, $Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$ is the completion of $I^\alpha(\pi)$ with respect to the inner product $(,)_1$. So as unitarized Harish-Chandra modules,

$$I^\alpha(\pi) \cong Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

Notice that here, we only assume $(,)_1$ is well-defined for $V(\pi) \otimes V(I^0)$. This will guarantee that $(,)_1$ is well-defined for $V(\pi) \otimes V(I^\alpha)$. Let us first analyze the restriction of I^α to $Mp_{2n_2}(\mathbb{R})$.

4.3.1 Parabolic Induction: the Compact Picture

Parabolic induction is a powerful tool in the study of representations of reductive Lie groups (see for example [KN] and [Wallach]). There are the compact picture and the noncompact picture. We will study the compact picture of I^α in various ways.

Notation 15 *Let $P_{n_2+n_1}$ be the Siegel parabolic subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$. Define a character on $P_{n_2+n_1} = GL(n_2 + n_1)N$ by*

$$\exp(-\rho_0(gn)) = (\det g)^{-\frac{n_1+n_2+1}{2}}.$$

Lift $\exp(-\rho_0)$ trivially to a character of $MP_{n_2+n_1}$. By abusing notation, denote this character by $\exp(-\rho_0)$.

The Hilbert space of I^α consists of L^2 sections of the linear bundle

$$Mp_{2n_1+2n_2}(\mathbb{R}) \times_{MGL(n_1+n_2)N} \mathbb{C}[\chi^\alpha \otimes \exp(-\rho_0)] \to X.$$

Equivalently,

Definition 25 *The Hilbert space of I^α consists of functions on $Mp_{2n_1+2n_2}(\mathbb{R})$ such that*

$$f(gh) = f(g)\chi(h)^\alpha \exp(-\rho_0(h)) \qquad (\forall\ h \in MP_{n_2+n_1}),$$

$$\int_K \|f(k)\|^2 dk < \infty,$$

where K is a maximal compact subgroup of $Mp_{2n_1+2n_2}(\mathbb{R})$. Notice that the restriction of f to K uniquely determines f. We call the restrictions of f to K the compact picture.

Restrict the L^2-sections to the open dense subset X_0. Fix the base point V. From Theorem 4.7, X_0 is a single $Sp_{2n_2}(\mathbb{R})$-orbit of the form

$$Sp_{2n_2}(\mathbb{R})/GL(n_2-n_1)N.$$

Hence f is uniquely determined by the function $\tilde{f} = f|_{Mp_{2n_2}(\mathbb{R})}$ up to a set of measure zero. Furthermore, $\tilde{f}$ satisfies

$$\tilde{f}(gh) = \tilde{f}(g)\chi(h)^{\alpha}\exp(-\rho_0(h)) \qquad (\forall\ h \in MGL(n_2-n_1)N).$$

Let us consider the action of $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$ on I^{α}. Keep in mind that $Mp_{2n_1}(\mathbb{R}) \cap Mp_{2n_2}(\mathbb{R}) = \{1, \epsilon\}$ and

$$I^{\alpha}(\epsilon, \epsilon) = Identity.$$

Lemma 4.4 *For $f \in I^{\alpha}$, $g_1 \in Mp_{2n_1}(\mathbb{R})$, $g_2 \in Mp_{2n_2}(\mathbb{R})$, (g_1, g_2) embedded in $Mp_{2n_1+2n_2}(\mathbb{R})$ diagonally,*

$$f((g_1,g_2)) = \tilde{f}(g_2\tau(g_1)^{-1}).$$

For every $h_1 \in Mp_{2n_1}(\mathbb{R})$,

$$(I^{\alpha}(h_1)\tilde{f})(g_2) = \tilde{f}(g_2\tau(h_1)).$$

Proof: By Theorem 4.7,

$$f((g_1,g_2)) = f((1, g_2\tau(g_1)^{-1})(g_1,\tau(g_1)) = \tilde{f}(g_2\tau(g_1)^{-1})\chi((g_1,\tau(g_1))^{\alpha}\exp(-\rho_0(g_1,\tau(g_1))) = \tilde{f}(g_2\tau(g_1)^{-1}).$$

As to the action of $h_1 \in Mp_{2n_1}(\mathbb{R})$,

$$(I^{\alpha}(h_1)\tilde{f})(g_2) = (I^{\alpha}(h_1)f)((1,g_2)) = f((h_1^{-1}, g_2)) = \tilde{f}(g_2\tau(h_1)).$$

□

Let $\tilde{f}$ with

$$\tilde{f}(gh) = \tilde{f}(g)\chi(h)^{\alpha}\exp(-\rho_0(h)) \qquad (\forall\ h \in MGL(n_2-n_1)N)$$

be an element in the Hilbert space of I^{α}.

Definition 26 *Let f be an element in the Hilbert space of I^{α}. Let $\tilde{f}$ be the restriction of f to $Mp_{2n_2}(\mathbb{R})$. We say that a vector f (or equivalently $\tilde{f}$) is compactly supported, if $\tilde{f}$ is compactly supported on $Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N$. Denote the set of compactly supported smooth vectors in I^{α} by $C_c^{\infty}(X_0, I^{\alpha})$.*

Theorem 4.9 *Let $\tilde{f}$ be a smooth and compactly supported vector in I^{α}. Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$ and $u \in V(\pi)$. Then*

$$\int_{Mp_{2n_1}(\mathbb{R})} I^{\alpha}(g_1)\tilde{f} \otimes \pi(g_1)u dg_1$$

is an element in the Hilbert space of $Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\pi^\tau \otimes \chi^\alpha$*. Furthermore, the (vector-valued) functions of the form*

$$\int_{Mp_{2n_1}(\mathbb{R})} I^\alpha(g_1)\tilde{f} \otimes \pi(g_1) u dg_1$$

are dense in the Hibert space of $Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\pi^\tau \otimes \chi^\alpha$*.*

Proof: Since $\tilde{f}$ is compactly supported,

$$\int_{Mp_{2n_1}(\mathbb{R})} I^\alpha(g_1)\tilde{f} \otimes \pi(g_1) u dg_1$$

is well-defined as a vector valued function on $Mp_{2n_2}(\mathbb{R})$. Notice that for $g_2 \in Mp_{2n_2}(\mathbb{R})$ and h_1 in the $Mp_{2n_1}(\mathbb{R})$ factor of $P_{n_2-n_1}$, the value of

$$\int_{Mp_{2n_1}(\mathbb{R})} I^\alpha(g_1)\tilde{f} \otimes \pi(g_1) u dg_1$$

at $g_2 h_1$ is equal to

$$\begin{aligned}
&\int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1)\tilde{f})(g_2 h_1)\pi(g_1) u dg_1 \\
=&\int_{Mp_{2n_1}(\mathbb{R})} \tilde{f}(g_2 h_1 \tau(g_1))\pi(g_1) u dg_1 \\
=&\int_{Mp_{2n_1}(\mathbb{R})} \tilde{f}(g_2 h_1 g_1)\pi(\tau(g_1)) u dg_1 \\
=&\int_{Mp_{2n_1}(\mathbb{R})} \tilde{f}(g_2 g_1)\pi(\tau(h_1^{-1} g_1)) u dg_1 \\
=&\int_{Mp_{2n_1}(\mathbb{R})} \tilde{f}(g_2 g_1)\pi(\tau(h_1)^{-1})\pi(\tau(g_1)) u dg_1 \\
=&\pi(\tau(h_1)^{-1})[\int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1)\tilde{f})(g_2)\pi(\tau(g_1)) u dg_1].
\end{aligned} \tag{4.3}$$

Since $\tilde{f}$ is smooth and compactly supported, the restriction of

$$g_2 \rightarrow \int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1)\tilde{f})(g_2)\pi(g_1) u dg_1$$

onto the maximal compact subgroup K must be continuous and bounded. Therefore

$$\int_{Mp_{2n_1}(\mathbb{R})} I^\alpha(g_1)\tilde{f} \otimes \pi(g_1) u dg_1$$

is an element in the Hilbert space of $Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\pi^\tau \otimes \chi^\alpha$. It follows easily that the functions of the form

$$\int_{Mp_{2n_1}(\mathbb{R})} I^\alpha(g_1)\tilde{f} \otimes \pi(g_1) u dg_1$$

are dense in $Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\pi^\tau \otimes \chi^\alpha$. $\square$

Notation 16 *Put*

$$\mathcal{L}(\tilde{f} \otimes u) = \int_{Mp_{2n_1}(\mathbb{R})} I^{\alpha}(g_1)\tilde{f} \otimes \pi(g_1) u dg_1.$$

if the integral is well-defined in a proper sense.

4.3.2 Parabolic Induction: Half Density Bundle Picture

Theorem 4.10 *Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$. For every $u, v \in V(\pi)$ and ϕ, ψ smooth and compactly supported in I^{α},*

$$2(u \otimes \phi, v \otimes \psi)_1 = \int_{Mp_{2n_1}(\mathbb{R})} (\pi(g_1)u, v)(I^{\alpha}(g_1)\phi, \psi) dg_1 = (\mathcal{L}(u \times \phi), \mathcal{L}(v \otimes \psi))_{Ind\, \pi^{\tau} \otimes \chi^{\alpha}}.$$

The main goal of this section is to prove this statement. Readers who are comfortable with this statement can move on to the next section.

Recall that the Hilbert inner product in I^{α} depends on the choice of the maximal compact group of $Mp_{2n_1+2n_2}(\mathbb{R})$. Within the process of restricting I^{α} from $Mp_{2n_1+2n_2}(\mathbb{R})$ to $Mp_{2n_2}(\mathbb{R})$, we lose the track of the Hilbert inner product. Somehow $(u \times \tilde{f}, u \times \tilde{f})_1$ becomes difficult to manipulate. Fortunately, there is a notion of half density bundle (see [GV]) that does not depend on the choice of the groups. In the half density model, the Hilbert norm of the unitary principle series takes an invariant form. It is abstract, independent of the group action, and easy to manipulate.

Let Y be a homogeneous space of Lie group G. Volume forms in Y can be regarded as sections of the determinant of the cotangent bundle (written as $\det T^*Y$ or $\wedge^{top} T^*Y$). The group action preserves the integration of the volume forms. However, we cannot use volume forms to form unitary representations directly. To overcome this obstacle, Graham and Vogan formulated the notion of half density bundle $\mathcal{D}^{\frac{1}{2}}$. Essentially, it is the real vector bundle $|\det T^*Y|^{\frac{1}{2}}$ with $\mathbb{R}^+$ as the transformation group. The manifold Y need not be oriented. With this in mind, one can complexify $\mathcal{D}^{\frac{1}{2}}$ and obtain $\mathcal{D}^{\frac{1}{2}}_{\mathbb{C}}$ for the purpose of taking complex sections. Notice that the transformation group of $\mathcal{D}^{\frac{1}{2}}_{\mathbb{C}}$ is still $\mathbb{R}^+$ and sections of $\mathcal{D}^{\frac{1}{2}}_{\mathbb{C}}$ are sections of $\mathcal{D}^{\frac{1}{2}}_{\mathbb{R}}$ tensored with $\mathbb{C}$. Let s_1, s_2 be two sections of $\mathcal{D}^{\frac{1}{2}}_{\mathbb{C}}$. s_1 and s_2 need not be continuous. Then $s_1 \overline{s_2}$ is a volume form. The Hilbert inner product is defined to be

$$(s_1, s_2) = \int s_1 \overline{s_2}.$$

Notice that the group G acts on the cotagent bundle. Therefore, G also acts on the determinant of T^*Y and on $\mathcal{D}^{\frac{1}{2}}_{\mathbb{C}}$. It is easy to see that the Hilbert inner product $(,)$ is preserved by the G action.

Let us now take a look at the degenerate principal series I^{α}. Let $\mathcal{D}^{\frac{1}{2}}_{\mathbb{C}}(X)$ be the complex half density bundle of X. Let $\chi^{\alpha}(X)$ be the homogeneous vector bundle defined by the unitary character χ^{α}. Let s_1, s_2 be two sections of $\mathcal{D}^{\frac{1}{2}}_{\mathbb{C}}(X) \otimes \chi^{\alpha}(X)$. χ^{α} being a unitary character of $MP_{n_1+n_2}$ implies that $s_1\overline{s}_2$ is a well-defined volume form on X and that the Hilbert inner product

$$(s_1, s_2) = \int_X s_1\overline{s_2}$$

is preserved by the action of $Mp_{2n_1+2n_2}(\mathbb{R})$. The Hilbert space I^α consists of square integrable $\mathbb{C}\chi^\alpha$-valued half densities.

By fixing a proper half density on X, every square integrable half density can be expressed as a function on X. This is how one identifies the half density picture with the compact picture. Namely, fix a half density $\mu^{\frac{1}{2}}$ on X such that μ is the $U(n_1+n_2)$-invariant measure on X. Then every sqaure integrable half density on X can be expressed as a square integrable function on X. We obtain the compact picture. More precisely, f in the compact picture corresponds to $f\mu^{\frac{1}{2}}$ in the half density model.

Now consider $C_c^\infty(X_0, I^\alpha)$. Identify X_0 with $Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N$. There is a principal fibration

$$Mp_{2n_1}(\mathbb{R})/\{1, \epsilon\} \rightarrow Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N \rightarrow Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1}.$$

The generalized flag variety $Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1}$ does not have a $Mp_{2n_2}(\mathbb{R})$-invariant measure. There is the standard K-invariant measure $d[k]$ with respect to a maximal compact subgroup K of $Mp_{2n_2}(\mathbb{R})$.

Proof of Theorem 4.10: Fix an invariant measure dg_1 on $Mp_{2n_1}(\mathbb{R})$. Fix the measure $d[k]dg_1$ on $Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N$. Fix the half density $[d[k]dg_1]^{\frac{1}{2}}$ and identify half densities on X_0 as functions on X_0. Lemma 4.4 remains valid under this interpretation. Now ϕ and ψ are $\mathbb{C}\chi^\alpha$-valued functions, and $\phi\overline{\psi}$ is a function on X_0. The inner product (ϕ, ψ) is the integration of $\phi\overline{\psi}$ with respect to $d[k]dg_1$. We have

$$
\begin{aligned}
&(u\otimes\phi, v\otimes\psi)_1\\
=&\int_{g_1\in Mp_{2n_1}(\mathbb{R})}(\pi(g_1)u,v)([I^{\alpha}(g_1)\phi](x),\psi(x))dg_1\\
=&\int_{g_1\in Mp_{2n_1}(\mathbb{R})}(\pi(g_1)u,v)\int_{x\in X_0}[I^{\alpha}(g_1)\phi](x)\overline{\psi}(x)d[k]dh_1dg_1\\
=&\frac{1}{2}\int_{g_1\in Mp_{2n_1}(\mathbb{R})}(\pi(g_1)u,v)\int_{[k]\in Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1}}\int_{h_1\in Mp_{2n_1}(\mathbb{R})}[I^{\alpha}(g_1)\phi](kh_1)\overline{\psi}(kh_1)\,dh_1\,d[k]\,dg_1\\
=&\frac{1}{2}\int_{g_1\in Mp_{2n_1}(\mathbb{R})}(\pi(g_1)u,v)\int_{[k]}\int_{h_1\in Mp_{2n_1}(\mathbb{R})}[I^{\alpha}(\tau(h_1))I^{\alpha}(g_1)\phi](k)\overline{[I^{\alpha}(\tau(h_1))\psi]}(k)\,dh_1\,d[k]\,dg_1\\
=&\frac{1}{2}\int_{[k]}\int_{h_1\in Mp_{2n_1}(\mathbb{R})}\int_{g_1\in Mp_{2n_1}(\mathbb{R})}(\pi(g_1)u,v)[I^{\alpha}(\tau(h_1)g_1)\phi](k)\overline{[I^{\alpha}(\tau(h_1))\psi]}(k)\,dg_1\,dh_1\,d[k]\\
=&\frac{1}{2}\int_{[k]}\int_{h_1\in Mp_{2n_1}(\mathbb{R})}\int_{g_1'\in Mp_{2n_1}(\mathbb{R})}(\pi(\tau(h_1)^{-1}g_1')u,v)[I^{\alpha}(g_1')\phi](k)\overline{I^{\alpha}(\tau(h_1))\psi}(k)\,dg_1'\,dh\,d[k]\\
=&\frac{1}{2}\int_{[k]}\int_{h_1\in Mp_{2n_1}(\mathbb{R})}\int_{g_1'\in Mp_{2n_1}(\mathbb{R})}(\pi(g_1')u,\pi(\tau(h_1))v)[I^{\alpha}(g_1')\phi](k)\overline{I^{\alpha}(\tau(h_1))\psi}(k)\,dg_1'\,dh\,d[k]\\
=&\frac{1}{2}(\mathcal{L}(u\otimes\phi),\mathcal{L}(v\times\psi))_{Ind\,\pi^{\tau}\otimes\chi^{\alpha}}
\end{aligned}
\tag{4.4}
$$

The integrals all converge absolutely. □.

Therefore, the form $(,)_1$, regarded as a Hilbert norm on $\mathcal{L}(V(\pi)\otimes C_c^{\infty}(X_0,I^{\alpha}))$, completes to the Hilbert norm of

$$Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\pi^{\tau}\otimes\chi^{\alpha}.$$

4.3.3 Proof of Theorem 4.8

Go back to the form $(,)_1$ restricted to $V(\pi)\otimes V(I^{\alpha})$. Since every element in $V(\pi)\otimes C_c^{\infty}(X_0,I^{\alpha})$ can be approximated by vectors in $V(\pi)\otimes V(I^{\alpha})$, it is reasonable to believe that $\mathcal{L}(V(\pi)\otimes V(I^{\alpha}))$ is dense in $Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\pi^{\tau}\otimes\chi^{\alpha}$. This may be a direct consequence of some theorem in functional analysis about the unbounded operator $\mathcal{L}$. Unfortunately, we are not aware of any such theorem. We are forced to go back and dig into the compact picture in which the Harish-Chandra module is visible.

Lemma 4.5 *Let K be a maximal compact subgroup of $Mp_{2n_1+2n_2}(\mathbb{R})$. Let 1_K be the spherical vector in I^0. Then $g\rightarrow(I^0(g)1_K,1_K)$ is positive and every K-finite matrix coefficient of I^{α} is bounded by a multiple of $(I^0(g)1_K,1_K)$.*

Proof: In the compact picture, 1_K is the constant function 1 on K. $I^0(g)1_K$ is also a postive function. Therefore $(I^0(g)1_K,1_K)\geq 0$. Furthermore, every K-finite function is bounded by a positive multiple of 1_K. Hence every K-finite matrix coefficient of I^{α} is bounded by a multiple of $(I^0(g)1_K,1_K)$. □

Lemma 4.6 *If $(u \otimes 1_K, v \otimes 1_K)_1$ converges absolutely for $u, v \in V(\pi)$ and 1_K in I^0, then $(,)_1$ is well-defined for $V(\pi) \otimes V(I^\alpha)$.*

Proof: This lemma follows directly from the previous lemma. □

Lemma 4.7 *Suppose that $(u \otimes 1_K, v \otimes 1_K)_1$ converges absolutely for $u, v \in V(\pi)$ and $1_K \in I^0$. Then $(u \otimes \phi, v \otimes \psi)_1$ converges absolutely for any $\phi \in V(I^\alpha)$, $\psi \in C_c^\infty(X_0, I^\alpha) \cup V(I^\alpha)$ and $u, v \in V(\pi)$. Furthermore $\mathcal{L}(V(\pi) \otimes V(I^\alpha))$ lies in the Hilbert space of*

$$Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

Proof: Notice that any $\phi \in V(I^\alpha), \psi \in V(I^\alpha) \cup C_c^\infty(X_0, I^\alpha)$ is bounded by a multiple of 1_K. Since

$$\int_{Mp_{2n_1}(\mathbb{R})} \int_K (\pi(g_1)u, v)[I^0(g_1)1_K](x)1_K(x)dg_1dk$$

converges absolutely, the integrals in the proof of Theorem 4.10 all converge absolutely for $\phi \in V(I^\alpha), \psi \in C_c^\infty(X_0, I^\alpha) \cup V(I^\alpha)$. By essentially the same argument, $\mathcal{L}(u \otimes \phi)$ is well-defined and

$$2(u \otimes \phi, v \otimes \psi)_1 = (\mathcal{L}(u \otimes \phi), \mathcal{L}(v \otimes \psi))_{Ind\, \pi^\tau \otimes \chi^\alpha}.$$

Taking $u = v$ and $\phi = \psi$, we have shown that $\mathcal{L}(u \otimes \phi)$ is in the Hilbert space of

$$Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

□

Proof of Theorem 4.8: It suffices to prove that $\mathcal{L}(V(\pi) \otimes V(I^\alpha))$ is dense in

$$Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

We shall prove that every element in $\mathcal{L}(V(\pi) \otimes C_c^\infty(X_0, I^\alpha)$ can be approximated by a sequence of elements in $\mathcal{L}(V(\pi) \otimes V(I^\alpha))$.

Consider $u \otimes \psi \in V(\pi) \otimes C_c^\infty(X_0, I^\alpha)$. Regard ψ as a smooth function on K, the maximal compact subgroup of $Mp_{2n_1+2n_2}(\mathbb{R})$. By the Stone-Weierstrass Theorem, there exists a sequence of functions $\phi_i \in V(I^\alpha)$, such that $\phi_i \to \psi$ under the sup norm. In other words, for i sufficiently large,

$$|\phi_i(k) - \psi(k)| \leq \delta 1_K(k)(\forall\ k \in K).$$

Now

$$(\mathcal{L}(u \otimes (\phi_i - \psi)), \mathcal{L}(u \otimes (\phi_i - \psi)))_{Ind\pi^\tau \otimes \chi^\alpha} = (u \otimes (\phi_i - \psi), u \otimes (\phi_i - \psi))_1$$

The later is no greater that $\delta^2(u \otimes 1_K, u \otimes 1_K)_1$ with respect to $\pi \otimes I^0$. Therefore, $\mathcal{L}(u \otimes \phi_i) \to \mathcal{L}(u \otimes \psi)$ under the Hilbert norm.

By an easy K-finiteness argument with respect to $Mp_{2n_2}(\mathbb{R})$,

$$\mathcal{L}(V(\pi) \otimes V(I^\alpha)) = V(Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha).$$

We have proved that

$$I^{\alpha}(\pi) \cong Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\pi^{\tau} \otimes \chi^{\alpha}.$$

□

The involution π^{τ} is not essential in Theorem 4.8 since we can always embed $g \in Mp_{2n_1}(\mathbb{R})$ into $Mp_{2n_2}(\mathbb{R})$ as $\tau(g)$.

Corollary 4.1 *Suppose $n_1 \leq n_2$. If $n_1 + n_2$ is odd,*

$$I^0(\pi) \cong Ind_{Sp_{2n_1}GL(n_2-n_1)N}^{Sp_{2n_2}(\mathbb{R})}\pi^{\tau} \otimes \text{trivial},$$

$$I^2(\pi) \cong Ind_{Sp_{2n_1}GL(n_2-n_1)N}^{Sp_{2n_2}(\mathbb{R})}\pi^{\tau} \otimes \text{sgn}(\det).$$

If $n_1 + n_2$ is even,

$$I^1(\pi) \cong Ind_{Mp_{2n_1}(\mathbb{R})MGL(n_2-n_1)N}^{Mp_{2n_2}(\mathbb{R})}\pi^{\tau} \otimes \chi;$$

$$I^3(\pi) \cong Ind_{Mp_{2n_1}(\mathbb{R})MGL(n_2-n_1)N}^{Mp_{2n_2}(\mathbb{R})}\pi^{\tau} \otimes \chi^3.$$

Combining this with Theorem 4.5, we obtain

Theorem 4.11 *If $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$ and $I^0(\pi)$ is well-defined, then*

$$Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\pi^{\tau} \otimes \chi^{\alpha} \cong \oplus_{p+q=n_1+n_2+1,\ p-q\equiv\alpha \pmod 4}\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$$

as unitary representations.

Of course, we must have $\alpha \equiv n_1 + n_2 + 1 \pmod 2$ in this situation.

4.4 Nonvanishing Theorem

Theorem 4.12 ([VO01]) *Let $P = MAN$ be a parabolic subgroup of a reductive group G. Let σ be a representation of MA. Then*

$$WF(Ind_P^G\sigma) = cl(Ind_{\mathfrak{l}_0}^{\mathfrak{g}_0}WF(\sigma)).$$

Notice that $Ind_{\mathfrak{l}_0}^{\mathfrak{g}_0}WF(\sigma))$ may contain several irreducible components of the same dimension. Applying Theorem 4.12 to I^{α} for $Mp_{2n}(\mathbb{R})$, we obtain

Corollary 4.2 *The wave front set $WF(I^{\alpha}) = Ind_{\mathfrak{gl}(n,\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})}\mathcal{O}_{[1^n]}$.*

Again assume $p + q = n + 1$. By the theorem of Kudla-Rallis-Lee-Zhu, we have

$$I^{\alpha} = \oplus_{p+q=n+1, p-q\equiv\alpha \pmod 4}\theta(p, q; 2n)(\text{trivial}).$$

Consider the wave front sets. From Corollary 2.7, $Ind_{\mathfrak{gl}(n,\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})}\mathcal{O}_{[1^n]}$ consists of $n+1$ components and is equal to the closure of $\mathcal{O}_{[2^n]} \cap \mathfrak{sp}_{2n}(\mathbb{R})$. This implies that $WF(\theta(p, q; 2n)(\text{trivial}))$ must be in the closure of $\mathcal{O}_{[2^n]} \cap \mathfrak{sp}_{2n}(\mathbb{R})$ ([BV80]). In fact, we are ready to compute $WF(\theta(p, q; 2n)(\text{trivial}))$.

Theorem 4.13 *Assume $p+q=n+1$. Let $[2^n]^{(i)}$ be the signed Young diagram with i rows starting with $+$. Then*

$$WF(\theta(p,q;2n)(\text{trivial})) = cl(\mathcal{O}_{[2^n]^{(p)}} \cup \mathcal{O}_{[2^n]^{(p-1)}}).$$

Here we take $\mathcal{O}_{[2^n]^{(-1)}}$ and $\mathcal{O}_{[2^n]^{(n+1)}}$ to be the empty set. In particular,

$$WF(I^\alpha) = \cup_{p+q=n+1, p-q\equiv\alpha \pmod 4} WF(\theta(p,q;2n)(\text{trivial})$$

Proof: From Lemma 3.3, only $cl(\mathcal{O}_{[2^n]^{(p)}} \cup \mathcal{O}_{[2^n]^{(p-1)}})$ can occur in the image of the moment map m_2 for $(O(p,q), Sp_{2n}(\mathbb{R}))$. Thus by Theorem 3.10 of Przebinda,

$$WF(\theta(p,q;2n)(\text{trivial})) \subseteq cl(\mathcal{O}_{[2^n]^{(p)}} \cup \mathcal{O}_{[2^n]^{(p-1)}}).$$

But by the decomposition Theorem 4.4,

$$\cup_{j=0}^n \mathcal{O}_{[2^n]}^{(j)} = WF(I^\alpha) = \cup_{p+q=n+1, p-q\equiv\alpha \pmod 4} WF(\theta(p,q;2n)(\text{trivial})) \subseteq \cup_{j=0}^n \mathcal{O}_{[2^n]^{(j)}}$$

We must have $WF(\theta(p,q;2n)(\text{trivial})) = cl(\mathcal{O}_{[2^n]^{(p)}} \cup \mathcal{O}_{[2^n]^{(p-1)}})$. $\square$

This theorem partly explains the decomposition theorem of Kudla-Rallis-Lee-Zhu in terms of the orbit philosophy. Generalizing this idea to $I^\alpha(\pi)$, we obtain

Theorem 4.14 (A Generic Nonvanishing Theorem) *Consider the group $Mp_{2n_1+2n_2}(\mathbb{R})$ with $n_1 \leq n_2$. Let $\pi \in \mathcal{R}_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1+n_2+1, 0; 2n_1))$ and π unitary. Suppose $\alpha \equiv n_1+n_2+1 \pmod 2$. Let $\mathcal{O}_{\mathbf{D}}$ be a nilpotent orbit of maximal dimension in*

$$Ind_{\mathfrak{sp}_{n_1}(\mathbb{R})\mathfrak{gl}(n_2-n_1)}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} WF(\pi^\tau).$$

If $\mathcal{O}_{\mathbf{D}}$ occurs only in the image of the moment map m_2 associated with $(O(p,q), Sp_{2n_2}(\mathbb{R}))$ for a finite set

$$S \subseteq \{(p,q) \mid p+q = n_1+n_2+1, p \text{ fixed parity}\},$$

then there exists $(p,q) \in S$ such that $\mathcal{Q}(2n_1;p,q;2n_2)(\pi)$ does not vanish.

Proof: Since $\pi \in \mathcal{R}_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1+n_2+1, 0; 2n_1))$, for every leading exponent v of π,

$$\Re(v) \preceq \frac{\mathbf{n_1+n_2+1}}{\mathbf{2}} - \mathbf{n_1} - \mathbf{1} - \rho(Sp_{2n_1}(\mathbb{R})).$$

Notice that I^α can be modeled on the space of L^2 functions on $\mathbb{R}^{\frac{(n_1+n_2+1)(n_1+n_2)}{2}}$. One can easily show that every matrix coefficient of I^α is bounded by a multiple of $a(g)^{\frac{-\mathbf{n_1}-\mathbf{n_2}-\mathbf{1}}{\mathbf{2}}}$. We obtain

$$\Re(v) - \frac{\mathbf{n_1+n_2+1}}{\mathbf{2}} + 2\rho(Mp_{2n_1}(\mathbb{R})) \preceq -\mathbf{n_1} - \mathbf{1} + \rho(Mp_{2n_1}(\mathbb{R})) \prec 0.$$

So $I^\alpha(\pi)$ is well-defined for any α. Theorem 4.11 applies. We have

$$Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha = \oplus_{p+q=n_1+n_2+1, p-q\equiv\alpha \pmod 4} \mathcal{Q}(2n_1;p,q;2n_2)(\pi).$$

It follows that

$$
\begin{aligned}
& Ind_{\mathfrak{sp}_{n_1}(\mathbb{R})\mathfrak{gl}(n_2-n_1)}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} WF(\pi^\tau) \\
= & \cup_{p+q=n_1+n_2+1, p-q\equiv\alpha \pmod 4} WF(\mathcal{Q}(2n_1;p,q;2n_2)(\pi)) \\
= & \cup_{p+q=n_1+n_2+1, p-q\equiv\alpha \pmod 4} WF(Q(2n_1;p,q;2n_2)(\pi)) \\
= & \cup_{p+q=n_1+n_2+1, p-q\equiv\alpha \pmod 4} WF(\theta_s(p,q;2n_2)\theta_s(2n_1;p,q)(\pi)).
\end{aligned}
\tag{4.5}
$$

By Theorem 3.10, there exists $(p,q) \in S$ such that

$$\mathcal{O}_{\mathbf{D}} \subseteq WF(\mathcal{Q}(2n_1;p,q;2n_2)(\pi)).$$

Hence $\mathcal{Q}(2n_1;p,q;2n_2)(\pi) \neq 0$. If follows that $\theta_s(2n_1;p,q)(\pi) \neq 0$. $\square$

As far as the conclusion is concerned, the parameter α is redundant. In fact $WF(I^\alpha(\pi))$ does not depend on α. Howerver, once one fixes a parity for p, α is then uniquely determined. Under certain favorable circumstances, $\mathcal{O}_{\mathbf{D}}$ only occurs in the image of m_2 for a unique (p,q) with $p+q = n_1+n_2+1$. In this case, we are able to determine for which (p,q), $\mathcal{Q}(2n_1;p,q;2n_2)(\pi) \neq 0$. Furthermore, we have Theorem 3.7 which guarantees that

$$\mathcal{Q}(2n_1;p,q;2n_2)(\pi) = \theta_s(p,q;2n_2)\theta_s(2n_1;p,q)(\pi).$$

Thus we arrive at $\theta_s(2n_1;p,q)(\pi) \neq 0$ under certain restrictions.

Chapter 5

Construction of Unipotent Representations

Start with a real nilpotent orbit $\mathcal{O} = \mathcal{O}_{\mathbf{D}}$ in $\mathcal{U}(O(p,q))$ or $\mathcal{U}(Mp_{2n}(\mathbb{R}))$. First, construct an alternating sequence of nilpotent orbits of metaplectic groups and nilpotent orbits of orthogonal groups:

$$\mathcal{O}(d_1) = \mathcal{O}_{\mathbf{D}}, \mathcal{O}(d_1 - 1) = \mathcal{O}_{\mathbf{D-1}}, \ldots, \mathcal{O}(1) = \mathcal{O}_{\mathbf{D-d_1+1}}.$$

Write

$$G(1) = G(\mathcal{O}(1)), G(2) = G(\mathcal{O}(2)), \ldots, G(d_1) = G(\mathcal{O}(d_1)).$$

Clearly, $\mathcal{O}(k) \in \mathcal{U}$ for every k (see Cor. 2.2).

Definition 27 *We define $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ inductively.*

1. *For $\mathcal{O}(1)$, let $\mathcal{N}(\mathcal{O}(1))$ be the set of all finite-dimensional irreducible unitary representation π of $G(1)$ such that π restricted to $G(1)_0$ contains a trivial constituent.*

2. *Suppose $\mathcal{N}(\mathcal{O}(k))$ is defined.*

3. *For $G(k+1)$ orthogonal, define $\mathcal{N}(\mathcal{O}(k+1))$ to be*

   $$\{\pi(k+1)\otimes\eta, \pi(k+1)^*\otimes\eta \mid \pi(k+1) = \theta_s(G(k), G(k+1))(\pi), \pi \in \mathcal{N}(\mathcal{O}(k)), \eta \text{ a character of } O(p,q)\}.$$

4. *For $G(k+1)$ metaplectic, define $\mathcal{N}(\mathcal{O}(k+1))$ to be*

   $$\{\pi(k+1), \pi(k+1)^\tau, \pi(k+1)^*, \pi(k+1)^{*\tau} \mid \pi(k+1) = \theta_s(G(k), G(k+1))(\pi), \pi \in \mathcal{N}(\mathcal{O}(k))\}.$$

Since $\mathcal{O}$ and $\tau(\mathcal{O})$ are indistinguishable geometrically, we include π^τ in $\mathcal{N}(\mathcal{O})$ for $G(\mathcal{O})$ metaplectic. Any of the operations in the definition of $\mathcal{N}$, tensoring with a character, involution by τ or $*$, do not alter the infinitesimal character, the real part of the leading exponent, or the associated variety. In most cases, we will ignore these operations.

Theorem 5.1 *For $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(G)$, the set $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ is not empty.*

For technical reasons, we will postpone the proof till the next chapter. In this chapter, we work under the assumption that $\mathcal{N}(\mathcal{O}_{\mathbf{D}}) \neq \emptyset$. We will show that

Theorem 5.2 *Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}(O(p,q))$ or $\mathcal{U}(Sp_{2n}(\mathbb{R}))$. Then any $\pi(k) \in \mathcal{N}(\mathcal{O}(k))$ is in $\mathcal{R}_{ss}(G(k),\omega)$ for the pair $(G(k), G(k+1))$. The set $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ is well-defined. The representations in $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ are all unitary.*

Thus $\mathcal{N}(\mathcal{O}_{\mathbf{D}}) \subset \Pi_u(G(\mathcal{O}_{\mathbf{D}}))$. We will further determine the infinitesimal character of $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. It turns out that $\mathcal{I}(\pi)$ only depends on $\mathbf{d}$ and consists of ρ-like segments.

5.1 Unitarity of $\mathcal{N}(\mathcal{O})$

Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}(O(p,q))$ or $\mathcal{U}(Sp_{2n}(\mathbb{R}))$. The group $G(\mathcal{O}_{\mathbf{D}})$ can be read off from D^+, the number of positive boxes in $\mathbf{D}$, and from D^-, the number of negative boxes in $\mathbf{D}$. Construct an alternating sequence of symplectic signed Young diagrams and orthogonal signed Young diagrams:

$$\mathbf{D}(d_1) = \mathbf{D}, \mathbf{D}(d_1-1) = \mathbf{D}(d_1) - \mathbf{1}, \ldots, \mathbf{D}(2) = \mathbf{D}(3) - \mathbf{1}, \mathbf{D}(1) = \mathbf{D}(2) - \mathbf{1}.$$

Let $\mathbf{D}(0) = 0$. Then we must have the following.

Lemma 5.1 *Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. Then*

- *For every $k \in [1, d_1 - 1]$,*

$$D(k+1)^+ \geq D(k)^-, \qquad D(k+1)^- \geq D(k)^+.$$

- *For every $k \in [1, d_1 - 1]$,*

  $$D(k+1)^+ - D(k)^+ \geq D(k)^- - D(k-1)^-, \qquad D(k+1)^- - D(k)^- \geq D(k)^+ - D(k-1)^+.$$

  Thus,

$$D(k+1)^+ + D(k-1)^- \geq ||\mathbf{d}(k)||, \qquad D(k+1)^- + D(k-1)^+ \geq ||\mathbf{d}(k)||.$$

- *For every $k \in [1, d_1 - 1]$,*

$$\|\mathbf{d}(k+1)\| + \|\mathbf{d}(k-1)\| \geq 2\|\mathbf{d}(k)\| + 2$$

  if $\mathbf{D}(k)$ is symplectic;

- *for every $k \in [1, d_1 - 1]$,*

$$\|\mathbf{d}(k+1)\| + \|\mathbf{d}(k-1)\| \geq 2\|\mathbf{d}(k)\|$$

  if $\mathbf{D}(k)$ is orthogonal.

- $\|\mathbf{d}(k+2)\| \equiv \|\mathbf{d}(k)\| \pmod 2$ *for all $k > 0$.*

Proof of Theorem 5.2: Consider the sequences $\mathbf{D}(k)$ and $\mathcal{O}(k)$. We shall prove that $\pi(k-1)$ lies in $\mathcal{R}_{ss}(G(k-1),\omega))$ for the pair $(G(k-1), G(k))$, and $\pi(k) = \theta_s(G(k-1), G(k))(\pi(k-1))$ is unitary. We will use induction.

1. For $k = 1$, by definition, every $\pi(1)$ in $\mathcal{N}(\mathcal{O}(1))$ is unitary.

2. For $k = 2$, if $G(1)$ is metaplectic, then we have

   $$D(1)^+ = D(1)^-, \qquad D(2)^+ \geq 2D(1)^-, \qquad D(2)^- \geq 2D(1)^+, \qquad D(2)^+ + D(2)^- \geq 4D(1)^- + 2;$$

   if $G(1)$ is orthogonal, then we have $D(2)^+ = D(2)^- \geq D(1)^- + D(1)^+$. It follows from the Definition 18, that every unitary representation is in $\mathcal{R}_{ss}(G(1), \omega)$ with respect to the dual pair $(G(1), G(2))$. A theorem of Li ([LI89]) says that $\pi(2) = \theta_s(G(1), G(2))(\pi(1))$ is unitary (see also Theorems 3.4 and 3.5). Therefore, $\mathcal{N}(\mathcal{O}(2))$ is well-defined and every $\pi \in \mathcal{N}(\mathcal{O}(2))$ is unitary.

3. Suppose that $\mathcal{N}(\mathcal{O}(k-1))$ is well-defined and $\mathcal{N}(\mathcal{O}(k-1)) \subseteq \Pi_u(G(k-1))$. Suppose that $\mathcal{N}(\mathcal{O}(k-1)) \subseteq \mathcal{R}_{ss}(G(k-1), \omega)$ with respect to the pair $(G(k-1), G(k))$. This is our induction hypothesis.

   If $G(k)$ is metaplectic, let

   $$p' = D(k+1)^+, q' = D(k+1)^-, n = D(k)^+ = D(k)^-, p = D(k-1)^+, q = D(k-1)^-.$$

   Then by Lemma 5.1, $p + q \equiv p' + q' \pmod 2$, $p' + q' - 2n \geq 2n - p - q + 2$, $p' \geq n$ and $q' \geq n$. Clearly $\max(p', q') > n$. If $n \neq \min(p', q')$, then Theorem 3.6 holds and $\pi(k) = \theta_s(\pi(k-1))$ is well-defined and unitary. Furthermore, $\pi(k) \in \mathcal{R}_{ss}(G(k), \omega)$ with respect to $(G(k), G(k+1))$. If $n = \min(p', q')$, without loss of generality, assume, $n = p'$. We shall prove that this case does not occur if $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$.

   Since $D(k)^- = n = p' = D(k+1)^+$, the diagram $\mathbf{D}(k+1)$ must have its first column marked with $-$. By the structure of signed Young diagram, $\mathbf{D}(k)$ must have its first column marked with $+$. $\mathbf{D}(k)$ must have its second column marked with $-$. Furthermore, the first column of $\mathbf{D}(k)$ and the second column of $\mathbf{D}(k)$ must have the same length. Otherwise, there are at least two rows of length one in $\mathbf{D}(k)$. At least one $\boxed{-}$ would occur in the first column of $\mathbf{D}(k)$ according to our definition of $\mathcal{YD}_-(n, n)$. This contradicts the fact that the first column of $\mathbf{D}(k)$ is all marked with $+$. Thus,

   $$\|\mathbf{d}(k)\| - \|\mathbf{d}(k-1)\| = \|\mathbf{d}(k-1)\| - \|\mathbf{d}(k-2)\|.$$

   By the same argument, $\mathbf{D}(k)$ must be of the following shape and sign pattern:

   $$\begin{array}{|c|c|c|c|c|c|}
   \hline + & - & + & - & + & - \\ \hline
   + & - & + & - & + & - \\ \hline
   + & - & + & - \\ \cline{1-4}
   + & - & + & - \\ \cline{1-4}
   + & - \\ \cline{1-2}
   + & - \\ \cline{1-2}
   \end{array}$$

   In other words, the j-th column of $\mathbf{D}(k)$ must be marked by $(-1)^j$ and the $2j+1$-th column of $\mathbf{D}(k)$ must be of the same length as the $2j+2$-th column. This kind of $\mathbf{D}(k)$

is excluded from $\mathcal{U}$. So the case $n = \min(p', q')$ does not occur.

If $G(k)$ is orthogonal, let

$$n' = D(k+1)^+ = D(k+1)^-, \qquad p = D(k)^+, q = D(k)^-, \qquad n = D(k-1)^+ = D(k-1)^-.$$

Then by Lemma 5.1, we have

$$2n' - p - q, \geq p + q - 2n > p + q - 2n - 2, \qquad p \geq n \qquad q \geq n.$$

By the same argument as for metaplectic $G(k)$, $n \neq \min(p, q)$. By Theorem 3.7, $\pi(k) = \theta_s(\pi(k-1))$ is well-defined, unitary and $\pi(k) \in \mathcal{R}_{ss}(G(k), \omega)$ with respect to the pair $(G(k), G(k+1))$.

4. It follows that every $\mathcal{N}(\mathcal{O}(k)) \subseteq \mathcal{R}_{ss}(G(k), \omega)$ for the pair $(G(k), G(k+1))$ and every $\pi(k) \in \mathcal{N}(\mathcal{O}(k))$ is unitary.

□

5.2 Infinitesimal Character

Theorem 5.3 (Przebinda, [PR96]) *Let $\mathcal{I}(\pi)$ be the infinitesimal character of π. Suppose that π occurs in the theta correspondence with respect to $(Sp_{2n}(\mathbb{R}), O(p, q))$.*

1. *Suppose $p + q < 2n$. Then $\mathcal{I}(\theta(p, q; 2n)(\pi))$ can be obtained by augmenting $\mathcal{I}(\pi)$ by*

$$(n - \frac{p+q}{2}, n - \frac{p+q}{2} - 1, \ldots, 1 + [\frac{p+q}{2}] - \frac{p+q}{2})$$

2. *Suppose $2n + 1 < p + q$. Then $\mathcal{I}(\theta(2n; p, q)(\pi))$ can be obtained by augmenting $\mathcal{I}(\pi)$ by*

$$(\frac{p+q}{2} - n - 1, \frac{p+q}{2} - n - 2, \ldots, \frac{p+q}{2} - [\frac{p+q}{2}])$$

3. *Suppose $p + q = 2n$ or $p + q = 2n + 1$. Then $\mathcal{I}(\theta(p, q; 2n)(\pi))$ is just $\mathcal{I}(\pi)$.*

Notation 17 *We define the orthogonal segment*

$$\mathcal{I}_+(m) = (\overbrace{\frac{m}{2} - 1, \frac{m}{2} - 2, \ldots, \frac{m}{2} - [\frac{m}{2}]}^{[\frac{m}{2}]})$$

and the symplectic segment

$$\mathcal{I}_-(m) = (\overbrace{\frac{m}{2}, \frac{m}{2} - 2, \ldots, \frac{m}{2} + 1 - [\frac{m+1}{2}]}^{[\frac{m+1}{2}]}).$$

The orthogonal segment $\mathcal{I}_+(m)$ is just $\rho(\mathfrak{o}(m, \mathbb{C}))$. For m even, $\mathcal{I}_-(m)$ is $\rho(\mathfrak{sp}_m(\mathbb{C}))$. For m odd, $\mathcal{I}_-(m)$ is the infinitesimal character of the oscillator representation of $Mp_{m+1}(\mathbb{R})$.

Theorem 5.4 *Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}(O(p,q))$ or $\mathcal{U}(Sp_{2n}(\mathbb{R}))$. Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$ and*

$$\mathbf{d}^t = (m_1 \geq m_2 \geq \ldots \geq m_{d_1}).$$

If $G(\mathbf{D})$ is an orthogonal group, then

$$\mathcal{I}(\pi) = (\mathcal{I}_+(m_1), \mathcal{I}_-(m_2), \mathcal{I}_+(m_3), \mathcal{I}_-(m_4), \ldots).$$

If $G(\mathbf{D})$ is a symplectic group, then

$$\mathcal{I}(\pi) = (\mathcal{I}_-(m_1), \mathcal{I}_+(m_2), \mathcal{I}_-(m_3), \mathcal{I}_+(m_4), \ldots).$$

Proof: Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. We prove this theorem by induction on d_1. If $d_1 = 1$, by definition of $\mathcal{N}(\mathcal{O}(1))$, π restricted to the identity component of $G(\mathcal{O}_{\mathbf{D}})$ must contain a trivial constituent. Thus $\mathcal{I}(\pi) = \mathcal{I}_{\pm}(m_1)$. Suppose our assertion holds for any $\mathbf{D}$ with $d_1 \leq k$. Let $d_1 = k+1$.

If $G(\mathbf{D})$ is $O(p,q)$, then according to our definition of π, $\pi|_{SO_0(p,q)}$ must be equivalent to $\theta(G(\mathcal{O}_{\mathbf{D}-1}), G(\mathcal{O}_{\mathbf{D}}))(\sigma)|_{SO_0(p,q)}$ or its contragredient for some $\sigma \in \mathcal{N}(\mathcal{O}_{\mathbf{D}-1})$. $\mathcal{I}(\pi)$ can be obtained by augmenting $\mathcal{I}(\sigma)$ with

$$(\overbrace{\frac{m_1}{2} - 1, \frac{m_1}{2} - 2, \ldots, \frac{m_1}{2} - [\frac{m_1}{2}]}^{[\frac{m_1}{2}]}) = \mathcal{I}_+(m_1).$$

If $G(\mathbf{D})$ is $Mp_{2n}(\mathbb{R})$, observe that $\mathcal{I}(\pi^{\tau}) = \mathcal{I}(\pi)$. Then

$$\mathcal{I}(\pi) = \mathcal{I}(\theta(G(\mathcal{O}_{\mathbf{D}-1}), G(\mathcal{O}_{\mathbf{D}}))(\sigma))$$

for some $\sigma \in \mathcal{N}(\mathcal{O}_{\mathbf{D}-1})$. $\mathcal{I}(\pi)$ can be obtained by augmenting $\mathcal{I}(\sigma)$ with

$$(\overbrace{\frac{m_1}{2}, \frac{m_1}{2} - 1, \ldots, \frac{m_1}{2} - [\frac{m_1 - 1}{2}]}^{[\frac{m_1+1}{2}]}) = \mathcal{I}_-(m_1).$$

$\square$

Notation 18 *Let*

$$\mathbf{d}^t = (m_1 \geq m_2 \geq \ldots \geq m_{d_1}).$$

For each orthogonal Young diagram $\mathbf{d}$, we define $\mathcal{I}_+(\mathbf{d})$ to be

$$(\mathcal{I}_+(m_1), \mathcal{I}_-(m_2), \mathcal{I}_+(m_3), \mathcal{I}_-(m_4), \ldots).$$

For each symplectic Young diagram $\mathbf{d}$, we define $\mathcal{I}_-(\mathbf{d})$ to be

$$(\mathcal{I}_-(m_1), \mathcal{I}_+(m_2), \mathcal{I}_-(m_3), \mathcal{I}_+(m_4), \ldots).$$

Theorem 5.4 states that $\mathcal{I}(\pi) = \mathcal{I}(\mathbf{d})$ for $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. Thus $\mathcal{I}(\pi)$ only depends on $\mathbf{d}$, not on the signs of $\mathbf{D}$.

5.3 $\mathcal{I}_{\pm}(\mathbf{d})$ and d: An algorithm

In this section, we will seek a direct way of obtaining $\mathcal{I}_{\pm}(\mathbf{d})$ from the Young diagram $\mathbf{d}$. We assume $\mathbf{d}^t$ is very odd or very even. The Young diagram $\mathbf{d}$ does not have to be pre-rigid or in $\mathcal{U}$. The algorithm can be described as follows.

1. First, cover the Young diagram $\mathbf{d}$ by horizontal and vertical dominos

   as follows. We cover each column of the Young diagram $\mathbf{d}$ by consecutive vertical dominos, starting from the bottom row. If $\mathbf{d}^t$ is very even, vertical dominos covers $\mathbf{d}$ completely. If $\mathbf{d}^t$ is very odd, we have the first row left uncovered. We cover the first row of the Young diagram $\mathbf{d}$ by consecutive horizontal dominos, starting from the *right*. We may have the leftmost block uncovered. In that case, cover it with a horizontal domino anyway. We call this domino an open domino.

2. For each vertical domino DO, we can enumerate the number of dominos above it. A horizontal domino will be counted as $\frac{1}{2}$ domino and a vertical domino will be counted as a full domino. Thus we obtain a number $n(DO)$.

3. If $\mathbf{d}^t$ is odd, fill the open domino with no number and fill the other horizontal dominos with $\frac{1}{2}$. For $\mathcal{I}_{+}(\mathbf{d})$, if a vertical domino DO is in the k-th column, we fill in DO with the number $n(DO) + \frac{1+(-1)^k}{2}$. For $\mathcal{I}_{-}(\mathbf{d})$, if a domino DO is in the k-th column, we fill in DO with the number $n(DO) + \frac{1-(-1)^k}{2}$. Extracting the numbers in all the dominos, we obtain $\mathcal{I}(\mathbf{d})$.

Notation 19 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$. We define $\overline{\lambda}$ to be the reordering of λ such that*

$$\overline{\lambda}_1 \geq \overline{\lambda}_2 \geq \dots \geq \overline{\lambda}_n.$$

5.4 Orderings and Reversal Phenomena

Recall from [CM] that complex nilpotent orbits of classical groups have a partial ordering $\preceq$.

Notation 20 *Write $\mathbf{d}_1 \preceq \mathbf{d}_2$ if $\mathcal{O}_{\mathbf{d}_1}$ is contained in the closure of $\mathcal{O}_{\mathbf{d}_2}$.*

Theorem 5.5 (Reversal Phenomena) *Let $\mathcal{O}_{\mathbf{D}_1}$ and $\mathcal{O}_{\mathbf{D}_2}$ be nilpotent orbits of a fixed orthogonal group or symplectic group. Suppose that $\mathbf{D}_1$ and $\mathbf{D}_2$ are either both very even or both very odd. If $\mathbf{d}_1 \preceq \mathbf{d}_2$ then $\overline{\mathcal{I}(\mathbf{d}_1)} \succeq \overline{\mathcal{I}(\mathbf{d}_2)}$.*

Later we will show that $\mathcal{V}(Ann\ \mathcal{N}(\mathbf{D})) = \mathcal{O}_{\mathbf{D}}$. This theorem suggests that, among the representations $\mathcal{N}$ of a fixed group G, the smaller the infinitesimal character, the bigger is the associated variety and conversely. For example, the trivial representation has the smallest associated variety, namely $\{0\}$. Its infinitesimal character is given by $\rho(G)$, the greatest among all $\mathcal{I}(\pi)$ with $\pi \in \mathcal{N}$.

We prove the reversal phenomena through a number of lemmas. The conditions in the theorem are always assumed in these lemmas.

Lemma 5.2 *If the Young diagram $\mathbf{d}_1$ can be obtained from the Young diagram $\mathbf{d}_2$ by moving one domino to its lower left without changing other dominos, then $\mathbf{d}_1 \preceq \mathbf{d}_2$.*

We call this procedure a *move*.

Lemma 5.3 *If the Young diagram $\mathbf{d}_1$ can be obtained from $\mathbf{d}_2$ by a finite number of moves, then $\mathbf{d}_1 \preceq \mathbf{d}_2$ and vice versa.*

This lemma holds only under the assumption of the theorem. If $\mathbf{d}_1^t$ is very even and $\mathbf{d}_2^t$ is very odd, this lemma is no longer valid.

Lemma 5.4 *If Young diagram $\mathbf{d}_1$ can be obtained from the Young diagram $\mathbf{d}_2$ by a move, then $\overline{\mathcal{I}_{\pm}(\mathbf{d}_1)} \succeq \overline{\mathcal{I}_{\pm}(\mathbf{d}_2)}$.*

Now it becomes evident that $\overline{\mathcal{I}_{\pm}(\mathbf{d}_1)} \succeq \overline{\mathcal{I}_{\pm}(\mathbf{d}_2)}$. By the above three lemmas, Theorem 5.5 is proved. $\square$

Corollary 5.1 *Let $\mathcal{O}_{\mathbf{D}_1}$ and $\mathcal{O}_{\mathbf{D}_2}$ be pre-rigid nilpotent orbits of a fixed orthogonal group or symplectic group. Suppose that $\mathbf{d}_1^t$ and $\mathbf{d}_2^t$ are both even or both odd. Then $\overline{\mathcal{I}(\mathbf{d}_1)} \succeq \overline{\mathcal{I}(\mathbf{d}_2)}$ if and only if $\mathbf{d}_1 \preceq \mathbf{d}_2$.*

Proof: For pre-rigid orbits, one can reconstruct $\mathbf{d}^t$ from $\mathcal{I}(\mathbf{d})$ in a unique way and the partial orderings are reversed in this reconstruction. $\square$

The ordering $\preceq$ for Young diagrams is different from the ordering $\preceq$ for numerical sequences (see Notation 2). If one regards $\mathbf{d}_1^t$ and $\mathbf{d}_2^t$ as numerical sequences arranged in descending orders, then $\mathbf{d}_1 \preceq \mathbf{d}_2$ if and only if $\mathbf{d}_1^t \succeq \mathbf{d}_2^t$. In what follows, all the ordering $\preceq$ will refer to numerical sequences.

Let $\mathcal{O}_{\mathbf{d}}$ be a nilpotent orbit of G. Let $\Pi_{\mathcal{O}_{\mathbf{d}}}(G)$ be the set of π such that $\mathcal{V}(Ann\ \pi) = cl(\mathcal{O}_{\mathbf{d}})$. In [BV85], Barbasch-Vogan conjectured that

$$\{\pi \in \Pi_{\mathcal{O}_{\mathbf{d}}}(G) \mid \|\mathcal{I}(\pi)\| = \min\{\|\mathcal{I}(\sigma)\| \mid \sigma \in \Pi_{\mathcal{O}_{\mathbf{d}}}(G)\}\} \subseteq \Pi_u(G).$$

Later on we shall prove that $\mathcal{N}(\mathcal{O}_{\mathbf{D}}) \subset \Pi_{\mathcal{O}_{\mathbf{d}}}(G)$. Thus $\mathcal{N}(\mathcal{O}_{\mathbf{D}}) \subset \Pi_{\mathcal{O}_{\mathbf{d}}}(G) \cap \Pi_u(G)$. We formulate the following conjecture.

Conjecture 3 *Let G be either an orthogonal group or a metaplectic group. Let π be an irreducible admissible representation of G such that $\mathcal{V}(Ann\ \pi) = \mathcal{O}_{\mathbf{d}}$ is rigid. Let λ be the shortest infinitesimal character among all $\mathcal{I}(\pi)$ with $\pi \in \Pi_{\mathcal{O}_{\mathbf{d}}}(G)$. Then $\lambda = \mathcal{I}(\mathbf{d})$.*

This conjecture is false if $\mathcal{V}(Ann\ \pi)$ fails to be rigid. In the next chapter, we will formulate another conjecture regarding the exhaustion of unitary $\Pi_{\mathcal{O}_{\mathbf{d}}}(G)$ by $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ for $\mathcal{O}_{\mathbf{D}}$ rigid.

Chapter 6

Matrix Coefficients and Associated Varieties

Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(G)$ and $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. In this chapter, we study the matrix coefficients and the associated variety of π. We first bound $\overline{\mathcal{I}(\pi)}$ by a multiple of $\rho(G)$. By Theorem 1.2, we obtain a bound on the matrix coefficients of π. Then we apply Przebinda's theorem to show that

$$\mathcal{V}(Ann\ \pi) = cl(\mathcal{O}_{\mathbf{d}}).$$

Let me briefly describe the proof of Theorem 5.1. Suppose $\pi(k) \in \mathcal{N}(\mathcal{O}(k))$. Then $\mathcal{I}(\pi(i))$ and $\mathcal{V}(Ann\ \pi(i))$ are all known for every $i \leq k$ based on our computation. If $G(\mathcal{O}(k))$ is orthogonal, then $\mathcal{N}(\mathcal{O}(k+1)) \neq \emptyset$ follows from Theorem 3.3. If $G(\mathcal{O}(k))$ is the metaplectic group, $\mathcal{N}(\mathcal{O}(k+1)) \neq \emptyset$ follows from Theorem 4.14. The details of the proof of 5.1 are given at the end of this chapter.

6.1 Estimates on Infinitesimal Characters: I

Let $G = Mp_{2n}(\mathbb{R})$. Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}(G)$. Let

$$\mathbf{d}^t = (m_1 \geq m_2 \geq \ldots \geq m_{d_1})$$

Then we have $2n = \|\mathbf{d}\|$ and $\rho(G) = (n, n-1, \ldots, 1)$.

Theorem 6.1 *Suppose $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(Mp_{2n}(\mathbb{R}))$. Let*

$$\mathbf{d}^t = (m_1, m_2, \ldots, m_{d_1}).$$

Then

$$\overline{\mathcal{I}_{-}(\mathbf{d})} \prec \frac{m_1 + 2}{2n} \rho(G).$$

A similar statement holds for $G = O(p,q)$.

Theorem 6.2 *Suppose $\mathcal{O}_{\mathbf{d}} \in \mathcal{U}(O(p,q))$. Let*

$$\mathbf{d}^t = (m_1, m_2, \ldots, m_{d_1}).$$

Then
$$\overline{\mathcal{I}_+(\mathbf{d})} \prec \frac{m_1+2}{p+q-2}(\frac{p+q}{2}-1, \frac{p+q}{2}-2, \ldots \frac{p+q}{2}-[\frac{p+q}{2}]).$$

In fact, we will have that
$$\overline{\mathcal{I}_-(\mathbf{d})} \preceq \frac{m_1}{2n}\rho(Sp_{2n}(\mathbb{R})).$$
$$\overline{\mathcal{I}_+(\mathbf{d})} \preceq \frac{m_1}{p+q-2}(\frac{p+q}{2}-1, \frac{p+q}{2}-2, \ldots \frac{p+q}{2}-[\frac{p+q}{2}])$$
for $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. These two statements are slightly stronger than those of Theorems 6.1 and 6.2. In the Appendix, we will give a proof for the first statement and Theorem 6.1. We skip the proof for Theorem 6.2.

6.2 Associated Varieties of $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$

Theorem 6.3 (Estimates on Leading Exponents) *Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}$. Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$.*

1. *If $G(\mathcal{O}_{\mathbf{D}})$ is $Mp_{2n}(\mathbb{R})$, then every leading exponent of π satisfies*
$$\Re(v) \prec (\frac{m_1+2}{2n}-1)\rho(Mp_{2n}(\mathbb{R})).$$

2. *If $G(\mathcal{O}_{\mathbf{D}})$ is $O(p,q)$, then every leading exponent of π satisfies*
$$\Re(v) \prec (\frac{m_1+2}{p+q-2}-1)\rho(O(p,q)).$$

Proof: Part (1) is a direct consequence of Theorem 6.1 and Theorem 1.1. Part (2) is a direct consequence of Theorem 6.2 and Theorem 1.1. □

Let me make one remark here concerning $\mathcal{N}(\mathcal{O}(1))$ in Definition 27. Theorem 6.3 can be established independently in the framework of [Heq] without resorting to the infinitesimal character estimate in Theorems 6.1 and 6.2. Thus Theorem 6.3 holds without the assumption that $\pi(1)|_{G(1)_0}$ contains the trivial representation. From now on we may allow any irreducible finite dimensional unitary representations in $\mathcal{N}(\mathcal{O}(1))$.

Theorem 6.4 *Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(G)$. Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. Then $\mathcal{V}(Ann\ \pi) = cl(\mathcal{O}_{\mathbf{d}})$.*

Proof: Notice that none of the operations in the definition of $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ changes $\mathcal{I}(\pi)$ or $\mathcal{V}(Ann\ \pi)$. Without loss of generality, we assume $\pi(k) = \theta_s(G(k-1), G(k))(\pi(k-1))$ and $\pi(d_1) = \pi$. Recall Theorem 3.8 and Theorem 3.9.

1. Condition (1) is automatic by the definition of $(G(k-1), G(k))$.

2. Conditions (2) from Theorem 3.8 and 3.9 are satisfied due to Theorem 6.3 and the definition of $(G(k-1), G(k))$.

3. Conditions (3) and (4) from Theorem 3.8 and Theorem 3.9, which basically say that $\theta_s(\pi(k-1))$ is well-defined and unitary, are readily verified by Theorem 5.2.

4. Conditions (5) are checked in Lemma 3.5 and Lemma 3.6.

Thus Theorem 3.8 and Theorem 3.9 hold. Therefore

$$\mathcal{V}(Ann\ \pi(k)) = \Theta(G(k-1), G(k))(\mathcal{V}(Ann\ \pi(k-1))).$$

By induction and By Lemma 3.2 of DKP, $\mathcal{V}(Ann\ \pi) = cl(\mathcal{O}_{\mathbf{d}})$. $\square$

Corollary 6.1 *Fix an $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(Mp_{2n}(\mathbb{R}))$. Let $O(p_0, q_0) = G(\mathcal{O}_{\mathbf{D}-1})$. Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. Then there exists a nilpotent orbit $\mathcal{O}_{\mathbf{D}_0}$ in $WF(\pi)$ such that*

1. $\mathbf{d}_0 = \mathbf{d}$.

2. $G(\mathcal{O}_{\mathbf{D}_0-1}) \cong O(p_0, q_0)$.

Proof: Suppose that $\pi = \theta_s(p_0, q_0; 2n)(\sigma)$ with $\sigma \in \mathcal{N}(\mathcal{O}_{\mathbf{D}-1})$. Then $\mathcal{V}(Ann\ \sigma) = \mathcal{O}_{\mathbf{d}-1}$ and $\mathcal{V}(Ann\ \pi) = \mathcal{O}_{\mathbf{d}}$. By Theorem 3.10, $WF(\pi)$ must be in the image of m_2 with respect to $(O(p_0, q_0), Sp_{2n}(\mathbb{R}))$. Then Lemma 3.3 says that every orbit $\mathcal{O}_{\mathbf{S}}$ in $WF(\pi)$ satisfies $G(\mathcal{O}_{\mathbf{S}-1}) = O(s, t)$ with $s \leq p_0$ and $t \leq q_0$. There must be at least one $\mathcal{O}_{\mathbf{D}_0}$ in $WF(\pi)$ such that $G(\mathcal{O}_{\mathbf{D}_0-1}) = O(p_0, q_0)$. Otherwise, $\mathcal{V}(Ann\ \pi)$, which is the complexification of $WF(\pi)$, will be strictly smaller than $cl(\mathcal{O}_{\mathbf{d}})$ and will not be equal to $cl(\mathcal{O}_{\mathbf{D}})$.

If $\pi = \theta_s(p_0, q_0; 2n)(\sigma)^{\tau}$, by a similar argument, there exists an $\mathcal{O}_{\mathbf{D}_0}$ in $WF(\pi)$ such that $G(\mathcal{O}_{\mathbf{D}_0-1}) = O(q_0, p_0)$. $\square$

6.3 Nonvanishing of $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$

Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. We have proved that π must be unitary and $\mathcal{V}(Ann\ \pi) = \mathcal{O}_{\mathbf{d}}$. In this section, we will prove that $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ is nonempty.

Lemma 6.1 *Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. Let $O(p_0, q_0) = G(k-1)$, $Mp_{2n_1}(\mathbb{R}) = G(k)$ and $O(p, q) = G(k+1)$. Suppose*

$$\pi(k-1) \in \mathcal{N}(\mathcal{O}(k-1)), \text{ and } \pi(k) = \theta_s(p_0, q_0; 2n_1)(\pi(k-1)) \neq 0.$$

Then $\pi(k+1) = \theta_s(2n_1; p, q)(\pi(k)) \neq 0$.

Proof: Write

$$\mathbf{d}(k+1)^t = (m_1 > m_2 \geq m_3 > m_4 \geq \ldots,).$$

Then

$$\mathbf{d}(k)^t = (m_2 \geq m_3 > m_4 \geq \ldots,), \qquad \mathbf{d}(k-1)^t = (m_3 > m_4 \geq \ldots,),$$

$$p + q = 2n_1 + m_1, \qquad 2n_1 = p_0 + q_0 + m_2.$$

By Theorem 6.4, $\mathcal{V}(Ann\, \pi(k)) = \mathcal{O}_{\mathbf{d}(k)}$ and $\mathcal{V}(Ann\, \pi(k-1)) = \mathcal{O}_{\mathbf{d}(k-1)}$. Let $n_2 = p+q-n_1-1$. Clearly $n_2 - n_1 = p + q - 2n_1 - 1 = m_1 - 1 > 0$. By Theorem 4.1, it suffices to show that

$$\mathcal{Q}(2n_1; p, q; 2n_2)(\pi(k)) \neq 0.$$

1. By Corollary 6.1, there exists a nilpotent orbit $\mathcal{O}_{\mathbf{D}_0}$ of $Sp_{2n_1}(\mathbb{R})$ such that
   (a) $\mathbf{d}_0 = \mathbf{d}(k)$;
   (b) $\mathbf{D}_0 - 1$ contains p_0 $\boxed{+}$'s and q_0 $\boxed{-}$'s;
   (c) $\mathcal{O}_{\mathbf{D}_0} \subseteq WF(\pi(k))$.

2. By Theorem 4.12,

$$Ind_{\mathfrak{sp}_{2n_1}(\mathbb{R})\oplus\mathfrak{gl}(n_2-n_1,\mathbb{R})}^{\mathfrak{sp}_{2n_2}(\mathbb{R})}\tau(\mathcal{O}_{\mathbf{D}_0}) \subseteq WF(Ind_{MP_{n_2-n_1}}^{Mp_{2n_2}(\mathbb{R})}\chi^{\alpha}\otimes\pi).$$

   Observe that $n_2 - n_1 = m_1 - 1$, and $m_1 - 1 \geq m_2$. m_2 is the first entry of $\mathbf{d}_0 = \mathbf{d}(k)$. By Theorem 2.8,

$$Ind_{\mathfrak{sp}_{2n_1}(\mathbb{R})\oplus\mathfrak{gl}(n_2-n_1,\mathbb{R})}^{\mathfrak{sp}_{2n_2}(\mathbb{R})}\tau(\mathcal{O}_{\mathbf{D}_0})$$

   contains $m_1 - m_2$ irreducible components

$$\cup_{j=0}^{m_1-m_2-1}\mathcal{O}_{\mathbf{S}^{(j)}}.$$

   And $\mathbf{S}^{(j)}$ can be obtained by

   - merging 2 columns of length $m_1 - 1$ to $\mathbf{D}_0$ from left;
   - extending the signs of $\mathbf{D}_0$ for the first m_2 rows;
   - assigning j $\boxed{-}\boxed{+}$'s and $m_1 - m_2 - 1 - j$ $\boxed{+}\boxed{-}$'s to rows of length 2.

3. Notice that for every j,

$$(\mathbf{s}^{(j)})^t = \mathbf{s}^t = (m_1 - 1, m_1 - 1, m_2, m_3, \ldots,);$$

$$(\mathbf{s}^{(j)} - \mathbf{1})^t = (\mathbf{s} - \mathbf{1})^t = (m_1 - 1, m_2, m_3, \ldots,).$$

   Consider the signature of $\mathbf{S}^{(j)} - \mathbf{1}$. For the first m_2 rows, there are $p_0 + m_2$ $\boxed{+}$'s and q_0+m_2 $\boxed{-}$'s. For the last m_1-m_2-1 rows of length 1, there are j $\boxed{-}$'s and m_1-m_2-1-j $\boxed{+}$'s. Thus the signature of $\mathbf{S}^{(j)} - 1$ is

$$(p_0 + m_2 + j, q_0 + m_2 + m_1 - m_2 - 1 - j) = (p_0 + m_2 + j, q_0 + m_1 - 1 - j)$$

4. Fix the parity of p. Let $j = p - p_0 - m_2$. Here $p - p_0$ is equal to the number of $\boxed{+}$ in the first two columns of $\mathbf{D}(k+1)$. It is greater or equal to the number of boxes in the second column of $\mathbf{D}(k+1)$. It follows that $j = p - p_0 - m_2 \geq 0$. Since $p+q = m_1+m_2+p_0+q_0$, the signature of $\mathbf{S}^{(j)} - \mathbf{1}$ equals

$$(p_0 + m_2 + j, q_0 + m_1 - 1 - j) = (p, q_0 + m_1 - 1 + p_0 + m_2 - p) = (p, q - 1).$$

   From Lemma 3.3, $\mathcal{O}_{\mathbf{S}^{(j)}}$ occurs in the image of m_2 associated with $(O(p,q), Sp_{2n_2}(\mathbb{R}))$. Also from Lemma 3.3, $\mathcal{O}_{\mathbf{S}^{(j)}}$ does not occurs in the image of m_2 associated with

$$(O(p-2i, q+2i), Sp_{2n_2}(\mathbb{R})), (O(p+2i, q-2i), Sp_{2n_2}(\mathbb{R}))$$

   for any $i \neq 0$.

5. By Theorem 5.2 and Theorem 3.7, $\pi(k) \in \mathcal{R}_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1+n_2+1, 0; 2n_1))$. So the generic nonvanishing theorem holds. Therefore,

$$\mathcal{Q}(2n_1; p, q; 2n_2)(\pi(k)) \neq 0.$$

and

$$\pi(k+1) = \theta_s(2n_1; p, q)(\pi(k)) \neq 0.$$

Q.E.D

Proof of Theorem 5.1: We prove it by induction. The existence of $\pi(1)$ is trivial. Suppose $\pi(k) \in \mathcal{N}(\mathcal{O}(k))$. If $G(k)$ orthogonal, from Theorem 3.3, $\mathcal{N}(\mathcal{O}(k+1))$ is nonempty. If $G(k)$ is metaplectic, from Lemma 6.1, $\pi(k+1)$ is not zero. Thus $\mathcal{N}(\mathcal{O}(k+1))$ is not empty. $\square$

Conjecture 4 *By allowing finite dimensional irreducible unitary representations in $\mathcal{N}(\mathcal{O}(1))$, for rigid $\mathcal{O} \in \mathcal{U}$, $\mathcal{N}(\mathcal{O})$ exhausts all irreducible unitary representations of $O(p,q)$ and $Mp_{2n}(\mathbb{R})$ with associated variety $cl(\mathcal{O})$.*

Notice that the rigid orbits in $\mathcal{U}$ include all special rigid orbits. For nonspecial rigid orbits of $O(p,q)$, by the works of Brylinski-Kostant ([BK]) and Huang-Li ([HL]), there might be unitary representations of the nontrivial covering of $SO_0(p,q)$ attached to them. Our construction does not cover their cases. Nevertheless, it is expected that the unipotent representations attached to nonspecial orbits must be representations of the nonsplitting covering of $SO_0(p,q)$. So for the linear groups, our construction should be exhaustive for rigid orbits. Of course, the problem of constructing unipotent representations attatched to nonspecial orbits remains open.

6.4 Perspectives

In this paper, we only treat $O(p,q)$ and $Mp_{2n}(\mathbb{R})$. Quantum induction for other classical group of type I can be defined similarly. Excluding the pair $(O(m,\mathbb{C}), Sp_{2n}(\mathbb{C}))$, the groups are connected and the metaplectic lift on these groups split. Thus the discussion on these groups should be considerably easier. We do not intend to work out the details here. Rather, I shall list some problems concerning quantum induction and unipotent representations for classical groups of type I. I shall also point out problems in connection with representation theory and the theory of automorphic forms. The main question is whether quantum induction can supplement parabolic induction (including complementary series) and cohomological induction to produce a complete classification of Π_u, Π_{auto} and Π_{rama}. In any case, this article should be viewed as a starting point for new development.

6.4.1 Wave Front Sets of $\mathcal{N}(\mathcal{O})$

Wave front sets under Howe's local theta correspondence are discussed in detail in [PR00]. There are still open questions which need to be answered. For the unipotent representations in $\mathcal{N}(\mathcal{O})$, we have showed that their associated varieties are the complexification of $cl(\mathcal{O})$. Now one may speculate that $cl(\mathcal{O})$ is the wave from set of $\pi \in \mathcal{N}(\mathcal{O})$. This is far from the case. In fact, there are irreducible representations in $\mathcal{N}(\mathcal{O})$ such that $WF(\pi) \neq cl(\mathcal{O})$ as demonstrated by Theorem 4.13. What should be true is that $\mathcal{O} \subseteq WF(\pi)$ geometrically. We fall short of proving it.

Conjecture 5 *For every $\pi \in \mathcal{N}(\mathcal{O})$, $\mathcal{O}$ can be embedded into $WF(\pi)$.*

We take special caution here since there are orbits that are geometrically identical but not algebraically identical.

Let me give some hint on how one may proceed. Przebinda started a program attempting to obtain the Harish-Chandra character of $\theta_s(\pi)$ from the Harish-Chandra character of π. If his program works in our situation, we will have a confirmation of this conjecture. Let me point out a critical step.

Conjecture 6 *Let $\pi \in \mathcal{R}_s(MG_1, MG_2)$. Then*

$$V(\theta_s(MG_1, MG_2)(\pi)) \cong [Hom_{\mathfrak{g}_1, MK_1}(\omega^c, \pi)]_{MK_2}.$$

Roughly speaking, this is saying that every MK_2-finite π^c-valued MG_1-equivariant distribution on ω can be obtained by integration over MG_1 as in [He00].

6.4.2 Unitary Dual, Automorphic Dual and Ramanujan Dual

Let G be a classical group of type I. Let $\Pi_{herm}(G)$ be the set of irreducible Harish-Chandra modules with an invariant Hermitian structure. Let $\Pi_{temp}(G)$ be the set of irreducible Harish-Chandra modules with almost L^2 matrix coefficients. Recall that

$$\Pi(G) \supset \Pi_{herm}(G) \supset \Pi_u(G) \supset \Pi_{temp}(G).$$

The classification of the admissible dual $\Pi(G)$ is due to Langlands. Langlands proved that every $\pi \in \Pi(G)$ occurs as the unique quotient of certain induced representation from a tempered representation. The quotients are often known as Langlands quotients. The classification of Π_{temp} is due to Knapp-Zuckerman. The classification of the Hermitian dual is also known.

With these classifications in hand, classification of unitary dual can be translated as the determination of unitarity of the Langlands quotients. But this approach is of great mathematical complexity. It often involves the positivity of certain analytically defined intertwining operator. In [VO86] and [VO87], Vogan envisioned a more geometric approach based on Kirillov-Kostant's orbit philosophy. We believe that quantum induction should be sufficient to supplement the existing techniques to produce a classification of the unitary dual for classical groups of type I.

Problem 1 *Can unitarity-preserving parabolic induction, cohomological induction and complementary series construction, combined with quantum induction, produces all irreducible unitary representations of classical groups of type I?*

Once this problem is solved, one can achieve the classification of the unitary dual by examining the Langlands-Vogan parameters of these representations ([LA], [Vogan79]).

Theta correspondences as formulated by Howe ([Ho79]) originated from the theory of theta series ([We65], [Si]). These ideas were further explored by Rallis, Li and many others ([Ra87], [Li94]). The upshot is that theta correspondences should produce automorphic representations of a bigger group from automorphic representations of a smaller group. Thus we are tempted to make the following conjecture.

Conjecture 7 *All representations in $\mathcal{N}(\mathcal{O})$ are automorphic.*

The evidence may also come from Arthur's conjectures. Based on the trace formula, Arthur's conjectures predict at least at the philosophical level which representations are automorphic unipotent representations. These "automorphic unipotent representations" are studied by Adams-Barbasch-Vogan from a representation theoretic viewpoint. Thus, if representations in $\mathcal{N}(\mathcal{O})$ agree with representations constructed in [ABV], very likely they will be automorphic.

In fact, we can even say more about $\mathcal{N}(\mathcal{O})$ and quantum induction. Recall that

$$\Pi_u(G) \supset \Pi_{auto}(G) \supset \Pi_{rama}(G).$$

We adopt the definitions from [BLS]. In principle, according to the conjectures of Ramanujan and Selberg ([Se], [Sa]), complementary series should not occur in $\Pi_{auto}(G)$ or $\Pi_{rama}(G)$. Assuming that,

Problem 2 *Can we construct every $\pi \in \Pi_{auto}(G)$ by unitary quantum induction, cohomological induction and parabolic induction?*

The argument that parabolic inductions should send Π_{auto} of a Levi subgroup to $\Pi_{auto}(G)$ was given in [BLS] with their H taking to be the parabolic subgroup rather than the Levi subgroup. Finally, for split classical groups, Barbasch showed that the irreducible spherical unitary representations can be constructed as a parabolically induced representation from a complementarily induced representation tensored with a special unipotent representation ([B01]). The Ramanujan dual is simply the intersection of the automorphic dual with the spherical dual. Thus one may ask

Problem 3 *For G a split classical group of type I, can one construct all $\pi \in \Pi_{rama}(G)$ by unitary quantum induction and parabolic induction?*

This exhaustion question is perhaps too difficult to answer. For $SL(2, \mathbb{R})$ where no quantum induction is involved, this is equivalent to the Ramanujan-Selberg conjecture.

Chapter 7

Appendix: Infinitesimal Characters for Mp

Recall that $\mathcal{I}_{\pm}(\mathbf{d})$ consists of an arithmetic sequence with multiplicities. For $\mathbf{d}$ odd, the sequence ends with $\frac{1}{2}$; for $\mathbf{d}$ even, the sequence ends with *zero* if $\mathbf{d} \neq [1]^{2n}$. In this chapter, we shall give a proof for Theorem 6.1. We start with a lemma.

Lemma 7.1 (Triviality)

1. *If $\overline{\lambda} \prec \overline{\mu}$ and $\overline{\lambda'} \preceq \overline{\mu'}$, then $\overline{\lambda} + \overline{\lambda'} \prec \overline{\mu} + \overline{\mu'}$.*
2. *If $\overline{\lambda} \prec \overline{\mu}$ and $\overline{\lambda'} \prec \overline{\mu'}$, then $\overline{(\lambda, \lambda')} \prec \overline{(\mu, \mu')}$.*
3. *If $\lambda_l \leq \mu_l$ for all l, then $\overline{\lambda} \preceq \overline{\mu}$.*
4. *If $\lambda_l < \mu_l$ for all l, then $\overline{\lambda} \prec \overline{\mu}$.*

Let $G = Sp_{2n}(\mathbb{R})$. Then

$$\rho(G) = (n, n-1, \ldots, 1) = \mathcal{I}_{-}(2n).$$

Lemma 7.2 *Suppose that $m \equiv r \pmod 2$. If $r \leq m$, then*

$$\overline{(\mathcal{I}_{-}(m), \mathcal{I}_{+}(r))} \preceq \frac{m}{m+r}\mathcal{I}_{-}(m+r).$$

Proof: Write

$$B = \overline{(\mathcal{I}_{-}(m), \mathcal{I}_{+}(r))}, \qquad A = \frac{m}{m+r}\mathcal{I}_{-}(m+r).$$

We prove our lemma by induction on $\frac{m-r}{2}$. If $\frac{m-r}{2} = 0$, $m = r$. Then

$$A = (\frac{m}{2}, \frac{m-1}{2}, \frac{m-2}{2}, \frac{m-3}{2}, \ldots, \frac{2}{2}, \frac{1}{2}),$$

$$B = (\frac{m}{2}, \frac{m-2}{2}, \frac{m-2}{2}, \frac{m-4}{2}, \frac{m-4}{2}, \ldots).$$

B ends with $(1,0)$ if m is even and $(\frac{1}{2},\frac{1}{2})$ if m is odd. Obviously, $A_k \geq B_k$. So $B \preceq A$. Assume that

$$\overline{(\mathcal{I}_-(m-2),\mathcal{I}_+(r))} \preceq \frac{m-2}{m-2+r}\mathcal{I}_-(m+r-2).$$

Notice that $\frac{m-2}{m-2+r} \leq \frac{m}{m+r}$. We have

$$\begin{aligned} B = \overline{(\mathcal{I}_-(m),\mathcal{I}_+(r))} &= \overline{(\frac{m}{2},\mathcal{I}_-(m-2),\mathcal{I}_+(r))} \\ &\preceq \overline{(\frac{m}{2},\frac{m-2}{m-2+r}\mathcal{I}_-(m+r-2))} \\ &\preceq \overline{(\frac{m}{m+r}\frac{m+r}{2},\frac{m}{m+r}\mathcal{I}_-(m+r-2))} \\ &= \frac{m}{m+r}\overline{\mathcal{I}_-(m+r)} = A. \end{aligned} \tag{7.1}$$

□

Lemma 7.3 *Consider a partition of* $2n$

$$\mathbf{d}^t = (\overbrace{m,m,m-2,m-2,\ldots,m-2j+2,m-2j+2}^{2j},m_0,r)$$

with $m \equiv m_0 \equiv r \pmod 2$ *and* $m-2j \geq m_0 \geq r \geq 0$. *Then*

$$\overline{\mathcal{I}_-(\mathbf{d})} \preceq \frac{m}{2n}\mathcal{I}_-(2n).$$

Proof: Clearly, $2n = m_0 + r + (2m-2j+2)j$. If $j=0$, $\mathbf{d}^t = (m_0,r)$. We have $\mathcal{I}_-(\mathbf{d}) \preceq \frac{m_0}{m_0+r}\mathcal{I}_-(m_0+r)$ by Lemma 7.2. Assume $j \geq 1$.

Use induction on m. When $m=2$, $B(2,0,0) = \mathcal{I}_-(2) \preceq I_-(2)$. So our lemma holds for $m=2$. Assume our lemma holds for $m-1$.

Suppose that $r \geq 1$. The case $r=0$ can be treated similarly. Obviously, $2n < 2m(j+1)$. So

$$m(2n-2j-2) = 2mn - m(2j+2) \leq 2nm - 2n = 2n(m-1).$$

It follows that $\frac{m}{2n} \leq \frac{m-1}{2n-2j-2}$. Consider

$$\mathbf{e}^t = (\overbrace{m-1,m-1,m-3,m-3,\ldots,m-2j+1,m-2j+1}^{2j},m_0-1,r-1).$$

Suppose that m is even. Then $\overline{\mathcal{I}_-(\mathbf{d})} = (\overline{\mathcal{I}_-(\mathbf{e})}+\mathbf{\frac{1}{2}},\overbrace{0,0,\ldots,0}^{j+1})$. By induction hypothesis,

$$\overline{\mathcal{I}_-(\mathbf{e})} \preceq \frac{m-1}{2n-2j-2}\mathcal{I}_-(2n-2j-2).$$

Then we have

$$
\begin{aligned}
\overline{\mathcal{I}_-(\mathbf{e})} + \mathbf{\frac{1}{2}} \preceq & \frac{m-1}{2n-2j-2}\mathcal{I}_-(2n-2j-2) + \mathbf{\frac{1}{2}} \\
= & \frac{\mathbf{m-1}}{\mathbf{2}} - \frac{m-1}{2n-2j-2}(0,1,\dots n-j-1) + \mathbf{\frac{1}{2}} \\
\preceq & \frac{\mathbf{m}}{\mathbf{2}} - \frac{m}{2n}(0,1,\dots,n-j-2) \\
= & \frac{m}{2n}\mathbf{n} - \frac{m}{2n}(0,1,\dots,n-j-2) \\
= & \frac{m}{2n}(\overbrace{n,n-1,\dots,j+2}^{n-j-1}).
\end{aligned}
\tag{7.2}
$$

We obtain $\overline{\mathcal{I}_-(\mathbf{d})} \preceq \frac{m}{2n}\mathcal{I}_-(2n)$.

Suppose m is odd. Then $\overline{\mathcal{I}_-(\mathbf{d})} = (\overline{\mathcal{I}_-(\mathbf{e})} + \mathbf{\frac{1}{2}}, \overbrace{\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}}^{j+1})$. We will again have

$$
\overline{\mathcal{I}_-(\mathbf{e})} + \mathbf{\frac{1}{2}} \preceq \frac{m}{2n}(\overbrace{n,n-1,\dots,j+2}^{n-j-1}).
$$

As l changes from $n-j$ to n, $\frac{m}{2n}\mathcal{I}_-(2n)_l - \frac{1}{2}$ decreases from positive to negative. So

$$
\frac{m}{2n}\sum_{i=1}^{l}\mathcal{I}_-(2n)_i - \sum_{i=1}^{l}\overline{\mathcal{I}_-(\mathbf{d})}_i
$$

increases and then decreases. It suffices to show that

$$
\frac{m}{2n}\sum_{i=1}^{n}\mathcal{I}_-(2n)_i - \sum_{i=1}^{n}\overline{\mathcal{I}_-(\mathbf{d})}_i \geq 0
$$

Notice that

$$
\sum_i (\mathcal{I}_-(m-2s))_i = \frac{(m-2s+1)^2}{8}, \qquad \sum_i (\mathcal{I}_+(m-2s))_i = \frac{(m-2s-1)^2}{8}
$$

We obtain

$$
\begin{aligned}
&\frac{m}{2n}\sum_{i=1}^{n}\mathcal{I}_-(2n)_i-\sum_{i=1}^{n}\overline{\mathcal{I}_-(\mathbf{d})}_i\\
=&\frac{m}{2n}\frac{(n+1)n}{2}-[\sum_{s=0}^{j-1}\frac{(m-2s+1)^2}{8}+\frac{(m-2s-1)^2}{8}]-\frac{(m_0+1)^2}{8}-\frac{(r-1)^2}{8}\\
=&\frac{1}{8}[m(2n+2)-(\sum_{s=0}^{j-1}2(m-2s)^2+2)-(m_0+1)^2-(r-1)^2]\\
=&\frac{1}{8}\{m(m_0+1+r-1+2+\sum_{s=0}^{j-1}2(m-2s))-(m_0+1)^2-(r-1)^2-2j-\sum_{s=0}^{j-1}2(m-2s)^2\}\\
=&\frac{1}{8}\{(m_0+1)(m-m_0-1)+(r-1)(m-r+1)+2(m-j)+2\sum_{s=0}^{j-1}(m-2s)(m-m+2s)\}\\
\geq&0
\end{aligned}
\tag{7.3}
$$

Therefore $\overline{\mathcal{I}_-(\mathbf{d})}\preceq\frac{m}{2n}\mathcal{I}_-(2n)$. By induction, $\overline{\mathcal{I}_-(\mathbf{d})}\preceq\frac{m}{2n}\mathcal{I}_-(2n)$ for all m. $\square$

Proof of Theorem 6.1: Fix n first. Let $\mathcal{O}_{\mathbf{d}}\in\mathcal{U}(Mp_{2n}(\mathbb{R}))$. Fix the number of row $m_1=m$. Consider

$$\mathbf{d}_0^t=(m,m,m-2,m-2,\dots,m-2j+2,m-2j+2,m_0,r)\qquad(j\geq 0)\tag{7.4}$$

with

$$0\leq r\leq m_0\leq m-2j-2,\qquad m\equiv m_0\equiv r\pmod 2.\tag{7.5}$$

Here $j=0$ means that $\mathbf{d}_0^t=(m_0,r)$. By Lemma 7.3,

$$\overline{\mathcal{I}_-(\mathbf{d}_0)}\preceq\frac{m}{2n}\mathcal{I}_-(2n).$$

Notice that $\mathcal{O}_{\mathbf{d}_0}$ is the minimal orbit in $\mathcal{U}(Mp_{2n}(\mathbb{R}))$ with a fixed m. Therefore, by Theorem 5.5, for any $\mathbf{d}$ with m rows, $\overline{\mathcal{I}_-(\mathbf{d})}\preceq\overline{\mathcal{I}_-(\mathbf{d}_0)}$. We obtain for any $\mathcal{O}_{\mathbf{d}}\in\mathcal{U}(Mp_{2n}(\mathbb{R}))$,

$$\overline{\mathcal{I}_-(\mathbf{d})}\preceq\frac{m_1}{2n}\rho(G)\prec\frac{m_1+2}{2n}\rho(G).$$

$\square$

Bibliography

[ABV] J. Adams, D. Barbasch, D. Vogan *The Langlands Classification and Irreducible Characters for Real Reductive Groups*, Progress in Mathematics, Birkhäuser, 1992.

[Ar] J. Arthur, "Unipotent Automorphic Representations: Conjectures "*Asterisque.*(v 171-172) 1989 (13-71).

[Ar83] J. Arthur, "On some Problems suggested by the trace formula ", *Lecture Notes in Mathematics, Vol 1041*, Springer, 1984, (1-50).

[Ar00] J. Arthur, "Harmonic Analysis and Group Representations ", *Notices of AMS*, Vol. 47, Number 1, (2000), 26-34.

[B01] Dan Barbasch, "Unitary Spherical Spectrum For Split Classical Groups ", preprint, 2001.

[B89] Dan Barbasch, "The unitary dual for complex classical groups " *Invent. Math. 89*, 1989, (103-176).

[BV80] Dan Barbasch, D. Vogan, "The Local Structure of Characters ", *Journal of Functional Analysis*, (37), 1980, (27-55).

[BV85] Dan Barbasch, David Vogan "Unipotent Representations of Complex Semisimple Groups"*Annals of Mathematics* (V. 121), 1985,(40-110).

[BZ] B. Binegar and R. Zierau, "Unitarization of a Singular Representation of $SO_e(p,q)$ ", *Commun. Math. Phys.* (Vol. 138), 1991, (245-258).

[BK] R. Brylinski and B. Kostant, "Minimal Representations, geometric quantization, and unitarity ", *Proc. Natl. Acad. Sci. USA* (Vol. 91), 1994, (6026-6029).

[BLS] M. Burger, J-S. Li, P. Sarnak, "Ramanujan Duals and Automorphic Spectrum ", *Bulletin of AMS*, (26), 1992, (253-257).

[CM] D. Collingwood, M. McGovern, *Nilpotent orbits on Semisimple Lie algebras*, Van Nostrand Reinhold (1994).

[DKP] A. Daszkiewicz, W. Kraskiewicz, Tomasz Przebinda, "Nilpotent Orbits and Complex Dual Pairs" *Journal of Algebra* (V. 190), 1997, (518-539).

[GV] W. Graham, D. Vogan "Geometric quantization for nilpotent coadjoint orbits ", *Geometry and representation theory of real and p-adic groups (Crdoba, 1995)*, Progr. Math., 158, Birkhuser Boston, Boston, MA, 1998, (69–137).

[HC] Harish-Chandra, "Invariant Eigendistributions on a semisimple Lie Group ", *Transactions of American Mathematical Society*, (V. 119), 1965, (457-508).

[Ho79] R. Howe, "θ-series and invariant theory" *Proc. Symp. Pure Math. 33*, AMS Providence, 1979 (275-285).

[Ho81] R. Howe "Wave front sets of representations of Lie groups "*Automorphic forms, representation theory and arithmetic* (Bombay, 1979), Tata Inst. Fundamental Res., Bombay, 1981 (117-140).

[Ho84] R. Howe, "Small Unitary Representations of Classical Groups "*Group representations, ergodic theory, operator algebras, and mathematical physics* (Berkeley, Calif., 1984), (121-150), Math. Sci. Res. Inst. Publ., 6, Springer, 1987.

[Ho89] R. Howe, "Transcending Classical Invariant Theory"*Journal of American Mathematical Society* (v2), 1989 (535-552).

[HT] R. Howe and Eng-Chye Tan, "Homogeneous Functions on Light Cones: The Infinitesimal Structure of Some Degenerate Principal Series Representations ", *Bull. Amer. Math. Soc.* (Vol. 28, Num. 1), 1993, (1-74).

[He99] Hongyu L. He "An Analytic Compactification of Symplectic Group"*Journal of Differential Geometry* (Vol.51) 1999, (375-399).

[He0] Hongyu He, *Compactification of Classical Groups*, *Communications in Analysis and Geometry*, 2002.

[He00] Hongyu He, "Theta Correspondence I-Semistable Range: Construction and Irreducibility ", *Communications in Contemporary Mathematics* (Vol 2), 2000, (255-283).

[Heu] Hongyu He, "Unitary Representations and Theta Correspondence for Classical Groups of Type I", to appear in *Journal of Functional Analysis*, 2002.

[He01] Hongyu He, "Nonvanishing of Certain Sesquilinear Form in Theta Correspondence ", *AMS Journal of Representation Theory*, 2001, (437-454).

[Heq] Hongyu He, "Theta Correspondence and Quantum Induction ", preprint.

[HL] Jing-Song Huang, Jian-Shu Li, "Unipotent Representations Attached to Spherical Nilpotent Orbits"*American Journal of Mathematics*(vol. 121), 1999, (497-517).

[KN] A. Knapp, *Representation Theory on Semisimple Groups: An Overview Based on Examples* Princeton University Press, 1986.

[KV] M. Kashiwara, M. Vergne, "On the Segal-Shale-Weil Representations and Harmonic Polynomials", *Invent. Math.* (44), 1978, (1-47).

[KR] Stephen Kudla and Stephen Rallis, "Degenerate Principal Series and Invariant Distribution ", *Israel Journal of Mathematics,* (Vol. 69, No. 1), 1990, (25-45).

[LA] R. Langlands, "On the Classification of Irreducible Representations of Real Algebraic Groups ", *Representation Theory and Harmonic Analysis on Semissimple Lie Groups,* Math. Surveys Monographs. 31, Amer. Math. Soc., Providence, RI 1989, (101-170).

[LZ] S. T. Lee, C-B. Zhu, "Degenerate Principal Series and Local Theta Correspondence II ", *Israel Journal of Mathematics, 100*, 1997, (29-59).

[LI89] J-S. Li, "Singular Unitary Representation of Classical Groups" *Inventiones Mathematicae* (V. 97),1989 (237-255).

[Li94] J-S. Li, "Singular Automorphic Forms ", *Proceedings of the International Congress of Mathematicians, Vol.1,2*, 1994, (790-799).

[Lus] G. Lusztig, "A Class of Irreducible Representations of a Weyl Group ", *Proc. Nederl. Akad. Series A*, Vol 82, 1979, (323-335).

[MI] D. Miličić, "Asymptotic Behaviour of Matrix Coefficients of The Discrete Series ", *Duke Mathematical Journal*, (44), 1977, (59-88).

[PAN] Shu-Yen Pan, "The Orbit Correspondences for Real and Complex Reductive Dual Pairs "(preprint).

[PR96] Tomasz Przebinda, "The Duality Correspondence of Infinitesimal Characters " *Colloq. Math.* (v 70), 1996 (93-102).

[PR93] Tomasz Przebinda, "Characters, Dual Pairs, and Unitary Representations" *Duke Journal of Mathematics* (V. 69), 1993 (547-592).

[PR00] T. Przebinda, "A Cauchy-Harish-Chandra Integral, for Real Reductive Dual Pair", *Invent. Math.*, (Vol. 141), 2000, (299-363).

[Ra87] S. Rallis, *L-functions and the oscillator representation*, LNM 1245, Springer-Verlag, 1987.

[Ro] W. Rossmann, "Picard-Lefschetz theory and characters of a semisimple Lie group, "*Invent. Math.*, (Vol. 121), 1995, (579-611).

[Sahi] S. Sahi, "Explicit Hilbert spaces for certain unipotent representations ", *Invent. Math.* , (Vol. 110), 1992, (409–418).

[SV] Salamanca-Riba, S. A., Vogan, David A., Jr., "On the classification of unitary representations of reductive Lie groups, "*Ann. of Math.* (No. 3) 148, 1998, (1067–1133).

[Sa] I. Satake, "Spherical Functions and Ramanujan Conjecture ", *Algebraic Groups and Discontinuous Subgroups*, (Proc. Sympos. Pure Math.,Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I.(258-264).

[Se] A. Selberg, "On the estimation of Fourier Coefficients of Modular Forms ", *Proc. Symposia in Pure Math. VIII*, A. M. S., Providence, 1965, (1-15).

[Si] C. L. Siegel, "Indefinite quadratische Formen und Funktionen Theorie I, II ", *Math. Ann.*, 124, (17-24, 364-387).

[SS] T. Springer, R. Steinberg, "Conjugacy Classes "in *Seminar on Algebraic Groups and Related Finite Groups*, Lecture Notes in Math. 131, Springer-Verlag, New York, 1970, (167-266).

[Vogan79] David Vogan, "The Algebraic Structure of the Representation of semisimple Lie groups " *Annals of Mathematics* (v 109), 1979, (1-60).

[VO87] D.Vogan, *Unitary Representations of Reductive Lie Groups*, Annals of Mathematics Studies, Princeton University Press, 1987.

[VO86] D. Vogan, "Representations of Reductive Lie Groups "*Proceedings of the International Congress of Mathematicians* Vol. 1, 2 (Berkeley, Calif.), 1986, (245-266).

[VO89] D. Vogan, "Associated Varieties and Unipotent Representations "in *Harmonic Analysis on Reductive Groups, Bowdoin College, 1989*, Progress in Mathematics, 101, Birkhäuser, Boston, 1991, (315-388).

[Vo86] D. Vogan, "The Unitary Dual of $GL(n)$ over an Archimedean field "*Invent. Math. 83*, 1986, (449-505).

[VO94] D. Vogan, "The Orbit Method and Unitary Representations for Reductive Lie Groups "*Algebraic and Analytic Methods in Representation Theory*, 1994, (243-339).

[VO00] D. Vogan, "A Langlands classification for unitary representations ", *Analysis on homogeneous spaces and representation theory of Lie groups*, Okayama–Kyoto 2000, (299-324).

[VO01] D. Vogan, email communication, Oct 5, (2001).

[WE65] A. Weil, "Sur certains groupes d'opérateurs unitaries ", *Acta Math.*, (113), 1965, (143-211).

[We65] A. Weil, "Sur la formule de Siegel dans la théorie des groupes classiques ", *Acta Math.*, (113), 1965, (1-87).

[Wallach] N. Wallach, *Real Reductive Groups: I, II* Academic Press, 1992.

[ZH] Chen-Bo Zhu, Huang-Jing Song, "On Certain Small Representations of Indefinite Orthogonal Groups", *Representation Theory* (Vol. 1), 1997, (190-206).