

AD-NILPOTENT IDEALS AND EQUIVALENCE RELATIONS

ABSTRACT. In this paper we study the set of ad-nilpotent ideals of a complex simple Lie algebra $\mathfrak{g}$ and its connections with affine Weyl groups and nilpotent orbits. We define a left equivalence relation for ad-nilpotent ideals based on their normalizer and generators, and prove that the equivalence relation is compatible with the left cell structure of affine Weyl group of $\mathfrak{g}$ and Lusztig's star operator for type $\tilde{A}_{n-1}$.

1. INTRODUCTION

Let G be a quasi-simple Lie group over $\mathbb{C}$ and $\mathfrak{g}$ be its Lie algebra. Fix a Borel subgroup B of G and let $\mathfrak{b}$ be the Lie algebra of B with nilradical $\mathfrak{n}$.

A subspace of $\mathfrak{n}$ is called an *ad-nilpotent ideal* if it is invariant under the adjoint action of B (sometimes it is also called a B -stable ideal when there is no confusion about the Borel subgroup B and for simplicity we will call it an ideal).

Ad-nilpotent ideals have close relations with affine Weyl groups and nilpotent orbits of $\mathfrak{g}$. In one aspect, the connection between ad-nilpotent ideals and affine Weyl group was studied by Cellini-Papi [7] [8], Panyushev [22] [23], Sommers [27], Shi [24], etc. Shi showed that there is a natural bijection between the set of ad-nilpotent ideals $\mathfrak{Ad}$ and the dominant sign types, and also the regions of the Catalan arrangement which are contained in the fundamental chamber. Later Cellini and Papi showed that there is a bijection between the ad-nilpotent ideals and certain elements of the affine Weyl group $\widehat{W}$ of $\mathfrak{g}$. These elements are called minimal elements of $\widehat{W}$. Panyushev observed that the minimal elements corresponding to the ideals are actually “minimal” ones contained in the dominant sign type corresponding to the same ideal. Sommers studied the “maximal” elements of $\widehat{W}$ and showed that they are in bijection with the set of strictly positive ideals.

The ad-nilpotent ideals also have natural connections with the nilpotent orbits via restriction of moment maps on the cotangent bundle. This involves the work of Kawanaka [13], Mizuno [19], Sommers [28] and Lawton [20]. The main purpose of this paper is to explore both the nilpotent orbits and affine Weyl groups for ad-nilpotent ideals and study some equivalence relations defined on them.

For any ad-nilpotent ideal I , consider the map $G \times_B I \rightarrow \mathfrak{g}$, which sends $(g, X) \in G \times I$ to $Ad(g)X$, where Ad means the adjoint action of

G . This is a restriction of the moment map and its image is the closure of one unique nilpotent orbit, which we denote by $\mathcal{O}_I$. This orbit is called the associated orbit of the ideal I .

There are various equivalence relations on $\mathfrak{Ad}$. The equivalence relation studied by Mizuno in [19] is simply that two ideals are equivalent if and only if they have the same associated orbit. To further distinguish ideals with same associated orbit but has different properties, Sommers defined in [28] an equivalence relation on $\mathfrak{Ad}$ based on some group cohomology of ad-nilpotent ideals. We define a new equivalence relation $\sim_L$ on $\mathfrak{Ad}$ based on its generators and normalizers. This turns out to be related to the star operator of affine Weyl groups in the case of type A .

The left, right and two-sided cells of affine Weyl groups were introduced by Kazhdan and Lusztig [15] to study Hecke algebra and Kazhdan-Lusztig polynomials. Two elements lies in the same left cell are left equivalent. The star operation on $\widehat{W}$ was defined by Lusztig, which induces an P_L -equivalence relation on $\widehat{W}$. Indeed P_L -equivalence implies left equivalence.

For affine Weyl group of type $\tilde{A}_{n-1}$, Shi used sign types (namely regions of the Shi arrangement) to describe combinatorially the cell structure of affine Weyl groups. Shi showed in [24] that the left cells (or two-sided cells) of $\widehat{W}$ are in bijection with the left cells (two-sided cells) of sign types. Also it was proved by Lusztig [21] and Shi [24] that the two-sided cells are parameterized by the partitions of n .

In this paper, we study the left-equivalence relations of ad-nilpotent ideals and the equivalence relation of affine Weyl groups for type $\tilde{A}_{n-1}$. Our main result shows that for two left equivalent ad-nilpotent ideals, there exists two dominant elements of $\widehat{W}$ corresponding to these two ideals and they lie in the same left cell.

Conversely, if two elements of $\widehat{W}$ are P_L -equivalent, one of which is dominant, then the other one is also dominant and their associated ideals are left equivalent.

The paper is organized as follows. We introduce the basic notations in section 2 and surveyed the known results about affine Weyl groups, the generator and normalizer of ad-nilpotent ideals. In section 3, we discussed mainly the moment maps and how to determine two ideals have the same associated orbits. We also introduced the left equivalence relations on ad-nilpotent ideals in this section. Section 4 mainly focused on dominant sign types. In section 5, we state some commutative diagram about the ad-nilpotent ideals and prove our main theorem.

2. NOTATIONS AND PRELIMINARIES

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{N}^+$ be the set of positive integers.

Let G be a complex semisimple algebraic group. Let $\mathfrak{g}$ be the Lie algebra of G . We denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the reduced root system associated to $(\mathfrak{g}, \mathfrak{h})$. For each root α , let $\mathfrak{g}_\alpha$ be the corresponding root space in $\mathfrak{g}$. Under the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$, there is a direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. Let W be the Weyl group of $\mathfrak{g}$.

We fix a positive root system Δ^+ inside Δ . The choice of the positive root system gives rise to a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and its opposite Borel subalgebra $\mathfrak{b}^-$. Let $\mathfrak{n}$ and $\mathfrak{n}^-$ be the nilradicals of $\mathfrak{b}$ and $\mathfrak{b}^-$. Let B be the Borel subgroup of G that corresponds to Δ^+ .

Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ be the set of simple roots of Δ . Then $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ is the root lattice. We set $V = \bigoplus_{i=1}^n \mathbb{R}\alpha_i = \mathfrak{h}_{\mathbb{R}}^*$. The Killing form on $\mathfrak{g}$ induces a W -invariant positive definite and symmetric bilinear form on V , which is denoted by $(\cdot, \cdot)$. For each root α , $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$ is the coroot for α and $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ is the coroot lattice.

Now we recall the definition of affine Weyl groups based on Kac's book [12].

Let $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$. We extend the bilinear form on V to the bigger space $\widehat{V}$ by letting $(\delta, \delta) = (\delta, v) = (\lambda, v) = (\lambda, \lambda) = 0$ for any $v \in V$ and $(\delta, \lambda) = 1$. This is a nondegenerate symmetric bilinear form on $\widehat{V}$ and by abuse of notation, we still denote it by $(\cdot, \cdot)$.

Let $\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ be the set of affine real roots. Let

$$\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^+\delta)$$

be the set of positive affine roots. We denote by $\alpha > 0$ when α is a positive affine root and $\alpha < 0$ when $\alpha \in \widehat{\Delta}$ is negative.

Let $\widehat{\Pi} = \Pi \cup \{-\alpha_0 + \delta\}$ be the set of affine simple roots, where α_0 is the highest root in Δ .

For each α in $\widehat{\Delta}$, we define the reflection s_α on $\widehat{V}$ by

$$s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha$$

for any $x \in \widehat{V}$. The affine Weyl group $\widehat{W}$ is generated by the set of reflections $\{s_\alpha\}_{\alpha \in \widehat{\Delta}}$. For simplicity, we write s_0 for $s_{-\alpha_0 + \delta}$ and s_i for the reflection s_{α_i} . Let S be the set of simple reflections $\{s_0, s_1, \dots, s_n\}$. Then $(\widehat{W}, S)$ is a Coxeter system.

Set

$$\begin{aligned} \mathcal{C}_0 &= \{\alpha \in V \mid (x, \alpha) > 0, \forall \alpha \in \Pi \text{ and } (x, \alpha_0) < 1\}, \\ \mathcal{C} &= \{x \in V \mid (x, \alpha) > 0, \forall \alpha \in \Pi\}. \end{aligned}$$

We call $\mathcal{C}_0$ the *fundamental alcove* and $\mathcal{C}$ the *(open) fundamental chamber*.

For any root $\alpha \in \Delta$ and any integer k , let $H_{\alpha, k} = \{x \in V \mid (x, \alpha) = k\}$ be the hyperplane that is determined by α and k .

Set

$$\begin{aligned} N(w) &= \{H_{\alpha,k} \mid H_{\alpha,k} \text{ separates } \mathcal{C}_0 \text{ and } w(\mathcal{C}_0)\} \\ &= \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) \in -\widehat{\Delta}^+\}. \end{aligned}$$

For each $w \in \widehat{W}$, let $l(w)$ be the cardinality of $N(w)$. We call l the length function of $\widehat{W}$.

Definition 2.1. We call a subset $\mathcal{I}$ of Δ^+ a (combinatorial) ideal of Δ^+ if for any $\alpha \in \mathcal{I}$, $\beta \in \Delta^+$ and $\alpha + \beta \in \Delta$, we have $\alpha + \beta \in \mathcal{I}$.

Lemma 2.2. The map $\mathcal{I} \mapsto \bigoplus_{\alpha \in \mathcal{I}} \mathfrak{g}_\alpha$ gives a bijection from the set of ideals of Δ^+ to the set of ad-nilpotent ideals of $\mathfrak{b}$.

Remark. Unless otherwise stated, we denote the ad-nilpotent ideal that corresponds to the ideal $\mathcal{I}$ of Δ^+ by I . The set of ad-nilpotent ideals of $\mathfrak{b}$ is denoted by $\mathfrak{Ad}$.

For ad-nilpotent ideals in $\mathfrak{g}$, we mainly discuss its geometric properties and for the ideals in the root system, we mainly discuss its combinatorial properties.

Let $\mathcal{I}$ be an ideal of Δ^+ . We call a root α a *generator* of $\mathcal{I}$ if $\alpha \in \mathcal{I}$ and for any $\beta \in \Delta^+$, $\alpha - \beta \notin \mathcal{I}$. The set of generators of $\mathcal{I}$ (resp. I) is denoted by $\Gamma(\mathcal{I})$ (resp. $\Gamma(I)$).

For two ad-nilpotent ideals I_1 and I_2 , the bracket relation is

$$[I_1, I_2] = \{[X, Y] \mid \forall X \in I_1 \text{ and } \forall Y \in I_2\}.$$

Similarly, for any ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ in Δ^+ , we define a bracket relation as follows:

$$[\mathcal{I}_1, \mathcal{I}_2] = \{\alpha + \beta \mid \alpha \in \mathcal{I}_1, \beta \in \mathcal{I}_2 \text{ and } \alpha + \beta \in \Delta\}.$$

Then $[\mathcal{I}_1, \mathcal{I}_2]$ is an ideal of Δ^+ and corresponds to the ad-nilpotent ideal $[I_1, I_2]$.

Then we can derive inductively a sequence of ideals $\{\mathcal{I}^1, \mathcal{I}^2, \dots, \mathcal{I}^k\}$ from $\mathcal{I}$ by letting $\mathcal{I}^1 = \mathcal{I}$, $\mathcal{I}^k = [\mathcal{I}^{k-1}, \mathcal{I}]$. This is a descending sequence of ideals.

Let

$$\widehat{W}_{dom} = \{w \in \widehat{W} \mid w(\alpha) \geq 0 \text{ for all } \alpha \in \Pi\}.$$

Elements that lie in the subset $\widehat{W}_{dom}$ are called dominant. By the definition of $N(w)$, the following conditions are equivalent:

- (1) $w \in \widehat{W}$ is dominant.
- (2) $N(w)$ is a subset of $\bigcup_{k \geq 1} (k\delta - \Delta^+)$.
- (3) there's no hyperplane $H_{\alpha,0}$ separating $\mathcal{C}_0$ from $w^{-1}(\mathcal{C}_0)$.
- (4) $w^{-1}(\mathcal{C}_0)$ lies in the fundamental chamber of V .

The first two properties of $\widehat{W}$ come from the linear action of W on $\widehat{W}$ and the last two properties of $\widehat{W}$ come from the affine transformation of $\widehat{W}$ on V .

Now define a map ϕ from $\widehat{W}_{dom}$ to the set of ideals in Δ^+ by

$$w \mapsto \{\mu \in \Delta^+ \mid \delta - \mu \in N(w)\}.$$

(The fact that $\mathcal{I}$ is an ideal of Δ^+ follows from dominance property of w .)

In fact it's a surjective map. For combinatorial ideal $\mathcal{I}$, there is a special dominant element w that corresponds to $\mathcal{I}$. These elements were introduced in [7].

Definition 2.3. *An element $w \in \widehat{W}$ is called minimal if*

- (1) $w(\alpha) \geq 0$ for all $\alpha \in \Pi$.
- (2) If $\alpha \in \widehat{\Pi}$ and $w^{-1}(\alpha) = k\delta + \mu$, then $k \geq -1$.

We denote the set of minimal elements by $\widehat{W}_{min}$. From part (1) of definition, it's obvious that $\widehat{W}_{min} \subset \widehat{W}_{dom}$. Moreover, we have the following result on the relations between the minimal elements and the ideals in Δ^+ .

Proposition 2.4. [7, Prop2.12] *There is a bijection between $\widehat{W}_{min}$ and the set of ideals in Δ^+ . The bijection is constructed as follows:*

- (a) *For each $w \in \widehat{W}_{min}$, its corresponding ideal is $\phi(w)$.*
- (b) *For each ideal $\mathcal{I}$ in Δ^+ , the corresponding w is determined by the set of affine roots*

$$N(w) = \cup_{k \geq 1} (k\delta - \mathcal{I}^k) \subset \widehat{\Delta}^+.$$

For each $w \in \widehat{W}_{min}$, we denote by $\mathcal{I}_w$ the ideal that is determined by the element w . Conversely, the minimal element that corresponds to $\mathcal{I}$ is denoted by $w_{min}(\mathcal{I})$, or simply by $w_{\mathcal{I}}$. Because of the bijection between the set of ad-nilpotent ideals and the set combinatorial ideals, we have similar notations I_w and w_I for the ad-nilpotent ideal I . We call the ad-nilpotent ideal I_w the *first layer ideal* of w .

It is obvious that for any positive root α in an ideal $\mathcal{I}$, then $w_{\mathcal{I}}^{-1}(-\alpha + \delta) > 0$.

Sommers gave a description of the set of generators of the ideal $\mathcal{I}$ in [27, Cor6.3] and Panyushev showed independently in [22, Thm2.2].

Proposition 2.5. *If $w \in \widehat{W}_{min}$, then any positive root $\alpha \in \mathcal{I}$ is a generator of the ideal $\mathcal{I}_w$ if and only if $w(\alpha - \delta)$ is an affine simple root in $\widehat{\Pi}$.*

The normalizer of an ad-nilpotent ideal I can also be described in terms of the minimal affine Weyl group element w_I . Since the Borel subalgebra $\mathfrak{b}$ normalize I , the normalizer of I must be a parabolic subalgebra containing $\mathfrak{b}$. Therefore the normalizer is determined by the simple root Levi subalgebras that it contains.

Suppose $\mathfrak{l}(\alpha)$ is the Levi subalgebra in $\mathfrak{g}$ corresponding to a simple root $\alpha \in \Pi$. Namely $\mathfrak{l}(\alpha) = \mathfrak{h} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}$. Panyushev showed in [23, Thm2.8] that:

Proposition 2.6. *If I is an ad-nilpotent ideal of $\mathfrak{b}$, then $\mathfrak{l}(\alpha) \subset N_{\mathfrak{g}}(I)$ if and only if $w_I(\alpha) \in \hat{\Pi}$, where $N_{\mathfrak{g}}(I)$ denotes the normalizer of I in $\mathfrak{g}$.*

3. THE MOMENT MAPS

Let $\mathcal{B}$ be the set of all Borel subalgebras in $\mathfrak{g}$, $T^*\mathcal{B}$ the cotangent bundle over $\mathcal{B}$ and $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$. Following [5, Lem1.4.9], if one identifies the Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via the Killing form of $\mathfrak{g}$, the moment map

$$m : T^*(\mathcal{B}) \longrightarrow \mathcal{N}$$

is equivalent to

$$m : G \times_B \mathfrak{n} \longrightarrow \mathcal{N}.$$

Therefore, for each ad-nilpotent ideal $\mathcal{I} \subset \mathfrak{n}$, $G \times_B \mathcal{I}$ may be considered as a G -invariant subbundle of cotangent bundle $T^*\mathcal{B}$. The image of $G \times_B \mathcal{I}$ under the moment map is the closure of a nilpotent orbit in $\mathcal{N}$.

Let $N(\mathfrak{g})$ be the set of nilpotent orbits of $\mathfrak{g}$. The moment map m induces a map

$$p : \mathfrak{Ad} \rightarrow N(\mathfrak{g})$$

by sending $I \in \mathfrak{Ad}$ to a nilpotent orbit $\mathcal{O}$, where the image of $G \times_B I$ under the moment map is the closure of $\mathcal{O}$. The nilpotent orbit $\mathcal{O}$ is called the associated orbit of the ideal I .

It is easy to see that the map is surjective. Indeed, if $e \in \mathcal{O}$, by Jacobson Morozov theorem, there exists an $\mathfrak{sl}_2$ -triple $\{e, h, f\}$ with $h \in \mathfrak{h}$. We conjugate the triple with elements of G such that $\alpha(h) \geq 0$ for positive root α . Then $I_h := \bigoplus_{i \geq 2} \mathfrak{g}_i$, where $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [h, X] = iX\}$ is an ad-nilpotent ideal and $\mathcal{O}_{I_h} = \mathcal{O}$.

Concerning this map p , there is several natural questions to ask:

When two different ideals give rise to the same nilpotent orbit?

Is it possible to describe combinatorially the equivalence relation on minimal elements of the affine Weyl group corresponding to $p(I_{w_1}) = p(I_{w_2})$?

In the special case that the two ideals differ by a single positive root, then the following result from [10] that partially answers the first question.

Proposition 3.1. *Let I be an ad-nilpotent ideal of $\mathfrak{n}$. Suppose that I is stable under the adjoint action of $\mathfrak{l}(\alpha)$ for some simple root $\alpha \in \Pi$ and β is a generator of I such that $s_{\alpha}(\beta) > \beta$. Let J be the ad-nilpotent ideal such that $I = J \oplus \mathfrak{g}_{\beta}$. Then I and J has the same image under p .*

If two ideals I and J satisfies the condition of Proposition 3.1, we denote by $I \sim_L J$ or $J \sim_L I$.

Definition 3.2. *Two ad-nilpotent ideals I and J are called left equivalent if either $I = J$ or there exists a sequence of ideals $I_1 = I, I_2, \dots, I_k = J$, such that $I_i \sim_L I_{i+1}$ for $i = 1, \dots, k-1$.*

It's easy to see that the relation defined above is actually an equivalence relation.

Proposition 3.1 is a sufficient condition for two ad-nilpotent ideals have a same associated orbits. Since the moment map sends the left-equivalent classes of ideals to the same orbit, one may ask whether it is true that two ad-nilpotent ideals have the same associated orbits only if these two ideals are left equivalent. For some cases, we can give an affirmative answer.

Example 3.3. *When $\mathfrak{g}$ is of type G_2 . Suppose that α_1 is the short simple root, and α_6 the long root. The other four roots are $\alpha_2 = \alpha_1 + \alpha_6$, $\alpha_3 = 2\alpha_1 + \alpha_6$, $\alpha_4 = 3\alpha_1 + \alpha_6$, $\alpha_5 = 3\alpha_1 + 2\alpha_6$. There are eight ideals and five equivalence classes:*

1. *The zero ideal, which corresponds to the zero orbit;*
2. *The ideal $\mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_5}$, which corresponds to the 8-dimensional orbit;*
3. *The maximal ideal, which corresponds to the principal orbit;*
4. *$\mathfrak{g}_{\alpha_5} \sim_L \mathfrak{g}_{\alpha_5} \oplus \mathfrak{g}_{\alpha_4}$, which corresponds to the minimal orbit;*
5. *$\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_5} \sim_L \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_5} \oplus \mathfrak{g}_{\alpha_6} \sim_L \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_5}$, which corresponds to the subregular orbit.*

In this example above, the left equivalence relation on the ideals completely determines the image of the moment map. For class cases other than type A, Proposition 3.1 is not necessary to determine the fiber of the moment map. The simplest counter example is in type B_3 .

Example 3.4. *Let $\mathfrak{g}$ be $\mathfrak{so}(7)$. Suppose that α_1, α_2 are the long simple roots of $\mathfrak{g}$ and α_3 is the short simple root. Let I be the ad-nilpotent ideal with generators $\alpha_2 + \alpha_3, \alpha_1 + \alpha_2$ and J be the ad-nilpotent ideal with generators by $\alpha_2 + 2\alpha_3$ and $\alpha_1 + \alpha_2$. Then $I = J \oplus \mathfrak{g}_{\alpha_2 + \alpha_3}$, and I, J have the same associated orbit. But no Levi subalgebra of $\mathfrak{g}$ normalizes I except $\mathfrak{h}$. I and J don't satisfy the condition of Proposition 3.1.*

In the type A case, we haven't found such a counter example yet. We would prove some results in the following section which may shed some light on whether Proposition 3.1 completely determines the fiber of the moment map.

Before directly approaching this problem, let us make a short digression and turn to the geometric description of the ideals via sign types. It turns out sign types are very useful for us to understand affine Weyl groups and are also connected with ad-nilpotent ideals.

4. SIGN TYPES

Recall that for any positive root $\alpha \in \Delta^+$, $H_{\alpha,k}$ is the hyperplane defined by $H_{\alpha,k} = \{v \in V \mid (v, \alpha) = k\}$. We denote three regions that

are separated by two hyperplanes $H_{\alpha,0}$ and $H_{\alpha,1}$ by:

$$\begin{aligned} H_{\alpha,+} &= \{v \in V \mid (v, \alpha) > 1\}, \\ H_{\alpha,0} &= \{v \in V \mid 0 < (v, \alpha) < 1\}, \\ H_{\alpha,-} &= \{v \in V \mid (v, \alpha) < 0\}. \end{aligned}$$

The non-empty connected simplex of $V - \bigcup_{\alpha \in \Delta^+, k=0,1} H_{\alpha,k}$ is called a *sign type* of V . The definition of sign type was first introduced by Shi in [24] to study the left cells and two-sided cells of affine Weyl group of type A_{n-1} and later was generalized by him to other classical groups in [25]. Notice that in Shi's original definition, he used the coroot system of $\mathfrak{g}$ while here we use the root system.

We denote the set of all sign types by $\mathcal{S}$. For any sign type $s \in \mathcal{S}$, it lies in one of the three regions $H_{\alpha,\epsilon}$, where $\epsilon \in \{0, +, -\}$. Therefore each sign type has the form $s = \bigcap_{\alpha \in \Delta^+, \epsilon_\alpha \in \{+, 0, -\}} H_{\alpha, \epsilon_\alpha}$, with $\alpha \mapsto \epsilon_\alpha$ a map from Δ^+ to the index set $\{0, +, -\}$. This map completely determines the sign type.

Since the walls $H_{\alpha,0}$ for $\alpha \in \Delta^+$ are used to define sign types, each sign type either lies in the fundamental chamber or has no intersection with the fundamental chamber. We call the special sign types that lie inside the fundamental chamber *$\mathcal{C}$ dominant sign types*. The set of dominant sign types in $\mathcal{S}$ is denoted by $\mathcal{S}_{dom}$. It's clear from the definition that the sign type s is dominant if and only if it lies in the region $H_{\alpha,\epsilon}$, for each $\alpha \in \Delta^+$ and $\epsilon \in \{+, 0\}$.

Proposition 4.1. [26] *The map*

$$I \mapsto R_I := \{x \in \mathcal{C} \mid (x, \alpha) > 1 \text{ if } \mathfrak{g}_\alpha \subset I \text{ and } 0 < (x, \alpha) < 1 \text{ if } \mathfrak{g}_\alpha \not\subset I\}$$

gives a bijection from the set $\mathfrak{Ad}$ of ad-nilpotent ideals in $\mathfrak{b}$ to $\mathcal{S}_{dom}$.

Also sign types are closely related to affine Weyl groups.

For each $w \in \widehat{W}$, w^{-1} maps the fundamental alcove $\mathcal{C}_0$ to another alcove. Thus $w^{-1}(\mathcal{C}_0)$ is contained in a unique sign type s . We obtain a map $\widehat{W} \rightarrow \mathcal{S}$ by sending w to s . It turns out that dominant elements in $\widehat{W}$ are mapped to dominant sign types. Namely, when restricted to $\widehat{W}_{dom}$, we have $\widehat{W}_{dom} \rightarrow \mathcal{S}_{dom}$.

Recall that for each dominant element $w \in \widehat{W}$, one can associate an ad-nilpotent ideal $I = \{\alpha \in \Delta^+ \mid w(\delta - \alpha) \leq 0\}$. This is equivalent to say that the hyperplane $H_{\alpha,1}$ separates $w^{-1}(\mathcal{C}_0)$ from $\mathcal{C}_0$ if and only if the positive root α lies in the ideal I . In other words, $w^{-1}(\mathcal{C}_0)$ lies in the region $H_{\alpha,+}$ for any positive root $\alpha \in I$ and lies in the region $H_{\alpha,0}$ for any positive root α that is not in the ideal I .

Comparing this with the map $\widehat{W} \rightarrow \mathcal{S}$ defined in the preceding paragraph and the bijection between dominant sign types and ad-nilpotent

ideals in Proposition 4.1, we have a commutative diagram.

$$(1.4.1) \quad \begin{array}{ccc} \widehat{W}_{dom} & \longrightarrow & \mathfrak{Ad} \\ \downarrow & \swarrow \simeq & \\ \mathcal{S}_{dom} & & \end{array}$$

Under this commutative diagram, it's possible for us to explore the relation between the equivalent classes of affine Weyl group and the equivalent classes of ad-nilpotent ideals given by Definition 3.2 in further detail.

5. THE CASE OF TYPE $\tilde{A}_{n-1}$

Now we focus on the affine Weyl group of type $\tilde{A}_{n-1}$. In this section, $\widehat{W}$ denotes an affine Weyl group of type $\tilde{A}_{n-1}$.

For affine Weyl groups of type $\tilde{A}_{n-1}$, there's a combinatorial description of the affine Weyl group elements. We regard $\widehat{W}$ as a set of permutations on $\mathbb{Z}$ as follows

$$\widehat{W} = \left\{ \sigma : \mathbb{Z} \rightarrow \mathbb{Z} \mid \begin{array}{l} \sigma(i+n) = \sigma(i) \text{ for } i \in \mathbb{Z}, \\ \sum_{t=1}^n (\sigma(t) - t) = 0 \pmod{n} \end{array} \right\}.$$

The simple reflection $s_0, s_1, \dots, s_{n-1}$ can be taken to the permutations as follows. For $0 \leq j \leq n-1$, j is mapped to:

$$s_i(j) = \begin{cases} j+1, & \text{if } j \equiv i \pmod{n}; \\ j-1, & \text{if } j \equiv i+1 \pmod{n}; \\ j, & \text{otherwise.} \end{cases}$$

We denote by $<$ the Bruhat order on $\widehat{W}$ and denote the set of partitions of n by $P(n)$.

Let $[n]$ be the index set $\{1, 2, \dots, n\}$. For any $w \in \widehat{W}$, the partial order $\succ_w$ of $[n]$ is defined by $i \succ_w j$ if and only if either $i < j$ and $w(i) > w(j)$ or $i > j$ and $w(i) > w(j) + n$. A chain of $[n]$ is a sequence of integers $\{i_1, \dots, i_h\}$ with the order $i_1 \succ_w i_2 \cdots \succ_w i_h$. A k -family is a subset of $[n]$ such that there's no chain of length $k+1$ in this subset.

Let d_k be the maximal cardinality of a k -family of $[n]$. Then $d_1 \leq d_2 \leq \dots \leq d_n = n$. Let $\lambda_1 = d_1$, $\lambda_j = d_j - d_{j-1}$ for $1 \leq j \leq n$. By a theorem of Green [9], $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $\sum_{i=1}^n \lambda_i = d_n = n$, which gives us a partition of n . Let $\mu = \{\mu_1 \geq \mu_2 \geq \dots \geq \mu_n\}$ be the conjugate partition of $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$, meaning that for any $h = 1, \dots, n$, μ_h is equal to the number of parts in λ of size h . This gives us a map $\phi : \widehat{W} \rightarrow P(n)$ with $\phi(w) = \mu$. This map is related to two-sided cells of $\widehat{W}$. (The reason that we choose the conjugate partition μ instead of λ is to get a commutative diagram 1.5.1 below).

Set

$$L(w) = \{s_j \in S \mid s_j w < w\} = \{\alpha_j \in \widehat{\Pi} \mid w^{-1}(\alpha_j) < 0\},$$

We follow the usual definition of left cells, right cells and two-sided cells in [15].

For any two elements $w, u \in \widehat{W}$, we denote by $w \sim_L u$ (resp, $w \sim_R u$; or $w \sim_{LR} u$) if w and u lie in the same left cell (resp, right cell; or two sided cell) of $\widehat{W}$.

It was proved by Lusztig in [21] and Shi in [24] that the two-sided cells of $\widehat{W}$ is parameterized by $P(n)$ and these two-sided cells coincide with fibers of ϕ . In particular, to describe the fiber of the map, Shi [24] constructed a map $\Phi : \mathcal{S} \rightarrow P(n)$ such that the following is a commutative diagram:

$$\begin{array}{ccc} \widehat{W} & \longrightarrow & \mathcal{S} \\ \phi \downarrow & \searrow \Phi & \\ P(n) & & \end{array}$$

For the map $\widehat{W} \rightarrow \mathcal{S}$, Shi proved in [24, Chap 18] that if two affine Weyl group elements are mapped to the same sign type, then these two elements lie in the same left cell. Because of the commutative diagram 1.4.1, two dominant elements of $\widehat{W}$ with the same associated ideal also lie in the same left cell.

When $\mathfrak{g} = \mathfrak{sl}(n)$, the set of nilpotent orbits of $\mathfrak{g}$ is parameterized by $P(n)$ (see [6]) and the moment map of $\mathfrak{g}$ gives us $p : \mathfrak{Ad} \rightarrow P(n)$. Since the set of dominant sign types $\mathcal{S}_{dom}$ is in bijection with the set of ad-nilpotent ideals, there might be some relation between the map $\Phi|_{\mathcal{S}_{dom}}$ and the map p .

Indeed the work of Shi (see [24]) and the work of Lawton (see [20]) imply that there exists a commutative diagram in the following form:

$$(1.5.1) \quad \begin{array}{ccccc} \widehat{W}_{dom} & \longrightarrow & \mathcal{S}_{dom} & \xrightarrow{\Phi} & P(n) \\ \downarrow id & & \downarrow & & \downarrow id \\ \widehat{W}_{dom} & \longrightarrow & \mathfrak{Ad} & \xrightarrow{p} & P(n) \end{array}$$

Recall that Proposition 3.1 gives us a criterion to determine when two ad-nilpotent ideals have the same associated orbits. One might hope that this criterion has some relations with the cell structure of affine Weyl groups and sign types. Here is the main result about these relations.

Let s, t be two simple reflections of $\widehat{W}$ such that st has order 3.

Definition 5.1.

$$D_L(s, t) = \{w \in \widehat{W} \mid L(w) \cap \{s, t\} \text{ contains only one element}\},$$

If w is in the set $D_L(s, t)$, then $\{tw, sw\} \cap D_L(s, t)$ contains only one element, which is denoted by $*w$. Then the map that sends w to $*w$ defines an involution on $D_L(s, t)$ and is called a left star operation. This involution depends on the two simple reflections s and t . For example, if w lies both in the sets $D_L(s_1, t_1)$ and $D_L(s_2, t_2)$ for simple reflections s_1, s_2, t_1, t_2 , then the involution $*$ may give two different $*w$.

The left star operation generates another equivalence relation on $\widehat{W}$. The element w is P_L equivalent to w' if and only if there is a succession of affine Weyl group elements $w_1 = w, w_2, \dots, w_n = w'$ such that each w_i lies in some set $D_L(s_i, t_i)$ and $w_{i+1} = *w_i$ under the left star operation in $D_L(s_i, t_i)$. We denote this equivalence relation by $\sim_{P_L}$.

It is proved in [15] that w and $*w$ lie in the same left cell, therefore $w \sim_{P_L} w'$ implies that $w \sim_L w'$.

Now we can come to the proof of the main theorems.

Theorem 5.2. *Suppose I_1 and I_2 are two ideals and $I_1 \sim_L I_2$, then there exists two dominant elements w_1 and w_2 , such that $I_{w_1} = I_1$, $I_{w_2} = I_2$ and $w_1 \sim_L w_2$.*

Proof. Since any elements of $\widehat{W}_{dom}$ with the same associated ideal lie in the same left cell, by the definition of left equivalence relation for the ideals, it suffices to prove the statement when I_1 and I_2 differs by only one positive root.

Suppose that $I_2 \subset I_1$, α is a simple root that normalize I_1 and β is a generator of I_1 , s.t. $I_1 = I_2 \oplus \mathfrak{g}_\beta$ and $s_\alpha(\beta) > \beta$. Let w_{I_1} be the minimal element in $\widehat{W}$ that corresponds to I_1 (By Proposition 4.1). Recall from Proposition 2.6, w_{I_1} maps α to a simple root in $\widehat{\Pi}$, which we denote by α_i . On the other hand, since β is a generator of I_1 , by Proposition 2.5, $w_{I_1}(\beta - \delta) = \alpha_j$ for some simple root α_j . The condition that $s_\alpha(\beta) > \beta$ implies that $(\alpha, \beta) < 0$. We denote s_{α_i} by s_i and s_{α_j} by s_j . Then s_i, s_j are two simple reflections of $\widehat{W}$. Since the inner product on $\widehat{V}$ is invariant under $\widehat{W}$, $(\alpha_i, \alpha_j) < 0$ and $s_i s_j$ has order 3. The fact that $w_{I_1}(\alpha) = \alpha_i$ implies that $s_i w_{I_1} > w_{I_1}$ and $w_{I_1}(\beta - \delta) = \alpha_j$ implies that $s_j w_{I_1} < w_{I_1}$. Therefore, w_{I_1} is an element that lies in the set $D_L(s_i, s_j)$. In this case, we can determine $*w_{I_1}$ explicitly. Indeed it's clear that $s_j(s_j w_{I_1}) = w_{I_1} > s_j w_{I_1}$ and $s_i(s_j w_{I_1}) < s_j w_{I_1}$ because:

$$(s_j w_{I_1})^{-1}(\alpha_i) = w^{-1} s_j(\alpha_i) = w_{I_1}^{-1}(\alpha_i + \alpha_j) = \alpha + \beta - \delta < 0$$

Hence $*w_{I_1} = s_j w_{I_1}$. In addition, $l(s_j w_{I_1}) < l(w_{I_1})$ and $N(w_{I_1}) = N(s_j w_{I_1}) \cup (\beta - \delta)$. Hence $N(s_j w_{I_1})$ is a subset of $\cup_{k \geq 1} (k\delta - \Delta^+)$,

which implies that $s_j w_{I_1}$ is still dominant. We have $I_{s_j w_{I_1}} \oplus \mathfrak{g}_\beta = I_1$, i.e. $I_{s_j w_{I_1}} = I_2$. $\square$

If two dominant elements in $\widehat{W}$ are P_L -equivalent, we can prove a converse version of Theorem 5.2. First we need to prove a lemma that states a special property for the P_L equivalence classes of elements in $\widehat{W}_{dom}$.

Lemma 5.3. *Suppose that w is dominant and α is a simple root in $\widehat{\Pi}$. If $w^{-1}(\alpha) < 0$, then $w^{-1}(\alpha) = -k\delta + \beta$, where k is positive and β is an element of I_w . In particular, if $k = 1$, then β is a generator of I_w .*

Proof. The element w is dominant implies that $w^{-1}(\alpha) = -k\delta + \beta$, where $k \geq 1$ and $\beta \in \Delta^+$. Since $w(\delta - \beta) = -\alpha - (k-1)\delta < 0$, β lies in the first layer ideal I_w . If $k = 1$, for any $\gamma \in \Delta^+$, $w(\delta - (\beta - \gamma)) = w(\delta - \beta) + w(\gamma) = -\alpha + w(\gamma)$. Since $w(\gamma)$ is a positive affine root and α is simple, $w(\delta - (\beta - \gamma)) > 0$ and $\beta - \gamma$ does not belong to the ideal I_w , which means that β is indeed a generator of I_w . $\square$

Lemma 5.4. *If $w \in \widehat{W}_{dom}$ and w lies in the set $D_L(s_i, s_j)$ for some simple reflections s_i and s_j with $s_i s_j s_i = s_j s_i s_j$, then *w is also a dominant element.*

Proof. Any element $u \in \widehat{W}$ is dominant if and only if $N(u)$ is contained in $\cup_{k \geq 1} (k\delta - \Delta^+)$. There are two possibilities for *w . The first case is $l({}^*w) = l(w) - 1$. Then $N({}^*w)$ is a subset of $N(w)$ and the fact that w is dominant implies *w is also dominant.

The other case is $l({}^*w) = l(w) + 1$. In this case, by the symmetry of i and j , suppose ${}^*w = s_j w$. Then $s_j w > w$, $s_i w < w$ and $s_i s_j w > s_j w$. Let a_i, a_j be the two simple roots of $\widehat{\Pi}$ that correspond to s_i and s_j . By the dominance property of w and the fact that $s_i w < w$, we have $w^{-1}(a_i) = t\delta - \beta$, where t is a positive integer and $\beta \in \Delta^+$. Similarly $s_j w > w$ means that $w^{-1}(\alpha_j) > 0$, namely $w^{-1}(\alpha_j) = k\delta + \gamma$, where $k \geq 0$ and $\gamma \in \Delta^+$ or $k \geq 1$ and $\gamma \in \Delta^-$.

Case (a): $k = 0$ and $\gamma \in \Delta^+$. From the facts that $s_i s_j w > s_j w$ and α_j has the same length as α_i , $w^{-1}(\alpha_i + \alpha_j) = (s_j w)^{-1}(\alpha_i) > 0$. On the other hand, $w^{-1}(\alpha_i + \alpha_j) = \beta + \gamma - k\delta < 0$, which is a contradiction.

Case (b): $k > 0$ and $\gamma \in \Delta^+$. Since w is dominant, $w(\gamma)$ is a positive root in $\widehat{\Delta}^+$. Also w fixes $k\delta$. Therefore, $\alpha_j = w(k\delta + \gamma)$ can not be a simple root in $\widehat{\Delta}^+$. Contradiction.

The only possible form for $w^{-1}(\alpha_j)$ is $k\delta + \gamma$, where $k \geq 1$ and $\gamma \in \Delta^-$. Notice that $N({}^*w)$ is the union of $N(w)$ and $w^{-1}(\alpha_j)$, then *w is again dominant. $\square$

Lemma 5.5. [23, Thm3.5] *Suppose $w \in \widehat{W}$ is dominant, and let I_w be the corresponding ideal of w . If there exist a simple root $\alpha \in \Pi$, such that $w(\alpha) \in \widehat{\Pi}$, then $\mathfrak{l}(\alpha)$ normalize I_w .*

Theorem 5.6. *Suppose that w_1 is dominant and $w_1 \sim_{P_L} w_2$. Then w_2 is also dominant and the corresponding ideals of w_1 and w_2 are left equivalent, i.e. $I_{w_1} \sim_L I_{w_2}$.*

Proof. The dominance of w_2 is shown in Lemma 5.4. By definition of P_L -equivalence relation, the problem can be reduced to the case when w_1 lies in the set $D_L(s_i, s_j)$ for some $i, j \in \{0, 1, \dots, n\}$ and $w_2 = {}^*w_1$. In addition, by the symmetry of w_1 and w_2 , as well as the symmetry of i and j , we may assume that $l({}^*w_1) < l(w_1)$ and ${}^*w_1 = s_i w_1$. Then $s_i w_1 < w_1$, $s_j w_1 > w_1$ and $s_j s_i w_1 < s_i w_1$. This means that $w_1^{-1}(\alpha_i) < 0$, $w_1^{-1}\alpha_j > 0$, and $w_1^{-1}(\alpha_i + \alpha_j) = (s_i w_1)^{-1}(\alpha_j) < 0$. By lemma 5.3, $w_1^{-1}(\alpha_i) = \beta - k\delta$, where β is an element of I_{w_1} , and $k \geq 1$.

If $k > 1$, then $k\delta - \beta$ belongs to $N(w_1)$ and any $t\delta - \beta$ for $1 \leq t < k$ also belongs to $N(w_1)$. That means $k\delta - \beta$ is the only element that lies in $N(w_1)$, but not in $N(s_i w_1)$. Since the definition of the corresponding ideals of $s_i w_1$ and w_1 involves only the first layer of positive roots in $s_i w_1$ and w_1 , $s_i w_1$ and w_1 have the same first layer ideal, i.e. $I_{w_1} = I_{s_i w_1}$.

Suppose $k = 1$. In this case, from Lemma 5.3, β is a generator of I_{w_1} . In addition, we have known that $w_1^{-1}(\alpha_j) > 0$ and $w_1^{-1}(\alpha_i + \alpha_j) < 0$. There are four possible cases for $w_1^{-1}(\alpha_j) > 0$.

Case(a): $w_1^{-1}(\alpha_j) = k\delta + \beta_1$, where $k \geq 1$ and $\beta_1 \in \Delta^+$.

In this case, $w_1^{-1}(\alpha_i + \alpha_j) = (k-1)\delta + \beta + \beta_1 > 0$. Contradiction.

Case(b): $w_1^{-1}(\alpha_j) = k\delta - \beta_1$, where $k \geq 2$ and $\beta_1 \in \Delta^+$.

Similar to case (a), $w_1^{-1}(\alpha_i + \alpha_j) = (k-1)\delta + \beta - \beta_1 > 0$. Contradiction.

Case(c): $w_1^{-1}(\alpha_j) = \delta - \beta_1$, where $\beta_1 \in \Delta^+$.

In this case, $w_1^{-1}(\alpha_i + \alpha_j) = \beta - \beta_1 \in \Delta^-$. Suppose that $\beta - \beta_1 = -\gamma$, where γ lies in Δ^+ . Then $\beta_1 = \beta + \gamma$, which implies that β_1 also lies in ideal I_{w_1} since β is a generator of I_{w_1} . This contradicts the fact that $w_1(\delta - \beta_1) = \alpha_j$, which is positive.

Case(d): $w_1^{-1}(\alpha_j) = \beta_1$, where $\beta_1 \in \Delta^+$.

This is the only possible form for $w_1^{-1}(\alpha_j)$. In this case, β_1 is a simple root of Δ^+ . Indeed, if β_1 is not simple, say $\beta_1 = \gamma_1 + \gamma_2$, where γ_1 and γ_2 are two positive roots in Δ^+ , then $w_1(\beta_1) = w_1(\gamma_1) + w_2(\gamma_2)$, where $w_1(\gamma_1)$ and $w_2(\gamma_2)$ are two positive roots in $\widehat{\Delta}^+$. This contradicts the fact that $\alpha_j = w_1(\beta_1)$ is a simple affine root. From lemma 5.5, $\mathfrak{l}(\beta_1)$ normalize the ideal I_{w_1} . Also β is a generator of I_{w_1} and $(\beta, \beta_1) = (\alpha_i, \alpha_j) < 0$. Since $N(w_1)$ is equal to $N(s_i w_1) \cup (\delta - \beta)$, β appears in the first layer ideal of w_1 , but not the first layer ideal of $s_i w_2$. By the definition of left equivalence of ideals, I_w is left equivalent to $I_{s_i w}$. $\square$

We have shown above that if two dominant elements in $\widehat{W}$ are P_L -equivalent, then their corresponding ideals are left equivalent (see Definition 3.2) and also two left equivalent ideals would give us two P_L equivalent elements. It was proved in [24, Chap 19] that two dominant

sign types lie in the same two-sided cell if and only if they lie in the left cell. Therefore, if we can prove that dominant elements in the same left cell give rise to left equivalent ideals, then Proposition 3.1 is the necessary condition to determine the two-sided cell structure of sign types. However, Shi defined two operations for elements in the same left cell. The first one is related to P_L -equivalence relation, which we showed above that it does give two left equivalent ideals. The other one is called raising layer operation, which we can not get its relation with the left equivalence relation of ad-nilpotent ideals.

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