θ-NILPOTENT SUBSPACES AND THE ASSOCIATED HYPERPLANE ARRANGEMENT

ABSTRACT. Suppose $G_{\mathbb{R}}$ is a real connected reductive Lie group with some special properties. Similar to ad-nilpotent ideals for complex reductive Lie groups, we define a new object called θ-nilpotent subspaces for $G_{\mathbb{R}}$. We prove that there's an analogous result of [34] about the closure relation of θ-nilpotent subspaces and their associated orbits.

To the set of θ-nilpotent subspaces, we can define a real hyperplane arrangement. We give a conjectural formula for the characteristic polynomials of the hyperplane arrangement which leads to the enumeration of the set of θ-nilpotent subspaces. We verify this formula for $G = SU(m, n)$ and $G = Sp(m, n)$. We also give a combinatorial calculation of the set of θ-nilpotent subspaces when $G = SU(m, n)$ and show that it's a Narayana number.

1. INTRODUCTION

Let $G_{\mathbb{R}}$ be a real reductive Lie group with $K_{\mathbb{R}}$ its maximal compact subgroup fixed by a Cartan involution θ. We restrict our discussion to the case when $rank(G_{\mathbb{R}}) = rank(K_{\mathbb{R}})$. Let G, K be the complexification of $G_{\mathbb{R}}, K_{\mathbb{R}}$ and $\mathfrak{g}, \mathfrak{k}$ be their Lie algebras. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Let B_K and B be the Borel subgroups of K and G such that B_K is contained in B and B is θ-stable.

The ad-nilpotent ideals of $\mathfrak{g}$ are the subspaces of the nilradical which are invariant under the adjoint action of B. We refer to the papers of Cellini-Papi [12] [13], Gunnells-Sommers [15], Panyushev [26] [28], Sommers [33] [34] and [9] [10] about the theory of ad-nilpotent ideals.

Among all the combinatorial properties of ad-nilpotent ideals, the enumeration of ad-nilpotent ideals is an interesting problem and involves many

The hyperplane arrangement plays an important role in the study of ad-nilpotent ideals.

In the study of sign types [32], Shi obtained the number of ad-nilpotent ideals by direct calculation. Later, in his thesis [1] and also in [2], Athanasiadis studied the hyperplane arrangement that was related to the ad-nilpotent ideal and got the characteristic polynomial of the Catalan arrangements by counting the number of points over finite fields that do not lie in any defining hyperplanes. By using a result of Zaslavsky, he got the number of dominant regions which was equal to the number of ad-nilpotent ideals. All the approaches above are based on case by case study of semisimple Lie algebras of different types. A case-free proof was first given by Cellini and Papa in [13]. They proved that there was a bijection between the set of ad-nilpotent ideals and the set of W-orbits in $Q^\vee/(h+1)Q^\vee$, where $Q^\vee$ is the coroot lattice and h is the Coxeter number of G. This led to a uniform formula for the number of ad-nilpotent ideals of G (as well as dominant regions of the Catalan arrangement):

$$\frac{1}{|W|}\prod_{i=1}^{n}(h+e_i+1),$$

where $e_1, e_2, \ldots, e_n$ are the exponents of G.

E. Sommers in [33] defined maximal dominant elements of the affine Weyl group associated to bounded dominant regions of the Catalan arrangement. Then he derived the formula for the number of bounded dominant regions of the Catalan arrangement:

$$\frac{1}{|W|}\prod_{i=1}^{n}(h+e_i-1).$$

Indeed, it was proved by Athanasiadis in [3] that the characteristic polynomial of the Catalan arrangement has a uniform formula:

$$\chi(Cat, q) = \prod_{i=1}^{n}(q-h-e_i).$$

The formulas of Cellini, Papi and Sommers are corollaries of his result.

The main purpose of this paper is to combine the ideas of [15] [34] In analogy, θ-nilpotent subspaces can be considered as certain subspaces of $\mathfrak{p}$ that are invariant under the adjoint action of B_K.

In the case of real groups, since there is no known bijection between the set of θ-nilpotent spaces and some special affine Weyl group elements, the method of Cellini and Papi fails and we could not get a uniform formula. However, the ideas in [1] and [2] can still be used in the study of the real hyperplane arrangement as well as the θ-nilpotent subspaces.

Suppose that $G_{\mathbb{R}}$ is semisimple, we conjecture a general formula for the characteristic polynomial of the real hyperplane arrangement $\mathcal{A}$ of $G_{\mathbb{R}}$. Suppose h is the Coxeter number of G, the complexification of $G_{\mathbb{R}}$ and $\{e_1, e_2, \ldots, e_m\}$ is the set of exponents of K. We have the following conjecture.

Conjecture 1.1. *Keep the notations from above. Then the characteristic polynomial of the real hyperplane arrangement $\mathcal{A}$ of $G_{\mathbb{R}}$ is*

$$\chi(\mathcal{A}_K, q) = \prod_{i=1}^{m}(q-h+e_i).$$

The rest of the paper is organized as follows.

2. θ-nilpotent subspaces

2.1. **Basic Notation.** We refer to the book of Onishchik and Vinberg [25] and the book of Knapp [17] about the basic structure theory of real reductive Lie algebras and Lie groups.

Let $G_{\mathbb{R}}$ be a connected linear reductive real Lie group. Let θ be the Cartan involution of $G_{\mathbb{R}}$. Let $K_{\mathbb{R}}$ be the fixed points of θ. It is a maximal compact subgroup of $G_{\mathbb{R}}$. We restrict our discussion to the situation when $rank(G_{\mathbb{R}}) = rank(K_{\mathbb{R}})$. Let $H_{\mathbb{R}}$ be a Cartan subgroup of $K_{\mathbb{R}}$ and H be its complexification. Under our assumption, $H_{\mathbb{R}}$ is also a maximally compact Cartan subgroup of $G_{\mathbb{R}}$. Let $\mathfrak{h}_{\mathbb{R}}$ be the Lie algebra of $H_{\mathbb{R}}$.

Let G (resp. K) be the corresponding complexification of $G_{\mathbb{R}}$ (resp. $K_{\mathbb{R}}$) and $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{g}$ be the Lie algebra of $G_{\mathbb{R}}$ and its complexification. We denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the complexified Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{h}$ be the complexification of $\mathfrak{h}_{\mathbb{R}}$ and Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$. The root space that corresponds to any root α is denoted by $\mathfrak{g}_\alpha$. Since $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}$, for any $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is θ-stable. Therefore either $\mathfrak{g}_\alpha \subset \mathfrak{k}$ or $\mathfrak{g}_\alpha \subset \mathfrak{p}$. In the first case, we call α a compact root and in the second case, we call α a noncompact root. We write $\Delta_{\mathfrak{k}}$ and $\Delta_{\mathfrak{p}}$ for the set of compact and noncompact roots respectively. Then $\Delta = \Delta_{\mathfrak{k}} \sqcup \Delta_{\mathfrak{p}}$.

We choose a set of positive roots $\Delta_{\mathfrak{k}}^+$ in $\Delta_{\mathfrak{k}}$ and a set of positive roots Δ^+ in Δ such that $\Delta_{\mathfrak{k}}^+ \subset \Delta^+$. Let B_K (resp. B) be the Borel subgroups of K and G corresponding to $\Delta_{\mathfrak{k}}^+$ and Δ^+. Then B is θ-stable and $B_K = B \cap K$. Let W and W_K be the Weyl groups of G and K. For any $w \in W$, let $\dot{w}$ be a representative of w in $N_G(H)$, where $N_G(H)$ is the normalizer of H in G.

Let Q be the weight lattice and $Q^\vee$ be the coweight lattice. Let $V = Q^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, the real span of the coweight lattice. Then the dimension of V is equal to the rank of K (and G) and all roots take real values on V. In fact $V = \mathfrak{h}_0 := i\mathfrak{h}_{\mathbb{R}}$.

Let $\mathcal{C}$ be the fundamental chamber and

$$\mathcal{C}_{\mathfrak{k}} = \{x \in V \mid \beta(x) > 0 \text{ for all } \beta \in \Delta_{\mathfrak{k}}^+\}.$$

We call $\mathcal{C}_{\mathfrak{k}}$ the (open) **real dominant chamber** of V.

For any left coset $c \in W_K \backslash W$, there exists a unique coset representative $w \in c$ mapping $\mathcal{C}$ into $\mathcal{C}_{\mathfrak{k}}$, i.e. $\dot{w} B \dot{w}^{-1} \cap K = B \cap K$. Let ${}^K W$ be the set of such coset representatives. Then ${}^K W$ is in bijection with $W_K \backslash W$.

2.2. **θ-nilpotent Subspaces.** Let $\mathcal{B}$ be the flag variety of G. Then $\mathcal{B}$ is isomorphic to G/B. The group G acts on $\mathcal{B}$ transitively, which induces a natural action of K on $\mathcal{B}$ simply by restriction. The flag variety $\mathcal{B}$ is a disjoint union of finitely many K-orbits. We denote by Σ the set of closed orbits of K in $\mathcal{B}$. As mentioned in [21, lem 5.8], the set of closed orbits is in bijection with the set of θ-stable Borel subalgebras (up to conjugacy), therefore, Σ is in bijection with ${}^K W$. Namely

$$\Sigma = \{K\dot{w}B\dot{w}^{-1} \mid w \in {}^K W\}.$$

Here notice that the K-orbit $K\dot{w}B\dot{w}^{-1}$ is independent of our choice of the representative $\dot{w} \in N_G(H)$, but is uniquely determined by $w \in {}^K W$. Therefore we denote the K-orbit by Q_w.

For the complex group G, recall that we have defined the moment map from the cotangent bundle of the flag variety to the nilpotent cone in section 1.3. We fix a non-degenerate symmetric G-invariant bilinear form on $\mathfrak{g}$ and identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via this bilinear form. There is a G-equivariant isomorphism $T^*\mathcal{B} \simeq G \times_B \mathfrak{n}$. Under this isomorphism, the moment map $m : T^*\mathcal{B} \to \mathfrak{g}^*$ can be identified with the projection map $G \times_B \mathfrak{n} \to \mathcal{N}$ (see [8, Lem 3.2.2 & Cor 3.2.3]).

For any $w \in {}^K W$, if we restrict the moment map m to the conormal bundle $T^*_{Q_w}(\mathcal{B})$ of some closed K-orbit Q_w on $\mathcal{B}$, the image $m(T^*_{Q_w}(\mathcal{B}))$ lies in the cone $\mathcal{N}^*_\theta$ of nilpotent elements in $(\mathfrak{g}/\mathfrak{k})^* = \{f \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } x \in \mathfrak{k}\}$. In this way, we have a restriction of moment maps

$$m_K : T^*_{Q_w}(\mathcal{B}) \longrightarrow \mathcal{N}^*_\theta.$$

Under the symmetric bilinear form on $\mathfrak{g}$, $(\mathfrak{g}/\mathfrak{k})^*$ is identified with $\mathfrak{p}$ and there is a K-equivariant isomorphism

$$T^*_{Q_w}(\mathcal{B}) \simeq K \times_{\dot{w}B\dot{w}^{-1}\cap K} (\dot{w}\mathfrak{n}\dot{w}^{-1} \cap \mathfrak{p}).$$

Then m_K is equivalent to

$$m_K : K \times_{\dot{w}B\dot{w}^{-1}\cap K} (\dot{w}\mathfrak{n}\dot{w}^{-1} \cap \mathfrak{p}) \longrightarrow \mathcal{N}_\mathfrak{p} := \mathcal{N} \cap \mathfrak{p}.$$

Notice that for any $w \in {}^K W$, $\dot{w}B\dot{w}^{-1} \cap K = B_K$ and there is a commutative diagram:

$$\begin{array}{ccc} T^*_{Q_w}(\mathcal{B}) & \longrightarrow & \mathcal{N}^*_\theta \\ \downarrow & & \downarrow{\scriptstyle i} \\ K \times_{B_K} (\dot{w}\mathfrak{n}\dot{w}^{-1} \cap \mathfrak{p}) & \longrightarrow & \mathcal{N}_\mathfrak{p} \end{array}.$$

Recall that when talking about an ad-nilpotent ideal I, we always need to specify a fixed Borel subalgebra $\mathfrak{b}$ and I is an ideal of $\mathfrak{b}$. Now if we pick any $w \in {}^K W$, then $\dot{w}\mathfrak{b}\dot{w}^{-1}$ is also a Borel subalgebra of $\mathfrak{g}$ and contains $\mathfrak{b}_\mathfrak{k}$. Let I be an ad-nilpotent ideal of $\dot{w}\mathfrak{n}\dot{w}^{-1}$, then $I \cap \mathfrak{p}$ is B_K-invariant and $K \times_{B_K} I \cap \mathfrak{p}$ is a subbundle of the conormal bundle $K \times_{B_K} (\dot{w}\mathfrak{n}\dot{w}^{-1} \cap \mathfrak{p})$.

Definition 2.1. *We call a subspace of $\mathfrak{p}$ a θ-nilpotent subspace if it is the intersection of an ideal of some Borel subalgebra and $\mathfrak{p}$, where the Borel subalgebra is of the form $\dot{w}\mathfrak{b}\dot{w}^{-1}$ for some $w \in {}^K W$.*

The name "θ-nilpotent" comes from the fact that any θ-nilpotent space is θ-stable and consists of nilpotent elements. We denote by $\mathfrak{Ad}_K$ the set of θ-nilpotent subspaces. Then

$$\mathfrak{Ad}_K = \{I \cap \mathfrak{p} \mid \text{ where } I \text{ is an ideal of } \dot{w}\mathfrak{b}\dot{w}^{-1}, \text{ and } w \in {}^K W\}.$$

This definition of θ-nilpotent subspaces looks tedious and not very natural. It's possible to have an ideal I of $\dot{w}\mathfrak{b}\dot{w}^{-1}$ and an ideal J of $\dot{w}_1\mathfrak{b}\dot{w}_1^{-1}$ for some $w, w_1 \in {}^K W$, such that $I \cap \mathfrak{p} = J \cap \mathfrak{p}$. We may try to interpret θ-nilpotent subspaces in a simpler way.

Let $\mathfrak{Ad}^K$ be the set of B_K-invariant subspaces of $\mathfrak{p}$, consisting no semisimple elements. Then any θ-nilpotent subspace is B_K-invariant and consists of nilpotent elements, hence lies in $\mathfrak{Ad}^K$ and $\mathfrak{Ad}_K$ is a subset of $\mathfrak{Ad}^K$. Now we can state our conjecture.

Conjecture 2.2. $\mathfrak{Ad}_K = \mathfrak{Ad}^K$. *Namely, for any subspace in* $\mathfrak{Ad}^K$, *there exists some* $w \in {}^K W$ *and some ad-nilpotent ideal* I *of* $\dot{w}\mathfrak{b}\dot{w}^{-1}$, *such that* $J = I \cap \mathfrak{p}$.

If Conjecture 2.2 is true, then we can have a simple and natural definition of θ-nilpotent subspaces. Indeed, as we may see in Chapter 4, we can prove this conjecture for $U(m, n)$ via direct calculation. Unfortunately, we don't have a general proof of this conjecture now.

Lemma 2.3. $[J, J] \subset \mathfrak{n}_{\mathfrak{k}}$.

Proof. Since J is a sum of root spaces, it is enough to look at two noncompact roots β_1 and β_2 so that X_{β_i} is in J, and to prove that the bracket of root spaces is in $\mathfrak{n}_{\mathfrak{k}}$. If $\beta_1 = -\beta_2$, then $X_{\beta_1} + X_{\beta_2}$ is a nonzero semisimple elements $\in J$, contradiction. So assume $\beta_1 + \beta_2$ is not zero. If it's not a root, then the bracket is zero. So suppose $\beta_1 + \beta_2$ is a root α. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, α has to be compact. If it's positive, we're done; so suppose it's negative. Then $\beta_1 - \alpha = -\beta_2$, so $[X_{\beta_1}, X_{-\alpha}]$ is a nonzero multiple of $X_{-\beta_2}$. But the bracket is in $[J, \mathfrak{n}_{\mathfrak{k}}]$, so it follows that $-\beta_2$ is a weight of J, and again we get a semisimple element in J. □

Corollary 2.4. *The space* $J + \mathfrak{n}_{\mathfrak{k}}$ *is an* $\mathfrak{h}$*-stable nilpotent subalgebra, so it is contained in the nilradical of one of the Borel subalgebras* $w\mathfrak{b}$, *for some* $w \in {}^K W$.

□

For any ad-nilpotent ideal I of a Borel subalgebra, the G-saturation of I is the closure of one nilpotent G-orbit. We denote this orbit by $\mathcal{O}_I$.

The K-saturation of a θ-nilpotent subspace J is always irreducible, closed and lies in the nilcone of $\mathfrak{p}$, hence is the closure of one unique nilpotent K-orbit, which we denote by $\mathcal{O}_J^K$.

We partially order the ad-nilpotent ideals of $\mathfrak{b}$ by inclusion, writing $I_1 \preceq I_2$ if I_1 is contained in I_2. Similarly, we partially order θ-nilpotent subspaces by inclusion.

We partially order nilpotent orbits in $\mathfrak{g}$ by inclusion of closures, writing $\mathcal{O}_1 \preceq \mathcal{O}_2$ if $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$. Similarly, for nilpotent orbits of K, we write $\mathcal{O}_1^K \preceq \mathcal{O}_2^K$ if $\mathcal{O}_1^K \subseteq \overline{\mathcal{O}_2^K}$.

First it's easy to see that if one ideal is contained in another, i.e. $I_1 \subset I_2$, then $\mathcal{O}_{I_1} \preceq \mathcal{O}_{I_2}$. On the other hand, if $\mathcal{O}_1$, $\mathcal{O}_2$ are two nilpotent G-orbits and $\mathcal{O}_1 \preceq \mathcal{O}_2$, the following theorem of Sommers showed that it's possible to find two ideals $I_1 \subset I_2$ and $\mathcal{O}_{I_1} = \mathcal{O}_1$, $\mathcal{O}_{I_2} = \mathcal{O}_2$ under the assumption that there's no intermediate orbit between $\mathcal{O}_1$ and $\mathcal{O}_2$.

Theorem 2.5. [34, Thm4.2] *Suppose $\mathcal{O}_1$, $\mathcal{O}_2$ are two nilpotent G-orbits such that $\mathcal{O}_1 \preceq \mathcal{O}_2$ and there's no intermediate orbit between $\mathcal{O}_1$ and $\mathcal{O}_2$. Then there exist two ad-nilpotent ideals I_1 and I_2, such that $I_1 \preceq I_2$ and $\mathcal{O}_{I_1} = \mathcal{O}_1$, $\mathcal{O}_{I_2} = \mathcal{O}_2$.*

We will prove a similar result for nilpotent K-orbits and θ-nilpotent subspaces. This is the precise statement.

Theorem 2.6. *Suppose $\mathcal{O}_1^K$ and $\mathcal{O}_2^K$ are two nilpotent K-orbits in $\mathfrak{p}$ and $\mathcal{O}_1^K \preceq \mathcal{O}_2^K$. There's no intermediate orbit between $\mathcal{O}_1^K$ and $\mathcal{O}_2^K$. Then there exist two θ-nilpotent subspaces J_1 and J_2, such that $J_1 \preceq J_2$ and $\mathcal{O}_{J_1}^K = \mathcal{O}_1^K$, $\mathcal{O}_{J_1}^K = \mathcal{O}_1^K$.*

2.3. Normal Triples.

Definition 2.7. *A normal triple $\{H, X, Y\}$ in $\mathfrak{g}$ is a standard triple with $\theta(X) = -X$, $\theta(Y) = -Y$ and $\theta H = H$.*

Given a normal triple $\{H, X, Y\}$, then $\mathfrak{g}$ can be decomposed into a direct sum of eigenspaces by the adjoint action of H.

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{H,i}, \qquad \text{where } \mathfrak{g}_{H,i} = \{Z \in \mathfrak{g} | [H, Z] = iZ\}.$$

We denote by $\mathfrak{g}_{H,\geqslant 2}$ the direct sum of eigenspaces of H with eigenvalues bigger or equal to 2 and define $\mathfrak{g}_{H,>2}$ in the same way. Since H is fixed by θ, each subspace $\mathfrak{g}_{H,2}$, $\mathfrak{g}_{H,\geqslant 2}$ is θ-invariant.

We write $\mathcal{O}_X$ for the G-orbit through X and $\mathcal{O}_X^K$ for the K-orbit through X.

Lemma 2.8. [20] *Let $\mathcal{O}^K$ be a nilpotent K orbit. Then there exists a normal triple $\{H, X, Y\}$ such that $\mathcal{O}_X^K = \mathcal{O}^K$, $H \in \mathfrak{h}_0$ and H is dominant for all the positive roots in $\mathfrak{k}$.*

If H is dominant, we call $I = \mathfrak{g}_{H,\geqslant 2}$ a Dynkin ideal. From Theorem 2.1.4, $\mathcal{O}_X \cap I$ is open and dense in I.

Lemma 2.9. [20] *Keep the notations as above. Then $\mathcal{O}_X^K \cap (I \cap \mathfrak{p})$ is open dense in $I \cap \mathfrak{p}$. This implies that $K(I \cap \mathfrak{p}) = \overline{\mathcal{O}}_X$.*

In particular, H is uniquely determined by the orbit $\mathcal{O}_X^K$ (see [20] and [16]).

2.4. **Proof of Theorem 2.6.** For simplicity, we write $\mathcal{O}_1$ for $\mathcal{O}_1^K$ and $\mathcal{O}_2$ for $\mathcal{O}_2^K$. First we show that if $\mathcal{O}_1 \prec \mathcal{O}_2$, there exist two ideals I and J stable under the same Borel subalgebra $\mathfrak{b}'$ with the property:

$$I \subset J;\ I, J \text{ are } \theta\text{-stable};\ K(I \cap \mathfrak{p}) = \overline{\mathcal{O}}_1 \text{ and } K(J \cap \mathfrak{p}) = \overline{\mathcal{O}}_2.$$

By Lemma 2.8, there is a normal triple $\{e_2, H, f\}$ with $e_2 \in \mathcal{O}_2$, $H \in \mathfrak{h}_0$, and H dominant for $\mathfrak{b}_{\mathfrak{k}}$. Define $J = \mathfrak{g}_{H,\geqslant 2}$. Then J is a θ-nilpotent ideal preserved by $\mathfrak{b}_{\mathfrak{k}}$ and $K(J \cap \mathfrak{p}) = \overline{\mathcal{O}}_2$.

Since $\mathcal{O}_1 \subset K(J \cap \mathfrak{p})$, there exists an element $e_1 \in \mathcal{O}_1 \cap (J \cap \mathfrak{p})$. There's a decomposition $e_1 = x + x'$ where $x \in \mathfrak{g}_{H,2} \cap \mathfrak{p}$, and $x' \in \mathfrak{g}_{H,>2} \cap \mathfrak{p}$.

Again there exists a normal triple $\{h, x, y\}$ for x. We have h and H are both semisimple elements in $\mathfrak{k}$ with eigenvalue 2 on x. We can write $H = h + z$, where $z \in Z_{\mathfrak{k}}(x)$ and $Z_{\mathfrak{k}}(x)$ is the centralizer of x in $\mathfrak{k}$. Then z is also semisimple. Since z is semisimple, it can be conjugated by some $k' \in Z_K(x, H)$ to $k'z \in \mathfrak{t}$, where $\mathfrak{t}$ is the Cartan subalgebra of $Z_K(X)$ and is contained in $\mathfrak{h}$. Here $Z_K(x, H)$ denotes the centralizer of x, H in K. In this case, $k'h = H - k'z \in \mathfrak{h}$ and we can replace the triple $\{h, x, y\}$ with $\{k'h, x, k'y\}$ and e_1 with $k'e_1 = x + k'x'$, where $x \in \mathfrak{g}_{H,2} \cap \mathfrak{p}$ and $k'x' \in \mathfrak{g}_{H,>2} \cap \mathfrak{p}$. By abuse of notation, we still use $\{h, x, y\}$ to denote the normal triple and e_1 to denote $k'e_1$. Then h and H lies in the same Cartan $\mathfrak{h}$.

Let I be $(\mathfrak{g}_{H,2} \cap \mathfrak{g}_{h,\geqslant 2}) \bigoplus \mathfrak{g}_{H,>2}$. It is θ-stable and contains e_1. It's obvious that $I \subset J$. In the proof of Theorem 4.2 in [34], Sommers showed that I does not intersect Ge_2. Then $K(I \cap \mathfrak{p}) \subsetneq K(J \cap \mathfrak{p})$.

Then we have the following inclusion:

$$\mathcal{O}_1 \subseteq K(I \cap \mathfrak{p}) \subsetneq K(J \cap \mathfrak{p}) = \overline{\mathcal{O}}_2.$$

Recall that there's no intermediate orbit between $\mathcal{O}_1$ and $\mathcal{O}_2$. Then it follows $\mathcal{O}_1 = K(I \cap \mathfrak{p})$.

Next we claim that I and J lie in the same nilradical of some Borel subalgebra $\mathfrak{b}'$ and they are the $\mathfrak{b}'$ stable ideals.

We choose a Weyl group element w in W so that H is dominant for the root system $w(\Delta^+)$. Since H is $\mathfrak{b}$ dominant, we can assume that $w\Delta^+$ contains $\Delta^+_{\mathfrak{k}}$. Then J is a $w(\mathfrak{b})$-stable ideal and lies in $w(\mathfrak{n})$.

Let L be the subgroup of G with Lie algebra $\mathfrak{g}_{H,0}$. Then $\mathfrak{h} \subset \mathfrak{g}_{H,0}$ and let W_L be the Weyl group of L. We conjugate $w\mathfrak{b}$ by some Weyl group element $w_{\mathfrak{l}}$ with the property that h is dominant for the root system $w_{\mathfrak{l}}(w\Delta^+ \cap \Delta(\mathfrak{l}))$. We can assume that $w_{\mathfrak{l}}\Delta^+_{\mathfrak{k}} = \Delta^+_{\mathfrak{k}}$. Since W_L fixes H, it also fixes the $w\mathfrak{b}$ ideal $\mathfrak{g}_{H,\geqslant 2}$. In other words, $\mathfrak{g}_{H,\geqslant 2}$ is a $w_{\mathfrak{l}}w\mathfrak{b}$-stable ideal.

Next by following an analogous statement in Sommers' paper [34], we can also prove that I is also a $w_{\mathfrak{l}}w\mathfrak{b}$-stable ideal.

Since $w_{\mathfrak{l}}w\Delta^+_{\mathfrak{k}} = \Delta^+_{\mathfrak{k}}$, $I \cap \mathfrak{p}$ and $J \cap \mathfrak{p}$ are the two θ-nilpotent subspaces we want. □

3. The Associated Hyperplane Arrangement for G

3.1. **Background.** The idea of θ-nilpotent subspaces in the real case is an analogue of ad-nilpotent ideals in the complex case. In the complex case, the ad-nilpotent ideals are closely related to the theory of affine Weyl groups and hyperplane arrangements. For example, Cellini and Papi showed in [12] that there exists a bijection between the set of dominant minimal affine Weyl group elements and the set of ad-nilpotent ideals. It was also proved by J. Y. Shi in [32] that the set of ad-nilpotent ideals is in bijection with the set of dominant regions of the Shi arrangement (and the set of dominant regions of the Catalan arrangement) (see the definition below).

When we turn to the situation of real groups, there is no known bijection between the set of θ-nilpotent spaces and some special affine Weyl group elements. This is because in general there might be several minimal elements in the affine Weyl group corresponding to one θ-nilpotent space. However, we will see that there still exists a bijection between the set of θ-nilpotent subspaces and the set of dominant regions of certain special hyperplane arrangements. Therefore, the theory of hyperplane arrangements can be applied to the case of real groups with some necessary modifications.

Definition 3.1. *A hyperplane arrangement is a finite collection of affine hyperplanes of* $\mathbb{R}^n$.

Definition 3.2. *A region of an arrangement* $\mathcal{A}$ *is a connected component of the complement* X *of the hyperplanes:*

$$X = \mathbb{R}^n - \cup_{H \in \mathcal{A}} H.$$

A bounded region is a region that is contained in a ball of finite radius.

The number of the regions of $\mathcal{A}$ is denoted by $r(\mathcal{A})$. The number of bounded regions is denoted by $b(\mathcal{A})$.

For instance, let G be a complex reductive Lie group of rank n. Let Δ be the set of roots and Δ^+ be the set of positive roots of G. Let $V = X_* \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, where X_* is the coweight lattice. The simplest example of hyperplane arrangement is $\{H_{\alpha,0} \mid \alpha \in \Delta\}$. Its regions are the Weyl chambers. We call it the Coxeter arrangement that corresponds to the root system Δ.

If moreover the arrangement $\mathcal{A}$ contains the Coxeter arrangement as a subarrangement, we call a region *dominant* when it is contained in the fundamental chamber of V.

By definition in [30] and [31], the collection of hyperplanes of the Shi arrangement is given by

$$Shi = \{H_{\alpha,k} \mid \alpha \in \Delta^+, k = 0, 1\}.$$

The collection of hyperplanes of the Catalan arrangement is given by

$$Cat = \{H_{\alpha,k} \mid \alpha \in \Delta^+, k = 0, 1, -1\}.$$

In the papers [3] and [4], Athanasiadis introduced the generalized Catalan arrangements. A generalized Catalan arrangement is defined by

$$gCat = \{H_{\alpha,k} \mid \alpha \in \Delta^+, k = 0, 1, \dots m\}$$

for some positive integer m. The characteristic polynomials of the general Catalan arrangements are similar to those of the usual Catalan arrangements.

From the definition above, the dominant regions of the Shi arrangement and the Catalan arrangement are exactly the same.

3.2. **Characteristic Polynomials.** For an arbitrary arrangement $\mathcal{A}$, let $L_{\mathcal{A}}$ be the set of non-empty intersections of hyperplanes in $\mathcal{A}$, including V itself as the intersection of empty set. The partial order on $L_{\mathcal{A}}$ is defined by the reverse inclusion principle. Namely, for any $x, y \in L_{\mathcal{A}}$, $x \preceq y$ if and only if $x \supseteq y$. Given by this partial order, $L_{\mathcal{A}}$ becomes an intersection poset. In particular, the space V is the minimal element in the poset $L_{\mathcal{A}}$ and is denoted by $\hat{0}$.

The Möbius function μ of $L_{\mathcal{A}}$ is defined recursively by

$$\mu(x,x) = 1, \text{ for any } x \in L_{\mathcal{A}};$$
$$\mu(x,y) = -\sum_{x \leqslant z < y} \mu(x,z), \text{ for all } x < y \text{ in } L_{\mathcal{A}}.$$

Using the Möbius function, the characteristic polynomial is defined by

$$\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{dim x}.$$

In geometry, let $\mathcal{M}$ be the complex manifold $\mathbb{C}^n - \cup_{H \in \mathcal{A}} H$, where $\mathcal{A}$ is considered as the set of complexified hyperplanes in $\mathbb{C}^n$. The Poincaré polynomial of $\mathcal{M}$ is

$$p(\mathcal{M}, t) = \sum_{p \geqslant 0} H^p(\mathcal{M}, \mathbb{C}) t^p,$$

where $H^p(\mathcal{M}, \mathbb{C})$ is the p-th cohomology of the complement $\mathcal{M}$. It was proved by Orlik and Solomon in [24] that the relation between the characteristic polynomial and the Poincaré polynomial is

$$p(\mathcal{M}, t) = (-t)^n \chi(\mathcal{A}, -t^{-1}).$$

The geometric method to compute the characteristic polynomial seems not straightforward. In combinatorics, a practical method to compute the characteristic polynomial is to restrict the hyperplanes in the finite field and count the number of some points, which would give us a polynomial. This is the finite field method.

The origin of finite field method was implicit in the work of Crapo and Rota (see [14]) about the Möbius inversion argument.

Motivated by their ideas and the work of Blass and Sagan in [7], Athanasiadis developed a systematic tool to compute characteristic polynomials for all hyperplane arrangements defined over integers. In particular, He used the abelian group $\mathbb{Z}_q$ of integers modulo q instead of a power prime.

An hyperplane arrangement $\mathcal{A}$ is called a *$\mathbb{Z}$-arrangement* if its hyperlanes are given by equations with integer coefficients. If we reduce the coefficients of $\mathcal{A}$ modulo q, then $\mathcal{A}$ defines an arrangement in $\mathbb{Z}_q^n$. Then Athanasiadis showed that

Theorem 3.3. [1] [2] [5] [6] *Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement in $\mathbb{R}^n$. There exist positive integer r, k which depend only on $\mathcal{A}$, such that for all q relatively prime to r with $q > k$,*

$$\chi(\mathcal{A}, q) = \sharp(\mathbb{Z}_q^n - \cup_{H\in\mathcal{A}} H).$$

A good exposition of finite field method is the lecture note [36] of Stanley.

Once we have obtained the characteristic polynomial of the arrangement $\mathcal{A}$, it can be applied immediately to count the number of regions $r(\mathcal{A})$ and the number of bounded regions $b(\mathcal{A})$ of the arrangement $\mathcal{A}$. This is based on a result of Zaslavsky [37, section 2].

Theorem 3.4. *For any hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$, we have*

$$r(\mathcal{A}) = (-1)^n \chi(\mathcal{A}, -1)$$

and

$$b(\mathcal{A}) = |\chi(\mathcal{A}, 1)| = |\sum_{x\in L_{\mathcal{A}}} \mu(\hat{0}, x)|,$$

where $|\cdot|$ denotes the absolute value.

Fix a positive root system Δ^+ and two integers $a \leqslant b$, we denote by $\hat{\mathcal{A}}^{[a,b]}(\Delta)$ the hyperplane arrangement defined by

$$(\alpha, x) = k \text{ for } \alpha \in \Delta^+ \text{ and } k = a, a+1, \dots, b.$$

In particular, if $\Delta = A_n$, we denote the hyperplane arrangement by $\hat{\mathcal{A}}_n^{[a,b]}$ and if $\Delta = BC_n$, we denote it by $\hat{\mathcal{BC}}_n^{[a,b]}$.

Remark. *In Theorem 3.3, the assumption for q is that q is relatively prime to an integer r and is big enough. As discussed in the remark after Theorem 2.1 of* [5]*, Athanasisiadis stated that if the arrangement $\mathcal{A}$ is contained in some $\hat{\mathcal{A}}_n^{[a,b]}$ and $\hat{\mathcal{BC}}_n^{[a,b]}$ respectively, then the choice of r is* 1 *or* 2*. We will use this fact in Lemma 3.12 and 3.18.*

3.3. **The Real Hyperplane Arrangements.** One motivation to study the real hyperplane arrangement comes from the paper [15] of Gunnells and Sommers and the paper [27] of Panyushev. In the complex case, there is a one-to-one correspondence between the set of dominant regions of the Shi (or the Catalan) arrangement and the set of ad-nilpotent ideals of $\mathfrak{g}$. In the real case, we will exhibit a one-to-one correspondence between the set of dominant regions of certain real hyperplane arrangement and the set of θ-nilpotent subspaces.

The general setting in this subsection is the same as in section 1.

Definition 3.5. *The real hyperplane arrangement $\mathcal{A}$ is the set of hyperplanes*

$$\{H_{\alpha,0} \mid \alpha \in \Delta_{\mathfrak{k}}^+\} \cup \{H_{\alpha,1} \mid \alpha \in \Delta_{\mathfrak{p}}\},$$

where $H_{\alpha,k} = \{v \in V \mid \alpha(v) = k\}$.

This set of hyperplanes cuts V into open regions. In particular, the boundary of the real dominant chamber is the set $\{H_{\alpha,0} \mid \alpha \in \Delta_{\mathfrak{k}}\}$, which is a subset of the real hyperplane arrangement. Hence the regions in the real dominant chamber are the open connected components of $\mathcal{C}_{\mathfrak{k}} - \cup_{\alpha\in\Delta_{\mathfrak{p}}} H_{\alpha,1}$. We denote by $\mathcal{R}$ the set of regions in $\mathcal{C}_{\mathfrak{k}}$. Given a region $R \in \mathcal{R}$, we define a subspace of $\mathfrak{p}$ by

$$I_R = \oplus_{\alpha\in\mathcal{I}_R}\mathfrak{g}_\alpha, \quad \text{where } \mathcal{I}_R = \{\alpha \in \Delta_{\mathfrak{p}} \mid \alpha(x) > 1 \text{ for any } x \in R\}.$$

Then we have the following lemma.

Lemma 3.6. *Let I_R be a θ-nilpotent subspace and $\mathcal{I}_R$ be its corresponding set of roots. Then*

(i) If $\mu \in \mathcal{I}_R$, $\gamma \in \Delta_{\mathfrak{k}}^+$ and $\mu + \gamma \in \Delta_{\mathfrak{p}}$, then $\mu + \gamma \in \mathcal{I}_R$.

(ii) The subspace I_R is $\mathfrak{b}_{\mathfrak{k}}$-stable, where $\mathfrak{b}_{\mathfrak{k}}$ is the Borel subalgebra of $\mathfrak{k}$ corresponding to $\Delta_{\mathfrak{k}}^+$.

(iii) The K-saturation of I_R is the closure of a unique nilpotent K-orbit.

Proof. Since the region R lies in the real dominant chamber, $\gamma(x) > 0$ for all $x \in R$. (i) holds. (ii) follows from (i).

To prove (iii), we choose a generic element $h \in R$, i.e. $\alpha(h) \neq 0$ for all $\alpha \in \Delta$. There exists an element $w \in W$, such that wh is dominant, i.e. $\alpha(wh) > 0$, for all $\alpha \in \Delta^+$. Let $J = \oplus_{\alpha\in\mathcal{J}}\mathfrak{g}_\alpha$, where $\mathcal{J} = \{\alpha \in \Delta^+ \mid \alpha(wh) > 1\}$. Then J is an ad-nilpotent ideal of $\mathfrak{b}$ and I_R is a equal to $w^{-1}J \cap \mathfrak{p}$. By its definition, I_R is θ-nilpotent.

Proposition 3.7. *There exists a bijection between the set of dominant regions of the real hyperplane arrangement $\mathcal{A}$ and the set of θ-nilpotent subspaces.*

Proof. Given a region $R \in \mathcal{R}$, in the proof of Lemma 3.17(iii), we actually constructed an ideal J of $\mathfrak{b}$, such that $I_R = w^{-1}J \cap \mathfrak{p}$ is a θ-nilpotent subspace. Conversely, given a θ-stable subspace I, and the corresponding set $\mathcal{I}$ of weights of I, the region is defined by $R = (\cap_{\alpha\in\mathcal{I}} H_{\alpha,+}) \cap \mathcal{C}_{\mathfrak{k}}$, where $H_{\alpha,+} = \{x \in V \mid \alpha(x) > 1\}$. Indeed, the region can be rewritten as

$$R = \{x \in \mathcal{C}_{\mathfrak{k}} \mid \gamma(x) > 1, \forall\gamma \in \mathcal{I} \text{ and } \gamma(x) < 1, \forall\gamma \in \Delta_{\mathfrak{p}} - \mathcal{I}\}.$$

We only need to check that R is nonempty. By the definition of θ-nilpotent subspaces, there exists an ad-nilpotent ideal J of $w\mathfrak{b}$ ($w \in {}^{K}W$), such that $I = J \cap \mathfrak{p}$. Then $w^{-1}J$ is an ad-nilpotent ideal of $\mathfrak{b}$ and by Proposition 1.4.1, there exists a sign type $\mathcal{S}$ corresponding to $w^{-1}J$. Then $w\mathcal{S}$ lies in R, which shows that R is nonempty. □

For any θ-nilpotent subspace I, we denote its corresponding region by R_I.

Recall that the affine Weyl group played an important role in the description of ad-nilpotent ideals. Here it also has some applications. In this subsection, we keep the same notation of $\mathcal{C}_0$ (the fundamental alcove), $\widehat{W}$, $\widehat{\Delta} = \{k\delta + \Delta \mid k \in \mathbb{Z}\}$ and $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$ as in [9]. Since the hyperplanes in $\mathcal{A}$ are among those defining the alcoves of V, any alcove of V is contained in some regions of $\mathcal{A}$.

Given a θ-nilpotent subspace I, pick an element $\hat{w} \in \widehat{W}$, such that $\hat{w}^{-1}(\mathcal{C}_0) \subset R_I$. That means that the hyperplane $H_{\gamma,1}$ separates $\mathcal{C}_0$ from $\hat{w}^{-1}(\mathcal{C}_0)$ for any $\gamma \in \mathcal{I}$ and $\mathcal{C}_0$ and $\hat{w}^{-1}(\mathcal{C}_0)$ lie in the same side of $H_{\gamma,1}$ when $\gamma \in \Delta_{\mathfrak{p}} - \mathcal{I}$. That is, if γ is a noncompact root, then $\hat{w}(\delta - \gamma) < 0$ if and only if $\gamma \in \mathcal{I}$.

Some results of ad-nilpotent ideals in [28] are still valid for the θ-nilpotent subspaces. For example, if I is θ-nilpotent subspace, set $|I| = \sum_{\mathfrak{g}_\gamma \subset I} \gamma$. Then we have

Lemma 3.8. *Let I_1, I_2 be two θ-nilpotent subspaces and $|I_1| = |I_2|$, then $I_1 = I_2$.*

Proof. Let $\mathcal{I}_1$, $\mathcal{I}_2$ be the corresponding sets of roots of I_1 and I_2. We write $\mathcal{I}_1 \backslash \mathcal{I}_2$ (resp. $\mathcal{I}_2 \backslash \mathcal{I}_1$) as the set of roots that is contained in $\mathcal{I}_1$ but not in $\mathcal{I}_2$ (resp. the roots in $\mathcal{I}_2$ but not in $\mathcal{I}_1$). Suppose that $I_1 \neq I_2$, then either $\mathcal{I}_1 \backslash \mathcal{I}_2$ or $\mathcal{I}_2 \backslash \mathcal{I}_1$ is nonempty or both. Since $|I_1| = |I_2|$, we have $\sum_{\beta\in\mathcal{I}_1\backslash\mathcal{I}_2} \beta = \sum_{\gamma\in\mathcal{I}_2\backslash\mathcal{I}_1} \gamma$. This equality can be rewritten as

$$\sum_{\beta\in\mathcal{I}_1\backslash\mathcal{I}_2} (\delta - \beta) - c\delta = \sum_{\gamma\in\mathcal{I}_2\backslash\mathcal{I}_1} (\delta - \gamma)$$

where $c = \dim I_1 - \dim I_2$. Without loss of generality, we assume that $c \geqslant 0$. Pick an element $\hat{w} \in \widehat{W}$ such that $\hat{w}^{-1}(\mathcal{C}_0) \subset R_{I_1}$. Then $\hat{w}(\delta - \beta) < 0$ for any $\beta \in \mathcal{I}_1 \backslash \mathcal{I}_2$ and $\hat{w}(\delta - \gamma) > 0$ for any $\gamma \in \mathcal{I}_2 \backslash \mathcal{I}_1$. Moreover $\hat{w}$ fixes $c\delta$. We apply $\hat{w}$ to the equation above and get a contradiction. □

The proof is basically the same as in [28] except that we need to change the minimal element in [28] to an arbitrary affine Weyl group element in R_{I_1}.

Given any region R in the real dominant chamber $\mathcal{C}_{\mathfrak{k}}$, and its corresponding θ-nilpotent subspace I_R, recall from section 1 that under the moment map, I_R is mapped to the closure of one unique nilpotent K-orbit, which is denoted by $\mathcal{O}_{I_R}$. Namely, the closure of the orbit $\mathcal{O}_{I_R}$ is the K-saturation of I_R.

Let $\mathcal{R}$ be the set of dominant regions of $\mathcal{A}$ in $\mathcal{C}_{\mathfrak{k}}$. Given a nilpotent K-orbit $\mathcal{O}$, define

$$N_{\mathcal{O}} = \{R \in \mathcal{R} \mid K \cdot I_R = \overline{\mathcal{O}}\}.$$

Recall in the subsection 2.3 of section 2, we have discussed normal triples. There exists a unique $H \in V = \mathfrak{h}_0$ that corresponds to the orbit $\mathcal{O}$. Then we have the following lemma.

Lemma 3.9. *$\frac{1}{2}H \in N_{\mathcal{O}}$ and for all $h \in N_{\mathcal{O}}$, $|h| \geqslant \frac{1}{2}|H|$, where $||$ denotes the length function. If $h \neq \frac{1}{2}H$, then $|h| > \frac{1}{2}|H|$.*

Proof. It's a special case of Proposition 2.4 in [27]. □

Remark. *The complex case is proved in* [15].

3.4. **The Real Hyperplane Arrangement for $U(m,n)$.** In this section, let $G_{\mathbb{R}} = U(m,n)$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{u}(m,n)$. The complexifications of $G_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}$ are $G = GL(m+n)$ and $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{gl}(m+n)$. The Cartan involution θ is given by

$$\theta(g) = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix} g \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}, \text{ for any } g \in GL(m+n).$$

Then $K = GL(m) \times GL(n)$ (embedded block diagonally into G). The space $V = i\mathfrak{h}_{\mathbb{R}}$ is isomorphic to $\mathbb{R}^{m+n}$.

The real hyperplane arrangement for $U(m,n)$ is the following set of hyperplanes

$$(1) \qquad x_i - x_j = 0, \text{ for } 1 \leqslant i,j \leqslant m, \text{ or } m+1 \leqslant i,j \leqslant m+n,$$

$$(2) \qquad x_i - x_j = 1, -1, \text{ for } 1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant m+n$$

and is denoted by $\mathcal{A}_{m,n}$.

Example 3.10. *The characteristic polynomial for $\mathcal{A}_{2,1}$ is $q(q^2-5q+6) = q(q-2)(q-3)$ (see Figure 4-1).*

Theorem 3.11. *The characteristic polynomial of $\mathcal{A}_{m,n}$ is given by*

$$\chi(\mathcal{A}_{m,n}, q) = q \prod_{i=1}^{m-1}(q-n-i) \prod_{j=0}^{n-1}(q-m-n+j).$$

To prove this theorem, we follow [1] [2] and apply the finite field method introduced in subsection 4.1.3.

Suppose $\mathcal{A}$ is an arrangement in $\mathbb{R}^{m+n}$ consisting of distinct hyperplanes of the form $a_1x_1 + a_2x_2 + \cdots + a_{m+n}x_{m+n} = a_0$ with all a_i in $\mathbb{Z}$

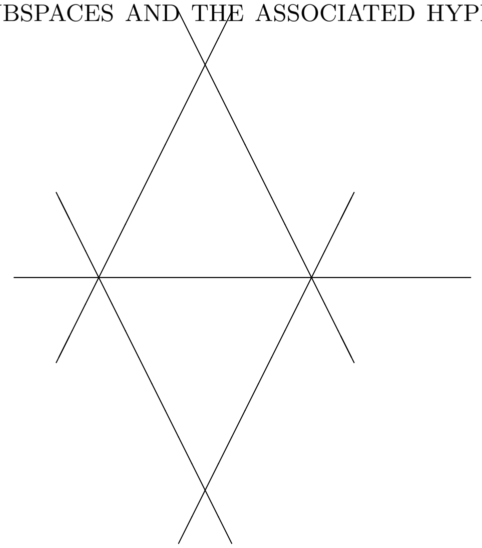

FIGURE 1. The real hyperplane arrangement for $\mathcal{A}_{2,1}$ on the space $x_1 + x_2 + x_3 = 0$

and $a_1, a_2, \ldots, a_n$ not all zero. Let q be a large enough integer satisfying the assumption of Theorem 3.3. By Theorem 3.3, $\chi(\mathcal{A}, q)$ counts the number of $(n+m)$-tuples $(x_1, x_2, \ldots, x_{m+n})$ in $\mathbb{Z}_q^{n+m}$ which satisfy conditions of the form

$$a_1 x_1 + \cdots + a_{m+n} x_{m+n} \neq a_0.$$

Each $(n+m)$-tuple $(x_1, x_2, \ldots, x_{n+m})$ in $\mathbb{Z}_q^{n+m}$ can be considered as a map from the index set $[n+m] = \{1, 2, \ldots, n+m\}$ to $\mathbb{Z}_q$, with the image of i being x_i.

Elements of $\mathbb{Z}_q$ can be arranged clockwise in a circle with q boxes. The zero class of $\mathbb{Z}_q$ is placed on the top of the circle and 1 mod $\mathbb{Z}_q$ is placed next to the zero class in the clockwise order, etc. Under this idea, an $(n+m)$-tuple $(x_1, x_2, \ldots, x_{m+n})$ in $\mathbb{Z}_q^{m+n}$ is a placement of integers from 1 to $m+n$ into q boxes on the circle with the integer i goes into the box x_i. The defining equations for $\mathcal{A}$ (which are not satisfied) give certain restrictions on the placement. For example, if $\mathcal{A}$ contains the Coxeter arrangement $\mathcal{A}_{m+n-1}$ as a subarrangement , it means that any distinct integers i and j should not be placed into the same box. If $\mathcal{A}$ contains the hyperplanes $x_i - x_j = 1$, then the integer i could not follow j immediately in the clockwise order.

Suppose that the equations in $\mathcal{A}$ are all of the form $x_i - x_j = m$ (This is the case for the real hyperplane arrangement $\mathcal{A}_{m,n}$). In the interpretation of elements of $\mathbb{Z}_q^{m+n}$ as placements of the integers from 1 to $m+n$ into q boxes on a circle, such an equation says that i should not be placed m boxes clockwise from j. Such a condition does not refer to which box is labeled 0, but only to relative position around the circle.

We may therefore consider a new formula

$$\tilde{\chi}(\mathcal{A}, q) \tag{4.1.5}$$

which counts placements of the integers from 1 to $m+n$ into q boxes on a circle; two placements differing by a rotation of the circle are now regarded as equal. Each such placement corresponds to q different placements of type $\mathcal{A}$, corresponding to the q possible boxes to label as 0. The relation between $\chi(\mathcal{A}, q)$ and $\tilde{\chi}(\mathcal{A}, q)$ is therefore:

$$\tilde{\chi}(\mathcal{A}, q) = \frac{1}{q}\chi(\mathcal{A}, q).$$

This shows in particular that $\chi(\mathcal{A}, q)$ always has q as a factor.

For large enough q, $\tilde{\chi}(\mathcal{A}, q)$ is the characteristic polynomial of $\mathcal{A}$ restricted in the hyperplane $x_1 + \cdots + x_{m+n} = 0$, therefore, it is the characteristic polynomial for the real group $SU(m, n)$.

For two integers i and j, if $x_i - x_j = 1$, we say that i is consecutive to j. We call i, j next to each other if $x_i - x_j = \pm 1$. Those boxes that are not filled with integers are called "unlabeled" boxes.

We divide the index set $[n+m]$ into two groups. The first group is the subset $\{1, 2, \ldots, m\}$ and the remaining integers $m + [n] = \{m + 1, m+2, \ldots, n+m\}$ form the second group. Indeed these two groups of integers are invariant under $S_n \times S_m$ (the Weyl group of K).

By defining equation (1) of $\mathcal{A}_{m,n}$, one can tell that two integers from the same group should not be placed into the same box and by equation (2) of $\mathcal{A}_{m,n}$, two integers from different groups should not be placed next to each other.

One difficulty to count the number $\chi(\mathcal{A}_{m,n}, q)$ directly is that two integers from different groups could be placed in the same box. To overcome such difficulty, we may apply some idea in [1]. That is, we may introduce a new arrangement that contains the Coxeter arrangement $\mathcal{A}_{m+n-1}$ as a subarrangement and find a relation between the characteristic polynomials of the two arrangements.

Theorem 3.13 follows from the following two lemmas.

Remark. *By the remark at the end of subsection 4.1.2, $r = 1$ and we can assume $q, q-m$ are both big enough and relatively prime to* 1.

Lemma 3.12. *Let $\tilde{\mathcal{A}}_{m,n}$ be the hyperplane arrangement that's defined by*

$$x_i - x_j = 0, 1, \text{ for } 1 \leqslant i, j \leqslant m, \tag{3}$$

$$x_i - x_j = 0, \text{ for } m+1 \leqslant i, j \leqslant m+n, \tag{4}$$

$$x_i - x_j = 0, 1, -1, \text{ for } 1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant m+n. \tag{5}$$

Suppose q and $q-m$ satisfies the requirements of Theorem 3.3. We have

$$\tilde{\chi}(\mathcal{A}_{m,n}, q-m) = \tilde{\chi}(\tilde{\mathcal{A}}_{m,n}, q).$$

Proof. A placement of type $\alpha(q-m)$ is a placement of the integers from 1 to $m+n$ into $q-m$ boxes arranged around a circle (modulo rotations of the circle), subject to the following requirements:

-two integers from the same group cannot be placed in the same box; and

-two integers from different groups cannot be placed next to each other.

The number of placements of type $\alpha(q-m)$ is $\tilde{\chi}(\mathcal{A}_{m,n}, q-m)$.

For example, if we cut the circle of boxes and unfold it, in the clockwise order, the possible form of a sub-string of the boxes could be :

$$\Box i_0/j_0 \Box i_1 i_2 \dots i_k \Box \dots \Box j_1 j_2 \dots j_l \Box\Box\Box$$

where $1 \leqslant i_0, \dots, i_k \leqslant m$, $m+1 \leqslant j_0, \dots, j_l \leqslant m+n$, $\Box$ denotes the unlabeled boxes, i_0/j_0 denotes that the integer i_0 and j_0 fill the same box in the string and $i_1 \dots i_k$ denotes that integers $i_1, \dots, i_k$ fill distinct boxes in the string.

A placement of type $\beta(q)$ is a placement of the integers from 1 to $m+n$ into q boxes around a circle (also modulo rotations of circle), subject to the following requirements:

-two integers from the first group cannot be placed in the same box or next to each other (equation (3));

-two integers from the same group cannot be placed in the same box (equation 4); and

-two integers from different groups cannot be placed in the same box or next to each other (equation (5)).

The number of placements of type $\beta(q)$ is $\tilde{\chi}(\tilde{\mathcal{A}}_{m,n}, q)$.

For example, one string of the placement of type $\beta(q)$ could be partly of form

$$\Box i_1 \Box \dots \Box i_2 \Box \dots \Box j_1 \dots j_l \Box$$

where $\Box$, i_1, i_2, and $j_1 \dots, j_l$ have the same meaning as before.

In order to prove the lemma, we construct a bijection between placements of type $\alpha(q-m)$ and placements of type $\beta(q)$.

For any placement of type $\beta(q)$, the boxes that are next to the integer i, when $1 \leqslant i \leqslant m$ are always boxes unfilled with integers. Remove each unfilled box that's consecutive to i in the clockwise order. The remaining $q-m$ boxes form a circle filled with $m+n$ integers and different integers are always in different boxes. After the removal of m boxes, in the clockwise order, the possible form of string of consecutive integers could only be

$$\Box i_1 i_2 \dots i_k j_1 j_2 \dots j_l \Box$$

where $1 \leqslant i_1, \dots, i_k \leqslant m$, $m+1 \leqslant j_1, \dots, j_l \leqslant m+n$. The string of integers always begins with integers from the first group because only the unlabeled boxes following integers from the first group are removed and sting of integers from the second group always end with an unlabeled box.

When $k=0$ or $l=0$, the string of integers satisfies the restrictions for placements of type $\alpha(q-m)$. If k and l are both nonzero, then two integers from different groups are consecutive to each other, which contradicts the restriction for $\mathcal{A}_{m,n}$. Then we need to rearrange this string of integers to get a placements of type $\alpha(q-m)$. If $k > l$, in the clockwise order, we rearrange the string in the following form

$$\Box i_1 \dots i_{k-l} \Box i_{k-l+1}/j_1 \Box \dots \Box i_k/j_l.$$

If $k < l$, we rearrange the string in the form

$$\Box i_1/j_1 \Box i_2/j_2 \Box \dots \Box i_k/j_k \Box j_{k+1} \dots j_l.$$

After this readjustments, two integers from different groups are not consecutive to each other, but it's possible to have such two integers into the same box. This gives a placement of type $\alpha(q-m)$.

On the other hand, given any placement of type $\alpha(q-m)$, we could reverse our operations and get a placement of type $\beta(q)$. This gives us a bijection between these two placements and the equality for the characteristic polynomial follows. □

Lemma 3.13.

$$\chi(\tilde{\mathcal{A}}_{m,n}, q) = q \prod_{i=1}^{m-1} (q-m-n-i) \prod_{j=0}^{n-1} (q-2m-n+j).$$

Proof. Notice that $\chi(\tilde{\mathcal{A}}_{m,n}, q) = q\tilde{\chi}(\tilde{\mathcal{A}}_{m,n}, q)$. For $\tilde{\chi}(\tilde{\mathcal{A}}_{m,n}, q)$, we need to count the number of placements of type $\beta(q)$ in lemma 3.12. Since $\tilde{\mathcal{A}}_{m,n}$ includes $\mathcal{A}_{m+n-1}$ as subarrangement, the $m+n$ integers are placed into distinct boxes and there are $q-m-n$ boxes unfilled with integers. Assume the $q-m-n$ boxes arranged around a circle and we need to insert integers from 1 to $m+n$ in between these unlabeled boxes. Because of cyclic symmetry, there is 1 choice to place 1. Integers from the first group could not be next to each other so there are $q-m-n-1$ possible choices for 2 and $q-m-n-i$ choices for the $i+1$ when $i < m$. Altogether there are $\prod_{i=1}^{m-1}(q-m-n-i)$ ways to insert the first m integers.

The integers from $m+1$ to $m+n$ could not be inserted into spaces between two unlabeled boxes that already contain integers from the first group. There are $q-2m-n$ choices for $m+1$. Since integers from the second group could be next to each other, there are $q-2m-n+j-1$ ways to insert the integer $m+j$. Combining these two kinds of insertions together, we get the desired formula for $\tilde{\mathcal{A}}_{m,n}$. □

Combining the two lemmas from above, we are able to get the characteristic polynomial of $\mathcal{A}_{m,n}$. □

Applying the theorem of Zaslasvky [37, section 2], we have the following consequence.

Corollary 3.14. *The number of regions of $\mathcal{A}_{m,n}$ is*

$$r(\mathcal{A}_{m,n}) = |\chi(\mathcal{A}_{m,n}, -1)| = \prod_{i=1}^{m-1}(n+1+i)\prod_{j=1}^{n}(m+1+j)$$

and the number of bounded regions of $\mathcal{A}_{m,n}$ is

$$b(\mathcal{A}_{m,n}) = |\chi(\mathcal{A}_{m,n}, 1)| = \prod_{i=1}^{m-1}(n-1+i)\prod_{j=1}^{n}(m+j-1).$$

□

Also combining Proposition 3.7 and Theorem 3.13 above, one can count the number of θ-nilpotent subspaces.

Corollary 3.15. *The number of θ-nilpotent subspaces is equal to*

$$N(m+n+1, m) = \frac{1}{m+n+1}\binom{m+n+1}{m}\binom{m+n+1}{m+1}.$$

Proof. The arrangement $\mathcal{A}_{m,n}$ is invariant under the Weyl group of K. Therefore, the number of regions in $\mathcal{C}_{\mathfrak{k}}$ is equal to $\frac{1}{\sharp(W_K)}|\chi_{\mathcal{A}_{m,n}}(-1)|$. It's an easy calculation to show that $\frac{1}{\sharp(W_K)}r(\mathcal{A}_{m,n}) = N(m+n+1, m)$. This corollary follows from Proposition 3.7. □

Remark. *The integer $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ is called a Narayana number.*

3.5. **The Real Hyperplane Arrangement for $Sp(m,n)$.** Let $G_{\mathbb{R}} = Sp(m,n)$. Then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(m,n)$ and $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C} = \mathfrak{sp}(m+n)$. Also $K = Sp(m) \times Sp(n)$ and the set of compact roots $\Delta_{\mathfrak{k}}$ is the disjoint union of the root systems of $\mathfrak{sp}(m)$ and $\mathfrak{sp}(n)$.

Let $\mathcal{C}_{m,n}$ be the real hyperplane arrangement with the set of hyperplanes:

$$\begin{aligned}
&2x_i = 0, \text{ for } 1 \leqslant i \leqslant m+n,\\
&x_i - x_j = 0, \text{ for } 1 \leqslant i,j \leqslant m, \text{ or } m+1 \leqslant i,j \leqslant m+n,\\
&x_i + x_j = 0, \text{ for } 1 \leqslant i,j \leqslant m, \text{ or } m+1 \leqslant i,j \leqslant m+n,\\
&x_i - x_j = 1, -1, \text{ for } 1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant m+n,\\
&x_i + x_j = 1, -1, \text{ for } 1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant m+n.
\end{aligned}$$

Example 3.16. *The characteristic polynomial for $\mathcal{C}(1,1)$ is $q^2 - 6q + 9 = (q-3)^2$.*

Theorem 3.17. *The characteristic polynomial for $\mathcal{C}_{m,n}$ is*

$$\chi(\mathcal{C}_{m,n}, q) = \prod_{i=1}^{m}(q - 2(m+n) + 2i - 1)\prod_{j=1}^{n}(q - 2(m+n) + 2j - 1).$$

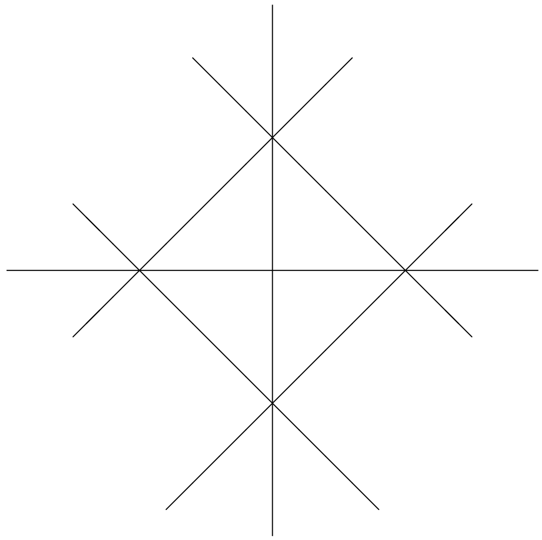

FIGURE 2. The real hyperplane arrangement for $\mathcal{C}_{1,1}$

The idea to prove the $U(m,n)$ case is still applicable here.

Let q be a large enough integer satisfying the assumption of Theorem 3.3. As in the previous case, the abelian group $\mathbb{Z}_q$ is arranged into a circle of boxes with each box denoting a class mod q. The zero class is placed on the top and all other classes in $\mathbb{Z}_q$ increase in the clockwise order. For arbitrary hyperplane arrangement $\mathcal{C}$ in type B, C, D, since we are dealing with both hyperplanes of the form $x_i \pm x_j = \alpha$, the $m{+}n$-tuples $(x_1, x_2, \ldots, x_{m+n})$ are considered as a map from $\pm[m{+}n] = \{\pm 1, \pm 2, \ldots, \pm(m+n)\}$ to $\mathbb{Z}^q$, sending i to the class $x_i \in \mathbb{Z}_q$ and $-i$ to the class $-x_i$.

Elements of $\pm[m{+}n]$ are called signed integers. The set of signed integers is divided into two groups. The first group is $\pm[m] = \{\pm 1, \ldots, \pm m\}$ and the second group is $\pm(m + [n]) = \{\pm(m+1), \ldots, \pm(m+n)\}$. The two groups of integers are invariant under the Weyl group of $K = Sp(m) \times Sp(n)$.

To count the number of $(n{+}m)$-tuples $(x_1, x_2, \ldots, x_{n+m})$ which does not satisfy the defining equation in $\mathcal{C}$, we need to count again the total number of placements of $2(m+n)$ signed integers into q boxes, with restrictions that come from the defining equation of $\mathcal{C}$. For example, the equation $2x_i = 0$ means that i should not placed into the zero class on the top of the circle. The equations $x_i \pm x_j = 0$ means that the signed integers i and $-i$ should not be placed to the same box as j.

Moreover, the signed integers are placed symmetrically around the zero class. If i is placed into the x_i class, then $-i$ is placed into the $-x_i$ class. Again we call boxes unfilled with integers "unlabeled" boxes.

The theorem follows from the following two lemmas.

Let $\tilde{\mathcal{C}}_{m,n}$ be the hyperplane arrangement defined by the following set of hyperplanes.

(1) $2x_i = 0, \text{ for } 1 \leqslant i \leqslant m+n,$

(2) $x_i \pm x_j = 0, \pm 1, \text{ for } 1 \leqslant i, j \leqslant m, \text{ and } i \neq j,$

(3) $x_i \pm x_j = 0, \text{ for } m+1 \leqslant i, j \leqslant m+n, \text{ and } i \neq j,$

(4) $2x_i = 0, \pm 1, \text{ for } 1 \leqslant i \leqslant m,$

(5) $x_i \pm x_j = 0, \pm 1, \text{ for } 1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant m+n.$

The relation between $\chi_{\mathcal{C}_{m,n}}(q)$ and $\chi_{\tilde{\mathcal{C}}_{m,n}}(q)$ is given by the following lemma.

Remark. *By the remark at the end of subsection 4.1.2, $r = 2$, if we assume q is odd and big enough, then q and $q-2m$ both satisfy the assumption of Theorem 3.3.*

Lemma 3.18. *Let q be an odd integer and $q-2m$ satisfy the condition of Theorem 3.3. Then*

$$\chi(\tilde{\mathcal{C}}_{m,n}, q) = \chi(\mathcal{C}_{m,n}, q-2m).$$

Proof. A placement of type $\alpha(q-2m)$ is a placement of the signed integers from ± 1 to $\pm(m+n)$ into $q-2m$ boxes around a circle, subject to the following requirements:

-the signed integers i and $-i$ should be placed symmetrically around the zero class;

-two integers from the same group cannot be placed to the same box; and

-two signed integers from distinct groups cannot be placed next to each other.

The number of placements of type $\alpha(q-2m)$ is $\chi_{\mathcal{C}_{m,n}}(q-2m)$.

A placement of type $\beta(q)$ is a placement of the signed integers from ± 1 to $\pm(m+n)$ into q boxes around a circle, subject to the following requirement:

-the signed integers i and $-i$ should be placed symmetrically around the zero class;

-any two signed integers cannot placed into the same box (equation (1)(2)(3)) and any two signed integers from the second group cannot be not placed next to each other (equation (2));

-all signed integers from the first group cannot be placed to the class $\frac{1}{2}(q-1)$ and $\frac{1}{2}(q+1)$ (equation (4)); and

-two signed integers from distinct groups cannot placed next to each other (equation (5)).

The number of placements of type $\beta(q)$ is $\chi_{\tilde{\mathcal{C}}_{m,n}}(q)$.

To get a bijection between these two kinds of placements, we need to remove $2m$ unlabeled boxes from the placements of type $\beta(q)$ and readjust the position of some signed integers to get a placement of type $\alpha(q-2m)$.

From the description of two placements above, i and $-i$ could not appear both on the right half circle from the class zero to the class $\frac{1}{2}(q-1) \ mod \ \mathbb{Z}_q$. It suffices to do the operation the right semicircle and do a symmetric operation on the left side.

Given a placement of type $\beta(q)$, remove one unlabeled box that is clockwise consecutive to a signed integer from the right half circle. Remove the same amount of unlabeled boxes on the left hand side in a symmetric way. This operation is possible because of equation (2) and (4). Equation (2) ensures that signed integers from the first group is not next to any signed integers. Equation (4) ensures that elements from the first group would not occupy the last position $\frac{1}{2}(q-1)$.

Similar to the $U(m,n)$ case, a consecutive string of integers on the right semicircle only has form

$$i_1 i_2 \dots i_k j_1 j_2 \dots j_l,$$

where $i_1, i_2, \dots, i_k$ come from the first group and $j_1, j_2, \dots, j_l$ come from the second group. To readjusting the position of these integers, we may choose a way that's slightly different from the previous case. When $k < l$, then in the clockwise order, the new string would become

$$i_1/j_1 \Box i_2/j_2 \dots i_k/j_k \Box j_{k+1} \dots j_l,$$

where i_s/j_s means that i_s and j_s are placed into the same box, $\Box$ means that it's an unlabeled box. The last part $j_{k+1} \dots j_l$ is a string of integers placed clockwise into distinct boxes that are consecutive to each other. When $k < l$, the new string is

$$i_1/j_1 \Box i_2/j_2 \dots i_k/j_k \Box i_{l+1} \dots i_k.$$

We do a symmetric operation on the left side. Under this adjustment, the last position on the right semicircle could not have two signed integers in the same box. This make sure that this placement satisfies all the equalities that are defined by $\chi_{\mathcal{C}_{m,n}}$ and we get a placement of type $\alpha(q-2m)$. The reverse operation is obvious and therefore we get a bijection of two kinds of placements, as well as the equality for the two characteristic polynomials. $\Box$

The number of placements of type $\beta(q)$ is easy to compute and we can derive the characteristic polynomial of $\tilde{\mathcal{C}}_{m,n}$.

Lemma 3.19. *The characteristic polynomial for $\tilde{\mathcal{C}}_{m,n}$ is equal to*

$$\chi(\tilde{\mathcal{C}}_{m,n}, q) = \prod_{i=1}^{m} \big(q - 2(m+n) - 2i + 1\big) \prod_{j=1}^{n} \big(q - 4m - 2n + 2j - 1\big).$$

Proof. To calculate the number of circular placements of type $\beta(q)$, again we only need to discuss the right semicircle. There are $m+n$

signed integers that should be placed on the right hand side and i, $-i$ could not appear both on the same side. The signed integers are placed into distinct boxes and the no signed integers appear on the zero class. Therefore we arrange the $\frac{1}{2}(q+1)-(m+n)$ unlabeled boxes (including the zero class on the top) around the right half circle such that $m+n$ signed integers are placed in between these boxes. Let us first consider the signed integer from the first group. They could not be placed into the $\frac{1}{2}(q-1)$ class. There are $\frac{1}{2}(q+1)-(m+n)-1$ ways to insert the first one. Since all signed integers from the first group are not next to each other, after we have inserted a signed integer, then the number of possible positions to insert next integer from the first group decreases by 1. Altogether there are

$$\prod_{i=1}^{m}(\frac{1}{2}(q+1)-(m+n)-i)$$

ways to insert signed integers from the first group.

Given a signed integer from the second group, then it could be placed to the class $\frac{1}{2}(q-1)$, but not next to any signed integers from first group. There are $\frac{1}{2}(q+1)-(2m+n)$ ways to place the first one and

$$\prod_{j=0}^{n-1}(\frac{1}{2}(q+1)-(2m+n)+j)$$

ways to place all signed integers from the second group.

Since we can choose either i or $-i$ when we place any signed integers, the total number of placements should be multiplied by 2^{m+n} and this gives us the characteristic polynomial we want. □

When $G_{\mathbb{R}}=SU(m,n)$, the Coxeter number of G is $m+n$ and the exponents of $K=S(GL(m)\times GL(n))$ are $\{0,1,\dots,m-1,1,\dots,n-1\}$. (The exponent 0 corresponds to the centralizer of K). The characteristic polynomial $\tilde{\chi}(\mathcal{A}_{m,n},q)$ for $SU(m,n)$ in Theorem 3.13 verifies the conjecture 1.1. This conjecture is not valid for $U(m,n)$.

When $G_{\mathbb{R}}=Sp(m,n)$, its Coxeter number is $2(m+n)$ and the exponents of $K=Sp(m)\times Sp(n)$ are $\{1,3,\dots,2m-1,1,\dots,2n-1\}$. Theorem 3.17 verifies this conjecture 1.1 for $Sp(m,n)$.

There are other real groups satisfying the assumption of conjecture 1.1. For example, $G_{\mathbb{R}}=Sp(n,\mathbb{R})$ or $G_{\mathbb{R}}=SO(m,n)$. We have verified the conjecture when $n=4$ or 6 for $Sp(n,\mathbb{R})$ and when $m=2$ and n is an odd integer for $SO(m,n)$.

Example 3.20. *This conjecture holds for G_2, where the short simple root α_1 is a compact root and the long simple root α_2 is a noncompact root. The characteristic polynomial for the real hyperplane arrangement of G_2 is $q^2-10q+25=(q-5)^2=(q-h+e_1)^2$, where h is the Coxeter number of G_2 and e_1 is the exponent of A_1.*

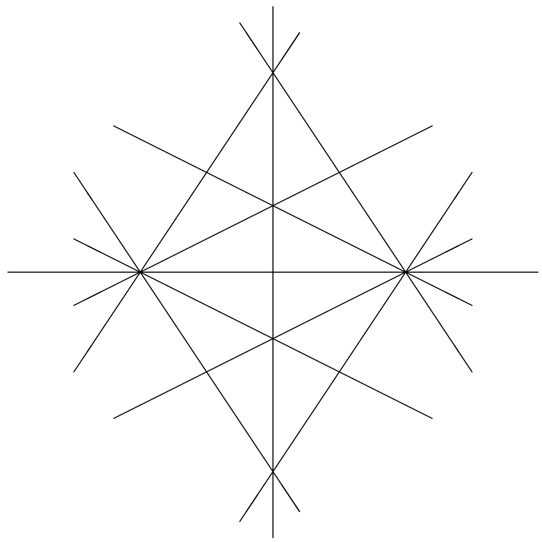

FIGURE 3. The real hyperplane arrangement for G_2

4. Some Combinatorics

4.1. **Calculation of the number of θ-nilpotent subspaces.** In this section, we will show that Conjecture 2.2 is valid for $U(m,n)$. It is proved in Corollary 3.15 that the number of θ-nilpotent subspaces is a Narayana number. Panyushev also showed in [26], the number of ad-nilpotent ideals with k generators in type A is a Narayana number.

Let $\mathfrak{Ad}^m_{m+n+1}$ be the set of ad-nilpotent ideals with m generators and $\mathfrak{Ad}_{m,n}$ be the set of θ-nilpotent subspaces. Let $\mathfrak{Ad}^{m,n}$ be the set of B_K-invariant subspaces of $\mathfrak{p}$, consisting no semisimple elements. As discussed in section 1, $\mathfrak{Ad}_{m,n}$ is a subset of $\mathfrak{Ad}^{m,n}$.

Next we will construct an explicit bijection between the ad-nilpotent ideals with m generators in type A_{m+n} and $\mathfrak{Ad}^{m,n}$. Then by using Panyushev's result, we can prove that $\mathfrak{Ad}_{m,n} = \mathfrak{Ad}^{m,n}$.

Lemma 4.1. [26] *The number of ad-nilpotent ideals in $\mathfrak{sl}(n)$ with k generators is equal to $N(n,k)$.*

Proposition 4.2. *There exists a bijection between $\mathfrak{Ad}^{m,n}$ and the set of ad-nilpotent ideals for $\mathfrak{sl}(m+n+1)$ with m generators.*

In this section, when we talk about ad-nilpotent ideals and subspaces in $\mathfrak{Ad}^{m,n}$, we always mean their underlying set of roots.

As shown in [26] and [10], we can use $[i,j]$ to denote the root $\alpha_{ij} = e_i - e_j$. When $i < j$, then $[i,j]$ is a positive root and when $i > j$, $[i,j]$ is a negative root.

For $\mathfrak{g} = \mathfrak{su}(m,n)$, the root $[i,j]$ is compact if and only if $1 \leqslant i,j \leqslant m$, or $m+1 \leqslant i,j \leqslant m+n$. And $[i,j]$ is noncompact if and only if $1 \leqslant i \leqslant m, m+1 \leqslant j \leqslant m+n$, or $1 \leqslant j \leqslant m, m+1 \leqslant i \leqslant m+n$.

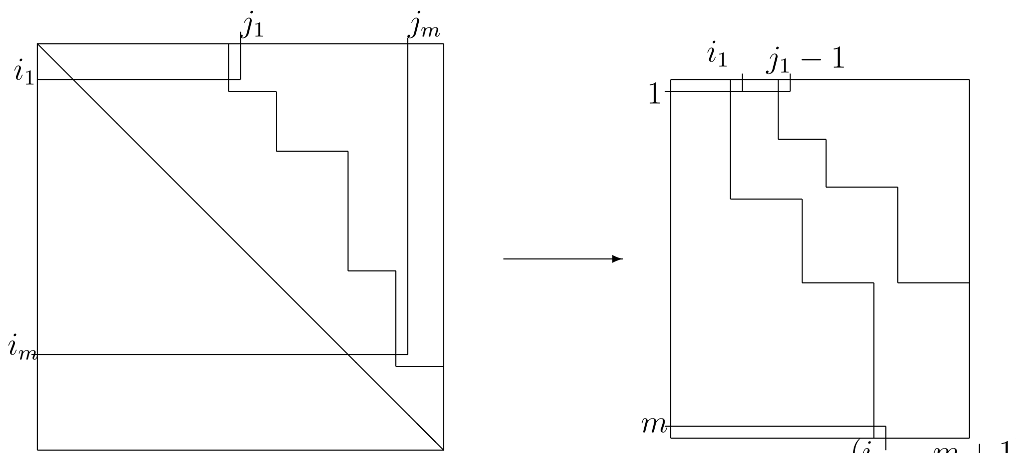

FIGURE 4. The bijection between $\mathfrak{Ad}^m_{m+n+1}$ and $\mathfrak{Ad}^{m,n}$

Let J be a subspace in $\mathfrak{Ad}^{m,n}$. Suppose $J^+ \subset \mathfrak{p}^+$ and $J^- \subset \mathfrak{p}^-$ are the positive and negative part of J. In the case of $U(m,n)$, J^+ and J^- are both B_K-invariant. The subspace J^+ is B_K-invariant means that J^+ is the northwest corner of $\mathfrak{p}^+$. Namely, if $[i,j] \subset J^+$, then for all integers k, l, such that $k \leqslant i$ and $l \geqslant j$, $[k,j]$ and $[i,l]$ also belong to J^+. In other words, J^+ is represented by a right-justified Young diagram with at most m rows and at most n boxes in each row.

The subspace J^- is also B_K-invariant, meaning that $\Delta(\mathfrak{p}^+) - J^-$ is also B_K-invariant and is represented by the right-justified Young diagram that's restricted to the $m \times n$ rectangle.

Since J consists no semisimple element, there's no pair of roots α and $-\alpha$ both appeared in J and vice versa. That means that J^+ is contained in $\Delta(\mathfrak{p}^+) - J^-$. The Young diagram that represents $\Delta(\mathfrak{p}^+) - J^-$ should include the Young diagram of J^+ as a sub-diagram.

From the descriptions above, one could conclude that the set $\mathfrak{Ad}^{m,n}$ is in bijection with the set of two restricted right-justified Young diagrams such that the first one contains in the second one.

Now we give the bijection between $\mathfrak{Ad}^{m,n}$ and $\mathfrak{Ad}^m_{m+n+1}$.

The ad-nilpotent ideal is completely determined by its generators. Suppose $I \in \mathfrak{Ad}^m_{m+n+1}$, then its generator $\Gamma(I)$ is the set

$$\{[i_1, j_1], [i_2, j_2] \ldots [i_m, j_m]\},$$

where

$$1 \leqslant i_1 \ldots i_m \leqslant m+n, 1 \leqslant j_1 \ldots j_m \leqslant m+n+1 \text{ and } i_k < j_k.$$

It's easy to see that the sequences $i_1 i_2 \ldots i_m$ and $j_1 j_2 \ldots j_m$ satisfy the inequalities

$$i_1 \geqslant 1,\ i_2 \geqslant 2,\ \cdots,\ i_m \geqslant m;$$
$$j_1 \leqslant m+2,\ j_2 \leqslant m+3,\ \cdots,\ j_m \leqslant m+n+1.$$

We have

$$0 \leqslant i_1 - 1 \leqslant i_2 - 2 \leqslant \ldots \leqslant i_m - m \leqslant n;$$
$$0 \leqslant j_1 - 2 \leqslant j_2 - 3 \leqslant \ldots \leqslant j_m - (m+1) \leqslant n;$$
$$\text{and } i_k - k \leqslant j_k - (k+1), \text{ for } 1 \leqslant k \leqslant m.$$

The right-justified Young diagram is completely determined by its left-most coordinate at each row. Then $[1, i_1], \ldots, [m, i_m - m + 1]$ and $[1, j_1 - 1], \ldots, [m, j_m - m]$ give us two right-justified Young diagrams with the first one containing the second one (If a Young diagram begins at $[k, m+1]$ at k's row , then it means that the Young diagram has length 0 at this row). That gives us a subspace in $\mathfrak{Ad}^{m,n}$ corresponding to I.

Combining Lemma 4.2.1 and Proposition 4.2.2 together, we have

Corollary 4.3. *The number of $\mathfrak{Ad}^{m,n}$ is equal to $N(m+n+1, m)$.*

Corollary 4.4. $\mathfrak{Ad}^{m,n} = \mathfrak{Ad}_{m,n}$.

Remark. *There is an isomorphism between real groups $U(m,n)$ and $U(n,m)$. In [26], Panyushev gave a natural bijection between $\mathfrak{Ad}^m_{m+n+1}$ and $\mathfrak{Ad}^n_{m+n+1}$. In the real case, it is compatible with a natural bijection between the sets of θ-nilpotent subspaces of $U(m,n)$ and $U(n,m)$.*

Example 4.5. *Suppose $G_{\mathbb{R}} = SU(2,2)$. The closed K-orbits are parameterized by*

$$\{+--+, ++--, +-+-,$$
$$-++-, -+-+, --++\}.$$

In this case, α_1, α_3 are the noncompact simple root and α_2 is the compact simple root. Suppose we use the generators of the θ-nilpotent subspace to specify the space. Then the set of θ-nilpotent subspaces is the following:

$$\emptyset, \{\alpha_2\}, \{\alpha_1+\alpha_2\}, \{\alpha_2+\alpha_3\}, \{-(\alpha_1+\alpha_2+\alpha_3)\},$$
$$\{-(\alpha_2+\alpha_3)\}, \{-(\alpha_1+\alpha_2)\}, \{-\alpha_2\}, \{\alpha_1+\alpha_2+\alpha_3\},$$
$$\{-(\alpha_1+\alpha_2), -(\alpha_2+\alpha_3)\}, \{\alpha_2+\alpha_3, \alpha_1+\alpha_2\},$$
$$\{\alpha_1+\alpha_2+\alpha_3, -\alpha_2\}, \{-(\alpha_1+\alpha_2), \alpha_1+\alpha_2+\alpha_3\},$$
$$\{\alpha_1+\alpha_2+\alpha_3, -(\alpha_2+\alpha_3)\}, \{-\alpha_2, \alpha_1+\alpha_2, \alpha_2+\alpha_3\},$$
$$\{-\alpha_2, \alpha_1+\alpha_2\}, \{\alpha_2+\alpha_3, -(\alpha_1+\alpha_2)\},$$
$$\{-(\alpha_2+\alpha_3), \alpha_1+\alpha_2\}, \{-\alpha_2, \alpha_2+\alpha_3\},$$
$$\{\alpha_1+\alpha_2+\alpha_3, -(\alpha_1+\alpha_2), -(\alpha_2+\alpha_3)\}.$$

There are altogether 20 θ-nilpotent subspaces.

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