QUATERNIONS AND REFLECTIONS*

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1. Introduction. It is just a hundred years since Cayley began to use quaternions for the discussion of rotations. He was followed by Boole, Donkin, Clifford, Buchheim, Klein, Hurwitz, Hathaway, Stringham, and Study. Apparently none of these men thought of considering first the simpler operation of *reflection* and deducing a rotation as the product of two reflections. This procedure will be described in §§3 and 5, and its consequences developed in the later sections.

Every quaternion $a=a_0+a_1i+a_2j+a_3k$ determines a point $\mathbf{P}_a=(a_0, a_1, a_2, a_3)$ in euclidean 4-space, and a hyperplane $a_0x_0+a_1x_1+a_2x_2+a_3x_3=0$. The reflection in that hyperplane is found to be the transformation $x\rightarrow -a\bar{x}a/\mathrm{N}a$. This leads easily to the classical expression

$$x \rightarrow axb \qquad (\mathrm{N}a = \mathrm{N}b = 1)$$

for the general displacement preserving the origin. Cayley obtained this elegant expression by "brute force" as early as 1855. It became somewhat more natural in the hands of Klein and Hurwitz, forty years later. The treatment in §7 will possibly serve to clarify it still further.

We begin with a few algebraic lemmas, mostly due to Hamilton.

2. Elementary properties of quaternions. A *quaternion* is a hyper-complex number $a=a_0+a_1i+a_2j+a_3k$, where a_0, a_1, a_2, a_3 are real numbers ("scalars"), and multiplication is defined by the rules

$$i^2 = j^2 = k^2 = ijk = -1,$$

which imply $jk=i=-kj$, $ki=j=-ik$, $ij=k=-ji$. Thus quaternions form an associative but non-commutative algebra.

It is often convenient to split a quaternion into its "scalar" and "vector" parts:†

$$a = \mathrm{S}a + \mathrm{V}a, \qquad \mathrm{S}a = a_0, \qquad \mathrm{V}a = a_1i + a_2j + a_3k.$$

We define also the *conjugate* quaternion‡

$$\bar{a} = \mathrm{S}a - \mathrm{V}a = a_0 - a_1i - a_2j - a_3k$$

and the *norm*

$$\mathrm{N}a = \bar{a}a = a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

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* This paper is an amplification of an invited address delivered at the annual meeting of the Mathematical Association of America in Chicago on November 24, 1945.

† W. R. Hamilton, Elements of Quaternions, vol. 1, London, 1899, pp. 177, 193. By his special convention, p. 186, Sxy means S(xy), not (Sx)y; similarly for Vxy and Nxy.

‡ Klein's symbol $\bar{a}$ seems preferable to Hamilton's Ka.

In terms of a and $\bar{a}$, we have $\mathrm{S}a=\frac{1}{2}(a+\bar{a})$, $\mathrm{V}a=\frac{1}{2}(a-\bar{a})$.

Since $\overline{ii}=-1=\bar{i}\bar{i}$ and $\overline{ij}=-k=\bar{j}\bar{i}$, *etc.*, we easily verify that

$$\overline{ab} = \bar{b}\bar{a},$$

whence $\mathrm{N}ab=\overline{ab}ab=\bar{b}\bar{a}ab=\bar{b}(\mathrm{N}a)b=\mathrm{N}a\mathrm{N}b$. If $\mathrm{N}a=1$, we call a a *unit* quaternion. To every non-vanishing quaternion a there corresponds a unit quaternion*

$$\mathrm{U}a = a/\sqrt{\mathrm{N}a}.$$

A quaternion p is said to be *pure* if $\mathrm{S}p=0$. Then $\bar{p}=-p$ and $p^2=-\mathrm{N}p$. Thus a pure unit quaternion is a square root of -1.

The point $\mathbf{P}_x=(x_1, x_2, x_3)$ in ordinary space may be represented by the pure quaternion $x=x_1i+x_2j+x_3k$. This representation resembles the Argand diagram in the plane, in that the distance $\mathbf{P}_x\mathbf{P}_y$ is given by $\mathbf{P}_x\mathbf{P}_y^2=\mathrm{N}(y-x)$. Since

$$xy = -(x_1y_1 + x_2y_2 + x_3y_3) + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k,$$

we see that $-\mathrm{S}xy$ and $\mathrm{V}xy$ are the ordinary "scalar product" and "vector product" of the two vectors $\mathbf{P}_0\mathbf{P}_x$ and $\mathbf{P}_0\mathbf{P}_y$. If x and y are pure *unit* quaternions, then $\mathbf{P}_x$ and $\mathbf{P}_y$ lie on the unit sphere around the origin $\mathbf{P}_0$, and we have $\angle\mathbf{P}_x\mathbf{P}_0\mathbf{P}_y=\theta$, where

$$\cos\theta = -\mathrm{S}xy = -\tfrac{1}{2}(xy + yx).$$

Thus the condition for $\mathbf{P}_x$ and $\mathbf{P}_y$ to lie in perpendicular directions from $\mathbf{P}_0$ is

$$xy + yx = 0.$$

LEMMA 2.1. *For any quaternion a we can find a unit quaternion y such that $ay=y\bar{a}$.*

Proof. Take any pure quaternion p for which $\mathbf{P}_0\mathbf{P}_p$ is perpendicular to $\mathbf{P}_0\mathbf{P}_{\mathrm{V}a}$.† Then

$$(\mathrm{V}a)p + p(\mathrm{V}a) = 0.$$

But since the scalar $\mathrm{S}a$ commutes with p,

$$(\mathrm{S}a)p - p(\mathrm{S}a) = 0.$$

By addition,

$$ap - p\bar{a} = 0.$$

The desired *unit* quaternion is $y=\mathrm{U}p$.

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* Hamilton, *op. cit.*, p. 137.

† If an explicit formula is desired, we may write $p=\mathrm{V}(\mathrm{V}a)q$, where q is any pure quaternion (other than the scalar multiples of $\mathrm{V}a$). In fact, if $p=\mathrm{V}\alpha q$, where $\alpha=\mathrm{V}a$, then $2p=\alpha q-q\alpha$, and

$$2(\alpha p + p\alpha) = \alpha(\alpha q - q\alpha) + (\alpha q - q\alpha)\alpha = \alpha^2 q - q\alpha^2 = 0,$$

α^2 being scalar. Thus our appeal to geometry could have been avoided.

LEMMA 2.2. *Let a and b be two quaternions having the same norm and the same scalar part.** *Then we can find a unit quaternion y such that $ay=yb$.*

Proof. If $b=\bar{a}$, this is covered by Lemma 2.1; so let us assume $b\neq\bar{a}$. Since $\mathrm{S}a=\mathrm{S}b$, we have

$$a+\bar{a}=b+\bar{b}, \qquad a-\bar{b}=b-\bar{a},$$
$$a(a-\bar{b})b=a(b-\bar{a})b,$$
$$a(ab-\mathrm{N}b)=(ab-\mathrm{N}a)b.$$

Thus $ac=cb$, where $c=ab-\mathrm{N}b=ab-\mathrm{N}a=a(b-\bar{a})$. Since $b\neq\bar{a}$, $c\neq 0$; so we can take $y=\mathrm{U}c$.

LEMMA 2.3. *Any quaternion is expressible as a power of a pure quaternion.*†

Proof. Since $\mathrm{N}p^t=(\mathrm{N}p)^t$, it will be sufficient to prove this for a *unit* quaternion, a. Since $(\mathrm{S}a)^2-(\mathrm{V}a)^2=\mathrm{N}a=1$, such a quaternion may be expressed as

$$a=\cos\alpha+p\sin\alpha,$$

where $\cos\alpha=\mathrm{S}a$ and p is a *pure* unit quaternion. Since $\mathrm{V}a=p\sin\alpha$, we have $p=\mathrm{UV}a$. Since $p^2=-1$, de Moivre's Theorem shows that

$$a^n=\cos n\alpha+p\sin n\alpha$$

for any real number n. In particular, $a^{\pi/2\alpha}=p$, so $a=p^{2\alpha/\pi}$. Thus

$$a=p^t,$$

where $p=\mathrm{UV}a$ and $\cos\frac{1}{2}t\pi=\mathrm{S}a$ (so that we may suppose $0\leqq t\leqq 2$).

3. Reflections and rotations in three dimensions. We have seen that $\mathbf{P}_x$ and $\mathbf{P}_y$ lie in perpendicular directions from the origin $\mathbf{P}_0$ if the pure quaternions x and y satisfy the relation

$$xy+yx=0.$$

If $\mathrm{N}y=1$, so that $y^2=-1$, this condition may be expressed as

$$x=yxy.$$

Since $\overline{yxy}=\bar{y}\bar{x}\bar{y}=-yxy$, yxy is pure for *any* position of $\mathbf{P}_x$. Thus the linear transformation $x\to yxy$ (where $\mathrm{N}y=1$) represents a collineation which leaves invariant every point $\mathbf{P}_x$ in the plane through $\mathbf{P}_0$ perpendicular to $\mathbf{P}_0\mathbf{P}_y$, *i.e.*, the plane

$$y_1x_1+y_2x_2+y_3x_3=0.$$

Moreover, it reverses the vector $\mathbf{P}_0\mathbf{P}_y$:

$$y\to y^3=-y.$$

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* Two such quaternions satisfy the same "rank equation" $x^2-2mx+n=0$, where $m=\mathrm{S}a=\mathrm{S}b$ and $n=\mathrm{N}a=\mathrm{N}b$. For a general discussion of the equation $ay=yb$, see Arthur Cayley, On the quaternion equation $qQ-Qq'=0$, Messenger of Mathematics, vol. 14, 1885, pp. 108–112.

† Hamilton, p. 399.

But the only collineation having these properties is the *reflection* in the plane of invariant points. Hence

THEOREM 3.1. *The reflection in the plane* $\sum y_\nu x_\nu = 0$ *is represented by the transformation*

$$x \to yxy \qquad (\mathrm{N}y = 1).$$

The product of two such reflections, $x \to yxy$ and $x \to zxz$ (with $\mathrm{N}y = \mathrm{N}z = 1$), is the transformation

$$x \to zyxyz.$$

Accordingly, this is a *rotation*, in the plane $\mathbf{P}_0\mathbf{P}_y\mathbf{P}_z$, through twice the angle $\mathbf{P}_y\mathbf{P}_0\mathbf{P}_z$. In other words, it is a rotation through ϕ about the line $\mathbf{P}_0\mathbf{P}_{\mathrm{V}yz}$, where

$$\cos \tfrac{1}{2}\phi = -\,\mathrm{S}yz.$$

By Lemma 2.1, any unit quaternion a may be expressed as $-zy$, where $ay = y\bar{a} = z$. (Here z, like y, is a pure unit quaternion; for,

$$z + \bar{z} = ay - y\bar{a} = 0, \quad \text{and} \quad \mathrm{N}z = \mathrm{N}a\mathrm{N}y = 1.)$$

Since $\bar{a} = -yz$, this shows that the transformation

$$x \to ax\bar{a} \qquad (\mathrm{N}a = 1)$$

is a rotation through ϕ about $\mathbf{P}_0\mathbf{P}_{\mathrm{V}a}$, where $\cos \frac{1}{2}\phi = \mathrm{S}a$.

In the notation of the proof of Lemma 2.3, we have $a = \cos\alpha + p\sin\alpha$, where $\mathbf{P}_0\mathbf{P}_p$ is the unit vector along the axis of rotation, and $\alpha = \pm\frac{1}{2}\phi$. The sign is a matter of convention. Taking the plus, we find that the rotation through $\frac{1}{2}\pi$ about the x_3-axis $\mathbf{P}_0\mathbf{P}_k$ transforms $\mathbf{P}_i$ into $\mathbf{P}_j$:

$$i \to ai\bar{a} = \frac{1+k}{\sqrt{2}}\, i\, \frac{1-k}{\sqrt{2}} = \frac{(i+j)(1-k)}{2} = j.$$

(The opposite convention would have given $i \to -j$.) We sum up in

THEOREM 3.2.* *The rotation through* ϕ *about the line with direction cosines* (p_1, p_2, p_3) *is represented by the transformation* $x \to ax\bar{a}$, *where*

$$a = \cos\tfrac{1}{2}\phi + (p_1 i + p_2 j + p_3 k)\sin\tfrac{1}{2}\phi.$$

Since the product of two rotations, $x \to ax\bar{a}$ and $x \to bx\bar{b}$, is another rotation, *viz.*, $x \to bax\overline{ba}$, we can immediately deduce

THEOREM 3.3. *All the rotations about lines through the origin in ordinary space form a group, homomorphic to the group of all unit quaternions.*

Since the rotation through ϕ is indistinguishable from the rotation through $\phi + 2\pi$ about the same axis, there are *two* quaternions, $\pm a$, for each rotation.

* Arthur Cayley, On certain results relating to quaternions, Philosophical Magazine (3), vol. 26, 1845, p. 142. George Boole, Notes on quaternions, *ibid.*, vol. 33, 1848, p. 279. W. F. Donkin, On the geometrical theory of rotation, *ibid.* (4), vol. 1, 1851, p. 189.

Alternatively, we may say that the group of rotations is *isomorphic* to the group of all "homogeneous quaternions," in accordance with the formula

$$x \to ax\bar{a}/\mathrm{N}a, \quad \text{or} \quad x \to axa^{-1}.$$

(A homogeneous quaternion is the class of all scalar multiples of an ordinary quaternion.) To make this an isomorphism rather than an anti-isomorphism, we must agree to multiply group elements from right to left.

Combining $x \to ax\bar{a}$ with the *inversion* $x \to -x$, we obtain the transformation

$$x \to -ax\bar{a} \qquad (\mathrm{N}a = 1)$$

which may be regarded either as a *rotatory-inversion* of angle ϕ or as a *rotatory-reflection* of angle $\phi + \pi$.

4. The general displacement. Two orthogonal trihedra at the same origin can be brought into coincidence by the successive application of the following reflections: one to interchange the two x_1-axes, another to interchange the two x_2-axes (without disturbing the x_1-axis), and a third (if necessary) to reverse the x_3-axis. Thus the general orthogonal transformation in three dimensions is the product of at most three reflections—an even or odd number according as the transformation preserves or reverses sense. In particular, any *displacement* (or "movement") leaving the origin fixed must be the product of only two reflections *i.e.*, a *rotation*. (This is a famous result, due to Euler.) Thus the group considered in Theorem 3.3 is the group of all such displacements, and is a subgroup of index 2 in the group of all orthogonal transformations. The latter contains also the rotatory-reflections $x \to -ax\bar{a}$, which are products of three reflections.

Similarly in four dimensions, the general orthogonal transformation (*i.e.*, congruent transformation with a fixed origin) is a product of at most four reflections (in hyperplanes). Thus the general displacement (leaving the origin fixed) is the product of two or four reflections. But the product of two reflections is a rotation (about the common plane of the two hyperplanes, through twice the angle between them). Hence a displacement is either a single rotation or the product of two rotations about distinct planes. If any point besides the origin is invariant, the displacement can only be a single rotation; for it operates essentially in the 3-space perpendicular to the line of invariant points. In the general case of a double rotation, where only the origin is fixed, it is well known (though far from obvious) that the axial planes of the two rotations may be chosen to be completely orthogonal. This was first proved by Goursat in 1889. We shall obtain a new proof in §9.

5. Reflections and rotations in four dimensions. A point $\mathbf{P}_x$ in euclidean 4-space has four Cartesian coordinates (x_0, x_1, x_2, x_3) which may be interpreted* as the constituents of a quaternion

$$x = x_0 + x_1 i + x_2 j + x_3 k.$$

* A. S. Hathaway, Quaternions as numbers of four-dimensional space, Bulletin of the American Mathematical Society, vol. 4, 1897, pp. 54–57.

The distance $\mathbf{P}_x\mathbf{P}_y$ is given by

$$\mathbf{P}_x\mathbf{P}_y^2 = (y_0 - x_0)^2 + (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 = \mathrm{N}(y - x).$$

If x and y are *unit* quaternions, then $\mathbf{P}_x$ and $\mathbf{P}_y$ lie on the unit hypersphere around the origin $\mathbf{P}_0$, and we have $\angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_y = \theta$, where

$$\cos\theta = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = \mathrm{S}x\bar{y} = \tfrac{1}{2}(x\bar{y} + y\bar{x}).$$

Thus the condition for $\mathbf{P}_x$ and $\mathbf{P}_y$ to lie in perpendicular directions from $\mathbf{P}_0$ is

$$x\bar{y} + y\bar{x} = 0.$$

If $\mathrm{N}y = 1$, this condition may be expressed as

$$x = -\, y\bar{x}y.$$

Thus the linear transformation $x \rightarrow -y\bar{x}y$ (where $\mathrm{N}y = 1$) represents a collineation which leaves invariant every point $\mathbf{P}_x$ in the hyperplane through $\mathbf{P}_0$ perpendicular to $\mathbf{P}_0\mathbf{P}_y$, *i.e.*, the hyperplane

$$y_0x_0 + y_1x_1 + y_2x_2 + y_3x_3 = 0.$$

But it reverses the vector $\mathbf{P}_0\mathbf{P}_y$:

$$y \rightarrow -\, y\bar{y}y = -\, y.$$

Hence

THEOREM 5.1. *The reflection in the hyperplane* $\sum y_\nu x_\nu = 0$ *is represented by the transformation*

$$x \rightarrow -\, y\bar{x}y \qquad (\mathrm{N}y = 1).$$

The product of two such reflections, $x \rightarrow -y\bar{x}y$ and $x \rightarrow -z\bar{x}z$ (with $\mathrm{N}y = \mathrm{N}z = 1$), is the transformation

$$x \rightarrow z\, \overline{y\bar{x}y}\, z = z\bar{y}x\bar{y}z.$$

Accordingly, this is a rotation, *in* the plane $\mathbf{P}_0\mathbf{P}_y\mathbf{P}_z$, through twice the angle $\mathbf{P}_y\mathbf{P}_0\mathbf{P}_z$. In other words, it is a rotation *about* the completely orthogonal plane

$$\sum y_\nu x_\nu = \sum z_\nu x_\nu = 0$$

through angle ϕ, where $\cos\tfrac{1}{2}\phi = \mathrm{S}y\bar{z} = \mathrm{S}z\bar{y} = \mathrm{S}\bar{y}z$. This proves

THEOREM 5.2. *The general rotation through angle* ϕ *(about a plane) is*

$$x \rightarrow axb,$$

where $\mathrm{N}a = \mathrm{N}b = 1$ and $\mathrm{S}a = \mathrm{S}b = \cos\tfrac{1}{2}\phi$.

Conversely, the transformation $x \rightarrow axb$ (where $\mathrm{N}a = \mathrm{N}b = 1$) is a rotation whenever $\mathrm{S}a = \mathrm{S}b$; for then, by Lemma 2.2, we can find unit quaternions y and z such that $ay = yb = z$, enabling us to write

$$a = z\bar{y}, \qquad b = \bar{y}z.$$

6. Clifford translations. The product of the two rotations $x \to axa$ and $x \to ax\bar{a}$ is the so-called *left translation**

$$x \to a^2 x,$$

while the product of $x \to axa$ and $x \to \bar{a}xa$ is the *right translation*

$$x \to xa^2.$$

Clearly, left translations $x \to ax$ form a group, so do right translations $x \to xb$, and any left translation commutes with any right translation. Every left translation has a *unique* expression $x \to ax$ $(\mathrm{N}a = 1)$; for, the equation $ax = bx$ would imply $a = b$. Similarly every right translation has a unique expression $x \to xb$. Hence

THEOREM 6.1. *The group of left (or right) translations is isomorphic to the group of unit quaternions.*

(In the case of right translations, we can make this a true isomorphism, and not merely an anti-isomorphism, by letting the quaternion b correspond to the translation $x \to x\bar{b}$, so that the product ab corresponds to $x \to x\overline{ab} = x\bar{b}\bar{a}$.)

Comparing this with Theorem 3.3, we see that the group of left (or right) translations is homomorphic to the group of rotations about a fixed origin in ordinary space, with two Clifford translations for each rotation.

A Clifford translation (*i.e.*, a left or right translation) has the remarkable property of turning every vector through the same angle. For, if $\mathrm{N}x = 1$, so that also $\mathrm{N}ax = 1$, the left translation $x \to ax$ transforms $\mathbf{P}_x$ into $\mathbf{P}_{ax}$, and

$$\cos \angle \mathbf{P}_x \mathbf{P}_0 \mathbf{P}_{ax} = \mathrm{S}ax\bar{x} = \mathrm{S}a,$$

which is the same for all vectors $\mathbf{P}_0\mathbf{P}_x$. Similarly, $x \to xb$ transforms $\mathbf{P}_x$ into $\mathbf{P}_{xb}$, and $\cos \angle \mathbf{P}_x \mathbf{P}_0 \mathbf{P}_{xb} = \mathrm{S}\bar{x}xb = \mathrm{S}b$.

Can a left translation be also a right translation? This would make $ax = xb$ for every x. The case $x = 1$ gives $a = b$. Now take x to be the y of Lemma 2.1. Then

$$ya = ay = y\bar{a}, \qquad y(a - \bar{a}) = 0, \qquad \mathrm{V}a = 0,$$

and so, since $\mathrm{N}a = 1$, $a = \pm 1$. Thus the only left translations that are also right translations† are the *identity* $x \to x$ and the *inversion* $x \to -x$.

Instead of deriving Theorem 5.1 from the condition for $\angle \mathbf{P}_x \mathbf{P}_0 \mathbf{P}_y$ to be a right angle, we might have observed that $x \to ax$ (where $\mathrm{N}a = 1$) must be some kind of congruent transformation (since $\mathrm{N}ax = \mathrm{N}x$), and that this transforms the special reflection $x \to -x$ into the general reflection

$$x \to -a\overline{\bar{a}x} = -a\bar{x}a.$$

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* Felix Klein, Vorlesungen über nicht-Euklidische Geometrie, Berlin, 1928, p. 240.

† William Threlfall and Herbert Seifert, Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes, Mathematische Annalen, vol. 104, 1931, p. 10.

7. The group of displacements. The product of a left translation and a right translation is, of course, $x \to axb$ (where $\mathrm{N}a = \mathrm{N}b = 1$). The product of two dis- displacements of this form is another of the same form. In particular, as we saw in Theorem 5.2, a *rotation* is of this form (with the special relation $\mathrm{S}a = \mathrm{S}b$). But the general displacement with a fixed origin is the product of two rotations. Hence

THEOREM 7.1.* *The general displacement preserving the origin is*

$$x \to axb \qquad (\mathrm{N}a = \mathrm{N}b = 1).$$

Of course, axb is the same as $(-a)x(-b)$. With this exception, each displacement has a unique expression $x \to axb$. For, the equation $axb = a'xb'$ would imply $a'^{-1}ax = xb'b^{-1}$ for every x, whence $a'^{-1}a = b'b^{-1} = \pm 1$. In other words, every displacement (with a fixed origin) is the product of a left translation and a right translation in just two ways. Thus the direct product of the groups of all left translations and of all right translations is homomorphic to the group of all four-dimensional displacements preserving the origin (with two elements of the direct product for each displacement), and this in turn is homomorphic to the direct square of the group of all three-dimensional displacements preserving the origin (with two displacements for each element of the direct square).

8. The general orthogonal transformation in four dimensions. The general "opposite" or "sense-reversing" transformation leaving the origin invariant is the product of an odd number of reflections. Hence, in four dimensions, it is either a single reflection or a product of three. But in the latter case the three reflecting hyperplanes intersect in a line of invariant points, and every hyperplane perpendicular to this line is invariant; so this scarcely differs from a rotatory-reflection in ordinary space. As such, it has an axis and a special reflecting plane. Its product with the special reflection $x \to -\bar{x}$ is a displacement $x \to axb$; so the rotatory-reflection itself must be $x \to -a\bar{x}b$, or, after changing the sign of a,

$$x \to a\bar{x}b.$$

Since $a\,\overline{a+b}\,b = a+b$, the line of invariant points is $\mathbf{P}_0\mathbf{P}_{a+b}$. Since $a\,\overline{a-b}\,b = -(a-b)$, the axis is $\mathbf{P}_0\mathbf{P}_{a-b}$.

We sum up our conclusion in

THEOREM 8.1. *Every orthogonal transformation in four dimensions is either*

$$x \to axb \quad \text{or} \quad x \to a\bar{x}b.$$

9. The general displacement expressed as a double rotation. By Theorem 5.2, the general *half-turn* is

$$x \to pxq,$$

* Arthur Cayley, Recherches ultérieures sur les déterminants gauches, Journal für die reine und angewandte Mathematik, vol. 50, 1855, p. 312.

where p and q are pure unit quaternions. This is the half-turn about the plane containing all points $\mathbf{P}_x$ for which $x = pxq$, or

$$px + xq = 0.*$$

Since $p - q$ and $1 + pq$ are particular solutions of this equation for x, we may describe the plane as $\mathbf{P}_0\mathbf{P}_{p-q}\mathbf{P}_{1+pq}$. A rotation through $t\pi$ about the same plane is, of course, $x \to p^t x q^t$. Replacing q by $-q$ $(= \bar{q} = q^{-1})$, we deduce that $x \to p^t x q^{-t}$ is a rotation through $t\pi$ about the completely orthogonal plane $\mathbf{P}_0\mathbf{P}_{p+q}\mathbf{P}_{1-pq}$. (The fact that these two planes *are* completely orthogonal is most easily verified by observing that the product of the half-turns $x \to pxq$ and $x \to pxq^{-1}$ is the inversion $x \to p^2 x = -x$.)

Thus the product of rotations through $t\pi$ and $u\pi$ about the respective planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$ is $x \to p^{t+u} x q^{t-u}$. Setting $t\pi = \alpha + \beta$, $u\pi = \alpha - \beta$, and observing that

$$\cos \alpha + p \sin \alpha = p^{2\alpha/\pi}$$

(see the proof of Lemma 2.3), we deduce that the product of rotations through angles $\alpha \pm \beta$ about planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$ is

$$x \to (\cos \alpha + p \sin \alpha) x (\cos \beta + q \sin \beta).$$

In other words,

THEOREM 9.1.† *The general displacement $x \to axb$ is the double rotation through angles $\alpha \pm \beta$ about planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$, where*

$$\cos \alpha = \mathrm{S}a, \qquad \cos \beta = \mathrm{S}b, \qquad p = \mathrm{UV}a, \qquad q = \mathrm{UV}b.$$

10. Lines in elliptic space. The above considerations can be translated into terms of *elliptic* geometry by identifying pairs of antipodal points on the hypersphere. Now all scalar multiples of a quaternion x represent the same point $\mathbf{P}_x$, whose coordinates (x_0, x_1, x_2, x_3) are homogeneous. The transformation

$$x \to -y\bar{x}y$$

is the reflection in the polar plane of $\mathbf{P}_y$, and this is the same as the inversion in $\mathbf{P}_y$ itself. We now say that the group of all displacements is precisely the direct product of the groups of left and right displacements; accordingly, it is isomorphic to the direct square of the group of displacements preserving the origin. Instead of Theorem 9.1, we say that the general displacement $x \to axb$ is the product of rotations through angles $\alpha \pm \beta$ about the respective *lines* $\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$, where

$$(1) \qquad \cos \alpha = \mathrm{S}a, \qquad \cos \beta = \mathrm{S}b, \qquad p = \mathrm{UV}a, \qquad q = \mathrm{UV}b.$$

* Irving Stringham, On the geometry of planes in parabolic space of four dimensions, Transactions of the American Mathematical Society, vol. 2, 1901, p. 194.

† Édouard Goursat, Sur les substitutions orthogonales et les divisions régulières de l'espace, Annales Scientifiques de l'École Normale Supérieure (3), vol. 6, 1889, p. 36.

The line $\mathbf{P}_{p-q}\mathbf{P}_{1+pq}$, which is the axis of the half-turn $x \to pxq$, is conveniently denoted by $\{p, q\}$,* or equally well by $\{-p, -q\}$. Thus *any two pure unit quaternions determine a line* $\{p, q\}$. The absolute polar line is $\{-p, q\}$ or $\{p, -q\}$.

Two lines $\{p, q\}$ and $\{p', q'\}$ have, in general, two common perpendicular lines, which are the transversals of the four lines $\{\pm p, q\}$ and $\{\pm p', q'\}$. These are preserved by either of the half-turns $x \to pxq$, $x \to p'xq'$, and so also by their product

$$x \to p'pxqq'. \tag{2}$$

Thus they are the two axes of this double rotation. Any point on either axis will be reflected first in $\{p, q\}$ and then in $\{p', q'\}$; altogether it will be translated along that axis through twice the distance between the lines. Hence *the two distances between the lines* $\{p, q\}$ *and* $\{p', q'\}$, *measured along their common perpendiculars, are* $\frac{1}{2}|\alpha \pm \beta|$, *where*

(3)† $$\cos \alpha = -\mathrm{S}p'p = -\mathrm{S}pp', \qquad \cos \beta = -\mathrm{S}qq';$$

and the common perpendiculars themselves are the lines

$$\{\pm \mathrm{UV}pp', \mathrm{UV}qq'\}. \tag{4}$$

It follows from (3) that the condition for $\{p, q\}$ and $\{p', q'\}$ to *intersect* is

$$\mathrm{S}pp' = \mathrm{S}qq', \tag{5}$$

and then the angle between them, being half the angle of the rotation (2), is arc cos $(\pm \mathrm{S}pp')$. Similarly, the condition for $\{p, q\}$ and $\{p', q'\}$ to be *perpendicular* (*i.e.*, for one to intersect the polar of the other) is

$$\mathrm{S}pp' + \mathrm{S}qq' = 0. \tag{6}$$

Thus the condition for them to intersect at right angles is

$$\mathrm{S}pp' = \mathrm{S}qq' = 0. \tag{7}$$

The common perpendicular lines (4) cease to be determinate if $\mathrm{V}pp'$ or $\mathrm{V}qq'$ vanishes, *i.e.*, if either $p' = \pm p$ or $q' = \pm q$. Lines $\{p, q\}$ and $\{p, q'\}$ are said to be *left parallel*. They have an infinity of common perpendiculars

$$\{\mathrm{UV}pp', \mathrm{UV}qq'\}, \tag{8}$$

where p' may range over *all* unit pure quaternions (except $\pm p$). To verify

* This notation was used by A. S. Hathaway, Quaternion space, Transactions of the American Mathematical Society, vol. 3, 1902, p. 53. It is closely associated with the representation of a line in elliptic space by an ordered pair of points on a sphere; see Eduard Study, Beiträge zur nichteuklidische Geometrie, American Journal of Mathematics, vol. 29, 1907, pp. 121–124.

† Here we are using (1) with $a = -p'p$, $b = -qq'$. Plus signs would have given the supplementary distances, which are equally valid; but it seems preferable to use the sign that makes α and β small when p' and q' are nearly equal to p and q.

this we merely have to observe that the line (8), which intersects $\{p, q\}$ at right angles, also intersects $\{p, q'\}$ at right angles. The distance between these left parallel lines, measured along any of the common perpendiculars, is $\frac{1}{2}$ arc cos $(-Sqq')$.

Similarly, *right parallel* lines $\{p, q\}$ and $\{p', q\}$ are distant $\frac{1}{2}$ arc cos $(-Spp')$ along any of an infinity of common perpendiculars (8), only now it is q' that can vary. Thus the common perpendicular lines of right parallels are left parallel, and vice versa.

By the remark at the beginning of §9, the condition for the line $\{p, q\}$ to contain the point $\mathbf{P}_x$ is

$$px + xq = 0. \tag{9}$$

Regarding this as an equation to be solved for q or p, we see that *we can draw through a given point* $\mathbf{P}_x$ *just one left and one right parallel to a given line* $\{p, q\}$, namely

$$\{p, -x^{-1}px\} \quad \text{and} \quad \{-xqx^{-1}, q\}.$$

Thus the set of all left (or right) parallels to a given line is an elliptic congruence: there is just one member of the set through every point of space.

If $\{p, q\}$ contains $\mathbf{P}_y$, its polar line lies in the polar plane of $\mathbf{P}_y$. Replacing q by $-q$, we deduce that the condition for the line $\{p, q\}$ to lie in the plane $\sum y_\nu x_\nu = 0$ is

$$py = yq. \tag{10*}$$

Instead of insisting that the p and q of the symbol $\{p, q\}$ shall be *unit* pure quaternions, we could just as well allow them to be any two pure quaternions of *equal* norm. Then $\{p, q\}$ is the same line as $\{\lambda p, \lambda q\}$ for any non-zero scalar λ; *i.e.*, the two pure quaternions are "homogeneous coordinates" for the line. Instead of (3) we must now write

$$\cos \alpha = -Spp'/\sqrt{Npp'}, \qquad \cos \beta = -Sqq'/\sqrt{Nqq'};$$

but instead of (4) we find that the common perpendiculars to $\{p, q\}$ and $\{p', q'\}$ are simply $\{\pm Vpp', Vqq'\}$, or

$$\{Vpp', Vqq'\} \quad \text{and} \quad \{Vp'p, Vqq'\}.$$

Formulas (5), (6), (7), (9), (10) remain valid, but (8) takes the simpler form $\{Vpp', Vqq'\}$.

To express the line $\mathbf{P}_a\mathbf{P}_b$ in the form $\{p, q\}$ we have to find pure quaternions p and q satisfying

$$pa + aq = pb + bq = 0.$$

We may take $p = a\bar{b} - b\bar{a}$ and $q = \bar{a}b - \bar{b}a$, or, halving these, $p = Va\bar{b}$ and $q = V\bar{a}b$. Thus *the line* $\mathbf{P}_a\mathbf{P}_b$ *is* $\{Va\bar{b}, V\bar{a}b\}$.

Similarly, the line of intersection of planes $\sum a_\nu x_\nu = 0$ and $\sum b_\nu x_\nu = 0$ is $\{Va\bar{b}, V\bar{b}a\}$.

* Hathaway, Quaternion Space, p. 52.