$\mathbb{H} = \mathbb{C}[S_2] \otimes Sym(x_1, x_2) \otimes Cliff(x_1, x_2)$ subject to (1)

$$(12)\cdot x_1 - x_2 \cdot (12) = -1 + c_1 c_2 \qquad c_i x_i = -x_i c_i$$
$$x_1 \cdot (12) - (12) x_2 = -1 - c_1 c_2 \qquad w c_i = c_{w(i)} w$$

GL(2): Dirac for Hecke-Clifford algebra

*-operator:

$$*(w) = w^{-1}$$
$$*(c_i) = -c_i \qquad \text{eg. } *(c_i c_j) = -c_i c_j.$$
$$*(x_i) = -w_0 (w_0 x_i) w_0.$$

Compute:

$$\begin{aligned} *(x_1) &= -(12) x_2 (12) \\ &= [-1 - c_1 c_2 - x_1 (12)](12) = \\ &= -x_1 - (1 + c_1 c_2)(12) \end{aligned}$$

$$\begin{aligned} *(x_2) &= -(12) x_1 (12) \\ &= -(-1 + c_1 c_2 + x_2 (12))(12) \\ &= -x_2 + (1 - c_1 c_2)(12) \end{aligned}$$

Define:

$$\tilde{x}_1 := x_1 + \tfrac{1}{2}(1 + c_1 c_2)(12).$$

Then

$$\begin{aligned} *(\tilde{x}_1) &= *(x_1) + \tfrac{1}{2} *(12) *(1 + c_1 c_2) \\ &= -x_1 - (1 + c_1 c_2)(12) + \tfrac{1}{2}(12)(1 - c_1 c_2) \\ &= -x_1 - (1 + c_1 c_2)(12) + \tfrac{1}{2}(1 + c_1 c_2)(12) \\ &= -x_1 - \tfrac{1}{2}(1 + c_1 c_2)(12) \\ &= -\tilde{x}_1 \end{aligned}$$

Similarly define:

$$\tilde{x}_2 := x_2 - \tfrac{1}{2}(1 - c_1 c_2)(12)$$

Then

$$\begin{aligned} *(\tilde{x}_2) &= -x_2 + (1 - c_1 c_2)(12) - \tfrac{1}{2}(1 - c_1 c_2)(12) \\ &= -x_2 + \tfrac{1}{2}(1 - c_1 c_2)(12) = \\ &= -\tilde{x}_2. \end{aligned}$$

Candidate Dirac ~~operator~~ element:

$$D = \tilde{X}_1 + \tilde{X}_2 \in \mathbb{H}.$$

$$D^2 = (\tilde{X}_1 + \tilde{X}_2)^2 = \tilde{X}_1^2 + \tilde{X}_2^2 + \tilde{X}_1\tilde{X}_2 + \tilde{X}_2\tilde{X}_1.$$

$$\tilde{X}_1^2 = \left[x_1 + \tfrac{1}{2}(1+c_1c_2)(12)\right]^2$$

$$= x_1^2 + \tfrac{1}{2}(1+c_1c_2)(12)x_1 + \tfrac{1}{2}x_1(1+c_1c_2)(12) + \tfrac{1}{2}.$$

$$= x_1^2 + \tfrac{1}{2}(1+c_1c_2)(12)x_1 + \tfrac{1}{2}(1-c_1c_2)x_1(12) + \tfrac{1}{2}.$$

$$\tilde{X}_2^2 = \left[x_2 - \tfrac{1}{2}(1-c_1c_2)(12)\right]^2$$

$$= x_2^2 - \tfrac{1}{2}x_2(1-c_1c_2)(12) - \tfrac{1}{2}(1-c_1c_2)(12)x_2 + \tfrac{1}{2}$$

$$= x_2^2 - \tfrac{1}{2}(1+c_1c_2)x_2(12) - \tfrac{1}{2}(1-c_1c_2)(12)x_2 + \tfrac{1}{2}.$$

$$\tilde{X}_1\tilde{X}_2 = \left(x_1 + \tfrac{1}{2}(1+c_1c_2)(12)\right)\left(x_2 - \tfrac{1}{2}(1-c_1c_2)(12)\right)$$

$$= x_1x_2 + \tfrac{1}{2}(1+c_1c_2)(12)x_2 - \tfrac{1}{2}(1+c_1c_2)x_1(12) + \tfrac{1}{2}c_1c_2$$

$$\tilde{X}_2\tilde{X}_1 = x_2x_1 - \tfrac{1}{2}(1-c_1c_2)(12)x_1 + \tfrac{1}{2}(1-c_1c_2)x_2(12) + \tfrac{1}{2}c_1c_2.$$

Altogether: $(x_1+x_2)^2 + c_1c_2(12)x_1 - c_1c_2x_1(12) - c_1c_2x_2(12) + c_1c_2(12)x_2$

$$= (x_1+x_2)^2 + c_1c_2\left[(12)x_1 - x_2(12)\right] - c_1c_2\left[x_1(12) - (12)x_2\right].$$

$$= (x_1+x_2)^2 + c_1c_2(-1+c_1c_2) - c_1c_2(-1-c_1c_2)$$

$$= (x_1+x_2)^2 - c_1c_2 + c_1c_2 + 2c_1c_2c_1c_2$$

$$= (x_1+x_2)^2 - 2.$$

$$\boxed{D^2 = (x_1+x_2)^2 - 2}$$

GL(2) revisited:

ker D is thus a module for ~~the~~ Sergeev algebra.

$$\left[\begin{array}{l} D \text{ commutes w/ } \mathbb{C} \\ Dc_1 = -c_1 D \\ Dc_2 = -c_2 D. \end{array}\right.$$

$$\Uparrow$$

$$\left.\begin{array}{l} (12)\tilde{x}_1 = \tilde{x}_2 (12) \\ (12)\tilde{x}_2 = \tilde{x}_1 (12) \end{array}\right]$$

$$D = \tilde{x}_1 c_1 + \tilde{x}_2 c_2 .$$

N.B.: $\tilde{x}_i c_i = -c_i x_i$

$\tilde{x}_i c_j = c_j \tilde{x}_i \quad i \neq j.$

$$D^2 = \tilde{x}_1 c_1 \tilde{x}_1 c_1 + \tilde{x}_2 c_2 \tilde{x}_2 c_2 + \tilde{x}_1 c_1 \hat{x}_2 c_2 + \hat{x}_2 c_2 \tilde{x}_1 c_1$$

$$= \tilde{x}_1^2 + \tilde{x}_2^2 + [\tilde{x}_1 \tilde{x}_2] c_1 c_2$$

$$\tilde{x}_1^2 = x_1^2 + \tfrac{1}{2}(1-c_1c_2)x_1(12) + \tfrac{1}{2}(1+c_1c_2)(12)x_1 + \tfrac{1}{2}$$

$$\tilde{x}_2^2 = x_2^2 - \tfrac{1}{2}(1-c_1c_2)(12)x_2 - \tfrac{1}{2}(1+c_1c_2)x_2(12) + \tfrac{1}{2} .$$

$$\tilde{x}_1 \tilde{x}_2 c_1 c_2 = [x_1 x_2 + \tfrac{1}{2}(1+c_1c_2)(12)x_2 - \tfrac{1}{2}(1+c_1c_2)x_1(12) - \tfrac{1}{4}(1+c_1c_2)$$

$$-\tilde{x}_2 \tilde{x}_1 c_1 c_2 = -[x_2 x_1 - \tfrac{1}{2}(1-c_1c_2)(12)x_1 + \tfrac{1}{2}(1-c_1c_2)x_2(12) - \tfrac{1}{4}(1-c_1c_2)^2$$

$$= x_1^2 + x_2^2 + \underbrace{\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}}_{\text{constant terms}} + (1-c_1c_2)[x_1(12) - (12)x_2]$$

$$+ (1+c_1c_2)[12(x_1) - x_2(12)]$$

$$= x_1^2 + x_2^2 + 2 - (1-c_1c_2)(1+c_1c_2) + (1-c_1c_2)(1+c_1c_2)$$

$$= x_1^2 + x_2^2 + 2. \qquad = \Omega_H + \rho(\Omega_{\tilde{w}}) \quad \text{up to scale.}$$