NOTES

Long Days on the Fibonacci Clock

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The rule generating the Fibonacci sequence is extraordinarily simple, but its repeated application produces rich mathematics. If we leave the rule alone and change the starting values, we again find sequences with interesting properties—such as the Lucas numbers. If we introduce modular arithmetic, there are new questions to answer. In what follows, I study Fibonacci sequences in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, the integers mod p, where p is a prime. I like to call these the "$\mathbb{F}_p$ibonacci numbers." Regardless of the starting pair, the sequence will repeat [**10**]. The question I want to answer is "What is the maximum period for any Fibonacci sequence in $\mathbb{Z}/p\mathbb{Z}$?"

In what follows, I present a particular point of view about the Fibonacci sequence in a way that gives some insight into both the standard sequence and its variations. Specifically, the Fibonacci sequence is interpreted in terms of a matrix acting on a finite set, an idea that is related to group actions and to (discrete) dynamical systems. The underlying set is the two-dimensional vector space ${\mathbb{F}_p}^2$; the matrix M from (2) provides the rule for the process. Iterations of the system correspond to powers of the matrix. Periods in the system are related, then, to powers of M that are equivalent modulo p to the identity matrix. The point of view works for all cases, even for the generalized $\mathbb{F}_p$ibonacci numbers, where weights are allowed in the recursion formula. For a thorough look at dynamical systems and number theory, Silverman's book [**8**] is an excellent source.

Most of what is contained here is not new. Searching *Mathematical Reviews* turns up dozens of articles about periods of Fibonacci numbers in $\mathbb{Z}/m\mathbb{Z}$, including many where m doesn't even have to be a prime or a power of a prime. The most-referenced article is Wall's article [**10**] in the MONTHLY in 1960. Wall established many fundamental results, and posed some tantalizing questions. In particular, he showed that the period divides $(p-1)$ when 5 is a quadratic residue mod p and divides $(2p+2)$ when it is not, but he did not find the maximal periods. Wall's investigation was motivated by a search for methods of generating pseudorandom numbers. Later, Brent [**1**], also motivated by pseudorandom numbers, considered the special properties of Fibonacci sequences modulo a power of 2. The story, however, begins even before the days of *Mathematical Reviews*. In the 1930s, Ward [**11**] considered periods, both minimal and maximal, and other characteristics of sequences arising from rather general recurrence relations, not just the Fibonacci relation. Kalman and Mena's article in an earlier issue of this MAGAZINE [**5**] examines many of the famous properties of the Fibonacci numbers as specific instances of properties of general second-order recurrences. Ward's results built on even earlier work by Carmichael [**2**] and others. For the early history of the subject, the curious reader should consult Dickson's history [**3**], particularly Volume I, Chapter XVII, where elements of the problem are traced back to Gauss and Lagrange. Earlier in the MAGAZINE, Vella and Vella [**9**] looked at possible periods in the generalized Fibonacci sequence modulo a prime. Their approach emphasized recursive formulas and led to similar results to those here, but are somewhat less precise when applied to the standard Fibonacci numbers.

This investigation stems from a homework assignment from my daughter's fourth-grade mathematics class. The students were taught the Fibonacci recursion relation and how to reduce mod 100. The assignment was to find two starting numbers that gave the longest sequence before it repeated mod 100. Being the child of a mathematician, my daughter tried to *solve* the problem. It turns out the teacher imagined that the students would try some numbers and make some guesses about what would work best. This article shows what to do when the reduction is modulo a prime. If you would like to complete the fourth-grade assignment, you may apply the prime case and a little extra work to find the longest possible period for reduction modulo a composite.

Some examples Let $p = 19$. Since order matters and repeats are allowed, there are $19^2 = 361$ possible choices for the starting pair a_0 and a_1. The standard sequence, which starts with $a_0 = 1$ and $a_1 = 1$, becomes

$$1, 1, 2, 3, 5, 8, 13, 2, 15, 17, 13, 11, 5, 16, 2, 18, 1, 0, 1, 1, \dots \pmod{19},$$

which has period 18. Using *Maple* to try all possible starting pairs shows that three hundred forty-two of them have period 18, eighteen of them have period 9, and one has period 1. In this case the maximum period is 18.

For $p = 23$, there are $23^2 = 529$ possible starting pairs. Direct computation shows that, other than the trivial sequence with $a_0 = a_1 = 0$, all the sequences have period equal to 48, making 48 the maximal period.

For $p = 29$, there are $29^2 = 841$ possible starting pairs. The standard sequence has period 14, as do eight hundred eleven other sequences. Twenty-eight sequences have period 7, and the trivial sequence has period 1.

Comment Since the sequences are periodic, it is a bit unnatural to say that, in the first example, the sequence starting with $a_0 = 1$ and $a_1 = 1$ is different from the sequence starting with $a_0 = 5$ and $a_1 = 8$, since they eventually come around to match up with each other. However, for these examples this is a convenient way to count.

At first glance, it seems that the periods are all over the map: Sometimes the period is $p - 1$, sometimes it is much less. Sometimes it is even bigger than p. However, by analyzing the sequences, in particular by examining the matrix that generates them all, certain features emerge that allow us to divide the problem into cases where the pattern becomes clear.

The problem

The sequences in question are

$$\begin{gathered} a_0, a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots, \\ a_{k+1} = a_k + a_{k-1} \qquad (1) \\ \text{with } a_N \equiv a_0 \pmod{p} \text{ and } a_{N+1} \equiv a_1 \pmod{p} \end{gathered}$$

and N is some number we don't know in advance. We will not restrict to the standard starting values of $a_0 = 0$ and $a_1 = 1$, which makes the question more interesting. We will assume that $p \neq 2$, since we often have to divide by 2. Most of the time, we will also assume that $p \neq 5$ to avoid a similar complication, as you will see. A few basic facts from number theory are used, and can be found in the classic texts by Hardy and Wright [**4**] or Niven and Zuckerman [**7**].

A convenient way of generating the sequence is to call on linear algebra. The standard trick is to write the recursion relation as:

$$\begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{k-1} \\ a_k \end{pmatrix}. \tag{2}$$

For convenience, let $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Note that this is the companion matrix of the polynomial $x^2 - x - 1$, which is important for Fibonacci numbers. We can now write:

$$\begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix} = M^k \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}. \tag{3}$$

In this formulation, the period of the Fibonacci sequence starting with a_0 and a_1 is the smallest positive integer k such that

$$M^k \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \equiv \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \pmod{p}. \tag{4}$$

Our problem, then, is to find a_0 and a_1 so that k is as large as possible. From (4), we see that k will always be less than any n such that

$$M^n \equiv I \pmod{p}, \tag{5}$$

where I is the 2×2 identity matrix. Moroever, the period of any Fibonacci sequence will be a divisor of n, meaning our k must divide n. As a result, the smallest such n, denoted $n(p)$, is an upper bound on the longest period.

Standard trick, part II: diagonalize the matrix:

$$M = A^{-1} D A. \tag{6}$$

The eigenvalues of M are $\mu = (1 + \sqrt{5})/2$ and $\overline{\mu} = (1 - \sqrt{5})/2$. Thus $D = \operatorname{diag}(\mu, \overline{\mu})$ and

$$A = \begin{pmatrix} 1 & 1 \\ \mu & \overline{\mu} \end{pmatrix}.$$

Then, $M^n \equiv I$ exactly when $\mu^n \equiv 1 \pmod{p}$ and $\overline{\mu}^n \equiv 1 \pmod{p}$. It's enough to figure out the minimum n for one of the two eigenvalues. I pick μ.

The solution

There are three cases to examine. We leave the case $p = 5$ to the end, as it is very different from the others. In each case, there are two tasks: compute $n(p)$ and determine whether any sequences of this maximal period occur.

Case 1: Suppose 5 is quadratic residue of *p* In this case, 5 has a square root in $\mathbb{F}_p$. By quadratic reciprocity, 5 is a quadratic residue of p when $p = 5N \pm 1$. Since the interesting values of p are odd, we actually have $p = 10N \pm 1$. Moreover, in this case, both μ and $\overline{\mu}$ are also elements of $\mathbb{F}_p$.

A *primitive element* of $\mathbb{F}_p^*$ is an element that generates $\mathbb{F}_p^*$ as a multiplicative group. If x is a primitive element of $\mathbb{F}_p^*$, then group theory tells us that $x^{p-1} \equiv 1$ and $p - 1$

is the least such exponent. Here $p-1$ is the the number of elements of $\mathbb{F}_p^*$. If x is not primitive, then the least such exponent is the order of x in the multiplicative group $\mathbb{F}_p^*$, which is necessarily a divisor of $p-1$.

PROPOSITION 1. *If 5 is a quadratic residue of p, then the smallest $n = n(p)$ satisfying* (5) *is the order of μ in $\mathbb{F}_p^*$. Moreover, there is at least one sequence with period $n(p)$.*

Proof. We have already seen that $n(p)$ equals the order of μ. For the second statement, we note that any nonmaximal period k corresponds to a nontrivial solution to (4), which means that $M^k - I$ has a nontrivial nullspace as a linear transformation on ${\mathbb{F}_p}^2$, the two-dimensional vector space over $\mathbb{F}_p$. The only way this nullspace can be nontrivial is for k to be a divisor of $n(p)$. We can now see that there won't be any sequences of maximal possible length if and only if the nullspaces of $M^k - I$, running over all *proper* divisors k of $n(p)$, exhaust ${\mathbb{F}_p}^2$. Now, since the nullspace is a vector space over $\mathbb{F}_p$, it will have either 1 or p elements. (We are already assuming that it's not the whole space.) However, $n(p)$ is either $p-1$ or a divisor of $p-1$, and there are fewer than p proper divisors of $p-1$. So by multiplying and counting, we see that the union of these nullspaces has fewer than p^2 elements, and cannot be all of ${\mathbb{F}_p}^2$. Hence, there must be at least one Fibonacci sequence with maximal period, $n(p)$. ■

Case 2: Suppose 5 is not a quadratic residue of p In this case, μ and $\overline{\mu}$ are not elements of $\mathbb{F}_p$. It is necessary, then, to work over the field $\mathbb{F}_p(\sqrt{5}) \cong \mathbb{F}_{p^2}$. The problem becomes finding the order of μ in $\mathbb{F}_p(\sqrt{5})^*$.

Write out

$$\mu^{p+1} = \left(\frac{1+\sqrt{5}}{2}\right)^{p+1} = \frac{1}{2^{p+1}}(1+\sqrt{5})^{p+1}$$

and reduce mod p. Reducing the denominator as $2^{p+1} = 2^p \cdot 2 \equiv 2 \cdot 2 \equiv 4$, since $x^p \equiv x \pmod{p}$ for all x, gives $1/2^{p+1} \equiv 1/4$. The second factor reduces as:

$$\begin{aligned}(1+\sqrt{5})^{p+1} &= \Big((\sqrt{5})^{p+1} + (p+1)(\sqrt{5})^p + \frac{p(p+1)}{2}(\sqrt{5})^{p-1} + \cdots \\ &\qquad + \frac{p(p+1)}{2}(\sqrt{5})^2 + (p+1)\sqrt{5} + 1\Big) \\ &\equiv \Big((\sqrt{5})^{p+1} + (\sqrt{5})^p + 0 + \cdots + 0 + \sqrt{5} + 1\Big) \pmod{p}.\end{aligned}$$

Now, compute $(\sqrt{5})^{p-1} = 5^{(p-1)/2}$: In general, if a is *any* quadratic nonresidue of p, then

$$(p-1)! \equiv a^{(p-1)/2} \pmod{p}, \tag{7}$$

which can be seen by multiplying together the (necessarily unequal) pairs of elements x and x' such that $x \cdot x' \equiv a \pmod{p}$. On the other hand, by doing a similar thing for $a = -1$ (and handling separately the cases when -1 is and is not a quadratic residue) we get Wilson's Theorem,

$$(p-1)! \equiv -1 \pmod{p}. \tag{8}$$

By combining (7) and (8), we see

$$(\sqrt{5})^{p-1} = 5^{(p-1)/2} \equiv -1 \pmod{p}.$$

Comment This formula was already known to Euler, but we will want to recall the method when considering generalized sequences in the last section. Now, substituting this into the expansion of $(1+\sqrt{5})^{p+1}$, we obtain

$$\begin{aligned}\mu^{p+1} &\equiv \frac{1}{2^{p+1}}\left((\sqrt{5})^{p+1}+(\sqrt{5})^{p}+\sqrt{5}+1\right)\\ &\equiv (1/4)\left((5\cdot 5^{(p-1)/2}+1)+\sqrt{5}(5^{(p-1)/2}+1)\right)\\ &\equiv (1/4)\left((5(-1)+1)+\sqrt{5}(-1+1)\right)\equiv (1/4)(-4)\equiv -1\end{aligned}$$

Then, $\mu^{2(p+1)}=1$ in $\mathbb{F}_p(\sqrt{5})$. Moreover, since $\mu^{p+1}=-1$ in $\mathbb{F}_p(\sqrt{5})$, we see that $2(p+1)$ is the least such exponent.

PROPOSITION 2. *If 5 is not a quadratic residue of p, then the smallest $n=n(p)$ satisfying* (5) *is $n(p)=2(p+1)$. There is at least one sequence with period $n(p)$.*

Proof. We have already computed the value of $n(p)$. The proof of the second statement is essentially the same sort of counting argument as in the first case, which shows that there aren't enough "short periods" to exhaust the ${\mathbb{F}_p}^2$ of possible sequences. Therefore, the maximum is attained in this case, too. ■

Case 3: $p = 5$ Since $5\equiv 0 \pmod 5$, we have $\mu\equiv(1+0)/2\equiv 1/2\equiv 3 \pmod 5$ and $\overline{\mu}\equiv(1-0)/2\equiv 1/2\equiv 3=\mu$. Thus, the matrix A in our earlier analysis is

$$A=\begin{pmatrix}1 & 1\\ \mu & \overline{\mu}\end{pmatrix}\equiv\begin{pmatrix}1 & 1\\ 3 & 3\end{pmatrix},$$

which is singular, i.e., there is no A^{-1}. The point is that we cannot diagonalize the Fibonacci matrix M in this case. However, $\mathbb{Z}/(5)$ is small, and it is not too hard to run through all the possibilities. Starting with $a_0=1$ and $a_1=1$ leads to the sequence

$$1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, \text{ repeat}$$

which has period 20. This is the period for most choices of initial values. However, for $a_0=1$ and $a_1=3$, the sequence is just

$$1, 3, 4, 2, \text{ repeat}$$

which has a period of just 4. The only other sequence is the trivial sequence $0, 0, 0, \ldots$.

Summary for Fibonacci sequences If 5 is a quadratic residue of p, then the maximal period of any Fibonacci sequence in $\mathbb{F}_p$ is the order of μ in $\mathbb{F}_p^*$. This is the maximal value of $p-1$ when μ is a primitive element. If μ is not primitive, then the order is some divisor of $p-1$, which needs to be determined by a direct calculation. By quadratic reciprocity, the primes p are of the form $p=10N\pm 1$.

If 5 is a quadratic nonresidue of p, then the maximal period of a Fibonacci sequence in $\mathbb{F}_p$ is $2(p+1)$. By quadratic reciprocity, these primes are of the form $p=10N+7$ and $p=10N+3$.

For $p=5$, the possible nontrivial periods are 4 and 20.

Generalized Fibonacci numbers

The Fibonacci recurrence relation can be generalized to allow for weights: $b_n=\alpha b_{n-1}+\beta b_{n-2}$, where α and β are integers. We can then ask for the maximal periods of sequences of generalized Fibonacci numbers, modulo a prime p:

$$\begin{gathered} b_0, b_1, \dots, b_{k-1}, b_k, b_{k+1}, \dots, b_n = \alpha b_{n-1} + \beta b_{n-2} \\ b_N \equiv b_0 \pmod{p} \text{ and } b_{N+1} \equiv b_1 \pmod{p}. \end{gathered} \tag{9}$$

The weights α and β need not be positive integers. However, in order to avoid accidental multiplication by zero, we should make sure the α and β are both relatively prime to p.

The arguments used for the standard Fibonacci now carry over, but become harder. The matrix becomes $M = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}$. The eigenvalues become

$$\mu = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2} \quad \text{and} \quad \overline{\mu} = \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2}.$$

Let $D = \alpha^2 + 4\beta$ be the discriminant of the polynomial $x^2 - \alpha x - \beta$, which plays the role of $x^2 - x - 1$ from the standard Fibonacci numbers. Assume, for the time being, that $D \neq 0$. If D is a quadratic residue of p, then the argument in Case 1 goes through *mutatis mutando*.

If D is not a quadratic residue of p, then we need to be more careful. We again need to work in an extension field of $\mathbb{F}_p$, this time the field is $\mathbb{F}_p(\sqrt{D})$. The essential problem is to determine the orders of μ and $\overline{\mu}$ in $\mathbb{F}_p(\sqrt{D})^*$. As before, the order of this multiplicative group is $p^2 - 1$, which factors as $(p-1)(p+1)$. Computing μ^{p+1} is a little more difficult now, since α and β are not explicit, meaning we don't have a tidy expression for μ. However, we can use the isomorphism $\mathbb{F}_p(\sqrt{D}) \cong \mathbb{F}_{p^2}$ and a standard fact about the map $x \mapsto x^p$ in a field of characteristic p. This map, denoted Fr, is called the *Frobenius map* or the Frobenius endomorphism, and it is very useful in number theory. In our setting, the Frobenius map has the following useful properties, as can be found in the book by Mullen and Mummert [**6**]:

$$\begin{aligned} \mathrm{Fr}(xy) &= \mathrm{Fr}(x)\mathrm{Fr}(y) \\ \mathrm{Fr}(x+y) &= \mathrm{Fr}(x) + \mathrm{Fr}(y) \\ \mathrm{Fr}(x) &= x \quad \text{if and only if } x \in \mathbb{F}_p \\ \mathrm{Fr}^2 &= \mathrm{Id}. \end{aligned}$$

The first two properties are just another way of saying that Fr is an endomorphism, that is, a homomorphism from $\mathbb{F}_p(\sqrt{D})$ to itself. The last property is special to the case of a quadratic extension, and can be deduced using reasoning similar to the computation presented in Case 2 above. The last two properties combine to imply $\mathrm{Fr}(\sqrt{D}) = -\sqrt{D}$.

We can now set about computing μ^{p+1}. Write out:

$$\mu^{p+1} = \left(\frac{\alpha + \sqrt{D}}{2} \right)^{p+1} = \frac{1}{2^{p+1}} (\alpha + \sqrt{D})^{p+1}$$

and reduce mod p. As before, the denominator reduces as $2^{p+1} = 2^p 2 \cong 2 \cdot 2 = 4$. The second factor reduces as:

$$\begin{aligned} (\alpha + \sqrt{D})^{p+1} &= (\alpha + \sqrt{D})^p (\alpha + \sqrt{D}) = \mathrm{Fr}(\alpha + \sqrt{D})(\alpha + \sqrt{D}) \\ &= (\alpha - \sqrt{D})(\alpha + \sqrt{D}) = \alpha^2 - D \end{aligned}$$

But $D = \alpha^2 + 4\beta$, so the second factor reduces to $\alpha^2 - (\alpha^2 + 4\beta) = -4\beta$ and

$$\mu^{p+1} \cong -\beta.$$

For the standard Fibonacci numbers, $\beta = 1$. So we knew that $\beta^2 = 1$ and could conclude that $\mu^{2(p+1)} \cong 1$. Now, however, we need to know the order d of $-\beta$ in $\mathbb{F}_p^*$. This is not an easy problem in general. All we really know is that d must divide $(p-1)$, the order of $\mathbb{F}_p^*$. Thus, the best we can conclude is only that the maximum period of the generalized Fibonacci sequence is $d(p+1)$, and we are left with separate computations for every case. Again, a counting argument verifies that the maximal period $n(p)$ does occur. For $\alpha = 3$, $\beta = 7$, and $p = 13$, the maximum *possible* period is $n(p) = p^2 - 1 = 168$. Using the starting values $b_0 = 0$ and $b_1 = 1$, a computation using *Maple* shows that this maximum period does indeed occur.

If the discriminant $D = \alpha^2 + 4\beta$ is zero, the situation is rather different. Observe that $\beta = -(\alpha/2)^2$ and, for notational convenience, let $\lambda = \alpha/2$. The recurrence relation now becomes:

$$b_n = 2\lambda b_{n-1} - \lambda^2 b_{n-2}$$

and the matrix becomes

$$M = \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 2\lambda \end{pmatrix}.$$

Unfortunately, M is not diagonalizable. It has Jordan form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \text{ so that } J^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}.$$

If k is a period for this generalized Fibonacci sequence, then we want to find k such that $J^k \equiv I$, which means

$$\begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p}.$$

Since $\mathbb{Z}/p\mathbb{Z}$ is a field, the equality of the $(2, 2)$ entries tells us that $\lambda^k \equiv 1 \pmod{p}$. This implies that k is a multiple of the order of λ in $(\mathbb{Z}/p\mathbb{Z})^*$, which is a divisor of $(p-1)$. Comparison of the $(1, 2)$ entries tells us that $k\lambda^{k-1} \equiv 0$. Again using that $\mathbb{Z}/p\mathbb{Z}$ is a field, we deduce that one of the factors must be zero. But a power of λ is zero only if λ itself is zero, so we are left with $k \equiv 0 \pmod{p}$, implying that k is a multiple of p. So the longest possible period is kp, where k is the order of λ in $(\mathbb{Z}/p\mathbb{Z})^*$. When λ is primitive, the longest possible period is $p(p-1)$.

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Fooling Newton's Method as Much as One Can

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We enjoyed reading how Horton [**1**] "fooled Newton's method" with an example where the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges but its limit does not satisfy $f(x) = 0$. Indeed, if

$$f(x) = \begin{cases} \pi - 2x \sin \frac{\pi}{x} & \text{for } x \neq 0, \\ \pi & \text{for } x = 0, \end{cases} \tag{1}$$

then the Newton sequence is

$$x_{n+1} = x_n - \frac{1}{2} \frac{\pi x_n - 2x_n^2 \sin \frac{\pi}{x_n}}{\pi \cos \frac{\pi}{x_n} - x_n \sin \frac{\pi}{x_n}},$$

and, starting from $x_1 = 1/2$, we have $x_2 = 1/4, x_3 = 1/8, \ldots, x_n = 1/2^n \to 0$, although $f(0) = \pi \neq 0$.

Can *f* be differentiable? Note that the function in (1) is not differentiable at $x = 0$. Since we thought that a differentiable function would fool the method even better, we wanted to know if such a function exists. Simply modifying Horton's function, we found an example that readers might find even more surprising:

$$f(x) = \begin{cases} \pi - x^2 \sin \frac{\pi}{x^2} & \text{for } x \neq 0, \\ \pi & \text{for } x = 0. \end{cases} \tag{2}$$