

THE STEINBERG VARIETY AND REPRESENTATIONS OF REDUCTIVE GROUPS

J. MATTHEW DOUGLASS AND GERHARD RÖHRLE

1. INTRODUCTION

Suppose G is a connected, reductive algebraic group defined over an algebraically closed field k , $\mathcal{B}$ is the variety of Borel subgroups of G , and u is a unipotent element in G . Let $\mathcal{B}_u$ denote the closed subvariety of $\mathcal{B}$ consisting of those Borel subgroups that contain u , let r denote the rank of G , and let C denote the conjugacy class of u .

In 1976, motivated by the problem of proving the equality conjectured by Grothendieck

$$(*) \quad \dim Z_G(u) = r + 2 \dim \mathcal{B}_u,$$

Steinberg [Ste76], in order to get the multiplicity 2 in the picture, introduced a variety of triples

$$S = \{ (v, B, B') \in C \times \mathcal{B} \times \mathcal{B} \mid v \in B \cap B' \}.$$

By analyzing the geometry of the variety S he was able to prove $(*)$ in most cases. In addition, by exploiting the fact that G -orbits on $\mathcal{B} \times \mathcal{B}$ are canonically indexed by elements of the Weyl group of G , he showed that S could be used to establish relationships between Weyl group elements and unipotent elements in G .

Now assume that the characteristic of k is good for G , let $\mathfrak{g}$ denote the Lie algebra of G , and let $\mathfrak{N}$ denote the variety of nilpotent elements in $\mathfrak{g}$. Then there is a G -equivariant isomorphism between $\mathfrak{N}$ and the variety of unipotent elements in G . The *Steinberg variety* of G is

$$Z = \{ (x, B, B') \in \mathfrak{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \operatorname{Lie}(B) \cap \operatorname{Lie}(B') \}.$$

In the thirty years since Steinberg first exploited the variety S , the Steinberg variety has played a key role in advancing our understanding of objects that at first seem to be quite unrelated:

- Representations of Weyl groups.
- The geometry of nilpotent orbits in $\mathfrak{g}$ and their covers.
- Differential operators on $\mathcal{B}$.
- Primitive ideals in the universal enveloping algebra of $\mathfrak{g}$.
- Representations of p -adic groups and the local Langlands program.

Date: February 8, 2008.

2000 *Mathematics Subject Classification.* Primary 22E46, 19L47, 20G05; Secondary 14F99, 20G99.

The authors would like to thank their charming wives for their unwavering support during the preparation of this paper.

In this paper we hope to give readers who are familiar with the some aspects of the representation theory of semisimple algebraic, or Lie, groups, but who are not specialists in this particular flavor of geometric representation theory, an overview of the main results that have been proved using the Steinberg variety. In the process we hope to make these results more accessible to non-experts and at the same time emphasize the unifying role played by the Steinberg variety.

We will more or less follow the historical development, beginning with concrete, geometric constructions and then progressing to increasingly more advanced and abstract notions.

In §2 we analyze the geometry of Z , including applications to characteristic varieties and primitive ideals.

In §3 we study the Borel-Moore homology of Z and the relation with representations of Weyl groups. Soon after Steinberg introduced his variety S , Kazhdan and Lusztig [KL80], defined an action of $W \times W$ on the top Borel-Moore homology groups of Z . Following a suggestion of Springer, they showed that the representation of $W \times W$ on the top homology group, $H_{4n}(Z)$, is the two-sided regular representation of W . Somewhat later, Ginzburg [CG97] defined a multiplication on the total Borel-Moore homology of Z . With this multiplication, $H_{4n}(Z)$ is a subalgebra isomorphic to the group algebra of W .

The authors [DR08a] [DR08b] have used Ginzburg's construction to describe the top Borel-Moore homology groups of the generalized Steinberg varieties $X_{0,0}^{\mathcal{P},\mathcal{Q}}$ and $X_{\text{reg,reg}}^{\mathcal{P},\mathcal{Q}}$ in terms of W , as well as to give an explicit, elementary, computation of the total Borel-Moore homology of Z : the total Borel-Moore homology of Z is isomorphic to the smash product of the coinvariant algebra of W and the group algebra of W .

Orbital varieties arise naturally in the geometry of the Steinberg variety. Using Ginzburg's formalism, Hinich and Joseph [HJ05] have used Ginzburg's construction to prove a conjecture of Joseph about inclusions of closures of orbital varieties.

In §4 we study the equivariant K -theory of Z and what is undoubtedly the most important result to date involving the Steinberg variety: the Kazhdan-Lusztig isomorphism [KL87] between $K^{G \times \mathbb{C}^*}(Z)$ and the extended, affine Hecke algebra $\mathcal{H}$. Using this isomorphism, Kazhdan and Lusztig were able to classify the irreducible representations of $\mathcal{H}$ and hence to classify the representations containing a vector fixed by an Iwahori subgroup of the p -adic group with the same type as the Langlands dual of G . In this way, the Steinberg variety plays a key role in the local Langlands program and also leads to a better understanding of the extended affine Hecke algebra.

Very recent work involving the Steinberg variety centers around attempts to categorify the isomorphism between the specialization of $K^{G \times \mathbb{C}^*}(Z)$ at p and the Hecke algebra of Iwahori bi-invariant functions on ${}^L G(\mathbb{Q}_p)$. Because of time and space constraints, we leave a discussion of this research to a future article.

2. GEOMETRY

For the rest of this paper, in order to simplify the exposition, we assume that the derived group of G is simply connected and that $k = \mathbb{C}$. Most of the results below hold, with obvious modifications, for an arbitrary reductive algebraic group when the characteristic of k is zero or very good for G .

Fix a Borel subgroup B in G and a maximal torus T in B . Define U to be the unipotent radical of B and define $W = N_G(T)/T$ to be the Weyl group of (G, T) . Set $n = \dim \mathcal{B}$ and $r = \dim T$.

We will use the convention that a lower case fraktur letter will denote the Lie algebra of the algebraic group denoted by the corresponding upper case roman letter.

For x in $\mathfrak{N}$, define $\mathcal{B}_x = \{gBg^{-1} \mid g^{-1}x \in \mathfrak{b}\}$, the *Springer fibre* at x .

2.1. Irreducible components of Z , Weyl group elements, and nilpotent orbits.

We begin analyzing the geometry of Z using ideas that go back to Steinberg [Ste76] and Spaltenstein [Spa82].

The group G acts on $\mathcal{B}$ by conjugation and on $\mathfrak{N}$ by the adjoint action. This latter action is denoted by $(g, x) \mapsto g \cdot x = gx$. Thus, G acts “diagonally” on Z .

Let $\pi: Z \rightarrow \mathcal{B} \times \mathcal{B}$ be the projection on the second and third factors. By the Bruhat Lemma, the elements of W parameterize the G -orbits on $\mathcal{B} \times \mathcal{B}$. An element w in W corresponds to the G -orbit containing (B, wBw^{-1}) in $\mathcal{B} \times \mathcal{B}$. Define

$$Z_w = \pi^{-1}(G(B, wBw^{-1})), \quad U_w = U \cap wUw^{-1}, \quad \text{and} \quad B_w = B \cap wBw^{-1}.$$

The varieties Z_w play a key role in the rest of this paper.

Since π is G -equivariant, G acts transitively on $G(B, wBw^{-1})$, and the fibre of π over (B, wBw^{-1}) is isomorphic to $\mathfrak{u}_w$, it follows that Z_w is isomorphic to the associated fibre bundle $G \times^{B_w} \mathfrak{u}_w$. Thus, Z_w is the image of an irreducible variety and so is irreducible and $\dim Z_w = \dim G - \dim B_w + \dim \mathfrak{u}_w = 2n$. Furthermore, each Z_w is locally closed in Z and so it follows that $\{\overline{Z_w} \mid w \in W\}$ is the set of irreducible components of Z .

Let $\mu_z: Z \rightarrow \mathfrak{N}$ denote the projection on the first component. For a G -orbit, $\mathfrak{C}$, in $\mathfrak{N}$, set $Z_{\mathfrak{C}} = \mu_z^{-1}(\mathfrak{C})$. Fix x in $\mathfrak{C}$. Then μ_z is G -equivariant, G acts transitively on $\mathfrak{C}$, and the fibre of μ_z over x is isomorphic to $\mathcal{B}_x \times \mathcal{B}_x$, and so $Z_{\mathfrak{C}} \cong G \times^{Z_G(x)} (\mathcal{B}_x \times \mathcal{B}_x)$. Spaltenstein [Spa82] has shown that the variety $\mathcal{B}_x$ is equidimensional and Steinberg and Spaltenstein have shown that $\dim Z_G(x) = r + 2 \dim \mathcal{B}_x$. This implies the following results due to Steinberg [Ste76, Proposition 3.1]:

- $\dim Z_{\mathfrak{C}} = \dim G - \dim Z_G(x) + 2 \dim \mathcal{B}_x = \dim G - r = 2n$.
- Every irreducible component of $Z_{\mathfrak{C}}$ has the form

$$G(\{x\} \times C_1 \times C_2) = G(\{x\} \times (Z_G(x)(C_1 \times C_2)))$$

where C_1 and C_2 are irreducible components of $\mathcal{B}_x$.

- A pair, (C'_1, C'_2) , of irreducible components of $\mathcal{B}_x$ determines the same irreducible component of $Z_{\mathfrak{C}}$ as (C_1, C_2) if and only if there is a z in $Z_G(x)$ with $(C'_1, C'_2) = (zC_1z^{-1}, zC_2z^{-1})$.

Thus, $Z_{\mathfrak{C}}$ is equidimensional with $\dim Z_{\mathfrak{C}} = 2n = \dim Z$ and there is a bijection between irreducible components of $Z_{\mathfrak{C}}$ and $Z_G(x)$ -orbits on the set of irreducible components of $\mathcal{B}_x \times \mathcal{B}_x$.

Now the closures of the irreducible components of $Z_{\mathfrak{C}}$ are closed, irreducible, $2n$ -dimensional subvarieties of Z and so each irreducible component of $Z_{\mathfrak{C}}$ is of the form $Z_{\mathfrak{C}} \cap \overline{Z_w}$ for some unique w in W . Define $W_{\mathfrak{C}}$ to be the subset of W that parameterizes the irreducible components of $Z_{\mathfrak{C}}$. Then w is in $W_{\mathfrak{C}}$ if and only if $Z_{\mathfrak{C}} \cap \overline{Z_w}$ is an irreducible component of $Z_{\mathfrak{C}}$.

We see that there is a bijection between $W_{\mathfrak{C}}$ and of $Z_G(x)$ -orbits on the set of pairs of irreducible components of $\mathcal{B}_x$ given as follows. If w is in $W_{\mathfrak{C}}$ and (C_1, C_2) is a pair of irreducible components of $\mathcal{B}_x$, then w corresponds to the $Z_G(x)$ -orbit of (C_1, C_2) if and only if $G(B, wBw^{-1}) \cap (C_1 \times C_2)$ is dense in $C_1 \times C_2$.

Using the isomorphism $Z_w \cong G \times^{B_w} \mathfrak{u}_w$ we see that $Z_{\mathfrak{C}} \cap Z_w \cong G \times^{B_w} (\mathfrak{C} \cap \mathfrak{u}_w)$. Therefore, w is in $W_{\mathfrak{C}}$ if and only if $\mathfrak{C} \cap \mathfrak{u}_w$ is dense in $\mathfrak{u}_w$. This shows in particular that $W_{\mathfrak{C}}$ is closed under taking inverses.

Clearly W is the disjoint union of the $W_{\mathfrak{C}}$'s as $\mathfrak{C}$ varies over the nilpotent orbits in $\mathfrak{N}$. The subsets $W_{\mathfrak{C}}$ are called *two-sided Steinberg cells* and have several properties in common with Kazhdan-Lusztig two-sided cells in W . Some of these properties will be described in the next subsection.

When $x = 0$ we have $Z_{\{0\}} = \overline{Z_{w_0}} = \{0\} \times \mathcal{B} \times \mathcal{B}$ where w_0 is the longest element in W . Therefore $W_{\{0\}} = \{w_0\}$. At the other extreme, let $\mathfrak{N}_{\text{reg}}$ denote the regular nilpotent orbit. Then it follows from the fact that every regular nilpotent element is contained in a unique Borel subalgebra that $W_{\mathfrak{N}_{\text{reg}}}$ contains just the identity element in W .

Notice that in general there are more two-sided Steinberg cells than two-sided Kazhdan-Lusztig cells. Two-sided Steinberg cells are in bijection with the set of nilpotent orbits in $\mathfrak{N}$ while two-sided Kazhdan-Lusztig cells are in bijection with the set of special nilpotent orbits in $\mathfrak{N}$.

2.2. Orbital varieties. Suppose that $\mathfrak{C}$ is a nilpotent orbit. An *orbital variety* for $\mathfrak{C}$ is an irreducible component of $\mathfrak{C} \cap \mathfrak{u}$. An *orbital variety* is a subvariety of $\mathfrak{N}$ that is orbital for some nilpotent orbit. (Warning: sometimes an orbital variety is defined as the closure of an irreducible component of $\mathfrak{C} \cap \mathfrak{u}$.)

Geometrically, orbital varieties can be used to decompose two-sided Steinberg cells into left and right Steinberg cells. This can be viewed as a geometric generalization of the Robinson-Schensted correspondence. We outline this construction using the Steinberg variety in this subsection.

Fix a nilpotent orbit $\mathfrak{C}$ and an element x in $\mathfrak{C} \cap \mathfrak{u}$. Define $p: G \rightarrow \mathfrak{C}$ by $p(g) = g^{-1}x$ and $q: G \rightarrow \mathcal{B}$ by $q(g) = gBg^{-1}$. Then $p^{-1}(\mathfrak{C} \cap \mathfrak{u}) = q^{-1}(\mathcal{B}_x)$. Spaltenstein [Spa82] has shown that

- if C is an irreducible component of $\mathcal{B}_x$, then $pq^{-1}(C)$ is an orbital variety for $\mathfrak{C}$,
- every orbital variety for $\mathfrak{C}$ has the form $pq^{-1}(C)$ for some irreducible component C of $\mathcal{B}_x$, and
- $pq^{-1}(C) = pq^{-1}(C')$ that for components C and C' of $\mathcal{B}_x$ if and only if C and C' are in the same $Z_G(x)$ -orbit.

It follows immediately that $\mathfrak{C} \cap \mathfrak{u}$ is equidimensional and all orbital varieties for $\mathfrak{C}$ have the same dimension: $n - \dim \mathcal{B}_x = \frac{1}{2} \dim \mathfrak{C}$.

Joseph [Jos84, §9] has refined the relation between orbital varieties for $\mathfrak{C}$ and the set $W_{\mathfrak{C}}$.

Suppose $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are orbital varieties for $\mathfrak{C}$. Choose irreducible components C_1 and C_2 of $\mathcal{B}_x$ so that $pq^{-1}(C_1) = \mathfrak{V}_1$ and $pq^{-1}(C_2) = \mathfrak{V}_2$. We have seen that there is a w in $W_{\mathfrak{C}}$ so that $Z_{\mathfrak{C}} \cap \overline{Z_w} = G(\{x\} \times Z_G(x)(C_1 \times C_2))$. Clearly $\overline{\mu_z^{-1}(x) \cap Z_w} \subseteq \mu_z^{-1}(x) \cap \overline{Z_w}$. Since both sides are closed and $Z_G(x)$ -stable, and the right hand side is the $Z_G(x)$ -saturation of $\{x\} \times C_1 \times C_2$, it follows that $\overline{\mu_z^{-1}(x) \cap Z_w} = \mu_z^{-1}(x) \cap \overline{Z_w}$.

Let p_2 denote the projection of $Z_{\mathfrak{C}}$ to $\mathcal{B}$ by $p_2(x, B', B'') = B'$. Then $pq^{-1}p_2(\mu_z^{-1}(x) \cap Z_w) = B(\mathfrak{C} \cap \mathfrak{u}_w)$. Also,

$$pq^{-1}p_2(\mu_z^{-1}(x) \cap \overline{Z_w}) = pq^{-1}p_2(\{x\} \times Z_G(x)(C_1 \times C_2)) = pq^{-1}(Z_G(x)C_1) = \mathfrak{V}_1.$$

Since $\mathfrak{C} \cap \mathfrak{u}_w$ is dense in $\mathfrak{u}_w$ we have $\overline{B\mathfrak{u}_w} \cap \mathfrak{C} = \overline{B(\mathfrak{C} \cap \mathfrak{u}_w)} \subseteq \mathfrak{V}_1$. However, since $\mu_z^{-1}(x) \cap Z_w$ is a dense, $Z_G(x)$ -stable subset of $\mu_z^{-1}(x) \cap \overline{Z_w}$, it follows that

$$\begin{aligned} \dim B(\mathfrak{C} \cap \mathfrak{u}_w) &= \dim pq^{-1}p_2(\mu_z^{-1}(x) \cap Z_w) \\ &= \dim p_2(\mu_z^{-1}(x) \cap \overline{Z_w}) + \dim B - \dim Z_G(x) \\ &= \dim \mathcal{B}_x + \dim B - r - 2 \dim \mathcal{B}_x \\ &= n - \dim \mathcal{B}_x \end{aligned}$$

and so $\overline{B\mathfrak{u}_w} \cap \mathfrak{C} = \mathfrak{V}_1$.

A similar argument shows that $\overline{B\mathfrak{u}_{w^{-1}}} \cap \mathfrak{C} = \mathfrak{V}_2$. This proves the following theorem.

Theorem 2.1. *If $\mathfrak{C}$ is a nilpotent orbit and $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are orbital varieties for $\mathfrak{C}$, then there is a w in $W_{\mathfrak{C}}$ so that $\mathfrak{V}_1 = \overline{B\mathfrak{u}_w} \cap \mathfrak{C}$ and $\mathfrak{V}_2 = \overline{B\mathfrak{u}_{w^{-1}}} \cap \mathfrak{C}$*

For w in W , define $\mathfrak{V}_l(w) = \overline{B\mathfrak{u}_{w^{-1}}} \cap \mathfrak{C}$ when w is in $W_{\mathfrak{C}}$. For w_1 and w_2 in W , define $w_1 \sim_l w_2$ if $\mathfrak{V}_l(w_1) = \mathfrak{V}_l(w_2)$. Then $\sim_l$ is an equivalence relation and the equivalence classes are called *left Steinberg cells*. Similarly, define $\mathfrak{V}_r(w) = \overline{B\mathfrak{u}_w} \cap \mathfrak{C}$ when w is in $W_{\mathfrak{C}}$ and $w_1 \sim_r w_2$ if $\mathfrak{V}_r(w_1) = \mathfrak{V}_r(w_2)$. The equivalence classes for $\sim_r$ are called *right Steinberg cells*.

Clearly, each two-sided Steinberg cell is a disjoint union of left Steinberg cells and is also the disjoint union of right Steinberg cells. Precisely, if w is in $W_{\mathfrak{C}}$, then

$$W_{\mathfrak{C}} = \coprod_{y \in \mathfrak{V}_r(w)} \mathfrak{V}_l(y) = \coprod_{y \in \mathfrak{V}_l(w)} \mathfrak{V}_r(y).$$

It follows from the theorem that the rule $w \mapsto (\mathfrak{V}_r(w), \mathfrak{V}_l(w))$ defines a surjection from W to the set of pairs of orbital varieties for the same nilpotent orbit. We will see below that the number of orbital varieties for a nilpotent orbit $\mathfrak{C}$ is the dimension of the Springer representation of W corresponding to the trivial representation of the component group of any element in $\mathfrak{C}$. Denote this representation of W by $\rho_{\mathfrak{C}}$. Then the number of pairs (V_1, V_2) , where V_1 and V_2 are orbital varieties for the same nilpotent orbit, is $\sum_{\mathfrak{C}} \dim \rho_{\mathfrak{C}}^2$. In general this sum is strictly smaller than $|W|$. Equivalently, in general, there are more irreducible representations of W than G -orbits in $\mathfrak{N}$.

However, if G has type A , for example if $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$, then every irreducible representation of W is of the form $\rho_{\mathfrak{C}}$ for a unique nilpotent orbit $\mathfrak{C}$. In this case $w \mapsto (\mathfrak{V}_r(w), \mathfrak{V}_l(w))$ defines a bijection from W to the set of pairs of orbital varieties for the same nilpotent orbit. Steinberg has shown that this bijection is essentially given by the Robinson-Schensted correspondence.

In more detail, using the notation in the proof of the theorem, suppose that $\mathfrak{C}$ is a nilpotent orbit, $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are orbital varieties for $\mathfrak{C}$, and C_1 and C_2 are the corresponding irreducible components in $\mathcal{B}_x$. In [Ste88] Steinberg defines a function from $\mathcal{B}$ to the set of standard Young tableau and shows that $G(B, wBw^{-1}) \cap (C_1 \times C_2)$ is dense in $C_1 \times C_2$ if and only if the pair of standard Young tableaux associated to generic (B', B'') in $C_1 \times C_2$ is the same as the pair of standard Young tableaux associated to w by the Robinson-Schensted correspondence. For more details, see also [Dou96].

An open problem, even in type A , is determining the orbit closures of orbital varieties. Some rudimentary information may be obtained by considering the top Borel-Homology group of Z (see §3 and [HJ05]).

2.3. Associated varieties and characteristic varieties. The Steinberg variety arises naturally in the theory of algebraic $(\mathfrak{g}, K)$ -modules. This was first observed by Borho and Brylinski [BB85] and Ginzburg [Gin86].

Recall that we have defined $Z = \{ (x, B', B'') \in \mathfrak{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \text{Lie}(B') \cap \text{Lie}(B'') \}$. If B' is a Borel subgroup of G , then using the Killing form on $\mathfrak{g}$, the cotangent space to $\mathcal{B}$ at B' may be identified with $\mathfrak{b}' \cap \mathfrak{N}$, the nilradical of $\mathfrak{b}'$. Define $\tilde{\mathfrak{N}} = \{ (x, B') \in \mathfrak{N} \times \mathcal{B} \mid x \in \mathfrak{b}' \}$ and let $\mu: \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ be the projection on the first factor. Then $\tilde{\mathfrak{N}} \cong T^*\mathcal{B}$ and it is easy to see that $Z \cong \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}} \cong T^*\mathcal{B} \times_{\mathfrak{N}} T^*\mathcal{B}$.

Using this description of Z as a fibred product, we see that it has an alternate description as the fibred product $Z \cong (\tilde{\mathfrak{N}} \times \tilde{\mathfrak{N}}) \times_{\mathfrak{N} \times \mathfrak{N}} \mathfrak{N}$ where $\mu \times \mu: \tilde{\mathfrak{N}} \times \tilde{\mathfrak{N}} \rightarrow \mathfrak{N} \times \mathfrak{N}$ and $\delta: \mathfrak{N} \rightarrow \mathfrak{N} \times \mathfrak{N}$ by $\delta(x) = (x, -x)$. Alternately, $Z \cong \{ (x, -x, B', B'') \mid x \in \mathfrak{b}' \cap \mathfrak{b}'' \}$.

For the rest of this subsection, we assume that G is a semisimple complex Lie group and K is a closed, connected, algebraic subgroup of G that acts on $\mathcal{B}$ with finitely many orbits. The two special cases we are interested in are the “highest weight” case, when $K = B$ is a Borel subgroup of G , and the “Harish-Chandra” case, when $K = G_d$ is the diagonal subgroup of $G \times G$. In the general setting, we suppose that W is a finite set that indexes the K -orbits on $\mathcal{B}$ by $w \leftrightarrow X_w$. Of course, in the examples we are interested in, the two definitions of W are compatible.

For w in W , let $T_w^*\mathcal{B}$ denote the conormal bundle to the K -orbit X_w in $T^*\mathcal{B}$. Then letting $\mathfrak{k}^\perp$ denote the subspace of $\mathfrak{g}$ orthogonal to $\mathfrak{k}$ with respect to the Killing form and using our identification of $T^*\mathcal{B}$ with pairs we may identify

$$T_w^*\mathcal{B} \cong \{ (x, B') \in \mathfrak{N} \times \mathcal{B} \mid B' \in X_w, x \in \mathfrak{b}' \cap \mathfrak{k}^\perp \}.$$

Define $Y_{\mathfrak{k}} = \mu^{-1}(\mathfrak{k}^\perp)$. Then $Y_{\mathfrak{k}}$ is closed, $Y_{\mathfrak{k}} = \coprod_{w \in W} T_w^*\mathcal{B} = \cup_{w \in W} \overline{T_w^*\mathcal{B}}$, and μ restricts to a surjection $Y_{\mathfrak{k}} \xrightarrow{\mu} \mathfrak{k}^\perp$.

Consider $U(\mathfrak{g})$ with the standard filtration. Then by the PBW Theorem, $\text{gr } U(\mathfrak{g}) \cong \text{Sym}(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, using the Killing form, we may identify $\text{gr } U(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{g}]$, the coordinate ring of the affine variety $\mathfrak{g}$.

A *finitely generated* $(\mathfrak{g}, K)$ -module is a finitely generated $\mathfrak{g}$ -module with a compatible algebraic action of K . If M is a finitely generated $(\mathfrak{g}, K)$ -module, then it has a “good” filtration such that the associated graded module, $\text{gr } M$, a module for $\text{gr } U(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}]$, is finitely generated. The *associated variety* of M , denoted by $V_{\mathfrak{k}^\perp}(M)$, is the support of the $\mathbb{C}[\mathfrak{g}]$ -module $\text{gr } M$ – a subvariety of $\mathfrak{g}$. It is known that $V_{\mathfrak{k}^\perp}(M)$ is independent of the choice of good filtration and is contained in $\mathfrak{k}^\perp$.

Let $\mathcal{D}_{\mathcal{B}}$ denote the sheaf of algebraic differential operators on $\mathcal{B}$. A *coherent* $(\mathcal{D}_{\mathcal{B}}, K)$ -module is a K -equivariant sheaf of $\mathcal{D}_{\mathcal{B}}$ -modules that is coherent as a $\mathcal{O}_{\mathcal{B}}$ -module. If $\mathcal{M}$ is a coherent $(\mathcal{D}_{\mathcal{B}}, K)$ -module, then it has a “good” filtration such that $\text{gr } \mathcal{M}$ is a coherent $\text{gr } \mathcal{D}_{\mathcal{B}}$ -module. The associated graded sheaf $\text{gr } \mathcal{D}_{\mathcal{B}}$ is isomorphic to the direct image $p_* \mathcal{O}_{T^*\mathcal{B}}$. Thus, $\text{gr } \mathcal{M}$ may be considered as an $\mathcal{O}_{T^*\mathcal{B}}$ -module. The *characteristic variety* of $\mathcal{M}$, denoted by $V_{Y_{\mathfrak{k}}}(\mathcal{M})$, is the support of the $\mathcal{O}_{T^*\mathcal{B}}$ -module $\text{gr } \mathcal{M}$ – a subvariety of $T^*\mathcal{B}$. It is known that $V_{Y_{\mathfrak{k}}}(\mathcal{M})$ is independent of the choice of good filtration and is contained in $Y_{\mathfrak{k}}$.

Unraveling the notation in the Harish-Chandra case we have $\mathfrak{k}^\perp = \mathfrak{g}_d^\perp = \{(x, -x) \mid x \in \mathfrak{g}\}$ is isomorphic to $\mathfrak{g}$. Therefore,

$$T_w^*(\mathcal{B} \times \mathcal{B}) = \{(x, y, B', B'') \mid (B', B'') \in G(B, wBw^{-1}), x \in \mathfrak{b}', y \in \mathfrak{b}'', (x, y) \in \mathfrak{g}_d^\perp\} \cong Z_w.$$

Thus, $Y_{\mathfrak{g}_d} = \coprod_{w \in W} T_w^*(\mathcal{B} \times \mathcal{B})$ is isomorphic to the Steinberg variety and we may identify the restriction of $\mu \times \mu$ to $Y_{\mathfrak{g}_d}$ with $\mu_z: Z \rightarrow \mathfrak{N}$.

Unraveling the notation in the highest weight case we have $\mathfrak{k}^\perp = \mathfrak{b}^\perp = \mathfrak{u}$. Hence, $Y_{\mathfrak{u}} = \mu^{-1}(\mathfrak{u}) \cong \{(x, B') \in \mathfrak{N} \times \mathcal{B} \mid x \in \mathfrak{u} \cap \mathfrak{b}'\}$. We denote $Y_{\mathfrak{u}}$ simply by Y and call it the *conormal variety*. For w in W , X_w is the set of B conjugates of wBw^{-1} and $T_w^*\mathcal{B} \cong \{(x, B') \in \mathfrak{N} \times \mathcal{B} \mid B' \in X_w, x \in \mathfrak{u} \cap \mathfrak{b}'\}$. The projection of $T_w^*\mathcal{B}$ to $\mathcal{B}$ is a B -equivariant surjection onto X_w and so $T_w^*\mathcal{B} \cong B \times^{B_w} \mathfrak{u}_w$. Therefore, $\dim T_w^*\mathcal{B} = \dim B - \dim B_w + \dim \mathfrak{u} = n$. Since $Y = \coprod_{w \in W} T_w^*\mathcal{B}$ and each $T_w^*\mathcal{B}$ is locally closed and n -dimensional, we see that the set $\{\overline{T_w^*\mathcal{B}} \mid w \in W\}$ is the set of irreducible components of Y .

Let $p_3: Z \rightarrow \mathcal{B}$ be projection on the third factor. Then p_3 is G -equivariant, G acts transitively on $\mathcal{B}$, and the fibre over B is isomorphic to Y . This gives yet another description of the Steinberg variety: $Z \cong G \times^B Y$.

Arguments in the spirit of those given above (see [HJ05, §3]) show that if we set $Y_w = T_w^*\mathcal{B}$ and $Y_{\mathfrak{C}} = \mu^{-1}(\mathfrak{C} \cap \mathfrak{u})$, then $\dim Y_{\mathfrak{C}} = n$, $\overline{Y_{\mathfrak{C}}}$ is equidimensional, and the set of irreducible components of $\overline{Y_{\mathfrak{C}}}$ is $\{\overline{Y_{\mathfrak{C}} \cap Y_w} \mid w \in W_{\mathfrak{C}}\}$.

Now consider the following three categories:

- coherent $(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)$ -modules, $\text{Mod}(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)^{\text{coh}}$;
- finitely generated $(\mathfrak{g} \times \mathfrak{g}, G_d)$ -modules with trivial central character, $\text{Mod}(\mathfrak{g} \times \mathfrak{g}, G_d)_{0,0}^{\text{fg}}$;
- and
- finitely generated $(\mathfrak{g}, B)$ -modules with trivial central character, $\text{Mod}(\mathfrak{g}, B)_0^{\text{fg}}$.

It is shown in [BB85, §3] that these three categories are naturally equivalent.

Suppose $\mathcal{H}$ is a coherent $(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)$ -module, H is the corresponding finitely generated $(\mathfrak{g} \times \mathfrak{g}, G_d)$ -module with trivial central character, and L is the corresponding finitely generated $(\mathfrak{g}, B)$ -module with trivial central character. Then it is shown in [BB85, §4] that there is a subset $\Sigma = \Sigma_{\mathcal{H}}$ of W so that when $\mu \times \mu: Y_{\mathfrak{g}_d} \rightarrow \mathfrak{g}_d^\perp$ is identified with $\mu_z: Z \rightarrow \mathfrak{N}$ we have:

- The characteristic variety of $\mathcal{H}$ is $V_Z(\mathcal{H}) = \cup_{y \in \Sigma} \overline{Z_y}$, a union of irreducible components of Z .
- The associated variety of H is $V_{\mathfrak{g}_d}(H) = \mu_z(V_{\mathfrak{g}_d}(\mathcal{H})) = \cup_{y \in \Sigma} \overline{G\mathfrak{u}_y} = G \cdot V_{\mathfrak{u}}(L)$, so the associated variety of H is image under μ_z of the characteristic variety of $\mathcal{H}$ and is also the G -saturation of the associated variety of L .
- The associated variety of L is $V_{\mathfrak{u}}(L) = \cup_{y \in \Sigma} \overline{B\mathfrak{u}_y}$, a union of closures of orbital varieties.

Moreover, the simple objects in each of these categories are indexed by W . If w is in W , and $\mathcal{H}_w$, H_w , and L_w are corresponding simple modules, then there is a nilpotent orbit, $\mathfrak{C}$, so that $V(H_w) = G \cdot V(L_w) = \overline{\mathfrak{C}}$.

For w in W , determining the subset $\Sigma = \Sigma(w)$ so that $V(L_w) = \cup_{y \in \Sigma} \overline{B\mathfrak{u}_y}$ is still an open problem (see [BB85] and [HJ05] for more information).

2.4. Generalized Steinberg varieties. When considering the restriction of a Springer representation to parabolic subgroups of W , Springer introduced a generalization of $\mathfrak{N}$ depending on a parabolic subgroup P and a nilpotent orbit in a Levi subgroup of P . Springer's

construction extends naturally to what we call “generalized Steinberg varieties.” The results in this subsection may be found in [DR04].

Suppose $\mathcal{P}$ is a conjugacy class of parabolic subgroups of G . The unipotent radical of a subgroup, P , in $\mathcal{P}$ will be denoted by U_P . A G -equivariant function, c , from $\mathcal{P}$ to the power set of $\mathfrak{N}$ with the properties

- $\mathfrak{u}_P \subseteq c(P) \subseteq \mathfrak{N} \cap \mathfrak{p}$ and
- the image of $c(P)$ in $\mathfrak{p}/\mathfrak{u}_P$ is the closure of a single nilpotent P/U_P -adjoint orbit

is called a *Levi class function* on $\mathcal{P}$. Define

$$\tilde{\mathfrak{N}}_c^{\mathcal{P}} = \{ (x, P) \in \mathfrak{N} \times \mathcal{P} \mid x \in c(P) \}.$$

Let $\mu_c^{\mathcal{P}}: \tilde{\mathfrak{N}}_c^{\mathcal{P}} \rightarrow \mathfrak{N}$ denote the projection on the first factor. Notice that $\mu_c^{\mathcal{P}}$ is a proper morphism.

If $\mathcal{Q}$ is another conjugacy class of parabolic subgroups of G and d is a Levi class function on $\mathcal{Q}$, then the *generalized Steinberg variety* determined by $\mathcal{P}$, $\mathcal{Q}$, c , and d is

$$X_{c,d}^{\mathcal{P},\mathcal{Q}} = \{ (x, P, Q) \in \mathfrak{N} \times \mathcal{P} \times \mathcal{Q} \mid x \in c(P) \cap d(Q) \} \cong \tilde{\mathfrak{N}}_c^{\mathcal{P}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}_d^{\mathcal{Q}}.$$

Since G acts on $\mathfrak{N}$, $\mathcal{P}$, and $\mathcal{Q}$, there is a diagonal action of G on $X_{c,d}^{\mathcal{P},\mathcal{Q}}$ for all $\mathcal{P}$, $\mathcal{Q}$, c , and d .

The varieties arising from this construction for some particular choices of $\mathcal{P}$, $\mathcal{Q}$, c , and d are worth noting.

- The special case when $c(P) = \mathfrak{u}_P$ and $d(Q) = \mathfrak{u}_Q$ is denoted by $X_{0,0}^{\mathcal{P},\mathcal{Q}}$. Then $X_{0,0}^{\mathcal{P},\mathcal{Q}} \cong T^*\mathcal{P} \times_{\mathcal{N}} T^*\mathcal{Q}$.
- When $\mathcal{P} = \mathcal{Q} = \mathcal{B}$, then $c(B) = d(B) = \{\mathfrak{u}_{B'}\}$ for every B' , in $\mathcal{B}$ and so $X_{0,0}^{\mathcal{B},\mathcal{B}} = Z$ is the Steinberg variety of G .
- When $\mathcal{P} = \mathcal{Q} = \{G\}$, $c(G) = \overline{\mathcal{O}}_1$, and $d(G) = \overline{\mathcal{O}}_2$, then $X_{c,d}^{\{G\},\{G\}} \cong \overline{\mathcal{O}}_1 \cap \overline{\mathcal{O}}_2$.
- The special case when $c(P) = \mathfrak{N} \cap \mathfrak{p}$ and $d(Q) = \mathfrak{N} \cap \mathfrak{q}$ is denoted simply by $X^{\mathcal{P},\mathcal{Q}}$.

Abusing notation slightly we let $\mu: X_{c,d}^{\mathcal{P},\mathcal{Q}} \rightarrow \mathcal{N}$ denote the projection on the first coordinate and $\pi: X_{c,d}^{\mathcal{P},\mathcal{Q}} \rightarrow \mathcal{P} \times \mathcal{Q}$ the projection on the second and third coordinates. We can investigate the varieties $X_{c,d}^{\mathcal{P},\mathcal{Q}}$ using preimages of G -orbits in $\mathcal{N}$ and $\mathcal{P} \times \mathcal{Q}$ under μ and π as we did for Z . Some special cases when at least one of $c(P)$ or $d(Q)$ is smooth turn out to be the most tractable. We will describe these cases in this subsection. We refer the reader to [DR04] for some results for arbitrary $\mathcal{P}$, $\mathcal{Q}$, c , and d .

Fix P in $\mathcal{P}$ and Q in $\mathcal{Q}$ with $B \subseteq P \cap Q$. Let W_P and W_Q denote the Weyl groups of (P, T) and (Q, T) respectively. We consider W_P and W_Q as subgroups of W .

Let $\pi_{\mathcal{P}}: \mathcal{B} \rightarrow \mathcal{P}$ by defining $\pi_{\mathcal{P}}(B')$ to be the unique subgroup in $\mathcal{P}$ containing B' . Now define

$$\eta: Z \rightarrow X^{\mathcal{P},\mathcal{Q}} \quad \text{by} \quad \eta(x, B', B'') = (x, \pi_{\mathcal{P}}(B'), \pi_{\mathcal{Q}}(B'')).$$

Then η depends on $\mathcal{P}$ and $\mathcal{Q}$ and is a proper, G -equivariant, surjective morphism.

Next, define $Z^{\mathcal{P},\mathcal{Q}} = \eta^{-1}(X_{0,0}^{\mathcal{P},\mathcal{Q}})$. We denote the restriction of η to $Z^{\mathcal{P},\mathcal{Q}}$ by η_1 . Then η_1 is also a proper, surjective, G -equivariant morphism. Moreover, the fibres of η_1 are all isomorphic to the smooth, complete variety $P/B \times Q/B$.

Finally, for w in W , define $Z_w^{\mathcal{P},\mathcal{Q}}$ to be the intersection $Z^{\mathcal{P},\mathcal{Q}} \cap Z_w$. Since $(0, B, wBw^{-1})$ is in $Z_w^{\mathcal{P},\mathcal{Q}}$ and η_1 is G -equivariant, it is straightforward to check that $Z_w^{\mathcal{P},\mathcal{Q}} \cong G \times^{B_w} (\mathfrak{u}_P \cap w\mathfrak{u}_Q)$. Thus $Z_w^{\mathcal{P},\mathcal{Q}}$ is smooth and irreducible.

The following statements are proved in [DR04].

- For w in W , $\dim \eta(Z_w) = \dim \mathcal{N}$ if and only if w has minimal length in $W_P w W_Q$. The set of irreducible components of $X^{\mathcal{P}, \mathcal{Q}}$ is

$$\{ \eta(\overline{Z_w}) \mid w \text{ has minimal length in } W_P w W_Q \}.$$

- For w in W , $Z_w^{\mathcal{P}, \mathcal{Q}} = Z_w$ if and only if w has maximal length in $W_P w W_Q$. The variety $Z^{\mathcal{P}, \mathcal{Q}}$ is equidimensional and the set of irreducible components of $Z^{\mathcal{P}, \mathcal{Q}}$ is

$$\{ \overline{Z_w} \mid w \text{ has maximal length in } W_P w W_Q \}.$$

- The variety $X_{0,0}^{\mathcal{P}, \mathcal{Q}}$ is equidimensional with dimension equal $\dim \mathfrak{u}_P + \dim \mathfrak{u}_Q$ and the set of irreducible components of $X_{0,0}^{\mathcal{P}, \mathcal{Q}}$ is

$$\{ \eta_1(\overline{Z_w}) \mid w \text{ has maximal length in } W_P w W_Q \}.$$

- For a Levi class function d on $\mathcal{Q}$, define ρ_d to be the number of irreducible components of $d(Q) \cap (\mathfrak{u} \cap \mathfrak{l}_Q)$ where L_Q is the Levi factor of Q that contains T . Notice that this is the number of orbital varieties for an L_Q -orbit in the variety of nilpotent elements in $\mathfrak{l}_Q$. Then the varieties $X_{0,d}^{\mathcal{B}, \mathcal{Q}}$ are equidimensional with dimension $\frac{1}{2}(\dim \mathfrak{u} + \dim d(Q) + \dim \mathfrak{u}_Q)$ and $|W : W_{\mathcal{Q}}| \rho_d$ irreducible components.

Notice that the first statement relates minimal double coset representatives to regular orbits in Levi subalgebras and the third statement relates maximal double coset representatives to the zero orbits in Levi subalgebras.

The quantity ρ_d in the fourth statement is the degree of an irreducible representation of $W_{\mathcal{Q}}$ (see §3.5) and so $|W : W_{\mathcal{Q}}| \rho_d$ is the degree of an induced representation of W . The fact that $X_{0,d}^{\mathcal{B}, \mathcal{Q}}$ has $|W : W_{\mathcal{Q}}| \rho_d$ irreducible components is numerical evidence for Conjecture 3.10 below.

3. HOMOLOGY

We now take up the rational Borel-Moore homology of the Steinberg variety and generalized Steinberg varieties. As mentioned in the Introduction, soon after Steinberg's original paper, Kazhdan and Lusztig [KL80], defined an action of $W \times W$ on the top Borel-Moore homology groups of Z . They constructed the action by defining an action of the simple reflections in $W \times W$ on $H_i(Z)$ and showed that the defining relations of $W \times W$ are satisfied. Then they proved that the representation of $W \times W$ on $H_{4n}(Z)$ is equivalent to the two-sided regular representation of W and gave a decomposition in terms of Springer representations of W .

In the mid 1990's Ginzburg [CG97] popularized a quite general convolution product construction that defines an algebra structure on $H_*(Z)$ and $K^{\overline{G}}(Z)$ (see the next section for $K^{\overline{G}}(Z)$). With this multiplication, $H_{4n}(Z)$ is a subalgebra isomorphic to the group algebra of W .

In this section, following [CG97], [DR08b], and [HJ05] we will first use some relatively elementary constructions to describe the algebra structure of $H_*(Z)$, the decomposition of $H_{4n}(Z)$ in terms of Springer representations, and the $H_{4n}(Z)$ -module structure on $H_{2n}(Y)$. Then we will use a more sophisticated sheaf theoretic construction of Borel-Moore homology to give an alternate description of $H_*(Z)$, a different version of the decomposition of $H_{4n}(Z)$

in terms of Springer representations, and to describe the Borel-Moore homology of some generalized Steinberg varieties.

3.1. Borel-Moore homology and convolution. Suppose that X is a d -dimensional, quasi-projective, complex algebraic variety (not necessarily irreducible). We denote the complex dimension of X by $\dim X$. Topological notions will refer to the Euclidean topology on X unless otherwise specified. We refer the reader to [CG97, Chapter 2, Chapter 3] for more information.

Let $X \cup \{\infty\}$ be the one-point compactification of X . Then the i^{th} Borel-Moore homology space of X , denoted by $H_i(X)$, is defined by $H_i(X) = H_i^{\text{sing}}(X, \{\infty\})$, the relative, singular homology with rational coefficients of the pair $(X \cup \{\infty\}, \{\infty\})$. Define $H_*(X) = \sum_{i \geq 0} H_i(X)$ – the *Borel-Moore homology of X* .

If $i > 2 \dim X$, then $H_i(X) = 0$, so $H_{2 \dim X}(X)$ is the top non-zero Borel-Moore homology space. Each d -dimensional irreducible component C of X gives rise to a homology class $[C]$ in $H_{2d}(X)$ and these classes form a basis of $H_{2d}(X)$.

In particular, for the Steinberg variety, $H_i(Z) = 0$ for $i > 4n$ and the set $\{[\overline{Z_w}] \mid w \in W\}$ is a basis of $H_{4n}(Z)$. For the conormal variety, $H_i(Y) = 0$ for $i > 2n$ and the set $\{[\overline{Y_w}] \mid w \in W\}$ is a basis of $H_{2n}(Y)$.

If X is smooth and A and B are closed subvarieties of X , there is an intersection pairing $\cap: H_i(A) \times H_j(B) \rightarrow H_{i+j-2 \dim X}(A \cap B)$. This pairing depends on (X, A, B) .

Fix a “base” variety, N . For $i = 1, 2, 3$, suppose that M_i is a smooth, connected, d_i -dimensional variety and $f_i: M_i \rightarrow N$ is a proper morphism. For $1 \leq i < j \leq 3$, let $p_{i,j}: M_1 \times M_2 \times M_3 \rightarrow M_i \times M_j$ denote the projection.

Suppose $Z_{1,2}$ is a closed subset of $M_1 \times M_2$ and $Z_{2,3}$ is a closed subvariety of $M_2 \times M_3$. Define $Z_{1,3} = Z_{1,2} \circ Z_{2,3}$ to be the composition of the relations $Z_{1,2}$ and $Z_{2,3}$. Then

$$Z_{1,3} = \{ (m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \text{ with } (m_1, m_2) \in Z_{1,2} \text{ and } (m_2, m_3) \in Z_{2,3} \}$$

Assume that the restriction, $p_{1,3}: p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3}) \rightarrow Z_{1,3}$ is a proper morphism. Then there is an associative *convolution product*, $H_i(Z_{1,2}) \times H_j(Z_{2,3}) \xrightarrow{*} H_{i+j-d_2}(Z_{1,3})$ defined by

$$c * d = (p_{1,3})! (p_{1,2}^*(c) \cap p_{2,3}^*(d))$$

where $\cap$ is the intersection pairing determined by the subsets $Z_{1,2} \times M_3$ and $M_1 \times Z_{2,3}$ of $M_1 \times M_2 \times M_3$.

When $M_1 = M_2 = M_3 = M$, $f_1 = f_2 = f_3 = f$, and $Z_{i,j} = M \times_N M$ for $1 \leq i < j \leq 3$, the convolution product defines a multiplication on $H_*(M \times_N M)$ so that $H_*(M \times_N M)$ is a $\mathbb{Q}$ -algebra with identity. The identity in $H_*(M \times_N M)$ is $[M_\Delta]$ where M_Δ is the diagonal in $M \times M$. If $d = \dim M$, then $H_i(M \times_N M) * H_j(M \times_N M) \subseteq H_{i+j-2d}(M \times_N M)$ and so $H_{2d}(M \times_N M)$ is a subalgebra and $\oplus_{i < 2d} H_i(M \times_N M)$ is a nilpotent, two-sided ideal.

If M and M' are smooth and $f: M \rightarrow N$ and $f': M' \rightarrow N$ are proper maps, then the convolution product defines a left $H_*(M \times_N M)$ -module structure on $H_*(M \times_N M')$. A special case is when $M' = A$ is a subset of N and $f': f^{-1}(A) \rightarrow N$ is the restriction of f . Then the convolution product defines a left $H_*(M \times_N M)$ -module structure on $H_*(f^{-1}(A))$.

Using the description $Z \cong \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \mathfrak{N}$, where $\mu: \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$, we get the following proposition.

Proposition 3.1. *The convolution product defines a $\mathbb{Q}$ -algebra structure on $H_*(Z)$ and left $H_*(Z)$ -module structures on $H_*(Y) = H_*(\mu^{-1}(\mathfrak{u}))$ and $H_*(\mathcal{B}_x)$ for x in $\mathfrak{N}$.*

3.2. The specialization construction and $H_{4n}(Z)$. Chriss and Ginzburg [CG97, §3.4] use a specialization construction to show that $H_{4n}(Z)$ is isomorphic to the group algebra $\mathbb{Q}[W]$. The specialization construction can also be used to show that $H_*(Z)$ is isomorphic to the smash product of the group algebra of W and the coinvariant algebra of W . To describe these results, we need some more notation.

Define $\tilde{\mathfrak{g}} = \{(x, B') \in \mathfrak{g} \times \mathcal{B} \mid x \in \text{Lie}(B')\}$ and $\widehat{Z} = \{(x, B', B'') \in \mathfrak{g} \times \mathcal{B} \times \mathcal{B} \mid x \in \text{Lie}(B') \cap \text{Lie}(B'')\}$. Abusing notation slightly, let $\mu: \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ and $\mu_z: \widehat{Z} \rightarrow \mathfrak{g}$ denote the projections on the first factors and let $\pi: \widehat{Z} \rightarrow \mathcal{B} \times \mathcal{B}$ denote the projection on the second and third factors.

For w in W define $\widehat{Z}_w = \pi^{-1}(G(B, wBw^{-1}))$. Then $\widehat{Z}_w \cong G \times^{B_w} \mathfrak{b}_w$ and so $\dim \widehat{Z}_w = \dim \mathfrak{g}$ and the closures of the $\widehat{Z}_w$'s for w in W are the irreducible components of $\widehat{Z}$.

As with Z , we have an alternate description of $\widehat{Z}$ as $(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}) \times_{\mathfrak{g} \times \mathfrak{g}} \mathfrak{g}$. However, in contrast to the situation in §2.3, where $\delta: \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$ by $\delta(x) = (x, -x)$, from now on we let $\delta: \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$ be the diagonal map. We will frequently identify $\widehat{Z}$ with the subvariety of $\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ consisting of all pairs $((x, B'), (x, B''))$ with x in $\mathfrak{b}' \cap \mathfrak{b}''$. Similarly, we will frequently identify Z with the subvariety of $\tilde{\mathfrak{N}} \times \tilde{\mathfrak{N}}$ consisting of all pairs $((x, B'), (x, B''))$ with x in $\mathfrak{N} \cap \mathfrak{b}' \cap \mathfrak{b}''$.

For (x, gBg^{-1}) in $\tilde{\mathfrak{g}}$, define $\nu(x, gBg^{-1})$ to be the projection of $g^{-1} \cdot x$ in $\mathfrak{t}$. For w in W , let $\Gamma_{w^{-1}} = \{(h, w^{-1} \cdot h) \mid h \in \mathfrak{t}\} \subseteq \mathfrak{t} \times \mathfrak{t}$ denote the graph of the action of w^{-1} on $\mathfrak{t}$ and define

$$\Lambda_w = \widehat{Z} \cap (\nu \times \nu)^{-1}(\Gamma_{w^{-1}}) = \{(x, B', B'') \in \widehat{Z} \mid \nu(x, B'') = w^{-1}\nu(x, B')\}.$$

In the special case when w is the identity element in W , we will denote Λ_w by Λ_1 .

The spaces we have defined so far fit into a commutative diagram with cartesian squares:

$$(3.2) \quad \begin{array}{ccccc} \Lambda_w & \xrightarrow{\quad} & \widehat{Z} & \xrightarrow{\mu_z} & \mathfrak{g} \\ \downarrow & & \downarrow & & \downarrow \delta \\ (\nu \times \nu)^{-1}(\Gamma_{w^{-1}}) & \xrightarrow{\quad} & \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} & \xrightarrow{\mu \times \mu} & \mathfrak{g} \times \mathfrak{g} \\ \downarrow & & \downarrow \nu \times \nu & & \\ \Gamma_{w^{-1}} & \xrightarrow{\quad} & \mathfrak{t} \times \mathfrak{t} & & \end{array}$$

Let $\nu_w: \Lambda_w \rightarrow \Gamma_{w^{-1}}$ denote the composition of the leftmost vertical maps in (3.2), so ν_w is the restriction of $\nu \times \nu$ to Λ_w .

For the specialization construction, we consider subsets of $\widehat{Z}$ of the form $\nu_w^{-1}(S')$ for $S' \subseteq \Gamma_{w^{-1}}$. Thus, for h in $\mathfrak{t}$ we define $\Lambda_w^h = \nu_w^{-1}(h, w^{-1}h)$. Notice in particular that $\Lambda_w^0 = Z$. More generally, for a subset S of $\mathfrak{t}$ we define $\Lambda_w^S = \coprod_{h \in S} \Lambda_w^h$. Then, $\Lambda_w^S = \nu_w^{-1}(S')$ where S' is the graph of w^{-1} restricted to S .

Let $\mathfrak{t}_{\text{reg}}$ denote the set of regular elements in $\mathfrak{t}$.

For w in W , define $\tilde{w}: G/T \times \mathfrak{t}_{\text{reg}} \rightarrow G/T \times \mathfrak{t}_{\text{reg}}$ by $\tilde{w}(gT, h) = (gwT, w^{-1}h)$. The rule $(gT, h) \mapsto (g \cdot h, gB)$ defines an isomorphism of varieties $f: G/T \times \mathfrak{t}_{\text{reg}} \xrightarrow{\cong} \tilde{\mathfrak{g}}_{\text{rs}}$, where $\tilde{\mathfrak{g}}_{\text{rs}} = \mu^{-1}(G \cdot \mathfrak{t}_{\text{reg}})$. We denote the automorphism $f \circ \tilde{w} \circ f^{-1}$ of $\tilde{\mathfrak{g}}_{\text{rs}}$ also by $\tilde{w}$.

Now fix a one dimensional subspace, ℓ , of $\mathfrak{t}$ so that $\ell \cap \mathfrak{t}_{\text{reg}} = \ell \setminus \{0\}$ and set $\ell^* = \ell \setminus \{0\}$. It is not hard to check that the variety $\Lambda_w^{\ell^*}$ is the graph of $\tilde{w}|_{\tilde{\mathfrak{g}}^{\ell^*}}: \tilde{\mathfrak{g}}^{\ell^*} \rightarrow \tilde{\mathfrak{g}}^{w^{-1}(\ell^*)}$. Thus, $\Lambda_w^{\ell^*}$ is an irreducible, $2n+1$ -dimensional variety and so $H_{4n+2}(\Lambda_w^{\ell^*})$ is one dimensional with basis $\{[\Lambda_w^{\ell^*}]\}$. Because $\Lambda_w^{\ell^*}$ is a graph, it follows easily from the definitions that for y in W , there

is a convolution product

$$H_*(\Lambda_w^{\ell*}) \times H_*(\Lambda_y^{w^{-1}\ell*}) \xrightarrow{*} H_*(\Lambda_{wy}^{\ell*})$$

and $[\Lambda_w^{\ell*}] * [\Lambda_y^{w^{-1}\ell*}] = [\Lambda_{wy}^{\ell*}]$.

Now $\Lambda_w^\ell = \Lambda_w^{\ell*} \coprod \Lambda_w^0 = \Lambda_w^{\ell*} \coprod Z$. Therefore there is a specialization map (see [FM81, §3.4], [CG97, §2.6.30]),

$$\lim: H_{i+2}(\Lambda_w^{\ell*}) \rightarrow H_i(Z).$$

Take $i = 4n + 2$ and define $\lambda_w = \lim([\Lambda_w^{\ell*}])$ in $H_{4n}(Z)$. Chriss and Ginzburg [CG97, §3.4] have proven the following:

- The element λ_w in $H_{4n}(Z)$ does not depend on the choice of ℓ .
- Specialization commutes with convolution. Therefore, $\lambda_w * \lambda_y = \lambda_{wy}$ for w and y in W .
- The expansion of λ_w as a linear combination of the basis elements $[\overline{Z_y}]$ of $H_{4n}(Z)$ has the form $\lambda_w = [\overline{Z_w}] + \sum_{y < w} a_{w,y} [\overline{Z_y}]$ where $<$ is the Bruhat order on W .

Combining these results we obtain the following theorem.

Theorem 3.3. *The assignment $w \mapsto \lambda_w$ extends to an algebra isomorphism $\mathbb{Q}[W] \xrightarrow{\cong} H_{4n}(Z)$.*

3.3. The Borel-Moore homology of Z and coinvariants. Now consider

$$Z_1 = \{ (x, B', B') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \text{Lie}(B') \}.$$

Then Z_1 may be identified with the diagonal in $\tilde{\mathfrak{N}} \times \tilde{\mathfrak{N}}$. It follows that Z_1 is closed in Z and isomorphic to $\tilde{\mathfrak{N}}$.

Since $\tilde{\mathfrak{N}} \cong T^*\mathcal{B}$, it follows from the Thom isomorphism in Borel-Moore homology that $H_{i+2n}(Z_1) \cong H_i(\mathcal{B})$ for all i . Since $\mathcal{B}$ is smooth and compact, $H_i(\mathcal{B}) \cong H^{2n-i}(\mathcal{B})$ by Poincaré duality. Therefore, $H_{4n-i}(Z_1) \cong H^i(\mathcal{B})$ for all i .

The cohomology of $\mathcal{B}$ is well-understood: there is an isomorphism of graded algebras, $H^*(\mathcal{B}) \cong \text{Coinv}(W)_*$ where $\text{Coinv}(W)_*$ is the coinvariant algebra of W with generators in degree 2. It follows that $H_j(Z) = 0$ if j is odd, $H_{4n-2i}(Z_1) \cong \text{Coinv}(W)_{2i}$ for $0 \leq i \leq n$.

The following is proved in [DR08b].

- (a) There is a convolution product on $H_*(Z_1)$. With this product, $H_*(Z_1)$ is a commutative $\mathbb{Q}$ -algebra and there is an isomorphism of graded $\mathbb{Q}$ -algebras

$$\beta: \text{Coinv}_*(W) \xrightarrow{\cong} H_{4n-*}(Z_1).$$

- (b) If $r: Z_1 \rightarrow Z$ denotes the inclusion, then the direct image map in Borel-Moore homology, $r_*: H_*(Z_1) \rightarrow H_*(Z)$, is an injective ring homomorphism.
- (c) If we identify $H_*(Z_1)$ with its image in $H_*(Z)$ as in (e), then the linear transformation given by the convolution product

$$H_i(Z_1) \otimes H_{4n}(Z) \longrightarrow H_i(Z)$$

is an isomorphism of vector spaces for $0 \leq i \leq 4n$.

The algebra $\text{Coinv}(W)_*$ has a natural action of W by algebra automorphisms and the isomorphism β in (a) is in fact an isomorphism of W -algebras. The W -algebra structure on $H_*(Z_1)$ is described as follows.

Fix w in W and identify $H_*(Z_1)$ with its image in $H_*(Z)$, then

$$\lambda_w * H_i(Z_1) * \lambda_{w^{-1}} = H_i(Z_1).$$

Therefore, conjugation by λ_w defines a W -algebra structure on $H_*(Z_1)$. With this W -algebra structure, the isomorphism $\beta: \text{Coinv}_*(W) \xrightarrow{\cong} H_{4n-*}(Z_1)$ in (a) is an isomorphism of W -algebras.

Suppose that $\text{Coinv}(W) \rtimes \mathbb{Q}[W]$ is graded by $(\text{Coinv}(W) \rtimes \mathbb{Q}[W])_i = \text{Coinv}(W)_i \otimes \mathbb{Q}[W]$. Then combining Theorem 3.3, (c), and the fact that β is an isomorphism of W -algebras we obtain the following theorem.

Theorem 3.4. *The composition*

$$\text{Coinv}(W)_* \rtimes \mathbb{Q}[W] \xrightarrow{\beta \otimes \alpha} H_{4n-*}(Z_1) \otimes H_{4n}(Z) \xrightarrow{*} H_{4n-*}(Z)$$

is an isomorphism of graded $\mathbb{Q}$ -algebras.

3.4. Springer representations of W . Springer [Spr76] has given a case-free construction of the irreducible representations of W . He achieves this by defining an action of W on $H^*(\mathcal{B}_x)$ for x in $\mathfrak{N}$. Define $d_x = \dim \mathcal{B}_x$ and let $C(x) = Z_G(x)/Z_G^0(x)$. Springer shows that if ϕ is an irreducible representation of $C(x)$ and $H^{2d_x}(\mathcal{B}_x)_\phi$ is the homogeneous component of $H^{2d_x}(\mathcal{B}_x)$ corresponding to ϕ , then $H^{2d_x}(\mathcal{B}_x)_\phi$ is W -stable and is either zero or affords an irreducible representation of W . He shows furthermore that every irreducible representation of W arises in this way. The convolution construction gives an alternate, more elementary approach to Springer's construction.

We have seen that for x in $\mathfrak{N}$, the convolution product defines a left $H_{4n}(Z)$ -module structure on $H_*(\mathcal{B}_x)$ and that $H_{4n}(Z) \cong \mathbb{Q}[W]$. Thus, we obtain a representation of W on $H_*(\mathcal{B}_x)$. Because $\mathcal{B}_x$ is projective, and hence compact, $H^*(\mathcal{B}_x)$ is the linear dual of $H_*(\mathcal{B}_x)$ and so we obtain a representation of W on $H^*(\mathcal{B}_x)$. Hotta [Hot82] has shown that up to sign, this is the representation constructed by Springer.

The centralizer of x acts on $\mathcal{B}_x$. Thus, the component group $C(x)$ acts on $H_*(\mathcal{B}_x)$. It is easy to check that the $C(x)$ -action and the $H_{4n}(Z)$ -action commute. Therefore, up to isomorphism, the representation of W on $H_*(\mathcal{B}_x)$ depends only the G -orbit of x .

We will give an alternate construction of Springer representations of W below in §3.6.

3.5. More on the top Borel-Moore homology of Z . Suppose $\mathfrak{C}$ is a G -orbit in $\mathfrak{N}$ and x is in $\mathfrak{C}$. Define $\partial\mathfrak{C} = \overline{\mathfrak{C}} \setminus \mathfrak{C}$. Set $Z_x = \mu_z^{-1}(x)$, $Z_{\partial\mathfrak{C}} = \mu_z^{-1}(\partial\mathfrak{C})$, and $Z_{\overline{\mathfrak{C}}} = \mu_z^{-1}(\overline{\mathfrak{C}})$. Clearly $Z_x \cong \mathcal{B}_x \times \mathcal{B}_x$ and so by the Künneth theorem $H_{4d_x}(Z_x) \cong H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x)$.

For suitable choices of $f_i: M_i \rightarrow N$ and $Z_{i,j}$ for $1 \leq i < j \leq 3$, the convolution product construction in §3.1 defines left and right $H_*(Z)$ -modules structures on $H_*(Z_x)$, $H_*(Z_{\partial\mathfrak{C}})$, and $H_*(Z_{\overline{\mathfrak{C}}})$.

The centralizer of x acts diagonally on Z_x . Thus, the component group $C(x)$ acts on $H_*(Z_x)$. As above, the $C(x)$ -action and the $H_{4n}(Z)$ -action commute and so $H_{4d_x}(Z_x)^{C(x)}$ is an $H_{4n}(Z)$ -submodule of $H_{4d_x}(Z_x)$.

It follows from §2.1 that $(H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x))^{C(x)}$ has a basis indexed by the set of $C(x)$ -orbits on $\mathcal{B}_x \times \mathcal{B}_x$. Since $H_{4d_x}(Z_x)^{C(x)} \cong (H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x))^{C(x)}$, we conclude that $\dim H_{4d_x}(Z_x) = |W_{\mathfrak{C}}|$.

Define $W_{\partial\mathfrak{C}} = \cup_{\mathfrak{D} \subseteq \partial\mathfrak{C}} W_{\mathfrak{D}}$ and $W_{\overline{\mathfrak{C}}} = \cup_{\mathfrak{D} \subseteq \overline{\mathfrak{C}}} W_{\mathfrak{D}}$. Then $\{\overline{[Z_w]} \mid w \in W_{\partial\mathfrak{C}}\}$ is a basis of $H_{4n}(Z_{\partial\mathfrak{C}})$ and $\{\overline{[Z_w]} \mid w \in W_{\overline{\mathfrak{C}}}\}$ is a basis of $H_{4n}(Z_{\overline{\mathfrak{C}}})$.

The inclusions $Z_{\partial\overline{\mathfrak{C}}} \subseteq Z_{\overline{\mathfrak{C}}} \subseteq Z$ induce injective, $H_{4n}(Z) \times H_{4n}(Z)$ -linear maps, $H_{4n}(Z_{\partial\overline{\mathfrak{C}}}) \rightarrow H_{4n}(Z_{\overline{\mathfrak{C}}}) \rightarrow H_{4n}(Z)$ and so we may identify $H_{4n}(Z_{\partial\overline{\mathfrak{C}}})$ and $H_{4n}(Z_{\overline{\mathfrak{C}}})$ with their images in $H_{4n}(Z)$ and consider $H_{4n}(Z_{\partial\overline{\mathfrak{C}}})$ and $H_{4n}(Z_{\overline{\mathfrak{C}}})$ as two-sided ideals in $H_{4n}(Z)$. Define $H_{\mathfrak{C}}$ to be the quotient $H_{4n}(Z_{\overline{\mathfrak{C}}})/H_{4n}(Z_{\partial\overline{\mathfrak{C}}})$. Then $H_{\mathfrak{C}}$ is a two-sided $H_{4n}(Z)$ -module with a $\mathbb{Q}$ -algebra structure inherited from the convolution product on $H_{4n}(Z)$.

Kazhdan and Lusztig [KL80] and Chriss and Ginzburg [CG97, §3.5] have proved the following proposition. An alternate argument has also been given by Hinich and Joseph [HJ05].

Let $\mathfrak{S}$ be a set of orbit representatives in $\mathfrak{N}$.

Proposition 3.5. *There are algebra isomorphisms*

$$H_{\mathfrak{C}} \cong H_{4n}(Z_x)^{C(x)} \cong (H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x))^{C(x)}.$$

Thus, the decomposition

$$H_{4n}(Z) \cong \bigoplus_{\mathfrak{C} \in \mathfrak{N}/G} H_{\mathfrak{C}} \cong \bigoplus_{x \in \mathfrak{S}} (H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x))^{C(x)}$$

is a decomposition of $H_{4n}(Z)$ into a direct sum of two-sided ideals with $\dim H_{\mathfrak{C}} = |W_{\mathfrak{C}}|$.

For $\mathfrak{C} = \{0\}$, the two-sided ideal $H_{\mathfrak{C}}$ corresponds to the trivial representation of W and for $\mathfrak{C}$ the regular nilpotent orbit, the two-sided ideal $H_{\mathfrak{C}}$ corresponds to the sign representation of W . However, in general $H_{\mathfrak{C}}$ is not a minimal two-sided ideal. To obtain the decomposition of $H_{4n}(Z)$ into minimal two-sided ideals we need to decompose each $H_{2d_x}(\mathcal{B}_x)$ into $C(x)$ -isotypic components.

For an irreducible representation of $C(x)$ with character ϕ , let $H_{2d_x}(\mathcal{B}_x)_{\phi}$ denote the ϕ -isotypic component. We've seen that $H_{2d_x}(\mathcal{B}_x)_{\phi}$ is an $H_{4n}(Z)$ -submodule of $H_{2d_x}(\mathcal{B}_x)$. Define $\widehat{C(x)}$ to be the set of ϕ with $H_{2d_x}(\mathcal{B}_x)_{\phi} \neq 0$. Then $\widehat{C(x)}$ contains the trivial representation of $C(x)$. In general, $\widehat{C(x)}$ does not contain all irreducible characters of $C(x)$.

The next theorem is proved directly in [KL80] and [CG97, §3.5]. It also follows from the sheaf-theoretic approach to Borel-Moore homology described below.

Theorem 3.6. *There is an isomorphism of $H_{4n}(Z)$ -modules,*

$$(H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x))^{C(x)} \cong \bigoplus_{\phi \in \widehat{C(x)}} \text{End}_{\mathbb{Q}}(H_{2d_x}(\mathcal{B}_x)_{\phi}).$$

Moreover, $H_{2d_x}(\mathcal{B}_x)_{\phi}$ is a simple $H_{4n}(Z)$ -module for every ϕ in $\widehat{C(x)}$ and the decomposition

$$H_{4n}(Z) \cong \bigoplus_{x \in \mathfrak{S}} \bigoplus_{\phi \in \widehat{C(x)}} \text{End}_{\mathbb{Q}}(H_{2d_x}(\mathcal{B}_x)_{\phi})$$

is a decomposition of $H_{4n}(Z)$ into minimal two-sided ideals.

Formulas for the action of a simple reflection on $H_{4n}(Z)$ analogous to Hotta's transformation formulas for the action of a simple reflection on $H_*(\mathcal{B}_x)$ have been given by Hinich and Joseph [HJ05]. The first two parts of the next theorem may be recovered from the more general (and more complicated) argument in [DR08a].

Theorem 3.7. *Suppose that s is a simple reflection in W and w is in W .*

- (a) $\lambda_s = [\overline{Z_s}] + 1$.
- (b) If $sw < w$, then $[\overline{Z_s}] * [\overline{Z_w}] = -2[\overline{Z_w}]$.
- (c) If $sw > w$, then there is a subset $F_{s,w}$ of $\{x \in W \mid x < w, sx < x\}$ so that $[\overline{Z_s}] * [\overline{Z_w}] = [\overline{Z_{sw}}] + \sum_{x \in F_{s,w}} n_x [\overline{Z_x}]$ with $n_x > 0$.

Using this result, Hinich and Joseph prove a result analogous to Proposition 3.5 for right Steinberg cells. Recall that for w in W we have defined $\mathfrak{V}_r(w) = \overline{Bu_w} \cap \mathfrak{C}$ when w is in $W_{\mathfrak{C}}$. For an orbital variety $\mathfrak{V}$, define $W_{\overline{\mathfrak{V}}} = \{y \in W \mid \mathfrak{V}_r(y) \subseteq \mathfrak{V}\}$.

Theorem 3.8. *For w in W , the smallest subset, S , of W with the property that $[\overline{Z_w}] * \lambda_y$ is in the span of $\{[\overline{Z_x}] \mid x \in S\}$ for all y in W is $\overline{\mathfrak{V}_r(w)}$. In particular, if $\mathfrak{V}$ is any orbital variety, then the span of $\{[\overline{Z_x}] \mid x \in W_{\overline{\mathfrak{V}}}\}$ is a right ideal in $H_{4n}(Z)$.*

3.6. Sheaf-theoretic Borel-Moore homology of fibred products. For a variety X , the $\mathbb{Q}$ -vector space $H_i(X)$ has an alternate description in terms of sheaf cohomology see [CG97, §8.3]. The properties of sheaves and perverse sheaves we use in this section may be found in [Dim04] and [Bor84].

Let $D(X)$ denote the full subcategory of the derived category of sheaves of $\mathbb{Q}$ -vector spaces on X consisting of complexes with bounded, constructible cohomology. Consider the constant sheaf, $\mathbb{Q}_X$, as a complex in $D(X)$ concentrated in degree zero. The dualizing sheaf $\mathbb{D}_X$ of X is $f^! \mathbb{Q}_{\{\text{pt}\}}$ where $f: X \rightarrow \{\text{pt}\}$. Then

$$H_i(X) \cong \text{Ext}_{D(X)}^{-i}(\mathbb{Q}_X, \mathbb{D}_X) = \text{Hom}_{D(X)}(\mathbb{Q}_X, \mathbb{D}_X[-i]).$$

Suppose we are in the convolution setup from §3.1 with proper morphisms $f_i: M_i \rightarrow N$ with $d_i = \dim M_i$ for $i = 1, 2, 3$. Consider the cartesian diagram

$$\begin{array}{ccc} M_1 \times_N M_2 & \xrightarrow{f_{1,2}} & N \\ \delta_1 \downarrow & & \downarrow \delta \\ M_1 \times M_2 & \xrightarrow{f_1 \times f_2} & N \times N \end{array}$$

where $f_{1,2}$ is the induced map. Using the argument in [CG97, §8.6] we have isomorphisms

$$\begin{aligned} H_i(M_1 \times_N M_2) &\cong \text{Ext}_{D(M_1 \times_N M_2)}^{-i}(\mathbb{Q}_{M_1 \times_N M_2}, \mathbb{D}_{M_1 \times_N M_2}) \\ &\cong \text{Ext}_{D(M_1 \times_N M_2)}^{-i}(f_{1,2}^* \mathbb{Q}_N, \delta_1^! \mathbb{D}_{M_1 \times M_2}) \\ &\cong \text{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, R(f_{1,2})_* \delta_1^! \mathbb{D}_{M_1 \times M_2}) && \text{(adjunction)} \\ &\cong \text{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, \delta^! R((f_1 \times f_2)_* \mathbb{D}_{M_1 \times M_2})) && \text{(base change)} \\ &\cong \text{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, \delta^! (R(f_1)_* \mathbb{D}_{M_1} \boxtimes R(f_2)_* \mathbb{D}_{M_2})) && \text{(K\"unneth)} \\ &\cong \text{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, \mathcal{H}om(R(f_1)_* \mathbb{Q}_{M_1}, R(f_2)_* \mathbb{D}_{M_2})) && [\text{Bor84, 10.25}] \\ &\cong \text{Ext}_{D(N)}^{2d_2-i}(R(f_1)_* \mathbb{Q}_{M_1}, R(f_2)_* \mathbb{Q}_{M_2}). \end{aligned}$$

It is shown in [CG97, §8.6] that the isomorphism

$$H_i(M_1 \times_N M_2) \cong \text{Ext}_{D(N)}^{2d_2-i}(R(f_1)_* \mathbb{Q}_{M_1}, R(f_2)_* \mathbb{Q}_{M_2})$$

intertwines the convolution product on the left with the Yoneda product (composition of morphisms) on the right.

3.7. Sheaf-theoretic decomposition of $H_{4n}(Z)$. Applying the computation above of the Borel-Moore homology of a fibred product to $Z \cong \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}$ we obtain

$$H_i(Z) \cong \text{Ext}_{D(\mathfrak{N})}^{4n-i}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}, R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}).$$

In particular, $H_{4n}(Z) \cong \text{End}_{D(\mathfrak{N})}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}})$.

Suppose $\mathcal{A}$ is an abelian category and $X = \sum_{i=1}^m S_i^{m_i}$ is a semisimple object $\mathcal{A}$, where $S_1, \dots, S_m$ are non-isomorphic simple objects in $\mathcal{A}$. Assume also that for $1 \leq i \leq m$, $\text{End}_{\mathcal{A}}(S_i) \cong \mathbb{Q}$. The following statements are easily checked.

- $\text{End}_{\mathcal{A}}(X) \cong \bigoplus_{i=1}^m \text{End}_{\mathcal{A}}(S_i^{m_i}) \cong \bigoplus_{i=1}^m M_{m_i}(\text{End}_{\mathcal{A}}(S_i)) \cong \bigoplus_{i=1}^m M_{m_i}(\mathbb{Q})$. In particular, $\text{End}_{\mathcal{A}}(X)$ is a semisimple, Artinian ring.
- $\text{Hom}_{\mathcal{A}}(S_i, X) \cong \text{Hom}_{\mathcal{A}}(S_i, S_i^{m_i}) \cong \text{End}_{\mathcal{A}}(S_i)^{m_i} \cong k^{n_i}$. Therefore, the simple $\text{End}_{\mathcal{A}}(X)$ -modules are the $\text{Hom}_{\mathcal{A}}(S_i, X)$ for $1 \leq i \leq n$.

Let $\mathcal{M}(\mathfrak{N})$ denote the full subcategory of $D(\mathfrak{N})$ consisting of perverse sheaves (with respect to the middle perversity). Then $\mathcal{M}(\mathfrak{N})$ is an abelian category and $\text{End}_{\mathcal{M}(\mathfrak{N})}(K) \cong \mathbb{Q}$ for each simple perverse sheaf K . The complex $R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}$ is a semisimple object in $\mathcal{M}(\mathfrak{N})$ and Borho and MacPherson [BM81] have shown that its decomposition into simple perverse sheaves is given by

$$R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}} \cong \bigoplus_{x, \phi} j_*^x IC(\overline{Gx}, L_{\phi})[-2d_x]^{n_{x, \phi}}$$

where x runs over a set of orbit representatives $\mathfrak{S}$ in $\mathfrak{N}$, and for each x , $j^x: \overline{Gx} \rightarrow \mathfrak{N}$ is the inclusion, ϕ is in $\widehat{C(x)}$, and $n_{x, \phi}$ is a non-negative integer.

Define $IC_{x, \phi} = j_*^x IC(\overline{Gx}, L_{\phi})[-2d_x]$. Then the formalism above implies that

- $H_{4n}(Z) \cong \bigoplus_{x, \phi} \text{End}_{D(\mathfrak{N})}(IC_{x, \phi}^{n_{x, \phi}})$ is the Wedderburn decomposition of $H_{4n}(Z)$ and
- the set $\{\text{Hom}_{D(\mathfrak{N})}(IC_{x, \phi}, R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}) \mid x \in \mathfrak{S}, \phi \in \widehat{C(x)}\}$ is a complete set of non-isomorphic simple $H_{4n}(Z)$ -modules.

If $\mathfrak{C}$ is a G -orbit in $\mathfrak{N}$ and x is in $\mathfrak{C}$, then it is straightforward to check that

$$H_{\mathfrak{C}} \cong \bigoplus_{\phi \in \widehat{C(x)}} \text{End}_{D(\mathfrak{N})}(IC_{x, \phi}^{n_{x, \phi}})$$

Also, if $i_x: \{x\} \rightarrow \mathfrak{N}$ denotes the inclusion, then $\mathcal{B}_x \cong \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \{x\}$. Thus,

$$H_i(\mathcal{B}_x) \cong \text{Ext}_{D(\mathfrak{N})}^{4n-i}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}, R(i_x)_*\mathbb{Q}_{\{x\}}) \cong \text{Ext}_{D(\{x\})}^{4n-i}(i_x^*R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}, \mathbb{Q}_{\{x\}})$$

and the diagram

$$\begin{array}{ccc} H_{4n}(Z) \times H_i(\mathcal{B}_x) & \xrightarrow{\quad * \quad} & H_i(\mathcal{B}_x) \\ \cong \downarrow & & \downarrow \cong \\ \text{End}_{D(\mathfrak{N})}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}) \times \text{Ext}_{D(\mathfrak{N})}^{4n-i}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}, R(i_x)_*\mathbb{Q}_{\{x\}}) & \xrightarrow{\quad \circ \quad} & \text{Ext}_{D(\mathfrak{N})}^{4n-i}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}, R(i_x)_*\mathbb{Q}_{\{x\}}) \end{array}$$

commutes. This gives an alternate description of the Springer representation of W on $H_*(\mathcal{B}_x)$.

3.8. Borel-Moore homology of generalized Steinberg varieties. Recall from §2.4 the generalized Steinberg variety

$$X^{\mathcal{P}, \mathcal{Q}} = \{ (x, P', Q') \in \mathfrak{N} \times \mathcal{P} \times \mathcal{Q} \mid x \in \mathfrak{p}' \cap \mathfrak{q}' \} \cong \tilde{\mathfrak{N}}_c^{\mathcal{P}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}_d^{\mathcal{Q}}$$

where $c(P') = \mathfrak{N} \cap \mathfrak{p}'$ and $d(Q') = \mathfrak{N} \cap \mathfrak{q}'$. Borho and MacPherson [BM83, 2.11] have shown that $\tilde{\mathfrak{N}}_c^{\mathcal{P}}$ and $\tilde{\mathfrak{N}}_d^{\mathcal{Q}}$ are rational homology manifolds. The constructions and results in §3.1 and §3.6 are valid when $Z_{i,j} = M_i \times_N M_j$ and the M_i 's are rational homology manifolds. Thus $H_*(X^{\mathcal{P}, \mathcal{Q}})$ may be computed as in §3.6.

Recall that $\eta: Z \rightarrow X^{\mathcal{P}, \mathcal{Q}}$ is a proper, G -equivariant surjection. The main result of [DR08a] is the following theorem describing the Borel-Moore homology of $X^{\mathcal{P}, \mathcal{Q}}$.

Theorem 3.9. *Consider $H_{4n}(Z)$ as a $W \times W$ -module using the isomorphism $H_{4n}(Z) \cong \mathbb{Q}[W]$. Then there is an isomorphism $\phi: H_*(X^{\mathcal{P}, \mathcal{Q}}) \xrightarrow{\sim} H_*(Z)^{W_P \times W_Q}$ so that the composition $\phi \circ \eta_*: H_*(Z) \rightarrow H_*(Z)^{W_P \times W_Q}$ is the averaging map.*

Let e_P (resp. e_Q) denote the primitive idempotent in $\mathbb{Q}[W_P]$ (resp. $\mathbb{Q}[W_Q]$) corresponding to the trivial representation. Then it follows immediately from the theorem that $H_{4n}(X^{\mathcal{P}, \mathcal{Q}}) \cong e_P \mathbb{Q}[W] e_Q$.

Next recall the generalized Steinberg variety $X_{0,0}^{\mathcal{P}, \mathcal{Q}} \cong T^*\mathcal{P} \times_{\mathfrak{N}} T^*\mathcal{Q}$. Set $m = \dim P/B + \dim Q/B$. Let ϵ_P (resp. ϵ_Q) denote the primitive idempotent in $\mathbb{Q}[W_P]$ (resp. $\mathbb{Q}[W_Q]$) corresponding to the sign representation. Then $\dim X_{0,0}^{\mathcal{P}, \mathcal{Q}} = 4n - 2m$ and it is shown in [DR08a, §5] that $H_{4n-2m}(X^{\mathcal{P}, \mathcal{Q}}) \cong \epsilon_P \mathbb{Q}[W] \epsilon_Q$.

Now suppose that c is a Levi class function on $\mathcal{P}$. Let L be a Levi subgroup of P and choose x in $c(P) \cap \mathfrak{l}$. Then we may consider the Springer representation of W_P on $H_{2d_x^L}(\mathcal{B}_x^L)^{C_L(x)}$ where $C_L(x)$ is the component group of $Z_L(x)$, $\mathcal{B}_x^L$ is the variety of Borel subalgebras of $\mathfrak{l}$ that contain x , and $d_x^L = \dim \mathcal{B}_x^L$. This is an irreducible representation of W_P . Let f_P denote a primitive idempotent in $\mathbb{Q}[W_P]$ so that $\mathbb{Q}[W_P] f_P \cong H_{2d_x^L}(\mathcal{B}_x^L)^{C_L(x)}$. Set $d_{c,d}^{\mathcal{P}, \mathcal{Q}} = \frac{1}{2}(\dim c(P) + \dim \mathfrak{u}_P + \dim d(Q) + \dim \mathfrak{u}_Q)$. Then it is shown in [DR04] that $\dim X_{c,d}^{\mathcal{P}, \mathcal{Q}} \leq d_{c,d}^{\mathcal{P}, \mathcal{Q}}$. We conjecture that the following statement is true.

Conjecture 3.10. *With the notation above, $H_{d_{c,d}^{\mathcal{P}, \mathcal{Q}}}(X_{c,d}^{\mathcal{P}, \mathcal{Q}}) \cong f_P \mathbb{Q}[W] f_Q$.*

4. EQUIVARIANT K -THEORY

Certainly the most important result to date involving the Steinberg variety is its application by Kazhdan and Lusztig to the Langlands program [KL87]. They show that the equivariant K -theory of Z is isomorphic to the two-sided regular representation of the extended, affine Hecke algebra $\mathcal{H}$. They use this representation of $\mathcal{H}$ to classify simple $\mathcal{H}$ -modules and hence to classify representations of ${}^L G(\mathbb{Q}_p)$, the $\mathbb{Q}_p$ points of the Langlands dual of G , containing a vector fixed by an Iwahori subgroup. As with homology, Chris and Ginzburg apply the convolution product formalism to the equivariant K -theory of Z and recast Kazhdan and Lusztig's results as an algebra isomorphism.

In this section, we assume that G is simply connected and describe the isomorphism $\mathcal{H} \cong K^{\overline{G}}(Z)$.

4.1. The generic, extended, affine Hecke algebra. We begin by describing the Bernstein-Zelevinski presentation of the extended, affine Hecke algebra following the construction in [Lus89].

Let $X(T)$ denote the character group of T . Since G is simply connected, $X(T)$ is the weight lattice of G . Define X^+ to be the set of dominant weights with respect to the base of the root system of (G, T) determined by B .

The *extended, affine, Weyl group* is $W_e = X(T) \rtimes W$. There is a “length function” ℓ on W_e that extends the usual length function on W . The *braid group of W_e* is the group $\mathcal{B}r$ with generators $\{T_x \mid x \in W_e\}$ and relations $T_x T_{x'} = T_{xx'}$ if $\ell(x) + \ell(x') = \ell(xx')$.

Let v be an indeterminate and set $A = \mathbb{Z}[v, v^{-1}]$. The *generic, extended, affine Hecke algebra* $\mathcal{H}$ is the quotient of the group algebra $A[\mathcal{B}r]$ by the two-sided ideal generated by the elements $(T_s + 1)(T_s - v^2)$ where s runs through the simple reflections in W .

Given λ in $X(T)$ one can write $\lambda = \lambda_1 - \lambda_2$ where λ_1 and λ_2 are in X^+ . Define E^λ in $\mathcal{H}$ to be the image of $v^{\ell(\lambda_1 - \lambda_2)} T_\lambda$. For w in W , denote the image of T_w in $\mathcal{H}$ again by T_w . Let $\mathcal{H}_W$ denote the Iwahori-Hecke algebra of W (an A -algebra) with standard basis $\{t_w \mid w \in W\}$. Lusztig [Lus89, §2] has proven the following theorem.

Theorem 4.1. *With the notation above we have:*

- (a) E^λ does not depend on the choice of λ_1 and λ_2 .
- (b) The mapping $A[X(T)] \otimes_A \mathcal{H}_W \rightarrow \mathcal{H}$ with $\lambda \otimes t_w \mapsto E^\lambda T_w$ is an isomorphism of A -modules.
- (c) For λ and λ' in X we have $E^\lambda E^{\lambda'} = E^{\lambda + \lambda'}$ and so the subspace of $\mathcal{H}$ spanned by $\{E^\lambda \mid \lambda \in X\}$ is a subalgebra isomorphic to $A[X(T)]$.
- (d) The center of $\mathcal{H}$ is isomorphic to $A[X(T)]^W$ via the isomorphism in (c).
- (e) The subspace of $\mathcal{H}$ spanned by $\{T_w \mid w \in W\}$ is a subalgebra isomorphic to $\mathcal{H}_W$.

Using part (b) of the theorem, we will identify $A[X(T)]$ with the subalgebra of $\mathcal{H}$ spanned by $\{E^\lambda \mid \lambda \in X\}$.

Let $\mathbb{C}^*$ denote the multiplicative group of non-zero complex numbers and set $\overline{T} = T \times \mathbb{C}^*$, $\overline{G} = G \times \mathbb{C}^*$. Define a $\mathbb{C}^*$ -action on $\mathfrak{g}$ by $(\xi, x) \mapsto \xi^2 x$. We consider $\mathcal{B}$ as a $\mathbb{C}^*$ -set with the trivial action. Then the action of G on $\mathfrak{N}$ and Z extends to an action of $\overline{G}$ on $\mathfrak{N}$ and Z and μ_z and μ are $\overline{G}$ equivariant.

For an algebraic group H , $R(H)$ denotes the representation ring of H .

By considering v as the trivial representation of $\mathbb{C}^*$, we may identify $R(\mathbb{C}^*)$ and $A = \mathbb{Z}[v, v^{-1}]$. Then

$$\begin{aligned} R(\overline{T}) &\cong R(T) \otimes R(\mathbb{C}^*) \cong R(T)[v, v^{-1}] \cong A[X(T)] \\ R(\overline{G}) &\cong R(G) \otimes R(\mathbb{C}^*) \cong R(G)[v, v^{-1}] \cong A[X(T)]^W. \end{aligned}$$

Therefore, we have a commutative diagram of A -algebra homomorphisms

$$\begin{array}{ccc} Z(\mathcal{H}) & \hookrightarrow & A[X(T)] \hookrightarrow \mathcal{H} \\ \cong \downarrow & & \cong \downarrow \\ R(\overline{G}) & \hookrightarrow & R(\overline{T}) \end{array}$$

where $Z(\mathcal{H})$ is the center of $\mathcal{H}$.

Let ${}^L G$ denote the Langlands dual of G , so ${}^L G$ is an adjoint group. Let ${}^L G_p$ denote the algebraic group over $\mathbb{Q}_p$ with the same type as ${}^L G$. Suppose that I is an Iwahori subgroup of ${}^L G_p$ and let $\mathbb{C}[I \backslash {}^L G_p / I]$ denote the space of all compactly supported functions ${}^L G_p \rightarrow \mathbb{C}$ that are constant on (I, I) -double cosets. Consider $\mathbb{C}$ as an A -module via the specialization $A \rightarrow \mathbb{C}$ with $v \mapsto \sqrt{p}$. The following theorem is due to Iwahori and Matsumoto [IM65].

Theorem 4.2. *The (I, I) -double cosets are parameterized by W_e . Moreover, if I_x is the double coset indexed by x in W_e , then the map which sends T_x to the characteristic function of I_x extends to an algebra isomorphism*

$$\mathbb{C}[I \backslash {}^L G_p / I] \cong \mathbb{C} \otimes_A \mathcal{H}.$$

4.2. Equivariant K -theory and convolution. Two beginning references for equivariant K -theory is described in [BBM89, Chapter 2] and [CG97, Chapter 5].

For a variety X , let $\mathcal{Coh}(X)$ denote the category of coherent $\mathcal{O}_X$ -modules. Suppose that H is a linear algebraic group acting on X . Let $a: H \times X \rightarrow X$ is the action morphism and $p: H \times X \rightarrow X$ is the projection. An H -equivariant coherent $\mathcal{O}_X$ -module is a pair $(\mathcal{M}, i)$, where $\mathcal{M}$ is a coherent $\mathcal{O}_X$ -module and $i: a^* \mathcal{M} \xrightarrow{\sim} p^* \mathcal{M}$ is an isomorphism (see [CG97, §5.1]). Let $\mathcal{Coh}^H(X)$ denote the abelian category of all H -equivariant $\mathcal{O}_X$ -modules. The Grothendieck group of $\mathcal{Coh}^H(X)$ is denoted by $K^H(X)$ and is called the H -equivariant K -group of X .

If $X = \{\text{pt}\}$ is a point, then $K^H(\text{pt}) \cong R(H)$ is the representation ring of H . For any X , $K^H(X)$ is naturally an $R(H)$ -module.

Suppose $f: X \rightarrow Y$ is a H -equivariant morphism. If f is proper it induces a direct image map in equivariant K -theory, $f_*: K^H(X) \rightarrow K^H(Y)$. If f is smooth it induces a pullback map in equivariant K -theory, $f^*: K^H(Y) \rightarrow K^H(X)$. If X is smooth and A and B are closed, H -stable subvarieties of X , there is an intersection pairing $\cap: K^H(A) \times K^H(B) \rightarrow K^H(A \cap B)$ (called a Tor-product in [Lus98, §6.4]). This pairing depends on (X, A, B) .

Now suppose we are in the convolution setup from §3.1 with $f_i: M_i \rightarrow N$ is proper for $i = 1, 2, 3$, $Z_{i,j}$ closed subvarieties of $M_i \times M_j$ for $1 \leq i < j \leq 3$, and $p_{1,3}: p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3}) \rightarrow Z_{1,3}$ is a proper morphism. Assume in addition that all maps are H -equivariant and the subvarieties $Z_{i,j}$ are H -stable. Then as in §3.1 the formula $c * d = (p_{1,3})_! (p_{1,2}^*(c) \cap p_{2,3}^*(d))$ where $\cap$ is the intersection pairing determined by the subsets $Z_{1,2} \times M_3$ and $M_1 \times Z_{2,3}$ of $M_1 \times M_2 \times M_3$ defines an associative convolution product, $K^H(Z_{1,2}) \otimes K^H(Z_{2,3}) \xrightarrow{*} K^H(Z_{1,3})$.

4.3. The Kazhdan-Lusztig isomorphism. Returning to the Steinberg variety, recall that $Z_1 = \{(x, B', B') \in \mathfrak{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b}'\}$. For suitable choices of $f_i: M_i \rightarrow N$ and $Z_{i,j}$, and using the embedding $A \subseteq R(\overline{G})$, the convolution product induces various A -linear maps:

- $K^{\overline{G}}(Z) \times K^{\overline{G}}(Z) \xrightarrow{*} K^{\overline{G}}(Z)$; with this multiplication, $K^{\overline{G}}(Z)$ is an A -algebra.
- $K^{\overline{G}}(Z_1) \times K^{\overline{G}}(Z_1) \xrightarrow{*} K^{\overline{G}}(Z_1)$; with this multiplication, $K^{\overline{G}}(Z_1)$ is a commutative A -algebra.
- $K^{\overline{G}}(Z) \times K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B}) \xrightarrow{*} K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$; this defines a left $K^{\overline{G}}(Z)$ -module structure on $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$.

The group $K^{\overline{G}}(Z_1)$ has a well-known description: First, the rule $(x, B') \mapsto (x, B', B')$ defines a $\overline{G}$ -equivariant isomorphism between $\tilde{\mathfrak{N}}$ and Z_1 and hence an isomorphism $K^{\overline{G}}(Z_1) \cong$

$K^{\overline{G}}(\tilde{\mathfrak{N}})$. Next, the projection $\tilde{\mathfrak{N}} \rightarrow \mathcal{B}$ is a vector bundle and so we have the Thom isomorphism $K^{\overline{G}}(\tilde{\mathfrak{N}}) \cong K^{\overline{G}}(\mathcal{B})$. Third, set $\overline{B} = B \times \mathbb{C}^*$. Then $\mathcal{B}$ is isomorphic to $G \times^B \{\text{pt}\}$ by a $\overline{G}$ -equivariant isomorphism and so $K^{\overline{G}}(\mathcal{B}) \cong K^{\overline{B}}(\{\text{pt}\}) \cong R(\overline{B})$. Finally, U is the unipotent radical of B and so $R(\overline{B}) \cong R(\overline{T}) \cong A[X(T)]$.

The composite isomorphism $A[X(T)] \cong K^{\overline{G}}(Z_1)$ is an A -algebra isomorphism and is given explicitly as follows. If λ is in $X(T)$, then λ lifts to representation of B . Denote the representation space by $\mathbb{C}_\lambda$. Then the sheaf of sections of the line bundle $G \times^B \mathbb{C}_\lambda$ on $\mathcal{B}$ is a $\overline{G}$ -equivariant, coherent sheaf of $\mathcal{O}_{\mathcal{B}}$ -modules that we will denote by L_λ . Pulling L_λ back first through the vector bundle projection $\tilde{\mathfrak{N}} \rightarrow \mathcal{B}$ and then through the isomorphism $Z_1 \cong \tilde{\mathfrak{N}}$, we get a $\overline{G}$ -equivariant, coherent sheaf of $\mathcal{O}_{Z_1}$ -modules we denote by $\mathcal{L}_\lambda$.

Let $i: Z_1 \rightarrow Z$ be the inclusion. Define $e^\lambda = i_*([\mathcal{L}_\lambda])$ in $K^{\overline{G}}(Z)$. Then $\lambda \mapsto e^\lambda$ defines an A -linear map from $A[X(T)]$ to $K^{\overline{G}}(Z)$.

A concentration theorem due to Thomason and the Cellular Fibration Lemma of Chris and Ginzburg can be used to prove the following proposition (see [CG97, 6.2.7] and [Lus98, 7.15]).

Proposition 4.3. *The closed embeddings $i: Z_1 \rightarrow Z$ and $j: Z \rightarrow \tilde{\mathfrak{N}} \times \mathcal{B}$ induce injective maps in equivariant K -theory,*

$$K^{\overline{G}}(Z_1) \xrightarrow{i_*} K^{\overline{G}}(Z) \xrightarrow{j_*} K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B}).$$

The map i_* is an A -algebra monomorphism and the map j_* is a $K^{\overline{G}}(Z)$ -module monomorphism. In particular, $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$ is a faithful $K^{\overline{G}}(Z)$ -module.

Using the proposition and the isomorphisms $K^{\overline{G}}(\{\text{pt}\}) \cong R(\overline{G}) \cong Z(\mathcal{H})$ we get a commutative diagram of A -algebra homomorphisms

$$\begin{array}{ccccc} Z(\mathcal{H}) & \hookrightarrow & A[X(T)] & \hookrightarrow & \mathcal{H} \\ \cong \downarrow & & \downarrow & & \\ K^{\overline{G}}(\{\text{pt}\}) & \hookrightarrow & K^{\overline{G}}(Z_1) & \hookrightarrow & K^{\overline{G}}(Z) \end{array}$$

We follow the argument in [Lus98, §7] to complete the diagram with an isomorphism of A -algebras $K^{\overline{G}}(Z) \cong \mathcal{H}$.

Fix a simple reflection, s , in W . Then there is a simple root, α , in $X(T)$ and a corresponding cocharacter, $\check{\alpha}: \mathbb{C}^* \rightarrow T$, so that if $\langle \cdot, \cdot \rangle$ is the pairing between characters and cocharacters of T , then $\langle \alpha, \check{\alpha} \rangle = 2$ and $s(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$ for λ in $X(T)$. Choose weights λ' and λ'' in $X(T)$ with $\langle \lambda', \check{\alpha} \rangle = \langle \lambda'', \check{\alpha} \rangle = -1$ and $\lambda' + \lambda'' = -\alpha$. Then $L_{\lambda'} \boxtimes L_{\lambda''}$ is in $\text{Coh}^{\overline{G}}(\mathcal{B} \times \mathcal{B})$. Lusztig [Lus98, 7.19] has shown that the restriction of $L_{\lambda'} \boxtimes L_{\lambda''}$ to $\overline{G(B, sBs)}$ does not depend on the choice of λ' and λ'' . Denote the restriction of $L_{\lambda'} \boxtimes L_{\lambda''}$ to $\overline{G(B, sBs)}$ by $\mathcal{L}_s$.

It is easy to check that $Z_1 \cap \overline{Z_s} = \{(x, gBg^{-1}, gBg^{-1}) \in Z_1 \mid g^{-1}x \in \mathfrak{u}_s\}$. It follows that $\overline{Z_s}$ is smooth and that $\pi: \overline{Z_s} \rightarrow \overline{G(B, sBs)}$ is a vector bundle projection with fibre $\mathfrak{u}_s$. Thus, $\pi^*([\mathcal{L}_s])$ is in $K^{\overline{G}}(\overline{Z_s})$. Let $i_s: \overline{Z_s} \rightarrow Z$ be the inclusion and recall that we are identifying A with a subspace of $R(\overline{G})$. We define $l_s = (i_s)_* \pi^*([\mathcal{L}_s])$ in $K^{\overline{G}}(Z)$.

Lusztig [Lus98, 7.24] has proved the following lemma.

Lemma 4.4. *There is a unique left $\mathcal{H}$ -module structure on $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$ with the property that for every k in $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$, λ in $X(T)$, and simple reflection s in W we have*

- $-(T_s + 1) \cdot k = l_s * k$ and
- $E^\lambda \cdot k = e^\lambda * k$.

Since $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$ is an $\mathcal{H}$ -module and a $K^{\overline{G}}(Z)$ -module, we have ring homomorphisms $\phi_1: \mathcal{H} \rightarrow \text{End}_A(K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B}))$ and $\phi_2: K^{\overline{G}}(Z) \rightarrow \text{End}_A(K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B}))$. It follows from Lemma 4.4 that the image of ϕ_1 is contained in the image of ϕ_2 . It follows from Proposition 4.3 that ϕ_2 is an injection. Therefore, $\phi_2^{-1} \circ \phi_1: \mathcal{H} \rightarrow K^{\overline{G}}(Z)$ is an A -algebra homomorphism. Denote the composition $\phi_2^{-1} \circ \phi_1$ simply by ϕ .

The following theorem is proved in [Lus98, §8] using construction that goes back to [KL87]. A different exposition may be found in [CG97, Chapter 7].

Theorem 4.5. *The A -algebra homomorphism $\phi: \mathcal{H} \rightarrow K^{\overline{G}}(Z)$ is an isomorphism.*

4.4. Equivariant K -theory of generalized Steinberg varieties. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are conjugacy classes of parabolic subgroups of G and recall the generalized Steinberg varieties $X^{\mathcal{P}, \mathcal{Q}}$ and $X_{0,0}^{\mathcal{P}, \mathcal{Q}}$ and the maps $\eta: Z \rightarrow X^{\mathcal{P}, \mathcal{Q}}$ and $\eta_1: Z^{\mathcal{P}, \mathcal{Q}} = \eta^{-1}(X_{0,0}^{\mathcal{P}, \mathcal{Q}}) \rightarrow X_{0,0}^{\mathcal{P}, \mathcal{Q}}$ from §2.4. We have a cartesian square

$$(4.6) \quad \begin{array}{ccc} Z^{\mathcal{P}, \mathcal{Q}} & \xrightarrow{k} & Z \\ \eta_1 \downarrow & & \downarrow \eta \\ X_{0,0}^{\mathcal{P}, \mathcal{Q}} & \xrightarrow{k_1} & X^{\mathcal{P}, \mathcal{Q}} \end{array}$$

where k and k_1 are the inclusions.

The morphism η_1 is smooth and so there is a pullback map in equivariant K -theory, $\eta_1^*: K^{\overline{G}}(X_{0,0}^{\mathcal{P}, \mathcal{Q}}) \rightarrow K^{\overline{G}}(Z^{\mathcal{P}, \mathcal{Q}})$. We can describe the $R(\overline{G})$ -module structure of $K^{\overline{G}}(Z^{\mathcal{P}, \mathcal{Q}})$ and $K^{\overline{G}}(X_{0,0}^{\mathcal{P}, \mathcal{Q}})$ using the argument in [Lus98, 7.15] together with a stronger concentration theorem due to Thomason [Tho92].

Theorem 4.7. *The homomorphisms $\eta_1^*: K^{\overline{G}}(X_{0,0}^{\mathcal{P}, \mathcal{Q}}) \rightarrow K^{\overline{G}}(Z^{\mathcal{P}, \mathcal{Q}})$ and $k_*: K^{\overline{G}}(Z^{\mathcal{P}, \mathcal{Q}}) \rightarrow K^{\overline{G}}(Z)$ are injective. Moreover, $K^{\overline{G}}(X_{0,0}^{\mathcal{P}, \mathcal{Q}})$ is a free $R(\overline{G})$ -module with rank $|W|^2/|W_P||W_Q|$ and $K^{\overline{G}}(Z^{\mathcal{P}, \mathcal{Q}})$ is a free $R(\overline{G})$ -module with rank $|W|^2$.*

In the rest of this subsection we consider the very special case when $\mathcal{P} = \mathcal{Q} = \{G\}$. In this case, we can use results of Lusztig to describe the map k_* in terms of $\mathcal{H}$ and results of Ostrik to describe the map η_* and $K^{\overline{G}}(\mathfrak{N})$.

Computing the spaces in (4.6) when $\mathcal{P} = \mathcal{Q} = \{G\}$ we obtain

$$X_{0,0}^{\{G\}, \{G\}} \equiv \{0\}, \quad Z^{\{G\}, \{G\}} = \overline{Z_{w_0}} = Z_{\{0\}} \cong \mathcal{B} \times \mathcal{B}, \quad X^{\{G\}, \{G\}} \equiv \mathfrak{N}.$$

Also, $\eta: Z \rightarrow X^{\{G\}, \{G\}}$ may be identified with $\mu_z: Z \rightarrow \mathfrak{N}$ and $k: Z^{\{G\}, \{G\}} \rightarrow Z$ may be identified with the closed embedding $\mathcal{B} \times \mathcal{B} \rightarrow Z$ by $(B', B'') \mapsto (0, B', B'')$.

It follows from Theorem 4.7 that $k_*: K^{\overline{G}}(\mathcal{B} \times \mathcal{B}) \rightarrow K^{\overline{G}}(Z)$ is injective. Notice that in contrast, even though the map $k_*: H_i(\mathcal{B} \times \mathcal{B}) \rightarrow H_i(Z)$ in Borel-Moore homology is an injection for $i = 4n$, it is not an injection in general. In particular, the inclusion of $\overline{Z_w}$ in Z for an arbitrary w does not induce an injective map on Borel-Moore homology.

Taking $M_i = \tilde{\mathfrak{N}}$, $Z_{1,2} = Z$ and $Z_{2,3} = Z^{\{G\},\{G\}}$ in the convolution construction, we have $Z_{1,3} = Z^{\{G\},\{G\}}$, and so $K^{\overline{G}}(Z^{\{G\},\{G\}})$ has a left $K^{\overline{G}}(Z)$ -module structure. Using the projection formula in equivariant K -theory [CG97, 5.3.12] it is straightforward to show that k_* is $K^{\overline{G}}(Z)$ -linear. Therefore, the image of k_* is a left ideal in $K^{\overline{G}}(Z)$. Similarly, the image of k_* is a right ideal in $K^{\overline{G}}(Z)$ and so the image of k_* is a two sided ideal of $K^{\overline{G}}(Z)$.

Define $E = \sum_{w \in W} T_w$ in $\mathcal{H}$. Then it is easy to see that $T_y E = v^{\ell(y)} E$ for y in W . Lusztig [Lus98, 8.11] has shown that E is in the image of k_* . This can be used to prove the next proposition.

Proposition 4.8. *The image of $k_*: K^{\overline{G}}(Z^{\{G\},\{G\}}) \rightarrow K^{\overline{G}}(Z)$ equals the image of the two-sided ideal $\mathcal{H}E\mathcal{H}$ of $\mathcal{H}$ under the Kazhdan-Lusztig isomorphism $\phi: \mathcal{H} \rightarrow K^{\overline{G}}(Z)$.*

Now consider $\eta_*: K^{\overline{G}}(Z) \rightarrow K^{\overline{G}}(X^{\{G\},\{G\}})$.

The extended, affine Weyl group W_e is not a Coxeter group in general, it is an extension of a Coxeter group, and so the algebra $\mathcal{H}$ is not the Hecke algebra of a Coxeter group. Nevertheless the $\mathcal{H}$ does have a Kazhdan-Lusztig basis, $\{c'_x \mid x \in W_e\}$ (see [Lus98]). Notice that for a simple reflection s in W we have $c'_s = T_s + 1$ and so by Lemma 4.4 we have $\phi(c'_s) = -l_s$.

Recall that $W_e = X(T) \rtimes W$. Using the computations in [IM65] it can be shown each (W, W) -double coset contains a unique element in X^+ and a unique element with minimal length. For λ in X^+ we let m_λ denote the element with minimal length in $W\lambda W$. Ostrik has proved the following theorem describing $K^{\overline{G}}(\mathfrak{N}) = K^{\overline{G}}(X^{\{G\},\{G\}})$.

Theorem 4.9. *For x in W_e , $\eta_*(c'_x) = 0$ unless $x = m_\lambda$ for some λ in X^+ . Moreover, the map $\eta_*: K^{\overline{G}}(S) \rightarrow K^{\overline{G}}(X^{\{G\},\{G\}})$ is surjective and $\{\eta_*(c'_{m_\lambda}) \mid \lambda \in X^+\}$ is an A -basis of $K^{\overline{G}}(X^{\{G\},\{G\}})$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON TX, USA 76203

E-mail address: douglass@unt.edu

URL: <http://hilbert.math.unt.edu>

FAKULTÄT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM, D-44780 BOCHUM, GERMANY

E-mail address: gerhard.roehrle@rub.de

URL: <http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Lehrstuhl-VI/rubroehrle.html>