AD-NILPOTENT IDEALS OF MINIMAL DIMENSION

ABSTRACT. We apply Jacobson-Morozov theorem and prove Sommers' conjecture about the lower bounds of ad-nilpotent ideals with the same associated orbit. More precisely, for each nilpotent orbit, we construct some minimal ad-nilpotent ideals for classical groups corresponding to their associated orbit.

For classical groups of type A_{n-1}, we get an explicit formula for the minimal dimension based on partitions of n.

1. INTRODUCTION

Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Fix a Borel subgroup of G. Let $\mathfrak{b}$ be the Lie algebra of B and $\mathfrak{n}$ the nilradical of $\mathfrak{b}$.

An ideal of $\mathfrak{b}$ is called *ad-nilpotent* if it is contained in the nilradical $\mathfrak{n}$ (sometimes it is also called a B-stable ideal or simply ideals). We refer to [4] for the general introduction of the history of ad-nilpotent ideals and its application.

One motivation to study ad-nilpotent ideals is their connection with nilpotent orbits. Suppose I is an ad-nilpotent ideal. Consider the map $G \times_B I \to \mathfrak{g}$, which is a restriction of the moment map $G \times_B \mathfrak{n} \to \mathfrak{g}$. Its image is the closure of one unique nilpotent orbit and we denote it by $\mathcal{O}_I$. This orbit is called the associated orbit of the ideal I. This induces a map from the set of ad-nilpotent ideals to the set of nilpotent orbits. By Jacobson-Morozov theorem, for each nilpotent orbit $\mathcal{O}$, there always exists an ad-nilpotent ideal associated to $\mathcal{O}$. Namely, the map is a surjection.

Ad-nilpotent ideal was a main tool of Mizuno [17] to study the conjugate classes of nilpotent elements for exceptional groups. Some further results in this direction were obtained by Kawanaka [12], Gunnells-Sommers [9] and Sommers [25].

The goal of this paper is to study the minimal dimension of ad-nilpotent ideals with the same associated orbit.

In [25] Sommers showed that the dimensions of the ideals with the same associated orbit $\mathcal{O}$ have a lower bound $m_{\mathcal{O}}$.

Proposition 1.1. [25, Prop5.1] *Let $\mathcal{O}$ be a nilpotent orbit. Let I be an ad-nilpotent ideal whose associated orbit is $\mathcal{O}$ and X be an element that lies both in the ideal I and $\mathcal{O}$. Then*

$$\dim I \geqslant \dim B - \dim B_{G_X}$$

where G_X is the centralizer of X in G and B_{G_X} is a Borel subgroup of G_X.

Notice that $\dim B_{G_X}$ is independent of the choice of X, but only depends on the orbit $\mathcal{O}$. Set $m_{\mathcal{O}} = \dim B - \dim B_{G_X}$. Then Sommers conjectured that

Conjecture 1.2. *For each nilpotent orbit $\mathcal{O}$, there exists an ideal I with $\mathcal{O}_I = \mathcal{O}$ and $\dim I = m_{\mathcal{O}}$.*

For exceptional groups, this conjecture was implicitly proved by the work of Kawanaka [12] and Mizuno [17]. In the rest of the paper, we will verify this conjecture for classical groups.

We give a brief outline of this paper. In section 2, we discuss about the standard triples for the nilpotent orbits and construct Dynkin ideals from the Dynkin element, which in most cases, are not ideals of minimal dimension, but can provide some useful information about the minimal ideals. In sections 3, we give a general strategy to construct the ideals of minimal dimension. We also construct some ideals of the minimal dimension for type A. Section 4 is an application of section 3, where we give a formula for the dimension of minimal ideals for the type A_{n-1} based on partitions of n and then derive that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are nilpotent orbits with $\mathcal{O}_1$ is contained in the closure of $\mathcal{O}_2$, then $m_{\mathcal{O}_1} \leqslant m_{\mathcal{O}_2}$. In sections 5, 6, and 7, the explicit construction of minimal ideals for type B, C and D is given.

2. Notation and Preliminaries

If V is a subspace of $\mathfrak{g}$ that's invariant under the action of $\mathfrak{h}$, we denote $\Delta(V) = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \in V\}$. If S is a finite set, we denote by $|S|$ the cardinality of S. For any $k \in \mathbb{R}$, let $\lfloor k \rfloor$ be the largest integer less than or equal to k and let $\lceil k \rceil$ be the smallest integer not less than k.

Notice that any ad-nilpotent ideal is completely determined by its underline set of roots. For any ad-nilpotent ideal I, let

Before we come to the proof of the conjecture of Sommers, let's first recall some results about standard triples.

Let $\{H, X, Y\}$ be a **standard** triple (see [6]) of $\mathfrak{g}$, satisfying:

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

We call X (resp. Y) the *nilpositive* (resp. *nilnegative*) element and H the *characteristic* of the triple $\{H, X, Y\}$. In particular, after conjugation by some element of G, we can assume that $H \in \mathfrak{h}$ and H is dominant, i.e. $\alpha(H) \geqslant 0$, for all $\alpha \in \Delta^+$. Such H is uniquely determined by the nilpotent orbit $\mathcal{O}_X$ and is called the *Dynkin element* for $\mathcal{O}_X$. There is an H-eigenspace decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_{H,i}, \text{ where } \mathfrak{g}_{H,i} = \{Z \in \mathfrak{g} \mid [H, Z] = iZ\}, i \in \mathbb{Z}.$$

It's also shown in [25] that

$$\dim B_{G_X} = \dim \mathfrak{g}_{H,1} + \frac{1}{2}[\dim(\mathfrak{g}_{H,0}) + \dim(\mathfrak{g}_{H,2}) + \operatorname{rank} G_X]. \tag{2.1.1}$$

Let $\mathfrak{q}_{H,i} = \oplus_{j\geqslant i}\mathfrak{g}_{H,j}$. Then, $\mathfrak{q}_{H,i}$ is an ad-nilpotent ideal and $X \in \mathfrak{q}_{H,2}$. We call the ideal $\mathfrak{q}_{H,2}$ the *Dynkin ideal* for the orbit $\mathcal{O}_X$. We may write X as $X_{\alpha_1} + X_{a_2} + \cdots + X_{\alpha_k}$, where X_{α_i} is a root vector and $\alpha_i(H) = 2$. Since H is dominant, each α_i is a positive root.

Consider the adjoint action $ad_X : \mathfrak{g}_{H,0} \to \mathfrak{g}_{H,2}$, which sends any $Z \in \mathfrak{g}_{H,0}$ to $[X, Z] \in \mathfrak{g}_{H,2}$. Since $\mathfrak{g}_{H,0}$ is a Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, there is a direct sum decomposition: $\mathfrak{g}_{H,0} = \mathfrak{h} \oplus \mathfrak{g}_0^+ \oplus \mathfrak{g}_0^-$, where $\mathfrak{g}_0^+ = \mathfrak{n} \cap \mathfrak{g}_{H,0}$ and $\mathfrak{g}_0^- = \mathfrak{n}^- \cap \mathfrak{g}_{H,0}$. It's obvious that $ad_X(\mathfrak{g}_{H,0}) = ad_X(\mathfrak{h}) + ad_X(\mathfrak{g}_0^+) + ad_X(\mathfrak{g}_0^-)$. If we impose some additional restrictions on the set $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$, we will have a direct sum decomposition of $\mathfrak{g}_{H,2}$. First we introduce the notion of antichain.

Definition 2.1. [26] *An antichain Γ of the root poset $(\Delta^+, <)$ is a set of pairwise incomparable elements, i.e: for any $\alpha, \beta \in \Gamma$, $\alpha - \beta \notin Q^+$, where $Q^+ = \{\sum_{i=1}^n n_i\alpha_i \mid n_i \in \mathbb{N}\}$ is the positive part of the root lattice.*

By the definition of generators of an ad-nilpotent ideal, a set $\Gamma = \{\gamma_1, \ldots, \gamma_l\}$ is a set of generators of some ad-nilpotent ideal if and only if $\gamma_i - \gamma_j \notin Q^+$. Therefore, the set of generators for ad-nilpotent ideals is in bijection with the set of antichains of the root poset.

We need the following result of Kostant. (see [14]):

Theorem 2.2. *Let $Q_{H,2}$, $G_{H,0}$ be the closed connected Lie subgroup of G with the Lie algebras $\mathfrak{q}_{H,2}, \mathfrak{g}_{H,0}$ respectively. Let $\mathcal{O}_X$ be the G-orbit of X and o_X the $G_{H,0}$-orbit of X. Then*

(1) o_X is open, dense in $\mathfrak{g}_{H,2}$.

(2) $o_X = \mathcal{O}_X \cap \mathfrak{g}_{H,2}$.

(3) $(Q_{H,2}G_{H,0})\cdot X = \mathcal{O}_X \cap \mathfrak{q}_{H,2} = o_X + \mathfrak{q}_{H,3}$. In particular, $(Q_{H,2}G_{H,0})\cdot X$ is open and dense in $\mathfrak{q}_{H,2}$.

As a consequence, the Dynkin ideal $\mathfrak{q}_{H,2}$ has associated orbit $\mathcal{O}_X$.

Lemma 2.3. [18, Prop2.10] *Let Γ be a subset of Δ^+. If for any roots $\alpha, \beta \in \Gamma$, $\alpha - \beta \notin \Delta$, then the elements of Γ are linearly independent and hence $|\Gamma| \leqslant dim(\mathfrak{h})$.*

Proof. Since $\alpha - \beta \notin \Delta$, $(\alpha, \beta) \leqslant 0$. This means that the angle between any pair of roots in Γ is non-acute. Since all the roots in Γ lie in the same open half-space of V, they are linearly independent. □

Remark. *If Γ is an antichain, then Γ satisfies the assumption of lemma 2.3, hence elements in an antichain are linearly independent. In Panyushev's original statement, he assumed that Γ is an antichain. But from his proof, the weaker condition that $\alpha - \beta \notin \Delta$ is sufficient for this lemma. In type A_{n-1} and D_n, we can get an antichain but in type C_n case, we can only get a set Γ satisfying the weaker condition of Lemma 2.3.*

Lemma 2.4. *Suppose Γ is a subset of Δ^+ as in Lemma 2.3. Let $\{H_\alpha, X_\alpha, Y_\alpha\}$ be a standard triple that corresponds to $\alpha \in \Gamma$. Suppose*

H lies in the span of all $\{H_\alpha\}_{\alpha\in\Gamma}$ so that $\alpha(H)=2$ for all $\alpha\in\Gamma$. Let $X=\sum_{\alpha\in\Gamma}X_\alpha$. There exists an element Y, such that $\{H,X,Y\}$ is a standard triple.

Remark. *The main idea of the proof comes from* [6, 4.1.6].

Proof. Since Γ is a subset as in Lemma 2.3, $\{H_\alpha\mid\alpha\in\Gamma\}$ is linearly independent. Also H lies in the subspace of $\mathfrak{h}$ that's spanned by all $\{H_\alpha\}_{\alpha\in\Gamma}$, therefore we may write H as $H=\sum_{\alpha\in\mathcal{C}}a_\alpha H_\alpha$ for some $a_\alpha\in\mathbb{R}$. Let $Y=\sum_{\alpha\in\Gamma}a_\alpha Y_\alpha$. Then $[H,X]=[H,\sum_{\alpha\in\Gamma}X_\alpha]=\sum_{\alpha\in\mathcal{C}}\alpha(H)X_\alpha=2X$. The last equality follows from (2.2.1). Similarly, $[H,Y]=-2Y$. Also

$$[X,Y]=\sum_{\alpha\in\Gamma}\sum_{\beta\in\Gamma}a_\beta[X_\alpha,Y_\beta]=\sum_{\beta\in\Gamma}a_\beta[X_\beta,Y_\beta]=\sum_{\beta\in\mathcal{C}}a_\beta H_\beta=H.$$

The second equality comes from the fact that $\alpha-\beta\notin\Delta$ and $[X_\alpha,Y_\beta]=0$ for $\alpha\neq\beta$. □

Proposition 2.5. *Suppose that the set $\{\alpha_1,\alpha_2,\dots\alpha_k\}$ is an antichain of $(\Delta^+,<)$. Then $\mathfrak{g}_{H,2}=ad_X(\mathfrak{g}_{H,0})$ and $\mathfrak{g}_{H,2}=ad_X(\mathfrak{h})\oplus ad_X(\mathfrak{g}_0^+)\oplus ad_X(\mathfrak{g}_0^-)$. Moreover, $ad_X(\mathfrak{h})$, $ad_X(\mathfrak{g}_0^+)$ and $ad_X(\mathfrak{h})$ are invariant under the adjoint action of $\mathfrak{h}$ and each can be written as a direct sum of root spaces.*

Proof. By Theorem 2.2, the image of ad_X is the whole space $\mathfrak{g}_{H,2}$. What remains to prove is that it is a direct sum decomposition. Suppose there exist three elements $H_1\in\mathfrak{h}$, $Z\in\mathfrak{g}_0^+$, $U\in\mathfrak{g}_0^-$ and $[X,H_1]+[X,U]+[X,Z]=0$.

Suppose $\alpha_i(H_1)\neq 0$ for some α_i. Since $[X,H_1]=\sum_{i=1}^k-\alpha_i(H_1)X_{\alpha_i}$, there is a nonzero summand in $[U,X]$ or $[Z,X]$ that lies in $\mathfrak{g}_{\alpha_i}$. Without loss of generality, we may assume $[U_\beta,X_{\alpha_j}]\in\mathfrak{g}_{\alpha_i}$, where U_β is a summand of U and X_{α_j} is a summand of X. Then $\alpha_i=\alpha_j+\beta$, which implies that $\alpha_i<\alpha_j$ and contradicts the assumption that α_i and α_j are incomparable.

Otherwise $\alpha_i(H)=0$ for $1\leqslant i\leqslant k$. By similar argument, we can prove that $ad_X(\mathfrak{g}_0^+)\cap ad_X(\mathfrak{g}_0^-)=0$, which shows that $\mathfrak{g}_{H,2}=ad_X(\mathfrak{h})\oplus ad_X(\mathfrak{g}_0^+)\oplus ad_X(\mathfrak{g}_0^-)$.

Suppose that X_α is a root vector in $\mathfrak{g}_{H,2}$. By lemma 2.3, $\{\alpha_1,\alpha_2,\dots,\alpha_k\}$ are linearly independent. If $\alpha\in\{\alpha_1,\alpha_2,\dots,\alpha_k\}$, then α lies in $ad_X(\mathfrak{h})$. If $\alpha=\alpha_i+\beta$, where $\beta\in\Delta(\mathfrak{g}_0^+)$, from the proof above, X_α can not appear in the summands of $ad_X(\mathfrak{h})$ and $ad_X(\mathfrak{g}_0^-)$. Similarly, elements in $ad_X(\mathfrak{g}_0^-)$ has the form $\sum X_{\alpha_i+\beta_i}$, where $\beta_i\in\Delta(\mathfrak{g}_0^-)$. Therefore, $X_\alpha\in ad_X(\mathfrak{g}_0^+)$ if $\alpha=\alpha_i+\beta$ and $\beta\in\Delta(\mathfrak{g}_0^+)$ and $X_\alpha\in ad_X(\mathfrak{g}_0)^-$ if $\alpha=\alpha_i+\beta$ and $\beta\in\Delta(\mathfrak{g}_0^-)$. This shows that $ad_X(\mathfrak{g}_0^-)$, $ad_X(\mathfrak{g}_0^+)$ and $ad_X(\mathfrak{h})$ are $\mathfrak{h}$-invariant, which completes the proof. □

Definition 2.6. *A* **partition** λ *is a sequence of positive integers*

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_p], \text{ where } \lambda_1 \geqslant \lambda_2 \ldots \geqslant \lambda_p > 0.$$

Each λ_i is a part of λ. If $\lambda_1 + \cdots + \lambda_p = n$, we say λ is a partition of n and write $\lambda \vdash n$.

We recall the parametrization of nilpotent orbits in classical groups.

Theorem 2.7. [6, 5.1] *(1)(Type A_{n-1}) Nilpotent orbits in $\mathfrak{sl}_n$ are in one-to-one correspondence with the set $P(n)$ of partitions of n.*

(2)(Type B_n) Nilpotent orbits in $\mathfrak{so}_{2n+1}$ are in one-to-one correspondence with the set $P_1(2n+1)$ of partitions of $2n+1$ in which even parts occur with even multiplicity.

(3)(Type C_n) Nilpotent orbits in $\mathfrak{sp}_{2n}$ are in one-to-one correspondence with the set $P_{-1}(2n)$of partitions of $2n$ in which odd parts occur with even multiplicity.

(4)(Type D_n) Nilpotent orbits in $\mathfrak{so}_{2n}$ are in one-to-one correspondence with the set $P_1(2n)$ of partitions of $2n$ in which even parts occur with even multiplicity, except that ('very even') partitions (those with only even parts; each having even multiplicity) correspond to two orbits.

For each partition λ, we denote by $\mathcal{O}_\lambda$ the nilpotent orbit that corresponds to λ except the very even case in type D_n, in which we denote the two orbits by $\mathcal{O}^I_\lambda$ and $\mathcal{O}^{II}_\lambda$.

We will prove conjecture 1.2 by constructing explicit minimal ideals in the classical groups. In addition, we have an explicit formula for the dimension of the minimal ideals in terms of partition.

Let's briefly discuss the main idea to construct minimal ideals. First let's recall the method to compute the weighted Dynkin diagram of a nilpotent orbit in [6]. Given a partition $\lambda = [\lambda_1, \ldots, \lambda_p]$ as a nilpotent orbit in type A_{n-1}, for each part λ_i, we take the set of integers $\{\lambda_i - 1, \ldots, 1 - \lambda_i\}$. Then we take the union of these sets and write it into a sequence $(h_1, h_2, \ldots, h_n)$, where $h_1 \geqslant \ldots \geqslant h_n$. We assign the value $h_i - h_{i+1}$ to the i-th node of the Dynkin diagram of A_{n-1}. This gives us the weighted Dynkin diagram corresponding to λ.

Coming back to the construction of ad-nilpotent ideals, for each h_i in the sequence $(h_1, \ldots, h_n)$, we have to specify which part of the partition it comes from. For example, if $h_{i_1}, \ldots, h_{i_k}$ come from the same part of λ in a descending order, then we may pick the roots $\{e_{i_j} - e_{i_{j+1}}\}_{j=1}^{k-1}$ to be generators of an ideal I. To make sure the ideal I is minimal, we have to choose carefully the positions of $\{\lambda_i - 1, \ldots, 1 - \lambda_i\}$ in the sequence $(h_1, \ldots, h_n)$.

Similar ideas apply to other types. For classical groups of types B, C and D, the construction of the weighted Dynkin diagram is a little different and we have to adjust our choice accordingly.

3. Minimal Ideals For Type A_{n-1}

Suppose $\mathfrak{g} = \mathfrak{sl}(n)$. Following standard notation, let $\mathfrak{b}$ be the standard upper triangular matrices and $\mathfrak{h}$ be the diagonal matrices. The root system of $\mathfrak{g}$ is $\{e_i - e_j \mid 1 \leqslant i, j \leqslant n, i \neq j\}$ and $\Delta^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n\}$. We denote by E_{ij} the elementary matrix with its ij-entry 1 and other entries 0. The root space for $e_i - e_j$ is spanned by the matrix E_{ij}.

By Theorem 2.7, let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_p]$ be a partition of n. Following the idea discussed at the end of previous section, we need to construct some maps to keep track of the positions of $\{\lambda_i - 1, \lambda_i - 3, \ldots, 1 - \lambda_i\}$.

Indeed, let σ_i, for $1 \leqslant i \leqslant p$, be a sequence of index maps

$$\sigma_i : \{\lambda_i - 1, \lambda_i - 3, \ldots, 1 - \lambda_i\} \to [n] = \{1, 2, \ldots, n\}$$

Lemma 3.1. *There exists a sequence of maps $\{\sigma_i\}_{i=1}^p$, satisfying the following properties:*

(1) Each σ_i is one-to-one and $Im(\sigma_i) \cap Im(\sigma_j) = \emptyset$, if $i \neq j$.

(2) If $k < l$ and $k \in Dom(\sigma_i)$, $l \in Dom(\sigma_j)$, then $\sigma_i(k) > \sigma_j(l)$.

(3) For any λ_i, λ_j and $k, l \in Dom(\sigma_i) \cap Dom(\sigma_j)$, if $\sigma_i(k) > \sigma_j(k)$, then $\sigma_i(l) > \sigma_j(l)$. Here $Dom(\sigma_i)$ denotes the domain of σ_i and $Im(\sigma_i)$ denotes the image of σ_i.

Proof. We form a sequence of integers $h = (h_1, \ldots, h_n)$ by placing $\lambda_i - 2s + 1$ in the position $\sigma_i(\lambda_i - 2s + 1)$ for $1 \leqslant i \leqslant n$. Property (1) and (2) make sure we indeed get a weighted Dynkin diagram from h. Property (3) gives some restriction on the positions of integers of the same value, but coming from different parts of λ. □

Remark. *1. If $\{\sigma_1, \ldots, \sigma_p\}$ is a sequence of maps as above, then $\sqcup_{i=1}^p Im(\sigma_i) = [n]$.*

2. As a special case of property (2), $\sigma_i(\lambda_i - 1) < \sigma_i(\lambda_i - 3) < \cdots < \sigma_i(1 - \lambda_i)$.

Example 3.2. *Let $\lambda = [4, 2]$ be a partition of 6. Let $h = (\underline{3}, \underline{1}, \overline{1}, \underline{-1}, \overline{-1}, \underline{-3})$ and $h' = (\underline{3}, \overline{1}, \underline{1}, \overline{-1}, \underline{-1}, \underline{-3})$, where $\bar{i}$ means that i comes from $\lambda_2 = 2$ and $\underline{i}$ means that i come from $\lambda_1 = 4$ of the partition λ. Then h, h' give rise to the same weighted Dynkin diagram and also show that the sequence of the maps $\{\sigma_i\}$ is not unique.*

For each σ_i, we attach a set of positive roots:

$$\mathcal{C}^+(\sigma_i) = \{e_{\sigma_i(\lambda_i-1)} - e_{\sigma_i(\lambda_i-3)}, e_{\sigma_i(\lambda_i-3)} - e_{\sigma_i(\lambda_i-5)}, \ldots, e_{\sigma_i(3-\lambda_i)} - e_{\sigma_i(1-\lambda_i)}\}.$$

For the partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_p]$, we set $\mathcal{C} = \cup_i \mathcal{C}^+(\sigma_i)$. Let X_α be a root vector that corresponds to the root $\alpha \in \mathcal{C}$ and define X be the sum of X_α.

Given λ, attach $\mathcal{C}_+(\lambda_i) = \{e_{N_i+1} - e_{N_i+2}, \ldots, e_{N_i+\lambda_i-1} - e_{N_i+\lambda_i}\}$ to λ_i. Here N_i is chosen so that $\sqcup_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant \lambda_i}(N_i + j) = [n]$. Let $\mathcal{C}_+$ be the union of $\mathcal{C}_+(\lambda_i)$ and let $\tilde{X} = \sum_{\alpha \in \mathcal{C}_+} X_\alpha$. It's showed in [6, section

5.2] that $\tilde{X}$ lies in the orbit $\mathcal{O}_\lambda$. From the construction of $\mathcal{C}$ and $\mathcal{C}_+$, we can see that X is conjugate to $\tilde{X}$ by some elements in S_n, therefore also lies in $\mathcal{O}_\lambda$.

Now let

$$H_{\mathcal{C}^+(\sigma_i)} = \sum_{s=1}^{\lambda_i} (\lambda_i - 2s + 1) E_{\sigma_i(\lambda_i - 2s+1), \sigma_i(\lambda_i - 2s+1)}$$

and

$$H = \sum_{i=1}^{n} H_{\mathcal{C}^+(\sigma_i)}.$$

Although each $H_{\mathcal{C}^+(\sigma_i)}$ is dependent on the choice of the map σ_i, by property (2) above, the diagonal entries of H are decreasing and H is independent of the series of the maps $\{\sigma_i\}$ we choose. Indeed, H is the Dynkin element of the orbit $\mathcal{O}_\lambda$.

For any root $\alpha = e_{\sigma_i(k)} - e_{\sigma_j(l)} \in \Delta$, $\alpha(H) = k - l$. In particular

$$\alpha(H) = 2, \qquad \text{for any } \alpha \in \mathcal{C}. \tag{2.2.1}$$

There are several things to show:

Lemma 3.3. *The set of positive roots $\mathcal{C}$ is an antichain in Δ^+.*

Proof. The set $\mathcal{C}$ is an antichain if and only if $\alpha - \beta \notin Q^+$ for any roots $\alpha, \beta \in \mathcal{C}$. Let $\alpha = e_{\sigma_i}(m) - e_{\sigma_i}(m-2)$ and $\beta = e_{\sigma_j}(l) - e_{\sigma_j}(l-2)$ be two roots in $\mathcal{C}$. Then by property (3) in lemma 3.1 above, $\alpha - \beta = (e_{\sigma_i}(m) - e_{\sigma_j}(l)) + (e_{\sigma_j}(l-2) - e_{\sigma_i}(m-2))$ can not lie in Q^+. Thus $\mathcal{C}$ is an antichain. □

Then the set $\mathcal{C}$ satisfies the assumption of Lemma 2.4 and we can find an appropriate nilnegative element Y such that $\{H, X, Y\}$ is a standard triple.

Since there is a canonical bijection between the antichains of the root poset and the ad-nilpotent ideals, we can construct an ad-nilpotent ideal $I_{\mathcal{C}}$ that is generated by $\mathcal{C}$.

Lemma 3.4. *The ideal $I_{\mathcal{C}}$ contains $\mathfrak{q}_{H,3}$.*

Proof. Both $I_{\mathcal{C}}$ and $\mathfrak{q}_{H,3}$ are direct sum of root spaces. Let α be a positive root and $\mathfrak{g}_\alpha \subset \mathfrak{q}_{H,3}$. Suppose $\alpha = e_{\sigma_i(k)} - e_{\sigma_j(l)}$. Then $\alpha(H) = k - l \geqslant 3$. There are two possible cases for k. If $k > 0$, then $k - 2 \in Dom(\sigma_i)$ and $\alpha = \beta + \gamma$, where $\beta = e_{\sigma_i(k)} - e_{\sigma_i(k-2)}$ and $\gamma = e_{\sigma_i(k-2)} - e_{\sigma_i(l)}$. By property 2 of the maps $\{\sigma_i,\ \sigma_j\}$, $\beta \in \mathcal{C}$ and $\gamma \in \Delta^+$, hence $\alpha \in \Delta(I_{\mathcal{C}})$. If $k \leqslant 0$, then $l < 0$ and $l + 2 \in Dom(\sigma_j)$. Therefore $\alpha = \beta + \gamma \in \Delta(I_{\mathcal{C}})$, where $\beta = e_{\sigma_i(k)} - e_{\sigma_i(l+2)} \in \Delta^+$ and $\gamma = e_{\sigma_i(l+2)} - e_{\sigma_i(l)} \in \mathcal{C}$. □

Proposition 3.5. *The associated orbit of the ideal $I_{\mathcal{C}}$ is $\mathcal{O}_\lambda$ and it is an ideal of minimal dimension.*

Proof. By formula (2.2.1), the ideal $I_{\mathcal{C}}$ is contained in the Dynkin ideal $\mathfrak{q}_{H,2}$. By Kostant's theorem 2.2, the associated orbit of the ideal $I_{\mathcal{C}}$ is contained in the closure of the orbit $\mathcal{O}_X$. On the other hand, $I_{\mathcal{C}}$ contains X, so $\mathcal{O}_{I_{\mathcal{C}}} = \mathcal{O}_X$. We only need to prove that $\dim I_{\mathcal{C}} = m_{\mathcal{O}}$.

The Dynkin ideal $\mathfrak{q}_{H,2}$ is contained in the Borel subalgebra $\mathfrak{b}$ and there is a decomposition $\mathfrak{b} = \mathfrak{g}_0^+ \oplus \mathfrak{h} \oplus \mathfrak{g}_{H,1} \oplus \mathfrak{q}_{H,2}$. By the formula 2.1.1 for $m_{\mathcal{O}}$,

$$\begin{aligned}
\dim \mathfrak{q}_{H,2} - m_{\mathcal{O}} &= \dim B - (\dim \mathfrak{g}_{H,1} + \dim \mathfrak{g}_0^+ + \dim \mathfrak{h}) - \dim B + \dim B_{G_X} \\
&= \dim B_{G_X} - (\dim \mathfrak{g}_{H,1} + \dim \mathfrak{g}_0^+ + \dim \mathfrak{h}) \\
&= \frac{1}{2}[\dim(\mathfrak{g}_{H,0}) + \dim(\mathfrak{g}_{H,2}) + \text{ rank } G_X] - \dim \mathfrak{g}_0^+ - \dim \mathfrak{h} \\
&= \frac{1}{2}(\dim \mathfrak{g}_{H,2} - \dim \mathfrak{h} + \text{rank } G_X).
\end{aligned}$$

The last equality follows from the fact that $\mathfrak{g}_{H,0} = \mathfrak{g}_0^+ \oplus \mathfrak{g}_0^- \oplus \mathfrak{h}$ and $\dim \mathfrak{g}_0^+ = \dim \mathfrak{g}_0^-$.

Let $\mathcal{C}^- = \{\alpha \in \Delta(\mathfrak{g}_{H,2}) \mid \alpha \notin \Delta(I_{\mathcal{C}})\}$ and $\mathcal{C}^+ = \{\alpha \in \Delta(\mathfrak{g}_{H,2}) \cap \Delta(I_{\mathcal{C}}) \mid \alpha \notin \mathcal{C}\}$.

Then $\Delta(\mathfrak{g}_{H,2}) = \mathcal{C} \sqcup \mathcal{C}^+ \sqcup \mathcal{C}^-$. By Lemma 2.3, since the set $\mathcal{C}$ consists of linearly independent roots, $|\mathcal{C}| = \dim ad_X(\mathfrak{h}) = \dim \mathfrak{h} - \dim Z_{\mathfrak{h}}(X) = \dim \mathfrak{h} - \text{rank } G_X$. Then

$$\dim \mathfrak{q}_{H,2} - m_{\mathcal{O}} = \frac{1}{2}(|\mathcal{C}^+| + |\mathcal{C}^-|).$$

From Lemma 3.4, $\mathfrak{q}_{H,3}$ is contained both in $\mathfrak{q}_{H,2}$ and $I_{\mathcal{C}}$. So

$$\dim \mathfrak{q}_{H,2} - \dim I_{\mathcal{C}} = \dim \mathfrak{g}_{H,2} - \dim I_{\mathcal{C}} \cap \mathfrak{g}_{H,2} = |\mathcal{C}^-|.$$

Now it suffices to prove that $\mathcal{C}^+$ and $\mathcal{C}^-$ have the same cardinality. This follows from Lemma 3.6 below, which proves this proposition. □

Notice that any root in $\Delta(\mathfrak{g}_{H,2})$ has the form $e_{\sigma_i(m)} - e_{\sigma_j(m-2)}$ for some i, j and $m \in Dom(\sigma_i)$ and $m - 2 \in Dom(\sigma_j)$. We define a map

$$\iota : \Delta(\mathfrak{g}_{H,2}) \to \Delta(\mathfrak{g}_{H,2})$$

by $e_{\sigma_i(m)} - e_{\sigma_j(m-2)} \mapsto e_{\sigma_j(2-m)} - e_{\sigma_i(-m)}$. Since the domain of σ_i and σ_j is symmetric with respect to 0, so $-m \in Dom(\sigma_i)$ and $2-m \in Dom(\sigma_j)$ and the map is well-defined.

Lemma 3.6. *Keep the notations as above, the map ι is an involution on $\Delta(\mathfrak{g}_{H,2})$. Moreover, ι maps $\mathcal{C}$ to itself and maps $\mathcal{C}^+$ to $\mathcal{C}^-$ and vice versa.*

Proof. Since $\iota^2 = id$, it's obvious that ι defines an involution on $\Delta(\mathfrak{g}_{H,2})$. If $\alpha = e_{\sigma_i(m)} - e_{\sigma_j(m-2)} \in \mathcal{C}$, then $m, m-2$ come from the same part of the partition λ, so $i = j$ and $\iota(\alpha) \in \mathcal{C}$. That means ι maps $\mathcal{C}$ to itself and also maps $\mathcal{C}^+ \sqcup \mathcal{C}^-$ to itself.

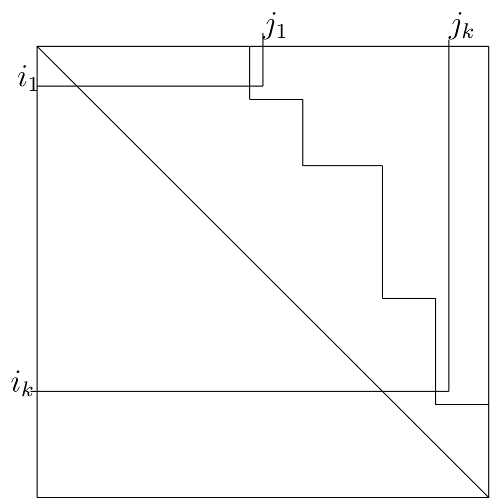

FIGURE 1. An ad-nilpotent ideal for type A_{n-1}

Suppose that $\alpha \in \mathcal{C}^+$. Since α lies in the ideal that's generated by $\mathcal{C}$, there exists a root $\beta \in \mathcal{C}$ and a positive root $\gamma \in \Delta^+$ such that $\alpha = \beta + \gamma$.

If $m > 0$, then $m - 2 \in Dom(\sigma_i)$. In this case $\alpha = \beta + \gamma$, where $\beta = e_{\sigma_i(m)} - e_{\sigma_i(m-2)} \in \mathcal{C}$ and $\gamma = e_{\sigma_i(m-2)} - e_{\sigma_j(m-2)} > 0$. Hence $\sigma_i(m-2) < \sigma_j(m-2)$. The domains of σ_i and σ_j are symmetric with respect to 0, so $2 - m, -m \in Dom(\sigma_i)$ and $2 - m \in Dom(\sigma_j)$. Then $\iota(\alpha) = e_{\sigma_j(2-m)} - e_{\sigma_i(-m)} = \beta' + \gamma'$, where $\beta' = e_{\sigma_j(2-m)} - e_{\sigma_i(2-m)}$ and $\gamma' = e_{\sigma_i(2-m)} - e_{\sigma_i(-m)}$. Then $\gamma' \in \mathcal{C}$ and by property (3) of the maps $\{\sigma_i, \sigma_j\}$, $\sigma_i(2-m) < \sigma_j(2-m)$, so $\beta' \in \Delta^-$ and $\iota(a) \in \mathcal{C}^-$.

If $m \leqslant 0$, then $m \in Dom(\sigma_j)$. In this case, $\beta = e_{\sigma_j(m)} - e_{\sigma_j(m-2)} \in \mathcal{C}$. With the same argument, $\alpha' = e_{\sigma_j(2-m)} - e_{\sigma_i(-m)}$ is the unique element that corresponds to α and lies in $\mathcal{C}^-$.

By the same reasoning, ι maps $\mathcal{C}^-$ to $\mathcal{C}^+$. The lemma is proved. $\square$

4. DIMENSION FORMULA FOR MINIMAL IDEALS OF TYPE A_{n-1}

For type A_{n-1}, $\mathfrak{n}$ is the set of strictly upper triangular matrices. Following [18], an ad-nilpotent ideal is represented by a right-justified Ferrers (or Young) diagram with at most $n-1$ rows, where the length of the i-th row is at most $n-i$. Namely, if any root space $\mathfrak{g}_\alpha$ lies in an ideal I, then any root subspace $\mathfrak{g}_\beta$ that's on the northeast side of $\mathfrak{g}_\alpha$ also lies this ideal. The generators of the ideal are the set of southwest corners of the diagram. We use the pair $[i, j]$ to denote the positive root $e_i - e_j$, where $1 \leqslant i < j \leqslant n$. Then the set of generators of the ideal I can be writen as (see Figure 2-1):

$$\Gamma(I) = \{[i_1, j_1], \dots, [i_k, j_k]\}, \text{where } 1 \leqslant i_1 < \cdots < i_k \leqslant n-1,$$
$$2 \leqslant j_1 < \cdots < j_k \leqslant n.$$

In order to compare the dimension of two minimal ideals, we first need to have an explicit formula for the dimension of the minimal ideal in terms of the partition λ. Suppose the sequence of maps $\{\sigma_i\}$ satisfies properties (1) and (2) in Lemma 3.1 plus an additional one:

(4) $\sigma_i(k) < \sigma_j(k)$, when $i < j$ and k lies in the domain of σ_i and σ_j.

The sequence of the maps exists and is uniquely determined by those restrictions. Moreover, $\{\sigma_i\}$ automatically satisfy property (3) of Lemma 3.1 so the ideal I constructed from these maps has minimal dimension. The Dynkin element $H = diag\{h_1, h_2, \dots h_n\}$ is the same as in last section.

Let $A(l)$ be the number of entries in H that are less or equal to l. Let $B(l)$ be the number of entries of H that are bigger than l. Then $A(l) + B(l) = n$. If we rewrite the partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_p]$ in the exponential form $\lambda = [t_1^{n_{t_1}}, t_2^{n_{t_2}}, \dots, t_k^{n_{t_k}}]$, then the minimal dimension has the following formula:

Proposition 4.1. *The dimension of the minimal ideals corresponding to the partition λ is equal to*

$$m_{\mathcal{O}_\lambda} = \frac{n(n+1)}{2} + \sum_{i=1}^{n}(-A(\lambda_i - 1) + A(-\lambda_i - 1)) + \sum_{i=1}^{k} \frac{n_{t_i}(n_{t_i} - 1)}{2}.$$

Proof: To calculate the dimension for the ideal I, we need to sum up the number of positive roots in I in each row.

From the construction of the maps $\{\sigma_i\}$, it is obvious that in the $\sigma_i(\lambda_i - 1), \sigma_i(\lambda_i - 3) \dots \sigma_i(3 - \lambda_i)'s$ rows, $\mathcal{C}_{\lambda_i}$ forms a subset of generators of the ideal I, therefore the Ferrers diagram begins with boxes $[\sigma_i(\lambda_i - 1), \sigma_i(\lambda_i - 3)], \dots, [\sigma_i(3 - \lambda_i), \sigma_i(1 - \lambda_i)]$.

On the other hand, there is no generator in row $\sigma_i(1-\lambda_i)$. Let $[s_i, t_i]$ be the generator that's below this row and row $\sigma_i(1 - \lambda_i)$. Then row s_{λ_i} share the same columns. Then row $\sigma_i(1-\lambda_i)$ of the Ferrers diagram begins with box $[\sigma_i(1 - \lambda_i), t_i]$. If there is no generator below this row, we simply say that the diagram begins with box $[\sigma_i(1-\lambda_i), n+1]$. This convention makes the formula (2.3.1) in the next paragraph give the correct number of positive roots in row $\sigma_i(1 - \lambda_i)$, which is zero.

Suppose that the Ferrers diagram corresponding to the ideal I begins with box $[s, t]$ in row s. Then the number of positive roots in this row is equal to $1 + n - t = (1 + n - s) + (s - t)$.

The total summation of positive roots in all rows is equal to:

(2.3.1)

$$\begin{aligned} |I| &= \sum_{i=1}^{p} \sum_{s=2}^{\lambda_i} \left[n + 1 - \sigma_i(\lambda_i - 2s + 1)\right] + \sum_{i=1}^{p} \left[n + 1 - t_i\right] \\ &= \sum_{i=1}^{p} \sum_{s=1}^{\lambda_i} (n + 1 - \sigma_i(\lambda_i - 2s + 1)) + \sum_{i=1}^{p} \sigma_i(\lambda_i - 1) - \sum_{i=1}^{p} t_i. \end{aligned}$$

Since $\cup_{i=1}^{p} Im(\sigma_i) = [n]$, the first term in the equation (2.3.1) is equal to $\sum_{j=1}^{n}(j) = \frac{n(n+1)}{2}$. We only need to know the value of $\sigma_i(\lambda_i - 1)$ and t_i.

Indeed, $\sigma_i(\lambda_i - 1)$ shows the position of $\lambda_i - 1$ in the Dynkin element $H = diag\{h_1, h_2, \dots h_n\}$. If λ_i satisfies that all $\lambda_j < \lambda_i$, when $j > i$, by property 4, $\sigma_i(\lambda_i - 1)$ is the first t, such that $h_t = \lambda_i - 1$. Namely $\sigma_i(\lambda_i - 1) = max\{t \mid h_t > \lambda_i - 1\} + 1 = B(\lambda_i - 1) + 1$.

If $\lambda_{i-1} = \lambda_i$, then $\sigma_{i-1}(\lambda_{i-1} - 1) = \sigma_i(\lambda_i - 1) + 1$. If we consider the exponential expression of the partition λ, the summation of all $\sigma_i(\lambda_i - 1)$, where $\lambda_i = t_j$ is equal to $n_j(1 + B(\lambda_i - 1)) - n_j(n_j + 1)/2$.

What remains to discuss is the value of t_i. We need to find the nearest corner of the Ferrers diagram that's below row $\sigma_i(1 - \lambda_i)$.

Case 1: Suppose that λ_l is the smallest integer such that $\lambda_l > \lambda_i$ and $\lambda_l \equiv \lambda_i (\text{mod } 2)$. Then $[\sigma_l(1-\lambda_i), \sigma_l(-\lambda_i - 1)]$ is a generator that's below row $\sigma_i(1 - \lambda_i)$. The column coordinate t_i is the smallest integer $\sigma_l(\lambda_i - 1)$ for such λ_l. Therefore it is equal to $B(-\lambda_i - 1)$.

Case 2: Suppose there's no such λ_l as in case (1). But there exist some λ_l such that $\lambda_l \geqslant \lambda_i + 2$. In this case $[\sigma_i(-\lambda_i), \sigma_i(-2 - \lambda_i)]$ is a generator that's below row $\sigma_i(1 - \lambda_i)$. Then similar to case (1), t_i is equal to $B(-2 - \lambda_i) + 1$. But under previous assumption, no $-1 - \lambda_i$ appears in the diagonal entries of H, so $B(-2 - \lambda_i) = B(-1 - \lambda_i)$.

Case 3: Suppose $\lambda_l - \lambda_i \leqslant 1$ when $1 \leqslant l \leqslant i$. Then all diagonal entries of H are bigger than $-l_i - 1$ and $B(-1 - \lambda_i) = n$. In this case, there's no generator below row $\sigma_i(1 - \lambda_i)$ so $(\star)_{\lambda_i} = n + 1 = B(-1 - \lambda_i + 1)$.

The formula for $m_{\mathcal{O}_\lambda}$ is derived if we use $A(l) = n - B(l)$ and put the values of $\sigma_i(1 - \lambda_i)$ and t_i into the equation (2.3.1). □

Lemma 4.2. $A(l) = \sum_{i=1}^{n} max(min(\lfloor \frac{\lambda_i + l + 1}{2} \rfloor, \lambda_i), 0)$.

Proof. The number of elements in the set $\{\lambda_i - 1, \dots, 1 - \lambda_i\}$ that are at most l is equal to a positive integer t, where $t \leqslant \lambda_i$ and $-\lambda_i - 1 + 2t \leqslant l$. The summation of all such t is $A(l)$. □

We write $\mathcal{O}_1 \leqslant \mathcal{O}_2$ (resp. $\mathcal{O}_1 < \mathcal{O}_2$) if the closure of the orbit $\mathcal{O}_1$ is (resp. strictly) contained in the closure of the orbit $\mathcal{O}_2$. This defines a partial order on nilpotent orbits. It is obvious that if $\mathcal{O}_1$ is smaller than $\mathcal{O}_2$, the dimension of $\mathcal{O}_1$ is smaller than the dimension of $\mathcal{O}_2$. It turns out that we also have the same relation for the dimension of minimal ideals.

Suppose that $\lambda = [\lambda_1, \lambda_2, \dots \lambda_p]$ and $\mathbf{d} = [d_1, d_2, \dots, d_q]$ are two partitions of n and correspond to the orbits $\mathcal{O}_\lambda$ and $\mathcal{O}_{\mathbf{d}}$ respectively. As shown in [6, 6.2.1], the partial order on $P(n)$ is defined as: $\mathbf{d} \leqslant \lambda$ if and only if $\sum_{1 \leqslant j \leqslant l} \lambda_j \leqslant \sum_{1 \leqslant j \leqslant l} d_j$ for $1 \leqslant l \leqslant n$.

Lemma 4.3. [6, Lem6.2.4] *(1). Suppose $\lambda, \mathbf{d} \in P(n)$. Then λ covers $\mathbf{d}$ in the order $\leqslant$ (meaning $\lambda < d$ and there is no partition e with*

$\lambda < e < d$) if and only if $\mathbf{d}$ can be obtained from λ by the following procedure. Choose an index i and let j be the smallest index greater than i with $0 \leqslant \lambda_j < \lambda_i - 1$. Assume that either $\lambda_j = \lambda_i - 2$ or $\lambda_k = \lambda_i$ whenever $i < k < j$. Then the parts of $\mathbf{d}$ are obtained from the λ_k by replacing λ_i, λ_j by $\lambda_i - 1$, $\lambda_j + 1$ respectively(and rearranging).

(2). $\mathcal{O}_\lambda \leqslant \mathcal{O}_{\mathbf{d}}$ if and only if $\lambda \leqslant \mathbf{d}$. Hence λ covers $\mathbf{d}$ iff $\mathcal{O}_{\mathbf{d}} < \mathcal{O}_\lambda$ and there is no nilpotent orbit $\mathcal{O}_e$, with $\mathcal{O}_{\mathbf{d}} < \mathcal{O}_e < \mathcal{O}_\lambda$.

Proposition 4.4. *If $\mathcal{O}_{\mathbf{d}} \leqslant \mathcal{O}_\lambda$ (resp $\mathcal{O}_{\mathbf{d}} < \mathcal{O}_\lambda$), then $m_{\mathcal{O}_{\mathbf{d}}} \leqslant m_{\mathcal{O}_\lambda}$ (resp. $m_{\mathcal{O}_{\mathbf{d}}} < m_{\mathcal{O}_\lambda}$).*

Proof. It suffices to prove the proposition under the assumption that λ covers $\mathbf{d}$. Since all λ_i are nonnegative, by Lemma 4.2,

$$A(\lambda_i - 1) = \sum_{j=1}^{n} min(\lfloor\frac{\lambda_i + \lambda_j}{2}\rfloor, \lambda_j) = \sum_{j\leqslant i}\lfloor\frac{\lambda_i + \lambda_j}{2}\rfloor + \sum_{j<i}\lambda_j$$

$$A(-1 - \lambda_i) = \sum_{j} max(min(\lfloor\frac{\lambda_j - \lambda_i}{2}\rfloor, j), 0) = \sum_{j<i}\lfloor\frac{\lambda_j - \lambda_i}{2}\rfloor.$$

Therefore

$$\begin{aligned} A(\lambda_i - 1) - A(-\lambda_i - 1) &= \sum_{j<i}(\lfloor\frac{\lambda_i + \lambda_j}{2}\rfloor - \lfloor\frac{\lambda_j - \lambda_i}{2}\rfloor) + \sum_{j\geqslant i}\lambda_j \\ &= \sum_{j<i}\lambda_i + \sum_{j\geqslant i}\lambda_j. \end{aligned}$$

Hence,

$$\sum_i(\sum_{j<i}\lambda_i + \sum_{j\geqslant i}\lambda_j) = \sum_i(\sum_{j<i}1)\lambda_i + \sum_j(\sum_{i\leqslant j}1)\lambda_j = \sum_i(2i-1)\lambda_i.$$

Therefore the formula for $m_{\mathcal{O}}$ can be written as

$$m_{\mathcal{O}} = \frac{n(n+1)}{2} + \sum_{i=1}^{n}(2i-1)\lambda_i + \sum_{i=1}^{k}\frac{n_{t_i}n_{t_i} - 1}{2}.$$

Since λ covers $\mathbf{d}$, as in lemma 4.3, for simplicity, we assume when $i < t < j$, $\lambda_i < \lambda_t < \lambda_j$. Then d differs from λ only when $d_i = \lambda_i - 1$ and $d_j = \lambda_j + 1$. The difference between the second term of $m_{\mathcal{O}_\lambda}$ and $m_{\mathcal{O}_{\mathbf{d}}}$ is equal to

$$\begin{aligned} &-(2i-1)\lambda_i - (2j-1)\lambda_j + (2i-1)(\lambda_i - 1) - (2j-1)(\lambda_j + 1) \\ &= 2(j-i) > 0. \end{aligned}$$

We need to discuss the exponential form of λ and $\mathbf{d}$. If $\lambda_i - \lambda_j > 2$, then because of the assumption, $i = j - 1$. Since $\lambda_i \neq \lambda_j$, suppose the exponential expression of λ is $\lambda = [\lambda_1^{n_{\lambda_1}}, \ldots, \lambda_i^{n_{\lambda_i}}, \lambda_j^{n_{\lambda_j}}, \ldots, \lambda_n^{n_{\lambda_n}}]$. Then the exponential form of $\mathbf{d}$ is $\lambda = [\lambda_1^{n_{\lambda_1}}, \ldots, \lambda_i^{n_{\lambda_i}-1}, \lambda_i - 1, \lambda_j +$

$1, \lambda_j^{n_{\lambda_j}-1}, \ldots, \lambda_n^{n_{\lambda_n}}]$. The difference between the third term of $m_{\mathcal{O}_\lambda}$ and $m_{\mathcal{O}_{\mathbf{d}}}$ is

$$\frac{n_{\lambda_i}(n_{\lambda_i}-1)}{2} + \frac{n_{\lambda_j}(n_{\lambda_j}-1)}{2} - \frac{(n_{\lambda_i}-1)(n_{\lambda_i}-2)}{2} - \frac{(n_{\lambda_j}-1)(n_{\lambda_j}-2)}{2}$$
$$= n_{\lambda_i} + n_{\lambda_j} - 2 \geqslant 0.$$

From the two inequalities above, it's easy to deduce that $m_{\mathcal{O}_\lambda} > m_{\mathcal{O}_{\mathbf{d}}}$.

If $\lambda_i - \lambda_j = 2$, then $\lambda_t = \lambda_i - 1 = \lambda_j + 1$, for any $i < t < j$. The exponential form of λ is $\lambda = [\lambda_1^{n_{\lambda_1}}, \ldots, \lambda_i^{n_{\lambda_i}}, \lambda_t^{n_{\lambda_t}}, \lambda_j^{n_{\lambda_j}}, \ldots, \lambda_n^{n_{\lambda_n}}]$ and d has exponential form $d = [\lambda_1^{n_{\lambda_1}}, \ldots, \lambda_i^{n_{\lambda_i}-1}, \lambda_t^{n_{\lambda_t}+2}, \lambda_j^{n_{\lambda_j}-1}, \ldots, \lambda_n^{n_{\lambda_n}}]$. The difference between the third term of $m_{\mathcal{O}_\lambda}$ and $m_{\mathcal{O}_{\mathbf{d}}}$ is

$$\frac{n_{\lambda_i}(n_{\lambda_i}-1)}{2} + \frac{n_{\lambda_j}(n_{\lambda_j}-1)}{2} + \frac{n_{\lambda_t}(n_{\lambda_t}-1)}{2}$$
$$- \frac{(n_{\lambda_i}-1)(n_{\lambda_i}-2)}{2} - \frac{(n_{\lambda_j}-1)(n_{\lambda_j}-2)}{2} - \frac{(n_{\lambda_t}+2)(n_{\lambda_t}+1)}{2}$$
$$= n_{\lambda_i} + n_{\lambda_j} - 2 - 2n_{\lambda_t} - 1.$$

Since $n_{\lambda_t} = j - i - 1$, we can compare the second and third term of $m_{\mathcal{O}_\lambda}$ and $m_{\mathcal{O}_{\mathbf{d}}}$ and still get strict inequality. □

We proved the existence of minimal ideals of dimension $m_{\mathcal{O}}$ for nilpotent $\mathcal{O}$. And Example 3.2 shows that minimal ideals are not unique. All the minimal ideals we have constructed above are contained in the Dynkin ideal $\mathfrak{q}_{H,2}$. However, it's possible to have minimal ideals that are not contained in the Dynkin ideal. If we know some information about a general ideal I, here is a criterion to see whether this ideal is minimal or not.

Corollary 4.5. *Suppose the ideal I contains a nilpotent element X and dim$I = m_{\mathcal{O}_X}$, then the associated orbit of I is $\mathcal{O}_X$.*

Proof. The ideal I contains X, so $\mathcal{O}_I \geqslant \mathcal{O}_X$. By the strict inequality in Proposition 4.4, it's not possible that $\mathcal{O}_I \gneq \mathcal{O}_X$. □

5. Minimal Ideals For Type C_n

Let $\mathfrak{g}$ be $\mathfrak{sp}_{2n}$. The set of positive roots in $\mathfrak{g}$ is $\Delta^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{2e_i \mid 1 \leqslant i \leqslant n\}$. Take a partition $\mathbf{d}$ of $2n$ with odd parts repeated with even multiplicity. By Theorem 2.7, it corresponds to a nilpotent orbit $\mathcal{O}_{\mathbf{d}}$ in $\mathfrak{g}$. The partition $\mathbf{d}$ can be written as

$$\mathbf{d} = [d^{r_d}, (d-1)^{r_{d-1}}, \ldots, 2^{r_2}, 1^{r_1}],$$

where r_k is even if k is odd. To distinguish the integers with with the same value but in different positions, we rewrite $\mathbf{d}$ as

$$\mathbf{d} = [d_1, \ldots, d_{r_d}, \ldots, 1_1 \ldots, 1_{r_1}],$$

where $\sum_{i=1}^{i}(r_i)d = 2n$ and r_i is even if $i \in 2\mathbb{N}+1$. Here the index d_i has value d and the subscript i distinguishes indexes with the same value but at different positions.

The procedure to get an ideal of minimal dimension for the nilpotent orbit $\mathcal{O}_{\mathbf{d}}$ is similar to what we did in the type A_{n-1} case. Let $\mathcal{A}$ be the index set $\sqcup_{k=1}^{d}\{k_1,\ldots,k_{r_k}\}$. First recall the procedures to get the weighted Dynkin diagram of $\mathbf{d}$. We take the union of integers $\{k_i-1,k_i-3,\ldots,1-k_i\}$ for any $k_i \in \mathcal{A}$ and rearrange the sequence in the form $h=(h_1,\ldots,h_n,-h_1,\ldots,-h_n)$, where $h_1 \geqslant h_2 \ldots \geqslant h_n \geqslant 0$. Again we need some index maps to keep track of the first n integers in h.

We define the following sequence of maps $\{\sigma_{k_i}\}_{k_i\in\mathcal{A}}$.

$$\sigma_{k_i} = \begin{cases} \{k-1,k-3,\ldots,2,0\} \to [n] \text{ if } k \text{ is odd and } 1 \leqslant i \leqslant \frac{r_k}{2}; \\ \{k-1,k-3,\ldots,2\} \to [n], \text{ if } k \text{ is odd and } \frac{r_k}{2} < i \leqslant r_k; \\ \{k-1,k-3,\ldots,1\} \to [n], \text{ if } k \text{ is even and } 1 \leqslant i \leqslant r_k. \end{cases}$$

For any $k_i \in \mathcal{A}$, let $Im(\sigma_{k_i})$ (resp. $Dom(\sigma_{k_i})$) be the image (resp. domain) of σ_{k_i}. For $1 \leqslant k \leqslant 2n, 1 \leqslant i \leqslant r_k$, write $\tilde{i}_k = r_k+1-i$.

Lemma 5.1. *There exists a set of maps $\{\sigma_\tau\}_{\tau\in\mathcal{A}}$ satisfying the following properties:*

(1) For any $\tau,\omega \in \mathcal{A}$, σ_τ is one-to-one and $Im(\sigma_\tau)\cap Im(\sigma_\omega)=\emptyset$.

(2) For any $\tau,\omega \in \mathcal{A}$, $m \in Dom(\sigma_\tau)$, $l \in Dom(\sigma_\omega)$, and $m > l$, $\sigma_\tau(m) < \sigma_\omega(l)$.

(3) If $1 \leqslant k \leqslant 2n$, $1 \leqslant i < j \leqslant r_k$, and $m \in Dom(\sigma_{k_i})\cap Dom(\sigma_{k_j})$, then $\sigma_{k_i}(m) < \sigma_{k_j}(m)$.

(4) If $k,l \in \mathbb{N}$, $1 \leqslant i \leqslant \lfloor\frac{r_k}{2}\rfloor$, $1 \leqslant j \leqslant \lfloor\frac{r_l}{2}\rfloor$ and $m > 0$, then either $\sigma_{l_j}(m) < \sigma_{k_i}(m) < \sigma_{k_{\tilde{i}_k}}(m) < \sigma_{l_{\tilde{j}_l}}(m)$ or $\sigma_{k_i}(m) < \sigma_{l_j}(m) < \sigma_{l_{\tilde{j}_l}}(m) < \sigma_{k_{\tilde{i}_k}}(m)$.

(5) Let k,l,i,j be the same as in (4), if $\sigma_{k_i}(2) < \sigma_{l_j}(2)$, then $\sigma_{k_i}(0) < \sigma_{l_j}(0)$.

(6) Let k,l be even integers, if r_k is odd, $i=\lceil\frac{r_k}{2}\rceil$ and $1 \leqslant j \leqslant \lfloor\frac{r_l}{2}\rfloor$, then $\sigma_{l_j}(m) < \sigma_{k_i}(m) < \sigma_{l_{\tilde{j}_l}}(m)$.

(7) Let k,l be even integers. If r_k, r_l are odd and $k < l$, then $\sigma_{k_{\lceil\frac{r_k}{2}\rceil}}(m) < \sigma_{l_{\lceil\frac{r_l}{2}\rceil}}(m)$.

The construction of index maps is similar to type A_{n-1} case except that here we need more restrictions for different types of indexes maps. If we put any $m \in Dom(\sigma_\tau)$ into the position $\sigma_\tau(m)$ and $-m$ into $\sigma_\tau(m)+n$, properties (1) and (2) make sure that we could get h as above. Property (3) and (4) gives orders for integers from different parts of $\mathbf{d}$. Properties 5 deal with odd parts of the partition $\mathbf{d}$. And properties (6) and (7) deal with even parts of $\mathbf{d}$.

Now it's possible to get a set of positive roots. If k is odd and $1 \leqslant i \leqslant \frac{r_k}{2}$, set

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \ldots, e_{\sigma_{k_i}(2)} - e_{\sigma_{k_i}(0)}\}.$$

If k is odd and $i > \frac{r_k}{2}$, set

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \ldots, e_{\sigma_{k_i}(4)} - e_{\sigma_{k_i}(2)}, e_{\sigma_{k_i}(2)} + e_{\sigma_{k_{\bar{i}_k}}(0)}\}.$$

If k is even and $i \leqslant \lfloor \frac{r_k}{2} \rfloor$, set

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \ldots, e_{\sigma_{k_i}(3)} - e_{\sigma_{k_i}(1)}, e_{\sigma_{k_i}(1)} + e_{\sigma_{k_{\bar{i}_k}}(1)}\}.$$

If k is even and $i > \lceil \frac{r_k}{2} \rceil$, set

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \ldots, e_{\sigma_{k_i}(3)} - e_{\sigma_{k_i}(1)}\}.$$

If k is even, r_k is odd and $i = \lceil r_k/2 \rceil$, set

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \ldots, e_{\sigma_{k_i}(3)} - e_{\sigma_{k_i}(1)}, 2e_{\sigma_{k_i}(1)}\}.$$

We define $\mathcal{C}$ (the union of $\{\mathcal{C}(\sigma_\tau)\}_{\tau \in \mathcal{A}}$), $\{X_\alpha\}_{\alpha \in \mathcal{C}}$ and $X = \sum_{\alpha \in \mathcal{C}} X_\alpha$ the same way as we did for $\mathfrak{sl}(n)$. Let

$$H = H_{\mathcal{C}} = \sum_{\tau \in \mathcal{A}} \sum_{m \in Dom(\sigma_\tau)} m(E_{\sigma_\tau(m), \sigma_\tau(m)} - E_{n+\sigma_\tau(m), n+\sigma_\tau(m)}).$$

Then H is the matrix realization of h, hence is the Dynkin element for the orbit $\mathcal{O}_{\mathbf{d}}$. And X is a nilpotent element that's in $\mathcal{O}_{\mathbf{d}}$ (The reason is similar to the case of type A_{n-1} and the reference is [6, chap5]).

Lemma 5.2. *For any roots $\alpha, \beta \in \mathcal{C}$, $\alpha - \beta \notin \Delta$.*

Proof. If a and β are positive roots such that $\alpha - \beta$ is a root, then we are in one of the following cases: 1) $\alpha = e_i \pm e_j, \beta = e_i \pm e_k$ $(i \neq j, i \neq k)$; 2) $\alpha = e_i - e_k$, $e_j - e_k$ $(j \neq l, i \neq k)$; 3) $\alpha = e_i \pm e_k$ and $\beta = 2e_i$ $(i \neq k)$. Because of the construction of the set $\mathcal{C}$, it does not contain two roots of the form mentioned above. □

Lemma 5.2 is a weaker condition than Lemma 3.3. Indeed, the set $\mathcal{C}$ is not antichain. For example, let $\mathbf{d} = [4, 2]$ and $H = diag\{\underline{3}, \overline{1}, \underline{1}\}$. For simplicity, we omit the negative part of the diagonal entries of H. Then $\mathcal{C} = \{e_1 - e_3, 2e_3\} \cup \{2e_2\}$ and $2e_2 > 2e_3$.

Notice in the construction of the standard triple for H and X, we only need the condition stated in Lemma 5.2, therefore we still can get a standard triple $\{H, X, Y\}$ associated to $\mathcal{C}$ as in Lemma 2.4. Thus $\mathfrak{g}_{H,i}$, $\mathfrak{q}_{H,i}$, $I_{\mathcal{C}}$, $\mathcal{C}^+$ and $\mathcal{C}^-$ are defined accordingly as in Section 2.2. Also we can prove that $\mathfrak{q}_{H,3}$ is contained in the ideal $I_{\mathcal{C}}$ in a similar way.

Proposition 5.3. *The dimension of the ideal $I_{\mathcal{C}}$ is equal to $m_{\mathcal{O}_{\mathbf{d}}}$ and $\mathcal{O}_{I_{\mathcal{C}}} = \mathcal{O}_{\mathbf{d}}$.*

Proof. The main part of the proof is basically the same as Proposition 3.5. The problem is reduced to construct a bijection between $\mathcal{C}^+$ and $\mathcal{C}^-$.

Since the root space of $\mathfrak{g}_{H,2}$ depends only on either the odd parts or the even parts of $\mathbf{d}$, we discuss odd partitions and even partitions separately.

Let k_i, l_j be odd parts of $\mathbf{d}$ and we always have the dual indexes $k_{\tilde{i}_k}$ and $l_{\tilde{j}_l}$. If $\alpha = e_{\sigma_{k_i}(m)} - e_{\sigma_{l_j}(m-2)} \notin \mathcal{C}$, where $m > 2$. Then we look at $\beta = e_{k_{\tilde{i}_k}(m)} - e_{l_{\tilde{j}_l}(m-2)}$. By condition 4 of Lemma 5.1, either α lies in $\mathcal{C}^+$ and β lies in $\mathcal{C}^-$ or the other way around.

By condition (5), the two roots $\alpha = e_{\sigma_{k_i}(2)} - e_{\sigma_{l_j}(0)}$ and $\beta = e_{\sigma_{k_{\tilde{i}_k}}(2)} + e_{l_j(0)}$ are in bijection with each other.

Suppose k_i, l_j are even parts of $\mathbf{d}$. The two roots $\alpha = e_{\sigma_{k_i}(m)} - e_{\sigma_{l_j}(m-2)}$ and $\beta = e_{k_{\tilde{i}_k}(m)} - e_{l_{\tilde{j}_l}(m-2)}$ are in bijection with each other when $m \geqslant 3$ and either $i \neq \tilde{i}_k$ or $j \neq \tilde{j}_l$.

If k_i and l_j are even and either $i \neq \tilde{i}_k$ or $j \neq \tilde{j}_l$,then $\alpha = e_{\sigma_{k_i}(1)} + e_{\sigma_{l_j}(1)}$ corresponds to $\beta = e_{\sigma_{k_{\tilde{i}_k}}(1)} + e_{\sigma_{l_{\tilde{j}_l}}(1)}$.

The only remaining part is $\alpha = e_{\sigma_{k_i}(m)} - e_{\sigma_{l_j}(m-2)}$ where k and l are even and $i = \tilde{i}_k$ and $j = \tilde{j}_l$. In this case, surely $k \neq l$.

If $k > l$ and $m > 5$, then α is in bijection with $\beta = e_{\sigma_{l_j}(m-2)} - e_{\sigma_{k_i}(m-4)}$ in $\Delta(\mathfrak{g}_{H,2}) \backslash \mathcal{C}$. If $m = 3$, α is in bijection with $\beta = e_{\sigma_{k_i}(1)} + e_{\sigma_{l_j}(1)}$. If $k < l$ and $m \geqslant 3$, α corresponds to root $\beta = e_{\sigma_{l_j}(m+2)} - e_{\sigma_{k_i}(m)}$.

If $i = \tilde{i}_k$ and $j = \tilde{j}_l$, then $\alpha = e_{\sigma_{k_i}(1)} + e_{\sigma_{l_j}(1)}$ is in bijection with $\beta = e_{\sigma_{l_j}(3)} - e_{\sigma_{k_i}(1)}$, if $l > k$ or $\beta = e_{\sigma_{k_i}(3)} - e_{\sigma_{l_j}(1)}$, if $l < k$.

Finally if k is even, $\alpha = 2e_{\sigma_{k_i}(1)}$ and $\beta = 2e_{\sigma_{k_{\tilde{i}_k}}(1)}$ are bijective with each other. □

Example 5.4. *Let* $\mathbf{d} = [5^2, 3^2]$ *and let* $H = diag\{\underline{4}, \overline{4}, \underline{2}, \tilde{2}, \dot{2}, \overline{2}, \underline{0}, \tilde{0}\}$ *be the Dynkin element. Here again we omit the negative half part of* H. σ_{5_1} *maps* i *to the position of* $\underline{i}$ *and* σ_{5_2} *maps* i *to the position of* $\overline{i}$ *and* σ_{3_1} *maps* i *to* $\tilde{i}$, σ_{3_2} *maps* i *to* $\dot{i}$.

6. Minimal Ideals For Type B_n

Let $\mathfrak{g} = \mathfrak{so}(2n+1)$. The set of positive roots is $\Delta^+ = \{e_i - e_j, e_i \mid 1 \leqslant i < j \leqslant n\} \cup \{e_i + e_j \mid 1 \leqslant i, j \leqslant n, i \neq j\} \cup \{e_i\}_{i=1}^n$. Let $\mathbf{d}$ be partition of $2n+1$ with even parts repeated with even multiplicity. As in Theorem 2.7, it corresponds to a nilpotent orbit $\mathcal{O}_{\mathbf{d}}$.

Suppose that $\mathbf{d} = [d^{r_d}, \ldots, 2^{r_2}, 1^{r_1}]$, then r_k is even when k is an even integer. We rewrite $\mathbf{d}$ in the form

$$\mathbf{d} = [d_1, \ldots, d_{r_d}, \ldots, 1_1, \ldots, 1_{r_1}]$$

where $\sum_{i=1}^{d} (r_i) i = 2n+1$ and r_i is even if $2|i$.

For each part of the partition $\mathbf{d}$, we need to attach a set of positive roots to it. Then this means again that we should choose the appropriate index map for each k_i, which would determine the corresponding positive roots. Let $\mathcal{A}$ be defined as in section 2.4.

For any $k_i \in \mathcal{A}$, let

$$\sigma_{k_i} = \begin{cases} \{k-1, k-3, \ldots, 2, 0\} \to [n] \text{ if } k \text{ is odd and } 1 \leqslant i \leqslant \lfloor \frac{r_k}{2} \rfloor; \\ \{k-1, k-3, \ldots, 2\} \to [n], \text{ if } k \text{ is odd and } \lceil \frac{r_k}{2} \rceil < i \leqslant r_k; \\ \{k-1, k-3, \ldots, 1\} \to [n], \text{ if } k \text{ is even and } 1 \leqslant i \leqslant r_k. \end{cases}$$

In the case of $\mathfrak{sp}(2n)$, the odd parts have even multiplicity, therefore, we could define a dual pair of indexes $(k_i, k_{\tilde{i}_k})$, where k is odd and $\tilde{i}_k := r_k + 1 - i$. The formula above defines index maps $\{\sigma_{k_i}\}$ for all any $k_i \in \mathcal{A}$ in the last section. However, for $\mathfrak{so}(2n+1)$, it's possible that r_k is odd for an odd part k of $\mathbf{d}$. In that case, we haven't defined the map σ_{k_i} when i is equal to $\lceil \frac{r_k}{2} \rceil$. Indeed, compared to the previous case, if the difficulty lies in the even parts of the partition $\mathbf{d}$ for $\mathfrak{sp}(2n)$, the most difficult part to construct the index map and to find the bijection for $\mathfrak{so}(2n+1)$ lies in its odd part.

Suppose $l^1, l^2, \ldots, l^r$ are the remaining indexes of $\mathcal{A}$ without index maps associated to them. Namely, $l^i = k_s$, where k, r_k are odd, and $s = \lceil \frac{r_k}{2} \rceil$. Moreover, we assume that $l^1 < l^2 \cdots < l^r$. Then r must be odd since the total summation of all parts of $\mathbf{d}$ is $2n+1$. Now it's possible to define the index maps for $\{l^i\}$.

If i is odd, set

$$\sigma_{l^i} : \{l^i - 1, l^i - 3, \ldots, 2\} \to [n].$$

If i is even, set

$$\sigma_{l^i} : \{l^i - 1, l^i - 3, \ldots, 2, 0\} \to [n].$$

The properties for these index maps are slightly different from the previous case.

Lemma 6.1. *There exits a sequence of maps $\{\sigma_\tau \mid \tau \in \mathcal{A}\}$, satisfying the first five properties as in lemma 5.1 with additional two:*

(6) Let k, l be odd integers, $1 \leqslant i \leqslant \lfloor r_k/2 \rfloor$ and $j = \tilde{j}_l$, then $\sigma_{k_i}(m) < \sigma_{l_j}(m) < \sigma_{k_{\tilde{i}_k}(m)}$. If $m = 0$, then $\sigma_{k_i}(0) < \sigma_{l_j}(0)$.

(7) If k, l are odd integers, $k < l$ and $i = \tilde{i}_k$, $j = \tilde{j}_l$, then $\sigma_{k_i}(m) < \sigma_{l_j}(m)$.

The last two properties give additional restrictions for the placements of the indexes that come from the odd parts of $\mathbf{d}$ with odd multiplicities.

For the even part k_i of $\mathbf{d}$, if $1 \leqslant i \leqslant \frac{r_k}{2}$, we can attach a set of positive roots to the map σ_{k_i}:

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \ldots, e_{\sigma_{k_i}(3)} - e_{\sigma_{k_i}(1)}, e_{\sigma_{k_i}(1)} + e_{\sigma_{k_{\tilde{i}_k}}(1)}.\}$$

If k is even and $i > \frac{r_k}{2}$, set:

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \dots, e_{\sigma_{k_i}(3)} - e_{\sigma_{k_i}(1)}\}.$$

For any odd part k_i of $\mathbf{d}$, if $i < \tilde{i}_k$, set

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \dots, e_{\sigma_{k_i}(2)} - e_{\sigma_{k_i}(0)}\}.$$

If $i > \tilde{i}_k$, the set of positive roots we attached to the map σ_{k_i} is

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \dots, e_{\sigma_{k_i}(2)} + e_{\sigma_{k_{\tilde{i}_k}}(0)}\}.$$

The remaining case is that there exists some i such that $i = \tilde{i}_k$. In this case, if $0 \in Dom(\sigma_{k_i})$, we can attach

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \dots, e_{\sigma_{k_i}(2)} + e_{\sigma_{k_i}(0)}, e_{\sigma_{k_i}(2)} - e_{\sigma_{k_i}(0)}\}$$

to the map σ_{k_i}. Otherwise, $0 \notin Dom(\sigma_{k_i})$, we can attach almost the same chunk of roots to σ_{k_i} except replacing the last two roots in $\mathcal{C}(\sigma_{k_i})$ with $e_{\sigma_{k_i}(2)}$. Namely

$$\mathcal{C}(\sigma_{k_i}) = \{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \dots, e_{\sigma_{k_i}(4)} - e_{\sigma_{k_i}(2)}, e_{\sigma_{k_i}(2)}\}$$

For each map σ_{k_i}, once we get the set of positive roots associated to σ_{k_i}, we can define $H, X, \mathcal{C}, I_{\mathcal{C}}, \mathfrak{g}_{H,i}, \mathfrak{q}_{H,i}, \mathcal{C}^+$ and $\mathcal{C}^-$ the same way as in previous section. The fact that $\mathcal{O}_X = \mathcal{O}_{\mathbf{d}}$ comes from [6, Chap 6]. In the case of type B_n, $\mathcal{C}$ is not an antichain and does not satisfy lemma 5.2. It's because that e_i, e_j can be both in $\mathcal{C}$ for some i, j. That means we cannot get a standard triple from H, X. However, we will prove

Proposition 6.2. *The ideal $I_{\mathcal{C}}$ has associated orbit $\mathcal{O}_{\mathbf{d}}$ and* $\dim I_{\mathcal{C}} = m_{\mathcal{O}_{\mathbf{d}}}$.

This follows from Proposition 6.3 below.

Proposition 6.3. *There exists a bijection between $\mathcal{C}^+$ and $\mathcal{C}^-$.*

Proof. We discuss the even parts and odd parts of the partition $\mathbf{d}$ separately. For simplicity, we will use k_i to denote the map σ_{k_i}.

Suppose k_i, l_j are even parts of the partition $\mathbf{d}$ and $k_i \neq l_j$ as index in $\mathcal{A}$. Then $\alpha = e_{k_i(m)} - e_{l_j(m-2)}$ is in bijection with $\beta = e_{k_{\tilde{i}_k}(m)} - e_{l_{\tilde{j}_l}(m-2)}$ when $m > 2$. Namely either $\alpha \in \mathcal{C}^+$ and $\beta \in \mathcal{C}^-$ or $\alpha \in \mathcal{C}^-$ and $\beta \in \mathcal{C}^-$ (by condition 4 of lemma 6.1). If $\alpha = e_{k_i(1)} + e_{l_j(1)}$, then it is in bijection with $\beta = e_{k_{\tilde{i}_k}(1)} + e_{l_{\tilde{j}_l}(1)}$ (also by lemma 6.1).

Suppose that k_i, l_j are odd parts of $\mathbf{d}$. If $m > 2$ and either $i \neq \tilde{i}_k$ or $j \neq \tilde{j}_l$, then the roots $\alpha = e_{k_i(m)} - e_{l_j(m-2)}$ and $\beta = e_{k_{\tilde{i}_k}(m)} - e_{l_{\tilde{j}_l}(m-2)}$ are bijective with each other. If $\alpha = e_{k_i(2)}$, where $i \neq \tilde{i}_k$, the root corresponding to α is $\beta = e_{k_{\tilde{i}_k}(2)}$. If $\alpha = e_{k_i(2)} - e_{\sigma_{l_j}(0)}$, then $\beta = e_{\sigma_{k_{\tilde{i}_k}}(2)} + e_{\sigma_{l_j}(0)}$ is the corresponding root.

The final step is to find the bijection when $k \neq l$, $i = \tilde{i}_k$ and $j = \tilde{j}_l$. Suppose that $m > 4$ and $k > l$, the root $\alpha = e_{k_i(m)} - e_{l_j(m-2)}$ lies in $\mathcal{C}^-$

while its corresponding root $\beta = e_{l_j(m-2)} - e_{k_i(m-4)}$ lies in $\mathcal{C}^+$. For any $\alpha = e_{k_i(2)} - e_{l_j(0)}$ that lies in $\mathcal{C}^-$, i.e. $k > l$, it corresponds to a positive root $\beta = e_{k_i(2)} + e_{l_j(0)}$ that lies in $\mathcal{C}^+$.

The remaining positive root that lies in the set $\mathcal{C}^-$ has the form $\alpha = e_{k_i(4)} - e_{l_j(2)}$, with $i = \tilde{i}_k$ and $j = \tilde{j}_l$. Then we need to switch our notation to $\alpha = e_{l^s(4)} - e_{l^t(2)}$, where $l^s = k_i$ and $l^t = l_j$. Since $\alpha \in \mathcal{C}^-$, by condition 7 of lemma 6.1, we have $l^s > l^t$ and $s > t$.

In this case, if $0 \notin Dom(l^t)$, then $e_{l^t(2)} \in \mathcal{C}$, and α corresponds to $\beta = e_{l^t(2)} - e_{l^s(0)} \in \mathcal{C}^+$ if $0 \in Dom(l^s)$ or α corresponds to $\beta = e_{l^t(2)} - e_{l^{s-1}(0)} \in \mathcal{C}^+$ if $0 \notin Dom(l^s)$.

If $0 \in Dom(l^t)$, then $e_{l^t(2)} \pm e_{l^t(0)} \in \mathcal{C}$, and α corresponds to $\beta = e_{l^t(2)} - e_{l^s(0)}$ if $0 \in Dom(l^s)$ or α corresponds to $\beta = e_{l^t(2)} - e_{l^{s-1}(0)}$ if $0 \notin Dom(l^s)$ and $s > t+1$. If $s = t+1$, then $0 \notin Dom(l^s)$ and a corresponds to $\beta = e_{l^t(2)} \in \mathcal{C}^+$. □

7. Minimal Ideals For Type D_n

Let $\mathfrak{g} = \mathfrak{so}(2n)$. The matrix realization of $\mathfrak{so}(2n)$ is

$$\{\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} \mid Z_i \in M_n(\mathbb{C}), Z_2, Z_3 \text{ skew-symmetric } \}.$$

The set of positive roots is $\Delta^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{e_i + e_j \mid 1 \leqslant i, j \leqslant n, i \neq j\}$. Let $\mathbf{d}$ be partition of $2n$ with even parts repeated with even multiplicity. If $\mathbf{d}$ is not a very even partition, as in Theorem 2.7, it corresponds to a nilpotent orbit $\mathcal{O}_{\mathbf{d}}$. If $\mathbf{d}$ is a very even partition, then it corresponds to two nilpotent orbits $\mathcal{O}^I_{\mathbf{d}}$ and $\mathcal{O}^{II}_{\mathbf{d}}$.

Suppose that $\mathbf{d} = [d^{r_d}, \ldots, 2^{r_2}, 1^{r_1}]$, then r_k is even when $k \in 2\mathbb{N}$. We rewrite $\mathbf{d}$ in the form

$$\mathbf{d} = [d_1, \ldots, d_{r_d}, \ldots, 1_1, \ldots, 1_{r_1}], \text{ where } \sum_{i=1}^{d} (r_i) i = 2n \text{ and } r_i \text{ is even if } 2|i.$$

Recall the procedure to get the weighted Dynkin diagram from the partition $\mathbf{d}$. We can obtain a sequence of integers from $\mathbf{d}$ the same way as we did for $\mathfrak{g} = \mathfrak{sp}(2n)$. The sequence takes the form

$$(h_1, \ldots, h_n, -h_1, \ldots, -h_n), \text{ where } h_1 \geqslant h_2 \geqslant \ldots \geqslant h_n.$$

If $\mathbf{d}$ is not a very even partition, following previous matrix realization of $\mathfrak{g}$, the Dynkin element for the orbit $\mathcal{O}_{\mathbf{d}}$ is

$$H = diag\{h_1, h_2, \ldots, h_n, -h_1, \ldots, -h_n\}.$$

However, if $\mathbf{d}$ is a very even partition, the Dynkin elements for the orbits $\mathcal{O}^I_{\mathbf{d}}$ and $\mathcal{O}^{II}_{\mathbf{d}}$ are

$$H_1 = diag\{h_1, \ldots, h_n, -h_1, \ldots, -h_n\}$$

and

$$H_2 = diag\{h_1, \ldots, h_{n-1}, -h_n, -h_1, \ldots, -h_{n-1}, h_n\},$$

both of which are dominant.

Let $\mathcal{A}$ be the same as in last section and we define the index maps $\{\sigma_{k_i}\}_{k_i\in\mathcal{A}}$ the same way as we did for $\mathfrak{g} = \mathfrak{so}(2n+1)$. The notations $l^1, l^2, \ldots, l^r$ have the same meaning as in last section. The only difference is that r is even.

Lemma 7.1. *There exits a sequence of maps $\{\sigma_\tau \mid \tau \in \mathcal{A}\}$, satisfying the first five properties as in lemma 5.1 with additional two:*

(6) Let k, l be odd integers, $1 \leqslant i \leqslant \lfloor r_k/2 \rfloor$ and $j = \tilde{j}_l$, then $\sigma_{k_i}(m) < \sigma_{l_j}(m) < \sigma_{k_{\tilde{i}_k}(m)}$. If $m = 0$, then $\sigma_{k_i}(0) < \sigma_{l_j}(0)$.

(7) If k, l are odd integers, $k < l$ and $i = \tilde{i}_k$, $j = \tilde{j}_l$, then $\sigma_{k_i}(m) < \sigma_{l_j}(m)$.

For every even part k_i of $\mathbf{d}$ and for odd part k_i such that $i \neq \tilde{i}_k := r_k - i + 1$, we attach the same set of positive roots $\mathcal{C}(\sigma_{k_i})$ to σ_{k_i} as in section 2.5.

For $l^1 \leqslant \ldots \leqslant l^r$, there is always a pair (l^i, l^{i+1}), where $1 \leqslant i \leqslant r$ and i is odd.

If i is odd, we attach the following set of roots to σ_{l^i}:

$$\mathcal{C}(\sigma_{l^i}) = \{e_{\sigma_{l^i}(l^i-1)} - e_{\sigma_{l^i}(l^i-3)}, \ldots, e_{\sigma_{l^i}(2)} \pm e_{\sigma_{l^{i+1}}(0)}\}.$$

If i is even, we attach the following set of roots to σ_{l^i}:

$$\mathcal{C}(\sigma_{l_i}) = \{e_{\sigma_{l^i}(l^i-1)} - e_{\sigma_{l^i}(l^i-3)}, \ldots, e_{\sigma_{l^i}(2)} - e_{\sigma_{l^i}(0)}\}.$$

Similarly, we have $\mathcal{C}, X, H, I_{\mathcal{C}}, \mathcal{C}^+, \mathcal{C}^-$ and $I_{\mathcal{C}}$. If $\mathbf{d}$ is not an even partition, $\mathcal{O}_X$ is the unique orbit $\mathcal{O}_{\mathbf{d}}$.

If $\mathbf{d}$ is a very even partition, then X corresponds to the first orbit $\mathcal{O}^I_{\mathbf{d}}$ and the Dynkin element H has the form

$$H = diag\{h_1, \ldots, h_n, -h_1, \ldots, -h_n\}.$$

To get a representative for another orbit, first notice that $n = \sigma_{k_i}(1)$ for some $k_i \in \mathcal{A}$ and k is even. Let $j = \tilde{i}_k = r_k - i + 1$. We let

$$\begin{aligned}\tilde{\mathcal{C}}(\sigma_{k_i}) \cup \tilde{\mathcal{C}}(\sigma_{k_j}) = &\{e_{\sigma_{k_i}(k-1)} - e_{\sigma_{k_i}(k-3)}, \ldots, e_{\sigma_{k_i}(3)} + e_{\sigma_{k_i}(1)}, e_{\sigma_{k_i}(1)} - e_{\sigma_{k_j}(1)}\} \\ &\cup \{e_{\sigma_{k_j}(k-1)} - e_{\sigma_{k_j}(k-3)}, \ldots, e_{\sigma_{k_j}(3)} - e_{\sigma_{k_j}(1)}\}.\end{aligned}$$

For any $\tau \in \mathcal{A}$, if $\tau \neq k_i, k_j$, let $\tilde{\mathcal{C}}(\sigma_\tau) = \mathcal{C}(\sigma_\tau)$ and $\tilde{\mathcal{C}} = \cup_{\tau\in\mathcal{A}} \tilde{\mathcal{C}}(s_\tau)$. Let $\tilde{X} = \sum_{\tau\in\tilde{\mathcal{C}}} X_\tau$, it corresponds to the orbit $\mathcal{O}^{II}_{\mathbf{d}}$. The Dynkin element $\tilde{H}$ has the form $\tilde{H} = diag\{h_1, \ldots, h_{n-1}, -h_n, -h_1, \ldots, -h_{n-1}, h_n\}$.

In both cases, it's easy to see that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are antichains and there exist triples $\{H, X, Y\}$ and $\{\tilde{H}, \tilde{X}, \tilde{Y}\}$.

Example 7.2. *Let* $\mathbf{d} = [4^2]$. *Then* $H = \mathrm{diag}(\overline{3}, \underline{3}, \overline{1}, \underline{1}, -3, -3, -1, -1)$, *where* $\{\overline{3}, \overline{1}\} = Im(\sigma_{4_1})$ *and* $\{\underline{3}, \underline{1}\} = Im(\sigma_{4_2})$. *Then* $\mathcal{C} = \{e_1 - e_3, e_3 + e_4, e_2 + e_4\}$ *and* $\tilde{\mathcal{C}} = \{e_1 - e_3, e_3 - e_4, e_2 + e_4\}$.

We can obtain Proposition 6.2 in a similar way for $\mathfrak{so}(2n)$. If $\mathbf{d}$ is a very even partition, then $I_{\mathcal{C}}$ is the minimal ideal for $\mathcal{O}^I_{\mathbf{d}}$ and $I_{\tilde{\mathcal{C}}}$ is the minimal ideal for $\mathcal{O}^{II}_{\mathbf{d}}$. Here we omit the proof.

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