For convenience we have omitted the superscripts λ_- and λ^- in the equations above. Now if we let

$$W=\begin{pmatrix} W & U \\ 0 & W \end{pmatrix},$$

we can write

$$AW=\begin{pmatrix} AW & AU \\ 0 & AW \end{pmatrix}, \qquad WB=\begin{pmatrix} WB & UB \\ 0 & WB \end{pmatrix}.$$

We must show that these matrices are equal. Since the diagonal blocks in AW and WB are equal by induction, this reduces to showing that the upper right blocks are equal. The most direct way to do this is to decompose all our matrices one step further:

$$A=\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}, \qquad W=\begin{pmatrix} W & U & 0 & 0 \\ 0 & W & W & U \\ 0 & 0 & W & U \\ 0 & 0 & 0 & W \end{pmatrix},$$

$$B=\begin{pmatrix} B & J & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & J \\ 0 & 0 & 0 & B \end{pmatrix}.$$

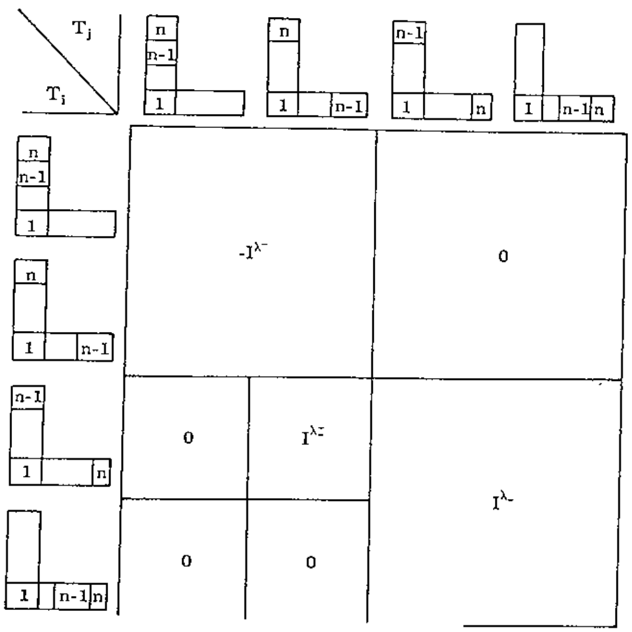


FIG. 7. Schematic diagram for computing $B^{\lambda}(n-1, n)$.

Here the matrix J (for "junk") is zero unless $k=n-2$, in which case it contains some non-zero elements whose nature need not concern us. In terms of these blocks we can then write

$$AW=\begin{pmatrix} AW & AU & 0 & 0 \\ 0 & AW & AW & AU \\ 0 & 0 & AW & AU \\ 0 & 0 & 0 & AW \end{pmatrix},$$

$$WB=\begin{pmatrix} WB & WJ+UB & 0 & 0 \\ 0 & WB & WB & WJ+UB \\ 0 & 0 & WB & WJ+UB \\ 0 & 0 & 0 & WB \end{pmatrix}.$$

The two non-zero blocks in the upper right corners of these matrices are equal to the blocks immediately below them. Since these blocks lie in the portions of the two matrices which have already been shown to be equal, it follows at once that the upper right corners of the two matrices are equal as well, so that $AW=WB$.

Case 2. $k=n-1$. Figures 2 and 7 then imply that

$$A=\begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \qquad W=\begin{pmatrix} W & U & 0 & 0 \\ 0 & W & W & U \\ 0 & 0 & W & U \\ 0 & 0 & 0 & W \end{pmatrix},$$

$$B=\begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & I & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Direct multiplication now gives

$$AW=\begin{pmatrix} -W & -U & 0 & 0 \\ 0 & W & W & U \\ 0 & W & W & U \\ 0 & 0 & 0 & 0 \end{pmatrix}=WB.$$

Case 3. $k=n-2$. This is just like Case 2 except that the matrices must be expanded slightly farther:

$$A=\begin{bmatrix} -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$W=\begin{bmatrix} W & U & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & W & W & U & 0 & 0 & 0 & 0 \\ 0 & 0 & W & U & W & U & 0 & 0 \\ 0 & 0 & 0 & W & 0 & W & W & U \\ 0 & 0 & 0 & 0 & W & U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W & W & U \\ 0 & 0 & 0 & 0 & 0 & 0 & W & U \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & W \end{bmatrix},$$

$$B=\begin{bmatrix} -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & I & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Again, direct multiplication yields

$$AW=\begin{bmatrix} -W & -U & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & W & U & W & U & 0 & 0 \\ 0 & W & W & U & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W & 0 & W & W & U \\ 0 & 0 & 0 & 0 & -W & -U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W & U \\ 0 & 0 & 0 & 0 & 0 & W & W & U \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & W \end{bmatrix}=WB.$$

Case 4. $k=1$. Referring to Figs. 3 and 5, and letting $N={}^tM$, we get the decompositions

$$A=\begin{pmatrix} A & N \\ 0 & A \end{pmatrix},\qquad W=\begin{pmatrix} W & U \\ 0 & W \end{pmatrix},\qquad B=\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

As a consequence,

$$AW=\begin{pmatrix} AW & AU+NW \\ 0 & AW \end{pmatrix}\qquad\text{and}\qquad WB=\begin{pmatrix} WB & UB \\ 0 & WB \end{pmatrix}.$$

The diagonal blocks of AW and WB are equal by induction. All that remains is to show that $AU+NW=UB$. As usual, this is done inductively by block decomposing all the matrices involved. Figures 3, 4, and 5 tell us that

$$A=\begin{pmatrix} A & N \\ 0 & A \end{pmatrix},\qquad N=\begin{pmatrix} -N & 0 \\ & 0 & N \end{pmatrix},\qquad B=\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},$$

and

$$U=\begin{pmatrix} 0 & 0 \\ W & U \end{pmatrix},\qquad W=\begin{pmatrix} W & U \\ 0 & W \end{pmatrix}.$$

We therefore have

$$UB=\begin{pmatrix} 0 & 0 \\ WB & UB \end{pmatrix}$$

and

$$AU+NW=\begin{pmatrix} 0 & 0 \\ AW & AU+NW \end{pmatrix},$$

so that $AU+NW=UB$ by induction.

This completes the fourth case, and with it the proof of the theorem. It is easy to see that the validity of the theorem for the pure row and column shapes $\lambda=n$ and $\lambda=1^n$, combined, if you like, with the explicit calculations in Section 3 for all $n\leqslant 7$, provides a basis for all these inductions.

Although the recurrences in Theorem 4.1 describe the transforming matrices rather neatly, one cannot help but feel that the proof we have given is not exactly illuminating. To generalize it to arbitrary shapes also does not seem too promising. We therefore turn in Section 5 to describing the K–L representation and the transforming matrices in more algebraic terms. This description leads not only to a proof of the upper triangularity of the transforming matrices for arbitrary shapes, but also to a new set of recurrences describing them.

5. Triangularity

The developments in this section depend crucially on Facts 8, 9, and 10. Unfortunately, the derivation of these facts from the few terse paragraphs dedicated to them in [13] is not an effortless matter. Fact 10 (the equality of Young's and K–L character labeling), for instance, is not even stated there, although readers familiar with the subject might suspect this particular result to be true.

We have personally benefited considerably from a sequence of seminar lectures given by A. Björner, who systematically filled in the combinatorics pertaining to the $W = S_n$ case of the Kazhdan–Lusztig work. Notes from these seminars are in preparation [1]. The readers should find there a more leisurely and detailed derivation of all the material needed here.

Nevertheless, pending the appearance of these notes, the reader is left with quite a bit of work to do for a thorough understanding of the present work. This given, it will be good to give here at least brief outlines of proofs. Before we can proceed, however, we need to go over a few more properties of the Robinson–Schensted correspondence.

Let $w = w_1 w_2 \cdots w_n$ be a permutation and let for a moment P_k be the left tableau resulting from the row insertion of $w_1 w_2 \cdots w_k$. Set

$$w^{(k)} = w(P_k)\, w_{k+1} \cdots w_n$$

In other words $w^{(k)}$ is the concatenation of the word of P_k with what is left of w. It is well known that the step

$$w^{(k)} \to w^{(k+1)}$$

can be carried out by a sequence of Knuth transformations [16, 22]. Thus assuming that

$$(\varnothing \leftarrow w) = (P, Q), \tag{5.1}$$

and that P and Q have shape I, we shall always have a sequence of indices $j_1, j_2, \ldots, j_s$ such that

$$w(P) = (P, R_I) = wR_{j_1} R_{j_2} \cdots R_{j_s}. \tag{5.2}$$

It will be convenient to identify a permutation with the pair of tableaux obtained by row insertion. Thus we shall write $w = (P, Q)$ instead of (5.1) or even write (P, Q) for w. We see then from (5.2) that given any triplet P, Q, Q' of standard tableaux of the same shape, we can always find a sequence of Knuth transformations connecting (P, Q) to (P, Q').

Let us recall that the *column superstandard* tableau of shape I, denoted by CS_I, is the tableau obtained by filling I with $1, 2, \ldots, n$ column by column from bottom to top and from left to right. A sequence $R_{j_1} R_{j_2} \cdots R_{j_m}$ giving

$$(P, CS_I) = (P, Q)\, R_{j_1} R_{j_2} \cdots R_{j_m} \tag{5.3}$$

will be referred to as *canonical for* (P, Q).

The tableau CS_I is of particular interest to us here since it has a number of peculiar properties. We shall state them in the form of propositions.

Proposition 5.1. *If T is any standard tableau of shape J and*

$$D(T) \supseteq D(CS_I) \tag{5.4}$$

then we can have only one of the following two possibilities:

(i) *$T = CS_I$ and of course $J = I$, or*

(ii) *$T \neq CS_I$ but then J is strictly below I in the dominance order.*

Proof. Let $c_1, c_2, \ldots, c_k$ be the lengths of the successive columns of I. It develops that the tableaux T of shape J satisfying the requirement in (5.4) are in bijection with the *row strict* tableaux of shape J and content $1^{c_1}2^{c_2} \cdots k^{c_k}$. These are the tableaux obtained by filling the shape J with c_1 ones, c_2 twos, etc., in succession so that no two equal letters fall in the same row. In fact, given any of the latter tableaux we can obtain one of our desired T's by simply replacing the 1's from bottom to top, then the 2's in the same manner, etc., by the successive letters $1, 2, \ldots, n$. Now it is easy to see that if $J = I$, the resulting row strict tableau is simply the one obtained by filling the ith column of I with i's whose bijective image is CS_I itself. On the other hand any other of our row strict tableaux with the desired property must have a shape J with enough room in its first i columns to contain the $c_1 + c_2 + \cdots + c_i$ letters less or equal to i. This implies that J must be dominated by I, which is our desired conclusion.

This proof establishes also another important fact for us here, namely:

Proposition 5.2. *The number of standard tableaux T of shape J satisfying the condition*

$$D(T) \supseteq D(CS_I)$$

is equal to the Kotska number

$$K_{I,J}.$$

Here the symbol "~" refers to conjugation. The definition of Kotsh numbers may be found in [20].

Given a tableau T the word obtained by reading the columns of T from top to bottom starting from the first column and proceeding to the right will be denoted by $cw(T)$ and referred to as the *column word* of T. In the figure below we illustrate the column superstandard tableau of shape (2, 2, 3), a standard tableau of the same shape, and its corresponding column word.

$$CS_{(2,3,3)} = \begin{matrix} 3 & 6 & \\ 2 & 5 & \\ 1 & 4 & 7 \end{matrix} \qquad T = \begin{matrix} 5 & 7 & \\ 3 & 6 & \\ 1 & 2 & 4 \end{matrix} \qquad \mathrm{cw}(T) = 5\ 3\ 1\ 7\ 6\ 2\ 4$$

The reader is invited to check that

$$(\varnothing \leftarrow 5\ 3\ 1\ 7\ 6\ 2\ 4) = \left(\begin{matrix} 5 & 7 & \\ 3 & 6 & \\ 1 & 2 & 4 \end{matrix}\ ,\ \begin{matrix} 3 & 6 & \\ 2 & 5 & \\ 1 & 4 & 7 \end{matrix} \right).$$

It should be clear that this is a general fact. That is, for any standard tableau T of shape I we have

$$(\varnothing \leftarrow \mathrm{cw}(T)) = (T, CS_I). \tag{5.5}$$

We are now a position to outline a proof of Facts 8, 9, and 10. It develops that the following Proposition simplifies considerably the derivation of Facts 8 and 9. The reader interested in the combinatorics of the Robinson–Schensted correspondence should find it a very welcome addition to the early tools introduced in the subject.

Note that the classes L_i and R_j can be defined entirely in terms of the left (resp. *right*) tableau of a permutation. More precisely, we see that L_i consists of the permutations whose left descent set contains one and only one of the two indices i, $i+1$. In other words we have

$$L_i = \{w = (P, Q) \colon |D(P) \cap \{i, i+1\}| = 1\}. \tag{5.6}$$

Similarly we see that

$$R_j = \{w = (P, Q) \colon |D(Q) \cap \{j, j+1\}| = 1\}. \tag{5.7}$$

Now, it is well known (see [16]) that the map R_j preserves the left tableau. From Schützenberger's result (Eq. (2.20)) and the identity

$$(wR_j)^{-1} = L_j w^{-1}$$

we can easily derive that likewise the map L_i preserves the right tableau. This means that both L_i and R_j map the set $L_i \cap R_j$ into itself. In fact, they

are both involutions on this set. The remarkable fact is that they commute. More precisely we have

PROPOSITION 5.3. *For any i, j and any permutation $w \in L_i \cap R_j$ we have*

$$L_i(wR_j) = (L_i w) R_j. \tag{5.8}$$

Proof. It is convenient to give separate arguments according to the cardinality of the set of entries of w that are affected by these two transformations. To this end, using the same notation as in Section 2, let us put

$$k = |\{i, i+1, i+2\} \cup \{a, b, c\}|.$$

When $k=6$ the result is completely trivial since L_i and R_j acts on different sets of entries and (5.1) is simply a consequence of associativity of permutation multiplication. The case $k=5$ is again a consequence of associativity as the reader can easily discover by simple experimentation. The main reason is that, because the indices i, $i+1$, $i+2$ are consecutive, even if one of them is equal to a, b, or c, the action of L_i does not alter the action of R_j. More precisely, the simple transpositions expressing these actions by left (resp. right) multiplication remain the same.

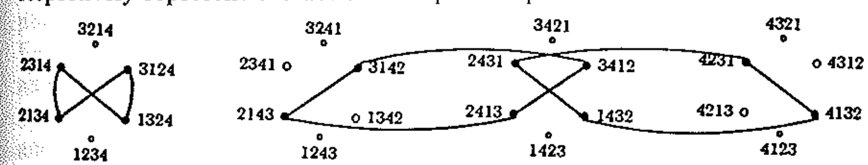
The case $k \leqslant 4$ is a little less obvious. We can shorten the argument considerably by making a few preliminary observations. First note that, since at most four entries are involved in this case, we can delete all the other entries and reduce w to a four-element permutation. We thus need only verify (5.8) in S_4. Second, because of the identities

$$\begin{cases} (1)\ w_0(wR_j) = (w_0 w) R_j \\ (2)\ (wR_j)\, w_0 = (ww_0) R_{n-j-1} \end{cases} \qquad \begin{cases} (1)\ w_0(L_i w) = L_{n-i-1}(w_0 w) \\ (2)\ (L_i w)\, w_0 = L_i(ww_0) \end{cases}$$

we can immediately see that once the identity (5.8) has been established for all i, j for a permutation w it will automatically hold for $w_0 w$, ww_0, and $w_0 w w_0$ as well (where w_0 is given by (2.14)). This observation permits us to assure that

$$\{i, i+1, i+2\} = \{j, j+1, j+2\} = \{1, 2, 3\}.$$

In the figure below we display the permutations of S_4 grouped according to the left cosets of the subgroup generated by (1, 2), (2, 3). Solid circles indicate the elements of $L_1 \cap R_1$. The straight and curved lines are to respectively represent the action of L_1 and R_1 on this set.



We see clearly from this picture that successive applications of L_1, R_1, L_1, R_1 always loop back to the point of departure. But this is precisely the result we want to establish.

This proposition can be translated into a beautiful result which entirely reveals the nature of the transformations L_i and R_j. To this end let us define a transformation acting directly on standard tableaux by setting

$$m_i T = \begin{cases} (i, i+1)\, T & \text{if } i+2 \text{ separates } i, i+1 \text{ in } \mathrm{cw}(T), \\ (i+1, i+2)\, T & \text{if } i \text{ separates } i+1, i+2 \text{ in } \mathrm{cw}(T). \end{cases} \tag{5.9}$$

We see that m_i is an involution on the class

$$T_i = \{T: |D(T) \cap \{i, i+1\}| = 1\}.$$

This given we have

THEOREM 5.1. *If $w = (P, Q)$ then*

$$\begin{aligned} &\text{(a)} \quad (\textit{when } w \in L_i)\ L_i w = (m_i P, Q) \\ &\text{(b)} \quad (\textit{when } w \in R_j)\ w R_j = (P, m_j Q). \end{aligned} \tag{5.10}$$

Proof. Let $w \in L_i$ and let P and Q be of shape I. Note that we have defined m_i in order to have

$$L_i \mathrm{cw}(P) = (m_i P, CS_I). \tag{5.11}$$

Let us now take a canonical sequence for (P, Q) giving

$$\mathrm{cw}(P) = w R_{j_1} R_{j_2} \cdots R_{j_m}.$$

Applying our commutativity result we then get

$$L_i \mathrm{cw}(P) = L_i(w R_{j_1} R_{j_2} \cdots R_{j_m}) = (L_i w)\, R_{j_1} R_{j_2} \cdots R_{j_m}.$$

Since Knuth transformations preserve the left tableau, we must conclude that the left tableaux of $L_i \mathrm{cw}(P)$ and $L_i w$ are the same. This proves (5.10a).

To prove (5.10b) we again use the relation

$$(w R_j)^{-1} = L_j w^{-1}$$

and get from (5.10a)

$$w R_j = [L_j w^{-1}]^{-1} = [(m_j Q, P)]^{-1} = (P, m_j Q).$$

This completes our proof.

Now let $w_1 = (P_1, Q)$ and $w_2 = (P_2, Q)$ be two permutations of shape I and in the same dual Knuth cell, and let Q' be any standard tableau of shape I. As observed above, we can find a sequence $R_{j_1} R_{j_2} \cdots R_{j_k}$ giving

$$(P_1, Q') = (P_1, Q)\, R_{j_1} R_{j_2} \cdots R_{j_k}.$$

From Theorem 5.1 we then deduce that

$$Q' = m_{j_k} \cdots m_{j_2} m_{j_1} Q,$$

which in turn (again from Theorem 5.1) implies that

$$(P_2, Q') = (P_2, Q)\, R_{j_1} R_{j_2} \cdots R_{j_k}.$$

This means that the change of label map corresponding to the replacement $Q \to Q'$ can be carried out by a sequence of elementary change of label maps of the form

$$(P, Q) \to (P, m_j Q).$$

In particular we see that the equality

$$\mu[x, y] = \mu[x', y'],$$

when x, y and x', y' correspond under a change of label map, is a consequence of the equality

$$\mu[x, y] = \mu[x R_j, y R_j],$$

In other words Theorem 5.1 reduces Fact 9 to Fact 11.

Remark 1.1. Fact 8 can also be reduced to Fact 11 in a similar manner. Some of the algebraic steps needed in this reduction, including a very leisurely proof of Fact 11, can be found in lecture notes by R. King [15]. The combinatorial steps were first written up by A. Björner. Note that Theorem 5.1 essentially says that the effect on the left (resp. right) tableau of the transformation L_i (resp. R_j) does not depend on the right (resp. left) tableau. The fact that the Kazhdan–Lusztig paper leads to this remarkable property of the Robinson–Schensted correspondence was first pointed out by Björner in the above-mentioned writeup. Our contribution here is a precise formulation of Proposition 5.1 and the reduction of the proof of Fact 8 to Theorem 5.1 and Proposition 5.1. We give here only our steps and refer the reader to [1, 15] for further details.

Let $w_1 = (P_1, Q_1)$ and $w_2 = (P_2, Q_2)$ be two permutations of shapes I and J. We are to show that if w_1 and w_2 are L-equivalent (as defined in

Section 2) then $I=J$ and $Q_1=Q_2$. To this end the first step is to show that if $x, y \in R_j$ then

$$x=L \Rightarrow y \rightarrow xR_j=L \Rightarrow yR_j.$$

The proof of this can be found in [13, 1, 15]. It is a consequence of Fact 11.

From this it is not difficult to derive that if x, y are L-equivalent and $x \in R_j$ then xR_j, yR_j are also L-equivalent. The fact that $x \in R_j \rightarrow y \in R_j$ is a consequence of Fact 7. This given, let $R_{j_1} R_{j_2} \cdots R_{j_k}$ be a canonical sequence for w_1. This gives

$$(P_1, CS_I)=(P_1, Q_1)\, R_{j_1} R_{j_2} \cdots R_{j_k}$$

and by Theorem 5.1

$$CS_I=m_{j_k} \cdots m_{j_2} m_{j_1} Q_1$$

or equivalently

$$Q_1=m_{j_1} m_{j_2} \cdots m_{j_k} CS_I.$$

From the remarks above, and Theorem 5.1 again, it then follows that (P_1, CS_I) and $(P_2, m_1 m_2 \cdots m_k Q_2)$ are also L-equivalent. But as observed in Section 2, this implies that these two permutations have the same right descent set. This means that

$$D(CS_I)=D(m_1 m_2 \cdots m_k Q_2).$$

However, Proposition 5.1 now tells us either that $I=J$ and

$$m_{j_1} m_{j_2} \cdots m_{j_k} Q_2=CS_I$$

and thus

$$Q_2=m_{j_k} \cdots m_{j_2} m_{j_1} CS_I=Q_1 \tag{5.12}$$

or that I strictly dominates J. Since we have assumed nothing special about our labeling of w_1 and w_2, the denial of the first alternative, and reversal of the roles of w_1, w_2, would also give that J strictly dominates I, which is absurd. Thus (5.12) is the only valid alternative and Fact 8 must necessarily hold true as asserted.

To establish Fact 10 and the triangularity result we need some preliminary remarks about the element c_w defined by (2.11).

Let I be a set of simple transpositions and let W_I be the subgroup of W generated by the elements of I. Set

$$V_I=\{v\colon vs>v \text{ for all } s \in I\}.$$

It is well known (see [2] or [23]) that the elements of V_I are the minimal elements of their left W_I-cosets. They may be taken as coset representatives, that is, every element $w \in W$ has a unique factorization

(i) $w=vu$ with $v \in V_I$ and $u \in W_I$.

Moreover, if we denote these factors by $v_I(w)$ and $u_I(w)$, respectively, we have

(ii) $l(w)=l(v_I(w))+l(u_I(w))$,
(iii) $w_1 \leqslant w_2 \rightarrow v_I(w_1) \leqslant v_I(w_2)$.

Now we let $w \in W$ be given and set

$$I=\{s \in S\colon ws<w\}=\{s_i\colon i \in D_R(w)\}. \tag{5.13}$$

It develops from properties (i) and (iii) that the set of elements below w decomposes into the product of the portion of V_I below w by all of W_I. More precisely we have

Proposition 5.4. *If $w \in W$ and I is as in (5.13) then an element x is below w if and only if it can be written in the form*

$$x=vu \quad \text{with} \quad v \in V_I,\ v \leqslant v_I(w) \quad \text{and} \quad u \in W_I. \tag{5.14}$$

Moreover this decomposition is unique.

Proof. The *only if* part is immediate from (i) and (iii) and we know that uniqueness holds for every element of W. To prove the *if* part let

$$x=vu \quad \text{with} \quad u \in W_I,\ v \in V_I \quad \text{and} \quad v \leqslant w$$

and let $u=a_1 a_2 \cdots a_k$ be a reduced decomposition of u with $a_i \in I$ for $i=1, 2, ..., k$. Set

$$x_0=v \quad \text{and} \quad x_i=va_1 \cdots a_i.$$

Now since $a_1 \cdots a_i$ is also reduced, uniqueness of factorization and property (ii) give that

$$l(x_i)=l(v)+i.$$

Thus $x_{i+1}=x_i a_{i+1}>x_i$. Now we see that, since $wa_{i+1}<w$, then $x_i<w$ and the ZZP imply that $x_{i+1} \leqslant w$. Note that we cannot have $x_{i+1}=w$ except maybe when $i+1=k$, for w is always brought down by any of the a_i's. Thus since $x_0=v<w$, all the x_i's must be below w. In particular the same must be true for $x_k=x$ itself. This completes the proof.

Note that this proposition implies that $u_I(w)$ is the top element of W_I. The immediate corollary of this proposition is a factorization of every element c_w.

PROPOSITION 5.5. *Let $w \in W$ and I be as in* (5.13), *then*

$$c_w = Q_w W'_I, \tag{5.15}$$

where

$$Q_w = \varepsilon_w \sum_{\substack{v \in V_I \\ v \leq v_I(w)}} \varepsilon_v p_{v,w} v \quad \textit{and} \quad W'_I = \sum_{u \in W_I} \varepsilon_u u. \tag{5.16}$$

Proof. Proposition 5.4 gives that

$$c_w = \sum_{\substack{v \in V_I \\ v \leq v_I(w)}} \sum_{u \in W_I} \varepsilon_w \varepsilon_v \varepsilon_u p_{vu,w} vu$$

and (5.15), (5.16) follow immediately from the fact that

$$ws < w \to p_{ys,w} = p_{y,w} \qquad \text{(for all } y\text{)},$$

which is Fact 6.

Now let μ be a fixed partition of n and let

$$T_1, T_2, ..., T_{n_\mu}$$

here and in the rest of this section denote the standard tableaux of shape μ labelled in the order of increasing *column words*. Set

$$w_i = (T_i, T_1).$$

It is easy to see that $T_1 = CS_\mu$, and thus we must have

$$w_i = \mathrm{cw}(T_i).$$

Specializing w to w_i in Proposition 5.5 and setting

$$I = \{s_i : i \in D(CS_\mu)\} \tag{5.17}$$

we immediately get the factorizations

$$c_{w_i} = Q_{w_i} N(T_1) \qquad \text{for} \quad i = 1, 2, ..., n_\mu. \tag{5.18}$$

The identification of W'_I with $N(T_1)$ (as defined in Section 1) is due to the fact that, when the right tableau Q of w is superstandard, the group W_I is identical with the column group of Q.

The identity in (5.18) has numerous consequences. In particular Fact 10 follows immediately from it. Indeed, note that for w_1 the factorization (5.16) reduces to

$$c_{w_1} = \varepsilon_{w_1} N(T_1) \tag{5.19}$$

This is because when I is as in (5.17) the representative $v_I(w)$ of any permutation w is simply obtained by rearranging in increasing order the elements of w which lie within unbroken strings of descents of w_1. An example will best get across what we are trying to say here. Let $\mu = (2, 3, 3)$. Then

$$w_1 = \mathrm{cw}(CS_\mu) = 3\ 2\ 1 \quad 6\ 5\ 4 \quad 8\ 7,$$

where we have separated by spaces the *unbroken strings of descents* of w_1. Now given a permutation $w \in S_8$, to construct $v_I(w)$ we simply rearrange in increasing order the elements of w occupying positions 1, 2, 3, then those occupying positions 4, 5, 6, and finally those in positions 7, 8. Thus

$$v_I(8\ 3\ 2 \quad 6\ 1\ 7 \quad 4\ 5) = 2\ 3\ 8 \quad 1\ 6\ 7 \quad 4\ 5.$$

Using this fact on w_1 immediately gives that

$$v_I(w_1) = \text{the identity permutation.}$$

Thus the sum in Q_{w_1} reduces to a single term and (5.19) follows.

Remark 5.2. We should point out that, from the remarks above, we can easily see that $v_I(w_i)$ is none other than the word obtained by reading T_i column from bottom to top rather than from top to bottom as is done to get w_i. It is also easy to derive (from property (iii)) that

$$w_i \leq_B w_j \leftrightarrow v_I(w_i) \leq_B v_I(w_j). \tag{5.20}$$

Let us now set, for any $f \in A(S_n)$

$$L_{xy}(f) = fc_y|_{c_x}.$$

Clearly, c_x is in the expansion of fc_y only if there is an L-path joining x to y. In view of Fact 7 we deduce that $L_{xy}(f) \neq 0$ implies that $D_R(x) \supseteq D_R(y)$. This gives that

$$A(S_n)\, c_w \subseteq L[c_x : D_R(x) \supseteq D_R(y)].$$

L here stands for *linear span.* Thus the character of the representation corresponding to the action on the left ideal $A(S_n)\, c_w$ is dominated by the sum of the characters of the representations B^λ corresponding to cells indexed by tableaux Q whose descent set contains the right descent set of w. That is,

$$\text{char } A(S_n)\, c_w \ll \sum_\lambda \xi^\lambda \ \# \ \{Q: D(Q) \supseteq D_R(w) \text{ and shape}(Q) = \lambda\}. \tag{5.21}$$

Here *domination* and "$\ll$" mean that corresponding coefficients of the irreducible characters are individually related by the same inequality.

This given, we see that the dimension of $A(S_n)\, c_w$ as a vector subspace of $A(S_n)$ satisfies the inequality

$$\dim A(S_n)\, c_w \leqslant \sum_\lambda n_\lambda \ \# \ \{Q: D(Q) \supseteq D_R(w) \text{ and shape}(Q) = \lambda\}. \tag{5.22}$$

If we now choose $w = w_1$, formula (5.19) gives us another way of calculating this dimension. Indeed, it is well known (see [7] or [9] for a simple proof), that the character of the representation corresponding to left action on the left ideal of an idempotent $N(T)$, for any tableau T of shape μ, has the expansion

$$\text{char } A(S_n)\, N(T) = \sum_\lambda K_{\tilde{\lambda}, \tilde{\mu}} \chi^\lambda, \tag{5.23}$$

where here the χ^λ is the irreducible character indexed by λ in the Young labeling (the character given by formula (1.19)).

Evaluating (5.23) at the identity gives

$$\dim A(S_n)\, N(T) = \sum_\lambda K_{\tilde{\lambda}, \tilde{\mu}} n_\lambda. \tag{5.24}$$

On the other hand, with this choice of w we have

$$D_R(w) = D(CS_\mu).$$

Thus we can use Proposition 5.2 and get that

$$\# \ \{Q: D(Q) \supseteq D_R(w) \text{ and shape}(Q) = \lambda\} = K_{\tilde{\lambda}, \tilde{\mu}}.$$

Substituting this in (5.22) yields

$$\dim A(S_n)\, c_w \leqslant \sum_\lambda K_{\tilde{\lambda}, \tilde{\mu}} n_\lambda.$$

Comparing with (5.24) we see that in this case (5.22), and thus (5.21) as well, must be equalities. Equating the right-hand sides of (5.21) and (5.23) yields the equation

$$\sum_\lambda K_{\tilde{\lambda}, \tilde{\mu}} \xi^\lambda = \sum_\lambda K_{\tilde{\lambda}, \tilde{\mu}} \chi^\lambda.$$

Now it is well know that the matrix $\|K_{\tilde{\lambda}, \tilde{\mu}}\|$ is invertible. This also follows from our proof of Proposition 5.2 since it gives that

$$K_{\tilde{\lambda}, \tilde{\mu}} \neq 0 \to \lambda \leqslant \mu. \tag{5.25}$$

Moreover, it is easily seen that all the diagonal elements are equal to one. We must then conclude that

$$\xi^\mu = \chi^\mu \qquad (\text{for all } \mu).$$

This establishes Fact 10.

To proceed with the proof of triangularity we need two auxiliary facts. The first is purely algebraic:

Proposition 5.6. *Let $\{A^\lambda\}$ be a complete set of irreducible representations of a finite group G and let $\{\chi^\lambda\}$ be the corresponding characters. Let θ be an arbitrary element of the group algebra $A(G)$. Then the character of the representation induced by the action of G on the left ideal $A(G)\,\theta$ is given by the formula*

$$\text{char } A(G)\,\theta = \sum_\lambda \text{rank } A^\lambda(\theta)\, \chi^\lambda. \tag{5.26}$$

The proof is quite straightforward and will be omitted.

The second fact is purely combinatorial; to state it we need some notation. Given a tableau T, let $v(T)$ denote the word obtained by reading T column by column from *bottom to top* (not from top to bottom as we do for $\mathrm{cw}(T)$) and let $c(T_1, T_2)$ be as defined in Section 1.

Proposition 5.7. *If T_1 and T_2 have the same shape and T_1 is standard then*

$$c(T_1, T_2) \neq 0 \to v(T_1) \leqslant_B v(T_2). \tag{5.27}$$

A proof is given in [8] and will not be repeated here. We should point out that originally Proposition 5.7 came out of efforts (see [8]) to develop a purely combinatorial treatment of skew representations. Curiously, it turns out that it now gives a crucial step in the proof of the triangularity result.

Going back to our tableaux $T_1, T_2, ..., T_{n_\mu}$ we note that when I is as defined in (5.17) we have that

$$v_I(w_i) = v(T_i). \tag{5.28}$$

Thus (5.27) combined with our convention on labeling the T_i's and Remark 5.2 in particular implies that the matrix

$$C(\varepsilon) = \|c(T_i, T_j)\|$$

is upper triangular. Again let σ_{ij} denote the permutation that sends T_j into T_i and set as before

$$F_{ij} = N(T_i)\,\sigma_{ij} P(T_j).$$

As observed in Section 1 (Remark 1.1) we have

$$\sigma\langle F_{1,s}, F_{2,s}, ..., F_{n_\mu,s}\rangle = \langle F_{1,s}, F_{2,s}, ..., F_{n_\mu,s}\rangle\, A^\mu(\sigma) \tag{5.29}$$

with

$$A^\mu(\sigma) = C^{-1}(\varepsilon)\, C(\sigma) \qquad \text{and} \qquad C(\sigma) = \|c(T_i, \sigma T_j)\|.$$

Let us now set

$$f_j = c_{w_j} P(T_1) = Q_{w_j} F_{1,1}. \tag{5.30}$$

It develops that these units afford the Kazhdan–Lusztig representation B^λ. More precisely, setting

$$B^\mu_{ij} = L_{w_i, w_j}, \qquad B^\mu = \|B^\mu_{ij}\|$$

we have

THEOREM 5.2.

$$\sigma\langle f_1, f_2, ..., f_{n_\mu}\rangle = \langle f_1, f_2, ..., f_{n_\mu}\rangle\, B^\mu(\sigma). \tag{5.31}$$

Proof. Note that we can write

$$\sigma c_{w_j} = \sum_{i=1}^{n_\mu} c_{w_i} B^\mu_{ij}(\sigma) + \sum\nolimits^{(*)} c_x L_{x, w_j}(\sigma),$$

where the $(\,*\,)$ is to indicate that this summation is to be carried out only over permutations x which lie outside the L-cell of w_1. Multiplying both sides of this equation by $P(T_1)$ gives

$$\sigma f_j = \sum_{i=1}^{n_\mu} f_i L^\mu_{ij}(\sigma) + \sum\nolimits^{(*)} c_x P(T_1)\, L_{x, w_j}(\sigma).$$

It is thus clear that to show (5.31) we need only establish that the second summation vanishes identically. For convenience, let us extend the meaning of the symbol "$x = L \Rightarrow y$" to mean that there is an L-path from x to y. This given, the following result, interesting in its own merits, is all that is needed to complete the proof of the theorem.

PROPOSITION 5.8. *If x is not in the L-cell of w_1 and $x = L \Rightarrow w_1$ then*

$$c_x P(T_1) = 0.$$

We break up the proof into three lemmas.

LEMMA 5.1. *Under the hypotheses of Proposition 5.8, a term involving c_y can occur in the expansion of an element fc_x ($f \in A(S_n)$) only if the shape of y is strictly dominated by μ.*

Proof. If such a term does occur then $y = L \Rightarrow x$. This combined with the hypothesis $x = L \Rightarrow w_1$ gives (Fact 7)

$$D_R(y) \supseteq D(CS_\mu). \tag{5.32}$$

Proposition 5.1 can thus be applied to the right tableau of y. However, we cannot have the first alternative here. For if the right tableau of y is CS_μ then y and w_1 are in the same L-cell and we would then have all three relations

$$x = L \Rightarrow w_1, \qquad w_1 = L \Rightarrow y, \qquad y = L \Rightarrow x,$$

which would imply that x and w_1 are in the same L-cell, contrary to our hypotheses. The second alternative gives that the shape of y is strictly dominated by μ as desired.

Let A^λ still denote Young's natural representation corresponding to shape λ.

LEMMA 5.2. *Under the hypotheses of Proposition 5.8, we have*

$$\operatorname{rank} A^\lambda(c_x) \leqslant \begin{cases} K_{\tilde\lambda, \tilde\mu} & \text{if } \lambda < \mu \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Lemma 5.1 and (5.32) we derive that

$$A(S_n)\, c_x \subseteq L(c_y : \text{shape } y < \mu \text{ and } D_R(y) \supseteq D(CS_\mu)).$$

This implies that the character of the representation corresponding to the action of S_n on the left ideal $A(S_n)\, c_x$ is dominated by the sum of the characters of the representations B^λ coming from cells indexed by standard tableaux Q satisfying the two conditions

$$\text{shape } Q < \mu \qquad \text{and} \qquad D(Q) \supseteq D(CS_\mu).$$

In other words, using Proposition 5.2 and Fact 10, we must have

$$\operatorname{char} A(S_n)\, c_x \ll \sum_{\lambda < \mu} K_{\tilde\lambda, \tilde\mu} \chi^\lambda,$$

where again, χ^λ denotes the character of A^λ. Comparing with (5.26) we see that the desired conclusion follows from Proposition 5.6.

The final step is given by

LEMMA 5.3. *For any tableau T of shape μ and any λ we have*

$$\operatorname{rank} A^\lambda(P(T)) = \begin{cases} K_{\lambda,\mu} & \text{if } \lambda \geqslant \mu \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows immediately from Proposition 5.6 and the identity

$$\operatorname{char} A(S_n)\,P(T) = \sum_{\lambda \geqslant \mu} K_{\lambda,\mu}\chi^\lambda$$

which is well known and is usually referred to as *Young's Rule* (see [20]). It is also essentially another version of formula (5.23).

We can now easily complete the proof of Proposition 5.8. Indeed, the combination of Lemmas 5.2 and 5.3 yields that for all λ

$$A^\lambda(c_x P(T_1)) = 0. \tag{5.33}$$

This is because in order for this matrix not to vanish both matrices

$$A^\lambda(c_x) \qquad \text{and} \qquad A^\lambda(P(T_1))$$

must have ranks different from zero. However, for the first (by Lemma 5.2) this happens only when $\lambda < \mu$ and for the second (by Lemma 5.3) only when $\lambda \geqslant \mu$. Thus (5.33) must hold true as asserted. But, since the $\{A^\lambda\}$ form a complete system of irreducibles for S_n, the vanishing of all the matrices in (5.33) implies the vanishing of $c_x P(T_1)$ itself. This completes the proof of Proposition 5.8 and Theorem 5.2.

Note now that (5.29) (for $s=1$) gives

$$\sigma F_{11} = \sum_{i=1}^{n_\mu} F_{i,1} A^\mu_{i,1}(\sigma).$$

Thus, if we extend by linearity the definition of A^μ to all of $A(S_n)$, formula (5.30) in terms of the natural representation becomes

$$f_j = \sum_{i=1}^{n_\mu} F_{i,1} A^\mu_{i,1}(Q_{w_j})$$

or in matrix form

$$\langle f_1, f_2, \ldots, f_{n_\mu}\rangle = \langle F_{1,1}, F_{2,1}, \ldots, F_{n_\mu,1}\rangle \| A^\mu_{i,1}(Q_{w_j})\|. \tag{5.34}$$

Equations (5.29), (5.31), and (5.34) give us a matrix transforming the Kazhdan–Lusztig representation into Young's natural, and a key to the triangularity result, namely:

THEOREM 5.3. *The matrix $W = \|w_{ij}\|$ with coefficients*

$$w_{ij} = A^\mu_{i1}(Q_{w_j}) = \varepsilon_{w_j} \sum_{\substack{v \in V_I \\ v \leqslant v_I(w_j)}} \varepsilon_v p_{v,w_j} A^\mu_{i,1}(v). \tag{5.35}$$

is the solution of the equation

$$A^\mu(\sigma) = WB^\mu(\sigma)\,W^{-1}. \tag{5.36}$$

Moreover, W is upper triangular with unit diagonal elements.

Proof. Eliminating the f_i's and the F_{i1}'s from (5.29), (5.31), and (5.34) we get (5.35) with

$$w_{ij} = A^\mu_{i1}(Q_{w_j}).$$

The second expression in (5.35) follows from (5.16) by linearity. To prove the final assertion, suppose $w_{ij} \neq 0$. Then from (5.35) it follows that there must be a $v \leqslant v_I(w_j)$ such that $A^\mu_{i1}(v) \neq 0$. But since

$$A^\mu_{i1}(v) = \sum_{s=1}^{n_\mu} C^{-1}_{is}(\varepsilon)\,C_{s1}(v) \tag{5.37}$$

there must be an s for which we have both

(a) $C^{-1}_{is}(\varepsilon) \neq 0$ and
(b) $C_{s1}(v) \neq 0$.

But (a) combined with the upper triangularity of $C(\varepsilon)$, implies $i \leqslant s$ while (b) means that

$$c(T_s, vT_1) \neq 0,$$

and this in turn (by Proposition 5.7) implies that

$$v(T_s) \leqslant_B v(vT_1).$$

Now we easily see that $v(vT_1) = v$, and since we have $v \leqslant_B v_I(w_j) = v(T_j)$ we finally get that

$$v(T_s) \leqslant_B v(T_j).$$

However, (5.20) and (5.28) combined with our convention on the labeling of the tableaux T_k give us then that $s \leqslant j$. Since we already had $i \leqslant s$, we deduce that $i \leqslant j$ as desired.

To complete the proof we need to find out what happens when $i=j$. Going through the above string of inequalities, we see that we must have $i=s=j$ as well, and thus also

$$v(T_i) \leqslant v \leqslant v(T_i),$$

which forces $v = v_I(w_i)$ to be the only contributor to the sum in (5.35). B
for this term we have

$$p_{v,\, w_i} = p_{w_i,\, w_i} = 1.$$

Moreover, since the term with $s = i$ is the only contributor to the sum i
(5.37) we derive that

$$A^{\mu}_{i1}(v) = A^{\mu}_{i1}(v(T_i)) = C^{-1}_{ii}(\varepsilon)\, c(T_i, T_i) = 1.$$

The only thing unaccounted for is the sign, which is

$$\varepsilon_{w_i} \varepsilon_{v_I(w_i)}.$$

However, since

$$w_i = v_I(w_i)\, u_0,$$

where u_0 is the top element of W_I, we see that this sign is the same for al
and is equal to the sign of u_0.

We end by showing that the elements w_{ij} satisfy recursions analogous t
the equations (2.16) satisfied by the p_{xw}'s themselves. To see this let u
change notation slightly and set, when $x = w_i$, $y = w_j$,

$$f_x = f_i, \qquad F_x = F_{i,1}, \qquad \text{and} \qquad q_{xy} = w_{ij}.$$

Note that with this notation

$$f_y = \sum_{x \leqslant y} F_x q_{xy}. \tag{5.3}$$

Similarly, (5.29) becomes

$$\sigma F_x = \sum_z F_z A^{\mu}_{zx}(\sigma). \tag{5.3}$$

This given we have

THEOREM 5.4. *If w is the column word of a standard tableau of shape and $y = sw < w$ is also such a word then*

$$q_{xw} = \sum_{z \leqslant y} A_{xz}(s)\, q_{zy} - q_{xy} - \sum_{sz < z} \mu(z, y)\, q_{xz}, \tag{5.4}$$

where all these sums are carried out only over column words of standa tableaux of shape μ.

Proof. Using the second equation in (2.12) with w replaced by y and sw replaced by w and multiplying both sides on the right by $P(T_1)$, Proposition 5.8 gives

$$sf_y = f_w + f_y + \sum_{sz < z} f_z \mu(z, y).$$

Substituting into it Eqs. (5.38) and (5.39) gives

$$\sum_{z \leqslant y} \Big(\sum_x F_x A^{\mu}_{xz}(s) \Big) q_{zy} = \sum_{x \leqslant w} F_x q_{xw} + \sum_{x \leqslant y} F_x q_{xy} + \sum_{sz < z} \sum_{x \leqslant z} F_x q_{xz} \mu(z, y).$$

Equating coefficients of F_x yields our desired recursion (5.40).

We should point out that, since Eq. (5.40) involves only elements of a single cell, it gives a considerably less laborious algorithm for the calculations of the transforming matrix W than the one described in Section 3.

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REFERENCES

1. A. BJÖRNER, The combinatorics of the Kazhdan–Lusztig representations, Lecture notes, in preparation.
2. N. BOURBAKI, "Groupes et algèbres de Lie," Chap. 4–6, Hermann, Paris, 1968.
3. C. W. CURTIS, N. IWAHORI, AND R. KILMOYER, Hecke algebras and characters of parabolic type of finite groups with (B, N) pair, *Inst. Hautes Etudes Sci. Publ. Math.* **40** (1972), 81–116.
4. C. W. CURTIS AND REINER, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.
5. H. GARNIR, Théorie de la représentation lineaire des groupes symétriques, *Mém. Soc. Roy. Belg. Cl. Sci. Mé. Coll.* in 8 (2) **26** (1951), No. 1615.
6. A. M. GARSIA AND J. B. REMMEL, Shuffles of permutations and the Kronecker product, *Graphs Combin.* **1** (1985), 217–263.
7. A. M. GARSIA AND J. REMMEL, Symmetric functions and raising operators, *Linear and Multilinear Algebra* **10** (1981), 15–43.

8. A. M. Garsia and M. Wachs, Combinatorial aspects of skew representations of the sy[mmetric] groups, to appear.
9. A. M. Garsia, Raising operators and Young's rule, *in* "Proceedings, Montreal Co[l]loquium in Enumerative Combinatorics, June 1985."
10. M. Goresky, Tables of Kazhdan–Lusztig polynomials, unpublished.
11. G. D. James, "The representation Theory of the Symmetric Groups," Lecture Note[s in] Mathematics, Vol. 682, Springer-Verlag, Berlin/New York/Heidelberg, 1978.
12. G. D. James and A. Kerber, The representation theory of the symmetric group, "Encyclopedia of Mathematics," No. 16, Addison–Wesley Reading, MA, 1981.
13. D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algeb[ras], *Invent. Math.* **53** (1979), 165–184.
14. D. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality, *in* Geometry [of] the Laplace Operator (R. Osserman and A. Weinstein, Eds.), Proc. Symp. Pure Ma[th.] Vol. 36, Amer. Math. Soc., Providence, RI, 1980.
15. R. King, Lecture notes, La Jolla, CA, 1983.
16. D. E. Knuth, "The Art of Computer Programming," Addison–Wesley, Reading, M[A], 1975.
17. A. Lascoux and M.-P. Schützenberger, La représentation de Kazhdan & Lusztig [du] groupe symétrique, preprint.
18. A. Lascoux and M.-P. Schützenberger, Polynomes de Kazhdan & Lusztig pour [les] Grassmanniennes, preprint.
19. G. Lusztig, On a theorem of Benson and Curtis, *J. Algebra* **71** (1981), 490–498.
20. I. G. MacDonald, "Symmetric Functions and Hall Polynomials," Oxford Univ. P[ress] (Clarendon), Oxford, 1979.
21. D. E. Rutherford, "Substitutional Analysis," Edinburgh Univ. Press, Edinburgh, 19[48].
22. M.-P. Schützenberger, La correspondance de Robinson, *in* "Combinatoire et repr[ésen]tation du groupe symétrique," Actes Table Ronde C.N.R.S. Strasbourg, 1976.
23. R. Steinberg, "Lectures on Chevalley Groups," Lecture Notes, Department [of] Mathematics, Yale University, 1968.
24. D. Vogan, A generalized τ-invariant for the primitive spectrum of a semisimple [Lie] algebra, preprint, Princeton University, 1978.
25. A. Young, "The Collected Papers of Alfred Young, 1873-1940," Math. Expositi[ons] No. 21, Univ. of Toronto. Toronto Press.

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Homology of Symplectic and Orthogonal Algebras

Jean-Louis Loday

Institut de Recherche Mathématique Avancée, C.N.R.S., 67084 Strasbourg, France

AND

Claudio Procesi

Dipartimento de Matematica, Università di Roma, 00185 Rome, Italy

In this paper we compute the homology of the infinite Lie algebra of orthogonal (resp. symplectic) matrices for an associative ring with involution over a characteristic zero field. It is shown (Theorem 5.5) to be the graded symmetric algebra over skew-dihedral homology. The surprising result is that the orthogonal and symplectic algebras have the same homology in the stable range.

Skew-dihedral homology is a variation of cyclic homology, which is studied in detail in [L]. The proof of the main theorem is based on computations in invariant theory (cf. [P1, P2, W]) for orthogonal and symplectic matrices. It follows the lines of proof of the analogue result for matrices given in [L–Q].

We also determine exactly the stable range (Theorem 7.1) and the first obstruction to stability (Theorem 7.3) in the symplectic case.

In the first part we give a comprehensive and detailed review of invariant theory for orthogonal and symplectic matrices. In particular we make explicit the relations between the hyperoctahedral group, the trace identities for matrices and the invariant space of the tensor algebra of matrices.

The homology result in the stable range was also announced in [F-T].

Conventions. Vector spaces will be abbreviated as spaces. If the group G is acting on the space V, the space of invariants is $V^G = \{v \in V \mid \forall g \in G, g \cdot v = v\}$ and the space of coinvariants is $V_G = V/\{g \cdot v - v \mid g \in G, v \in V\}$. The signature of a permutation σ is denoted $\varepsilon(\sigma) = \pm 1$.

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