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Abstract

The category of highest weight representations is of special interest within the full set of representations of a real semisimple Lie group. This memoir describes the structure of the generalized Verma modules as well as the Kazhdan-Lusztig data for the simple modules in this category for the classical groups. In particular, explicit formulas for composition factors of generalized Verma modules and Kazhdan-Lusztig polynomials are given.

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§1. Introduction and summary of results

Let $\underline{g}$ be a semisimple Lie algebra over $\mathbb{C}$. To study any category $\mathcal{C}(\underline{g})$ of $\underline{g}$-modules the standard procedure is to first study those with regular generalized infinitesimal character $\mathcal{C}_{\text{reg}}(\underline{g})$ and then to pass to those with singular generalized infinitesimal character $\mathcal{C}_{\text{sing}}(\underline{g})$. In this article we invert this usual procedure. The structure of modules in $\mathcal{C}_{\text{reg}}(\underline{g})$ is determined from the corresponding information for $\mathcal{C}_{\text{sing}}(\underline{g})$. In turn the latter category is analyzed using an induction argument on the rank of $\underline{g}$.

The article is organized into two parts. The first part (sections two through seven) analyzes $\mathcal{C}_{\text{sing}}(\underline{g})$ in terms of $\mathcal{C}_{\text{reg}}(\underline{g}')$ where rank$(\underline{g}')$ is less than rank$(\underline{g})$. The second part (sections eight through sixteen) is an application of part one to the categories of highest weight modules for classical Hermitian symmetric pairs. The results of part one are quite general while those of part two are very explicit. In particular, in part two we obtain explicit formulas for the the composition factors of generalized Verma modules as well as explicit formulas for the Kazhdan-Lusztig polynomials.

We now describe the results in some detail, beginning with the necessary notation for highest weight modules. For undefined terms and greater detail see sections two and three. Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and $\underline{b} = \underline{h} \oplus \underline{n}$ a Borel subalgebra. Let $\underline{p} = \underline{m} \oplus \underline{u}$ be a parabolic subalgebra of $\underline{g}$ with nilradical $\underline{u}$, $\underline{u} \subseteq \underline{n}$ and with $\underline{h} \subseteq \underline{m}$. Denote by Δ (repectively $\Delta(\underline{m})$, $\Delta(\underline{u})$) the $\underline{h}$-roots of $\underline{g}$ (resp. $\underline{m}$, $\underline{u}$). Let Δ^+ be the positive roots which are the roots of $\underline{n}$ and put $\Delta^+(\underline{m}) = \Delta^+ \cap \Delta(\underline{m})$. Let ρ (resp. $\rho(\underline{m})$) equal half the sum of the elements

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of Δ^+ (resp. $\Delta^+(\underline{m})$). Let $\mathcal{W}$ and $\mathcal{W}_{\underline{m}}$ be the Weyl groups of $(\underline{g}, \underline{h})$ and $(\underline{m}, \underline{h})$ respectively. Put $\mathcal{W}^{\underline{m}} = \{w \in \mathcal{W} | w\Delta^+ \supset \Delta^+(\underline{m})\}$. Then $\mathcal{W} = \mathcal{W}_{\underline{m}}\mathcal{W}^{\underline{m}}$. Let $\mathcal{P}_{\underline{m}}$ equal the set of $\lambda \in \underline{h}^*$ which are Δ-integral and $\Delta^+(\underline{m})$-dominant and regular.

Let $\mathcal{O}(\underline{g}, \underline{p})$ be the category of $\underline{g}$-modules which are: (i) finitely generated over $U(\underline{g})$, (ii) $U(\underline{p})$-locally finite and (iii) completely reducible over $U(\underline{m})$. For $\lambda \in \underline{h}^*$, let $\mathcal{O}(\underline{g}, \underline{p}, \lambda)$ be the full subcategory of $\mathcal{O}(\underline{g}, \underline{p})$ of modules with generalized infinitesimal character parameterized by the $\mathcal{W}$-orbit of λ. Let $M(\underline{g}, \lambda)$ be the Verma module with highest weight $\lambda - \rho$ and $L(\underline{g}, \lambda)$ its irreducible quotient. For $\lambda \in \mathcal{P}_{\underline{m}}$, let $F(\underline{m}, \lambda - \rho)$ be the finite dimensional irreducible $\underline{m}$-module with highest weight $\lambda - \rho$. Let $N(\underline{g}, \underline{p}, \lambda)$ denote the generalized Verma module with highest weight $\lambda - \rho$. Let $P(\underline{g}, \underline{p}, \lambda)$ denote the projective cover of $L(\underline{g}, \lambda)$ in $\mathcal{O}(\underline{g}, \underline{p})$. When possible we delete the indices $\underline{g}$ and $\underline{p}$ and write $M(\lambda)$, $L(\lambda)$, $N(\lambda)$ and $P(\lambda)$ respectively for these modules. Define the truncated category $\mathcal{O}_t(\underline{g}, \underline{p}, \lambda)$ to be the full subcategory of $\mathcal{O}(\underline{g}, \underline{p})$ whose simple modules are the $L(\underline{g}, \xi)$ with ξ less than or equal to λ in the Bruhat order.

In section four we give several results relating $\mathcal{C}_{\text{sing}}(\underline{g})$ and $\mathcal{C}_{\text{reg}}(\underline{g})$. Let ν and μ be Δ^+-dominant elements in $\underline{h}^*$ with μ integral and let $\phi = \phi_\nu^{\nu+\mu}$ be the Zuckerman translation functor (cf. section two). Then ϕ is a functor from $\mathcal{O}(\underline{g}, \underline{p}, \nu)$ to $\mathcal{O}(\underline{g}, \underline{p}, \nu + \mu)$ which is an equivalence when ν and $\nu + \mu$ have the same stabilizer in $\mathcal{W}$. However, in all cases we prove: for $w \in \mathcal{W}$ and $w\nu \in \mathcal{P}_{\underline{m}}$, $\phi P(w\nu)$ is an indecomposable projective module. Also $P(w\nu)$ is self-dual if and only if $\phi P(w\nu)$ is. These and similar results are given in (4.1), (4.5) and (4.6).

Section five reviews the necessary material on Zuckerman derived functors. Proofs are given for some known facts which however are not available in the literature.

Section six includes the first result relating categories of modules for Lie algebras of different rank. Let $\underline{q} = \underline{l} \oplus \underline{u}_{\underline{q}}$ be a maximal parabolic subalgebra of $\underline{g}$ with $\underline{h} \subset \underline{l}$ and $\underline{q}$ not containing $\underline{p}$. Suppose that $\lambda \in \underline{h}^*$ is Δ-integral and $\Delta(\underline{l})$-regular. Put $\mathcal{O}_{\underline{g}} = \mathcal{O}(\underline{g}, \underline{p}, \lambda)$ and $\mathcal{O}_{\underline{l}} = \mathcal{O}_t(\underline{l}, \underline{l} \cap \underline{p}, \lambda)$. The main result of section six, (6.6), asserts that if the set of highest weights plus ρ in $\mathcal{O}_{\underline{g}}$ equals the set of highest weights plus $\rho(\underline{l})$ in $\mathcal{O}_{\underline{l}}$ then these two categories are equivalent.

The main result in section seven is similar, however, with different hypotheses. Put $\underline{r} = \underline{l} \cap \underline{m}$. Let w_1 be the longest element in $\mathcal{W}_{\underline{l}}$ and w_0^{-1} the longest element in $\mathcal{W}^{\underline{r}}$ where $\mathcal{W}_{\underline{m}} = \mathcal{W}_{\underline{r}} \mathcal{W}^{\underline{r}}$. Put $\nu = -w_1 \rho$ and $\mu = w_0 \nu$. Now set $\mathcal{O}_{\underline{l}} = \mathcal{O}(\underline{l}, \underline{l} \cap \underline{p}, \nu)$ and $\mathcal{O}_{\underline{g}} = \mathcal{O}_t(\underline{g}, \underline{p}, \mu)$; and note, the truncation operation is on the other side in this case. Then these two categories are equivalent (7.1). The proofs of both (6.6) and (7.1) rely on the use of Zuckerman derived functors. However, their application is somewhat different in the two cases. Section six uses derived functors in the top dimension while section seven uses them in the middle dimension. These two results are the main results in part one of the article. At the level of the Grothendieck group the equivalence established in section seven has been proved by Boe and Collingwood ([6] Scholium 2.6) for the Hermitian symmetric setting. Their argument uses the Zuckerman derived functors in the middle dimension to identify the Hasse diagram of $\mathcal{O}_{\underline{l}}$ with a subdiagram of the Hasse diagram for $\mathcal{O}(\underline{g}, \underline{p}, \rho)$.

Let (G,K) be a classical Hermitian symmetric pair. The highest weight representations for the simply connected covering group of G are infinitesimally equivalent to the modules in the category $\mathcal{O}(\underline{g}, \underline{p})$ where $\underline{g}$ is the complexified Lie algebra of G and $\underline{p}$ is a maximal parabolic subalgebra of $\underline{g}$. In part two of this article, we apply the results from part one to analyze these categories. The first results describe the composition factors $L(x\rho)$ in the generalized Verma module $N(y\rho)$, $x, y \in \mathcal{W}^{\underline{m}}$. We give explicit formulas both for x in terms of

y and y in terms of x. Also we determine the socle of $N(y\rho)$. These results and others are stated with greater precision in Theorems 8.4 and 8.5. All answers are expressed in terms of sets of orthogonal roots. Their proofs occupy sections nine through thirteen. In related work the multilicity free nature of the decompositions above was also obtained by Boe and Collingwood [6]. For other special cases of multiplicity one, we cite [9], [12] and [21].

In sections fourteen and fifteen we turn to questions regarding Kazhdan-Lusztig polynomials and $\mathrm{Ext}^*(N(\nu), L(\xi))$. Here a standard lemma from algebraic topology (the mapping cone lemma) combines with the equivalences of categories from part one to give simple recursion formulas for Ext. This result is given as Theorem 14.4. From it we immediately derive explicit formulas for $\mathrm{Ext}^i(N(y\rho), L(w\rho))$, $y, w \in \mathcal{W}^{\underline{m}}$, $i \in \mathbb{N}$. These results appear as Theorems 14.9 and Corollary 14.14.

Following Kazhdan and Lusztig [23] and Vogan [34], we define what we call KLV polynomials $Q_{y,w}(q)$, $y, w \in \mathcal{W}^{\underline{m}}$, in section fifteen. The correspondence with the standard Kazhdan-Lusztig polynomials is given as Lemma 15.3. The recursion formulas for Ext described above then lead to corresponding recursion formulas for the KLV polynomials. These recursion formulas uniquely determine the polynomials. This result (Theorem 15.4) generalizes to the Hermitian symmetric cases $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SO}^*(2n)$ the recursion relations found by Lascoux and Schützenberger [26] for the case $\mathrm{SU}(p,q)$. Finally in Theorem 15.5 we give the solutions to these recursion formulas. This gives explicit formulas for the KLV polynomials in terms of a combinatorial notion which we call chains (cf. Definition 14.8).

In [16], canonical decompositions are given for self-dual modules which are $U(\underline{u}^-)$-free. The results depend on two hypotheses: simplicity of the socle of $N(\nu)$ and self-duality of $D(\nu)$ (cf. (2.7)). We have verified both of these properties for $\mathcal{O}(\underline{g}, \underline{p})$; and so, all the results in [16] apply here as well. These results are summarized in section sixteen. Possibly the most interesting is Proposition 16.5 which asserts uniqueness of signature of nondegenerate Hermitian forms on the modules $D(\nu)$. In addition we give a characterization of self-duality for $U(\underline{u}^-)$-free modules X in $\mathcal{O}$. We prove X admits a nondegenerate symmetric invariant form if and only if the contravariant dual module $X^\vee$ is a free $U(\underline{u}^-)$-module (cf. (16.3)).

Section eight includes a more detailed summary of the main results of part two as well as additional introductory remarks for that part of the article. The reader primarily interested in the applications to Hermitian symmetric pairs should review the table of contents and then turn to section eight.

This article is the culmination of a project begun some years ago. It has undergone a number of changes in perspective and has benefited from the comments and suggestions of a number of our friends. The early develepmont of the project was strongly influenced by the work of R. Irving [20]. In particular, the simple structure of the category $\mathcal{O}$ for SU(1,n) was pointed out to us by him. This example was the starting point for all which followed. Our colleagues G. Carlsson and L. Small pointed out to us that one of the techniques we employed in section fourteen is well known in topology and is called the algebraic mapping cone lemma. The connection between "Poincaré" polynomials for $\mathcal{O}(\underline{g},\underline{p})$ and those for $\mathcal{O}(\underline{g},\underline{b})$ was pointed out to us by T. Springer. Finally we thank Neola Crimmins for her excellent preparation of part one of this manuscript. Her work is much appreciated.

§2. Notation

In this section we set down our notation and conventions for Lie algebras, root systems and highest weight modules. Also, we describe the BGG reciprocity theorem for the category $\mathcal{O}$ as well as generalizations of it to what we call truncated categories.

Throughout this article $\underline{g}$ will denote a semisimple Lie algebra over $\mathbb{C}$, the field of complex numbers. Let $\underline{h}$ be a Cartan subalgebra (CSA) of $\underline{g}$ and $\underline{b}$ a Borel subalgebra with $\underline{h} \subset \underline{b}$. Let $\underline{p}$ be a parabolic subalgebra of $\underline{g}$ containing $\underline{b}$ and let $\underline{n}$ (resp. $\underline{u}$) be the nilradical of $\underline{b}$ (resp. $\underline{p}$). So we have Levi decompositions $\underline{b} = \underline{h} \oplus \underline{n}$ and $\underline{p} = \underline{m} \oplus \underline{u}$. Let $\underline{n}^-$ (resp $\underline{u}^-$) denote the opposite nilradicals with $\underline{g} = \underline{n}^- \oplus \underline{b}$ (resp $\underline{u}^- \oplus \underline{p}$). Let Δ denote the set of roots of $(\underline{g}, \underline{h})$ and let Δ^+ be the positive root system determined by $\underline{b}$. For any $ad(\underline{h})$-stable subspace $\underline{a}$ in $\underline{g}$, let $\Delta(\underline{a})$ denote the set of roots whose root spaces lie in $\underline{a}$ and put $\Delta^+(\underline{a}) = \Delta^+ \cap \Delta(\underline{a})$. Also put $\rho(\underline{a})$ equal to one half the sum of the positive roots in $\Delta^+(\underline{a})$. For convenience we write ρ instead of $\rho(\underline{g})$. If $\underline{l}$ is any Lie algebra, let $U(\underline{l})$ denote the universal enveloping algebra of $\underline{l}$.

Let $\mathcal{W}$ denote the Weyl group of $(\underline{g}, \underline{h})$ and for $\alpha \in \Delta$, let s_α be the reflection corresponding to α. Let $\mathcal{W}_{\underline{a}}$ denote the subgroup of $\mathcal{W}$ generated by the reflections s_α, $\alpha \in \Delta(\underline{a})$. For $\underline{m}$ as above we identify the Weyl group of $(\underline{m}, \underline{h})$ with $\mathcal{W}_{\underline{m}}$. Also define $\mathcal{W}^{\underline{m}} = \{w \in \mathcal{W} | w\Delta^+ \supset \Delta^+(\underline{m})\}$ and let ${}^{\underline{m}}\mathcal{W} = \{x | x^{-1} \in \mathcal{W}^{\underline{m}}\}$. So $\mathcal{W} = {}^{\underline{m}}\mathcal{W} \cdot \mathcal{W}_{\underline{m}} = \mathcal{W}_{\underline{m}} \cdot \mathcal{W}^{\underline{m}}$. Let $\ell(\cdot)$ denote the length function on $\mathcal{W}$.

Next we describe the categories of highest weight modules used in this article. We begin with the Bernstein, Gelfand and Gelfand (BGG) category $\mathcal{O}$. Let $\mathcal{O}(\underline{g},\underline{p})$ denote the category of all $\underline{g}$-modules X which satisfy the conditions: (i) X is finitely generated over $U(\underline{g})$, (ii) X is $U(\underline{p})$-locally finite and (iii) X is completely reducible over $U(\underline{m})$. If $\underline{h}^*$ is the algebraic dual of $\underline{h}$ and $\lambda \in \underline{h}^*$, let $M(\underline{g},\lambda)$ denote the Verma $\underline{g}$-module with highest weight $\lambda-\rho$. Let $L(\underline{g},\lambda)$ be the unique irreducible quotient of $M(\underline{g},\lambda)$. Write χ_λ for the infinitesimal character of $M(\underline{g},\lambda)$ and let $\mathcal{O}(\underline{g},\underline{p},\lambda)$ be the full subcategory of $\mathcal{O}(\underline{g},\underline{p})$ of modules with generalized infinitesimal character χ_λ. For $Y \in \mathcal{O}(\underline{g},\underline{p})$, let Y_{χ_λ} be the maximal submodule of Y with generalized infinitesimal character χ_λ. Then Y is the direct sum of its submodules Y_{χ_λ}.

Let $\mathcal{P}_{\underline{m}}$ denote the set of elements in $\underline{h}^*$ which are Δ-integral and $\Delta^+(\underline{m})$-dominant and regular. For $\lambda \in \mathcal{P}_{\underline{m}}$, let $F(\underline{m},\lambda-\rho)$ be the finite dimensional irreducible $\underline{m}$-module with highest weight $\lambda-\rho$. By letting $\underline{u}$ act by zero, $F(\underline{m},\lambda-\rho)$ becomes a $\underline{p}$-module. For $\lambda \in \mathcal{P}_{\underline{m}}$, define the generalized Verma module $N(\underline{g},\underline{p},\lambda)$ by:

$$N(\underline{g},\underline{p},\lambda) = U(\underline{g}) \bigotimes_{U(\underline{p})} F(\underline{m},\lambda-\rho). \tag{2.1}$$

In $\mathcal{O}(\underline{g},\underline{p})$ each irreducible module $L(\underline{g},\lambda)$ admits a unique indecomposable projective cover which we denote by $P(\underline{g},\underline{p},\lambda)$ (cf. [31]). When there is little chance of confusion, the modules $M(\underline{g},\lambda)$, $L(\underline{g},\lambda)$, $N(\underline{g},\underline{p},\lambda)$ and $P(\underline{g},\underline{p},\lambda)$ will be denoted by $M(\lambda)$, $L(\lambda)$, $N(\lambda)$ and $P(\lambda)$ respectively.

A module A in $\mathcal{O}(\underline{g},\underline{p})$ is said to have a Verma flag if A admits a flag of submodules $A = A_1 \supset A_2 \supset \cdots \supset A_{n+1} = 0$ with $A_i/A_{i+1} \cong N(\lambda_i)$ for some $\lambda_i \in \mathcal{P}_{\underline{m}}$, $1 \le i \le n$. In this case we say A has a Verma flag of length n or that A has index n. For $\lambda \in \mathcal{P}_{\underline{m}}$, let $[A : N(\lambda)]$ denote the number of indices i, $1 \le i \le n$, with $\lambda = \lambda_i$. This number is independent of the Verma flag of A and we call it the multiplicity of $N(\lambda)$ in a Verma flag of A. We say A has a multiplicity free Verma flag if the multiplicities $[A : N(\lambda)]$ are all one or zero. For any module B in $\mathcal{O}(\underline{g}, \underline{p})$, let $(B : L(\lambda))$ denote the multiplicity of $L(\lambda)$ in any Jordan-Hölder series for B.

The role of projective covers in $\mathcal{O}(\underline{g}, \underline{p})$ is made more fundamental by the following reciprocity theorem.

Proposition 2.2 [31] *Let λ and ν be in $\mathcal{P}_{\underline{m}}$. Then:*

(a) $P(\lambda)$ admits a Verma flag.

(b) $[P(\lambda) : N(\nu)] = (N(\nu) : L(\lambda))$.

This result was first proved by Bernstein, Gelfand and Gelfand for the case $\underline{p} = \underline{b}$ (cf. [4]), and so we refer to identity (b) as the BGG reciprocity theorem. The generalization to $\mathcal{O}(\underline{g}, \underline{p})$ is due to Rocha-Caridi.

The space $\underline{h}^*$ has two standard partial orderings defined as follows. For $\lambda, \nu \in \underline{h}^*$ we write $\lambda \leq \nu$ if $\nu - \lambda$ is a nonnegative integral sum of elements in Δ^+. We write $\lambda \prec \nu$ if $\mathrm{Hom}_{\underline{g}}(M(\lambda), M(\nu)) \neq 0$. This second partial ordering is the Bruhat ordering (cf. [13]) and is the one most frequently used in this article. We now use it to define what we call truncated categories.

Let $\mu \in \mathcal{P}_{\underline{m}}$ and define $\mathcal{O}_t(\underline{g}, \underline{p}, \mu)$ to be the full subcategory of $\mathcal{O}(\underline{g}, \underline{p}, \mu)$ of modules X with the property:

(2.3) $$\text{If } \nu \in \mathcal{P}_{\underline{m}} \text{ and } (X : L(\nu)) \neq 0 \text{ then } \nu \prec \mu.$$

When $\underline{p}$ is fixed and little confusion can arise we write $\mathcal{O}$, $\mathcal{O}(\mu)$ and $\mathcal{O}_t(\mu)$ in place of $\mathcal{O}(\underline{g}, \underline{p})$, $\mathcal{O}(\underline{g}, \underline{p}, \mu)$ and $\mathcal{O}_t(\underline{g}, \underline{p}, \mu)$ respectively. We call $\mathcal{O}_t(\mu)$ the truncated category $\mathcal{O}$ with highest weight $\mu - \rho$. Suppose $A \in \mathcal{O}(\mu)$. Then there is a unique minimal submodule B of A with $A/B \in \mathcal{O}_t(\mu)$. Define $T_\mu(A) = A/B$.

Lemma 2.4 *Let μ and ν be in $\mathcal{P}_{\underline{m}}$.*

(a) If $\nu \prec \mu$ then $N(\nu)$ is in $\mathcal{O}_t(\mu)$.

(b) T_μ is a covariant right exact functor from $\mathcal{O}$ to $\mathcal{O}_t(\mu)$.

Proof: If $(N(\nu) : L(\xi)) \neq 0$ then $\xi \prec \nu$. This gives (a). Let $A, A' \in \mathcal{O}$ and $f \in \mathrm{Hom}(A, A')$ and choose B and B' with $T_\mu A = A/B$ and $T_\mu A' = A'/B'$. To prove T_μ is a covariant functor it is sufficient to check $f(B) \subseteq B'$. We prove this inclusion by induction on the length n of a composition series for B. If $n = 0$ then $B = 0$ and the inclusion holds. Now assume $n > 0$ and let $\xi - \rho$ be a maximal weight of B with $\xi \not\prec \mu$. Choose a nonzero map $\psi \in \mathrm{Hom}(N(\xi), B)$. Then $f\psi N(\xi)$ is either zero or contains a cyclic vector of weight $\xi - \rho$, and thus $f\psi N(\xi) \subseteq B'$. Now put $C = \psi N(\xi)$ and $C' = fC$. This shows that f induces a map $\bar{f} : A/C \to A'/C'$. One checks easily that $T_\mu(A/C) \cong A/B$ and $T_\mu(A'/C') \cong A'/B'$. So by the induction hypothesis, $\bar{f}(B/C) \subseteq B'/C'$. This implies $f(B) \subseteq B'$, and thus f induces a map $T_\mu f : A/B \to A'/B'$.

To prove right exactness consider an exact sequence $A^0 \xrightarrow{g} A \xrightarrow{f} A' \to 0$. Now f induces the map $T_\mu f$, so $T_\mu f$ is surjective. We now claim: image$(T_\mu g) =$ kernel$(T_\mu f)$. Define B^0 with $A^0/B^0 = T_\mu A^0$ and let B and B' be as above. Since f is surjective, $A'/f(B) = f(A)/f(B) \in \mathcal{O}_t(\mu)$. Thus $f(B) \supseteq B'$, and so $f(B) = B'$. This implies the claim and completes the proof of (2.4).

The categories $\mathcal{O}_t(\mu)$ are structurally quite similar to $\mathcal{O}(\mu)$. In particular, simple modules admit unique indecomposable projective covers and we have a BGG type reciprocity theorem.

Proposition 2.5 *Let ν and μ be in $\mathcal{P}_{\underline{m}}$.*

(i) If $\nu \not\prec \mu$ then $T_\mu P(\nu) = 0$.

(ii) If $\nu \prec \mu$ then $T_\mu P(\nu)$ is the unique indecomposable projective cover of $L(\nu)$ in $\mathcal{O}_t(\mu)$.

(iii) If $\nu \prec \mu$ then $T_\mu P(\nu)$ has a Verma flag and we have the reciprocity formula:

$$[T_\mu P(\nu) : N(\xi)] = \begin{cases} (N(\xi) : L(\nu)) & \text{if } \xi \in \mathcal{P}_{\underline{m}} \text{ and } \xi \prec \mu \\ 0 & \text{otherwise.} \end{cases}$$

Proof: $L(\nu)$ is the unique simple quotient of $P(\nu)$ and so also of $T_\mu P(\nu)$. This proves (i) and also proves $T_\mu P(\nu)$ is indecomposable. To complete (ii) we need only show $T_\mu P(\nu)$ is projective. Let $\phi : B \to C$ be a surjection with $B, C \in \mathcal{O}_t(\mu)$ and $\gamma : T_\mu P(\nu) \to C$. Let ε denote the natural projection of $P(\nu) \to T_\mu P(\mu)$. Then $P(\nu)$ is projective, and so there exists a map δ : $P(\nu) \to B$ with $\phi \circ \delta = \gamma \circ \varepsilon$. Now apply the functor T_μ. We obtain a map $T_\mu \delta : T_\mu P(\nu) \to B$ with $\phi \circ T_\mu \delta = \gamma$. So $T_\mu P(\nu)$ is projective, indecomposable and has quotient $L(\nu)$. This proves (ii).

To prove (iii) we chose a Verma flag for $P(\nu) = A_1 \supset \cdots \supset A_{n+1} = 0$ with $A_i/A_{i+1} \cong N(\lambda_i)$. From Lemma 2.1 in [16] we may assume $\lambda_i \prec \mu$ if and only if $1 \leq i < a$, for some integer a. Then $T_\mu P(\nu) = A_1/A_a$ and the reciprocity formula follows from (2.2).

Let A be any $\underline{g}$-module and let Soc$A$ denote the sum of all the simple submodules of A. Soc$A$ is called the socle of A. Let ν be in $\mathcal{P}_{\underline{m}}$. If Soc$N(\nu)$ is a simple module then we reserve the notation $\nu^\#$ to indicate :

$$\text{Soc}N(\nu) \cong L(\nu^\#). \tag{2.6}$$

In this setting we define modules $D(\nu) = D(\underline{g}, \underline{p}, \nu)$ by:

$$D(\nu) = T_\nu P(\nu^\#). \tag{2.7}$$

For an equivalent definition of these modules see [16].

Translation functors are needed at several points in this article. Let $\mu, \lambda \in \underline{h}^*$ with λ Δ-integral. Let F be the finite dimensional irreducible $\underline{g}$-module with extreme weight λ and let F^* denote the contragradient $\underline{g}$-module. Let $\phi = \phi_\mu^{\mu+\lambda}$ and $\psi = \psi_{\mu+\lambda}^\mu$ denote the functors on $\mathcal{O}(\underline{g}, \underline{b})$ as well as on $\mathcal{O}(\underline{g}, \underline{p})$ given by:

$$\phi A = (F \otimes A_{\chi_\mu})_{\chi_{\mu+\lambda}}, \qquad \psi A = (F^* \otimes A_{\chi_{\mu+\lambda}})_{\chi_\mu}. \tag{2.8}$$

These functors are adjoints of each other and are called the Zuckerman translation functors when λ and the real part of μ lie in the same Weyl chamber (cf. [38]). In this case, if μ and $\mu + \lambda$ have the same stabilizer in $\mathcal{W}$ we say ϕ and ψ are equisingular. For the now standard properties of ϕ and ψ the reader should consult [22] or [36].

The Verma modules in category $\mathcal{O}(\underline{g}, \underline{p})$ all admit contravariant forms (in the sense of Jantzen, cf. [22]). We shall call a module X in $\mathcal{O}(\underline{g}, \underline{p})$ self-dual if it admits a symmetric nondegenerate contravariant form.

We will need the following lemma on the Bruhat order.

Lemma 2.9 *Let λ and μ be elements of $\mathcal{P}_{\underline{m}}$ with $\lambda \prec \mu$. Then there exist $\alpha_i \in \Delta^+$, $1 \leq i \leq t$, with $\mu_i = s_{\alpha_i} \cdots s_{\alpha_1} \mu \in \mathcal{P}_{\underline{m}}$ and $\lambda = \mu_t \prec \cdots \prec \mu_1 \prec \mu$.*

This follows from Theorem 2.22(b) in [22].

At times it will be convenient to write for $\lambda \in \underline{h}^*$ and $\alpha \in \Delta$, $\lambda_\alpha = 2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$. Also, for any module X in $\mathcal{O}(\underline{g}, \underline{p})$, let chX denote the formal character of X as in [22].

§3. Preliminary results

In this section we collect several technical results on Weyl groups and categories of modules. The reader is encouraged to postpone reading this section and instead refer to the results here when needed.

For any Δ^+-dominant $\xi \in \underline{h}^*$, let $S(\xi) = \{\alpha \in \Delta^+ | \langle \xi, \alpha \rangle = 0\}$ and let $\mathcal{W}_\xi$ be the subgroup of $\mathcal{W}$ generated by the s_α, $\alpha \in S(\xi)$. The element ξ determines a parabolic subalgebra and $\mathcal{W}_\xi$ is the Weyl group of the Levi component. However, this fact will not play a role here, and so we won't pursue the connection with the parabolic subalgebra in this section.

Fix $\lambda, \xi \in \underline{h}^*$ both Δ^+-dominant integral. Let $F = F(\underline{g}, \lambda)$ and let Λ denote the set of weights of F.

Lemma 3.1 *Set $\Theta = (\xi + \Lambda) \cap \mathcal{W}(\xi + \lambda)$. Then*

$$\Theta = \xi + \mathcal{W}_\xi \cdot \lambda = \mathcal{W}_\xi(\xi + \lambda) \ .$$

Proof: Fix $\mu \in \Lambda$ and suppose $\xi + \mu = w(\xi + \lambda)$ for some $w \in \mathcal{W}$. It is suficient to prove: $w \in \mathcal{W}_\xi$. Since $\mu \in \Lambda$, $||\mu|| \leq ||\lambda||$ and $\mu \leq \lambda$. Also $||\xi + \lambda|| = ||\xi + \mu||$, and so we obtain: $0 \leq ||\lambda||^2 - ||\mu||^2 \leq 2\langle \mu - \lambda, \xi \rangle \leq 0$. So we have equality: $||\lambda|| = ||\mu||$ and $\langle \mu, \xi \rangle = \langle \lambda, \xi \rangle$. But then $\mu \in \mathcal{W}\lambda$ and if $\mathcal{C}$ is the positive Weyl chamber, $\mu \in \mathcal{W}_\xi \cdot \mathcal{C}$. In turn this gives $\mu = s\lambda$ for some $s \in \mathcal{W}_\xi$. Then $w(\xi + \lambda) = \xi + \mu = s(\xi + \lambda)$. This gives $s^{-1}w \in \mathcal{W}_{\xi+\lambda} \subset \mathcal{W}_\xi$ and finally $w \in \mathcal{W}_\xi$.

Set ${}^\xi\mathcal{W} = \{w \in \mathcal{W} | wS(\xi) \subset \Delta^+\}$.

Lemma 3.2 *Let $v, w \in {}^{\xi}\mathcal{W}$ and $\mu, \nu \in \mathcal{W}_\xi \cdot \lambda$.*

(i) $\xi + \mu \prec \xi + \nu$ if and only if $w(\xi + \mu) \prec w(\xi + \nu)$.

(ii) $v(\xi + \mu) \prec w(\xi + \lambda)$ if and only if $v\xi \prec w\xi$.

Proof: Suppose $\xi + \mu \prec \xi + \nu$. Then choose $\alpha_i \in \Delta^+$, $1 \leq i \leq k$, such that with $s_i = s_{\alpha_i}$, $\xi + \mu = s_k \cdots s_1(\xi + \nu)$ and $\langle a_{i+1}, s_i \cdots s_1(\xi + \nu)\rangle \geq 0$, $k > i \geq 0$. We claim $\alpha_i \in S(\xi)$ for all i.

Choose $r \in \mathcal{W}_\xi$ with $\mu = r\lambda$. Since $\Delta^+ \setminus S(\xi)$ is $\mathcal{W}_\xi$-stable, if $\alpha_k \notin S(\xi)$ then $\langle s_k(\xi + \mu), \alpha_k\rangle = \langle \xi + \lambda, -r^{-1}\alpha_k\rangle < 0$. However, $\langle s_k(\xi + \mu), a_k\rangle = \langle s_{k-1} \cdots s_1(\xi + \nu), \alpha_k\rangle \geq 0$. This contradiction proves $\alpha_k \in S(\xi)$. Assume $\alpha_i \in S(\xi)$ for $k \geq i > a \geq 1$ and put $\overline{\mu} = s_{a+1} \cdots s_1\mu$. Then $\xi + \overline{\mu} = s_a \cdots s_1(\xi + \nu)$ and $\xi + \overline{\mu} \prec \xi + \nu$. Now argiung as above with μ replaced by $\overline{\mu}$ we obtain $\alpha_a \in S(\xi)$. This completes the induction and proves the claim.

For $1 \leq i \leq k$ set $\beta_i = w\alpha_i$. Then $\beta_i \in \Delta^+$ and $s_{\beta_i}w = ws_i$. Also, let $\overline{s}_i = s_{\beta_i}$. Then $\overline{s}_k \cdots \overline{s}_1 w(\xi + \nu) = w(\xi + \mu)$ and $\langle \overline{s}_j \cdots \overline{s}_1 w(\xi + \nu), \beta_{j+1}\rangle = \langle s_j \cdots s_1(\xi + \nu), \alpha_{j+1}\rangle \geq 0$, $1 \leq j < k$. This proves $w(\xi + \mu) \prec w(\xi + \nu)$.

The converse of (i) is proved in a similar fashion. Assume $w(\xi + \mu) \prec w(\xi+\nu)$. Choose $\beta_i \in \Delta^+$, $1 \leq i \leq k$, with $s_i = s_{\beta_i}$, $w(\xi+\mu) = s_k \cdots s_1 w(\xi+\nu)$ and $\langle s_j \cdots s_1 w(\xi + \nu), \beta_{j+1}\rangle > 0$, $0 \leq j < k$. Now arguing as above one can show $w^{-1}\beta_i \in S(\xi)$. Put $\alpha_i = w^{-1}\beta_i$ and $\overline{s}_i = s_{\alpha_i}$. Then as above, $\xi + \mu = \overline{s}_k \cdots \overline{s}_1(\xi + \nu)$ and $\xi + \mu \prec \xi + \nu$.

We now prove (ii). Suppose $v\xi \prec w\xi$. By (i), it suffices to prove $v(\xi + \lambda) \prec w(\xi + \lambda)$. Choose $\alpha_i \in \Delta^+$, $1 \leq i \leq k$, with $w\xi = s_k \cdots s_1 v\xi$ and $\langle s_j \cdots s_1 v\xi, \alpha_{j+1}\rangle < 0$, $0 \leq j < k$. Then $-v^{-1}s_1 \cdots s_j\alpha_{j+1} \in \Delta^+$. Since $\xi + \lambda$ is dominant this gives: $\langle s_j \cdots s_1 v(\xi + \lambda), \alpha_{j+1}\rangle < 0$, and thus if we put $\lambda' = w^{-1}s_k \cdots s_1 v\lambda$ then $v(\xi + \lambda) \prec s_k \cdots s_1 v(\xi + \lambda) = w(\xi + \lambda')$. By (3.1), $w^{-1}s_k \cdots s_1 v \in \mathcal{W}_\xi$ and $\lambda' \in \mathcal{W}_\xi\lambda$. Then by (i), $v(\xi + \lambda) \prec w(\xi + \lambda') \prec w(\xi + \lambda)$ giving the first half of (ii).

For the converse suppose $v(\xi+\mu) \prec w(\xi+\lambda)$. Choose $\alpha_i \in \Delta^+$, $1 \le i \le k$, with $s_i = s_{\alpha_i}$, $s_k \cdots s_1 v(\xi+\mu) = w(\xi+\lambda)$ and $\langle s_j \cdots s_1 v(\xi+\mu), a_{j+1}\rangle < 0$, $0 \le j < k$. Since $\xi+\mu$ and ξ lie in the same (closed) Weyl chamber, we have $\langle s_j \cdots s_1 v\xi, \alpha_{j+1}\rangle \le 0$. Also if $y = s_k \cdots s_1 v$ then $w^{-1}y(\xi+\mu) = \xi+\lambda$, and so by (3.1), $w^{-1}y \in \mathcal{W}_\xi$. So $y\xi = w\xi$, and thus $v\xi \prec w\xi$. This proves (3.2).

In section two we defined both truncation functors and translation functors. They satisfy the following identities.

Lemma 3.3 *Fix $w \in {}^\xi\mathcal{W}$, λ, ξ Δ^+-dominant integral and put $\mu = w\xi$, $\nu = w\lambda$. Let ϕ and ψ denote the translation functors $\phi_\xi^{\xi+\lambda}$ and $\psi_{\xi+\lambda}^\xi$ respectively. For any $B \in \mathcal{O}(\underline{g}, \underline{p}, \xi)$ and $E \in \mathcal{O}(\underline{g}, \underline{p}, \xi+\lambda)$ we have:*

(i) $T_{\mu+\nu}\phi B \cong \phi T_\mu B$.

(ii) $T_\mu \psi E \cong \psi T_{\mu+\nu} E$.

Proof: We argue by induction on the length of a composition series for B. Put $C = T_\mu B$ and consider the short exact sequence: $0 \to A \to B \to C \to 0$. First assume $A \ne 0$ and $C \ne 0$. Then by the induction hypothesis, $T_{\mu+\nu}\phi A \cong \phi T_\mu A$. Since $T_\mu A = 0$, ϕ is exact and $T_{\mu+\nu}$ is right exact we obtain: $0 \to T_{\mu+\nu}\phi B \to T_{\mu+\nu}\phi C \to 0$ is exact. So $T_{\mu+\nu}\phi B \cong T_{\mu+\nu}\phi C \cong \phi T_\mu C \cong \phi T_\mu B$ by the induction hypothesis applied to C.

Now suppose $A = 0$. Let $\gamma - \rho$ be a maximal weight of B. Then $\gamma \prec \mu$ and there is a nonzero $f \in \mathrm{Hom}(N(\gamma), B)$. Set $D = fN(\gamma)$ and choose r in ${}^\xi\mathcal{W}$ with $\gamma = r\xi$. By (3.1), $\phi N(\gamma)$ has a Verma flag with Verma factors of the form $N(\gamma+\lambda')$ with $\lambda' \in r\mathcal{W}_\xi \cdot \lambda$. Then (3.2(ii)) implies $\gamma + \lambda' \prec \mu + \nu$ for each λ', and so $\phi N(\gamma) \in \mathcal{O}_t(\mu+\nu)$. This implies $\phi D \in \mathcal{O}_t(\mu+\nu)$. The induction hypothesis gives: $T_{\mu+\nu}\phi(B/D) \cong \phi T_\mu(B/D) = \phi(B/D)$ and $\phi(B/D) \in \mathcal{O}_t(\mu+\nu)$. Combining these, $\phi B \in \mathcal{O}_t(\mu+\nu)$, and hence $T_{\mu+\nu}\phi B = \phi B \cong \phi T_\mu B$.

Finally, suppose $T_\mu B = 0$. Let $\gamma - \rho$ be a maximal weight of B with $\gamma \not\prec \mu$. Let f be a nonzero element of $\mathrm{Hom}(N(\gamma), B)$ and set $D = fN(\gamma)$. By

(3.2(ii)), $T_{\mu+\nu}\phi N(\gamma) = 0$, and so $T_{\mu+\nu}\phi D = 0$. The induction hypothesis gives $T_{\mu+\nu}\phi(B/D) \cong \phi T_\mu(B/D) = 0$. By right exactness of $T_{\mu+\nu}$, $T_{\mu+\nu}\phi B = 0$. This completes the proof of (i).

The proof of (ii) is similar and we omit the details.

We end this section with two results on categories of modules.

Lemma 3.5 *Fix $\mu \in \mathcal{P}_{\underline{m}}$ and let $\mathcal{C} = \mathcal{O}_t(\underline{g}, \underline{p}, \mu)$. Let T and S be covariant exact functors from $\mathcal{C}$ to $\mathcal{C}$ and let f_X ($X \in \mathcal{C}$) be a natural transformation from T to S. If f_N is an isomorphism for each Verma module N then f_X is an equivalence.*

Proof: We must prove f_X is an isomorphism for all X. Choose λ Δ^+-dominant with $\mu \in \mathcal{W}\lambda$ and define $\mathcal{C}_i$ to be the full subcategory of $\mathcal{C}$ consisting of modules whose composition factors all lie in $\{L(s\lambda) | s \in \mathcal{W} \text{ and } \ell(s) \geq i\}$. Thus $\mathcal{C}_0 = \mathcal{C}$ and we proceed by downward induction on i. Assume $i \in \mathbb{N}$ and f_X is an isomorphism for all $X \in \mathcal{C}_{i+1}$. Assume $s\lambda \in \mathcal{P}_{\underline{m}}$ and $\ell(s) = i$. Then set $L = L(s\lambda)$ and $N = N(s\lambda)$ and define J to give the short exact sequence: $0 \to J \to N \to L \to 0$. Then $J \in \mathcal{C}_{i+1}$, and by hypothesis f_N is an isomorphism. Therefore the exactness of T and S and the five lemmas imply f_L is an isomorphism. This proves f_X is an isomorphism for all simple modules in $\mathcal{C}_i$. Now let X be any module in $\mathcal{C}_i$. Using the five lemmas again and inducting on the length of a composition series for X shows f_X is an isomorphism for all $X \in \mathcal{C}_i$. This completes the first induction argument and proves (3.5).

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories and F a covariant functor form $\mathcal{C}$ to $\mathcal{C}'$. We say F is full (resp. faithful) if F maps $\mathrm{Hom}(X, Y)$ surjectively (resp. injectively) to $\mathrm{Hom}(FX, FY)$.

Proposition 3.6 [30] *The functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence if and only if F is full and faithful and for all $X \in \mathcal{C}'$ there exists $Y \in \mathcal{C}$ with X and FY isomorphic.*

§4. Reduction of singularities

We retain the previous notation. The translation functors ϕ and ψ carry projective modules to projective modules. In this section we describe a setting where ϕ carries indecomposable projective modules to indecomposable projective modules. Let ξ and λ be Δ^+-dominant integral elements of $\underline{h}^*$ and put $\phi = \phi_\xi^{\xi+\lambda}$. We are especially interested in the non equisingular cases, i.e. when ξ and $\xi + \lambda$ have different stabilizers in $\mathcal{W}$.

Let $\Theta = \xi + \mathcal{W}_\xi \cdot \lambda$ and let w_0 be the element of maximal length in $\mathcal{W}_\xi$. Then $\xi + w_0\lambda$ is the minimal element in Θ.

Proposition 4.1 *Suppose $w \in {}^\xi\mathcal{W}$ with $w\xi \in \mathcal{P}_{\underline{m}}$. Then*

(i) $\phi N(w\xi)$ is indecomposable.

(ii) $\phi P(w\xi) \cong P(w(\xi + w_0\lambda))$.

Proof: First we check: $w\Theta \subseteq \mathcal{P}_{\underline{m}}$. Since $w\xi \in \mathcal{P}_{\underline{m}}$ and ξ is dominant, $w^{-1}\Delta^+(\underline{m}) \subseteq \Delta^+ \setminus S(\xi)$. But $\Delta^+ \setminus S(\xi)$ is $\mathcal{W}_\xi$-stable, and so $\mathcal{W}_\xi w^{-1}\Delta^+(\underline{m}) \subseteq \Delta^+ \setminus S(\xi) \subseteq \Delta^+ \setminus S(\xi+\lambda)$. This gives $w\Theta \subseteq \mathcal{P}_{\underline{m}}$. Let $N = N(w\xi)$. Then using (3.1), ϕN has a Verma flag $A_1 \supset A_2 \supset \cdots \supset A_{d+1} = 0$ with $A_i/A_{i+1} \cong N(\nu_i)$ and $\{\nu_1, \ldots, \nu_d\} = w\Theta$.

For each $\nu \in w\Theta$, $\psi N(\nu) \cong N$ and since ϕ and ψ are adjoint functors we obtain:

$$(4.2) \qquad \dim(\mathrm{Hom}(N(\nu), \phi N)) = \dim(\mathrm{Hom}(N, N)) = 1 \ .$$

By (3.2), $\nu \prec w(\xi + \lambda)$. We claim:

$$(4.3) \qquad \mathrm{Hom}(N(\nu), N(w(\xi + \lambda))) \neq 0 \ .$$

Suppose for some $\alpha \in \Delta^+(\underline{m})$, $\nu \prec s_\alpha w(\xi+\lambda)$. Choose $v \in {}^\xi\mathcal{W}$, $r \in \mathcal{W}_\xi$ with $s_\alpha w = vr$. Then for some $\lambda' \in \mathcal{W}_\xi \cdot \lambda$, $w(\xi+\lambda') = \nu \prec v(\xi + r\lambda) \prec v(\xi+\lambda)$, by (3.2(i)). Now by (3.2(ii)), $w\xi \prec v\xi = s_\alpha w\xi$. This contradicts $w\xi \in \mathcal{P}_{\underline{m}}$, and so $\nu \not\prec s_\alpha w(\xi+\lambda)$ for any $\alpha \in \Delta^+(\underline{m})$. But then the inclusion of Verma modules $M(\nu) \hookrightarrow M(w(\xi+\lambda))$ induces a nonzero map (called the standard map) of $N(\nu)$ into $N(w(\xi+\lambda))$. This proves the claim (4.3).

Now suppose ϕN is decomposable, say $\phi N = B \oplus C$. Since $w(\xi+\lambda)$ is maximal in $w\Theta$, we may assume $N_1 = N(w(\xi+\lambda))$ is a submodule of B. Let $\nu - \rho$ be a highest weight of C. Then $\dim(\mathrm{Hom}(N(\nu), \phi N)) \geq \dim(\mathrm{Hom}(N(\nu), N_1 \oplus C)) \geq 2$ by (4.3). This contradicts (4.2), and so ϕN is indecomposable proving (i).

Put $P = P(w\xi)$ and let $\gamma = w(\xi + w_0\lambda)$. Now ϕP is projective, and so we choose $\gamma_i \in \mathcal{W}(\xi+\lambda)$ with $\phi P = \bigoplus_{0 \leq i \leq k} P(\gamma_i)$. By (3.2(i)), γ is the minimal element of $w\Theta$. Then $N(\gamma)$ is a quotient of ϕN and also of ϕP. Thus we may assume $\gamma_0 = \gamma$. Recall the truncation functors from section two and put $\zeta = w\xi$ and $\nu = w(\xi+\lambda)$. Then (3.2) and (3.3) give:

$$\phi N = \phi T_\zeta P \cong T_\nu \phi P \cong \bigoplus_{0 \leq i \leq k} T_\nu P(\gamma_i)\ . \tag{4.4}$$

By (i), ϕN is indecomposable, and since $T_\nu P(\gamma_0) \neq 0$ we conclude that $T_\nu P(\gamma_i)$ is zero for all $i \geq 1$. So in particular, $\gamma_i \not\prec \nu$ for $i \geq 1$. Finally suppose $\gamma_i \in w_i\Theta$, $w_i \in {}^\xi\mathcal{W}$. Then by (3.2), $w_i\xi \not\prec w\xi$, and so $\mathrm{Hom}(\phi P, L(\gamma_i)) \cong \mathrm{Hom}(P, \psi L(\gamma_i)) \subseteq \mathrm{Hom}(P, L(w_i\xi)) = 0$ for $i \geq 1$. But then $P(\gamma_i)$ is not a summand of ϕP. This contradiction proves $k = 0$ and $\phi P = P(\gamma_0)$. This proves (ii).

Proposition 4.5 *Let $w \in {}^\xi\mathcal{W}$ with $w\xi \in \mathcal{P}_{\underline{m}}$ and put $\zeta = w\xi$ and $\gamma = w(\xi + w_0\lambda)$. Then*

(i) $P(\zeta)$ is self-dual if and only $P(\gamma)$ is self-dual.

(ii) $P(\zeta)$ has a multiplicity free Verma flag if and only if $P(\gamma)$ does.

Proof: Self-dual modules are preserved under the functors ϕ and ψ, and so by (4.1) if $P(\zeta)$ is self-dual then so is $P(\gamma)$. Conversely, if $P(\gamma)$ is self-dual then $\psi P(\gamma)$ is self-dual. However, if r is the index of the stabilizer of γ in $\mathcal{W}$ in the stabilizer of ζ then the formal character of $\psi P(\gamma)$ is r times the formal character of $P(\zeta)$. But then $\psi P(\gamma)$ is the direct sum of r copies of $P(\zeta)$. By Proposition 4.2 in [20], $P(\zeta)$ is self-dual. This proves (i).

If $P(\zeta)$ has a multiplicity free Verma flag then (3.1), (4.1) and (4.2) show $P(\gamma)$ has a multiplicity free Verma flag. Conversely if $P(\gamma)$ has a multiplicity free Verma flag then by the previous paragraph, $[\psi P(\gamma) : N(\beta)]$ equals zero or r for any β. Since $\psi P(\gamma)$ is the direct sum of r copies of $P(\zeta)$, $P(\zeta)$ has a multiplicity free Verma flag.

Recall from section two the notation $\xi^{\#}$ reserved for the case when the socle of $N(\zeta)$ is $L(\zeta^{\#})$. Also recall the modules $D(\zeta)$.

Proposition 4.6 *Suppose $w \in {}^{\xi}\mathcal{W}$ with $w\xi \in \mathcal{P}_{\underline{m}}$. Put $\zeta = w\xi$ and $\nu = w(\xi + \lambda)$.*

(i) If $Soc(N(\zeta))$ is simple and $D(\zeta)$ is self-dual then $Soc(N(\nu))$ is simple and $D(\nu)$ is self-dual. Moreover in this case, $\phi D(\zeta) \cong D(\nu)$.

(ii) If $Soc(N(\nu)) = L(\nu^{\#})$ with $\nu^{\#} = u(\xi + w_0\lambda)$ for some $u \in {}^{\xi}\mathcal{W}$ and if $D(\nu)$ is self-dual then $Soc(N(\zeta))$ is simple and $D(\zeta)$ is self-dual.

Proof: Suppose $Soc(N(\zeta)) = L(\zeta^{\#})$ and $D(\zeta)$ is self-dual. Then $D(\zeta) \cong T_{\zeta}P(\zeta^{\#})$. Choose $v \in {}^{\xi}\mathcal{W}$ with $\zeta^{\#} = v\xi$ and put $\overline{\nu} = v(\xi + w_0\lambda)$. By (4.1), $\phi P(\zeta^{\#}) = P(\overline{\nu})$. Using (3.3) we obtain: $\phi D(\zeta) \cong \phi T_{\zeta} P(\zeta^{\#}) \cong T_{\nu} P(\overline{\nu})$. Thus $\phi D(\zeta)$ has a unique irreducible quotient $L(\overline{\nu})$. By self-duality we have: $Soc(\phi D(\zeta)) \cong L(\overline{\nu})$. Finally, ν is a maximal weight of $\phi D(\zeta)$. Therefore $N(\nu)$ is a submodule of $\phi D(\zeta)$ and $Soc(N(\nu)) \cong Soc(\phi D(\zeta)) \cong L(\overline{\nu})$. In turn, $\overline{\nu} = \nu^{\#}$ and $D(\nu) \cong T_{\nu}(P(\overline{\nu})) \cong \phi D(\zeta)$. This proves (i).

Now suppose $Soc(N(\nu)) \cong L(\nu^{\#})$ with $\nu^{\#} = u(\xi + w_0\lambda)$ for some $u \in {}^{\xi}\mathcal{W}$ and $D(\nu)$ is self-dual. Set $\overline{\zeta} = u\xi$. As in the proof of (4.5), $\psi P(\nu^{\#}) \cong$

$\psi\phi P(\bar{\zeta})$ is isomorphic to the direct sum of r copies of $P(\bar{\zeta})$. Using (3.3), $\psi D(\nu) \cong \psi T_\nu P(\nu^\#) \cong T_\zeta \psi P(\nu^\#)$, i.e. $\phi D(\nu)$ is isomorphic to the direct sum of r copies of $T_\zeta P(\bar{\zeta})$. Since any quotient of $P(\bar{\zeta})$ is indecomposable and $\psi D(\nu)$ is self-dual, $T_\zeta P(\bar{\zeta})$ is self-dual. Then as above, $Soc(T_\zeta P(\bar{\zeta})) \cong L(\bar{\xi})$ and since $N(\zeta) \hookrightarrow T_\zeta P(\bar{\zeta})$, $Soc(N(\zeta)) \cong L(\bar{\zeta})$. This proves (ii) with $\zeta^\# = \bar{\zeta}$.

§5. The Zuckerman derived functors

In this section we reecall the basic facts about Zucerman derived functors as presented in [17] (see also [36]). The main result is Proposition 5.5 which we suppose is known. However a good reference for this result is not available, and so we give a proof. As well we point out an error in the statement of Lemma 3.3 in [17]. The correct formulation of this lemma is given here as Lemma 5.15.

Let notation be as in earlier sections. Define a second parabolic subalgebra $\underline{q}$ of $\underline{g}$ with $\underline{b} \subseteq \underline{q} \subseteq \underline{p}$. Let $\underline{q} = \underline{l} \oplus \underline{u}(\underline{q})$ be the Levi decomposition of $\underline{q}$ with $\underline{u}(\underline{q})$ the nilradical of $\underline{q}$ and $\underline{h} \subseteq \underline{l} \subseteq \underline{m}$. Fix an involutive automorphism θ of $\underline{g}$ with $\theta = -1$ on $\underline{h}$ and set $\underline{p}^- = \theta \underline{p}$. Now $\underline{m} \cap \underline{q}$ is a parabolic subalgebra of $\underline{m}$ with Levi decomposition $\underline{m} \cap \underline{q} = \underline{l} \oplus (\underline{m} \cap \underline{u}(\underline{q}))$. Also $\underline{q} = (\underline{m} \cap \underline{q}) \oplus \underline{u}$. Now by inducing in stages, the reader can easily verify:

Lemma 5.1 *Let $\nu \in \mathcal{P}_{\underline{l}}$. Set $N = N(\underline{g}, \underline{q}, \nu)$ and M equal to the $\underline{p}^-$-module $U(\underline{p}^-) \bigotimes_{U(\underline{m})} N(\underline{m}, \underline{m} \cap \underline{q}, \nu + \rho(\underline{m}) - \rho)$. Then N and M are isomorphic as $\underline{p}^-$-modules.*

For any Lie algebras $\underline{k} \subseteq \underline{a}$ we let $\mathcal{C}(\underline{a}, \underline{k})$ denote the category of all $\underline{a}$-modules which are $U(\underline{k})$-locally finite and completely reducibe as $\underline{k}$-modules. Let $\mathcal{A}(\underline{a}, \underline{k})$ be the full subcategory of $\mathcal{C}(\underline{a}, \underline{k})$ of finitely generated $\underline{a}$-modules with finite dimensional $\underline{k}$-isotypic subspaces; i.e., the so called admissible modules. Fix a subalgebra $\underline{m}$ with $\underline{k} \subseteq \underline{m} \subseteq \underline{a}$ and for $X \in \mathcal{C}(\underline{a}, \underline{k})$ define $\Gamma_{\underline{a}} X$ to be the span of the finite dimensional simple $\underline{m}$-submodules of X. In the next three propositions we summarize the basic propoerties of the right derived functors $\Gamma^i_{\underline{a}}$ of $\Gamma_{\underline{a}}$. The proofs can be found in [17].

Suppose $\underline{k}$ and $\underline{m}$ are reductive and $\underline{k}$ is reductive in both $\underline{m}$ and $\underline{a}$ and $\underline{m}$ is reductive in $\underline{a}$.

Proposition 5.2 *(i) For $i > \dim(\underline{m}/\underline{k})$, $\Gamma^i_{\underline{a}} \equiv 0$.*

(ii) For finite dimensional $F \in \mathcal{C}(\underline{a}, \underline{m})$ and $X \in \mathcal{C}(\underline{a}, \underline{k})$, $X \mapsto \Gamma^i_{\underline{a}}(F \otimes X)$ and $X \mapsto F \otimes \Gamma^i_{\underline{a}}(X)$ are naturally equivalent functors on $\mathcal{C}(\underline{a}, \underline{k})$ for all $i \in \mathbb{N}$.

Part (ii) is Lemma 3.3 in [17]. In that article the finite dimensionality of F is not a hypothesis; however, it is required for the proof given there to be correct.

For $X \in \mathcal{C}(\underline{a}, \underline{k})$, let $X^{\sim}$ (resp. $X^{\approx}$) denote the set of $U(\underline{k})$ (resp. $U(\underline{m})$) locally finite vectors in the algebraic dual of X.

Proposition 5.3 *Put $d = \dim(\underline{m}/\underline{k})$. Then the functors $X \mapsto \Gamma^i_{\underline{a}}(X)$ and $X \mapsto \Gamma^{d-i}_{\underline{a}}(X^{\sim})^{\approx}$ are naturally equivalent.*

We now return to the special setting of this article. Let f denote the forgetful functor from $\mathcal{C}(\underline{g}, \underline{l})$ to $\mathcal{C}(\underline{p}^-, \underline{l})$.

Proposition 5.4 *For all $i \in \mathbb{N}$, $f \circ \Gamma^i_{\underline{g}}$ and $\Gamma_{\underline{p}^-} \circ f$ are naturally equivalent functors on $\mathcal{C}(\underline{g}, \underline{l})$.*

This result is proved in [17] with $\underline{p}^-$ replaced by $\underline{m}$. However, since $\underline{m}$ is reductive in $\underline{p}^-$, essentially the same proof gives (5.4). We do not repeat the argument.

We now prove a basic result for this article. The derived functors $\Gamma^i_{\underline{g}}$ map Verma modules for $(\underline{g}, \underline{q})$ to Verma modules for $(\underline{g}, \underline{p})$.

Proposition 5.5 *Let $d = \dim(\underline{m}/\underline{l})$ and let $\nu \in \mathcal{P}_{\underline{m}}$. Suppose $w \in \mathcal{W}_{\underline{m}}$ and $w\nu \in \mathcal{P}_{\underline{l}}$. Then , for $i \in \mathbb{N}$*

$$\Gamma^i_{\underline{g}} N(\underline{g}, \underline{q}, w\nu) = \begin{cases} N(\underline{g}, \underline{p}, \nu) & \text{if } i = d - \ell(w) \\ 0 & \text{otherwise .} \end{cases}$$

Proof: In the case where $\underline{g} = \underline{m}$, this result is the basic calculation for derived functors (cf. [17, Proposition 6.3] and (5.3)). Then $N(\underline{g}, \underline{p}, \nu)$ is the irreducible finite dimensional $\underline{g}$-module with extreme weight $\nu - \rho$.

Let $N = N(\underline{g}, \underline{q}, w\nu)$, $A = N(\underline{m}, \underline{m} \cap \underline{q}, w\nu + \rho(\underline{m}) - \rho)$ and let f be the forgetful functor from $\mathcal{C}(\underline{g}, \underline{l})$ to $\mathcal{C}(\underline{p}^-, \underline{l})$. Consider A to be a $\underline{p}^-$-module by letting $\underline{u}^- = \theta \underline{u}$ act by zero. By (5.4) and (5.1) we have:

$$(5.6) \qquad f\Gamma^i_{\underline{g}} N \cong \Gamma^i_{\underline{p}^-} fN \cong \Gamma^i_{\underline{p}^-}(U(\underline{u}^-) \otimes A) \ .$$

Here $U(\underline{u}^-)$ is a $\underline{p}^-$-module with $\underline{u}^-$ acting by left multiplication and $\underline{m}$ acting by the adjoint action. Now suppose we have an isomorphism of $\underline{p}^-$-modules:

$$(5.7) \qquad \Gamma^i_{\underline{p}^-}(U(\underline{u}^-) \otimes A) \cong U(\underline{u}^-) \otimes \Gamma^i_{\underline{p}^-}(A).$$

By Proposition 6.3 in [17] and (5.3) we have:

$$(5.8) \qquad \Gamma^i_{\underline{p}^-}(A) \cong \begin{cases} F(\underline{m}, \nu - \rho) & \text{if } i = d - \ell(w) \\ 0 & \text{otherwise .} \end{cases}$$

Here $F(\underline{m}, \nu - \rho)$ is the irreducible finite dimensional $\underline{m}$-module with highest weight $\nu - \rho$ and we let $\underline{u}^-$ act by zero. Now combining (5.6), (5.7) and (5.8), for $i = d - \ell(w)$, $f\Gamma^i_{\underline{g}} N$ and $fN(\underline{g}, \underline{p}, \nu)$ are isomorphic. Moreover, by (5.7), the $\underline{m}$-modules in $\underline{u} \otimes F(\underline{m}, \nu - \rho)$ do not occur in $\Gamma^i_{\underline{g}} N$, and so $\underline{u}$ acts on$F(\underline{m}, \nu - \rho)$ by zero. Thus there is a $\underline{g}$-module map $\chi \neq 0$, $\chi : N(\underline{g}, \underline{p}, \nu) \to \Gamma^i_{\underline{g}} N$. By (5.7), χ is surjective and hence an isomorphism. For $i \neq d - \ell(w)$, the identities give: $\Gamma^i_{\underline{g}} N = 0$. This completes the proof except for the verification of (5.7). This verification will be given in a series of lemmas. Our argument depends critically on the admissibility of the modules in (5.7)

Lemma 5.9 **[18]** *Let F be any $\underline{a}$-module and let $\{f_i | i \in I\}$ be a basis for F over $\mathbb{C}$. Consider $U(\underline{a})$ as an $\underline{a}$-module by right multiplication. Then $F \otimes U(\underline{a})$ is a free $\underline{a}$-module with basis $\{f_i \otimes 1 | i \in I\}$.*

The proof of (5.9) is easy, using only the natural filtration of $U(\underline{a})$.

Let R and S be rings with multiplicative identity. Let $B = {}_S B_R$ be a left S-module and a right R-module. Then $B \otimes_R \cdot$ and $\mathrm{Hom}_S(B, \cdot)$ form an adjoint pair; i.e., for X a left R-module and Y a left S-module the map $f \mapsto \overline{f}$ defined by $\overline{f}(x)(b) = f(b \otimes x)$ is an isomorphism giving:

$$(5.10) \qquad \mathrm{Hom}_S(B \otimes_R X, Y) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_S(B, Y)) \ .$$

We refer to (5.10) as the adjoint isomorphism.

We now recall some of the notation from [17]. For any $\underline{a}$-module X, let $X[\underline{a}]$ denote the submodule of X spanned by the finite dimensional simple submodules of X. For any Lie algebras $\underline{a}$, $\underline{b}$ with $\underline{b} \subseteq \underline{a}$ and any $\underline{b}$-module W we put $I(\underline{a}, \underline{b}, W) = \mathrm{Hom}_{\underline{b}}(U(\underline{a}), W)[\underline{b}]$. Here the actions are defined as follows. Let f be a linear map from $U(\underline{a})$ to W. Then $f \in \mathrm{Hom}_{\underline{b}}(U(\underline{a}), W)$ if $f(ba) = bf(a)$ for all $a \in U(\underline{a})$, $b \in U(\underline{b})$. We consider $\mathrm{Hom}_{\underline{b}}(U(\underline{a}), W)$ as an $\underline{a}$-module by defing $(a \cdot f)(x) = f(xa)$, $x \in U(\underline{a})$, $a \in \underline{a}$. If $\underline{b}$ is reductive and reductive in $\underline{a}$ then $I(\underline{a}, \underline{b}, W)$ is an $\underline{a}$-submodule of $\mathrm{Hom}_{\underline{b}}(U(\underline{a}), W)$. For any $\underline{a}$-modules $X$and Y put:

$$(5.11) \qquad H(\underline{a}, X, Y) = \mathrm{Hom}_{\mathbb{C}}(X, Y)[\underline{a}], \qquad I(\underline{a}, X, Y) = \mathrm{Hom}_{\underline{a}}(X, Y)[\underline{a}] \ .$$

The reader can verify that the adjoint isomorphism induces an isomorphism: for $F \in \mathcal{C}(\underline{a}, \underline{b})$, $W \in \mathcal{C}(\underline{b}, \underline{b})$;

$$(5.12) \quad H(\underline{b}, F, I(\underline{a}, \underline{b}, W)) \cong H(\underline{b}, F, \mathrm{Hom}_{\underline{b}}(U(\underline{a}), W)) \cong I(\underline{b}, F \otimes U(\underline{a}), W) \ .$$

Lemma 5.13 *Let $F, X \in \mathcal{C}(\underline{a}, \underline{b})$, $W \in \mathcal{C}(\underline{b}, \underline{b})$ and define a map $\phi \mapsto \overline{\phi}$ by $\overline{\phi}(x)(f) = \phi(x)(f \otimes 1)$. Then this map gives a natural isomorphism:*

$$Hom_{\underline{a}}(X, Hom_{\underline{b}}(F \otimes U(\underline{a}), W)) \cong Hom_{\underline{b}}(X, Hom_{\underline{b}}(F, W)) \ .$$

Moreover this isomorphism induces isomorphisms:

$$Hom_{\underline{a}}(X, I(\underline{b}, F \otimes U(\underline{a}), W)) \cong Hom_{\underline{b}}(X, I(\underline{b}, F, W))$$

and

$$Hom_{\underline{a}}(X, H(\underline{b}, F \otimes U(\underline{a}), W)) \cong Hom_{\underline{b}}(X, H(\underline{b}, F, W)) \ .$$

The proof of (5.13) is straightforward and relies only on the freeness of $F \otimes U(\underline{a})$ (cf. (5.9)). We omit the details. This Lemma is a slight reformulation of Lemma 6.2.10 in [36].

Lemma 5.14 *For any injective module I in $\mathcal{C}(\underline{a}, \underline{b})$ and any $F \in \mathcal{C}(\underline{a}, \underline{b})$, $H(\underline{b}, F, I)$ is an injective module in $\mathcal{C}(\underline{a}, \underline{b})$.*

Proof: From [17] we know I is a summand of $I(W)$ for some $W \in \mathcal{C}(\underline{b}, \underline{b})$. So we may suppose $I = I(\underline{a}, \underline{b}, W)$. Let T denote the functor on $\mathcal{C}(\underline{a}, \underline{b})$ defined by $T(X) = \mathrm{Hom}_{\underline{a}}(X, H(\underline{b}, F, I))$. By (5.12) T is equivalent to: $X \mapsto \mathrm{Hom}_{\underline{a}}(X, I(\underline{b}, F \otimes U(\underline{a}), W))$. In turn, by (5.13) T is equivalent to the functor S with $S(X) = \mathrm{Hom}_{\underline{b}}(X, I(\underline{b}, F, W))$. Now modules in $\mathcal{C}(\underline{a}, \underline{b})$ are completely reducible as $\underline{b}$-modules and so S is exact. Thus T is exact and $H(\underline{b}, F, I)$ is injective.

Note this lemma is similar to Lemma 6.1.24 in [36].

We now return to the hypotheses asnd setting for (5.2) through (5.4); $\underline{k} \subseteq \underline{m} \subseteq \underline{a}$.

Lemma 5.15 *Let $F \in \mathcal{C}(\underline{a}, \underline{m})$, $A \in \mathcal{C}(\underline{a}, \underline{k})$. Then for all i, $A \mapsto \Gamma^i H(\underline{k}, F, A)$ and $A \mapsto H(\underline{m}, F, \Gamma^i A)$ are naturally equivalent functors.*

Proof: Let $0 \to A \to I^*$ be an injective resolution of A. Since F is completely reducible as an $\underline{m}$ and hence $\underline{k}$-module, $A \mapsto H(\underline{k}, F, A)$ is exact and by(5.14), takes injectives to injectives. So $0 \to H(\underline{k}, F, A) \to H(\underline{k}, F, I^*)$ is an injective resolution. Therefore $\Gamma^i H(\underline{k}, F, A)$ is the $i^{\underline{th}}$ cohomology group of the complex $\Gamma H(\underline{k}, F, I^*)$. Now by hypothesis, $\Gamma H(\underline{k}, F, I^*) = H(\underline{m}, F, I^*) = H(\underline{m}, F, \Gamma I^*)$. Finally, using exactness of $H(\underline{m}, F, \cdot)$ on $\mathcal{C}(\underline{a}, \underline{m})$, the cohomology group of the complex is $H(\underline{m}, F, \cdot)$ applied to the cohomology group of ΓI^*. This gives $\Gamma^i H(\underline{k}, F, A) \cong H(\underline{m}, F, \Gamma^i A)$. The naturality of these isomorphisms is easy to verify and we omit the details.

We can now prove identity (5.7). The produced modules are the duals of induced modules (cf. [13]). Recall the notation of (5.3) and the proof of (5.5). If F denotes the dual of the finite dimensional irreducible $\underline{l}$-module with extreme weight $w\nu - \rho$, then

$$(5.16) \qquad N^\sim \cong \mathrm{Hom}_{\underline{q}}(U(\underline{g}), F)[\underline{l}], \qquad A^\sim \cong \mathrm{Hom}_{\underline{m} \cap \underline{q}}(U(\underline{m}), F)[\underline{l}]$$

Consider these modules as $\underline{p}^-$-modules by restriction. Then

$$(5.17) \qquad N^\sim \cong H(\underline{l}, U(\underline{u}^-(\underline{q})), F), \qquad A^\sim \cong H(\underline{l}, U(\underline{m} \cap \underline{u}^-(\underline{q})), F) \ .$$

The adjoint isomorphism gives a $\underline{p}^-$-isomorphism: $N^\sim \cong H(\underline{l}, U(\underline{u}^-), A^\sim)$. Now by (5.15) for all i we have: $\Gamma^i N^\sim \cong H(\underline{m}, U(\underline{u}^-), \Gamma^i A^\sim)$. Applying the duality theorem, $\Gamma^{d-i} N \cong U(\underline{u}^-) \otimes \Gamma^{d-i} A$ and the proof of (5.7) is complete.

§6. An equivalence of categories

We keep the notation of sections two through four. Let $\underline{q}$ be a maximal parabolic subalgebra of $\underline{g}$ with Levi decomposition $\underline{q} = \underline{l} \oplus \underline{u}_{\underline{q}}$, $\underline{u}_{\underline{q}}$ equaling the nilradical of $\underline{q}$. Further, suppose $\underline{q}$ does not contain $\underline{p}$ and $\underline{l}$ contains $\underline{h}$ and $\underline{q}$ contains $\underline{b}$. Let $\underline{c}$ equal the center of $\underline{l}$. Recall from section two the categories of highest weight modules and the related truncated categories. Fix $\lambda \in \underline{h}^*$ and put $\mathcal{O}_{\underline{g}} = \mathcal{O}(\underline{g}, \underline{p}, \lambda)$ and $\mathcal{O}_{\underline{l}} = \mathcal{O}_t(\underline{l}, \underline{l} \cap \underline{p}, \lambda)$. In this section we establish an equivalence of categories between $\mathcal{O}_{\underline{g}}$ and $\mathcal{O}_{\underline{l}}$ whenever the sets of highest weights plus ρ (resp. $\rho(\underline{l})$) are equal.

Throughout this section we assume λ is Δ^+-integral and $\Delta(\underline{l})$-regular. Also we assume the sets of highest weights plus ρ in $\mathcal{O}_{\underline{g}}$ equals the set of highest weights plus $\rho(\underline{l})$ in $\mathcal{O}_{\underline{l}}$. Let Θ denote this set. Let L be the one-dimensional $\underline{q}$-module with weight $-\rho(\underline{u}_{\underline{q}})$. For any $\underline{l}$-module X, consider X as a $\underline{q}$-module by letting $\underline{u}_{\underline{q}}$ act by zero. Define the exact covariant functor U by:

$$U(X) = U(\underline{g}) \bigotimes_{U(\underline{q})} (X \otimes L) \, . \tag{6.1}$$

Set $d = \dim(\underline{m}/\underline{m} \cap \underline{l})$ and let Γ denote the $\underline{m}$-finite submodule functor on $\mathcal{C}(\underline{g}, \underline{m} \cap \underline{l})$ described in section five. Let $\mathcal{C}$ denote the category $\mathcal{O}(\underline{l}, \underline{l} \cap \underline{p})$ and define a functor τ on $\mathcal{C}$ by:

$$\tau X = \Gamma^d U(X), \qquad X \in \mathcal{C} \, . \tag{6.2}$$

Lemma 6.3 *For $X \in \mathcal{C}$, τX is the unique maximal $U(\underline{m})$-locally finite and $U(\underline{m})$-completely reducible quotient of $U(X)$.*

Proof: We use the notation surrounding the duality theorem (5.3). Note that all modules in $\mathcal{C}$ are admissible. Let A be a $U(\underline{m})$-locally finte and $U(\underline{m})$-completely reducible quotient of $Y = U(X)$. We have: $Y \xrightarrow{f} A \to 0$. Dualizing, we get the exact sequence: $0 \to A^{\sim} \xrightarrow{f^{\sim}} Y^{\sim}$. Define injections h and i with $i \circ h = f^{\sim}$, $h : A^{\sim} \to \Gamma Y^{\sim}$, $i : \Gamma Y^{\sim} \to Y^{\sim}$. Dualizing again, and identifying $Y \cong (Y^{\sim})^{\sim}$, $A \cong (A^{\sim})^{\sim}$ and $f \cong (f^{\sim})^{\sim}$ we have surjections $h^{\sim}$ and $i^{\sim}$ with $f = h^{\sim} \circ i^{\sim}$, $i^{\sim} : Y \to (\Gamma Y^{\sim})^{\sim}$, $h^{\sim} : (\Gamma Y^{\sim})^{\sim} \to A$. Now $(\Gamma Y^{\sim})^{\sim} = (\Gamma Y^{\sim})^{\approx} \cong \Gamma^d Y$. This proves the lemma.

We denote by p_X the surjection $i^{\sim}$ defined above. So p_X is the natural surjection : $p_X : U(X) \to \tau X$. The naturality in X is expressed as follows: for $X, Y \in \mathcal{C}$, $f \in \mathrm{Hom}(X, Y)$

$$\tau(f) \circ p_X = p_Y \circ U(f) \ . \tag{6.4}$$

The center $\underline{c}$ is one dimensional with basis vector say H. Note that $\rho(H) = \rho(\underline{u}_{\underline{q}})(H)$. Let $a = \lambda(H)$ and for any $\underline{c}$-module Z, let Z^a denote the eigensubspace of Z where H acts by eigenvalue a. Define a covariant functor σ on $\mathcal{C}$ by:

$$\sigma Y = (Y \otimes L^*)^a, \qquad Y \in \mathcal{C} \ . \tag{6.5}$$

The main result of this section is:

Theorem 6.6 *The functor τ gives an equivalence of $\mathcal{O}_{\underline{l}}$ onto $\mathcal{O}_{\underline{g}}$ with inverse σ.*

We shall prove this theorem through a series of lemmas.

Lemma 6.7 *Let $\mu \in \Theta$ and put $A = N(\underline{l}, \underline{l} \cap \underline{p}, \mu)$, $B = N(\underline{g}, \underline{p} \cap \underline{q}, \mu)$ and $C = N(\underline{g}, \underline{p}, \mu)$. Then (i) $U(A) \cong B$, (ii) $\tau A \cong C$ and (iii) $\sigma C \cong A$.*

Proof: Induction in stages proves (i) while (i) and (5.5) prove (ii). We now prove (iii). Since μ lies in the $\mathcal{W}_{\underline{l}}$-orbit of ρ, $\mu(H) = a$. Moreover, we may suppose for $\alpha \in \Delta$, $\alpha(H) > 0$ if and only if $\alpha \in \Delta(\underline{u}_{\underline{q}})$. Let β be the unique simple root of Δ^+ which is not in $\Delta(\underline{l})$. By assumption $\underline{q} \not\supset \underline{p}$ and so β is a simple root of $\Delta^+(\underline{m})$. This implies $s_\beta\mu \in \mathcal{P}_{\underline{m}\cap\underline{l}}$ and we put $N = N(\underline{g}, \underline{p} \cap \underline{q}, s_\beta\mu)$. Let D denote the image of N in B. Since $\underline{q} \cap \underline{m}$ is a maximal parabolic subalgebra of $\underline{m}$ and β is the unique complementary simple root, B/D is $U(\underline{m})$-locally finite and $U(\underline{m})$-completely reducible, and thus $C \cong B/D$. Now $\sigma N = 0$ and so $\sigma D = 0$. Again by exactness of σ, $\sigma C \cong \sigma B$. Finally, $B \cong U(\underline{u}^-(\underline{q}) \oplus (\underline{u}^- \cap \underline{l})) \otimes F(\underline{m} \cap \underline{l}, \mu - \rho)$ which gives: $\sigma B \cong U(\underline{u}^- \cap \underline{l}) \otimes F(\underline{m} \cap \underline{l}, \mu - \rho) \otimes L^*$. In turn, since $B \to \sigma B$ is an $\underline{l}$-module projection, $\sigma B \cong A$. This completes the proof.

Now to prove (6.6) we must establish:

(6.8) $\sigma \circ \tau$ is naturally equivalent to the identity functor on $\mathcal{O}_{\underline{l}}$.

(6.9) $\tau \circ \sigma$ is naturally equivalent to the identity functor on $\mathcal{O}_{\underline{g}}$.

First we prove (6.8). Let ℓ (resp ℓ^*) denote a basis vector for L (resp. L^*). Recall the natural surjections $p_X : U(X) \to \tau X$. For $X \in \mathcal{O}_{\underline{l}}$ define a map f_X by $f_X(v) = \sigma \circ p_X(1 \otimes v \otimes l)$, $v \in X$. From (6.4) we obtain the following commutative diagram: for $X, Y \in \mathcal{O}_{\underline{l}}$, $\phi \in \mathrm{Hom}_{\underline{l}}(X, Y)$

$$(6.10) \qquad \begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow f_X & & \downarrow f_Y \\ \sigma\tau X & \xrightarrow{\sigma\tau\phi} & \sigma\tau Y \end{array}$$

This shows $X \mapsto f_X$ is a natural transformation from the identity functor to $\sigma \circ \tau$. To complete the proof of (6.8) we show f_X is an isomorphism for all $X \in \mathcal{O}_{\underline{l}}$.

Each $\mu \in \Theta$ has the form $\mu = w\lambda$ with $w \in \mathcal{W}_{\underline{l}}$. We set $N_w = N(\underline{l}, \underline{l} \cap \underline{p}, \mu)$, $L_w = L(\underline{l}, \mu)$. For $i \in \mathbb{N}$ let $\mathcal{O}_i$ be the full subcategory of $\mathcal{O}_{\underline{l}}$ of all X such that $(X : L_w) \neq 0$ implies $\ell(w) \geq i$. If $j = \ell(w)$ is maximal then $N_w = L_w$ and for all $X \in \mathcal{O}_j$, (6.7) implies that f_X is an isomorphism. Also, by (5.5) $\sigma \circ \Gamma^t \circ U(X) = 0$ for all $t < d$ and $X \in \mathcal{O}_j$. We now proceed inductively. Suppose $X \mapsto f_X$ is a natural equivalence between the identity functor and $\sigma \circ \tau$ on $\mathcal{O}_i$ and $\sigma \Gamma^t U(X) = 0$ for all $X \in \mathcal{O}_i$ and $t < d$. Fix $\mu \in \Theta$ with $\mu = w\lambda$ and $\ell(w) = i - 1$. We have a short exact sequence $0 \to J \to N_w \to L_w \to 0$ with $J \in \mathcal{O}_i$. Now σ and U being exact, this sequence induces a long exact sequence:

$$(6.11) \qquad \cdots \to \sigma\Gamma^j U(J) \to \sigma\Gamma^j U(N_w) \to \sigma\Gamma^j U(L_w) \to \sigma\Gamma^{j+1} U(N_w) \to \cdots$$

By induction hypothesis, $\sigma\Gamma^j U(J) = 0$ for $j < d$. Then by (5.5), $\sigma\Gamma^j U(L_w) \cong \sigma\Gamma^j U(N_w) = 0$ for $j < d - 1$. Applying f_X to the last six terms of (6.11) we obtain: for $L = L_w$, and $N = N_w$

(6.12)
$$\begin{array}{ccccccccc} 0 & \to & \sigma\Gamma^{d-1}U(L) & \to & \sigma\Gamma^d U(J) & \to & \sigma\Gamma^d U(N) & \to & \sigma\Gamma^d U(L) & \to & 0 \\ & & & & f_J \uparrow\cong & & f_N \uparrow\cong & & f_L \uparrow & & \\ & & 0 & \to & J & \to & N & \to & L & \to & 0 \end{array}$$

Now by hypothesis f_J is an isomorphism while (6.7) proves f_N is an isomorphism. Since the diagram is commutative $\sigma\Gamma^{d-1}U(L) = 0$ and then by the five lemmas, f_L must be an isomorphism. A similar argument by induction on the length of a composition series for $X \in \mathcal{O}_{i-1}$ shows that f_X is an isomorphism and $\sigma\Gamma^t U(X) = 0$ for $t < d$. This completes the induction step. We conclude that f_X is an isomorphism for all $X \in \mathcal{O}_{\underline{l}}$ proving (6.8) and also:

$$(6.13) \qquad \sigma\Gamma^t U(X) = 0 \ \text{ for all } \ X \in \mathcal{O}_{\underline{l}}, \quad t < d\,.$$

Lemma 6.14 *(i) τ is an exact functor from $\mathcal{O}_{\underline{l}}$ to $\mathcal{O}_{\underline{g}}$.*

(ii) For $\mu \in \Theta$, $\tau L(\underline{l}, \mu) \cong L(\underline{g}, \mu)$.

Proof: τ is right exact since $\Gamma^i = 0$ for all $i > d$. Suppose $\phi : A \to B$ is an injection, $A, B \in \mathcal{O}_{\underline{l}}$. Let K equal the kernel of the map $\tau\phi$. By (6.13), $\sigma K = 0$. However, for $Y \in \mathcal{O}$, if $\sigma Y = 0$ then $Y = 0$. This proves $K = 0$, and so τ is exact. This fact also implies that $Y \in \mathcal{O}_{\underline{g}}$ is simple if σY is simple in $\mathcal{O}_{\underline{l}}$. By (6.7), (6.8) and exactness of τ, $\tau L(\underline{l}, \mu)$ is a simple quotient of $N(\underline{g}, \underline{p}, \mu)$. This proves (ii)

We now turn to the proof of (6.9). For $X \in \mathcal{O}_{\underline{g}}$ define a map $g_X : U(\sigma X) \to X$ by $g_X(y \otimes v) = y \cdot v$, $y \in U(\underline{g})$, $v \in \sigma X$. Note that the shifts by L and L^* cancel. Since $\sigma Y = 0$ implies $Y = 0$ in $\mathcal{O}_{\underline{g}}$, g_X is a $\underline{g}$-module surjection. Since $X \in \mathcal{O}_{\underline{g}}$, the map g_X factors through $\tau\sigma X$. Let h_X denote the induced surjection: $h_X : \tau\sigma X \to X$. We have: $g_X = h_X \circ p_{\sigma X}$. Also $X \mapsto h_X$ is a natural transformation from $\tau \circ \sigma$ to the identity functor on $\mathcal{O}_{\underline{g}}$. FInally by (6.7), h_X is an isomorphism for all generalized Verma modules X, and so by (3.5), $X \mapsto h_X$ is an eqivalence of $\tau \circ \sigma$ and the identity. This proves (6.9) and completes the proof of our main result (6.6).

Lemma 6.15 *Let $X \in \mathcal{O}_{\underline{l}}$. Then X is self-dual if and only if τX is self-dual.*

Proof: Suppose X is self-dual; i.e., X admits a symmetric nondegenerate contravariant form ϕ. Let ϕ_L (resp. ϕ_{L^*}) denote a nondegenerate form on L (resp. L^*). Then Proposition 6.12 in [14] asserts that $\phi \otimes \phi_L$ has a unique extension $\overline{\phi}$ to a contaravariant form on $U(X)$. Let Y be any $\underline{g}$-submodule of $U(X)$ with $Y \cap (1 \otimes X \otimes L) = 0$. Then invariance of $\overline{\phi}$ implies Y is contained in the radical of $\overline{\phi}$. Let K be the kernel of $p_X : U(X) \to \tau X$. Since $K \cap (1 \otimes X \otimes L) = 0$, $\overline{\phi}$ induces a form $\tau\phi$ on τX. The radical R of $\tau\phi$ is a $\underline{g}$-submodule of τX and since σR is contained in the radical of ϕ, which is zero, $\sigma R = 0$. But this gives $R = 0$ and so τX is self-dual.

Conversely suppose τX is self-dual with a symmetric nondegenerate contravariant form ψ. By contravariance, eigenspaces for H are orthogonal and so $\psi \otimes \phi_{L^*}$ restricts to a nondegenerate contravariant form on $\sigma\tau X \cong X$. Thus X is self-dual.

§7. A second equivalence of categories

We retain the notation from previous sections. In particular $\underline{q} = \underline{l} \oplus \underline{u}_{\underline{q}}$ is a maximal parabolic subalgebra of $\underline{g}$ which does not contain $\underline{p}$. In this section we obtain a result similar to that of section six. However, here truncation will appear on the $\underline{g}$-module category instead of the $\underline{l}$-module category. The main result is Proposition 7.1. As in section six this equivalence will use Zuckerman derived functors. However, unlike section six which used derived functors in the top dimension, here we use derived functors in the middle dimension.

Put $\underline{r} = \underline{m} \cap \underline{l}$. Let w_1 be the element of maximal length in $\mathcal{W}_{\underline{l}}$ and put $\nu = -w_1\rho$. Recall the decomposition $\mathcal{W}_{\underline{m}} = {}^{\underline{r}}\mathcal{W} \cdot \mathcal{W}_{\underline{r}}$ and let w_0 be the element in ${}^{\underline{r}}\mathcal{W}$ of maximal length. Then put $\mu = w_0\nu$ and define $\mathcal{O}_{\underline{l}} = \mathcal{O}(\underline{l}, \underline{l} \cap \underline{p}, \nu)$, $\mathcal{O}_\nu = \mathcal{O}_t(\underline{g}, \underline{p} \cap \underline{q}, \nu)$ and $\mathcal{O}_\mu = \mathcal{O}_t(\underline{g}, \underline{p}, \mu)$.

Proposition 7.1 *The categories $\mathcal{O}_{\underline{l}}$, $\mathcal{O}_\nu$ and $\mathcal{O}_\mu$ are all equivalent.*

We shall prove this result through a series of lemmas keeping careful track of the induced maps on highest weight vectors. Put $\Psi = \{w\nu | w \in \mathcal{W}_{\underline{l}},\ w\nu \in \mathcal{P}_{\underline{r}}\}$. Then Ψ parameterizes the set of simple modules in $\mathcal{O}_{\underline{l}}$ via highest weights plus $\rho(\underline{l})$.

Lemma 7.2 *Let $\xi \in \Psi$, $\lambda \in \mathcal{P}_{\underline{r}}$ and suppose $\lambda \prec \xi$. Then $\lambda \in \Psi$.*

Proof: Choose $\alpha_i \in \Delta^+$, $1 \leq i \leq t$, and let $\mu_i = s_{\alpha_i} \cdots s_{\alpha_1}\xi$ with $\lambda = \mu_t \prec \cdots \prec \mu_0 = \xi$. Suppose $\lambda \notin \Psi$ then choose j maximal with $\alpha_j \in \Delta(\underline{u}_{\underline{q}})$. But $\mu_j \in -\mathcal{W}_{\underline{l}} \cdot \rho$, and so $\langle \mu_j, \beta \rangle < 0$ for all $\beta \in \Delta(\underline{u}_{\underline{q}})$. This contradiction proves (7.2).

This lemma shows that Ψ also parameterizes the set of simple modules in $\mathcal{O}_\nu$. We now turn to the equivalence of $\mathcal{O}_{\underline{l}}$ and $\mathcal{O}_\nu$. Recall from (6.1) the functor U.

Proposition 7.3 *(i) For $\xi \in \Psi$, $U(L(\underline{l}, \xi)) \cong L(\underline{g}, \xi)$.*

(ii) $A \mapsto U(A)$ induces an equivalence of $\mathcal{O}_{\underline{l}}$ onto $\mathcal{O}_\nu$.

Proof: By (7.2) any nontrivial $\underline{g}$-submodule of $U(A)$ has a nontrivial intersection with $1 \otimes A$. This gives (i). Let σ be the functor defined in (6.5) however with λ replaced by ν. The natural map $A \to \sigma U(A)$ is an $\underline{l}$-module isomorphism, say f_A. Moreover $A \mapsto f_A$ is a natural transformation from 1 to $\sigma \circ U$ on $\mathcal{O}_{\underline{l}}$. To complete the proof we now show $U \circ \sigma$ is naturally equivalent to 1 on $\mathcal{O}_\nu$.

Let a (resp. a^*) be a basis vector for L (resp. L^*). Let $B \in \mathcal{O}_\nu$ and define $g_B \in \mathrm{Hom}_{\underline{g}}(U\sigma B, B)$ by the formula: for $v \otimes a^* \in \sigma B$ and $x \in U(\underline{g})$, $g_B(x \otimes v \otimes a^* \otimes a) = x \cdot v$. Again by (7.2) it follows that g_B is an isomorphism for all B, and thus $B \mapsto g_B$ is an equivalence of $U \circ \sigma$ and 1 on $\mathcal{O}_\nu$. This proves (7.3).

We now come to the more difficult part of (7.1): the equivalence of $\mathcal{O}_\nu$ and $\mathcal{O}_\mu$. Let Γ denote the $U(\underline{m})$-locally finite submodule functor on $\mathcal{C}(\underline{g}, \underline{r})$ as defined in section five. Put $2s = \dim(\underline{m}/\underline{r})$ and recalling w_0, put $\Theta = w_0 \Psi$.

Lemma 7.4 *(i) $\Theta \subseteq \mathcal{P}_{\underline{m}}$. (ii) For $\xi \in \underline{h}^*$, $\xi \in \Psi$ if and only if $w_0\xi \in \mathcal{P}_{\underline{m}}$ and $w_0\xi \prec \mu$..*

Note: This lemma implies that Θ parameterizes the set of simple modules in $\mathcal{O}_\mu$.

Proof: By construction, elements of Ψ lie in $\mathcal{P}_{\underline{r}}$ and have negative inner product with elements of $\Delta(\underline{u}_q)$. Thus $w_0 \Psi \subset \mathcal{P}_{\underline{m}}$ giving (i).

Now suppose $\xi, \zeta \in \Psi$ with $\xi \prec \zeta$. Then we claim: $w_0\xi \prec w_0\zeta$. Choose $\alpha_i \in \Delta^+$, $1 \le i \le t$, and let $\zeta_i = s_{\alpha_i} \cdots s_{\alpha_1}\zeta$ with $\xi = \zeta_t \prec \cdots \prec \zeta_0 = \zeta$ and

$\langle\alpha_i, \zeta_{i-1}\rangle > 0$. By (7.2), $\alpha_i \in \Delta^+(\underline{l})$. Since $\Delta^+(\underline{l}) \subset \Delta^+(\underline{r}) \cup \Delta^+(\underline{u})$, $w_0\alpha_i \in \Delta^+$. Put $\gamma_i = w_0\alpha_i$. Then $w_0\zeta_i = s_{\gamma_i}w_0\zeta_{i-1}$, $1 \le i \le t$, and $\langle\gamma_i, w_0\zeta_{i-1}\rangle = \langle\alpha_i, \zeta_{i-1}\rangle$, and so $w_0\xi \prec w_0\zeta$. This proves the only if part of (ii).

Finally, suppose $\xi \in \underline{h}^*$, $w_0\xi \in \mathcal{P}_{\underline{m}}$ and $w_0\xi \prec \mu$. By (2.9) we choose $\gamma_i \in \Delta^+$, $1 \le i \le t$, with $\mu_i = s_{\gamma_i}\cdots s_{\gamma_1}\mu \in \mathcal{P}_{\underline{m}}$. and $w_0\xi = \mu_t \prec \cdots \prec \mu_1 \prec \mu$. Since each μ_i is $\Delta^+(\underline{m})$-dominant, $\gamma_i \in \Delta(\underline{u})$ and thus $\beta_i = w_0^{-1}\gamma_i \in \Delta(\underline{u})$ also. Now $w_0^{-1}\mu_i = s_{\beta_i}w_0^{-1}\mu_{i-1}$ and $\langle\beta_i, w_0^{-1}\mu_{i-1}\rangle = \langle\gamma_i, \mu_{i-1}\rangle$. Therefore $\xi \prec w_0^{-1}\mu_t \prec \cdots \prec w_0^{-1}\mu = \nu$, and so by (7.2) $\xi \in \Psi$. This proves (7.4).

The calculation of derived functors in (5.5) gives:

Lemma 7.5 *Let $\xi \in \Psi$. Then*

$$\Gamma^i N(\underline{g}, \underline{p} \cap \underline{q}, \xi) \cong \begin{cases} N(\underline{g}, \underline{p}, w_0\xi) & \text{if } i = s \\ 0 & \text{otherwise}. \end{cases}$$

By the duality theorem (5.3) we obtain:

Lemma 7.6

(a) *Let $A \in \mathcal{O}_\nu$ and suppose A admits a nondegenerate invariant form. Then so does $\Gamma^s A$.*

(b) *Let $A\hat{}$ denote the contravariant dual module in $\mathcal{O}_\nu$. Then $\hat{}$ and Γ^s commute; i.e., $\Gamma^s A\hat{}$ and $(\Gamma^s A)\hat{}$ are isomorphic.*

Lemma 7.7 *Γ^s is an exact functor on $\mathcal{O}_\nu$.*

Proof: Every module A in $\mathcal{O}_\nu$ is free over $U(\underline{m} \cap \underline{u}_{\underline{q}}^-)$, and so by (5.5), $\Gamma^i A = 0$ for all $i < s$. For any simple module B in $\mathcal{O}_\nu$, B is self-dual, and thus by duality (5.3), $\Gamma^i B = 0$ for all $i \ne s$. Using long exact sequences we conclude $\Gamma^i A = 0$ if $i \ne s$. This proves (7.7).

Proposition 7.8 *(i) For $\xi \in \Psi$, $\Gamma^s L(\underline{g}, \xi) \cong L(\underline{g}, w_0\xi)$.*

(ii) Γ^s gives an equivalence of $\mathcal{O}_\nu$ onto $\mathcal{O}_\mu$.

Proof: Let $A = \Gamma^s L(\underline{g}, \xi)$. By exactness and (7.5), A is a quotient of $B = N(\underline{g}, \underline{p}, w_0\xi)$. By (7.6), A admits a nondegenerate invariant form. However, the only quotient of B to admit a nondegenerate invariant form is $L(\underline{g}, w_0\xi)$. This gives (i).

To prove (ii) it is sufficient by (3.6) to prove:

(7.9) Γ^s is full and faithful.

(7.10) For any $X \in \mathcal{O}_\mu$ there is a $Y \in \mathcal{O}_\nu$ with $X \cong \Gamma^s Y$.

To establish these facts we must first prove that Γ^s maps projective objects in $\mathcal{O}_\nu$ to projective objects in $\mathcal{O}_\mu$. The proof of this is somewhat delicate and involves wall crossing and an inductive argument.

Recall from section two the truncation functors T_ν and T_μ.

Lemma 7.11 *Let $\xi \in \Psi$. Then*

$$\Gamma^s T_\nu P(\underline{g}, \underline{p} \cap \underline{q}, \xi) \cong T_\mu P(\underline{g}, \underline{p}, w_0\xi) .$$

Proof: To simplify notation, for $\xi \in \Psi$ put $N(\xi) = N(\underline{g}, \underline{p} \cap \underline{q}, \xi)$, $P(\xi) = P(\underline{g}, \underline{p} \cap \underline{q}, \xi)$, $N(w_0\xi) = N(\underline{g}, \underline{p}, w_0\xi)$ and $P(w_0\xi) = P(\underline{g}, \underline{p}, w_0\xi)$. We now prove (7.11) by downward induction on the Bruhat ordering of Ψ. If ξ is maximal then $\xi = \nu$ and since $T_\nu P(\nu) = N(\nu)$ and $T_\mu P(w_0\nu) = N(w_0\nu)$, (7.5) gives the isomorphism (7.11) in this case. Now let $\xi \in \Psi$ and suppose we have:

$$\Gamma^s T_\nu P(\xi) \cong T_\mu P(w_0\xi) . \tag{7.12}$$

Choose $r \in \mathcal{W}$ with $\xi = r\rho$ and suppose $\alpha \in \Delta^+$ is simple, $\ell(rs_\alpha) = \ell(r) + 1$ and $\overline{\xi} = rs_\alpha\rho \in \Psi$. We now prove (7.12) holds with ξ replaced by $\overline{\xi}$.

Let ω be the fundamental wieght corresponding to α and let ϕ and ψ be the Zuckerman translation functors; $\phi = \phi^{\rho}_{\rho-\omega}$ and $\psi = \psi^{\rho-\omega}_{\rho}$. Let θ denote the wall crossing composite; $\theta = \phi \circ \psi$. Now (3.3) asserts that θ and T_ν commute on $\mathcal{O}(\underline{g}, \underline{p} \cap \underline{q}, \rho)$ and θ and T_μ commute on $\mathcal{O}(\underline{g}, \underline{p}, \rho)$. Since θ also commutes with Γ^s we obtain:

$$\Gamma^s T_\nu \theta P(\xi) \cong T_\mu \theta P(w_0 \xi) \ . \tag{7.13}$$

Note that this module is projective in $\mathcal{O}_\mu$. Now since $P(\overline{\xi})$ is a summand of $\theta P(\xi)$ (cf. [20, Proposition 2.3]), (7.13) implies that $\Gamma^s T_\nu P(\overline{\xi})$ is projective in $\mathcal{O}_\mu$. On the other hand, by (2.5) and (7.5), $\Gamma^s T_\nu P(\overline{\xi})$ and $T_\mu P(w_0\overline{\xi})$ have the same character. Thus, both being projective, they are isomorphic. This proves (7.12) for $\overline{\xi}$ and completes the induction step. In turn this proves (7.11).

We now prove Γ^s is full and faithful. Let $A, B \in \mathcal{O}_\nu$ and $\phi \in \mathrm{Hom}(A, B)$. By (7.7) and (7.8), $\mathrm{image}(\Gamma^s \phi) \cong \Gamma^s(\mathrm{image}(\phi)) \neq 0$. So Γ^s is faithful.

Next we prove (7.10). For $\zeta \in \underline{h}^*$, let $L(\zeta) = L(\underline{g}, \zeta)$. For $\xi \in \Psi$ and $A \in \mathcal{O}_\nu$, $\dim(\mathrm{Hom}(T_\nu P(\xi), A)) = (A : L(\xi)) = (\Gamma^s A : L(w_0\xi)) = \dim(\mathrm{Hom}(T_\mu P(w_0\xi), \Gamma^s A)$ by (7.8). Since Γ^s is faithful, we obtain: for any projective $P \in \mathcal{O}_\nu$,

$$\Gamma^s \mathrm{Hom}(P, A) \cong \mathrm{Hom}(\Gamma^s P, \Gamma^s A) \ . \tag{7.14}$$

Now let $X \in \mathcal{O}_\nu$ and choose projective modules P and P' with $P' \xrightarrow{\pi} P \to X \to 0$ exact. By (7.11), choose projective modules Q and Q' in $\mathcal{O}_\nu$ with $P \cong \Gamma^s Q$ and $P' \cong \Gamma^s Q'$. Now identyify these modules and using (7.14) choose $\pi_1 \in \mathrm{Hom}(Q', Q)$ with $\Gamma^s \pi_1 = \pi$. Then by the exactness of Γ^s, $X \cong \Gamma^s(Q/\pi_1 Q')$. This proves (7.10).

Finally we prove Γ^s is full. Let A, Y, Q and Q' be in $\mathcal{O}_\nu$ with Q, Q' projective and $Q' \xrightarrow{\pi} Q \xrightarrow{\varepsilon} Y \to 0$ exact. Applying $\mathrm{Hom}(\cdot, A)$ and Γ^s we obtain the commutative diagram:

(7.15)
$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}(Y,A) & \to & \mathrm{Hom}(Q,A) & \to & \mathrm{Hom}(Q',A) \\ & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ 0 & \to & \mathrm{Hom}(\Gamma^s Y, \Gamma^s A) & \to & \mathrm{Hom}(\Gamma^s Q, \Gamma^s A) & \to & \mathrm{Hom}(\Gamma^s Q', \Gamma^s A) \end{array}$$

The two right hand maps are isomorphisms by (7.14), and so (adding two zeros on the left) the five lemma implies all maps in (7.15) are isomorphisms. Thus Γ^s is full. This completes the proof of (7.8).

Propositions 7.3 and 7.8 combine to prove (7.1). For reference we now summarize the properties we have for the composite functor $I = \Gamma^s \circ U$ on $\mathcal{O}_{\underline{l}}$.

Proposition 7.16 *I induces an equivalence of categories $\mathcal{O}_{\underline{l}}$ onto $\mathcal{O}_\mu$ with the properties: for $\xi \in \Psi$,*

(i) $I(L(\underline{l}, \xi)) \cong L(\underline{g}, w_0\xi)$;

(ii) $I(N(\underline{l}, \underline{l} \cap \underline{p}, \xi)) \cong N(\underline{g}, \underline{p}, w_0\xi)$;

(iii) $I(P(\underline{l}, \underline{l} \cap \underline{p}, \xi)) \cong T_\mu P(\underline{g}, \underline{p}, w_0\xi)$.

Moreover, $A \in \mathcal{O}_{\underline{l}}$ admits a nondegenerate $\underline{l}$-invariant form if and only if IA admits a nondegenerate $\underline{g}$-invariant form.

Proof: The properties (i), (ii) and (iii) follow from the results above. The result on invariant forms follows from the proof of (6.15) and (7.6).

Part II - Highest Weight Modules for Hermitian Symmetric Pairs.

§8. Statement of the Main Results

In Part II we turn our attention to the highest weight modules of classical Hermitian symmetric pairs. Our principal results are explicit descriptions of the composition factors of a generalized Verma module, as well as, explicit formulas for the Kazhdan-Lusztig polynomials in this setting.

We refer to each classical Hermitian symmetric pair $(\underline{g},\underline{p})$ by HS.i with $1 \leq i \leq 5$. These are defined by Table 8.1 below. Throughout Part II, $(\underline{g},\underline{p})$ will be one of these Hermitian symmetric pairs. To each pair we attach a constant p which is also given in the table. In the equal root length cases p is just the split rank of the pair. In the non-equal root length cases p is the greatest integer in one half of the split rank plus one.

Table 8.1

$(\underline{g},\underline{p})$	$\underline{g}$	$[\underline{m},\underline{m}]$	constant p
HS.1	A_n	$\mathrm{A}_{r-1} \times \mathrm{A}_{s-1}$	$\min(r,s) \quad (r+s=n+1)$
HS.2	B_n	B_{n-1}	1
HS.3	C_n	A_{n-1}	$[(n+1)/2]$
HS.4	D_n	D_{n-1}	2
HS.5	D_n	A_{n-1}	$[n/2]$

The nonclassical Hermitian symmetric pairs E_6 and E_7 have been studied in detail by D. Collingwood [12] so we omit discussion of them here.

We begin by describing the combinatorics which gives the solution(s) to the composition factors problem for $\mathcal{O}(\underline{g}, \underline{p}, \rho)$. We make the convention that if all of the roots of $\underline{g}$ are of the same length then they are all called short roots.

Definition 8.2 *Set $\mathcal{M} = \{(\gamma,\nu) \mid \gamma,\nu \in \Delta^+$ and either $\langle\gamma,\nu\rangle \neq 0$ or both γ and ν are long roots $\}$. Let $\mathcal{S}(\Delta^+)$ denote the collection of all subsets Ω of Δ^+ which satisfy the following conditions:*

i) If γ, ν are in $\Omega, \gamma \neq \nu$, then $(\gamma,\nu) \notin \mathcal{M}$.

ii) If γ is in Ω and ξ is in Δ^+ with $\gamma \neq \xi$, $(\gamma,\xi) \in \mathcal{M}$ and $\xi \leq \gamma$ then there is a ζ in Ω with $\zeta \neq \gamma$, $(\zeta,\xi) \in \mathcal{M}$ and $\zeta \leq \gamma$.

We note that $\mathcal{S}(\Delta^+)$ is defined for any positive system of any root system. When there is no chance of confusion we will denote this set simply as $\mathcal{S}$.

Fix x in $\mathcal{W}^{\underline{m}}$. Then $\mathcal{S}_x$ will denote the set of all Ω in $\mathcal{S}$ which satisfy $x\Omega \subseteq \Delta(\underline{u}) \cup -\Delta(\underline{u})$. By $\mathcal{E}_x$ we will denote the collection of Ω in $\mathcal{S}_x$ which satisfy the additional condition:

iii) If γ is in Ω then there is a ζ in Ω with $\gamma \leq \zeta$ and $x\zeta \in \Delta(\underline{u})$.

Two distinguished elements of $\mathcal{S}_x$ have an alternative and more computable description. For any root γ we set $\gamma^\vee = 2\gamma/\langle\gamma,\gamma\rangle$. For λ in $\underline{h}^*$ set $\Sigma_0(\lambda) = \{\gamma \in \Delta(\underline{u}) \mid \langle\lambda,\gamma^\vee\rangle = 0\}$. We define $\Sigma_i(\lambda)$, for $i \in N$ inductively as follows. $\Sigma_i(\lambda) = \{\gamma \in \Delta(\underline{u}) | \langle\lambda,\gamma^\vee\rangle = -i, \gamma$ is orthogonal to $\bigcup_0^{i-1} \Sigma_j(\lambda)$ and if there is a long root in $\bigcup_0^{i-1} \Sigma_j(\lambda)$ then γ is a short root $\}$.

Definition 8.3 *For x in $\mathcal{W}^{\underline{m}}$ we set:*

$$\Sigma_x = \bigcup_{j>0} \Sigma_j(x\rho) \quad \text{and} \quad \Sigma_x^+ = \bigcup_{j>0} \Sigma_j(-x\rho)$$

We note that Σ_x and Σ_x^+ are sets of mutually orthogonal roots in $\Delta(\underline{u})$ with $\Sigma_x \subset -x\Delta^+$ and $\Sigma_x^+ \subset x\Delta^+$.

For Ω in $\mathcal{S}_x$, set $\Omega^+ = \{\gamma \in \Omega \mid x\gamma \in \Delta(\underline{u})\}$ and $\Omega^- = \{\gamma \in \Omega \mid x\gamma \in -\Delta(\underline{u})\}$. Note that the definitions of Ω^+ and Ω^- depend on x. Let $\mathcal{S}_x^+$ (resp. $\mathcal{S}_x^-$) be the collection of $\Omega \in \mathcal{S}_x$ with $\Omega = \Omega^+$ (resp. $\Omega = \Omega^-$). Define $r_\Omega \in \mathcal{W}$ by $r_\Omega = \prod_{\gamma\in\Omega^+} s_\gamma$. If λ is in $\underline{h}^*$, let $\bar{\lambda}$ denote the $\Delta^+(\underline{m})$-dominant element of the $\mathcal{W}_{\underline{m}}$-orbit of λ. Write $\mathcal{W} = \mathcal{W}_{\underline{m}}\mathcal{W}^{\underline{m}}$ as usual and let $w \to \overline{w}$ denote the projection of $\mathcal{W}$ onto $\mathcal{W}^{\underline{m}}$ "perpendicular" to $\mathcal{W}_{\underline{m}}$. Then $\overline{w}\rho = \overline{w\rho}$. Suppose $\Sigma = \{\gamma_1, \dots, \gamma_t\}$ is a subset of Σ_x. Put $s_\Sigma(x) = s_\Sigma = \overline{s_{\gamma_1}\cdots s_{\gamma_t}x}$ and $\mathcal{A}_x = \{s_\Sigma | \Sigma \subseteq \Sigma_x\}$ We can now state our main results on composition factors.

Theorem 8.4 *Fix x in $\mathcal{W}^{\underline{m}}$.*

i) $-x^{-1}\Sigma_x$ is in $\mathcal{S}_x$.

ii) $L(x\rho)$ is a composition factor of $N(\mu)$ if and only if $\mu \in \mathcal{A}_x\rho$. Also, the map $\Sigma \to s_\Sigma$ is a bijection from the set of subsets of Σ_x onto $\mathcal{A}_x$.

iii) The composition factors of $N(x\rho)$ occur with multiplicity one and are exactly $\{L(\overline{xr_\Omega\rho}), \Omega \in \mathcal{E}_x\}$. Moreover, the map $\Omega \to \overline{xr_\Omega\rho}$ is injective on $\mathcal{E}_x$.

The multiplicity one part of (8.4) was originally observed by Boe and Collingwood [6]. We will see in section 9 that (8.4 iii) follows easily from (8.4 i) and (8.4 ii).

The techniques which prove theorem (8.4) are sufficient to prove several other interesting results. These are listed in the next theorem.

Theorem 8.5 *Fix x in $\mathcal{W}^{\underline{m}}$ and set $r_{\Sigma_x^+} = \prod_{\gamma\in\Sigma_x^+} s_\gamma$.*

i) $x^{-1}\Sigma_x^+$ is in $\mathcal{E}_x$ and Socle$(N(x\rho)) \cong L(\overline{r_{\Sigma_x^+}x\rho})$.

ii) $D(x\rho)$ is self-dual (cf. (2.7)).

iii) If Ω is in $\mathcal{S}_x$, then the cardinality of Ω is less than or equal to the constant p for $(\underline{g}, \underline{p})$ (cf. (8.1)).

iv) $P(x\rho)$ is self dual if and only if the cardinality of Σ_x equals the constant p for $(\underline{g}, \underline{p})$.

If Ω is in $\mathcal{S}_x$, then Ω is said to be positive (respectively negative) if $\Omega = \Omega^+$ (resp. Ω^-). If Ω is a negative set then we put $t_\Omega = \prod_{\gamma\in\Omega} s_\gamma$. For completeness we state the following result.

Theorem 8.6 *Let x and y be in $\mathcal{W}^{\underline{m}}$.*

(i) $Hom(N(x\rho), N(y\rho)) \cong \mathbb{C}$ or zero depending as $x\rho = \overline{yr_\Omega\rho}$ for some positive set $\Omega \in \mathcal{S}_y$ or not.

(ii) $Hom(N(x\rho), N(y\rho)) \cong \mathbb{C}$ or zero depending as $y\rho = \overline{xt_\Omega\rho}$ for some negative set $\Omega \in \mathcal{S}_x$ or not.

This theorem, proved by Brian Boe and the authors, will appear in [8]. The result relies on the work of R. Irving [20] and B. Boe and D. Collingwood [7].

The proofs of Theorems 8.4 and 8.5 occupy sections nine through thirteen. We will briefly outline the arguments. If $\underline{g}$ is of type HS.2 or HS.4 then the structure of the category $\mathcal{O}(\underline{g}, \underline{p}, \rho)$ is very simple and is given explicitly in [16]. Theorems (8.4) and (8.5) can be read off directly from there. In this article we will complete the proofs of (8.4) and (8.5) by addressing the cases HS.1, HS.3 and HS.5. Most of the arguments will proceed by induction on the constant p for $(\underline{g}, \underline{p})$ using the equivalences of categories proved in Part I.

In section 9 we give some technical lemmas needed to keep track of our combinatorics. We also prove that (8.4 i) and (8.4 ii) imply the first claim of (8.4 iii) (cf (9.6)) and we prove the last claim of (8.4 iii) directly (cf. (9.7)). Section 10 contains an equivalence of categories based on a result of D. Vogan and section 11 gives another equivalence of categories based on the results in Part I. Theorem (8.4) is proved in section 12. Also an interesting corollary on the structure of the semiregular category $\mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_\beta)$ is given when $(\underline{g}, \underline{p})$ is of type HS.3 and ω_β is the fundamental weight corresponding to the long simple root. Here a parity is uncovered and the category splits as the sum of even and odd parts. Section 13 contains the proof of (8.5).

In sections fourteen and fifteen we turn to calculation of Ext groups and Kazhdan-Lusztig polynomials. Section fourteen includes the proof of:

Theorem 8.7 *Let y and w be in $\mathcal{W}^{\underline{m}}$. Then:*

$$Ext^1(N_y, L_w) = \begin{cases} \mathbb{C} & \text{if } w = \overline{s_\gamma y} \text{ for some } \gamma \in \Sigma_y \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently,

$$Ext^1(N_y, L_w) = \begin{cases} \mathbb{C} & \text{if } y = \overline{wr_\Omega} \text{ for some } \Omega \in \mathcal{E}_w \text{ with } card(\Omega^+)=1 \\ 0 & \text{otherwise.} \end{cases}$$

This theorem is a special case of a much more general result given as Theorem 14.9 which presents formulas for $\mathrm{Ext}^j(N_y, L_w)$ for all $j \in \mathbb{N}$.

Following the articles of Kazhdan and Lusztig [23] and Vogan [34], we define what we call KLV polynomials $Q_{y,w}(q)$ in section fifteen. Roughly speaking these are the Poincaré polynomials for the category $\mathcal{O}(\underline{g}, \underline{p}, \rho)$. The correspondence with the standard Kazhdan-Lusztig polynomials is given as Lemma 15.3. There are two main results here. The first is the determination of a simple recursion formula which uniquely defines these polynomials. Let $y, w \in \mathcal{W}^{\underline{m}}$ and let β be a simple root. Suppose $ys_\beta \prec y$. Then in Theorem 15.4 we describe explicitly the two options: either $Q_{y,w}$ equals $Q_{ys_\beta,w}$ or the difference $Q_{y,w}$ minus $Q_{ys_\beta,w}$ equals q^r times a KLV polynomial for a lower rank Hermitian symmetric pair. This generalizes to the cases HS.3 and HS.5, the recursion relations found by Lascoux and Schützenberger [26] for the case HS.1. The second result is Theorem 15.5 which gives explicit formulas for the KLV polynomials in terms of a combinatorial notion called chains (cf. Definition 14.8).

The article ends with a description of the decomposition theory of self-dual $U(\underline{u}^-)$-free $\underline{g}$-modules. Here the main results concern canonical decompositions into indecomposable submodules and signature results for Hermitian forms.

§9. Additional notation and preliminary results

In this section we collect some technical information to be used in later proofs. We assume throughout that $(\underline{g}, \underline{p})$ is one of the Hermitian symmetric pairs HS.1, HS.3 or HS.5 with rank n. We continue with the notation of the previous section.

We will use the usual Bourbaki notation for the roots of $\underline{g}$. If $\underline{g}$ is of type HS.1 then we have:

$$\begin{gathered}\Delta^+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\} \\ \Delta(\underline{u}) = \{e_i - e_j \mid 1 \leq i \leq p, p+1 \leq j \leq n+1\}.\end{gathered}$$

For $\underline{g}$ of type HS.3 we have:

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leq i \leq j \leq n\} \setminus \{0\} \text{ and } \Delta(\underline{u}) = \{e_i + e_j \mid 1 \leq i \leq j \leq n\}.$$

And finally for $\underline{g}$ of type HS.5 we have:

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \text{ and } \Delta(\underline{u}) = \{e_i + e_j \mid 1 \leq i < j \leq n\}.$$

In each case the first $n-1$ simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$. If $\underline{g}$ is HS.1 (respectively HS.3, HS.5) then the last simple root is $\alpha_n = e_n - e_{n+1}$ (resp. $2e_n$, $e_{n-1} + e_n$).

Let x be in $\mathcal{W}^{\underline{m}}$. We set $\Psi_x^- = \Delta(\underline{u}) \cap -x\Delta^+$ and $\Psi_x^+ = \Delta(\underline{u}) \cap x\Delta^+$. For any γ in Δ, set $\hat{\gamma} = \gamma$ if $\gamma \in \Delta^+$ and $\hat{\gamma} = -\gamma$ if $\gamma \in -\Delta^+$. We define a partial order $\leq_x$ on $\Delta(\underline{u})$ by $\gamma \leq_x \nu$ if and only if $\widehat{x^{-1}\gamma} \leq \widehat{x^{-1}\nu}$.

Lemma 9.1 *Fix y in $\mathcal{W}^{\underline{m}}$. Let Ω be any subset of Ψ_y^- (resp. Ψ_y^+) that is maximal with respect to inclusion and satisfies the following conditions:*

(a) If γ and ν are in Ω and $\gamma \neq \nu$ then $(\gamma, \nu) \notin \mathcal{M}$.

(b) If γ is in Ω and ξ is in Ψ_y^- (resp. Ψ_y^+) with $(\gamma, \xi) \in \mathcal{M}$ and $\xi \leq_y \gamma$ then there is a ζ in Ω with $(\zeta, \xi) \in \mathcal{M}$ and $\zeta \leq_y \xi$.
Then $\Omega = \Sigma_y$ (resp. Σ_y^+).

Proof: Assume that Ω is contained in Ψ_y^-. Set $\Omega_i = \{\gamma \in \Omega | \ \langle y\rho, \gamma^\vee \rangle = -i\}$. Then $\Omega = \cup_{i>0}\Omega_i$. Suppose that Ω is not Σ_y. Let t be the smallest positive integer for which $\Omega_t \neq \Sigma_t$ so that $\cup_{0<i<t}\Omega_i = \cup_{0<i<t}\Sigma_i$. Fix γ in Ω_t. By a.) and (8.2), Ω is an orthogonal set with at most one long root. Thus γ is orthogonal to $\cup_{0<i<t}\Sigma_i$ and there is at most one long root in $\cup_{0<i<t}\Sigma_i$. So γ is in Σ_t and $\Omega_t \subset \Sigma_t$. Next choose a λ in $\Sigma_t \setminus \Omega_t$. We will show that the set $\Omega' = \Omega \cup \{\lambda\}$ satisfies conditions a.) and b.). This will contradict the maximality of Ω and complete the proof of (9.1).

First, suppose that for some i there is a γ in Ω_i with $(\gamma, \lambda) \in \mathcal{M}$. If $i \leq t$ then γ is in Σ_i and so $\langle \gamma, \lambda \rangle = 0$ and γ and λ are not both long roots. This contradicts (8.2). Thus $i \geq t$. It follows that $\lambda \leq_y \gamma$. Using b.), there is a $\zeta \in \Omega$ with $(\zeta, \lambda) \in \mathcal{M}$ and $\zeta \leq_y \lambda$. But this implies that ζ is in Ω_j for some $j < t$. In turn both λ and ζ lie in Σ_y and $(\zeta, \lambda) \in \mathcal{M}$. This contradiction shows that Ω' satisfies condition a.).

Now suppose that there is a ξ in Ψ_y^- with $\xi \leq_y \lambda$, $\xi \neq \lambda$ and $(\xi, \lambda) \in \mathcal{M}$. Set $s = -\langle y\rho, \xi \rangle$ and note that $s < t$. If ξ is in Σ_y then set $\zeta = \xi$. By minimality of t, ζ would then be in Ω'. On the other hand, if ξ is not in Σ_y then either ξ is not orthogonal to $\cup_{0<i<s}\Sigma_i$ or ξ is long and there is another long root in $\cup_{0<i<s}\Sigma_i$. Either way, by (8.2) there is a ζ in $\cup_{0<i<s}\Sigma_i$ with $(\zeta, \xi) \in \mathcal{M}$. Since $(\zeta, \xi) \in \mathcal{M}$ and $-\langle y\rho, \zeta \rangle < s$, $\zeta \leq_y \xi$. This shows that Ω' satisfies condition b.) and completes the proof of (9.1) when Ω is contained in Ψ_y^-. If Ω is contained in Ψ_y^+ the proof is exactly the same.

Definition 9.2 *If α is in Δ we define $\Delta(\underline{u},\alpha) = \{\gamma \in \Delta(\underline{u}) | (\gamma,\alpha) \notin \mathcal{M}\}$. If α is a simple root in Δ^+, let ω_α be the corresponding fundamental weight and set $\mathcal{W}_\alpha = \{w \in \mathcal{W}^{\underline{m}} | w \prec ws_\alpha$ and $w(\rho - \omega_\alpha) \in \mathcal{P}_{\underline{m}}\} = \{w \in \mathcal{W}^{\underline{m}} \mid w \prec ws_\alpha, ws_\alpha \in \mathcal{W}^{\underline{m}}\}$. $\mathcal{W}_\alpha$ parameterizes the simple modules in $\mathcal{O}(\underline{g},\underline{p},\rho-\omega_\alpha)$. Fix $y \in \mathcal{W}_\alpha$. Then we take $\Psi^{\pm}_{y,\alpha} = \Psi^{\pm}_y \cap \Delta(\underline{u},-y\alpha)$, $\Sigma_{y,\alpha} = \Sigma_y \cap \Delta(\underline{u},-y\alpha)$, and $\Sigma^+_{y,\alpha} = \Sigma^+_{ys_\alpha} \cap \Delta(\underline{u},-y\alpha)$. Set $\mathcal{A}_{y,\alpha} = \{\overline{s_{\gamma_1}\cdots s_{\gamma_t} y} | \gamma_1,\ldots,\gamma_t \in \Sigma_{y,\alpha}\}$. Note that $\Sigma_y = \{-y\alpha\} \cup \Sigma_{y,\alpha}$.*

Lemma 9.3 *Let α be a simple root in Δ and fix y in $\mathcal{W}_\alpha$. Let Ω be any subset of $\Psi^-_{y,\alpha}$ (resp. $\Psi^+_{y,\alpha}$) that is maximal and satisfies the following conditions:*

(a) If γ,ν are in Ω, $\gamma \neq \nu$ then $(\gamma,\nu) \notin \mathcal{M}$.

(b) If γ is in Ω and ξ is in $\Psi^-_{y,\alpha}$ (resp. $\Psi^+_{y,\alpha}$) with $(\gamma,\xi) \in \mathcal{M}$ and $\xi \leq_y \gamma$ then there is a ζ in Ω with $(\zeta,\xi) \in \mathcal{M}$ and $\zeta \leq_y \xi$.

Then $\Omega = \Sigma_{y,\alpha}$ (resp. $\Sigma^+_{y,\alpha}$).

Proof: Assume that Ω is contained in $\Psi^-_{y,\alpha}$. Set $\Omega' = \Omega \cup \{-y\alpha\}$. By a. and the definition of $\Psi^-_{y,\alpha}$ it is clear that Ω' satisfies condition a. of (9.1). We must show that Ω' satisfies condition b. of (9.1). Suppose that γ is in Ω' and ξ is in Ψ^-_y with $\xi \leq_y \gamma$, $\xi \neq \gamma$ and $(\xi,\gamma) \in \mathcal{M}$. Since α is simple, $\gamma \neq -y\alpha$ and thus γ is in Ω. If ξ is in $\Delta(\underline{u}, y\alpha)$ then by b., there is a ζ in Ω with $\zeta \leq_y \xi$ and $(\zeta,\xi) \in \mathcal{M}$. If $\xi \notin \Delta(\underline{u},-y\alpha)$ then $\zeta = -y\alpha$ satisfies $(\xi,\zeta) \in \mathcal{M}$. If $\langle\xi,\zeta\rangle \neq 0$ then $\langle\xi,\zeta\rangle > 0$ since both roots are in $\Delta(\underline{u})$ and hence, since α is simple, $\zeta \leq_y \xi$. If ξ and ζ are orthogonal then by (8.2) they must both be long roots. In this case $\zeta \leq_y \xi$ again by the simplicity of ζ. Either way, Ω' satisfies condition b. of (9.1), and so by (9.1), $\Omega' = \Sigma_y$. Thus $\Omega = \Sigma_{y,\alpha}$. The argument for $\Psi^+_{y,\alpha}$ is the same. This completes the proof of (9.3).

Definition 9.4 *Let α be a simple root and recall the definition of $\mathcal{S}(\Delta^+) = \mathcal{S}$. Set $\Delta(\alpha) = \{\gamma \in \Delta \mid (\gamma,\alpha) \notin \mathcal{M}\}$. From (8.2) we see that $\Delta^+(\alpha) = \Delta(\alpha) \cap \Delta^+$ is a positive system for a root system. We define $\mathcal{S}_\alpha = \mathcal{S}(\Delta^+(\alpha))$. If x is in $\mathcal{W}^{\underline{m}}$ then we define $\mathcal{S}_{x,\alpha} = \{\Omega \in \mathcal{S}_\alpha \mid x\Omega \subseteq \Delta(\underline{u}) \cup -\Delta(\underline{u})\}$.*

We remark that if Δ is A_n (resp. D_n) then $\Delta(\alpha)$ is A_{n-2} (resp. $\mathrm{D}_{n-2}\times\mathrm{A}_1$) for every simple root α. If Δ is C_n then $\Delta(\alpha)$ is $\mathrm{C}_{n-2}\times\mathrm{A}_1$ for every short simple root and D_{n-1} when α is the long simple root.

Lemma 9.5 *Let α be a simple root and fix x in $\mathcal{W}_\alpha$. Then the map $\Omega \to \Omega \cup \{\alpha\}$ is an injection from $\mathcal{S}_{x,\alpha}$ to $\mathcal{S}_x$.*

Proof: Fix Ω in $\mathcal{S}_{x,\alpha}$. Set $\Omega' = \Omega \cup \{\alpha\}$. Choose γ and ν in Ω' with $\gamma \neq \nu$. We must first show that $(\gamma,\nu) \notin \mathcal{M}$. By the definition of $\Delta(\alpha)$ we may assume that $\gamma \neq \alpha$ and $\nu \neq \alpha$, i.e. $\gamma,\nu \in \Omega$. But then $(\gamma,\nu) \notin \mathcal{M}$ by the definition of $\mathcal{S}_{x,\alpha}$.

Now suppose that γ is in Ω' and ξ is in Δ^+ with $\xi \neq \gamma, (\xi,\gamma) \in \mathcal{M}$ and $\xi \leq \gamma$. We must find a $\zeta \in \Omega'$ with $\zeta \neq \gamma$, $(\zeta,\xi) \in \mathcal{M}$ and $\zeta \leq \gamma$. By minimality, $\gamma \neq \alpha$. Suppose that $(\alpha,\xi) \notin \mathcal{M}$, i.e. $\xi \in \Delta^+(\alpha)$. Then, since $(\xi,\gamma) \in \mathcal{M}$ and $\xi \leq \gamma$, either $\gamma - \xi$ is a root in $\Delta^+(\alpha)$ or γ and ξ are both long roots. Either way, $\xi \leq \gamma$ in $\Delta^+(\alpha)$. Thus by hypotheses, there is a ζ in Ω with $\zeta \neq \gamma, (\zeta,\xi) \in \mathcal{M}$ and $\zeta \leq \gamma$ in $\Delta^+(\alpha)$ (and thus also in Δ^+). Thus we may assume that $(\alpha,\xi) \in \mathcal{M}$. Set $\gamma = \sum a_\beta \beta$ and $\xi = \sum b_\beta \beta$ where both sums are taken over all simple roots β. Then since $\xi \leq \gamma$, $a_\beta \geq b_\beta \geq 0$ for all β. We claim that $\alpha \leq \gamma$. Suppose not, i.e. $a_\alpha = 0$. Then since α and γ are orthogonal, $a_\beta = b_\beta = 0$ for all β that are not orthogonal to α. This tells us that α and ξ are orthogonal. By (8.2) $(\underline{g},\underline{p})$ must be HS.3 and α and ξ must both be long roots. But then $\alpha = 2e_n$ and $\xi = 2e_k$ for some k and we have $\alpha \leq \xi \leq \gamma$. Hence $\alpha \leq \gamma$ and we may take $\zeta = \alpha$. This shows that Ω' is in $\mathcal{S}_x$ and completes the proof of (9.5).

We next show that the first claim of part iii of Theorem 8.4 is a consequence of parts i and ii. For ξ and λ in $\mathcal{P}_{\underline{m}}$ we denote the multiplicity $(N(\lambda) : L(\xi))$ by $m(\xi,\lambda)$.

Proposition 9.6 *Suppose that for each x in $\mathcal{W}^{\underline{m}}$, $-x^{-1}\Sigma_x$ is in $\mathcal{S}_x$ and $m(x\rho,\mu)$ is one or zero depending as $\mu \in \mathcal{A}_x\rho$ or not. Fix y in $\mathcal{W}^{\underline{m}}$. Then $m(\zeta, y\rho)$ is one or zero depending as $\zeta = \overline{yr_\Omega\rho}$ for some Ω in $\mathcal{E}_y$ or not.*

Proof: It suffices to show that $m(x\rho, y\rho) = 1$ if and only if $x = \overline{yr_\Omega}$ for some Ω in $\mathcal{E}_y$. Suppose first that $m(x\rho, y\rho) = 1$. Then y is in $\mathcal{A}_x$. That is, there are $\gamma_1, \ldots, \gamma_t$ in Σ_x with $y = \overline{s_{\gamma_1} \cdots s_{\gamma_t} x}$. Define Ω to be $\{\zeta \in -x^{-1}\Sigma_x \mid \zeta \leq -x^{-1}\gamma_i \text{ for some } i\}$. We claim that Ω is in $\mathcal{E}_y$ and $x = \overline{yr_\Omega}$.

Since Ω is contained in $-x^{-1}\Sigma_x$ and $-x^{-1}\Sigma_x$ is in $\mathcal{S}_x$, Ω immediately satisfies condition (8.2 i). Let γ be in Ω and ξ be in Δ^+ with $\gamma \neq \xi$, $(\gamma, \xi) \in \mathcal{M}$ and $\xi \leq \gamma$. Since $-x^{-1}\Sigma_x$ is in $\mathcal{S}_x$ there is a ζ in $-x^{-1}\Sigma_x$ with $\zeta \neq \gamma$, $(\zeta, \xi) \in \mathcal{M}$ and $\zeta \leq \gamma$. But $\gamma \leq -x^{-1}\gamma_i$ for some i and so $\zeta \leq -x^{-1}\gamma_i$. Thus ζ is in Ω and Ω satisfies (8.2 ii). Thus Ω is in $\mathcal{S}$.

Fix γ in Ω and fix r in $\mathcal{W}_{\underline{m}}$ with $y = rs_{\gamma_1} \cdots s_{\gamma_t} x$. Then $-x\gamma$ is in $\Delta(\underline{u})$, and so $-rx\gamma$ is also in $\Delta(\underline{u})$. By the orthogonality of Σ_x, we have $y\gamma = -yx^{-1}(-x\gamma) = \pm rx\gamma$, and so $y\gamma$ is in $\pm\Delta(\underline{u})$. Moreover $y\gamma \in \Delta(\underline{u})$ if and only if $\gamma = -x^{-1}\gamma_i$ for some i, $1 \leq i \leq t$. Also, $\gamma \leq -x^{-1}\gamma_i$ for some i. But $y(-x^{-1}\gamma_i) = r\gamma_i$ which is in $\Delta(\underline{u})$. These two observations show that Ω is in $\mathcal{E}_y$. By the definition of y, we have: $x = r^{-1} y s_{x^{-1}\gamma_1} \cdots s_{x^{-1}\gamma_t}$. But $\Omega^+ = \{-x^{-1}\gamma_1, \ldots, -x^{-1}\gamma_t\}$ and so $x = \overline{yr_\Omega}$.

Conversely, suppose that $x = \overline{yr_\Omega}$ for some Ω in $\mathcal{E}_y$. Set $w = \prod_{\gamma\in\Omega^+} s_{x\gamma}$. Then $y = \overline{wx}$ and it suffices to show that $\overline{wx}$ is in $\mathcal{A}_x$. Set $\Omega' = -x\Omega$. We claim that $\Omega' \subseteq \Sigma_x$. This claim will prove that $\overline{wx}$ is in $\mathcal{A}_x$. To prove the claim we use Lemma (9.1). Choose $m \in \mathcal{W}_{\underline{m}}$ so that $x = myr_\Omega$. Then $\Omega' = -myr_\Omega\Omega = -my(\Omega^- \cup -\Omega^+)$ and thus $\Omega' \subseteq \Delta(\underline{u})$. Since $\Omega \subseteq \Delta^+$, we get $\Omega' \subseteq \Psi_x^-$. It is clear that Ω' satisfies (9.1 a) since Ω is in $\mathcal{E}_y$. We must verify (9.1 b) for Ω' (with respect to x). Fix $\gamma \in \Omega'$ and $\xi \in \Psi_x^-$ and suppose that $(\gamma, \xi) \in \mathcal{M}$, $\gamma \neq \xi$ and $\xi \leq_x \gamma$. Then $-x^{-1}\xi \leq -x^{-1}\gamma$, $-x^{-1}\xi \neq -x^{-1}\gamma$ and $(-x^{-1}\xi, -x^{-1}\gamma) \in \mathcal{M}$. Thus by (8.2) there is a ζ in Ω with $\zeta \neq -x^{-1}\gamma$,

$(\zeta, -x^{-1}\xi) \in \mathcal{M}$ and $\zeta \leq -x^{-1}\gamma$. Choose ζ to be minimal with respect to $\leq$ among the elements of Ω which have these three properties. Set $\zeta' = -x\zeta$. Then $\zeta' \neq \gamma$, $(\zeta', \xi) \in \mathcal{M}$ and $\zeta' \leq_x \gamma$. For (9.1 b) it remains to see that $\zeta' \leq_x \xi$. Since ζ' and ξ are both in Ψ_x^- ($\Omega' \subseteq \Psi_x^-$) and $(\zeta', \xi) \in \mathcal{M}$, we see from (8.2) that either $\langle \zeta', \xi \rangle = 1$ or ζ' and ξ are both long roots. In particular, either $\zeta' \leq_x \xi$ or $\xi \leq_x \zeta'$. If the later occurs then we may repeat the argument with ζ' in place of γ and obtain a contradiction to the minimality of ζ. Thus $\zeta' \leq_x \xi$ and Ω' satisfies (9.1 b). By (9.1), $\Omega' \subseteq \Sigma_x$. This completes the proof of (9.6).

Finally we prove the second assertion in (8.4 ii and iii).

Proposition 9.7 *Fix x and y in $\mathcal{W}^{\underline{m}}$.*

(i) The mapping of subsets of Σ_x into $\mathcal{A}_x$ given by $\{\gamma_1, \ldots, \gamma_t\} \longmapsto \overline{\prod_i s_{\gamma_i} x}$ is bijective.

(ii) The mapping $\Omega \longmapsto \overline{yr_\Omega}$ is injective on $\mathcal{E}_y$.

Proof: Let $\{\gamma_1, \ldots, \gamma_t\}$ and $\{\zeta_1, \ldots, \zeta_s\}$ be subsets of Σ_x with $\overline{\prod s_{\gamma_i} x} = \overline{\prod s_{\zeta_j} x}$. Then $m = \prod s_{\gamma_i} \prod s_{\zeta_j}$ is in $\mathcal{W}_{\underline{m}}$ and is a product of orthogonal reflections from $\Delta(\underline{u})$. If s_β is one of these reflections that is not repeated then $m\beta = -\beta$ which contradicts $m \in \mathcal{W}_{\underline{m}}$. Thus $m = 1$ and $\{\gamma_1, \ldots, \gamma_t\} = \{\zeta_1, \ldots, \zeta_s\}$. This proves (i).

Suppose that Ω, Ω_1 are in $\mathcal{E}_y$ and $x = \overline{yr_\Omega} = \overline{yr_{\Omega_1}}$. Then $y = \overline{xr_\Omega} = \overline{xr_{\Omega_1}}$ and, as in the proof of (9.6), $-x\Omega$ and $-x\Omega_1$ are subsets of Σ_x. By part i we see $-x\Omega^+ = -x\Omega_1^+$. Thus $-x\Omega = \{\gamma \in \Sigma_x \mid -x^{-1}\gamma \leq \nu \text{ for some } \nu \in \Omega^+\} = -x\Omega_1$. This completes the proof of (9.7).

Remark 9.8 We wish to draw some further conclusions from (9.7) and the proof of (9.6). Let y be in $\mathcal{W}^{\underline{m}}$ and choose $\Omega \in \mathcal{S}_y$. Set $x = \overline{yr_\Omega}$. Then the last paragraph of the proof of (9.6) shows:

(a) $\Omega \in \mathcal{S}_x$ and $-x\Omega \subseteq \Sigma_x$.

Choose $\Gamma \subseteq -y^{-1}\Sigma_y$. Set $\hat{\Gamma} = \{\zeta \in -y^{-1}\Sigma_y | \zeta \leq \gamma$ for some $\gamma \in \Gamma\}$. The second paragraph of the proof of (9.6) shows:

(b) If $-y^{-1}\Sigma_y \in \mathcal{S}_y$ then $\hat{\Gamma} \in \mathcal{S}_y$.

Suppose that $-y^{-1}\Sigma_y \in \mathcal{S}_y$. Set $w = \overline{y \prod_{\gamma \in \Gamma} s_\gamma}$. Then, again by the proof of (9.6), $\hat{\Gamma} \in \mathcal{E}_w$. However, if Γ is in $\mathcal{S}_y$ then clearly Γ is in $\mathcal{E}_w$ also. Thus, by (9.7 ii):

(c) If $-y^{-1}\Sigma_y \in \mathcal{S}_y$ then $\Gamma \in \mathcal{S}_y$ if and only if $\Gamma = \hat{\Gamma}$.

The upshot of these observations is this. Once we have established that $-y^{-1}\Sigma_y$ is in $\mathcal{S}_y$ for all $y \in \mathcal{W}^{\underline{m}}$ (cf. section 12) then statements about the sets in $\mathcal{S}_y$ can be converted into statements about the (computable) sets Σ_x.

§10. Wall shifting

In this section we use a result of Vogan [32] to give equivalences of categories between categories $\mathcal{O}(\underline{g}, \underline{p}, \lambda)$ where λ varies among the integral semiregular points in $\underline{h}^*$. We then give the formulas that allow one to transfer specific information from one wall to another. We continue with the notation of the previous sections. We take $(\underline{g}, \underline{p})$ to be one of the pairs given in Table 8.1.

Throughout this section we fix two adjacent (nonorthogonal) simple roots α and β in Δ^+. If there is more than one root length in Δ, then we assume that α and β are both short roots.

For any simple root μ, ω_μ will denote the fundamental weight associated to μ. We set $\phi_\mu = \phi^\rho_{\rho-\omega_\mu}$ and $\psi_\mu = \psi_\rho^{\rho-\omega_\mu}$. These are the translation functors from and to the "μ-wall". Set $\mathcal{O}_\mu = \mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_\mu)$. Recall from (9.2) the definition of $\mathcal{W}_\mu$ ($\subseteq \mathcal{W}^{\underline{m}}$) which parameterizes the simple modules in $\mathcal{O}_\mu$.

Lemma 10.1 *Set $\pi = \psi_\beta \circ \phi_\alpha$. Then π is an equivalence of categories from $\mathcal{O}_\alpha$ to $\mathcal{O}_\beta$ with natural inverse $\pi' = \psi_\alpha \circ \phi_\beta$.*

Proof: Set $\lambda = \rho - \omega_\alpha$ and $\mu = \rho - \omega_\beta$. Fix x in $\mathcal{W}_\alpha$. Then $x\lambda, x\rho$ and $xs_\alpha\rho$ are all in $\mathcal{P}_{\underline{m}}$ and $x\rho \prec xs_\alpha\rho$. So $\psi_\alpha(L(x\rho)) = L(x\lambda)$. Since $\underline{u}$ is Abelian and $x\alpha \in -\Delta(\underline{u})$, we see $x\beta \notin -\Delta(\underline{u})$ and so $x \notin \mathcal{W}_\beta$. By a theorem of Vogan ([32], 3.2) there is a unique composition factor M in $\phi_\alpha(L(x\lambda))$ with $\psi_\beta\phi_\alpha(L(x\lambda)) = \psi_\beta(\mathrm{M}) \neq 0$. M has multiplicity one in $\phi_\alpha(L(x\lambda)$. Furthermore, $\mathrm{M} \cong L(xs_\alpha\rho)$ if $xs_\alpha \in \mathcal{W}_\beta$ and $\mathrm{M} \cong L(xs_\beta\rho)$ if $xs_\beta \in \mathcal{W}_\beta$. In either case, $\pi(L(x\lambda)) = \psi_\beta(\mathrm{M})$ is simple and nonzero and $\pi'(\pi(L(x\lambda)) \cong L(x\lambda))$. By the adjointness of ϕ_α and ψ_α (respectively ϕ_β and ψ_β), for any X in $\mathcal{O}_\alpha$,

$\mathrm{Hom}(X, \pi' \circ \pi X) \cong \mathrm{Hom}(\pi X, \pi X)$. Let $f_X \in \mathrm{Hom}(X, \pi' \circ \pi X)$ be the element corresponding to the identity in $\mathrm{Hom}(\pi X, \pi X)$. For any simple module L in $\mathcal{O}_\alpha$, $\pi L \neq 0$ and thus f_L is nonzero. Since $\pi' \circ \pi L$ is simple, f_L is thus an isomorphism for each simple module L in $\mathcal{O}_\alpha$. Arguing by induction on length of a composition series and using the exactness of $\pi' \circ \pi$, it is easily seen that f_X is an isomorphism for all X. A short exercise shows that f_X is also natural and is thus a natural equivalence between the identity functor and $\pi' \circ \pi$. The same argument shows $\pi' \circ \pi$ is naturally equivalent to the identity on $\mathcal{O}_\beta$. This proves (10.1).

Definition 10.2 *Recall that α and β are adjacent short simple roots. Define a bijection $\pi : \mathcal{W}_\alpha \to \mathcal{W}_\beta$ by:*

$$\pi(L(y(\rho - \omega_\alpha))) \cong L(\pi(y)(\rho - \omega_\beta))$$

where $y \in \mathcal{W}_\alpha$ and $\pi(y) \in \mathcal{W}_\beta$. As in the proof of (10.1), we can give a formula for $\pi(y)$. Note that if $y \in \mathcal{W}_\alpha$ then since $\underline{u}$ is Abelian, $y\beta \notin -\Delta(\underline{u})$. If $y\beta \in \Delta(\underline{m})$ then $y \in \mathcal{W}^{\underline{m}}$ implies $y\beta \in \Delta^+(\underline{m})$. Therefore $y\beta \in \Delta^+$ and $y \notin \mathcal{W}_\beta$.

(10.3)

Case A: *If $y\beta \in \Delta^+(\underline{m})$ then $\pi(y) = ys_\alpha$.*

Case B: *If $y\beta \in \Delta(\underline{u})$ then $\pi(y) = ys_\beta$.*

Note: In both cases, $\pi(y) = \overline{ys_\beta s_\alpha}$.

Lemma 10.4 *Fix α and β as above.*

(i) $s_\alpha s_\beta(\Delta^+(\alpha)) = \Delta^+(\beta)$.

(ii) *If (γ, ν) is in $\mathcal{M}$ and $x \in \mathcal{W}$ then $(x\gamma, x\nu)$ is in $\mathcal{M}$.*

(iii) *If γ, ν are in $\Delta^+(\alpha)$ with $\gamma \le \nu$ (with respect to $\Delta^+(\alpha)$) then $s_\alpha s_\beta \gamma \le s_\alpha s_\beta \nu$ (with respect to $\Delta^+(\beta)$).*

(iv) $s_\alpha s_\beta \mathcal{S}_\alpha = \mathcal{S}_\beta$.

Proof: The first two parts of (10.4) are obvious. The third part follows from part (i) and the last assertion follows from the first three.

We can now check the action of wall shifting on our combinatorial sets.

Lemma 10.5 *Fix α and β as above and let y be in $\mathcal{W}_\alpha$.*

(i) $s_\alpha s_\beta \mathcal{S}_{y,\alpha} = \mathcal{S}_{\pi(y),\beta}$.

(ii) Set $r = \pi(y)s_\alpha s_\beta y^{-1}$. *Then r is a reflection in $\mathcal{W}_{\underline{m}}$ and* $r\Sigma_{y,\alpha} = \Sigma_{\pi(y),\beta}$. *In particular, if $-y^{-1}\Sigma_{y,\alpha}$ is in $\mathcal{S}_{y,\alpha}$ then $-\pi(y)^{-1}\Sigma_{\pi(y),\beta}$ is in $\mathcal{S}_{\pi(y),\beta}$. Similar statements hold for $\Sigma^+_{y,\alpha}$.*

(iii) Let $\gamma_1, \ldots, \gamma_t$ be mutually orthogonal roots in $\Delta(\underline{u}, y\alpha)$ and set $w = s_{\gamma_1}\cdots s_{\gamma_t}$. *Then $\overline{wy} \in \mathcal{W}_\alpha$ and* $\pi(\overline{wy}) = \overline{rwr\pi(y)} = \overline{wys_\beta s_\alpha}$. *In particular,* $\pi(\mathcal{A}_{y,\alpha}) = \mathcal{A}_{\pi(y),\beta}$.

Proof: If y is in case A; $y\beta \in \Delta(\underline{m})$ and $r = ys_\beta y^{-1} = s_{y\beta} \in \mathcal{W}_{\underline{m}}$. In case B; $y(\alpha+\beta) \in \Delta(\underline{m})$ and $r = ys_\beta s_\alpha s_\beta y^{-1} = s_{y(\alpha+\beta)} \in \mathcal{W}_{\underline{m}}$. This observation, with (10.4), proves (i) and the first part of (ii).

To prove that $r\Sigma_{y,\alpha} = \Sigma_{\pi(y),\beta}$ we use lemma (9.3) (with y and α replaced by $\pi(y)$ and β respectively). Set $\Omega = r\Sigma_{y,\alpha}$. Since r is in $\mathcal{W}_{\underline{m}}$ and $ry\alpha = \pi(y)\beta$, $r\Delta(\underline{u}, y\alpha) = \Delta(\underline{u}, \pi(y)\beta)$. So $\Omega \subseteq \Psi^-_{\pi(y),\beta}$. Ω clearly satisfies (9.3 a). From the definition of r we see that $r\Psi^-_{y,\alpha} = \Psi^-_{\pi(y),\beta}$. By (10.4 iii), for $\xi, \gamma \in \Delta(\underline{u}, y\alpha)$, $\xi \le_y \gamma$ if and only if $r\xi \le_{\pi(y)} r\gamma$. Thus Ω satisfies (9.3 b) (with y and α replaced by $\pi(y)$ and β respectively). Since r gives a bijection of $\Psi^-_{y,\alpha}$ onto $\Psi^-_{\pi(y),\beta}$, Ω is maximal and (9.3) gives $\Omega = \Sigma_{\pi(y),\beta}$. This proves that $r\Sigma_{y,\alpha} = \Sigma_{\pi(y),\beta}$. The rest of part (ii) follows immediately.

Finally, let w be as in part (iii). By orthogonality, $wy\alpha = y\alpha \in -\Delta(\underline{u})$, and so $\overline{wy(\rho - \omega_\alpha)}$ has singularity outside of $\Delta(\underline{m})$. Thus $\overline{wy} \in \mathcal{W}_\alpha$. Since $r \in \mathcal{W}_{\underline{m}}$, $\overline{rwr\pi(y)} = \overline{wys_\beta s_\alpha}$. If $\overline{wy}$ is in Case A, then $wy\beta \in \Delta(\underline{m})$ and $\overline{wys_\beta s_\alpha} = \overline{wys_\alpha} = \pi(\overline{wy})$. If $\overline{wy}$ is in Case B, then $wy(\alpha+\beta) \in \Delta(\underline{m})$ and $\overline{wys_\beta s_\alpha} = \overline{wys_\beta} = \pi(\overline{wy})$. This proves the last assertion and completes (10.5).

In section fourteen we will require a linear ordering $\underset{lin}{\leq}$ on Δ^+ and an analogue of (10.4) for this ordering. Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be the usual indexing of the simple roots of Δ^+ as in Bourbaki. For $\gamma, \zeta \in \Delta^+$ write $\gamma = \sum_i a_i\alpha_i$, $\zeta = \sum_i b_i\alpha_i$. Then we write $\gamma \underset{lin}{\leq} \zeta$ when either $\gamma = \zeta$ or, for the largest index j with $a_j \neq b_j$, $a_j < b_j$.

Lemma 10.6 *Suppose $\underline{g}$ is of type HS.3 and $\alpha = e_r - e_{r+1}$. Let $\gamma, \zeta \in \Delta^+(\alpha) \setminus \{e_r + e_{r+1}\}$. Then $\gamma \underset{lin}{\leq} \zeta$ if and only if $s_\alpha s_\beta \gamma \underset{lin}{\leq} s_\alpha s_\beta \zeta$.*

Proof: We may assume $\beta = e_{r+1} - e_{r+2}$. Since γ and ζ are fixed by s_α, if $s = s_{\alpha+\beta}$ then we must show: $\gamma \underset{lin}{\leq} \zeta$ if and only if $s\gamma \underset{lin}{\leq} s\zeta$. Here s acts on roots by the permutation of indices $(r, r+2)$. Write $\zeta - \gamma = \sum_i c_i\alpha_i$, $c_i \in \mathbb{Z}$. Then $\gamma \underset{lin}{\leq} \zeta$ if and only if all c_i are zero or the largest j with $c_j \neq 0$ gives $c_j > 0$. The action of s on the simple roots α_i is as follows. If $r \leq n-3$ then s fixes all α_i except those α_i with $r-1 \leq i \leq r+2$. For these $s\alpha_{r-1} = \alpha_{r-1} + \alpha + \beta$, $s\alpha = -\beta$, $s\beta = -\alpha$ and $s\alpha_{r+2} = \alpha + \beta + \alpha_{r+2}$. For $r = n-2$, the formulas are the same except $s\alpha_n = 2\alpha + 2\beta + \alpha_n$. Now ζ and γ lie in $\Delta^+(\alpha) \setminus \{e_r + e_{r+1}\}$; and so, a short calculation yields $c_{r-1} = c_r = c_{r+1}$. It follows from the formulas above that the largest index j with $c_j \neq 0$ is positive if and only if the same property holds for $s\sum_i c_i\alpha_i = \sum_i d_i\alpha_i$. This proves the lemma.

§11. Induction from lower rank.

In this section we use the results of Part I to give an equivalence between categories of highest weight modules for two Hermitian symmetric pairs of different rank. This theorem will be the inductive step in the proofs of our main theorems. Throughout this section $(\underline{g}, \underline{p})$ is of type HS.1, HS.3 or HS.5 with rank n . We use the notation for roots of $\underline{g}$ given in section nine.

We must first define two standard parabolic subalgebras $\underline{q}_1$ and $\underline{q}_2$ of $\underline{g}$ with $\underline{q}_1 \supseteq \underline{q}_2$. Let $\underline{q}_1 = \underline{l} \oplus \underline{u}_1$ and $\underline{q}_2 = \underline{g}' \oplus \underline{u}_2$ be Levi decompositions with $\underline{u}_i = \mathrm{nilrad}(\underline{q}_i)$ and $\underline{h} \subseteq \underline{g}' \subseteq \underline{l}$. We define $\underline{q}_1$ and $\underline{q}_2$ by giving the simple roots complementary to the simple roots of $\underline{l}$ and $\underline{g}'$ (see Table 11.1). Set $\underline{p}' = \underline{g}' \cap \underline{p}$. Then $\underline{p}'$ is a maximal parabolic subalgebra of $\underline{g}'$. If $\underline{c}'$ is the center of $\underline{g}'$ then $(\underline{g}'/\underline{c}', \underline{p}'/\underline{c}')$ is also of Hermitian symmetric type and this type is also given in Table 11.1. In each case, $\underline{g}'$ is of the same type as $\underline{g}$ but with rank $n-2$. The constant p for $\underline{g}'$ (cf. (8.1)) is always exactly one less than the constant p for $\underline{g}$.

Table 11.1

$(\underline{g}, \underline{p})$	A_n, $A_{p-1} \times A_{n-p}$	C_n, A_{n-1}	D_n, A_{n-1}
Simple roots in $\Delta(\underline{u}_1)$	α_1	α_1	α_1
Simple roots in $\Delta(\underline{u}_2)$	α_1, α_n	α_1, α_2	α_1, α_2
$(\underline{g}'/\underline{c}', \underline{p}'/\underline{c}')$	A_{n-2}, $A_{p-2} \times A_{n-p-1}$	C_{n-2}, A_{n-3}	D_{n-2}, A_{n-3}

Let Δ' denote the set of roots of $\underline{g}'$ and set $\Delta'^{+} = \Delta^{+} \cap \Delta'$. Set $\underline{m}' = \underline{g}' \cap \underline{m}$ and recall the decomposition $\mathcal{W}_{\underline{l} \cap \underline{m}} = {}^{\underline{m}'}\mathcal{W}_{\underline{l} \cap \underline{m}} \cdot \mathcal{W}_{\underline{m}'}$. Let w_0 be the longest element in ${}^{\underline{m}'}\mathcal{W}_{\underline{l} \cap \underline{m}}$ and let r_0 be the longest element of $\mathcal{W}_{\underline{l}}^{\underline{g}'}$. Let ω_1 be the first fundamental weight of $\underline{g}$. Set $\mathcal{W}' = \mathcal{W}_{\underline{g}'}$.

Proposition 11.2 *There is a covariant equivalence of categories Λ:*

$$\Lambda : \mathcal{O}\,(\,\underline{g}',\underline{p}',\rho(\underline{g}')) \longrightarrow \mathcal{O}(\underline{g},\underline{p},\rho-\omega_1).$$

For each x in $\mathcal{W}'^{\underline{m}'}$:

$$\Lambda(L(\underline{g}',x\rho(\underline{g}'))) \cong L(\underline{g},w_0xr_0(\rho-\omega_1)).$$

Similar formulas hold for generalized Verma modules and projective modules. Furthermore, Λ preserves self-duality.

Proof: Set $\rho' = \rho(\underline{g}')$. Let ζ be the maximal weight in $\mathcal{W}(\rho-\omega_1)\cap\mathcal{P}_{\underline{m}}$. Set $\mathcal{O}_1 = \mathcal{O}(\underline{g}',\underline{p}',\rho')$, $\mathcal{O}_2 = \mathcal{O}_t(\underline{l},\underline{l}\cap\underline{p},\zeta)$ and $\mathcal{O}_3 = \mathcal{O}(\underline{g},\underline{p},\rho-\omega_1) = \mathcal{O}(\underline{g},\underline{p},\zeta)$. Let w_1 be the longest element of $\mathcal{W}_{\underline{g}'}$, and put $\nu = -w_1\rho(\underline{l})$ and $\mu = w_0\nu$. Set $\mathcal{O}_1{}' = \mathcal{O}(\underline{g}',\underline{p}',\nu)$ and $\mathcal{O}_2{}' = \mathcal{O}_t(\underline{l},\underline{l}\cap\underline{p},\mu)$. Let Θ_2 (respectively Θ_3) be the set of highest weights plus $\rho(\underline{l})$ (resp. ρ) which parameterize the simple modules in $\mathcal{O}_2$ (resp. $\mathcal{O}_3$).

We express root systems in their usual Euclidean coordinates as in Bourbaki, except in one case. For HS.1, we find it convenient to shift by $t(1,1,..,1)$, $t\in\mathbb{R}$. We write $\rho = (n,n-1,\ldots,0)$ in place of the more common $(\frac{n}{2},\frac{n}{2}-1,\ldots,-\frac{n}{2})$ and $\omega_1 = (1,0,\ldots,0)$ in place of $\frac{1}{n+1}(n,-1,-1,\ldots,-1)$. Our definitions give:

$$(11.3)\qquad \rho' = \begin{cases} ((n-2)/2,n-2,n-3,\ldots,0,(n-2)/2) & \text{if } \underline{g} \text{ is HS.1}\\ (0,0,n-2,\ldots,1) & \text{if } \underline{g} \text{ is HS.3}\\ (0,0,n-3,\ldots,0) & \text{if } \underline{g} \text{ is HS.5.}\end{cases};$$

$$(11.4)\qquad \nu = \begin{cases} ((n-1)/2,n-2,n-3,\ldots,0,n-1) & \text{if } \underline{g} \text{ is HS.1}\\ (0,-n+1,n-2,\ldots,2,1) & \text{if } \underline{g} \text{ is HS.3}\\ (0,-n+2,n-3,\ldots,1,0) & \text{if } \underline{g} \text{ is HS.5}\end{cases};$$

(11.5)
$$\mu = \begin{cases} ((n-1)/2,n-2,\ldots,n-p,n-1,n-p-1,\ldots,0) & \text{if } \underline{g} \text{ is HS.1}\\ (0,n-2,n-3,\ldots,1,-n+1) & \text{if } \underline{g} \text{ is HS.3}\\ (0,n-3,n-4,\ldots,0,-n+2) & \text{if } \underline{g} \text{ is HS.5}\end{cases}$$

and

$$(11.6)\quad \zeta = \begin{cases} (n-1, n-2, \dots, n-p, n-1, n-p-1, \dots, 0) & \text{if } \underline{g} \text{ is HS.1} \\ (n-1, n-2, \dots, 2, 1, -n+1) & \text{if } \underline{g} \text{ is HS.3} \\ (n-2, n-3, \dots, 1, 0, -n+2) & \text{if } \underline{g} \text{ is HS.5.} \end{cases}$$

Let C (respectively D) be the one dimensional representation of the center $\underline{c}'$ of $\underline{g}'$ (resp. the center of $\underline{l}$) whose weight is the restriction of $\nu - \rho'$ (resp. $\zeta - \mu$). Let Λ_1 (respectively Λ_3) be the functor of tensoring with C (resp. D). From the above computations of the weights, we see that Λ_1 is an equivalence from $\mathcal{O}_1$ to $\mathcal{O}_1{}'$ and Λ_3 is an equivalence from $\mathcal{O}_2{}'$ to $\mathcal{O}_2$. The category $\mathcal{O}_1{}'$ satisfies the hypotheses of section seven. Let Λ_2 be the equivalence I from $\mathcal{O}_1{}'$ to $\mathcal{O}_2{}'$ given in (7.16) with $\underline{g} = \underline{l}$, $\underline{l} = \underline{g}'$ and $\underline{q} = \underline{q}_2 \cap \underline{l}$. Finally, an easy check in coordinates shows that $\Theta_2 = \Theta_3$; and so, by (6.6) there is an equivalence $\Lambda_4 = \tau$ from $\mathcal{O}_2$ to $\mathcal{O}_3$. Set $\Lambda = \Lambda_4 \circ \Lambda_3 \circ \Lambda_2 \circ \Lambda_1$. Then Λ is an equivalence of categories and the properties of Λ follow from (7.16), (6.7), (6.14) and (6.15). In particular, suppose that $\Lambda(L(\underline{g}', x\rho')) = L(\underline{g}, y(\rho - \omega_1))$ with

$$x\rho' = \begin{cases} ((n-2)/2, a_1, a_2, \dots, a_{n-1}, (n-2)/2) & \text{if } \underline{g} \text{ is HS.1} \\ (0, 0, a_1, a_2, \dots, a_{n-2}) & \text{if } \underline{g} \text{ is HS.3 or HS.5.} \end{cases}$$

Then

$$y(\rho - \omega_1) = \begin{cases} (n-1, a_1, \dots, a_p, n-1, a_{p+1}, \dots, a_{n-1}) & \text{if } \underline{g} \text{ is HS.1} \\ (n-1, a_1 \quad, \dots, a_{n-2}, -n+1) & \text{if } \underline{g} \text{ is HS.3} \\ (n-2, a_1 \quad, \dots, a_{n-2}, -n+2) & \text{if } \underline{g} \text{ is HS.5} \end{cases}$$

That is, $y = w_0 x r_0$ or $y = w_0 x r_0 s_{\alpha_1}$. This completes the proof of (11.2).

Let $\underline{p}' = \underline{m}' \oplus \underline{u}'$ be the Levi decomposition of $\underline{p}'$ with $\underline{u}' = \underline{u} \cap \underline{g}'$. Let $\mathcal{W}'$ be the Weyl group of $\underline{g}'$, Δ' the roots of $\underline{g}'$. We identify $\mathcal{W}'$ with the subgroup of $\mathcal{W}$ generated by the simple reflections s_α for $\alpha \in \Delta'$. For λ in $\underline{h}^*$, we set $L(\lambda) = L(\underline{g}, \lambda)$ and $\mathrm{L}'(\lambda) = L(\underline{g}', \lambda)$. Similar conventions will hold for generalized Verma modules and their projective covers. In general, we denote objects computed with respect to the pair $(\underline{g}', \underline{p}')$ with a "prime" ($'$) as superscript. In particular, for y in $\mathcal{W}'^{\underline{m}'}$ we have the sets Σ'_y, Σ'^+_y, $\mathcal{A}'_y$, $\mathcal{S}'_y$ and $\mathcal{E}'_y$ as defined in section eight.

For $x \in \mathcal{W}'^{\underline{m}'}$ define $\Lambda(x)$ in $\mathcal{W}_{\alpha_1}$ by: $\Lambda(L(\underline{g}', x\rho')) = L(\underline{g}, \Lambda(x)(\rho - \omega_{\alpha_1}))$.

Lemma 11.7 *Fix $x \in \mathcal{W}'^{\underline{m}'}$ and put $\alpha = \alpha_1$ and $y = w_0 x r_0 s_\alpha$. Then $y = \Lambda(x)$ and we have:*

(i) $w_0\Sigma'_x = \Sigma_{y,\alpha}$ and $w_0{\Sigma'_x}^+ = \Sigma^+_{y,\alpha}$.

(ii) $r_0^{-1}\mathcal{S}'_x = \mathcal{S}_{y,\alpha}$.

(iii) If $-x^{-1}\Sigma'_x$ (resp. $x^{-1}{\Sigma'_x}^+$) is in $\mathcal{S}'_x$, then $-y^{-1}\Sigma_{y,\alpha}$ (resp. $y^{-1}\Sigma^+_{y,\alpha}$) is in $\mathcal{S}_{y,\alpha}$.

(iv) For $\gamma_1, \ldots, \gamma_t \in \Sigma'_x$, $\Lambda(\overline{s_{\gamma_1} \cdots s_{\gamma_t} x}) = \overline{s_{w_0\gamma_1} \cdots s_{w_0\gamma_t}\Lambda(x)}$. In particular, $\Lambda(\mathcal{A}'_x) = \mathcal{A}_{y,\alpha}$.

Proof: We know from (11.2) that $y = w_0 x r_0$ or $y = w_0 x r_0 s_\alpha$, whichever is in $\mathcal{W}_\alpha$. In case HS.1, $w_0 x r_0 \alpha = e_1 - e_{p+1}$ (independent of x) and in cases HS.3 and HS.5, $w_0 x r_0 \alpha = e_1 + e_n$. Thus $w_0 x r_0 \alpha \in \Delta(\underline{u})$. This proves that $y = w_0 x r_0 s_\alpha$.

We prove parts i) and ii) by case by case coordinate computations. Parts iii) and iv) follow immediately from i) and ii). A simple check with coordinates shows that $w_0\Delta(\underline{u}') = \Delta(\underline{u}, -y\alpha)$. For example, if $(\underline{g}, \underline{p})$ is HS.3, then $-y\alpha = e_1 + e_n$ and $w_0\Delta(\underline{u}') = w_0\{e_i + e_j \mid 3 \le i \le j \le n\} = \{e_{i-1} + e_{j-1} \mid 3 \le i \le j \le n\} = \Delta(\underline{u}, -y\alpha)$. HS.1 and HS.5 are similar. Another case by case check in coordinates shows that $r_0 s_\alpha \rho = \rho' + \zeta$ where ζ is a weight in $\underline{h}^*$ that is orthogonal to Δ'. Thus for any γ in $\Delta(\underline{u}')$, $\langle y\rho, w_0\gamma\rangle = \langle x r_0 s_\alpha \rho, \gamma\rangle = \langle x\rho', \gamma\rangle$. Now $\Sigma_{y,\alpha} = \Sigma_y \setminus \{-y\alpha\}$. Therefore, since α is simple, $w_0\Sigma'_x = \Sigma_{y,\alpha}$. This proves part i.

If $\underline{g}$ is of type HS.1 then $r_0^{-1}\Delta' = r_0^{-1}\{\pm(e_i - e_j) \mid 2 \le i < j \le n\} = \{\pm(e_i - e_j) \mid 3 \le i < j \le n+1\} = \Delta(\alpha)$. Moreover, this map induced by r_0^{-1} preserves the respective sets of positive roots. Therefore r_0^{-1} maps $\mathcal{S}'$ onto $\mathcal{S}_\alpha$. Similarly, if $\underline{g}$ is HS.3 or HS.5 then $r_0^{-1}\Delta' = \Delta(\alpha) \setminus \{e_1 + e_2\}$. Therefore, in these cases r_0^{-1} maps $\mathcal{S}'$ onto $\{\Omega \in \mathcal{S}_\alpha | e_1 + e_2 \notin \Omega\}$. Note that in the cases of HS.3 and HS.5, since $-y\alpha \in \Delta(\underline{u})$, $y(e_1 + e_2)$ cannot lie in $\Delta(\underline{u}) \cup -\Delta(\underline{u})$ and thus $e_1 + e_2$ is never an element of an element of $\mathcal{S}_{y,\alpha_1}$. Now for all three of the cases HS.1, HS.3 and HS.5, for $\Omega \in \mathcal{S}'$, $x\Omega \subseteq \Delta(\underline{u}') \cup -\Delta(\underline{u}')$ if and only if $yr_0^{-1}\Omega = w_0 x\Omega \subseteq \Delta(\underline{u}) \cup -\Delta(\underline{u})$, since $w_0 \in \mathcal{W}_{\underline{m}}$. And so, r_0^{-1} maps $\mathcal{S}'_x$ onto $\mathcal{S}_{y,\alpha}$. This proves part ii) and completes the proof of (11.7).

§12. Proof of Theorem 8.4

In this section we prove Theorem 8.4. As a corollary of the proof we give a splitting of the category $\mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_\beta)$ when $(\underline{g}, \underline{p})$ is of type HS.3 and β is the long simple root α_n.

Retain the notation of previous sections. For $1 \leq i \leq n$, let ω_i be the fundamental weight of $\underline{g}$ corresponding to the simple root α_i. Set $\mathcal{O}_i = \mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_i)$ for $1 \leq i \leq n$, $\mathcal{O} = \mathcal{O}(\underline{g}, \underline{p}, \rho)$ and $\mathcal{O}' = \mathcal{O}(\underline{g}', \underline{p}', \rho(\underline{g}'))$. If α is any simple root, we have the translation functors $\phi_\alpha = \phi^{\rho}_{\rho-\omega_\alpha}$ and $\psi_\alpha = \psi^{\rho-\omega_\alpha}_{\rho}$.

We proceed to prove (8.4) by induction on the constant p of from Table (8.1). If $(\underline{g}, \underline{p})$ is HS.1 then the induction starts with the case $(\mathrm{A}_n, \mathrm{A}_{n-1})$. In case HS.3 the induction starts with the case $(\mathrm{C}_2, \{\alpha_1\})$ (which is equivalent to the case $(\mathrm{B}_2, \{\alpha_2\})$) or with the case $(\mathrm{A}_1, \emptyset)$. For HS.5 the induction starts with either $(\mathrm{D}_3, \mathrm{A}_2)$ (which is equivalent to $(\mathrm{A}_3, \mathrm{A}_2)$) or $(\mathrm{D}_2, \mathrm{A}_1)$ (which is actually $(\mathrm{A}_1 \times \mathrm{A}_1, \mathrm{A}_1)$). In all of these $p = 1$ cases, (8.4) follows immediately from [16].

For the remainder of the section we assume inductively that (8.4) holds for the pair $(\underline{g}', \underline{p}')$ which has constant $p - 1$. In view of (9.6) and (9.7), it suffices to show the following two claims for each x in $\mathcal{W}^{\underline{m}}$:

$(12.1)_x$ $$-x^{-1}\Sigma_x \in \mathcal{S}_x$$

$(12.2)_x$ $$m(x\rho, \mu) = 1 \text{ or } 0 \text{ depending as } \mu \in \mathcal{A}_x\rho \text{ or not.}$$

We begin by proving the analogue of (8.4) for the semiregular categories $\mathcal{O}_i$ when α_i is a short root.

Lemma 12.3 *Let β be a short simple root and set $\omega = \omega_\beta$. Fix x in $\mathcal{W}_\beta$ and set $\lambda = \rho - \omega$.*

(i) $-x^{-1}\Sigma_{x,\beta}$ is in $\mathcal{S}_{x,\beta}$.

(ii) $m(x\lambda, \mu) = 1$ or zero depending as μ is in $\mathcal{A}_{x,\beta}\lambda$ or not.

Proof: First suppose that β is α_1. In this case part i) follows from (11.7 iii) and our inductive hypothesis. Part ii) follows immediately from the equivalence of categories Λ of (11.2) and the formulas in (11.7 iv).

Next suppose that β is not α_1. Working inductively, we may assume that there is a short simple root α adjacent to β and that (12.3) holds with α in place of β. By (10.1) there is an equivalence of categories π from $\mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_\alpha)$ to $\mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_\beta)$. Part i) for β now follows from (10.5 ii). Part ii) follows from (10.5 iii). This completes the proof of (12.3).

Lemma 12.4 *Let x be in $\mathcal{W}^{\underline{m}}$ and suppose that either x is the identity element or there is a short simple root α with $x \in \mathcal{W}_\alpha$. Then $(12.1)_x$ and $(12.2)_x$ both hold.*

Proof: If x is the identity, then $\Sigma_x = \emptyset \in \mathcal{S}$. Also $P(x\rho) = N(x\rho)$. This gives $(12.1)_x$ and $(12.2)_x$. Suppose that $x \in \mathcal{W}_\alpha$ and α is short. By (12.3), $-x^{-1}\Sigma_{x,\alpha}$ is in $\mathcal{S}_{x,\alpha}$ and by definition (9.2), $\Sigma_x = \Sigma_{x,\alpha} \cup \{-x\alpha\}$. From (9.5) we find $-x^{-1}\Sigma_x$ is in $\mathcal{S}_x$. Again by (12.3), $m(x(\rho - \omega_\alpha), \mu) = 1$ or 0 depending as $\mu \in \mathcal{A}_{x,\alpha}(\rho - \omega_\alpha)$ or not. However, $\phi_\alpha(P(x(\rho - \omega_\alpha)) \cong P(x\rho)$ by (4.1); and so, the Verma flag factors of $P(x\rho)$ are exactly the $N(\mu)$ for $\mu \in \mathcal{A}_{x,\alpha}\rho$ or $\mu \in \mathcal{A}_{x,\alpha}s_\alpha\rho$. Since $\Sigma_x = \Sigma_{x,\alpha} \cup \{-x\alpha\}$ we get ; $\mathcal{A}_{x,\alpha} \cup \mathcal{A}_{x,\alpha}s_\alpha = \mathcal{A}_x$. Thus $m(x\rho, \mu) = 1$ or 0 depending as $\mu \in \mathcal{A}_x\rho$ or not. This completes the proof of (12.4).

Remark 12.5 *Lemma 12.4 completes the proof of (8.4) if $(\underline{g}, \underline{p})$ is HS.1 or HS.5 since in those two cases all roots are short. Thus for the remainder of this section we assume that $(\underline{g}, \underline{p})$ is of type HS.3 with rank n. We set $\alpha = \alpha_{n-1}$ and $\beta = \alpha_n = 2e_n$.*

Let Θ_i, $1 \leq i \leq n$ (respectively Θ) be the set of highest weights plus ρ for simple modules in the category $\mathcal{O}_i$ (resp. $\mathcal{O}$). Identify $\underline{h}^*$ with $\mathbb{C}^n$ and recall the usual Bourbaki notation for weights of $\underline{g}$. Then

(12.6) $\Theta = \{(a_1, \ldots, a_n) |\ \{|a_k|\}_{k=1}^n = \{1, \ldots, n\} \text{ and } a_1 > a_2 > \ldots > a_n\}$,

(12.7) $\Theta_{n-1} = \{(b_1, \ldots, b_n) |\ \{|b_k|\}_{k=1}^n = \{n-1, \ldots, 1, 1\} \text{ and } b_1 > \ldots > b_n\}$,

(12.8) $\Theta_n = \{(b_1, \ldots, b_n) | \{|b_k|\}_{k=1}^n = \{n-1, \ldots, 1, 0\} \text{ and } b_1 > \ldots > b_n\}$.

We distinguish a set of elements ζ_j, $1 \leq j \leq n$ of Θ as follows:

(12.9) $$\zeta_j = (n, \ldots, j+1, -1, -2, \ldots, -j)$$

Define x_j, $1 \leq j \leq n$, in $\mathcal{W}^{\underline{m}}$ by $x_j \rho = \zeta_j$. Notice that the set $\{x_1, \ldots, x_n\}$ is the complement in $\mathcal{W}^{\underline{m}}$ of the elements considered in (12.4) and $x_j \in \mathcal{W}_\beta$ for each j. In particular it remains to prove $(12.1)_x$ and $(12.2)_x$ only for $x = x_j$, $1 \leq j \leq n$.

Lemma 12.10 *Let x be in $\mathcal{W}_\beta$ and assume that $x \neq x_j$, $2 \leq j \leq n$. Then*

(i) $-x^{-1}\Sigma_{x,\beta} \in \mathcal{S}_{x,\beta}$

(ii) $m(x(\rho - \omega_\beta), \mu) = 1$ or 0 depending as $\mu \in \mathcal{A}_{x,\beta}(\rho - \omega_\beta)$ or not.

Proof: If $x = x_1$ then $x(\rho - \omega_\beta)$ is dominant and i and ii are true automatically. Thus we assume $x \neq x_j$ for any j. Since $x \neq x_j$ there must be a short simple root γ with $x \in \mathcal{W}_\gamma$. By (12.4), $-x^{-1}\Sigma_x \in \mathcal{S}_x$ and $m(x\rho, \zeta) = 1$ or 0 depending as $\zeta \in \mathcal{A}_x\rho$ or not. We see immediately from this that $-x^{-1}\Sigma_{x,\beta}$ is in $\mathcal{S}_{x,\beta}$, since $\Sigma_{x,\beta} = \Sigma_x \setminus \{-x\beta\}$. Since the stabilizer of $\rho - \omega_\beta$ has order two,

$$\psi_\beta(P(x\rho)) = \psi_\beta\phi_\beta(P(x(\rho - \omega_\beta)) = P(x(\rho - \omega_\beta)) \oplus P(x(\rho - \omega_\beta)).$$

The Verma flag factors of $P(x(\rho - \omega_\beta))$ are thus exactly $N(y(\rho - \omega_\beta))$ for $y \in \mathcal{A}_x \cap \mathcal{W}_\beta$. But $\mathcal{A}_x \cap \mathcal{W}_\beta = \mathcal{A}_{x,\beta}$. This completes the proof of (12.10).

Definition 12.11 *Let $\zeta = (b_1, b_2, \ldots, b_n)$ be an element of Θ_n (see (12.8)). We say that ζ has even (respectively odd) parity if the number of negative b_i is even (resp. odd). If $\zeta = (b_1, \ldots, b_k, 0, -1, \ldots, b_n)$ then we define $\tilde{\zeta} = (b_1, \ldots, b_k, 1, 0, \ldots, b_n)$ and $\tilde{\tilde{\zeta}} = \zeta$. We remark that ζ and $\tilde{\zeta}$ always have opposite parity. If $x \in \mathcal{W}_\beta$ then we define $\tilde{x} \in \mathcal{W}_\beta$ as follows: if $x(\rho - \omega_\beta) = \zeta$ then $\tilde{x}(\rho - \omega_\beta) = \tilde{\zeta}$. We see that $\tilde{x}$ is either $xs_{\alpha+\beta}$ or $xs_{\alpha+\beta}s_\beta$.*

Set $\xi_j = x_j(\rho - \omega_\beta) = (n-1, \ldots, j, 0, -1, \ldots, -(j-1))$ for $1 \leq j \leq n$. The following lemma will complete the proof of Theorem 8.4.

Lemma 12.12 *For $1 \leq j \leq n$;*

(i) $-x_j^{-1}\Sigma_{x_j,\beta} \in \mathcal{S}_{x_j,\beta}$

(ii) $m(\xi_j, \mu) = 1$ *or 0 depending as* $\mu \in \mathcal{A}_{x_j,\beta}(\rho - \omega_\beta)$ *or not.*

(iii) $-x_j^{-1}\Sigma_{x_j} \in \mathcal{S}_{x_j}$.

(iv) $m(\zeta_j, \mu) = 1$ *or 0 depending as* $\mu \in \mathcal{A}_{x_j}\rho$ *or not.*

Proof: Fix j, $1 \leq j \leq n$, and set $x = x_j$. By direct computation,

$$-x^{-1}\Sigma_x = \{\alpha_n, e_{n-1} + e_{n-2}, e_{n-3} + e_{n-4}, \ldots, e_{k+1} + e_k\},$$

$$\text{where } k = \begin{cases} n+1-j, & \text{if } j \text{ is odd;} \\ n+2-j, & \text{if } j \text{ is even.} \end{cases}$$

Set $\Omega = -x^{-1}\Sigma_x$. We prove part (iii) by showing that Ω satisfies conditions (8.2 i) and (8.2 ii). It is clear that Ω satisfies (8.2 i) since it consists of orthogonal roots and has only one long root. Fix γ in Ω and suppose that there is a ξ in Δ^+ with $\gamma \neq \xi$, $(\gamma, \xi) \in \mathcal{M}$ and $\xi \leq \gamma$. We must produce a $\zeta \in \Omega$ with $\zeta \neq \gamma$, $(\zeta, \xi) \in \mathcal{M}$ and $\zeta \leq \gamma$. Since $\xi \leq \gamma$, $\gamma \neq \alpha_n$ and so we may write $\gamma = e_i + e_{i+1}$ for some i. If ξ is a long root then ξ must be $2e_{i+1}$ and so we may take $\zeta = \alpha_n$. If ξ is a short root then we may write $\xi = e_r \pm e_s$ where $r = i$ or $i+1$ and $s > i+1$. By inspection, there is a unique ζ in Ω equal to one of: $e_s + e_{s+1}$ or $e_{s-1} + e_s$ or $2e_s$. This is the desired ζ. So Ω satisfies (8.2 ii). This proves part (iii).

Part (i) follows at once from part (iii).

We next prove part ii. If $j = 1$ then ξ_j is dominant and part ii is clear. Thus we assume that $j \geq 2$. Set $\pi = \psi_\beta \phi_\alpha$ and $\pi' = \psi_\alpha \phi_\beta$. Then $\pi : \mathcal{O}_{n-1} \longrightarrow \mathcal{O}_n$, $\pi' : \mathcal{O}_n \longrightarrow \mathcal{O}_{n-1}$ and π and π' are adjoint to one another with respect to Hom. If ζ or $\tilde{\zeta}$ is $(b_1, \ldots, b_k, 0, -1, \ldots, b_n)$ in Θ_n then we define $\pi'(\zeta) = \pi'(\tilde{\zeta}) = (b_1, \ldots, b_k, 1, -1, \ldots, b_n)$ in Θ_{n-1}. By a theorem of Vogan ([32], 3.2), $\pi'(L(\zeta)) = \pi'(L(\tilde{\zeta})) = L(\pi'(\zeta))$ for each ζ in Θ_n; and moreover, the composition factors of $\pi(L(\pi'(\zeta)))$ are $L(\zeta)$ and $L(\tilde{\zeta})$. By the adjoint property we have $\mathrm{Hom}(\pi\pi' L(\zeta), L(\zeta)) \cong \mathrm{Hom}(\pi' L(\zeta), \pi' L(\zeta)) \neq 0$ and similarly $\mathrm{Hom}(\pi\pi' L(\zeta), L(\tilde{\zeta})) \cong \mathrm{Hom}(\pi' L(\zeta), \pi' L(\tilde{\zeta})) \neq 0$. Thus $\pi L(\pi'\zeta) = L(\zeta) \oplus L(\tilde{\zeta})$. It follows at once that $\pi\pi' P(\zeta) = P(\zeta) \oplus P(\tilde{\zeta})$ and $\pi\pi' N(\zeta) = N(\zeta) \oplus N(\tilde{\zeta})$. Combining these two facts, $m(\zeta, \mu) = m(\tilde{\zeta}, \tilde{\mu})$ for all $\zeta, \mu \in \Theta_n$.

We apply these observations to the weights ξ_j, $2 \leq j \leq n$. We have $\tilde{x}_j(\rho - \omega_\beta) = \tilde{\xi}_j = (n-1, \ldots, j, 1, 0, -2, \ldots, -(j-1))$ and $\tilde{x}_j \rho = (n, \ldots, j+1, 2, -1, -3, \ldots, -j)$. Thus by direct computation:

$$-\tilde{x}_j^{-1} \Sigma_{\tilde{x}_j} \setminus \{e_{n-2} - e_{n-1}\} = -x_j^{-1} \Sigma_{x_j} \setminus \{e_{n-2} + e_{n-1}\}.$$

Set $\lambda = 2\alpha + \beta = 2e_{n-1}$. The above formula gives: $s_\lambda(-x_j^{-1}\Sigma_{x_j}) = -\tilde{x}_j^{-1}\Sigma_{\tilde{x}_j}$. Also, for any $\mu \in \Theta_n$ and $y \in \mathcal{W}^{\underline{m}}$, $\mu = y(\rho - \omega_\beta)$ if and only if $\tilde{\mu} = \overline{y s_\lambda (\rho - \omega_\beta)}$. The last two statements together say that the assignment $y \rightarrow \overline{y s_\lambda}$ is a bijection from $\mathcal{A}_{x_j,\beta}$ to $\mathcal{A}_{\tilde{x}_j,\beta}$. In particular, μ is in $\mathcal{A}_{x_j,\beta}(\rho - \omega_\beta)$ if and only if $\tilde{\mu}$ is in $\mathcal{A}_{\tilde{x}_j,\beta}(\rho - \omega_\beta)$. However, $\tilde{x}_j$ is one of the parameters considered in Lemma (12.10). Thus $m(\tilde{\xi}_j, \tilde{\mu}) = 1$ or 0 depending as $\tilde{\mu}$ is in $\mathcal{A}_{\tilde{x}_j,\beta}(\rho - \omega_\beta)$ or not. By the previous paragraph, $m(\xi_j, \mu) = m(\tilde{\xi}_j, \tilde{\mu})$; and thus, $m(\xi_j, \mu) = 1$ or 0 depending as μ is in $\mathcal{A}_{x_j,\beta}(\rho - \omega_\beta)$ or not. This proves part (ii).

Part (iv)follows from part (ii) exactly as in the proof of (12.4). This completes the proof of (12.12).

Lemma 12.12 marks the completion of the proof of Theorem 8.4.

Definition 12.13 *Let $\mathcal{O}_\beta(even)$ (respectively $\mathcal{O}_\beta(odd)$) be the full subcategory of $\mathcal{O}_\beta = \mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_\beta)$ consisting of all modules whose composition factors all have highest weights plus ρ of even (resp. odd) parity.*

As before set $\pi = \psi_\beta \phi_\alpha$ and $\pi' = \psi_\alpha \phi_\beta$.

Corollary 12.14 *The restriction of the functor π' to $\mathcal{O}_\beta(even)$ (respectively $\mathcal{O}_\beta(odd)$) is an equivalence of categories onto $\mathcal{O}_\alpha$. Moreover, the functor Γ : $\mathcal{O}_\beta(even) \times \mathcal{O}_\beta(odd) \longrightarrow \mathcal{O}_\beta$ given by $\Gamma(M, M') = M \oplus M'$ is an equivalence of categories.*

Proof: Let ζ and μ be in Θ_n. From (12.10 ii) and (12.12 ii), if $m(\zeta, \mu) \neq 0$ then ζ and μ must have the same parity. This implies $N(\zeta)$ and $P(\zeta)$ are both objects in $\mathcal{O}_\beta(even)$ or $\mathcal{O}_\beta(odd)$ depending as ζ is even or odd. Define o:$\mathcal{O}_\beta \longrightarrow \mathcal{O}_\beta(odd)$ by taking oM to be the maximal submodule of M that is an object in $\mathcal{O}_\beta(odd)$. Define e:$\mathcal{O}_\beta \longrightarrow \mathcal{O}_\beta(even)$ similarly. Now for any projective module P in $\mathcal{O}_\beta$, P = eP$\oplus$oP and thus, using a projective cover, M = eM$\oplus$oM for any module M in $\mathcal{O}_\beta$. In particular e and o are exact. Since there can be no nonzero morphisms between modules in $\mathcal{O}_\beta(even)$ and modules in $\mathcal{O}_\beta(odd)$, Γ is an equivalence with natural inverse (e,o).

Arguing exactly as in (10.1), using the adjointness of π and π', we see that the functors $\pi' \circ e \circ \pi$ and $e \circ \pi \circ \pi'$ are naturally equivalent to the identity functors on $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta(even)$ respectively. This shows that $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta(even)$ are equivalent. The argument for $\mathcal{O}_\beta(odd)$ is the same. This completes (12.14).

The functor $\pi' : \mathcal{O}_\beta \to \mathcal{O}_\alpha$ induces a two-to-one surjection $\tau : \mathcal{W}_\beta \to \mathcal{W}_\alpha$ given by $\pi'(L(y(\rho - \omega_\beta))) \cong L(\tau y(\rho - \omega_\alpha))$. This map satisfies $\tau y = \tau \tilde{y}$ for all $y \in \mathcal{W}_\beta$. If $y\alpha \in \Delta(\underline{u})$ then $\tau y = y s_\alpha$ and if $y\alpha \in \Delta(\underline{m})$ then $\tau y = y s_\beta$. We define a surjective map, also denoted τ, from $\Delta^+(\beta)$ to $\Delta^+(\alpha) \setminus \{e_{n-1} + e_n\}$ as follows:

$$\text{For } 1 \leq i < j \leq n-1, \quad \tau(e_i \pm e_j) = \begin{cases} e_i \pm e_j & \text{if } j \neq n-1 \\ 2e_i & \text{if } j = n-1 \end{cases}.$$

(Recall that $\Delta^+(\beta)$ contains only short roots.)

Lemma 12.15 *Let $(\underline{g}, \underline{p})$ be of type HS.3 with rank $n \geq 2$. Fix $y \in \mathcal{W}_\beta$ and put $w = \tau y$. Then τ is a bijection from $-y^{-1}\Sigma_{y,\beta}$ (respectively $y^{-1}\Sigma^+_{y,\beta}$) to $-w^{-1}\Sigma_{w,\alpha}$ (resp. $w^{-1}\Sigma^+_{w,\alpha}$).*

Proof: We prove the first assertion by induction on the rank of $\underline{g}$. If $n = 2$ then the statement is clear since both of the sets in question are empty. If $n = 3$ then the two sets are also empty unless $y\rho = (2, -1, -3)$ or $(-1, -2, -3)$. In these two cases $w\rho = (1, -2, -3)$ and the assertion is easily verified.

Assume that $n \geq 4$. Then the rank of the subalgebra $\underline{g}'$ is at least 2 and so we may assume inductively that (12.15) holds for the pair $(\underline{g}', \underline{p}')$. Let τ' denote the map τ computed for the pair $(\underline{g}'.\underline{p}')$. Suppose that there is a simple root γ in $-y^{-1}\Sigma_y$ that is orthogonal to α (as well as to β). Then $\tau\gamma = \gamma$, $w(\tau\gamma) = y\gamma \in -\Delta(\underline{u})$ and γ is in $-w^{-1}\Sigma_w$. By composing the equivalence of categories from $\mathcal{O}'$ to $\mathcal{O}_{\alpha_1}$ with the equivalences between the categories $\mathcal{O}_{\alpha_i}$, α_i short, we may construct an equivalence $\Lambda_\gamma : \mathcal{O}' \to \mathcal{O}_\gamma$. Fix $x \in \mathcal{W}'^{\underline{m}'}$ with $\Lambda_\gamma(L(x(\rho(\underline{g}')))) \cong L(y(\rho - \omega_\gamma))$. By the formulas of sections 10 and 11, in particular (10.5) and (11.7), Λ_γ induces an injection $\Lambda_\gamma : \Delta'^+ \to \Delta^+(\gamma)$ which is a bijection from $\mathcal{S}'_x$ to $\mathcal{S}_{y,\gamma}$. Recall from section eleven that we identify the root system Δ' of $\underline{g}'$ with a subsystem of Δ. If $\gamma = \alpha_i = e_i - e_{i+1}$ then the formula for Λ_γ is obtained by extending linearly the formula:

$$\Lambda_\gamma(e_j) = \begin{cases} e_j & \text{if } j \geq i+2 \\ e_{j-2} & \text{if } 3 \leq j \leq i+1 \ . \end{cases}$$

Note that $\Lambda_\gamma(\alpha) = \alpha$ and $\Lambda_\gamma(\beta) = \beta$. Let τ' denote both the restriction of τ to $\Delta'^+(\beta) = \Delta' \cap \Delta^+(\beta)$ and the restriction ot τ to $\mathcal{W}'_\beta$. A short computation shows that the diagram:

$$\begin{array}{ccc} \Delta^+(\beta) \cap \Delta^+(\gamma) & \xrightarrow{\tau} & \Delta^+(\alpha) \cap \Delta^+(\gamma) \\ \Lambda_\gamma \Big\uparrow & & \Lambda_\gamma \Big\uparrow \\ \Delta'^+(\beta) & \xrightarrow{\tau'} & \Delta'^+(\alpha) \end{array}$$

commutes. Another short computation shows $\Lambda_\gamma(L(\tau' x(\rho(\underline{g}'))) \cong L(\tau y(\rho - \omega_\gamma))$. By induction, τ' is a bijection from $-x^{-1}\Sigma'_{x,\beta}$ to $-(\tau' x)^{-1}\Sigma'_{\tau' x,\alpha}$. Thus

$$\tau(-y^{-1}\Sigma_{y,\beta}\backslash\{\gamma\}) = \tau\circ\Lambda_\gamma(-x^{-1}\Sigma'_{x,\beta}) = \Lambda_\gamma\circ\tau'(-x^{-1}\Sigma'_{x,\beta}) = -w^{-1}\Sigma_{w,\alpha}\backslash\{\gamma\}.$$

This proves the first assertion of the lemma whenever such a γ exists.

Suppose next that β is the only simple root in $-y^{-1}\Sigma_y$. Then recalling the notation of (12.9), $y = x_i$ for some i, $1 \leq i \leq n$. In this case, $-y^{-1}\Sigma_y$ was computed in (12.12). If $j = 1$ then $y = s_\beta$, $w = s_\beta s_\alpha$ and both of the sets in the first assertion of the lemma are empty. Similarly, if $j = 2$ then both sets are empty. If $j > 2$ then $w = ys_\beta$ and $w\rho = (n, \ldots, j+1, 1, -2, -3, \ldots, -j)$. By inspection: $-w^{-1}\Sigma_w = (-y^{-1}\Sigma_y \setminus \{\beta, e_{n-2} + e_{n-1}\}) \cup \{\alpha, 2e_{n-2}\}$. Thus $\tau(-y^{-1}\Sigma_{y,\beta}) = -w^{-1}\Sigma_{w,\alpha}$.

Finally suppose that the only simple root γ in $-y^{-1}\Sigma_y$ other than β is not orthogonal to α. Then γ must be $e_{n-2} - e_{n-1}$ and for some $j > 2$, $y\rho$ must take the form $y\rho = (n, \ldots, j+1, 2, -1, -3, \ldots, -j)$. So, $w\rho = (n, \ldots, j+1, 1, -2, \ldots, -j)$ and by inspection: $-w^{-1}\Sigma_w = (-y^{-1}\Sigma_y\backslash\{\beta, \gamma\})\cup\{\alpha, 2e_{n-2}\}$. Thus $\tau(-y^{-1}\Sigma_{y,\beta}) = -w^{-1}\Sigma_{w,\alpha}$. The proof for $y^{-1}\Sigma^+_{y,\beta}$ is similar. We omit the details. This completes the proof of the lemma.

We warn the reader that although the formula given for τ on $\Delta^+(\beta)$ is independent of $y \in \mathcal{W}_\beta$, any formula for the inverse of τ from $-(\tau y)^{-1}\Sigma_{\tau y,\alpha}$ to $-y^{-1}\Sigma_{y,\beta}$ must involve y.

Lemma 12.16 *Fix $y \in \mathcal{W}_\beta$ and put $w = \tau y$. Then for any γ, ν in $-y^{-1}\Sigma_{y,\beta}$, $\gamma \leq \nu$ in $\Delta^+(\beta)$ if and only if $\tau\gamma \leq \tau\nu$ in $\Delta^+(\alpha)$. Moreover, if $\Omega \subseteq -y^{-1}\Sigma_{y,\beta}$ then $\Omega \in \mathcal{S}_{y,\beta}$ if and only if $\tau\Omega \in \mathcal{S}_{w,\alpha}$.*

Proof: Fix γ, ν in $-y^{-1}\Sigma_{y,\beta}$. We first claim that $\gamma \leq \nu$ in $\Delta^+(\beta)$ if and only if $\gamma \leq \nu$ in Δ^+. Obviously if $\gamma \leq \nu$ in $\Delta^+(\beta)$ then $\gamma \leq \nu$. So assume that $\gamma \leq \nu$ in Δ^+. By inspection, $\gamma \leq \nu$ in $\Delta^+(\beta)$ unless $\gamma = e_j - e_{n-1}$ and $\nu = e_k + e_{n-1}$

for some $k \leq j < n-1$. But γ and ν are orthogonal; and so, $j = k$. However, $e_j - e_{n-1}$ and $e_j + e_{n-1}$ can not both be in $-y^{-1}\Sigma_{y,\beta}$. Thus $\gamma \leq \nu$ in $\Delta^+(\beta)$. This proves the claim. A similar proof will show that $\tau\gamma \leq \tau\nu$ in $\Delta^+(\alpha)$ if and only if $\tau\gamma \leq \tau\nu$ in Δ^+. Thus, to prove the lemma it suffices to show that $\gamma \leq \nu$ if and only if $\tau\gamma \leq \tau\nu$ (both inequalities in Δ^+).

If $\tau\gamma = \gamma$ and $\tau\nu = \nu$ then there is nothing to prove. There are two other cases to consider.

Case one: $\tau\nu = 2e_j$ for some $j < n-1$. If $\gamma \leq \nu$ then $\tau\gamma = \gamma \leq \nu \leq \tau\nu$. Conversely, suppose that $\tau\gamma \leq \tau\nu$. Then $\tau\gamma = \gamma = e_k \pm e_l$, for some $j < k < l < n-1$. We claim that $2e_j$ and $e_k + e_l$ cannot both be in $-w^{-1}\Sigma_{w,\alpha}$. Otherwise, $2e_l \leq e_k + e_l$ and $(2e_l, e_k + e_l) \in \mathcal{M}$. But $2e_j$ is the only other root ζ in $-w^{-1}\Sigma_{w,\alpha}$ with $(\zeta, 2e_l) \in \mathcal{M}$ and $2e_j \not\leq e_k + e_l$. This implies, from (8.2), that $-w^{-1}\Sigma_{w,\alpha}$ is not in $\mathcal{S}_{w,\alpha}$ which we know is false. This proves the claim and thus $\gamma = e_k - e_l$ and $\gamma \leq \nu$. This completes case one.

Case two: $\tau\gamma = 2e_j$ for some $j < n-1$. The argument here is similar to the argument in case one and we omit the details.

This completes the proof of the first assertion of (12.16). The second assertion follows immediately from the first assertion, (9.8 c) and (12.15).

Lemma 12.17 *Fix y and x in $\mathcal{W}_\beta$ and $\Omega \subseteq \Delta^+(\beta) \cap y^{-1}(\Delta(\underline{u}) \cup -\Delta(\underline{u}))$. Set $r_\Omega = \prod_{\zeta\in\Omega\cap y^{-1}\Delta(\underline{u})} s_\zeta$ and $r_{\tau\Omega} = \prod_{\zeta\in\tau\Omega\cap(\tau y)^{-1}\Delta(\underline{u})} s_\zeta$. Suppose that $x(\rho-\omega_\beta)$ and $y(\rho-\omega_\beta)$ have the same parity. Then*

$$\Omega \in \mathcal{S}_{y,\beta} \text{ and } x = \overline{yr_\Omega} \text{ if and only if } \tau(\Omega) \in \mathcal{S}_{\tau y,\alpha} \text{ and } \tau x = \overline{\tau y r_{\tau(\Omega)}}.$$

Proof: Suppose that Ω is in $\mathcal{S}_{y,\beta}$ and $x = \overline{yr_\Omega}$. Set $a = r_\Omega = \prod_{\mu\in\Omega^+} s_\mu$ and $b = \prod_{\mu\in\Omega^+} s_{\tau\mu}$. Then $x = \overline{ya}$. We claim that $\tau x = \overline{(\tau y)b}$. There are two cases to check.

Case one: suppose that $y\alpha$ is in $\Delta(\underline{u})$. Then $\tau y = ys_\alpha$. If Ω^+ is orthogonal to α then $x\alpha \in \Delta(\underline{u})$, $\tau x = xs_\alpha$ and $a = b$. The claim follows immediately from this. So we may assume that Ω^+ is not orthogonal to α. Then there is a unique γ in Ω^+ with $\langle\gamma,\alpha\rangle \neq 0$. Since $y\gamma$ and $y\alpha$ are both short roots in $\Delta(\underline{u})$, $\langle\gamma,\alpha\rangle = 1$ and $\gamma = e_j + e_{n-1}$ for some $j < n-1$. In particular, $x\alpha \in \Delta(\underline{m})$ and thus $\tau x = xs_\beta = \overline{yas_\beta}$. To prove the claim it suffices to show: $\overline{yas_\beta bs_\alpha} = y$. However, $as_\beta bs_\alpha = abs_{\beta+2\alpha}s_{\alpha+\beta} = s_\gamma s_{\tau\gamma}s_{2e_{n-1}}s_{\alpha+\beta} = s_{e_j+e_{n-1}}s_{2e_j}s_{2e_{n-1}}s_{\alpha+\beta} = s_{e_j-e_{n-1}}s_{\alpha+\beta}$. Also, $y(e_j - e_{n-1}) = y(\gamma - 2\alpha - \beta) \in \Delta(\underline{m})$ and $y(\alpha+\beta) \in \Delta(\underline{m})$. Thus $\overline{yas_\beta bs_\alpha} = \overline{ys_{e_j-e_{n-1}}s_{\alpha+\beta}} = y$. This proves the claim in case one.

Case two: suppose that $y\alpha$ is in $\Delta(\underline{m})$. Then $\tau y = ys_\beta$. If Ω^+ is orthogonal to α then, as in case one, the claim is obvious. Assume that Ω^+ is not orthogonal to α. Then there is a unique γ in Ω^+ with $\langle\gamma,\alpha\rangle \neq 0$. This γ must have the form $e_j \pm e_{n-1}$. Now $x\alpha = ys_\gamma\alpha = s_{y\gamma}y\alpha$. Since $y\alpha \in \pm\Delta(\underline{u})$, $y\alpha \in \Delta(\underline{m})$ and both roots are short, $x\alpha = y\alpha \pm y\gamma \in \pm\Delta(\underline{u})$. But $y \in \mathcal{W}_\beta$ and so $x\alpha \in \Delta(\underline{u})$. Thus, $\tau x = \overline{xs_\alpha}$. To prove the claim it suffices to show that $\overline{yas_\alpha bs_\beta} = y$. Set $\mu = \beta + 2\alpha = 2e_{n-1}$. Then, $as_\alpha bs_\beta = abs_\mu s_\alpha = s_\gamma s_{2e_j}s_\mu s_\alpha = s_{s_\mu\gamma}s_\alpha$. Also, $ys_\mu\gamma \in \Delta(\underline{m})$ (since $y\gamma \in \Delta(\underline{u})$) and $y\alpha \in \Delta(\underline{m})$. Thus $\overline{yas_\alpha bs_\beta} = y$. This proves the claim in case two.

By (9.8), $\Omega \in \mathcal{S}_{x,\beta}$ and $\Omega \subseteq -x^{-1}\Sigma_{x,\beta}$. From (12.15) and (12.16), $\tau\Omega \subseteq -(\tau x)^{-1}\Sigma_{\tau x,\alpha}$ and $\tau\Omega \in \mathcal{S}_{\tau x,\alpha}$. By the claim above, $\tau x = \overline{(\tau y)b}$. Thus $\tau\Omega \in \mathcal{S}_{\tau y,\alpha}$ and $r_{\tau\Omega} = b$. This proves the only if part of the lemma.

Conversely, suppose that $\tau\Omega \in S_{\tau y,\alpha}$ and $\tau x = \overline{\tau y r_{\tau\Omega}}$ Set $\tau\Omega^+ = \tau\Omega \cap (\tau y)^{-1}\Delta(\underline{u})$. By (9.8), $\tau\Omega \subseteq -(\tau x)^{-1}\Sigma_{\tau x,\alpha}$ and so by (12.15), there is a subset Γ of $-x^{-1}\Sigma_{x,\beta}$ with $\tau\Gamma = \tau\Omega$. By (12.16), $\Gamma \in \mathcal{S}_{x,\beta}$. Set $a = \prod_{\zeta\in\Gamma,\tau\zeta\in\tau\Omega^+} s_\zeta$ and $y' = \overline{xa}$. Then $\Gamma \in \mathcal{S}_{y',\beta}$ and $x = \overline{y'a}$. Applying the first claim of this proof to y' and Γ, we have $\tau x = \overline{(\tau y')b}$ where $b = r_{\tau\Omega}$. Thus, $\overline{(\tau y')b} = \tau x = \overline{(\tau y)b}$ and so $\tau y' = \tau y$. By parity we must have $y = y'$ and thus also $\Gamma = \Omega$ (since

τ is one-to-one on $\Delta^+(\beta) \cap y^{-1}(\Delta(\underline{u}) \cup -\Delta(\underline{u}))$. This completes the proof of (12.17).

In section fourteen we will need an analogue of (12.16), with the usual ordering of Δ^+ replaced by a linear ordering. We include this result here. Let $\underset{lin}{\leq}$ denote the linear ordering introduced for (10.6). Note that with this ordering any root of the form $2e_i$ or $e_i + e_j$ will be greater than any root of the form $e_r - e_s$, $r < s$. Also for any $\gamma, \zeta \in \Delta^+$, $\gamma \leq \zeta$ implies $\gamma \underset{lin}{\leq} \zeta$; i. e., $\underset{lin}{\leq}$ is compatible with $\leq$.

Lemma 12.18 *Let $x \in \mathcal{W}_\beta$ and $\gamma, \nu \in -x^{-1}\Sigma_{x,\beta}$. Then $\gamma \underset{lin}{\leq} \nu$ if and only if $\tau\gamma \underset{lin}{\leq} \tau\nu$.*

Proof: At most one element in $-x^{-1}\Sigma_{x,\beta}$ is changed by τ. So we need only consider two cases: $\tau\gamma \neq \gamma$ and $\tau\nu \neq \nu$. However, this ordering is linear and therefore $\gamma \underset{lin}{\leq} \nu$ if and only if γ, ν do not satisfy $\nu \underset{lin}{\leq} \gamma$. Thus we need only prove the lemma for $\tau\gamma \neq \gamma$, $\tau\nu = \nu$.

Fix $j \leq n-2$ with $\gamma = e_j \pm e_{n-1}$. Since $\xi \underset{lin}{\leq} \gamma$ for all roots ξ of the form $e_r - e_t$, $r < t \leq n-2$, $\xi \underset{lin}{\leq} \gamma$ and $\xi \underset{lin}{\leq} \tau\gamma$ for all such ξ. Thus (12.18) holds if ν has this form. So we assume otherwise: $\nu = e_r + e_s$ with $r < s \leq n-2$, $r, s \neq j$. Now directly from (8.2) and (8.4), we find $s \leq j$; and thus, both $\gamma \underset{lin}{\leq} \nu$ and $\tau\gamma \underset{lin}{\leq} \nu$. This completes the proof.

§13. Proof of Theorem 8.5

In this section we prove Theorem 8.5. We continue to assume that $(\underline{g}, \underline{p})$ is of type HS.1, HS.3 or HS.5 with rank n and constant p (as in Table 8.1). The techniques used here are essentially the same as those used to prove (8.4). We proceed by induction on p. If $p = 1$ then (8.5) follows immediately from [16]. We assume $p > 1$ and (8.5) holds for the pair $(\underline{g}', \underline{p}')$.

Lemma 13.1 *Let α be a simple root and fix $x \in \mathcal{W}_\alpha$. Set $r = \prod_{\gamma \in \Sigma^+_{x,\alpha}} s_\gamma$ Then:*

(i) $x^{-1}\Sigma^+_{x,\alpha}$ is in $\mathcal{S}_{x,\alpha}$ and Socle$(N(x(\rho - \omega_\alpha))) \cong L(\overline{rx(\rho - \omega_\alpha)})$.

(ii) $D(x(\rho - \omega_\alpha))$ is self-dual.

(iii) If Ω is in $\mathcal{S}_{x,\alpha}$ then $|\Omega| \leq p - 1$.

(iv) $P(x(\rho - \omega_\alpha))$ is self-dual if and only if $|\Sigma_{x,\alpha}| = p - 1$.

Proof: Suppose that α is a short root and let $\Lambda_\alpha : \mathcal{O}' \to \mathcal{O}_\alpha$ be the equivalence of categories given by (10.1) and (11.2). Then all of the claims of (13.1) follow from the inductive hypotheses and the formulas of (10.5) and (11.7). If β is the long simple root of HS.3, then let α be the adjacent short simple root. Then i) through iv) hold for α. By the formulas of (12.15), (12.16) and (12.17) and the equivalence of categories in (12.14), i) through iv) must also hold for β. This completes the proof of (13.1).

(13.2) Proof of Theorem 8.5

Fix x in $\mathcal{W}^{\underline{m}}$. If $x\rho$ is not $\Delta(\underline{u})$−antidominant then there is a simple root α with $xs_\alpha \in \mathcal{W}_\alpha$. Then $\Sigma^+_x = \Sigma^+_{xs_\alpha,\alpha} \cup \{x\alpha\}$ and thus $x^{-1}\Sigma^+_x \in \mathcal{S}_x$ by

(13.1) and (9.5). If $x\rho$ is antidominant then $\Sigma_x^+ = \emptyset$. Combining parts i and ii of (13.1) with (4.6) we also have: $\operatorname{Socle}(N(x\rho)) \cong L(\overline{r_{\Sigma_x^+} x\rho})$ and $D(x\rho)$ is self-dual. This proves i and ii of (8.5)

Let Ω be in $\mathcal{S}_x$. If Ω is nonempty then there must be a simple root α in Ω. Since $x\alpha \in \pm\Delta(\underline{u})$, either x or xs_α is in $\mathcal{W}_\alpha$. Thus $\Omega \setminus \{\alpha\}$ is in $\mathcal{S}_{x,\alpha}$ or $\mathcal{S}_{xs_\alpha,\alpha}$. By (13.1 iii) we have $|\Omega| \leq p$. This is (8.5 iii).

If $x\rho$ is dominant then $\Sigma_x = \emptyset$ and $P(x\rho)$ is never self dual. Assume that $x\rho$ is not dominant. Fix a simple root α with $x \in \mathcal{W}_\alpha$. Then by (4.5), (13.1 iv) and (9.5), $P(x\rho)$ is self-dual if and only if $P(x(\rho - \omega_\alpha))$ is self-dual if and only if $|\Omega \setminus \{-x\alpha\}| = p - 1$ if and only if $|\Omega| = p$. This proves (8.5 iv) and completes the proof of (8.5).

§14. Projective resolutions and Ext

In this section we give formulas for the groups $\mathrm{Ext}^*(N(x\rho), L(y\rho))$ These formulas arise as a consequence of a special type of projective resolution for generalized Verma modules which exists in $\mathcal{O}(\underline{g}, \underline{p}, \rho)$. As a consequence of the Ext^1 result, we also give an explicit formula for Vogan's U_α functor (cf. [34]). Throughout this section, unless otherwise stated, $(\underline{g}, \underline{p})$ is any of the pairs of Table 8.1.

If x is in $\mathcal{W}^{\underline{m}}$ then we will denote by L_x, N_x and P_x the modules $L(x\rho)$, $N(x\rho)$ and $P(x\rho)$ respectively. If α is any simple root and x or xs_α is in $\mathcal{W}_\alpha$ then we write L_x^α, N_x^α and P_x^α for the modules $L(x(\rho - \omega_\alpha))$, $N(x(\rho - \omega_\alpha))$ and $P(x(\rho - \omega_\alpha))$ respectively. As usual, ψ_α and ϕ_α are the translation functors to and from the α-wall. Set $\mathcal{O} = \mathcal{O}(\underline{g}, \underline{p}, \rho)$, $\mathcal{O}_\alpha = \mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_\alpha)$ and $\mathcal{O}' = \mathcal{O}(\underline{g}', \underline{p}', \rho(\underline{g}'))$.

Definition 14.1 *Let A be in $\mathcal{O}$ and let $P_* \to A$ be a projective resolution of A in $\mathcal{O}$. For $j \in \mathbb{N}$ there are $m_j \in \mathbb{N}$ and elements $w_{ij} \in \mathcal{W}^{\underline{m}}$, $1 \le i \le m_j$, with $P_j \cong \bigoplus_{1 \le i \le m_j} P_{w_{ij}}$. We say that the resolution has even (resp. odd) parity if $j - \ell(w_{ij}) \equiv 0$ (resp. 1) mod 2 for all i, j.*

Lemma 14.2 *Let y be in $\mathcal{W}^{\underline{m}}$ and assume $P_* \to A$ is a projective resolution in $\mathcal{O}$ with even (resp odd) parity. Then with notation as in 14.1, $\mathrm{Ext}^j(A, L_y) \cong \mathrm{Hom}(P_j, L_y)$ and thus $\dim \mathrm{Ext}^j(A, L_y) = \mathrm{card}\{i \mid w_{ij} = y\}$. Moreover, $\mathrm{Ext}^j(A, L_y) = 0$ unless $j - \ell(y) \equiv 0$ (resp. 1) mod 2.*

Proof: For each indecomposable projective, $\mathrm{Hom}(P_w, L_y) = \mathbb{C}$ or 0 depending as $w = y$ or not. Thus the parity condition implies that every other term in $\mathrm{Hom}(P_*, L_y)$ is zero. Thus $\mathrm{Ext}^j(A, L_y) = \mathrm{H}^j(\mathrm{Hom}(P_*, L_y)) = \mathrm{Hom}(P_j, L_y)$. This proves the lemma.

The previous lemma describes the importance of projective resolutions with parity. We will prove that any generalized Verma module admits such a resolution. The technical basis for our argument is a basic result called the algebraic mapping cone.

Proposition 14.3 [28, p.46] *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules and let $P_* \to A$ and $Q_* \to B$ be projective resolutions. Then there is a projective resolution $R_* \to C$ with $R_0 = Q_0$ and $R_i = Q_i \oplus P_{i-1}$ for $i \in \mathbb{N}^*$.*

To describe our recursive formula we will need some notation. For the moment assume that $(\underline{g}, \underline{p})$ is one of the three types HS.1, HS.3 or HS.5. Fix a simple root α. If α is short then combining (10.1) with (11.2) there is an equivalence of categories $\Lambda_\alpha : \mathcal{O}' \to \mathcal{O}_\alpha$. If α is long then by (12.14) there are equivalences $e\Lambda_\alpha : \mathcal{O}' \to \mathcal{O}_\alpha(even)$ and $o\Lambda_\alpha : \mathcal{O}' \to \mathcal{O}_\alpha(odd)$. Fix $y \in \mathcal{W}_\alpha$ so that L_y^α is in $\mathcal{O}_\alpha$ (resp. $\mathcal{O}_\alpha(even)$, $\mathcal{O}_\alpha(odd)$). Set $\Lambda = \Lambda_\alpha$ (resp. $e\Lambda_\alpha$, $o\Lambda_\alpha$). For $x \in \mathcal{W}'^{\underline{m}'}$, set $L'_x = L(\underline{g}', x\rho')$ and $N'_x = N(\underline{g}', \underline{p}', x\rho')$. If $x \in \mathcal{W}_\alpha$ (with an even or odd condition on $x(\rho - \omega_\alpha)$ if α is long) define $x' \in \mathcal{W}'^{\underline{m}'}$ by the formula: $L_x^\alpha = \Lambda(L'_{x'})$. It is clear from our equivalences that if $\mathrm{Ext}^*_{\mathcal{O}'}$ denotes Ext groups computed in the category $\mathcal{O}'$ then $\mathrm{Ext}^k(N_x^\alpha, L_y^\alpha) \cong \mathrm{Ext}^k_{\mathcal{O}'}(N'_{x'}, L'_{y'})$, for all $x, y \in \mathcal{W}_\alpha$ (with $x(\rho - \omega_\alpha)$ and $y(\rho - \omega_\alpha)$ having the same parity if relevant).

Proposition 14.4 *Let y and w be in $\mathcal{W}^{\underline{m}}$.*

(a) *N_y admits a projective resolution with the same parity as $\ell(y)$.*

(b) *For any j, $\mathrm{Ext}^j(N_y, L_w) = 0$ unless $y \prec w$ and $j \equiv \ell(y) - \ell(w)$ mod 2.*

(c) Let α be any simple root with $y \in \mathcal{W}_\alpha$. Then for all $k \geq 1$:

$$Ext^k(N_y, L_w) \cong Ext^{k-1}(N_{ys_\alpha}, L_w) \oplus Ext^k(N_y^\alpha, \psi_\alpha L_w)$$

Moreover, the last term in this equation is zero when $w \notin \mathcal{W}_\alpha$ or when α is long and $y(\rho - \omega_\alpha)$ and $w(\rho - \omega_\alpha)$ have opposite parity (cf. (12.11)).

(d) Suppose $(\underline{g}, \underline{p})$ is of type HS.1, HS.3 or HS.5, $p > 1$ and $w \in \mathcal{W}_\alpha$. Also for HS.3 and α long, assume $y(\rho - \omega_\alpha)$ and $w(\rho - \omega_\alpha)$ have the same parity. Then: $Ext^k(N_y^\alpha, \psi_\alpha L_w) \cong Ext^k_{\mathcal{O}'}(N'_{y'}, L'_{w'})$.

Proof: Suppose first that $(\underline{g}, \underline{p})$ is of type HS.1 with constant $p = 1$, i.e. $(\mathrm{A}_n, \mathrm{A}_{n-1})$. Using the notation of [16, section 6], we have $N_y = N(\nu(i))$ for some i, $0 \leq i \leq n$. By (6.4) of [16] there is a projective resolution $P_* \to N_y$ with $P_k \cong P(\nu(i+k))$. This is easily seen to be a projective resolution with the same parity as $\ell(y) = n - i$. This proves (a). Suppose that $L_w = L(\nu(t))$ for some $0 \leq t \leq n$. Then $\mathrm{Ext}^k(N_y, L_w) \cong \mathrm{Hom}(P(\nu(i+k)), L(\nu(t))) = 0$ unless $t = i + k$. This proves (b). Finally, since $N_{ys_\alpha} = N(\nu(i+1))$, $\mathrm{Ext}^{k-1}(N_{ys_\alpha}, L_w) \cong \mathrm{Hom}(P(\nu(i+k)), L(\nu(t))) \cong \mathrm{Ext}^k(N_y, L_w)$. From [16, (6.2)], $\mathrm{Ext}^k(N_y, \psi_\alpha L_w) = 0$ for $k \geq 1$. This proves (c) and completes the proof in the case HS.1, $p = 1$.

As discussed at the beginning of section twelve, for HS.1, HS.3 and HS.5 the $p = 1$ cases are either type A_1 (for which the assertions are obvious) or are included in sections six and seven of [16]. That article also treats the case HS.2. In all of these cases the proof of (14.4) is precisely that of the previous paragraph, and so we omit the details.

Next we assume that $(\underline{g}, \underline{p})$ is of type HS.1, HS.3 or HS.5 with constant $p > 1$ and proceed by induction on p.

Begin a secondary induction on $\ell(y)$. If $\ell(y) = 0$ then $N_y = N(\rho) = P(\rho)$. Thus N_y trivially has the required resolution. So we assume $\ell(y) > 0$. Fix a simple root α with $y \in \mathcal{W}_\alpha$. Then $\ell(ys_\alpha) = \ell(y)-1$, and so by induction, N_{ys_α} has a projective resolution $P_* \to N_{ys_\alpha}$ with even or odd parity depending as $\ell(y)$ is odd or even. Choose $w_{ik} \in \mathcal{W}^{\underline{m}}$ so that $P_k \cong \oplus_i P_{w_{ik}}$.

Let Λ and y' be as above. By induction on p, $N'_{y'}$ has a projective resolution $Q'_* \to N'_{y'}$ with the same parity as $\ell(y')$. Fix t_{ik} in $\mathcal{W}'^{\underline{m}'}$ with $Q'_k = \bigoplus_i P(\underline{g}', \underline{p}', t_{ik}\rho')$. Choose y_{ik} in $\mathcal{W}_\alpha$ with $t_{ik} = y'_{ik}$ (and even or odd condition if relevant). Set $Q_i = \phi_\alpha \Lambda(Q'_i)$. Then Q_* is a projective resolution of $\phi_\alpha(N^\alpha_y)$ with terms $Q_k = \bigoplus_i P_{y_{ik}}$ (cf. (4.1)). Applying the algebraic mapping cone to the short exact sequence $0 \to N_{ys_\alpha} \to \phi_\alpha N^\alpha_y \to N_y \to 0$, we obtain a projective resolution $R_* \to N_y$ with $R_0 = Q_0$ and $R_k = \bigoplus_i P_{y_{ik}} \oplus \bigoplus_j P_{w_{j,k-1}}$. To check the parity it suffices to show:

$$k - \ell(y_{ik}) \equiv \ell(y) \bmod 2 \tag{14.5}$$

and

$$k - \ell(w_{i,k-1}) \equiv \ell(y) \bmod 2. \tag{14.6}$$

But (14.6) follows at once from $\ell(y) = \ell(ys_\alpha)+1$. So it suffices to prove (14.5). To check (14.5) we need the following lemma:

Lemma 14.7(a) *Let $(\underline{g}, \underline{p})$ be of type HS.1. If $x \in \mathcal{W}^{\underline{m}}$ with $x\rho = (a_1, \dots a_{n+1})$ $(\sum a_i = 0)$ then $\ell(x) = \sum_{1\le i\le p}(\frac{n}{2}+1-i-a_i) = \frac{1}{2}p(n+1-p) - \sum_{1\le i\le p} a_i$. In particular, if $x \in \mathcal{W}_\alpha$ with $\alpha = \alpha_j$ and $-x\alpha = e_t - e_r$ then $\ell(x') = \ell(x)+2t+C_j$ where C_j is a constant that depends only on α.*

(b) Let $(\underline{g}, \underline{p})$ be of type HS.3. If $x \in \mathcal{W}^{\underline{m}}$ with $x\rho = (a_1, \dots, a_n)$ (where $\rho = (n, \dots, 1)$) then $\ell(x) = \sum_{a_i \le -1} -a_i$. If $x \in \mathcal{W}_\alpha$ with $\alpha = \alpha_i$, $1 \le i \le n-1$, and $-x\alpha = e_r + e_t$ $(r < t)$ then $\ell(x') = \ell(x) + i + 2t - 3n - 1$. If $x \in \mathcal{W}_\alpha$ with $\alpha = \alpha_n$ and $-x\alpha = 2e_t$ then $\ell(x') = \ell(x) - 2(n-t) - 1$.

(c) Let $(\underline{g}, \underline{p})$ be of type HS.5. If $x \in \mathcal{W}^{\underline{m}}$ with $x\rho = (a_1, \dots, a_n)$ (where $\rho = (n-1, \dots, 0)$) then $\ell(x) = \sum_{a_i \le 0} -a_i$. If $x \in \mathcal{W}_\alpha$ with $\alpha = \alpha_j$ and

$-x\alpha = e_r + e_t$ $(r < t)$ *then* $\ell(x') = \ell(x) + 2t + C_j$ *where* C_j *is a constant that depends only on* α.

Proof: The first claim of part (a) is an easy exercise which we omit. The second claim follows by observing that

$$x'\rho(\underline{g}') = \begin{matrix}(a_1 - 1, a_2 - 1, \ldots, a_{t-1} - 1, a_{t+1} + 1, \ldots, a_p + 1, \\ a_{p+1} - 1, \ldots, a_{r-1} - 1, a_{r+1} + 1, \ldots, a_{n+1} + 1)\end{matrix}$$

and then applying the first claim.

The first part of (b) we also leave as an exercise. If $(\underline{g}, \underline{p})$ is HS.3 and $-x\alpha = e_r + e_t$, $r < t$, then $x'\rho(\underline{g}') = (a_1 - 2, \ldots, a_{r-1} - 2, a_{r+1}, \ldots, a_{t-1}, a_{t+1} + 2, \ldots, a_n + 2)$. If $-x\alpha = 2e_t$ then $x'\rho(\underline{g}') = (a_1 - 2, \ldots, a_{t-2} - 2, a_{t+1} + 2, \ldots, a_n + 2)$ if $x\alpha_{n-1} \in \Delta(\underline{u})$ and $x'\rho(\underline{g}') = (a_1 - 2, \ldots, a_{t-1} - 2, a_{t+2} + 2, \ldots, a_n + 2)$ if $x\alpha_{n-1} \in \Delta(\underline{m})$. These observations prove the second claim of (b). The proof of part (c) is similar to the proof of (b). We omit the details. This completes (14.7).

Returning to the proof of (14.4) (a), we see immediately from (14.7) that for any $x \in \mathcal{W}_\alpha$, $\ell(y) - \ell(x) \equiv \ell(y') - \ell(x')$ mod 2. Thus we have for each i, k; $k \equiv \ell(y') - \ell(y'_{ik}) \equiv \ell(y) - \ell(y_{ik})$ mod 2. This proves (14.5) and completes the proof of part (a) for HS.1, HS.3 and HS.5.

We next complete the proof of (c) and (d). We assume that α is a simple root with $y \in \mathcal{W}_\alpha$. By the definition and parity of the resolution $R_* \to N_y$ given in the proof of part (a) we have:

$$\mathrm{Ext}^k(N_y, L_w) \cong \mathrm{Ext}^k(\phi_\alpha N_y^\alpha, L_w) \oplus \mathrm{Ext}^{k-1}(N_{ys_\alpha}, L_w).$$

By the adjoint property we have $\mathrm{Ext}^k(\phi_\alpha N_y^\alpha, L_w) = \mathrm{Ext}^k(N_y^\alpha, \psi_\alpha L_w)$. This proves the first claim of (c).

Set $A = \mathrm{Ext}^k(N_y^\alpha, \psi_\alpha L_w)$. If $w \notin \mathcal{W}_\alpha$ then $\psi_\alpha L_w = 0$, and so $A = 0$. If $w \in \mathcal{W}_\alpha$, α is long and $y(\rho - \omega_\alpha)$ and $w(\rho - \omega_\alpha)$ have opposite parity, then by (12.14) we still have $A = 0$. Otherwise, by our equivalences of categories in sections 10 and 11 and by (12.14), $A \cong \mathrm{Ext}^k(N_y^\alpha, L_w^\alpha) \cong \mathrm{Ext}^k_{\mathcal{O}'}(N'_{y'}, L'_{w'})$. This completes the proof of (c)and (d).

Finally, if $\mathrm{Ext}^k(N_y, L_w) \neq 0$ then by (14.2), $k - \ell(w) \equiv \ell(y) \bmod 2$. Now proceed by induction on p and $\ell(y)$. If $\ell(y) = 0$ then the result is clear as N_y is projective. If $\ell(y) > 0$ then using (c), either $ys_\alpha \prec w$ or $y(\rho - \omega_\alpha) \prec w(\rho - \omega_\alpha)$. Either way, $y \prec w$. This proves (b) and completes the proof of (14.4) for HS.1,HS.3 or HS.5.

Only the case HS.4 remains. Here the argument is essentially the same as that described above for HS.1, HS.3 and HS.5. However, there are some simplifying features. In place of the inductive hypothesis, we use the explicit structure of the categories $\mathcal{O}(\underline{g}, \underline{p}, \rho - \omega_\alpha)$ given in [16]. We omit the details of this verification. This completes the proof of (14.4).

The recursion formulas in (14.4) have solutions expressed in the language of Σ_x and $\mathcal{S}_x$. We now describe these solutions. We say a linear ordering on Δ^+, $\underset{lin}{\leq}$, is weakly compatible with $\leq$ if for all simple roots α, all $y \in \mathcal{W}_\alpha$ and all $\gamma, \nu \in -y^{-1}\Sigma_{y,\alpha}$, $\gamma \leq \nu$ implies $\gamma \underset{lin}{\leq} \nu$. Fix a linear ordering $\underset{lin}{\leq}$ on Δ^+ weakly compatible with $\leq$. In the case HS.3, we let $\underset{lin}{\leq}$ be given as in (10.6) or (12.18).

Definition14.8 a) *Recall the notation surrounding (8.2) and (8.3). For* $x \in \mathcal{W}^{\underline{m}}$, *define a* <u>*chain*</u> *(of length* t *associated to* x*) to be an indexed set of positive roots* $\{\gamma_i\}_{1 \leq i \leq t}$ *with the following properties. For* $1 \leq i \leq t+1$, *put* $x_i = \overline{xs_{\gamma_1} \cdots s_{\gamma_{i-1}}}$. *Define* Ω_i, $0 \leq i \leq t+1$ *inductively by* $\Omega_0 = \emptyset$ *and* $\Omega_i = \Omega_{i-1} \cup \{\delta \in -x_i^{-1}\Sigma_{x_i} | \delta \underset{lin}{\leq} \gamma_i,\ \delta \neq \gamma_i\}$, $1 \leq i \leq t+1$. *Put* $\overline{\Omega_i} = \Omega_i \cup \{\gamma_i\}$. *Then*

(i) $\overline{\Omega_i} \in \mathcal{S}^-_{x_i}$, $1 \leq i \leq t$

(ii) $\gamma_i \notin \Omega_i$, $0 \leq i \leq t$.

Note that a chain is determined by the flag of sets $\{\Omega_j\}$; *and moreover, these* Ω_i *all lie in* $-x_{t+1}^{-1}\Sigma_{x_{t+1}}$.

b) For any $y, w \in \mathcal{W}^{\underline{m}}$, $t \in \mathbb{N}$, let $C(y,w,t)$ denote the set of chains $\{\gamma_i\}$ of length t associated to y with $w = \overline{ys_{\gamma_1} \cdots s_{\gamma_t}}$. By convention, $C(y,w,0)$ is the set containing the empty chain or the empty set depending as $y = w$ or not. Let $C(y,w) = \bigcup_{t \geq 0} C(y,w,t)$. We refer to the elements of $C(y,w)$ as the chains from y to w.

c) For $x \in \mathcal{W}^{\underline{m}}$, define a positive chain (of length t associated to x) to be an indexed set of positive roots $\{\gamma_i\}_{1 \leq i \leq t}$ which satisfy the following. For $1 \leq i \leq t$, put $x_i = \overline{xs_{\gamma_1} \cdots s_{\gamma_i}}$, $\Omega_{t+1} = \emptyset$, $\Omega_i = \{\gamma \in -x^{-1}\Sigma_x | \gamma \underset{lin}{\leq} \gamma_i, \gamma \neq \gamma_i\}$ and $\overline{\Omega_i} = \Omega_i \cup \{\gamma_i\}$. Then

(i) $\overline{\Omega_i} \in \mathcal{S}^-_{x_i}$, $1 \leq i \leq t$,

(ii) $\gamma_i \notin \Omega_i$.

d) For $y, w \in \mathcal{W}^{\underline{m}}$, $t \in \mathbb{N}$, let $C^+(y,w,t)$ denote the positive chains $\{\gamma_i\}$ of length t associated to y with $w = \overline{ys_{\gamma_1} \cdots s_{\gamma_t}}$. By convention $C^+(y,w,0)$ is the empty positive chain or the empty set depending as $y = w$ or not. Let $C^+(y,w)$ be the union of $C(y,w,t)$ for $t \in \mathbb{N}$. For any positive chain $\{\gamma_i\}$ put $\hat{\gamma}_j = \gamma_{t+1-j}$. Then the map $\{\gamma_i\} \mapsto \{\hat{\gamma}_j\}$ defines a bijection from $C^+(y,w,t)$ to $C(w,y,t)$.

Theorem 14.9 *Let $y, w \in \mathcal{W}^{\underline{m}}$ and $j \in \mathbb{N}$. Then*

$$\dim Ext^j(N_y, L_w) = card(C(y,w,j)).$$

Proof: We proceed by induction on the integer p and the integer $\ell(y) - \ell(w)$. If $\ell(y) < \ell(w)$ then all the Ext groups are zero by (14.4) and by (14.8 ii) $C(y,w)$ is empty. So we may assume $\ell(y) - \ell(w) \in \mathbb{N}$.

We begin with the $p = 1$ cases. From [16] we know $\mathrm{Ext}^j(N_y, L_w) = \mathbb{C}$ or zero depending as $j = \ell(y) - \ell(w)$ or not. A short calculation using the notation of [16] shows that $C(y,w,j)$ is empty except when $j = \ell(y) - \ell(w)$

and, in that case, there is one chain from y to w of length j. This verifies the proposition when $p = 1$.

Next we turn to the cases HS.1, HS.3 and HS.5. Suppose $p > 1$ and assume (14.9) is true for smaller p. Fix $y, w \in \mathcal{W}^{\underline{m}}$, $j \in \mathbb{N}^*$ (the case $j = 0$ is trivial) and put $C = C(y, w, j)$. If $\ell(y) = 0$ then N_y is projective, C is empty and the proposition is true. So we assume $\ell(y) \neq 0$. Write C as a disjoint union: $C = A \cup B$, where A is the set of chains $\{\gamma_i\}$ with Ω_1 empty and B is the complement. Let δ be the unique minimal element in $-y^{-1}\Sigma_y$ with respect to $\underset{lin}{\leq}$. Then a chain in C lies in A when $\gamma_1 = \delta$ and in B when $\delta \in \Omega_1$.

The map $\{\gamma_i\}_{1 \leq i \leq j} \mapsto \{\gamma_i\}_{2 \leq i \leq j}$ gives a bijection:

$$A \longrightarrow C(ys_\delta, w, j-1). \tag{14.10}$$

To analyze B we recall some equivalences of categories. Let $\mathcal{O}'$ equal $\mathcal{O}'(\underline{g}', \underline{p}', \rho(\underline{g}'))$ and let $\mathcal{O}''$ equal $\mathcal{O}_\delta$, $\mathcal{O}_\delta(even)$ or $\mathcal{O}_\delta(odd)$ depending as δ is short, δ is long and $y(\rho - \omega_\delta)$ is even or δ is long and $y(\rho - \omega_\delta)$ is odd. The results of sections ten, eleven and twelve give an equivalence $\Lambda : \mathcal{O}'' \to \mathcal{O}'$. Let $\nu \mapsto \nu'$ and $x \mapsto x'$ denote the maps from $\Delta^+(\delta)$ to Δ' and $\mathcal{W}_\delta$ to $\mathcal{W}'^{\underline{m}'}$. Then (10.5), (12.15) and (11.7) combine to assert that for all $x \in \mathcal{W}_\delta$, $\Sigma_{x,\delta}$ is carried to $\Sigma'_{x'}$ and $\mathcal{S}_{x,\delta}$ to $\mathcal{S}'_{x'}$. Suppose $\{\gamma_i\}_{1 \leq i \leq t}$ is a chain in B. We claim $\{\gamma_i'\}_{1 \leq i \leq t}$ is a chain from y' to w'. Let Ω_i and $\overline{\Omega_i}$ be as in (14.8). Both Ω_i and $\overline{\Omega_i}$ are subsets of $-x_i^{-1}\Sigma_{x_i}$, and so $\Omega_i \setminus \{\delta\}$ and $\overline{\Omega_i} \setminus \{\delta\}$ lie in $-x_i^{-1}\Sigma_{x_i,\delta}$. Since δ lies in each Ω_i, $x_i' = \overline{x' s_{\gamma_1'} \cdots s_{\gamma_{i-1}'}}$. Put $\Psi_i = \{\nu' | \nu \in \Omega_i \setminus \{\delta\}\}$ and $\overline{\Psi}_i = \Psi_i \cup \{\gamma_i'\}$. Since $\{\gamma_i\} \in B$, the γ_i are orthogonal to δ and thus $x_i \in \mathcal{W}_\delta$ for all i, $1 \leq i \leq t$. So we may recall the formulas (10.5), (11.7) and (12.15) for each pair (x_i, δ). From this we conclude: $\overline{\Psi}_i$ lies in $-x_i'^{-1}\Sigma'_{x_i'}$; i.e., $\overline{\Psi}_i \in \mathcal{S}'^{-}_{x_i'}$, $1 \leq i \leq t$. Thus the sets Ψ_i and $\overline{\Psi}_i$ satisfy properties i) and ii) respectively of Definition 14.8.

Now suppose δ is a short root. In cases HS.3 and HS.5, if $\delta = e_i - e_{i+1}$ put $\bar{\delta} = e_i + e_{i+1}$. Then $\nu \mapsto \nu'$ is a bijection of $\Delta^+(\delta)$ (resp. $\Delta^+(\delta) \setminus \{\bar{\delta}\}$) onto Δ'^+ in the case HS.1 (resp. cases HS.3 and HS.5). Let $\underset{lin}{\leq'}$ denote the linear ordering on Δ'^+ given by $\nu \underset{lin}{\leq} \gamma$ if and only if $\nu' \underset{lin}{\leq'} \gamma'$ for $\nu, \gamma \in \Delta^+(\delta)$ (resp. $\Delta^+(\delta) \setminus \{\bar{\delta}\}$). Define chains from y' to w' with respect to this ordering and let $C'(y', w', t)$ denote the chains from y' to w' of length t. Then $\Psi_i = \{\zeta \in -{x'_i}^{-1}\Sigma'_{x'_i} | \zeta \underset{lin}{\leq'} \gamma'_i$ and $\zeta \neq \gamma'_i\} \cup \Psi_{i-1}$. This proves $\{\gamma'_i\} \in C'(y', w', t)$ when δ is a short root.

Next suppose that δ is a long root. Then δ is the unique long simple root. In this case, HS.3, let $\underset{lin}{\leq}$ denote the ordering defined as in (12.18). Recall the notation surrounding (12.15) through (12.18) with β replaced by δ. Set $R = \Delta^+(\alpha) \setminus \{e_{n-1} + e_n\}$. Then the map τ is a surjection of $\Delta^+(\delta)$ onto R. The assignment $\nu \mapsto \nu'$ is the composition of this map τ with an order preserving bijection onto Δ'^+ by (10.6). Thus $\nu \mapsto \nu'$ preserves the linear order relation, and so $\Psi_i = \{\zeta \in -{x'_i}^{-1}\Sigma'_{x'_1} | \zeta \underset{lin}{\leq'} \gamma'_i$ and $\zeta \neq \gamma'_i\} \cup \Psi_{i-1}$. This proves $\{\gamma'_i\}$ is a chain from y' to w' when δ is a long root and completes the proof of the claim.

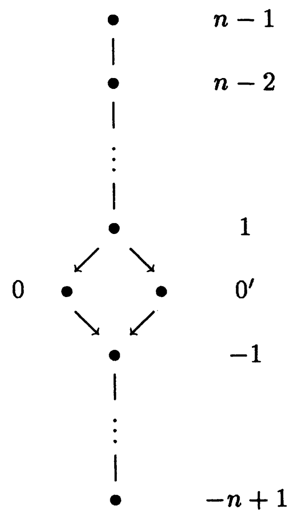
Reversing the argument which proves the claim above, and using the fact that δ is minimal in $-y^{-1}\Sigma_y$, we see that the mapping $B \to C'(y', w', t)$, $\{\gamma_i\} \mapsto \{\gamma'_i\}$ is also surjective.

By the induction hypothesis on $\ell(y) - \ell(w)$, $\mathrm{card}(A) = \mathrm{card} C(ys_\delta, w, j-1)$ which in turn equals $\dim \mathrm{Ext}^{j-1}(N_{ys_\delta}, L_w)$. By induction on p, $\mathrm{card}(B) = \mathrm{card} C'(y', w', j)$ which equals $\dim \mathrm{Ext}^j(N'_{y'}, L'_{w'})$. Combining these with (14.4 c and d), $\mathrm{card} C = \dim \mathrm{Ext}^j(N_y, L_w)$. This completes the proof for HS.1, HS.3 and HS.5.

The case HS.4 is the remaining case. Here the proposition can be verified using the results in [16]. We outline the verification in some detail. Suppose $\underline{g}$

is of type HS.4. The reader should refer to [16] for additional details. The set $\mathcal{W}^{\underline{m}}$ for this case $(\mathrm{D}_n, \mathrm{D}_{n-1})$ can be described by the diagram:

(14.11)



We use integers i, $-n < i < n$, and $0'$ to denote the corresponding elements of $\mathcal{W}^{\underline{m}}$. In this setting the semiregular integral categories are equivalent to the regular integral category for $sl(2,\mathbb{C})$. Therefore, $\mathrm{Ext}^k(N_y^\alpha, \psi_\alpha L_w)$ is always zero for $k \geq 2$. Then (14.4 c) and the results of [16] imply the next Lemma.

Lemma 14.12 *i) For $0 < j < t < n$, $Ext^k(N_{-t}, L_j) \cong \mathbb{C}$ when k equals either $t+j$ or $t-j$. For other k, this space is zero.*

ii) For all pairs (y,w) not covered in i), $Ext^k(N_y, L_w) \cong \mathbb{C}$ when $k = \ell(y) - \ell(w) \geq 0$ and is zero otherwise.

A calculation in the coordinates of [16] yields:

Lemma 14.13*i) For $0 < j < t < n$, $C(-t,j)$ contains two chains:*

$$\{\alpha_{n-t}, \alpha_{n-t+1}, \ldots, \alpha_{n-2}, \alpha_{n-1}, \alpha_n, \alpha_{n-2}, \ldots, \alpha_{n-j}\}$$

and

$$\{\alpha_{n-t}, \alpha_{n-t+1}, \ldots, \alpha_{n-j-2}, \alpha_\nu\}$$

where $\nu = \alpha_{n-j-1} + 2\alpha_{n-j} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. The first has length $t+j$ while the second has length $t-j$.

ii) For all pairs (y, w) not covered in i), $C(y, w)$ contains one chain of length $\ell(y) - \ell(w)$ if $\ell(y) - \ell(w) \geq 0$ and is empty otherwise.

These two lemmas combine to verify the theorem in the case HS.4. This completes the proof of Theorem 14.9.

Corollary 14.14 *Let y and w be in $\mathcal{W}^{\underline{m}}$. Then:*

$$Ext^1(N_y, L_w) = \begin{cases} \mathbb{C} & \text{if } w = \overline{s_\gamma y} \text{ for some } \gamma \in \Sigma_y; \\ 0 & \text{otherwise} \end{cases}$$

and

$$Ext^1(N_y, L_w) = \begin{cases} \mathbb{C} & \text{if } y = \overline{wr_\Omega} \text{ for some } \Omega \in \mathcal{E}_w \text{ with } \mid \Omega^+ \mid = 1; \\ 0 & \text{otherwise.} \end{cases}.$$

Proof: The first formula follows from (14.9), while the second follows from the first as in the proof of (9.6).

Next we turn to a description of the functor U_α introduced by Vogan. Let α be a simple root and set $\Theta_\alpha = \phi_\alpha \circ \psi_\alpha$. If $w \in \mathcal{W}_\alpha$ then $\Theta_\alpha(L_w)$ is a self dual module with unique irreducible quotient L_w and socle L_w. Define $U_\alpha(L_w)$ to be the maximal submodule of $\Theta_\alpha(L_w)/L_w$.

Proposition 14.15 *If w is in $\mathcal{W}_\alpha$ then:*

$$U_\alpha(L_w) \cong L_{ws_\alpha} \oplus \bigoplus_{\substack{\Omega \in \mathcal{E}_w, \mid \Omega^+ \mid = 1 \\ \overline{wr_\Omega} \notin \mathcal{W}_\alpha}} L_{\overline{wr_\Omega}}.$$

We must first establish that $U_\alpha(L_w)$ is semisimple. This fact is implied by the Kazhdan-Lusztig conjectures. However in our setting we give a more direct proof.

Lemma 14.16 *If $w \in \mathcal{W}_\alpha$ and L_z is a composition factor of both N_w and N_{ws_α} then $z \in \mathcal{W}_\alpha$.*

Proof: If $(\underline{g}, \underline{p})$ is of type HS.2 then the only composition factor common to both N_w and N_{ws_α} is L_w. Thus we may assume that $(\underline{g}, \underline{p})$ is not of type HS.2. Note that if γ and ν are in $\Delta(\underline{u})$ and γ is a short root not orthogonal to ν and $\gamma \neq \nu$ then $s_\nu \gamma \in \Delta(\underline{m})$.

Suppose that $z \notin \mathcal{W}_\alpha$. For $\Omega \subseteq \Sigma_z$, set $s_\Omega = \prod_{\gamma \in \Omega} s_\gamma$. By (8.4) there are Ω, $\Gamma \subseteq \Sigma_z$ with $w = \overline{s_\Omega z}$ and $ws_\alpha = \overline{s_\Gamma z}$. By rearranging terms and cancelling z in the equation $ws_\alpha = \overline{s_\Gamma z} = \overline{s_\Omega z} s_\alpha$, we see:

$$(*) \qquad s_{s_\Omega z\alpha} s_\Omega s_\Gamma \in \mathcal{W}_{\underline{m}} \qquad \text{and} \qquad s_{s_\Gamma z\alpha} s_\Gamma s_\Omega \in \mathcal{W}_{\underline{m}}$$

Suppose that there is a short root γ in either Ω or Γ with $\langle \gamma, z\alpha \rangle \neq 0$. Say $\gamma \in \Gamma$. Then $\langle \gamma, s_\Gamma z\alpha \rangle = \langle -\gamma, z\alpha \rangle \neq 0$ and, since $w \in \mathcal{W}_\alpha$, $s_\Gamma z\alpha \in \Delta(\underline{u})$. Thus $s_{s_\Gamma z\alpha} s_\Gamma s_\Omega(\gamma) = s_{s_\Gamma z\alpha}(\pm\gamma) \in \Delta(\underline{m})$ This contradicts (*) and thus there is no γ as above.

Since $z \notin \mathcal{W}_\alpha$ and $w = \overline{s_\Omega z} \in \mathcal{W}_\alpha$ there must be a root $\gamma \in \Omega$ with $\langle \gamma, z\alpha \rangle \neq 0$. By the previous paragraph, γ must be a long root and γ is the only root in Ω not orthogonal to $z\alpha$. Since $z\alpha \notin -\Delta(\underline{u})$, $z\alpha \neq \gamma$ and $z\alpha$ cannot be a long root. In particular, $s_\gamma z\alpha = s_\Omega z\alpha \in -\Delta(\underline{u})$ and thus $z\alpha \in \Delta(\underline{m})$. Since $ws_\alpha \alpha \in \Delta(\underline{u})$, there must be some $\lambda \in \Gamma$ with $\langle \lambda, z\alpha \rangle \neq 0$. Now, arguing as above, λ must be a long root. So $\lambda = \gamma$ and all other roots in Ω and Γ are orthogonal to $z\alpha$. However, $s_\gamma z\alpha \in -\Delta(\underline{u})$ and since $ws_\alpha\alpha \in \Delta(\underline{u})$, $s_\lambda z\alpha \in \Delta(\underline{u})$. This contradicts $\gamma = \lambda$ and completes the proof of (14.16).

Proof of (14.15): Let L_z be a composition factor of $U_\alpha(L_w)$ which occurs with multiplicity at least two. The character of $U_\alpha(L_w)$ is dominated by the character of $\Theta_\alpha(N_w)$. Since $ch(\Theta_\alpha(N_w)) = ch(N_w) + ch(N_{ws_\alpha})$ and since both N_w and N_{ws_α} are multiplicity free, L_z must be a composition factor of both N_w and N_{ws_α}. This contradicts (14.16) since $z \notin \mathcal{W}_\alpha$. Thus, $U_\alpha(L_w)$ is multiplicity free. Since it is a self-dual module it must also be semisimple.

It remains only to compute the composition factors of $U_\alpha(L_w)$. Suppose that L_y is a composition factor of $U_\alpha L_w$. Then, by the semisimplicity of $U_\alpha L_w$, $\mathrm{Ext}^1(L_y, L_w) \neq 0$. Let J_y be the maximal submodule of N_y. If $y \prec w$ then $\mathrm{Hom}(J_y, L_w) = 0$ and thus, by a long exact sequence, $\mathrm{Ext}^1(L_y, L_w)$ injects into $\mathrm{Ext}^1(N_y, L_w)$ and $\mathrm{Ext}^1(N_y, L_w) \neq 0$. Thus, by (14.14), if $y \prec w$ then $y = \overline{wr_\Omega}$ for some $\Omega \in \mathcal{E}_w$ with $\mathrm{card}(\Omega^+) = 1$. On the other hand, if $y \not\prec w$ then $\mathrm{Hom}(J_w, L_y) = 0$ and thus $\mathrm{Ext}^1(N_w, L_y) \neq 0$. By (14.14) this could only happen if $y = ws_\alpha$, since we must have $y \prec ws_\alpha$.

Conversely, suppose that $y = \overline{wr_\Omega}$ for some $\Omega \in \mathcal{E}_w$ with $card(\Omega^+) = 1$ and $y \notin \mathcal{W}_\alpha$. Let $Q_\alpha L_w$ be the maximal quotient of $\Theta_\alpha L_w$. From the short exact sequence $0 \to L_w \to \Theta_\alpha L_w \to Q_\alpha L_w \to 0$, we obtain the long exact sequence:

(**)

$$\mathrm{Hom}(N_y, \Theta_\alpha L_w) \to \mathrm{Hom}(N_y, Q_\alpha L_w) \to \mathrm{Ext}^1(N_y, L_w) \to \mathrm{Ext}^1(N_y, \Theta_\alpha L_w)$$

The first term in (**) must be zero, since $y \neq w$ and $y \neq ws_\alpha$. If $y\alpha \in \Delta(\underline{m})$ then $\psi_\alpha N_y = 0$, and so the last term is also zero. Suppose that $y\alpha \in \Delta(\underline{u})$, ($y\alpha \notin -\Delta(\underline{u})$ since $y \notin \mathcal{W}_\alpha$). For some γ in Σ_y, $w = \overline{s_\gamma y}$. Since $w\alpha \in -\Delta(\underline{u})$ and $y\alpha \in \Delta(\underline{u})$, this can only happen if α is a long root and γ is not orthogonal to $y\alpha$. A simple check in coordinates then shows that $w(\rho - \omega_\alpha)$ and $y(\rho - \omega_\alpha)$ have different parity (cf. (12.11)). Thus $\mathrm{Ext}^1(N_y, \Theta_\alpha L_w) \cong \mathrm{Ext}^1(\psi_\alpha N_y, \psi_\alpha L_w) = 0$ by (12.14). In any case, the last term of (**) is zero, and thus the second and the third terms are both nonzero (by (14.14)). Since L_y is the unique simple quotient of N_y, this shows that L_y must be a composition factor of $Q_\alpha L_w$. The exact sequence $0 \to U_\alpha L_w \to Q_\alpha L_w \to L_w \to 0$ implies that L_y must actually be a composition factor of $U_\alpha L_w$. Finally, L_{ws_α} must be a composition factor of $U_\alpha L_w$ because of the isomorphism $\mathrm{Hom}(N_{ws_\alpha}, \Theta_\alpha L_w) \cong \mathrm{Hom}(\psi_\alpha N_w, \psi_\alpha L_w) \cong \mathbb{C}$. This completes the proof of (14.15).

§15. Kazhdan-Lusztig polynomials

The Kazhdan-Lusztig conjecture [23] gives a recursion relation which determines the character formula for irreducible highest weight modules for any semisimple Lie algebra. This conjecture was proved in [2] using the theory of $\mathcal{D}$-modules and in [11] using the theory of holonomic systems. In the setting of classical Hermitian symmetric pairs, as one might guess, there are easier proofs of this conjecture. In [34], the semisimplicity of the modules $U_\alpha L(\nu)$ implies the validity of the conjecture. In turn, this semisimplicity is established in (14.15). This gives another proof for our setting.

The work of Lascoux and Schützenberger gives a combinatorial description of the Kazhdan-Lusztig polynomials in the case HS.1. They define generating functions $\sum_\nu q^{|\nu|}$ and then prove these polynomials are solutions to the Kazhdan-Lusztig recursion relations. However, these polynomials satisfy a much simpler recursion relation determined by the combinatorics. This relation is given in [26] and asserts (in their notation):

$$(15.1) \qquad Q^{v'\alpha\beta v''}_{w'\alpha\beta w''} = Q^{v'\beta\alpha v''}_{w'\alpha\beta w''} + q^c Q^{v'v''}_{w'w''} \quad \text{where} \quad c = |v'|_\alpha - |w'|_\alpha.$$

There are two main results in this section. The first is Theorem 15.4 which generalizes the relation (15.1) to the Hermitian symmetric cases HS.3 and HS.5. The second is Theorem 15.5 which gives explicit formulas in all the classical Hermitian symmetric cases for the Kazhdan-Lusztig polynomials in the language of chains as introduced in (14.8). Of course our hope is that similar representation theoretic techniques will yield similar results for $\mathcal{O}(\underline{g}, \underline{p})$ where $\underline{p}$ is any maximal parabolic subalgebra of a semisimple Lie algebra $\underline{g}$.

Following the work of Kazhdan and Lusztig [23] and Vogan [34], we define polynomials which we call KLV polynomials. Let $w_{\underline{g}}$ (resp. $w_{\underline{m}}$) be the element of maximal length in $\mathcal{W}$ (resp. $\mathcal{W}_{\underline{m}}$) and for $w \in \mathcal{W}$ define $\dot{w} = w_{\underline{m}} w w_{\underline{g}}$. For

$w \in \mathcal{W}^{\underline{m}}$, put $M_{\dot{w}} = M(\dot{w}\rho)$, $N_{\dot{w}} = N(\dot{w}\rho)$ and $L_{\dot{w}} = L(\dot{w}\rho)$. For $y, w \in \mathcal{W}^{\underline{m}}$, define a polynomial $Q_{y,w}(q)$ by :

$$
\begin{aligned}
Q_{y,w}(q) &= \sum_{j\geq 0} q^j \ \dim \operatorname{Ext}^{\ell(w)-\ell(y)-2j}(N_{\dot{y}}, L_{\dot{w}}) \\
&= \sum_{i\geq 0} q^{\frac{\ell(w)-\ell(y)-i}{2}} \ \dim \operatorname{Ext}^i(N_{\dot{y}}, L_{\dot{w}}).
\end{aligned}
\tag{15.2}
$$

From the parity (14.4), we know the sum over j can be taken over the integers or half integers. The result is the same. Let $P_{y,w}(q)$ denote the corresponding polynomials with Ext computed in $\mathcal{O}(\underline{g}, \underline{b})$ instead of $\mathcal{O}(\underline{g}, \underline{p})$ and with $N_{\dot{y}}$ and $L_{\dot{w}}$ replaced by $M(-y\rho)$ and $L(-w\rho)$ respectively. It is well known that these polynomials are related. We have:

Lemma 15.3 *For $y, w \in \mathcal{W}^{\underline{m}}$ and $r \in \mathcal{W}_{\underline{m}}$,*

$$Q_{y,w}(q) = P_{ry,w_{\underline{m}}w}(q).$$

Proof: Let $\underline{b} = \underline{h}\oplus\underline{n}$ and $\underline{n} = \underline{n}_{\underline{m}}\oplus\underline{u}$. Then $\underline{u}$ is an ideal of $\underline{n}$. Put $\xi = -w_{\underline{m}}y\rho - \rho$ and let F denote the irreducible finite dimensional $\underline{m}$-module with extreme weight ξ. For any $\underline{h}$-module E, let E^ξ denote the ξ-weight space of E. The Ext groups are related to cohomology as follows: $\operatorname{Ext}^i(N_{\dot{y}}, L_{\dot{w}}) \cong \operatorname{Hom}_{\underline{m}}(F, \operatorname{H}^i(\underline{u}, L_{\dot{w}}))$ and $\operatorname{Ext}^i(M_{r\dot{y}}, L_{\dot{w}}) \cong \operatorname{H}^i(\underline{n}, L_{\dot{w}})^{r(\xi+\rho)-\rho}$. Now the modules $\operatorname{H}^i(\underline{u}, L_{\dot{w}})$ are finite dimensional $\underline{m}$-modules, and so we may use Kostant's formula for computing the $\underline{n}_{\underline{m}}$-cohomology of these modules [25]. Then the Lyndon spectral sequence ([28], p. 351) proves the equality of these polynomials.

In the cases HS.2 and HS.4, the KLV polynomials can be computed explicitly from [16], as can the cases HS.1,HS.3 and HS.5 when $p = 1$. We remark that in all the $p = 1$ cases, the polynomials $Q_{y,w}$ are either one or zero depending as $w \prec y$ or not. So we now restrict to the remaining cases where recursion formulas are needed. Let $(\underline{g}, \underline{p})$ be of type HS.1, HS.3 or HS.5 and let notation be as in the remarks surrounding (14.4). Recall the notation $x \mapsto x'$ defined by: $L_x^\alpha = \Lambda(L'_{x'})$. Let $Q'_{y',w'}$ denote the KLV polynomial defined as above with $\underline{g}$, $\underline{p}$, $\underline{m}$ and $\mathcal{W}^{\underline{m}}$ replaced by $\underline{g}'$, $\underline{p}'$, $\underline{m}'$ and $\mathcal{W}'^{\underline{m}'}$ respectively.

Theorem 15.4 *Let y and w be in $\mathcal{W}^{\underline{m}}$.*

(i) If y is the element of maximal length in $\mathcal{W}^{\underline{m}}$ then $Q_{y,w}$ equals the constant polynomial one or zero depending as $y = w$ or not.

(ii) If y is not of maximal length then choose a simple root β with $ys_\beta \in \mathcal{W}_\beta$, i.e. $y\beta \in \Delta(\underline{u})$. We have the following two cases:

(a) Suppose that either $ws_\beta \notin \mathcal{W}_\beta$ or $(\underline{g}, \underline{p})$ is of type HS.3, β is a long root and $y(\rho - \omega_\beta)$ and $w(\rho - \omega_\beta)$ have opposite parity (cf. (12.11)). Then

$$Q_{y,w} = Q_{ys_\beta,w}.$$

(b) Suppose that $ws_\beta \in \mathcal{W}_\beta$. Also, for HS.3 and β long, assume that $y(\rho - \omega_\beta)$ and $w(\rho - \omega_\beta)$ have the same parity. Then

$$Q_{y,w} = Q_{ys_\beta,w} + q^r Q'_{y',w'}$$

with $2r = \ell(w) - \ell(y) - \ell'(w') + \ell'(y')$ and $\ell(\ )$ (resp. $\ell'(\ )$) denoting the length function on $\mathcal{W}$ (resp. $\mathcal{W}'$).

Proof: If y has maximal length in $\mathcal{W}^{\underline{m}}$ then $N_{\dot{y}}$ is projective. This proves (i). Now suppose y and β satisfy (ii). Put $\alpha = -w_{\underline{g}}\beta$. From (9.2) we obtain: for y, β as above, $ys_\beta \in \mathcal{W}_\beta$ if and only if $\dot{y} \in \mathcal{W}_\alpha$. Also, if β is long then $\alpha = \beta$ and $y(\rho - \omega_\beta)$ and $w(\rho - \omega_\beta)$ have the same parity if and only if $\dot{y}(\rho - \omega_\beta)$ and $\dot{w}(\rho - \omega_\beta)$ do. Therefore, case a) follows directly from (14.4) c) while case b) follows from (14.4) c) and d). This completes the proof.

After a change of notation from that used here to that of Lascoux and Schützenberger [26], we find that the identity (15.4, ii, b) for HS.1 is precisely the identity (15.1). Therefore the combinatorial generating functions $\sum_\nu q^{|\nu|}$ in [26] are the polynomials $Q_{y,w}$ for HS.1.

The notion of chain introduced in Definition 14.8 generalizes these results of Lascoux and Schützenberger . The formulas for Ext given in (14.9) translate as follows for KLV polynomials.

Theorem 15.5

(i) Let $y, w \in \mathcal{W}^{\underline{m}}$. The coefficient of q^j in the KLV polynomial $Q_{y,w}$ equals the number of chains from $\dot{y}$ to $\dot{w}$ of length $\ell(w) - \ell(y) - 2j$.

(ii) For $w \in \mathcal{W}^{\underline{m}}$ expand the character $chL_w = \sum_y (-1)^{\ell(y)-\ell(w)} a_y chN_y$ with $a_y \in \mathbb{Z}$. Then a_y equals the number of chains from y to w (of all lengths), or equivalently the number of positive chains from w to y.

§16. Decompositions of $U(\underline{u}^-)$-free self-dual $\underline{g}$-modules

We keep the notation and assumptions of section eight. So $(\underline{g}, \underline{p})$ is a Hermitian symmetric pair of classical type (cf. (8.1)) and $\mathcal{O} = \mathcal{O}(\underline{g}, \underline{p}, \rho)$. Let $\mathcal{D}$ be the full subcategory of $\mathcal{O}$ consisting of all modules which admit both a Verma flag and a nondegenerate $\underline{g}$-invariant bilinear form. In this section we give a complete description of the modules in $\mathcal{D}$. This description then yields several corollaries concerning the signatures of Hermitian forms.

Recall the modules $D(\nu)$ from (2.7). Let $D_w = D(w\rho)$ for $w \in \mathcal{W}^{\underline{m}}$.

Proposition 16.1 *Let $X \in \mathcal{D}$. Then X admits a symmetric nondegenerate $\underline{g}$-invariant bilinear form ϕ. Furthermore, there is an orthogonal direct sum decomposition (with respect to ϕ): $X = \bigoplus_{i\in I} X_i$ where $X_i = D_{w_i}$ for some $w_i \in \mathcal{W}^{\underline{m}}$, $i \in I$.*

Proof: This is exactly Theorem 1.8 in [16]. The hypotheses (1.7) and (1.4) of [16] have been verified here in (8.5).

Corollary 16.2 *Let $X, Y \in \mathcal{D}$ and assume $chX = chY$. Then X and Y are isomorphic.*

For any module X in $\mathcal{O}(\underline{g}, \underline{p})$, let $X^\vee$ denote the $\underline{h}$-finite contravariant dual module to X. Then $X^\vee$ is in $\mathcal{O}(\underline{g}, \underline{p})$ and $chX = chX^\vee$.

Proposition 16.3 *Let $X \in \mathcal{O}$ and assume that both X and $X^\vee$ are free $U(\underline{u}^-)$-modules. Then X admits a symmetric nondegenerate form; i.e., X lies in $\mathcal{D}$.*

Proof: Put $Y = X \oplus X^\vee$. Then $Y \in \mathcal{D}$ and by (16.1), $Y = \bigoplus_w Y_w$ where each summand Y_w is itself a direct sum of copies of D_w, $w \in \mathcal{W}^{\underline{m}}$. Therefore Y has a decomposition $Y = \bigoplus_{w,i} Y_{w,i}$ where $Y_{w,i} \cong D_w$. We say this decomposition satisfies (C, z) for $z \in \mathcal{W}^{\underline{m}}$ if for all $w \succ z$ and all i, $Y_{w,i}$ is a submodule of either X or $X^\vee$.

Suppose the decomposition of Y does not satisfy (C, z) for some $z \in \mathcal{W}^{\underline{m}}$. Choose $u \in \mathcal{W}^{\underline{m}}$ maximal in the Bruhat order with (C, u) not satisfied. Set $Y' = \bigoplus Y_w$ with the sum taken over all $w \in \mathcal{W}^{\underline{m}}$ for which (C, w) is satisfied. By maximality of u choose an invariant symmetric form ϕ on Y which has nondegenerate restrictions to both $X \cap Y'$ and $X^\vee \cap Y'$. Set $\underline{Y}$(resp. $\underline{X}$, $\underline{X}^\vee$) equal to the orthogonal complement of Y' (resp. $X \cap Y'$, $X^\vee \cap Y'$) in Y (resp. X, $X^\vee$). The maximality of u gives:

$$\underline{Y} = \underline{X} \oplus \underline{X}^\vee \quad , \quad Y = Y' \oplus \underline{Y}. \tag{16.4}$$

Put $\nu = u\rho - \rho$ and let L denote the weight space $\underline{Y}_\nu$. Then ν is a maximal weight of $\underline{Y}$. Fix any nondegenerate symmetric bilinear form ψ' on L which has nondegenerate restrictions to both $L \cap X$ and $L \cap X^\vee$. The module $\underline{Y}$ has a decomposition as above: $\underline{Y} = \bigoplus_w \underline{Y}_w$. Since ν is a maximal weight, L equals $L \cap \underline{Y}_u$. In turn from [16, (3.1)], this implies that ψ' extends to an invariant symmetric form ψ on $\underline{Y}$.

Decompose $\underline{Y}_u$ into a sum $E \oplus F$ where E and F are each direct sums of D_u and $L \cap E = L \cap X$, $L \cap F = L \cap X^\vee$. Note that the socles of E and F are contained in X and $X^\vee$ respectively. Therefore if π and $\pi^\vee$ denote the projections of Y onto X and $X^\vee$, then π and $\pi^\vee$ induce isomorphisms: $E \xrightarrow{\sim} \pi E$, and $F \xrightarrow{\sim} \pi^\vee F$. By (16.4), these modules lie in $\underline{Y}$. Moreover, from [16, (3.1)], the restrictions of ψ to πE, $\pi^\vee F$ and $\pi E \oplus \pi^\vee F$ are nondegenerate, since $(\pi E)_\nu = L \cap X$ and $(\pi^\vee F)_\nu = L \cap X^\vee$. Taking complements with respect to ψ-orthogonality we obtain $\underline{Y} = \pi E \oplus \pi^\vee F \oplus G$. Recall the original decomposition of Y. Replacing the summands $Y_{u,i}$ by the summands of πE and $\pi^\vee F$ and the summands $Y_{w,i}$ not in Y' by summands of G, we obtain a decomposition of Y which satisfies (C, u). By induction we obtain a decomposition of Y which satisfies (C, w) for all $w \in \mathcal{W}^{\underline{m}}$. This completes the proof of (16.3).

If ϕ is a nondegenerate Hermitian form on a finite dimensional vector space E then for some basis of E, ϕ is represented by a diagonal matrix with a 1's and b -1's. We call the pair (a, b) the signature of ϕ. Let $X \in \mathcal{O}$ and let ψ be a nondegenerate invariant Hermitian form on X. Define the signature of ψ, $S(\psi)$, to be a map $S(\psi)$: weights of $X \rightarrow \mathbb{N} \times \mathbb{N}$. For a weight λ of X, $S(\psi)(\lambda)$ is the signature of ψ restricted to the finite dimensional λ-weight space of X.

Proposition 16.5 [16, Theorem 1.10] *D_w admits a nondegenerate invariant Hermitian form. Moreover, if ϕ and ψ are two such forms then either*

$$S(\phi) = S(\psi) \qquad \text{or} \qquad S(\phi) = -S(\psi).$$

For any $X \in \mathcal{D}$, by (16.1) we may write: $X = \bigoplus X_w$ with X_w isomorphic to a direct sum of d_w-copies of D_w, $w \in \mathcal{W}^{\underline{m}}$. If ϕ is a nondegenerate $\underline{g}$-invariant Hermitian form on X then we may assume this decomposition is orthogonal with respect to ϕ. Put ϕ_w equal to the restriction of ϕ from X to X_w.

Proposition 16.6 [16, Theorem 1.12] *Let X and Y be in $\mathcal{D}$ with nondegenerate invariant Hermitian forms ϕ and ψ respectively. Assume $S(\phi) = S(\psi)$. Then X and Y are isomorphic and for all $w \in \mathcal{W}^{\underline{m}}$, $S(\phi_w) = S(\psi_w)$.*

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