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Complex geometry and the asymptotics of Harish-Chandra modules for real reductive Lie groups II

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1. Introduction and reformulations

Let $G_{\mathbb{R}}$ be a connected semisimple real matrix group. It is now apparent that the representation theory of $G_{\mathbb{R}}$ is intimately connected with the complex geometry of the flag variety $\mathscr{B}$. By studying appropriate orbit structures on $\mathscr{B}$, we are naturally led to representation theory in the category of Harish-Chandra modules $\mathscr{HC}$, or the representation theory of category $\mathcal{O}'$. The Jacquet functor J: $\mathscr{HC} \to \mathcal{O}'$ has proved a useful tool in converting "$\mathscr{HC}$ problems" into "$\mathcal{O}'$ problems", which are often more tractable. The full potential of this transference philosophy is obstructed by the notoriously obscure $\mathfrak{g}$-module structure of the Jacquet modules $J(X)$, $X \in \mathscr{HC}$. In this paper, we advance the philosophy that the complex geometry of $\mathscr{B}$, associated to $\mathscr{HC}$ and $\mathcal{O}'$, interacts in a natural way with the functor J, leading to deep new information on the structure of Jacquet modules. Indeed, our main theorem will establish the existence of a computable $\mathfrak{g}$-module filtration of $J(X)$ having semisimple subquotients, whenever X is an irreducible Harish-Chandra module with integral infinitesimal character. Our techniques depend upon the combinatorial Hecke algebra formalism of our previous paper [8], analysis of certain "weights of Frobenius" in the setting of positive characteristic and Bernstein's geometric construction of Zuckerman's "K-finite functor".

In more details, we fix, once and for all, a reductive affine algebraic group G over $\mathbb{R}$ ($\mathbb{R}$ the field of real numbers). We look at G as the set of zeros in $\mathrm{GL}(n, \mathbb{C})$ of a finite set of polynomials in the matrix entries with real coefficients. Assume $G_{\mathbb{R}}$ has finite index in the set of real points of G; then $G_{\mathbb{R}}$ is a *real reductive group*, in the sense of [22]. Fix a maximal compact subgroup $K_{\mathbb{R}}$ with complexification K, $B = H_m N$ an *Iwasawa-Borel subgroup* of G with complexified Lie algebra $\mathfrak{b}$ (where $H_{m,\mathbb{R}}$ is a maximally split θ-stable Cartan subgroup (θ being the Cartan involution) and N is the nilradical) and $\bar{N} = \theta(N)$,

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the opposite nilradical. Denote by $\mathfrak{g}$ the complexified Lie algebra of $G_{\mathbb{R}}$ and define $\mathscr{B}$ to be the usual flag variety of all Borel subalgebras of $\mathfrak{g}$. Both K and N act upon $\mathscr{B}$ in a natural way, with finitely many orbits. Fix, once and for all, a finite dimensional irreducible representation F of $G_{\mathbb{R}}$. We introduce the category $\mathscr{HC}_F$ (resp. $\mathcal{O}'_F$) to consist of all Harish-Chandra modules (resp. all finitely generated $\mathfrak{b}$-finite $\mathfrak{U}(\mathfrak{g})$-modules) with the same infinitesimal character as F. Let $H=K$ or N and define

$$\mathscr{D}_H=\left\{\begin{matrix}(\gamma,\mathcal{O}_\gamma)\colon \mathcal{O}_\gamma \text{ is an } H\text{-orbit on } \mathscr{B} \text{ and } \gamma \text{ is an } H\text{-homogeneous}\\ \text{line bundle on } \mathcal{O}_\gamma \text{ with a flat connection.}\end{matrix}\right\}$$

Given $(\gamma,\mathcal{O}_\gamma)\in\mathscr{D}_H$, we will often abbreviate $\gamma=(\gamma,\mathcal{O}_\gamma)$ and to each such object there is attached a natural notion of "size", given by

$$\textit{length of } \gamma=\ell(\gamma)=\dim\mathcal{O}_\gamma.$$

Our next result recalls the formal connection between representation theory and the sets $\mathscr{D}_H$, $H=K$ or N.

(1.1) **Proposition** [1, 17, 21]. *The set of irreducible representations in* $\mathscr{HC}_F$ (resp. $\mathcal{O}'_F$) *is in one-to-one correspondence with the set* $\mathscr{D}_K$ (resp. $\mathscr{D}_N$).

Given $\gamma\in\mathscr{D}_K$, we may attach a standard induced from discrete series $\pi(\gamma)$ and its unique irreducible quotient $\bar{\pi}(\gamma)$, [21]. The set $\mathscr{D}_N$ may be identified with the Weyl group W of (G,B) and in this sense each $w\in\mathscr{D}_N$ may be associated to a Verma module M_w and its unique irreducible quotient L_w. Here, we follow the conventions of [15]: $M_e=L_e$ and M_{w_0} contains F as a quotient, where w_0 = the longest element of W. A refinement of (1.1) asserts:

(1.2) The Grothendieck group $K(\mathscr{HC}_F)$ (resp. $K(\mathcal{O}'_F)$) has a basis $\{\pi(\gamma)\colon\gamma\in\mathscr{D}_K\}$ and $\{\bar{\pi}(\gamma)\colon\gamma\in\mathscr{D}_K\}$ (resp. $\{M_w\colon w\in\mathscr{D}_N\}$ and $\{L_w\colon w\in\mathscr{D}_N\}$).

We recall the *Jacquet functor* $J\colon\mathscr{HC}_F\to\mathcal{O}'_F$, defined by

(1.3) $$J(X)=\gamma_{\mathfrak{n}}((\tilde{X})^*);$$

here, $\gamma_{\mathfrak{n}}(\ldots)$ = $\mathfrak{n}$-locally nilpotent vectors in $(\ldots)$, $\sim$(resp. $*$) denoting the admissible (resp. full algebraic) dual. J is a faithful exact covariant functor commuting with the natural duality operations in category $\mathscr{HC}_F$ and $\mathcal{O}'_F$; in particular, $J(\bar{\pi}(\gamma))$ is a self-dual object in $\mathcal{O}'_F$ [9, 14, 19, 23]. Our main result (1.11) can be loosely stated as follows

(1.4) **The filtration algorithm.** *Fix an irreducible Harish-Chandra module* $\bar{\pi}(\gamma)$, $\gamma\in\mathscr{D}_K$. *Then there exists an explicit combinatorial algorithm to compute a* $\mathfrak{g}$-*module filtration*

$$0=E_0\subsetneqq E_1\subsetneqq E_2\subsetneqq\ldots\subsetneqq E_{N-1}\subsetneqq E_N=J(\bar{\pi}(\gamma)),$$

with the property that each E_i/E_{i-1} $(1\leqq i\leqq N)$ *is completely reducible in* $\mathcal{O}'$.

(1.5) *Remarks.* (a) By the term "explicit", we mean the following: Given Langlands data $\bar{\pi}(\gamma)$, our algorithm will describe the composition factors of each E_i *and* the irreducible summands of each semisimple subquotient E_i/E_{i-1}, in terms of various L_w, $w \in W$.

(b) In practice, the algorithm of (1.4) may be very tedious to complete by hand. Roughly speaking, one must use the Kazhdan-Lusztig formulas for $\mathscr{H}\mathscr{C}$ and $\mathcal{O}'$, simultaneously. The computational difficulty increases (rapidly) with the real rank of $G_{\mathbb{R}}$. If $G_{\mathbb{R}}$ has real rank one, these filtrations were computed in [8, sec. 4] or [10]; in this case, $J(\bar{\pi}(\gamma))$ contains one, two, three or (at most) four irreducible composition factors. In the first paper, we compute the filtration of (1.4) for $\bar{\pi}(\gamma_f)$ = an irreducible fundamental series representation of the real rank two group $\mathrm{SL}_3\mathbb{C}$; in this case, $J(\bar{\pi}(\gamma_f))$ contains 44 composition factors arranged in seven different levels!

(c) The case of $G_{\mathbb{R}} = \mathrm{SL}_2\mathbb{R}$ is described in (1.15) below.

(d) Before the appearance of (1.4), the best structural result concerning $J(\bar{\pi}(\gamma))$, was knowledge of its formal character; i.e. the irreducible composition factors of $J(\bar{\pi}(\gamma))$ were previously known. This uses the *Kazhdan-Lusztig conjectures* (which are theorems [1, 6, 21]) and *Osborne's conjecture* (which was proved in [13]).

(e) Typically, $J(\bar{\pi}(\gamma))$ is highly reducible, as can be seen from the examples cited in (b). Nevertheless, the self-duality of $J(\bar{\pi}(\gamma))$ imposes interesting symmetry properties upon the $\mathfrak{g}$-module filtration of (1.4). In all examples we have studied, the filtration $\{E_i\}$ of $J(\bar{\pi}(\gamma))$ has an odd number of *levels* (i.e. successive non-zero subquotients E_i/E_{i-1}). However, we have been unable to prove this in general (cf. remarks below). Presumably, this parity condition on the levels is true and the term(s) L_w defining the Langlands data of $\bar{\pi}(\gamma)$ are all attached to the "middle row" of $J(\bar{\pi}(\gamma))$. Otherwise put, we do not know $\{E_i\}$ of (1.4) is a *self-dual filtration*; though we conjecture it is such (cf. (1.23) below).

(1.6) **Corollary**[1]. *There exists an explicit combinatorial algorithm to compute a completely reducible submodule of* $J(\bar{\pi}(\gamma))$, *which must contribute to* $H_0(\bar{\mathfrak{n}}, \bar{\pi}(\gamma))$ *and will define the same primitive ideal as* $\bar{\pi}(\gamma)$.

Of course, the corollary may produce several irreducible submodules of $J(\bar{\pi}(\gamma))$, depending upon the length of the completely reducible $\mathfrak{g}$-module E_1 of (1.4). Using *Frobenius reciprocity* [9], we are led to (typically) new embeddings of $\bar{\pi}(\gamma)$ into principal series representations. These embeddings, which we term *buried embeddings*, are attached to the bottom level of the filtration $\{E_i\}$ in (1.4); by contrast, the *leading embedding*(*s*) (given by asymptotics [9, 13]) are typically attached to an intermediary level "near the middle". (In fact, if the parity assertion of (1.5e) held, they would be attached to the middle level). In principle, (1.6) describes the primitive ideals attached to objects in $\mathscr{H}\mathscr{C}_F$, by using the known classification in category $\mathcal{O}'$.

We now turn toward a discussion of our proof. Just as with the proof of the Kazhdan-Lusztig conjectures, a key idea is to translate (1.4) into both a combinatorial and a geometric setting. Then, within this translated setting, our problem will be solved.

[1] Further connections with $\bar{\mathfrak{n}}$-homology appear in [24]

To motivate our combinatorial reformulation of (1.4), we make a few observations. The Grothendieck groups $K(\mathscr{H}\mathscr{C}_F)$ and $K(\mathcal{O}'_F)$ carry a W-module action, via the coherent continuation representation. The Jacquet functor extends to a map $\mathbb{J}^0$ of Grothendieck groups and

(1.7) $\mathbb{J}^0\colon K(\mathscr{H}\mathscr{C}_F)\to K(\mathcal{O}'_F)$ *is a W-module map.*

From a result such as (1.7), the best one could hope to extract would be composition factor information for $J(\bar{\pi}(\gamma))$. Indeed, this is exactly why (1.5d) holds. How can we possibly hope to extract the filtration $\{E_i\}$ of (1.4) from (1.7)? Quite simply, the answer is that we cannot. To proceed further, we need an analogue of (1.7), where the Grothendieck groups are replaced by "finer" entities and $\mathbb{J}^0$ interacts nicely with this extra information.

To carry out the program of our previous paragraph will require the formalism of the Hecke algebra, Hecke modules, etc. Let $H=K$ or N and define

(1.8) $$\mathscr{M}_H=\mathbb{Z}[q^{-1/2},q^{1/2}]\otimes_{\mathbb{Z}}\mathbb{Z}[\mathscr{D}_H],$$

which is referred to as the *Hecke module* attached to $\mathscr{D}_H$. Identifying $\pi(\gamma)\leftrightarrow\gamma$ (or $M_w\leftrightarrow w$), we may view (1.8) as a q-deformation of the usual Grothendieck groups. Moreover, recall the *Hecke algebra* $\mathscr{H}$ associated to our pair (W,B), [15, 17]. Then the Hecke modules $\mathscr{M}_H$ carry an action of $\mathscr{H}$ [8, 17].

We noted in (1.2) that $K(\mathscr{H}\mathscr{C}_F)$ and $K(\mathcal{O}'_F)$ carry two natural basis. Moreover, the irreducibles in (1.2) form a *self-dual basis*. By analogy, $\{\gamma\colon\gamma\in\mathscr{D}_K\}$ or $\{w\colon w\in W=\mathscr{D}_N\}$ form basis for $\mathscr{M}_H$, $H=K$ or N. There is an analogue of the self-dual basis given by

(1.9) $$\{\hat{C}_\gamma\colon\gamma\in\mathscr{D}_K\}\quad\text{and}\quad\{\hat{C}_w\colon w\in W\},$$

as defined in [8]. The objects in (1.9) are self-dual with respect to the natural duality operation on $\mathscr{M}_H$.

Given $X\in\mathscr{M}_N$, write

(1.10) $$X=\sum_{n=-M}^{M}q^{n/2}\underbrace{\Big(\sum_{n(i)}a_w(n(i))\,\hat{C}_{w(n(i))}\Big)}_{X_n},$$

where $a_w(n(i))\in\mathbb{Z}$ and X_n is called the n^{th} *level* of the *weight filtration data* of X. We caution the reader that the indexing in (1.10) is not meant to suggest any kind of self-duality; i.e. some X_n may be zero. Schematically, we may represent (1.10) as

$$X=X_{-M}<X_{-M+1}<\ldots<X_{M-1}<X_M,$$

or

$$X=\begin{array}{|c|}\hline X_M\\\hline \vdots\\\hline X_{-M}\\\hline\end{array}\qquad\begin{array}{c}q^{M/2}\\ \vdots\\ q^{-M/2}\end{array}$$

levels *weights*

Our main theorem is best stated (and proved) in the following context.

(1.11) **Theorem.** *There exists a Hecke algebra map of Hecke modules* $\mathbb{J}$: $\mathscr{M}_K \to \mathscr{M}_N$, *which satisfies the following two properties:*

(I) *Let* $\gamma \in \mathscr{D}_K$ *and consider the weight filtration data*

$$\mathbb{J}(\hat{C}_\gamma) = \sum_{n=-M}^{M} q^{n/2} \Big(\sum_{n(i)} a_w(n(i))\, \hat{C}_{w(n(i))}\Big). \tag{1.12}$$

Then there exists a $\mathfrak{g}$*-module filtration* $\{E_n\}$ *of* $J(\bar{\pi}(\gamma))$, *having semisimple subquotients* (*as in* (1.4)), *and satisfying*

$$E_n/E_{n-1} = \bigoplus_i |a_w(n(i))|\, L_{w(n(i))}. \tag{1.13}$$

(II) *There exists an explicit combinatorial algorithm to compute the weight filtration data of* $\mathbb{J}(\hat{C}_\gamma)$; *hence, the semisimple subquotients of our* $\mathfrak{g}$*-module filtration* $\{E_n\}$ of $J(\bar{\pi}(\gamma))$.

(1.14) *Remarks.* (a) We view this as the "correct" analogue of (1.7), in a context where data as in (1.4) may be extracted.

(b) In the examples we have computed (cf. [8]), all levels X_n of (1.12) $-M \leqq n \leqq M$, are non-zero. This suggests that the weight filtration data of $\mathbb{J}(\hat{C}_\gamma)$ is self-dual *and* weights attached to successive levels differ by $q^{\pm 1/2}$; this will be discussed further near the end of the introduction.

(c) In (1.13), we must take the absolute value of $a_w(n(i))$. This is related to the connection between $\mathscr{M}_N$ and a characteristic p version of $K(\mathscr{O}'_F)$; see [8] or [3].

(1.15) *Example.* To fully orient the reader, we now describe in full details the case of $G_{\mathbb{R}} = \mathrm{SL}_2\mathbb{R}$. In this case, $\mathscr{B} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Thus, $\mathscr{B}$ has three K-orbits: $\{0\}$, $\{\infty\}$ and $\mathbb{C}^x =$ the open K-orbit. The isotropy groups K_x for x in one of the three orbits, are K, K and $\{\pm I\}$, respectively. The set $\mathscr{D}_K$ contains four elements; the constant sheaves on the orbits and a "mobius band" coming from the double cover of $\mathbb{C}^x$. From (1.1), each of these elements may be identified with a standard representation: $\delta_+ =$ principal series which has F as a quotient (corresponds to the trivial sheaf on $\mathbb{C}^x$); $\delta_- =$ irreducible principal series (corresponds to the "mobius band" on the open orbit); $\delta_h =$ a holomorphic discrete series (corresponds to the constant sheaf on $\{0\}$); and $\delta_a =$ an antiholomorphic discrete series (corresponds to the constant sheaf on $\{\infty\}$). Notice that

$$\ell(\delta_+) = \ell(\delta_-) = 1 \quad \text{and} \quad \ell(\delta_h) = \ell(\delta_a) = 0.$$

In a similar way, one can identify $\mathscr{D}_N$ with a two-element set: $\delta_s =$ Verma module with F as a quotient (corresponds to the open N orbit in $\mathscr{B}$) and δ_e $=$ an irreducible Verma module (corresponds to the closed N orbit). Furthermore, we have $\ell(\delta_s) = 1$ and $\ell(\delta_e) = 0$.

Using (1.11), lets see how to compute the weight filtration data for $\mathbb{J}(\hat{C}_\delta)$. Begin by setting

$$\mathbb{J}(C_{\delta_-}) = a\, C_s + b\, C_e,$$

where a, $b \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$; we are identifying $s \leftrightarrow \delta_s$, $e \leftrightarrow \delta_e$ and $W = \{e, s\}$. From [15, 17 and 21],

(a) $(T_s + 1) \cdot \delta_- = 0$

(b) $(T_s + 1) \cdot C_s = (1 + q)\, C_s$

(c) $(T_s + 1) \cdot C_e = C_s$.

We find

$$0 = \mathbb{J}((T_s + 1) \cdot \delta_-) = (T_s + 1)\, \mathbb{J}(C_{\delta_-}) = (T_s + 1)(a\, C_s + b\, C_e) = (a + a\, q + b)\, C_s.$$

This forces $b = -(1 + q) a$ and so

$$\mathbb{J}(C_{\delta_-}) = a(C_s - (1 + q)\, C_e).$$

Since δ_- is a largest growth representation, [12] insures that $a(1) = 1$. *Assume* that $a \equiv 1$, then

$$\begin{aligned} \mathbb{J}(\hat{C}_{\delta_-}) &= \mathbb{J}(q^{-1/2}\, C_{\delta_-}) \\ &= q^{-1/2}\, \mathbb{J}(C_{\delta_-}) \\ &= q^{-1/2}\, C_s - q^{-1/2}\, C_e - q^{1/2}\, C_e \\ &= -q^{-1/2}\, \hat{C}_e + q^0\, \hat{C}_s - q^{1/2}\, \hat{C}_e. \end{aligned}$$

This gives weight filtration data

$$\mathbb{J}(\hat{C}_{\delta_-}) = \begin{array}{|c|} \hline -\hat{C}_e \\ \hline \hat{C}_s \\ \hline -\hat{C}_e \\ \hline \end{array} \quad \begin{array}{c} 1/2 \\ 0 \\ -1/2. \end{array} \tag{1.16}$$

The observation to be made is that (1.16) will correspond to the socle filtration of $J(\bar{\pi}(\delta_-))$. Precisely, as is described in [14], we have

$$J(\bar{\pi}(\delta_-)) = \begin{array}{|c|} \hline L_e \\ \hline L_s \\ \hline L_e \\ \hline \end{array}\,,$$

where $L_e = \operatorname{socle} J(\bar{\pi}(\delta_-))$, $L_s = \operatorname{socle}(J(\bar{\pi}(\delta_-))/L_e)$, $L_e = \operatorname{top} J(\bar{\pi}(\delta_-))$. The correspondence can be even more precise, by inserting the appropriate ± 1's in $\hat{C}_{\delta_-}$. This "match up" was our original motivation for studying the fine structure of Jacquet modules. Similar calculations show $\mathbb{J}(\hat{C}_{\delta_h}) = J(\hat{C}_{\delta_a}) = \hat{C}_e$ and $J(\bar{\pi}(\delta_a)) = J(\bar{\pi}(\delta_h)) = L_e$.

The above example suggests that the knowledge of $\mathbb{J}$ being a Hecke algebra map (1.11), leads to an explicit algorithm for computing the weight filtration data of $\mathbb{J}(\hat{C}_\gamma)$, $\gamma \in \mathscr{D}_K$, *modulo* a priori knowledge of certain pieces of large growth modules. (This indeterminacy entered through "a" of (1.15).) More carefully, let

$$\mathscr{D}_K^0 = \{\gamma \in \mathscr{D}_K : \dim \mathcal{O}_\gamma = \dim \mathscr{B}\}.$$

This set parametrizes the *largest growth representations* of Hecht [12]. The important property of such representations is the fact: $J(\bar{\pi}(\gamma))$ contains F as a subquotient exactly once, whenever $\gamma\in\mathscr{D}_K^0$. Define a map A: $\mathscr{D}_K^0\to\mathbb{Z}[q^{-1/2},q^{1/2}]$, $\mathbb{J}(\hat{C}_\gamma)=\sum_{w\in W} a_w(\gamma)\hat{C}_w$, $a_{w_0}(\gamma)=A(\gamma)$, $\gamma\in\mathscr{D}_K^0$ and w_0 = the long element of W. Our work in [8] and Sect. 6 shows

(1.17) **Observation.** *Modulo the knowledge of A, there exists an explicit combinatorial algorithm to compute the weight filtration data of $\mathbb{J}(\hat{C}_\gamma)$, $\gamma\in\mathscr{D}_K$.*

The proof of (1.11) is now reduced to a priori understanding of the map A in (1.17), plus justification that the weight filtration data of $\mathbb{J}(\hat{C}_\gamma)$ corresponds to an actual $\mathfrak{g}$-module filtration $\{E_n\}$ of $J(\bar{\pi}(\gamma))$ having semisimple subquotients. It is at this juncture we must invoke the philosophy of "passing to positive characteristic". Using the Beilinson-Bernstein localization theory [1] and the Riemann-Hilbert correspondence [3], we obtain a functor

(1.18) $$J_q\colon \mathscr{H}\mathscr{C}_q\to\mathscr{O}'_q.$$

Here, $q=p^r$ is almost any prime power and $\mathscr{H}\mathscr{C}_q$, $\mathscr{O}'_q$ are certain (abelian) categories of perverse $\bar{\mathbb{Q}}_\ell$ sheaves supported on K_q or N_q orbits of the mod q flag variety $\mathscr{B}_q$ defined over the finite field $\mathbb{F}_q$; see [8, Sect. 3] for unexplained notation or terminology. We can now translate the problem solved by (1.4) into this characteristic p setting; i.e. we want to compute a filtration of the sheaf $J_q(\bar{\pi}(\gamma))$, $\gamma\in\mathscr{D}_K$. As it stands, this looks just as formidable (if not more so). However, our new setting admits an underlying structure which was invisible in characteristic zero. In particular, the *Frobenius morphism* Fr: $\mathscr{B}_q\to\mathscr{B}_q$ becomes a homeomorphism (in the étale topology) and acts upon the stalks of $\bar{\mathbb{Q}}_\ell$-sheaves which carry a Frobenius action. That is, given a point $x\in\mathscr{B}_q$ fixed by Fr and a sheaf $\mathscr{E}$ carrying an action of Frobenius, Fr: $\mathscr{E}_x\to\mathscr{E}_x$ and we can ask to compute the eigenvalues of this endomorphism. From this viewpoint, our main theorem has the following geometric reformulation.

(1.19) *We compute the action of Frobenius on $J_q(\bar{\pi}(\gamma))$, modulo roots of unity.*

(1.20) *Remarks.* (a) Implicit in (1.19) is the fact that $J_q(\bar{\pi}(\gamma))$ carries an action of Frobenius; something which is not a priori clear. Moreover, we are showing these weights have the form $\varepsilon q^{i/2}$ ($i\in\mathbb{Z}$), ε a root of unity *and* we compute these weights. Typically, such calculations are very deep problems in algebraic geometry (eg. the Riemann hypothesis portion of the Weil conjectures, as proved by Deligne and generalized by Gabber [2]). However, we are blessed with a priori knowledge of the weights of Frobenius on the intersection cohomology of the standard irreducibles in $\mathscr{H}\mathscr{C}_F$ and $\mathscr{O}'_F$ [3, 17]; these facts use the Weil conjectures! With these known weights in hand, Theorem (1.11) allows us to compute the weights in (1.19).

(b) Computing the map A of (1.17) becomes important in (1.11).

(c) If $\mathscr{E}$ carries a Frobenius action and the weights of Fr (i.e. the eigenvalues on stalks above fixed points) have complex absolute value $q^{i/2}$ for some

$i \in \mathbb{Z}$, then we say $\mathscr{E}$ is a *mixed sheaf*. If $\mathscr{E}$ carries a Frobenius action (with no condition on the eigenvalues), then $\mathscr{E}$ is called a *Weil sheaf*.

As described in [2] or [8, Sect. 3], a result of Gabber's asserts that any mixed *perverse* sheaf has a filtration by perverse subsheaves and each successive subquotient will be a direct sum of intersection cohomology sheaves with Fr acting by a fixed weight of absolute value $q^{i/2}$, some i. From the formalism in [17], we see that this filtration will correspond to the weight filtration data of $\mathbb{J}(\hat{C}_\gamma)$. Thus, these remarks reduce our entire proof to the computation of A in (1.17), plus showing $J_q(\bar{\pi}(\gamma))$ is mixed. These two facts are proved simultaneously in Sect. 6. In this sense, everything has now been reduced to proving $\mathbb{J}$ is a Hecke algebra map.

In many respects, everything we have said thus far is present in [8]. However, showing commutativity of $\mathbb{J}$ with $\mathscr{H}$ proves to be extremely subtle. Originally, our approach was to geometrically construct J in such a way that it would obviously commute with $\mathscr{H}$. This was attempted in [8] and encapsulated thru the notion of a geometric Jacquet functor J_{geo}. However, there we were unable to show $J_{\text{geo}} = J$. When the results of [8] were presented in a seminar talk at MIT during the Fall of 1984, Wilfried Schmid suggested the following tact: By the work of Bernstein, Zuckerman's "K-finite functor" $\mathscr{L}L(\ldots) = R(K) \overset{\mathscr{L}}{\underset{\mathfrak{k}}{\otimes}} (\ldots)$ ($R(K)$ = regular functions on K) is geometrically constructible [4] and satisfies the adjoint relation

$$\text{(1.21)} \qquad \operatorname{Hom}(M, J(X)) = \operatorname{Hom}(\mathscr{L}^0 L(M), X),$$

$M \in \mathcal{O}'_F$ and $X \in \mathscr{H}\mathscr{C}_F$. On moral grounds, nice properties of $\mathscr{L}L$ should carry over to J. In particular, (1.21) will suffice to insure $\mathbb{J}$ is a Hecke algebra map; actually, we need a slight variant of (1.21), which holds when M is a complex with cohomology in $\mathcal{O}'_F$ (Sect. 3). In this sense, Schmid's suggested use of (1.21) and Bernstein's construction of $\mathscr{L}L$ are the central keys to our entire paper.

(1.22) *Singular case*. Fix a *singular* integral infinitesimal character χ. Using translation functors "to the walls", (1.4) leads to a $\mathfrak{g}$-filtration with semisimple subquotients of $J(X)$, where X is an irreducible Harish-Chandra module having infinitesimal character χ. Of course, some composition factors may go to zero, but the exactness of ψ_α (as in [20]) insures that the number of levels will not increase (moreover, the number of composition factors cannot increase [20]). In this sense, we have solved the singular integral case, by reduction (logically) to the regular case. Of course, combinatorially this approach is somewhat dissatisfying.

(1.23) *Connections with* [8]. We wish to clearly indicate the similarity and difference between (1.4) (or (1.11)) and our work in [8], since there are some rather subtle distinctions. Firstly, the philosophy in [8] was to geometrically construct J. With this in mind, we axiomatized a functor J_{geo} with properties as in (1.11), then showed such exists. However, except in special cases, we were unable to precisely relate the $\mathfrak{g}$-filtration of $J_{\text{geo}}(\bar{\pi}(\gamma))$ with a filtration of $J(\bar{\pi}(\gamma))$. One property of this J_{geo} was the fact that $\mathbb{J}_{\text{geo}}(\hat{C}_\gamma)$ would have self-dual weight filtration data. On the other hand, our current work shows Theorem (1.11), but we have *not* geometrically constructed J, nor do we know the weight filtration data of $\mathbb{J}(\hat{C}_\gamma)$ is self-dual.

In addition to these remarks, there exists a deeper and more subtle difference between these two approaches, related to the map A of (1.17). In [8], the map A amounts to $A(\delta)=1$, $\delta\in\mathscr{D}_K^0$. A clear connection between [8] and (1.11) is given by

(1.24) **Corollary** (to [8] and (1.11)). *Assume A is constant on blocks* (cf. [22, (9.2)]). *Then the weight filtration data attached to the geometrically constructed $J_{\mathrm{geo}}(\bar{\pi}(\gamma))$ coincides (up to a shift by a specifiable $q^{r/2}$, $r\in\mathbb{Z}$) with the weight filtration data attached to $J(\bar{\pi}(\gamma))$. In particular, the filtration $\{E_n\}$ of* (1.4) *is self-dual.*

For example, the hypothesis of (1.24) would be satisfied whenever $G_{\mathbb{R}}$ has connected Cartan subgroups; this is the case if $G_{\mathbb{R}}$ is a connected semisimple complex group or if $G_{\mathbb{R}}$ is a connected rank one group, other than $\mathrm{SL}_2\mathbb{R}$. The case of $\mathrm{PSP}_2\mathbb{R}$ in [8] indicates an example where (1.24) holds, yet $|\mathscr{D}_K^0|>1$. Proving A is constant on blocks roughly amounts to computing the lengths of a Gabber filtration on all principal series representations with largest growth Langlands data and showing these lengths are the same for two such principal series in the same block. (Here, principal series means "induced from the minimal parabolic subgroup".) The work in [17] solves this problem (in principle); the combinatorics are very complicated and do not obviously prove or disprove the constancy of A.

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2. Preliminaries and a construction of Bernstein

We first introduce some conventions and notation. Throughout this section we will have to view an algebraic variety over $\mathbb{C}$ as an object endowed with, respectively, three different structures: $Y^{\mathrm{alg}}=Y$ is Y the algebraic variety with its Zariski topology; Y^{an} consists of the complex points of Y viewed as a complex analytic set and Y^{et} is Y with its étale topology [18]. Often we will consider derived categories over Y^{et}, Y^{an} consisting of complexes of sheaves with constructible cohomology. If π is an algebraic map between two varieties π_*, $\pi_!$ etc, will denote the direct image, direct image with proper supports etc. as in [2] taken in the derived category, instead of using the notation $\mathscr{R}\pi_*$, $\mathscr{R}\pi$ etc.

Recall that we are fixing throughout this paper a finite dimensional representation F of $G_{\mathbb{R}}$. Let $\mathscr{L}_F$ be a homogeneous line bundle over $\mathscr{B}^{\mathrm{alg}}$ realizing F on its global sections (as in the Borel-Weil theorem), $\mathcal{O}_F$ the sheaf of sections of $\mathscr{L}_F$ and D_F the sheaf of twisted differential operators on $\mathcal{O}_F$.

Let χ_F be the infinitesimal character of F. Set $R_F = R = \mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{g}) \ker \chi_F$. The algebra R can be identified with the global sections of D_F [1]. We will be using the machinery of the Kazhdan and Lusztig conjectures [1, 2, 3, 6], i.e. localization, the Riemann-Hilbert correspondence (RHC) and passage to positive characteristic (PPC). This machinery consists mainly of a dictionary between "nice" R-modules and $\bar{\mathbb{Q}}_\ell$-perverse sheaves over the analogue $\mathscr{B}_q$ of $\mathscr{B}$ defined over a finite field $\mathbb{F}_q$ with q elements. The first dictionary, localization [1], gives a correspondence between R-modules and D_F-modules over $\mathscr{B}^{\mathrm{alg}}$. The second, RHC, is between regular singular D_F-modules and perverse sheaves over $\mathscr{B}^{\mathrm{an}}$. The third step, PPC, is subtle and is described in [2]. In our situation it will allow us to associate to perverse sheaves on $\mathscr{B}^{\mathrm{an}}$, arising from R-modules in $\mathscr{H}\mathscr{C}_F$ or $\mathcal{O}'_F$, $\bar{\mathbb{Q}}_\ell$-perverse sheaves over $\mathscr{B}_q^{\mathrm{et}}$. Keeping in mind all the above dictionaries, we will be using several versions of the sets $\mathscr{D}_K$, $\mathscr{D}_N$ introduced in Sect. 1. A new set $\mathscr{D}_H^{\mathrm{new}}$, $H = K$ or N, will always be in some canonical bijective correspondence to $\mathscr{D}_H^{\mathrm{old}}$, but for convenience both will simply be denoted $\mathscr{D}_H$. For instance, if we view R-modules as perverse sheaves over $\mathscr{B}^{\mathrm{an}}$ via RHC, $\mathscr{D}_H$ becomes

$$(2.1) \qquad \mathscr{D}_H = \left\{ \begin{array}{l} (\mathcal{O}, \mathscr{L}),\ \mathcal{O} \text{ is an } H\text{-orbit in } \mathscr{B}^{\mathrm{an}} \text{ and } \mathscr{L} \text{ is an } H\text{-equivariant} \\ \text{sheaf of } \mathbb{C}\text{-vector spaces with stalks of dimension one.} \end{array} \right\}$$

Reduction to positive characteristic

Let $\mathbb{F}_q$ be the field with q elements $q = p^r$, with p a prime number. Let $k = k_q$ be an algebraic closure of $\mathbb{F}_q$. If Y is a complex variety defined over a finite extension of $\mathbb{Q}$ by a certain set of polynomials S_Y (by equations and inequalities), it is possible to obtain new varieties Y_q defined over $\mathbb{F}_q$, by reducing modulo p the coefficients involved in the polynomials of S_Y. By this procedure, one can obtain, starting from Y, new varieties Y_q having similar properties as Y for almost all prime powers $p^r = q$. In particular, from our group G, we obtain G_q defined over $\mathbb{F}_q$ and

(a) each class of parabolic subgroups in G_q is defined over $\mathbb{F}_q$ (including our fixed group B_q)

(b) K_q and θ (the Cartan involution) are defined over $\mathbb{F}_q$;

(c) each K_q-orbit on $\mathscr{B}_q$ is defined over $\mathbb{F}_q$.

(d) Each N_q-orbit on $\mathscr{B}_q$ is defined over $\mathbb{F}_q$.

The Weyl group W_q defined with respect to an $\mathbb{F}_q$-split torus in B_q can be identified to the Weyl group W of G, so we just denote it by W. We also recall our set $S \subset W$ of simple reflections making (W, S) a Coxeter group.

Let ℓ be a prime number, prime to q. The sets $\mathscr{D}_K$, $\mathscr{D}_N$ now become H-equivariant $\bar{\mathbb{Q}}_\ell$-sheaves over H-orbits on $\mathscr{B}_q$, $H = K_q$ or N_q. We assume Fr acts trivially on $\mathscr{D}_K$ and $\mathscr{D}_N$; recall that Fr denotes the Frobenius map.

We now define q-analogues of the categories $\mathscr{HC}_F$, $\mathcal{O}'_F$. Since H is assumed to be connected, an object in any of these categories is determined by its R-action. Therefore we will ignore for the moment the H-actions on all our objects.

Let $\mathscr{C}^1_{H,q}$, $H=K_q$ or N_q be the category of constructible sheaves $\mathfrak{s}$ over $\mathscr{B}_q$, which are locally constant along H-orbits. Moreover, we ask that along each H-orbit $\mathcal{O}$, $\mathfrak{s}|_{\mathcal{O}}$ is a successive extension of $\bar{\mathbb{Q}}_\ell$-sheaves from the set $\mathscr{D}_H$. The categories $\mathscr{C}^1_{H,q,\mathrm{Weil}}$ and $\mathscr{C}^1_{H,q,\mathrm{mix}}$ are defined as follows: the category of *Weil sheaves* denoted $\mathscr{C}^1_{H,q,\mathrm{Weil}}$ consists of pairs $(\mathfrak{s},\Phi)$, where $\mathfrak{s}\in Ob\,\mathscr{C}^1_{H,q}$ and Φ is an isomorphism $\mathrm{Fr}^*(\mathfrak{s})\xrightarrow{\Phi}\mathfrak{s}$. The morphisms are morphisms in $\mathscr{C}^1_{H,q}$, compatible with the various maps Φ. We define $\mathscr{C}^1_{H,q,\mathrm{mix}}$ as a subcategory of $\mathscr{C}^1_{H,q,\mathrm{Weil}}$, consisting of *mixed sheaves* (recall (1.20)(c)).

Let $D^b(\mathscr{HC}_F)$, $D^b(\mathcal{O}'_F)$ be the derived categories of bounded complexes of R-modules. With cohomology in $\mathscr{HC}_F$ or $\mathcal{O}'_F$. Using the dictionaries provided by localization, RHC and PPC, we can identify $D^b(\mathscr{HC}_F)$ and $D^b(\mathcal{O}'_F)$ as subcategories of $D^b_H(\mathscr{B}_q,\bar{\mathbb{Q}}_\ell)$, $H=K_q$ or N_q. This is the derived category of $\bar{\mathbb{Q}}_\ell$-sheaves whose cohomology sheaves are in $\mathscr{C}^1_{H,q}$ (defined as a projective limit as in [2]). We denote these new incarnations of $D^b(\mathscr{HC}_F)$ and $D^b(\mathcal{O}'_F)$ as derived categories of $\bar{\mathbb{Q}}_\ell$-sheaves by $D^b(\mathscr{HC})_q$ and $D^b(\mathcal{O}')_q$. We have, therefore, equivalences of categories (for almost all q)

$$D^b(\mathscr{HC}_F)\approx D^b(\mathscr{HC})_q \tag{2.2}$$

$$D^b(\mathcal{O}'_F)\approx D^b(\mathcal{O}')_q.$$

Also

(2.3) **Theorem** [2]. *Under the equivalences of* (2.2), *objects in* $\mathscr{HC}_F$ *or* $\mathcal{O}'_F$ *correspond to perverse sheaves inside* $D^b(\mathscr{HC})_q$ *or* $D^b(\mathcal{O}')_q$ *respectively.*

We will denote the subcategories of $D^b(\mathscr{HC})_q$ and $D^b(\mathcal{O}')_q$ corresponding to $\mathscr{HC}_F$ and $\mathcal{O}'_F$ respectively by $\mathscr{HC}_q$ and $\mathcal{O}'_q$.

If we consider sheaves with Frobenius actions instead (Weil sheaves), we can define the categories $\mathscr{HC}_{q,\mathrm{Weil}}$, $\mathcal{O}'_{q,\mathrm{Weil}}$ and their mixed versions $\mathscr{HC}_{q,\mathrm{mix}}$, $\mathcal{O}'_{q,\mathrm{mix}}$. These are imbedded inside $D^b_{H,\mathrm{Weil}}(\mathscr{B}_q,\bar{\mathbb{Q}}_\ell)$, $H=K_q$ or N_q, the derived category of complexes $\mathfrak{s}$ of $\bar{\mathbb{Q}}_\ell$-sheaves with a Frobenius action, defined as in [2]. As before the subscript H denotes that along H-orbits the cohomology of $\mathfrak{s}$ is in $\mathscr{C}^1_{H,q,\mathrm{Weil}}$.

Hecke modules

We now recall the geometric description of the modules $\mathscr{M}_K$ and $\mathscr{M}_N$ from [17] or [15]. We need to define, for technical reasons, something slightly bigger than $\mathscr{M}_K$ and $\mathscr{M}_N$. The following definitions are taken, with slight variations from [17].

(2.4) *Definition.* Let $\mathscr{C}_{H,\mathrm{Weil}}$, $H=K_q$ or N_q be the category whose objects are sheaves in $\mathscr{C}^1_{H,q,\mathrm{Weil}}$ with some identifications. Recall that a typical object in $\mathscr{C}^1_{H,q,\mathrm{Weil}}$ is a pair $(\mathfrak{s},\Phi)$ with Φ: $\mathrm{Fr}^*(\mathfrak{s})\to\mathfrak{s}$ an isomorphism. We identify

$(\mathfrak{s}, \Phi_1) \approx (\mathfrak{s}, \Phi_2)$. If $\Phi_1^n = \Phi_2^n$ for some $n \in \{1, 2, \ldots\}$. The morphisms in $\mathscr{C}_{H,\mathrm{Weil}}$ are only required to be compatible with the corresponding Φ^n for some n. We now define $\mathscr{C}_H$ in a similar way, starting from sheaves in $\mathscr{C}^1_{H,q,\mathrm{Weil}}$ such that all the eigenvalues of Φ over a point x defined by Fr, are of the form $q^{n/2}\varepsilon$, with ε a root of unity, and $n \in \{0, 1, 2, \ldots\}$.

Let B be defined by setting

$$B_1 = \bar{\mathbb{Q}}_\ell - \{0\} \tag{2.5}$$
$$B_0 = \text{roots of unity in } B_1$$
$$B = B_1/B_0.$$

(2.6) *Definition.* Write $\mathbb{Z}[B]$ for the group ring of B and $\mathscr{M}'_H$ for the free $\mathbb{Z}[B]$-module with basis $\mathscr{D}_H$. We can identify $\mathscr{M}_H = \mathbb{Z}[q^{1/2}, q^{-1/2}] \bigotimes_{\mathbb{Z}} \mathbb{Z}[\mathscr{D}_H]$ as a submodule of $\mathscr{M}'_H$ by sending $q^{1/2}$ to its image in B_1/B_0. We have

$$K(\mathscr{C}_{H,\mathrm{Weil}}) \approx \mathscr{M}'_H \tag{2.7}$$
$$K(\mathscr{C}_H) \approx \mathscr{M}_H.$$

As in [17], $\mathscr{M}_N$ is the *Hecke algebra* and $\mathscr{M}'_N$ is obtained by extension of scalars, i.e.

$$\mathscr{M}'_N = \mathbb{Z}[B] \bigotimes_{\mathbb{Z}} \mathbb{Z}[\mathscr{D}_N]$$

with its algebra structure defined by the same relations satisfied in $\mathscr{M}_N$. Recall the operators T_w, $w \in W \approx \mathscr{D}_N$ of the Hecke algebra; each T_w can be imbedded in $\mathscr{M}'_N \approx K(\mathscr{C}_{N,\mathrm{Weil}})$ by identifying T_w with the sheaf $i^w_! 1_{\mathscr{O}_w}$. Here $\mathscr{O}_w$ is the N_q-orbit in $\mathscr{B}_q$ attached to w and i^w denotes the inclusion, with $1_{\mathscr{O}_w}$ being the trivial $\bar{\mathbb{Q}}_\ell$-sheaf on $\mathscr{O}_w$ with trivial Fr-action. The action of $T_s + 1$, $s \in S$ on $\mathscr{M}'_H$ or $\mathscr{M}_H$ can be described in geometric terms as in [17] by

$$(T_s + 1)[\gamma] = \sum_i (-1)^i [\pi_s^* \mathscr{R}^i \pi_{s!} \gamma] \tag{2.8}$$

with γ a representative in $\mathscr{C}_{H,\mathrm{Weil}}$. Here π_s denotes the projection $\mathscr{B}_q \to (\mathscr{P}_s)_q$ to the variety of parabolics of type s.

We now recall the maps of $\mathbb{Z}[B]$ or $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-modules

$$K(\mathscr{H}\mathscr{C}_{q,\mathrm{Weil}}) \to \mathscr{M}'_K \tag{2.9}$$
$$K(\mathscr{O}'_{q,\mathrm{Weil}}) \to \mathscr{M}'_N.$$

These maps are defined by sending the class $[\mathfrak{s}]$ of $\mathfrak{s}$ to the element

$$[\mathfrak{s}] \to \sum (-1)^i [\mathscr{H}^i \mathfrak{s}]. \tag{2.10}$$

The signs in (2.10) are responsible for the signs in our map of Hecke algebras $\mathbb{J}$ in example (1.15).

A construction of Bernstein

We now describe a geometric construction of the functor $\mathscr{L}L$ due to Bernstein [4]. We will distinguish between two functors $L=L_{\mathrm{I}}$ and L_{II} whose only difference is essentially the categories on which they are defined.

Let $B\subset G$ be a Borel subgroup containing $A_{\mathbb{R}}N_{\mathbb{R}}$ of the Iwasawa decomposition of $G_{\mathbb{R}}$ (an Iwasawa Borel subgroup). Denote $T=H_m\cap K$ and T^0 the component of the identity. Consider the category

(2.11) $\mathcal{O}'_{F,T^0}=\mathcal{O}'_F$-modules with a T^0-action compatible with the action of R.

(2.12) *Definition.* We consider functors

$$L_{\mathrm{I}}\colon\ \mathcal{O}'_F\to\mathscr{H}\mathscr{C}_F$$

$$L_{\mathrm{II}}\colon\ \mathcal{O}'_{F,T^0}\to\mathscr{H}\mathscr{C}_F$$

both given by the formula

$$Z\to R(K)\bigotimes_{\mathfrak{t}} Z$$

as vector spaces, with $R(K)$ the regular functions on K. As in [11], $L_j(Z)$, $j=\mathrm{I}$, II acquires an R-action (see also our appendix).

Given Y a non-singular algebraic variety and an algebraic group T acting on Y, we denote by (D_Y,T)-mod (respectively D_Y-mod), the categories of D_Y-modules with a T-action (resp. D_Y-modules). The category (D_Y,T)-mod is as in [1], and corresponds to the notion used in [17] or [8] for perverse sheaves.

We have

(2.13) **Lemma** (Bernstein). *Let Y be a non-singular algebraic variety with an algebraic free action of an algebraic group T. Then there is an equivalence of categories*

$$((D_Y,T)\text{-mod})\xrightarrow{\ \xi\ }(D_{Y/T}\text{-mod}).$$

(2.14) *Remark.* The inverse ξ^{-1}, is given by π^*, the operation on D-modules that corresponds to $\pi^*[\dim T]$ on constructible sheaves by the RHC.

Recall $T_q=H_{m,q}\cap K_q$. Let T_q^0 be the component of the identitiy. Let $\mathscr{A}_{T^0}$ be the category of $\bar{\mathbb{Q}}_\ell$-perverse sheaves on $K_q\times\mathscr{B}_q$ which are constant along $K_q\times N_q$-orbits and which carry a T_q^0-action. Here T_q^0 acts by $t(k,x)=(kt^{-1},t\cdot x)$. Let $\mathscr{A}$ be the category of $\bar{\mathbb{Q}}_\ell$-perverse sheaves on $K_q\times_{T_q^0}\mathscr{B}_q$ which are constant along $K_q\times N_q$-orbits. We denote by $\mathscr{A}_{T^0,\mathrm{Weil}}$, $\mathscr{A}_{\mathrm{Weil}}$, $\mathscr{A}_{T^0,\mathrm{mix}}$, $\mathscr{A}_{\mathrm{mix}}$ the version of these categories with actions of Frobenius, following the above conventions. We obtain by passage to positive characteristic the following version of (2.13)

(2.15) **Proposition.** *There are equivalences of categories*

$$\mathscr{A}_{T^0}\xrightarrow{\ \xi\ }\mathscr{A}$$

$$\mathscr{A}_{T^0,\mathrm{Weil}}\xrightarrow{\ \xi\ }\mathscr{A}_{\mathrm{Weil}}$$

$$\mathscr{A}_{T^0,\mathrm{mix}}\xrightarrow{\ \xi\ }\mathscr{A}_{\mathrm{mix}}.$$

Also, ξ^{-1} is given by π^*, where π is the projection $\pi\colon K_q\times\mathscr{B}_q\to K_q\times_{T_q^0}\mathscr{B}_q$.

Consider the following maps $p\colon K_q \times \mathscr{B}_q \to \mathscr{B}_q$, $p(k,x)=x$, act: $K_q \times \mathscr{B}_q \to \mathscr{B}_q$, $\operatorname{act}(k,x)=k\cdot x$, the projection $\pi\colon K_q \times \mathscr{B}_q \to K_q \underset{T_q^0}{\times} \mathscr{B}_q$ as in (2.15), and finally act′: $K_q \underset{T_q^0}{\times} \mathscr{B}_q \to \mathscr{B}_q$ defined in the same way as act. Recall the notation $\mathscr{H}\mathscr{C}_q$, $\mathcal{O}'_q$ and denote by $\mathcal{O}'_{q,T^0}$ the category of perverse sheaves in $\mathcal{O}'_q$ with a T^0-action. Since T^0 is connected $\mathcal{O}'_{q,T^0}$ can be embedded as a subcategory of $\mathcal{O}'_q$. The analogous statement for R-modules is that $\mathcal{O}'_F$ contains as a subcategory, the category $\mathcal{O}'_{F,T^0}$ of R-modules in $\mathcal{O}'_F$ with a compatible T^0-action. In the context of R-modules it becomes clear that the compatibility condition forces the T^0-action to be unique.

We define functors

$$\mathscr{L}L_q = \mathscr{L}L_{\mathrm{I},q}\colon \mathcal{O}'_q \to D^b(\mathscr{H}\mathscr{C})_q$$

$$\mathscr{L}L_{\mathrm{II},q}\colon \mathcal{O}'_{q,T^0} \to D^b(\mathscr{H}\mathscr{C})_q$$

$$\mathscr{L}L_{\mathrm{I},q}(M) \overset{\mathrm{def}}{=} (\operatorname{act}_* p^* M)[\dim K] \otimes L^{-\dim K/2}$$

$$\mathscr{L}L_{\mathrm{II},q}(M) \overset{\mathrm{def}}{=} (\operatorname{act}'_* \cdot \xi \cdot p^* M)[\dim K/T] \otimes L^{-\dim(K/T)/2}.$$

Here we remark that if M is in $\mathcal{O}'_{q,T_q^0}$, p^*M is in $\mathscr{A}_{T^0}$, thus we can apply ξ. The symbol "L" denotes a *Tate sheaf.*

Denote by S and S_{T^0} the equivalence of categories obtained by localization, the RHC and PPC, between the appropriate category of R-modules (or complexes of R-modules) and of $\bar{\mathbb{Q}}_\ell$-constructible sheaves. We have

(2.16) **Theorem** (Bernstein [4]). *There are commutative diagrams*

(a)
$$\begin{array}{ccc} \mathcal{O}'_F & \xrightarrow{\mathscr{L}L_{\mathrm{I}}} & D^b(\mathscr{H}\mathscr{C}_F) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ \mathcal{O}'_q & \xrightarrow{\mathscr{L}L_{\mathrm{I},q}} & D^b(\mathscr{H}\mathscr{C})_q \end{array}$$

and

(b)
$$\begin{array}{ccc} \mathcal{O}'_{F,T^0} & \xrightarrow{\mathscr{L}L_{\mathrm{II}}} & D^b(\mathscr{H}\mathscr{C}_F) \\ {\scriptstyle S_{T^0}}\downarrow & & \downarrow{\scriptstyle S} \\ \mathcal{O}'_{q,T^0} & \xrightarrow{\mathscr{L}L_{\mathrm{II},q}} & D^b(\mathscr{H}\mathscr{C})_q. \end{array}$$

We refer the reader to our appendix for a sketch of proof. The reader should note that only (2.16)(a) is used in this paper (proved in the appendix), plus the remark (2.18) concerning $\mathscr{L}L_{\mathrm{II}}$.

(2.17) *An example.* Consider the case when $G_{\mathbb{R}} = \mathrm{SL}_2\mathbb{R}$. Then $\mathscr{B}_q = \mathbb{P}^1$ and there are three K_q-orbits that can be identified with $\{0\}$, $\{\infty\}$ and $\mathbb{P}^1 - \{0,\infty\}$. Note that the map $f\colon K_q \to \mathbb{P}^1 - \{0,\infty\}$ is a double cover, and if 1_{K_q} is the trivial sheaf on K_q, $f_*(1_{K_q}) = a \oplus b$, where a is a trivial sheaf and b is a non-trivial sheaf. Therefore, if M is a skyscraper sheaf supported on $x_0 \in \mathbb{P}^1 - \{0,\infty\}$, $p^*M = 1_{K_q} \boxtimes M$ and $\operatorname{act}_* p^* M = i_* f_*(1_{K_q})$, where i is the inclusion $\mathbb{P}^1 - \{0,\infty\} \hookrightarrow \mathbb{P}^1$. Therefore we obtain $\operatorname{act}_* p^* M = i_*(a \oplus b)$. This agrees with the fact that if we view M as an irreducible Verma module M_e, then $\mathscr{L}L_{\mathrm{I}}(M_e)$ is the sum of the two principal series in $\mathscr{H}\mathscr{C}_F$.

(2.18) *Remark.* Let For: $\mathcal{O}'_{q,T^0} \to \mathcal{O}'_q$ be the forgetful functor ignoring the T^0-action. Then

$$\mathscr{L}^0 L_{\mathrm{I},q} \cdot \mathrm{For} \approx \mathscr{L}^0 L_{\mathrm{II},q}.$$

However, in general

$$\mathscr{L}^q L_{\mathrm{I},q} \cdot \mathrm{For} \neq \mathscr{L}^q L_{\mathrm{II},q}.$$

Here we caution the reader that $\mathscr{L}^0$ refers to the zero-cohomology in the perverse sense, corresponding to the zero-cohomology in the category of D-modules [2].

3. An adjoint formula

In this section, we realize J as the right adjoint to Zuckerman's "K-finite functor" L. For technical reasons, we need this adjointness in the derived category; we use [5] as a general reference on the derived category. For emphasis, recall that an object $A \in D^b(R)$ is a complex of R-modules and a morphism $A \Rightarrow B$ means the following: there exists $C \in D^b(R)$ and a diagram of chain maps

$$A \underset{qi}{\longleftarrow} C \longrightarrow B,$$

where "qi" denotes a *quasi-isomorphism* (i.e. a chain map inducing an isomorphism on cohomology). Recall the full subcategories $D^b(\mathscr{H}\mathscr{C}_F)$ and $D^b(\mathcal{O}'_F)$, as discussed in §2. The main result of this section is

(3.1) **Proposition.** *If $X \in \mathscr{H}\mathscr{C}_F$ and $M \in D^b(\mathcal{O}'_F)$, then we have an isomorphism of complex vector spaces*

$$\mathrm{Hom}_{D^b(\mathcal{O}'_F)}(M, \mathscr{R}J(X)) \cong \mathrm{Hom}_{D^b(\mathscr{H}\mathscr{C}_F)}(\mathscr{L}L(M), X).$$

By [5, (9.13)] and the exactness of J [9], (3.1) easily implies the adjointness formula (1.21) of the introduction. To prove (3.1), we may as well assume $F = \mathbb{C}$. This context, $A \to A^* =$ algebraic dual of A, is an exact functor on R-modules. Let $\gamma_{\mathfrak{n}}$ (resp. Γ) denote the functor of taking b-finite (resp. K-finite) vectors. Appealing to the fact L and Γ are adjoint functors (see (3.11)), under the hypothesis of (3.1) we are reduced to establishing an isomorphism

$$\mathrm{Hom}_{D^b(\mathcal{O}'_F)}(M, \mathscr{R}\gamma_{\mathfrak{n}} X^*) \cong \mathrm{Hom}_{D^b(\mathscr{H}\mathscr{C}_F)}(X, \mathscr{R}\Gamma M^*). \tag{3.2}$$

To this end, we define two maps as follows:

$$\mathrm{Hom}_{D^b(\mathcal{O}'_F)}(M, \mathscr{R}\gamma_{\mathfrak{n}} X^*) \xrightarrow{\Phi} \mathrm{Hom}_{D^b(\mathscr{H}\mathscr{C}_F)}(X, \mathscr{R}\Gamma M^*); \tag{3.3}$$

$$\mathrm{Hom}_{D^b(\mathscr{H}\mathscr{C}_F)}(X, \mathscr{R}\Gamma M^*) \xrightarrow{\Psi} \mathrm{Hom}_{D^b(\mathcal{O}'_F)}(M, \mathscr{R}\gamma_{\mathfrak{n}} X^*). \tag{3.4}$$

Definition of Φ. We are given a morphism $M \overset{T}{\Rightarrow} \mathscr{R}\gamma_{\mathfrak{n}} X^*$, represented by a diagram

$$M \underset{qi}{\longleftarrow} C \longrightarrow \mathscr{R}\gamma_{\mathfrak{n}} X^*.$$

Replacing X with a (quasi-isomorphic) projective R-resolution $p_* X$ and applying $*$, we arrive at

$$M^* \underset{qi}{\longrightarrow} C^* \longleftarrow (\gamma_{\mathfrak{n}}(p_* X)^*)^* \longleftarrow p_* X \underset{qi}{\longrightarrow} X. \tag{3.5}$$

In $D^b(R)$, (3.5) yields $X \Rightarrow M^*$, leading to

$$\mathscr{R}\Gamma X \Rightarrow \mathscr{R}\Gamma M^*. \tag{3.6}$$

The map in (3.6) is represented by taking injective resolutions and applying Γ. But, since we have an actual map $X \to I_* X =$ injective resolution of X, we obtain the following diagram defining Φ

$$\begin{tikzcd}
\Gamma I_* p_* X \arrow[r] \arrow[d] & \Gamma I_* C^* & \Gamma I_* M^* \arrow[l] \arrow[d, equal, "\mathrm{def}"] \\
\Gamma I_* X \overset{\mathrm{def}}{=} \mathscr{R}\Gamma X \arrow[rr, Rightarrow] & & \mathscr{R}\Gamma M^* \\
\Gamma X \underset{(1)}{=} X \arrow[u] \arrow[urr, Rightarrow, "\Phi(T)"'] & &
\end{tikzcd} \tag{3.7}$$

Here, (1) is using the fact X is a module (not a complex!) in $\mathscr{H}\mathscr{C}_F$.

Definition of Ψ. Suppose we are given $X \overset{T}{\Rightarrow} \Gamma M^*$ represented by the diagram

$$X \underset{qi}{\longleftarrow} C \longrightarrow \mathscr{R}\Gamma M^*.$$

Replacing M by a projective resolution $p_* M$ and applying $*$ we obtain

$$X^* \underset{qi}{\longrightarrow} C^* \longleftarrow (\Gamma(p_* M)^*)^* \longleftarrow p_* M \longrightarrow M. \tag{3.8}$$

We have now produced $M \Rightarrow X^*$, leading to

$$\mathscr{R}\gamma_{\mathfrak{n}} M \Rightarrow \mathscr{R}\gamma_{\mathfrak{n}} X^*. \tag{3.9}$$

There exists a convergent spectral sequence $\gamma_{\mathfrak{n}}^p H^q(M) \to H^{p+q}(\mathscr{R}\gamma_{\mathfrak{n}} M)$. By hypothesis, $M \in D^b(\mathcal{O}'_F)$ implies that $H^q(M)$ are all $\mathfrak{n}$-locally finite. This shows $H^q(M) \cong H^q(\mathscr{R}\gamma_{\mathfrak{n}} M)$ and since we have an actual map $M \to \mathscr{R}\gamma_{\mathfrak{n}} M$, the map becomes a quasi-isomorphism. We now arrive at a diagram defining Ψ, via (3.9)

$$\begin{tikzcd}
\mathscr{R}\gamma_{\mathfrak{n}} M \arrow[r, Rightarrow] & \mathscr{R}\gamma_{\mathfrak{n}} X^* \\
M. \arrow[u, "qi"] \arrow[ur, Rightarrow, "\Psi(T)"'] &
\end{tikzcd}$$

Proof of (3.2). We must show $\Psi \circ \Phi =$ identify and $\Phi \circ \Psi =$ identity. In both cases, the philosophy is the same. For this reason, we will argue $\Psi \circ \Phi =$ identity, leaving the second composition to the reader. Given $M \overset{T}{\Rightarrow} \mathscr{R}\gamma_{\mathfrak{n}} X^*$, we make the observation that T is represented by a diagram

(3.10)
$$\begin{tikzcd}
M & C \arrow[l, "qi"'] \arrow[r, "T"] & \gamma_{\mathfrak{n}} X^* \\
& p_* C, \arrow[ul, "qi"] \arrow[u, "qi"'] \arrow[ur, "T"'] &
\end{tikzcd}$$

where $p_* C$ is a projective resolution of C. This observation is justified by the exactness of $\gamma_{\mathfrak{n}}$ [9]. (Here, the fact that $\mathfrak{n}$ is attached to the "Iwasawa Borel subalgebra" is used. For other nilradicals $\mathfrak{n}'$ attached to other Borel subalgebras, $\gamma_{\mathfrak{n}'}$, may have higher derived functors; this phenomena is investigated in [7].) From (3.10), we are led to

$$\begin{tikzcd}
(p_* C)^* & (\gamma_{\mathfrak{n}} X^*)^* \arrow[l] & X \arrow[l] \arrow[dl, "T'"] \\
& \Gamma(p_* C)^*. \arrow[ul] &
\end{tikzcd}$$

Taking duals again,

$$\begin{tikzcd}
X^* & (\Gamma(p_* C)^*)^* \arrow[l] & p_* C \arrow[l] \arrow[ll, bend left, "T''"]
\end{tikzcd}$$

For our explicit map T, we can check that $T = T''$ and T'' factors thru $\gamma_{\mathfrak{n}} X^*$.

We have just described how our actual map T of (3.10), after two applications of duality, gives back T. If we can show $T'' = \Psi(T') = \Psi \circ \Phi(T)$, using the definitions of Φ and Ψ, our proposition is proved.

To show $\Phi(T) = T'$, we begin with an injective resolution $X \to I_* X$ and form the diagram

$$\begin{tikzcd}
(p_* C)^* & X \arrow[l, "T'"'] \arrow[d, "\underset{\sim}{b}"] \arrow[dl, phantom, "\circlearrowleft"] \\
& I_* X. \arrow[ul, "\underset{\sim}{a}"]
\end{tikzcd}$$

By the injectivity of $(p_* C)^*$, we have the existence of $\underset{\sim}{a}$ (which is unique up to homotopy). Apply Γ to obtain

$$\begin{tikzcd}
\mathscr{R}\Gamma M^* = \Gamma(p_* C)^* & \Gamma X = X \arrow[l, "T'"'] \arrow[d, "\underset{\sim}{b}"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \Gamma I_* X. \arrow[ul, "\underset{\sim}{a}"]
\end{tikzcd}$$

According to our definitions, $\Phi(T) = \underset{\sim}{a} \cdot \underset{\sim}{b} = T'$.

The next step is slightly more involved. Recalling the construction of $\Psi(T')$, we are led to a diagram

$$\begin{tikzcd}
 & \gamma_n X^* \arrow[ddl, Rightarrow, "b"'] \arrow[d, "qi"] & p_* C \approx M \arrow[l, "T''"'] \arrow[d, "qi"] \arrow[ddll, Rightarrow, bend left=60, "\Psi(T')"] \\
 & I_*(\gamma_n X^*) & I_*(p_* C) \arrow[l, "qi"] \\
\mathscr{R}\gamma_n X^* = \gamma_n I_* X^* & \gamma_n I_*(\gamma_n X^*) \arrow[l, "qi"] \arrow[u, "qi"] & \gamma_n I_*(p_* C) \arrow[l, "qi"] \arrow[u, "qi"]
\end{tikzcd}$$

Now, the difference between T'' and $\Psi(T')$ is accounted for by b; this just amounts to reversing our initial observation that morphisms of M into $\gamma_n X^*$ and morphisms of M into $\mathscr{R}\gamma_n X^*$ coincide. Q.E.D.

We now have the following adjointness relation between $\mathscr{L}L$ and $\mathscr{R}\Gamma$ in the derived category (proved similarly to (3.2))

(3.11) **Proposition.** *Let $X \in \mathscr{H}\mathscr{C}_F$ and $M \in D^b(\mathcal{O}'_{F^*})$. Then there is a natural isomorphism of $\mathbb{C}$-vector spaces*

$$\mathrm{Hom}_{D^b(\mathscr{H}\mathscr{C}_F)}(X, \mathscr{R}\Gamma(M^*)) \approx \mathrm{Hom}_{D^b(\mathscr{H}\mathscr{C}_{F^*})}(\mathscr{L}L(M), \tilde{X})$$

where $\tilde{X}$ is the K-finite dual of X.

If we invoke base change, as discussed in Sect. 2, then (3.1) and (3.11) give

(3.12) **Corollary.** *Let $X \in \mathscr{H}\mathscr{C}_q$ and $M \in D^b(\mathcal{O}')_q$. Then for almost all prime powers $q = p^r$, there exists an isomorphism of $\bar{\mathbb{Q}}_\ell$ vector spaces*

$$\mathrm{Hom}_{D^b(\mathcal{O}')_q}(M, \mathscr{R}J_q(X)) \cong \mathrm{Hom}_{D^b(\mathscr{H}\mathscr{C})_q}(\mathscr{L}L_q(M), X).$$

4. An adjoint formula respecting Frobenius actions

In this section, we argue that the base changed Jacquet modules $J_q(X)$ carry a natural action of Frobenius, whenever X has such. Moreover, the q-analogue of (1.21) holds in the category of Weil sheaves. These results are established using Bernstein's construction of $\mathscr{L}L_q$ and our adjoint formula (1.21). We emphasize that the derived category adjointness of Sect. 3 is not needed here; it becomes a technical necessity in Sect. 5.

We begin by stating the base changed version of (1.21).

(4.1) **Corollary.** *If $X \in \mathscr{H}\mathscr{C}_{q,\mathrm{Weil}}$ and $M \in \mathcal{O}'_{q,\mathrm{Weil}}$, then there exists an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}'_q}(M, J_q(X)) \cong \mathrm{Hom}_{\mathscr{H}\mathscr{C}_q}(\mathscr{L}^0 L_q(M), X).$$

It is important to realize that the isomorphism of (4.1) is *natural*. This means the following: Given a commutative diagram as in (4.2)(a), then (4.1) implies the adjoint diagram (4.2)(b) is commutative, and vice versa.

(4.2)
$$
\begin{tikzcd}
\mathscr{L}^0 L_q M_1 \arrow[r, "\phi"] & X_1 \\
\mathscr{L}^0 L_q M_2 \arrow[r, "\psi"'] \arrow[u, "L_q(f)"] \arrow[ur, phantom, "\circlearrowleft"] & X_2 \arrow[u, "a"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M_1 \arrow[r, "\Phi(\phi)"] & J_q(X_1) \\
M_2 \arrow[r, "\Phi(\psi)"'] \arrow[u, "f"] \arrow[ur, phantom, "\circlearrowleft"] & J_q(X_2). \arrow[u, "J_q(a)"']
\end{tikzcd}
$$
(a) (b)

(4.3) **Lemma.** *There is a natural isomorphism* $J_q \mathrm{Fr}^* \approx \mathrm{Fr}^* J_q$.

Proof. Recall that $\mathscr{L}L_q = (\mathrm{act}_* p^*)[\dim K]$, from Sect. 2. This tells us that $\mathscr{L}L_q$ will commute with Fr, i.e.

$$\mathscr{L}L_q \mathrm{Fr}^* \approx \mathrm{Fr}^* \mathscr{L}L_q \quad \text{and} \quad \mathscr{L}L_q \mathrm{Fr}_* \approx \mathrm{Fr}_* \mathscr{L}L_q.$$

From (4.1) we find (dropping the $\mathcal{O}'_q$, $\mathscr{HC}_q$ subscripts on Hom)

$$
\begin{aligned}
\mathrm{Hom}(M, J_q \mathrm{Fr}^*(X)) &= \mathrm{Hom}(\mathscr{L}^0 L_q(M), \mathrm{Fr}^* X) = \mathrm{Hom}(\mathrm{Fr}_* \mathscr{L} L_q(M), X) \\
&= \mathrm{Hom}(\mathscr{L} L_q(\mathrm{Fr}_* M), X) = \mathrm{Hom}(\mathscr{L}^0 L_q(\mathrm{Fr}_* M), X) \\
&= \mathrm{Hom}(\mathrm{Fr}_* M, J_q(X)) = \mathrm{Hom}(M, \mathrm{Fr}^* J_q(X)),
\end{aligned}
$$

for all $X \in \mathscr{HC}_q$, $M \in \mathcal{O}'_q$. This implies the existence of non-zero maps

$$\mathrm{Fr}^* J_q(X) \rightleftarrows J_q \mathrm{Fr}^*(X),$$

which are necessarily isomorphisms. Q.E.D.

(4.4) **Corollary.** *If* $X \in \mathscr{HC}_{q,\mathrm{mix}}$, *then* $J_q(X) \in \mathcal{O}'_{q,\mathrm{Weil}}$.

This result is the start of our odyssey, but far from the end of the journey. We still must show $J_q(X)$ is mixed and has a computable filtration by weights of Frobenius. Crucial to this is the following improvement of (4.1).

(4.5) **Proposition.** *If* $X \in \mathscr{HC}_{q,\mathrm{mix}}$ *and* $M \in \mathcal{O}'_{q,\mathrm{mix}}$, *then there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{O}'_{q,\mathrm{Weil}}}(M, J_q(X)) \cong \mathrm{Hom}_{\mathscr{HC}_{q,\mathrm{Weil}}}(\mathscr{L}^0 L_q(M), X).$$

Proof. If we disregard the action of Frobenius, this is just (4.1). Our task is to show the following: If diagram (4.6)(a) is commutative, then the adjoint diagram (4.6)(b) commutes, and vice versa

(4.6)
$$
\begin{tikzcd}
\mathscr{L}^0 L_q(M) \arrow[r, "f"] & X \\
\mathrm{Fr}^* \mathscr{L}^0 L_q(M) \arrow[r, "\mathrm{Fr}^* f"'] \arrow[u, "\approx"] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Fr}^* X \arrow[u, "\approx"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\Phi(f)"] & J_q(X) \\
\mathrm{Fr}^* M \arrow[r, "\mathrm{Fr}^* \Phi(f)"'] \arrow[u, "\approx"] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Fr}^* J_q(X). \arrow[u, "\approx"']
\end{tikzcd}
$$
(a) (b)

Here, $\approx$ denotes the natural isomorphisms induced by Fr^*. Our proof is broken into two steps. Only the second step requires a bit of work.

Step 1. There exists a commutative diagram

(4.7)
$$\begin{tikzcd}
M \arrow[r, hook, "\Phi(\mathrm{id}_{LM})"'] \arrow[rr, bend left, "\Phi(f)"] & J_q\mathscr{L}^0L_qM \arrow[r, "J_q(f)"'] \arrow[dr, phantom, "\circlearrowright"] & J_qX \\
\mathrm{Fr}^*M \arrow[u, "a"] \arrow[r, "\Phi(m)"'] \arrow[ur, phantom, "\circlearrowright"] & J_q\mathrm{Fr}^*\mathscr{L}^0L_qM \arrow[u, "J(c)"'] \arrow[r, "J_q(\mathrm{Fr}^*f)"'] \arrow[dr, phantom, "\circlearrowright"] & J_q\mathrm{Fr}^*X \arrow[u, "\approx"'] \\
& \mathrm{Fr}^*J_q\mathscr{L}^0L_qM \arrow[u, "x"'] \arrow[r, "\mathrm{Fr}^*J_q(f)"'] & \mathrm{Fr}^*J_qX. \arrow[u, "\approx"']
\end{tikzcd}$$

Step 2. There exists a commutative triangle

(4.8)
$$\begin{tikzcd}
\mathrm{Fr}^*M \arrow[r, "\Phi(m)"'] \arrow[dr, "\mathrm{Fr}^*\Phi(\mathrm{id}_{LM})"'] & J_q\mathrm{Fr}^*\mathscr{L}^0L_qM \arrow[d, phantom, "\circlearrowright"] \\
& \mathrm{Fr}^*J_q\mathscr{L}^0L_qM. \arrow[u, "x"']
\end{tikzcd}$$

Assuming these two steps are established, pasting (4.8) into the lower left-hand corner of (4.7) will produce the desired diagram (4.6)(b).

Proof of Step 1. Begin with the commuting diagram

(4.9)
$$\begin{tikzcd}
\mathscr{L}^0L_qM \arrow[r, "\mathrm{id}_{LM}"] & \mathscr{L}^0L_qM \arrow[r, "f"] & X \\
\mathscr{L}^0L_q\mathrm{Fr}^*M \arrow[u, "L(a)", "\approx"'] \arrow[r, "m", "\approx"'] & \mathrm{Fr}^*\mathscr{L}^0L_qM \arrow[u, "\approx", "c"'] \arrow[r, "\mathrm{Fr}^*f"] & F^*M \arrow[u, "\approx"']
\end{tikzcd}$$

here, the right most diagram is (4.6)(a), the left diagram commutes by definition of the Frobenius action on $\mathscr{L}^0L_qM$. Functoriality of J_q (applied to (4.6)(a)) and naturality of the adjoint (applied to the left of (4.9)) produces the upper two rows of (4.7). The bottom block in (4.7) comes from the naturality of $J_q\mathrm{Fr}^*\approx\mathrm{Fr}^*J_q$.

Proof of Step 2. From the bottom line of (4.9) we obtain a diagram

(4.10)
$$\begin{tikzcd}
\mathrm{Fr}^*M \arrow[r, "\Phi(\mathrm{id}_{LF^*M})"] \arrow[rr, bend left, "\Phi(m)"] \arrow[dr, "\mathrm{Fr}^*\Phi(\mathrm{id}_{LM})"'] & J_q\mathscr{L}^0L_q\mathrm{Fr}^*M \arrow[r, "J(m)"] & J_q\mathrm{Fr}^*\mathscr{L}^0L_qM \\
& \mathrm{Fr}^*J_q\mathscr{L}^0LM \arrow[u, "T"'] \arrow[ur, "x"'] \arrow[u, phantom, shift left=6ex, "\circlearrowleft"] &
\end{tikzcd}$$

where $T=J(m)^{-1}x$. We must show the left-hand triangle commutes. The key idea is to realize how the isomorphisms $\mathscr{L}^0L_q\mathrm{Fr}^*\approx\mathrm{Fr}^*\mathscr{L}^0L_q$ and $J_q\mathrm{Fr}^*\approx\mathrm{Fr}^*J_q$ are naturally induced by an identity map; recall the proof of (4.3).

For $A, M \in \mathcal{O}'_{q,\mathrm{mix}}$, we have natural isomorphisms

$$
(4.11)\qquad
\begin{array}{l}
\mathrm{Hom}(\mathscr{L}^0 L_q A, \mathscr{L}^0 L_q M) = \mathrm{Hom}(A, J_q \mathscr{L}^0 L_q M) = \mathrm{Hom}(\mathrm{Fr}^* A, \mathrm{Fr}^* J_q \mathscr{L}^0 L_q M)\\
\quad \|\\
\mathrm{Hom}(\mathrm{Fr}^* \mathscr{L}^0 L_q A, \mathrm{Fr}^* \mathscr{L}^0 L_q M)\\
\quad \|\\
\mathrm{Hom}(\mathscr{L}^0 L_q \mathrm{Fr}^* A, \mathscr{L}^0 L_q \mathrm{Fr}^* M)\\
\quad \|\\
\mathrm{Hom}(\mathrm{Fr}^* A, J_q \mathscr{L}^0 L_q \mathrm{Fr}^* M).
\end{array}
$$

Putting $A = J_q \mathscr{L}^0 L_q M$, the identity will induce a natural isomorphism

$$\mathrm{Fr}^* J_q \mathscr{L}^0 L_q M \xrightarrow{\hat{T}} J_q \mathscr{L}^0 L_q \mathrm{Fr}^* M,$$

analogous to the argument in the proof of (4.3). Put $A = M$ in the above equality of Hom's and obtain a commutative diagram

$$
(4.12)\qquad
\begin{tikzcd}
\mathrm{Hom}(\mathscr{L}^0 L_q M, \mathscr{L}^0 L_q M) \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] \arrow[r, equal] & \mathrm{Hom}(\mathrm{Fr}^* M, \mathrm{Fr}^* J_q \mathscr{L}^0 L_q M) \arrow[dl, "\hat{T}"] \\
\mathrm{Hom}(\mathrm{Fr}^* M, J_q \mathscr{L}^0 L_q \mathrm{Fr}^* M). &
\end{tikzcd}
$$

Using (4.11) and (4.12) we may trace thru what happens to id_{LM}: $\mathscr{L}^0 L_q M \to \mathscr{L}^0 L_q M$ and obtain (by definition, since $\hat{T}$ is defined by (4.12))

$$
(4.13)\qquad
\begin{tikzcd}
\mathrm{Fr}^* M \arrow[r, "\mathrm{Fr}^* \Phi(\mathrm{id}_{LM})"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Fr}^* J_q \mathscr{L}^0 L_q M \arrow[d, "\hat{T}"] \\
\mathrm{Fr}^* M \arrow[r, "\Phi(\mathrm{id}_{LF^*M})"'] & J_q \mathscr{L}^0 L_q \mathrm{Fr}^* M.
\end{tikzcd}
$$

Thus, in view of these remarks, the proof of step 2 is reduced to showing $T = \hat{T}$; T as in (4.10).

The following sequence of Hom diagrams will commute. This sequence of diagrams corresponds to a chain of commutative squares involving the objects inside the Hom symbols, which "explicitly" construct $\hat{T}$ and x

(4.14)

$$
\begin{array}{lcl}
\mathrm{Hom}(J_q \mathscr{L}^0 L_q M, J_q \mathscr{L}^0 L_q M) & = & \mathrm{Hom}(J_q \mathscr{L}^0 L_q M, J_q \mathscr{L}^0 L_q M)\\
\quad \wr\| & & \|\wr\\
\mathrm{Hom}(\mathscr{L}^0 L_q J_q \mathscr{L}^0 L_q M, \mathscr{L}^0 L_q M) & = & \mathrm{Hom}(\mathscr{L}^0 L_q J_q \mathscr{L}^0 L_q M, \mathscr{L}^0 L_q M)\\
\quad \wr\| & & \|\wr\\
\mathrm{Hom}(\mathrm{Fr}^* \mathscr{L}^0 L_q J_q \mathscr{L}^0 L_q M, \mathrm{Fr}^* \mathscr{L}^0 L_q M) & = & \mathrm{Hom}(\mathrm{Fr}^* \mathscr{L}^0 L_q J_q \mathscr{L}^0 L_q M, \mathrm{Fr}^* \mathscr{L}^0 L_q M)\\
\quad \wr\| & & \|\wr\\
\mathrm{Hom}(\mathscr{L}^0 L_q \mathrm{Fr}^* J_q \mathscr{L}^0 L_q M, \mathscr{L}^0 L_q \mathrm{Fr}^* M) & \cong & \mathrm{Hom}(\mathscr{L}^0 L_q \mathrm{Fr}^* J_q \mathscr{L}^0 L_q M, \mathrm{Fr}^* \mathscr{L}^0 L_q M)\\
\quad \wr\| & & \|\wr\\
\mathrm{Hom}(\mathrm{Fr}^* J_q \mathscr{L}^0 L_q M, J_q \mathscr{L}^0 L_q \mathrm{Fr}^* M) & \overset{\textcircled{1}}{\cong} & \mathrm{Hom}(\mathrm{Fr}^* J_q \mathscr{L}^0 L_q M, J_q \mathrm{Fr}^* \mathscr{L}^0 L_q M).
\end{array}
$$

If we begin with id_{LM}: $\mathscr{L}^0 L_q M \to \mathscr{L}^0 L_q M$, apply J_q and follow the left-hand column down, we arrive at the map $\hat{T}$ above. A similar procedure on the right-hand column leads us to x (of (4.7) or (4.10)). The isomorphism ① in (4.14) implies the existence of a commutative square as (4.15). The diagram (4.14) insures that the two horizontal arrows are induced (through adjointness) from corresponding identity maps; that is, they are (by definition) $\hat{T}$ and x, respectively

$$\text{(4.15)} \qquad \begin{tikzcd} \mathrm{Fr}^* J_q \mathscr{L}^0 L_q M \arrow[r, "\hat{T}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & J_q \mathscr{L}^0 L_q \mathrm{Fr}^* M \arrow[d, "J(m)"] \\ \mathrm{Fr}^* J_q \mathscr{L}^0 L_q M \arrow[r, "x"] & J_q \mathrm{Fr}^* \mathscr{L}^0 L_q M. \end{tikzcd}$$

Thus $J(m)^{-1} x = \hat{T}$. But, from (4.10), $J(m)^{-1} x = T$. Q.E.D.

5. Maps of Hecke-modules

We now describe a geometric analogue of the functor U_α defined by Vogan in [20] and [21] for semisimple Harish-Chandra modules. This functor, denoted $\tilde{U}_\alpha$, plays a role only because it is designed to induce the operator $T_{s_\alpha}+1$, $s_\alpha \in S$ on $\mathscr{M}'_K$ and $\mathscr{M}'_N$ (introduced in Sect. 2).

Let $s_\alpha = s \in S$. We define $\tilde{U}_\alpha$ by

$$\text{(5.1)} \qquad \tilde{U}_\alpha(\mathfrak{s}) = \pi_s^* \pi_{s!}(\mathfrak{s}[1]) \otimes L^{-1/2}.$$

Here π_s: $\mathscr{B}_q \to (\mathscr{P}_s)_q$ is the projection introduced in Sect. 2, L is a Tate sheaf which corresponds to twisting by $q^{-1/2}$ in the Grothendieck groups $\mathscr{M}'_H$. The symbol "[1]" denotes, as usual, a shift in degree. The functor $\tilde{U}_\alpha$ applied to an object in $\mathscr{H}\mathscr{C}_q$ or $\mathscr{O}'_q$ has three perverse-cohomology groups $\tilde{U}_\alpha^{-1}$, $\tilde{U}_\alpha^0$ and $\tilde{U}_\alpha^{+1}$. If we interpret $\tilde{U}_\alpha$ as acting on Harish-Chandra modules via all our dictionaries, abusing notation, we have that for $X \in 0b\mathscr{H}\mathscr{C}_F$ irreducible with $s_\alpha \notin \tau(X)$ (the τ-invariant of X), $\tilde{U}_\alpha^0 X \approx U_\alpha X$. This follows by the decomposition theorem and 5.3 of [17]. In general $\tilde{U}_\alpha$ is defined on the derived category $D_H^b(\mathscr{B}_q, \bar{\mathbb{Q}}_\ell)$, inducing functors

$$\text{(5.2)} \qquad \begin{aligned} \tilde{U}_\alpha &: D^b(\mathscr{H}\mathscr{C})_q \to D^b(\mathscr{H}\mathscr{C})_q \\ \tilde{U}_\alpha &: D^b(\mathscr{O}')_q \quad \to D^b(\mathscr{O}')_q. \end{aligned}$$

Using (2.8), the map induced by $\tilde{U}_\alpha$ on the Grothendieck groups $\mathscr{M}'_H$ corresponds to the operator $-(T_{s_\alpha}+1)$ of the Hecke algebra (the minus sign comes from the shift). For example, we obtain a commutative diagram

$$\text{(5.3)} \qquad \begin{tikzcd} K(\mathscr{O}'_{q,\mathrm{Weil}}) \arrow[r] \arrow[d, "\tilde{U}_\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{M}'_N \arrow[d, "-(T_{s_\alpha}+1)"] \\ K(\mathscr{O}'_{q,\mathrm{Weil}}) \arrow[r] & \mathscr{M}'_N. \end{tikzcd}$$

Similarly, we obtain maps on the level of Grothendieck groups, induced by the functors $\mathscr{L}L_q$ and J_q, which we denote by $\mathbb{L}$ and $\mathbb{J}$. These are at least $\mathbb{Z}[B]$-map

$$\text{(5.4)} \qquad \begin{aligned} \mathbb{L}&: \mathcal{M}'_N \to \mathcal{M}'_K \\ \mathbb{J}&: \mathcal{M}'_K \to \mathcal{M}'_N. \end{aligned}$$

The main result of this section is

(5.5) **Proposition.** *The $\mathbb{Z}[B]$-linear map $\mathbb{J}$ is a map of Hecke algebra modules.*

Since $\tilde{U}_\alpha$ induces the operator $-(T_{s_\alpha}+1)$, $s_\alpha = s \in S$, it is enough to show that J_q commutes with $\tilde{U}_\alpha$. We will prove something slightly weaker than this, which will suffice to finish the proof of (5.5).

We first prove:

(5.6) **Lemma.** *The $\mathbb{Z}[B]$-linear map $\mathbb{L}$ is a map of Hecke algebra modules; moreover, $\mathscr{L}L_q$ commutes with $\tilde{U}_\alpha$.*

Proof of (5.6). We use base change as in [8]. First consider the base-change diagram

$$\text{(5.7)} \qquad \begin{array}{ccc} K_q \times \mathscr{B}_q & \xrightarrow{p} & \mathscr{B}_q \\ \Big\downarrow {\scriptstyle 1\times\pi_s} & & \Big\downarrow {\scriptstyle 1\times\pi_s} \\ K_q \times (\mathscr{P}_s)_q & \xrightarrow{p} & (\mathscr{P}_s)_q \end{array}$$

which implies that $(1\times\pi_s)_* p^* = p^*(1\times\pi_s)_*$ because $1\times\pi_s$ is proper. Also $(1\times\pi_s)^* p^* = p^*(1\times\pi_s)^*$ follows from diagram (5.7), therefore $(1\times\pi_s)_*$ and $(1\times\pi_s)^*$ commute with p^*.

In order to deal with the map act, note that act can be factored as act $= p\circ\varphi$, with $\varphi: K_q\times\mathscr{B}_q \to K_q\times\mathscr{B}_q$ given by $\varphi(k,x)=(k,k\cdot x)$. We then use the commutative squares

$$\text{(5.8)} \qquad \begin{array}{ccc} K_q \times \mathscr{B}_q & \xrightarrow{\varphi} & K_q \times \mathscr{B}_q \\ \Big\downarrow {\scriptstyle 1\times\pi_s} & & \Big\downarrow {\scriptstyle 1\times\pi_s} \\ K_q \times (\mathscr{P}_s)_q & \xrightarrow{\varphi} & K_q \times (\mathscr{P}_s)_q \end{array}$$

and

$$\text{(5.9)} \qquad \begin{array}{ccc} K_q \times \mathscr{B}_q & \xrightarrow{p} & \mathscr{B}_q \\ \Big\downarrow {\scriptstyle 1\times\pi_s} & & \Big\downarrow {\scriptstyle \pi_s} \\ K_q \times (\mathscr{P}_s)_q & \xrightarrow{p} & (\mathscr{P}_s)_q \end{array}$$

to conclude (by base change) that $\mathrm{act}_*(1\times\pi_s)^* = \pi_s^*\,\mathrm{act}_*$ and $\mathrm{act}_*(1\times\pi_s)_* = (\pi_s)_*\,\mathrm{act}_*$. In diagram (5.9) it is important to realize that π_s is a smooth map. We now conclude that $\mathscr{L}L_q\tilde{U}_\alpha = \tilde{U}_\alpha\mathscr{L}L_q$ since all the functors involved in the definition of $\mathscr{L}L_q$ commute with π_s^* and π_{s*}. Q.E.D.

(5.10) *Remark.* The forgetful functor For: $\mathcal{O}'_{F,T^0} \to \mathcal{O}'_F$ embedds $\mathcal{O}'_{F,T^0}$ as a full subcategory of $\mathcal{O}'_F$. Using the equivalence of categories $\mathcal{O}'_{F,T^0} \approx \mathcal{O}'_{q,T^0}$ and

$\mathcal{O}'_F \approx \mathcal{O}'_q$ the forgetful functor For: $\mathcal{O}'_{q,T^0} \to \mathcal{O}'_q$ has to embedd $\mathcal{O}'_{q,T^0}$ as a full subcategory of $\mathcal{O}'_q$. Similarly for the versions of $\mathcal{O}'_q$, $\mathcal{O}'_{q,T^0}$ having Frobenius actions; namely, $\mathcal{O}'_{q,\mathrm{Weil}}$, $\mathcal{O}'_{q,\mathrm{mix}}$, $\mathcal{O}'_{q,T^0,\mathrm{Weil}}$, $\mathcal{O}'_{q,T^0,\mathrm{mix}}$ (using the conventions of Sect. 2).

Note that For induces isomorphisms on the level of Grothendieck groups

(5.11)
$$K(\mathcal{O}'_{q,T^0,\mathrm{Weil}}) \xrightarrow{\eta} K(\mathcal{O}'_{q,\mathrm{Weil}})$$
$$K(\mathcal{O}'_{q,T^0,\mathrm{mix}}) \xrightarrow{\eta} K(\mathcal{O}'_{q,\mathrm{mix}}).$$

By composing with η^{-1}, the map of Grothendieck groups induced by $\mathscr{L}L_{\mathrm{II},q}$ gives rise to a map $K(\mathcal{O}'_{q,\mathrm{Weil}}) \to K(\mathscr{H}\mathscr{C}_{q,\mathrm{Weil}})$ and a map of $\mathbb{Z}[B]$-modules denoted $\mathbb{L}_{\mathrm{II}}$

$$\mathbb{L}_{\mathrm{II}}\colon \mathscr{M}'_N \to \mathscr{M}'_K.$$

In general $\mathbb{L}_{\mathrm{II}} \neq \mathbb{L}_{\mathrm{I}}$.

(5.12) **Lemma.** *The map* $\mathbb{L}_{\mathrm{II}}$ *is a Hecke algebra module map.*

Proof. The proof of (5.6) applies except at the moment when we apply the functor ξ. Recall that ξ^{-1} is given by π^*, where π is the projection $\pi\colon K_q \times \mathscr{B}_q \to K_q \times_{T_q^0} \mathscr{B}_q$. We have a commutative diagram

(5.13)
$$\begin{array}{ccc} K_q \times \mathscr{B}_q & \xrightarrow{\pi} & K_q \times_{T_q^0} \mathscr{B}_q \\ \Big\downarrow{\scriptstyle 1\times\pi_s} & & \Big\downarrow{\scriptstyle 1\times\pi_s} \\ K_q \times (\mathscr{P}_s)_q & \xrightarrow{\pi} & K_q \times_{T_q^0} \mathscr{P}_s \end{array}$$

and by base change

(5.14)
$$\pi^*(1\times\pi_s)_* = (1\times\pi_s)_*\pi^*,$$
$$\pi^*(1\times\pi_s)^* = (1\times\pi_s)^*\pi^*.$$

By interpreting π^* as the functor $\xi^{-1}\colon \mathscr{A}_{\mathrm{Weil}} \to \mathscr{A}_{T^0,\mathrm{Weil}}$ of (2.15), we obtain an isomorphism
$$\pi^* = \xi^{-1}\colon K(\mathscr{A}_{\mathrm{Weil}}) \to K(\mathscr{A}_{T^0,\mathrm{Weil}}).$$

Let $\mathscr{A}_{s,\mathrm{Weil}}$, $\mathscr{A}_{s,T^0,\mathrm{Weil}}$ be the categories defined in the same way as $\mathscr{A}_{\mathrm{Weil}}$, $\mathscr{A}_{T^0,\mathrm{Weil}}$ substituting $\mathscr{P}_s$ instead of $\mathscr{B}$. Then we obtain from (5.14) the equations $\xi^{-1}(1\times\pi_s)_* = (1\times\pi_s)_*\xi^{-1}$ and $\xi^{-1}(1\times\pi_s)^* = (1\times\pi_s)^*\xi^{-1}$ which in turn imply $\xi(1\times\pi_s)_* = (1\times\pi_s)_*\xi$ and $\xi(1\times\pi_s)^* = (1\times\pi_s)^*\xi$ in the Grothendieck group. The rest of the proof proceeds as (5.6). Q.E.D.

(5.15) **Lemma.** *For any* X *in* $\mathscr{H}\mathscr{C}_q$ *and* M *in* $D^b(\mathcal{O}')_q$ *there is a natural isomorphism of* $\bar{\mathbb{Q}}_\ell$*-vector spaces*

$$\mathrm{Hom}_{D^b(\mathcal{O}'_q)}(M, \tilde{U}_\alpha J_q X) \xrightarrow{\Phi} \mathrm{Hom}_{D^b(\mathcal{O}')_q}(M, J_q \tilde{U}_\alpha X).$$

Proof. We have (dropping the subscripts $D^b(\mathcal{O}'_q)$ on $D^b(\mathscr{H}\mathscr{C})_q$)

$$\operatorname{Hom}(M,\tilde U_\alpha J_q X)\overset{①}{\approx}\operatorname{Hom}(\tilde U_\alpha M, J_q X)\overset{②}{\approx}\operatorname{Hom}(\mathscr{L}L_q(\tilde U_\alpha M),X)\overset{③}{\approx}\operatorname{Hom}(\tilde U_\alpha \mathscr{L}L_q M,X)$$
$$\overset{④}{\approx}\operatorname{Hom}(\mathscr{L}L_q M,\tilde U_\alpha X)\overset{⑤}{\approx}\operatorname{Hom}(M,J_q(\tilde U_\alpha x)).$$

Here we obtained ① from adjointness relations satisfied by π_s^*, π_{s*}; ② from the adjoint formula of Sect. 3 in the derived category; ③ by (5.6). Similarly for ④ and ⑤. Q.E.D.

(5.16) *Remark.* By naturality of Φ if we have

$$M_1 \xrightarrow{b} M_2 \xrightarrow{a} \tilde U_\alpha J_q X,$$

then $\Phi(a\circ b)=\Phi(a)\circ b$. This is what it means for the following diagram to be commutative

$$\begin{tikzcd}
\operatorname{Hom}_{D^b(\mathcal{O}')_q}(M_2,\tilde U_\alpha J_q X) \arrow[r] \arrow[d, "\Phi"] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{Hom}_{D^b(\mathcal{O}')_q}(M_1,\tilde U_\alpha J_q X) \arrow[d, "\Phi"] \\
\operatorname{Hom}_{D^b(\mathcal{O}')_q}(M_2,J_q\tilde U_\alpha X) \arrow[r] & \operatorname{Hom}_{D^b(\mathcal{O}')_q}(M_1,J_q\tilde U_\alpha X).
\end{tikzcd}$$

(5.17) **Corollary.** *If $\tilde U_\alpha X$ is non-zero, then there is a non-zero map*

$$d\colon\ \tilde U_\alpha J_q X\to J_q\tilde U_\alpha X.$$

Proof. The identity $\tilde U_\alpha J_q X\to\tilde U_\alpha J_q X$ produces the desired map by (5.15)

(5.18) **Corollary.** *Let X be irreducible in $\mathscr{H}\mathscr{C}_q$ with $s=s_\alpha\notin\tau(X)$, the τ-invariant of X. Then the map*

$$d\colon\ \tilde U_\alpha J_q X\to J_q\tilde U_\alpha X$$

is an isomorphism. Moreover, $\tilde U_\alpha J_q X$ is perverse.

Proof. Let $\tilde U_\alpha^{-1}$, $\tilde U_\alpha^0$, $\tilde U_\alpha^{+1}$ denote the three perverse-cohomology groups of the functor $\tilde U_\alpha$. Assume that $\tilde U_\alpha^{-1}J_q X\neq 0$. We obtain a non-zero map S

$$\tilde U_\alpha^{-1}J_q X[1]\xrightarrow{S}\tilde U_\alpha J_q X\xrightarrow{d}J_q\tilde U_\alpha X.$$

However, $d\circ S=0$ because $\tilde U_\alpha^{-1}J_q X[1]$ has all its perverse-cohomology groups in negative degrees and $J_q\tilde U_\alpha X$ has only perverse cohomology in zero degree. Therefore, we obtain $\operatorname{Hom}_{D^b(\mathcal{O}')_q}(\tilde U_\alpha^{-1}J_q X, J_q\tilde U_\alpha X)=0$. By (5.16) this means that $\Phi(S)=0$ (because $d=\Phi(\text{identity})$). Therefore, $S\equiv 0$ by (5.15). We conclude $\tilde U_\alpha^{-1}J_q X=0$. Using that J, the usual Jacquet functor, commutes with duality [14], we obtain that $J_q X$ is self-dual, therefore $\tilde U_\alpha^{+1}J_q X\equiv 0$ (since $\tilde U_\alpha$ commutes with duality). We now show that the map d is injective. Assume that d has a kernel (note that we are now in an abelian category since $\tilde U_\alpha^{\pm 1}J_q X=0$ and $\tilde U_\alpha^{\pm 1}X=0$ by our assumption $\alpha\notin\tau(X)$) we obtain

$$K\xrightarrow{i}\tilde U_\alpha J_q X\xrightarrow{d}J_q\tilde U_\alpha X.$$

Again $d\circ i=0$ and this implies $\Phi(i)=0$. By (5.15) we conclude $i=0$ and $K=0$. In order to prove that d is surjective, it is enough to obtain a map d':

$J_q \tilde{U}_\alpha X \to \tilde{U}_\alpha J_q X$ and prove that d' is injective again. We leave this to the reader. Q.E.D.

(5.19) **Lemma.** *Let X be irreducible in $\mathscr{HC}_{q,\mathrm{Weil}}$. Then the map d: $\tilde{U}_\alpha J_q X \to J_q \tilde{U}_\alpha X$ is a map of Weil-complexes (it commutes with the corresponding Frobenius actions).*

Proof. We construct d in a way that makes it clear that d commutes with Frobenius actions. By (4.5) we have

$$\mathrm{Hom}_{\mathscr{O}'_{q,\mathrm{Weil}}}(J_q X, J_q X) \approx \mathrm{Hom}_{\mathscr{HC}_{q,\mathrm{Weil}}}(\mathscr{L}^0 L_q J_q X, X).$$

Therefore, the identity of $J_q X$ induces a non-trivial map $\mathscr{L}^0 L_q J_q X \to X$ in $\mathscr{HC}_{q,\mathrm{Weil}}$. Recall here that $\mathscr{L}^0$ refers here to the 0-perverse-cohomology. Since $\mathscr{L} L_q J_q X$ has its perverse-cohomology concentrated in non-positive degrees, there is a map $\mathscr{L} L_q J_q X \to \mathscr{L}^0 L_q X$ giving $\mathscr{L} L_q J_q X \to X$ in $D^b(\mathscr{HC})_{q,\mathrm{Weil}}$. Now apply $\tilde{U}_\alpha$ to obtain

$$\begin{tikzcd} \tilde{U}_\alpha \mathscr{L} L J_q X \arrow[r] \arrow[d, "\text{①}\wr"'] & \tilde{U}_\alpha X \\ \mathscr{L} L \tilde{U}_\alpha J_q X. \arrow[ur, dashed, "\text{②}"'] & \end{tikzcd}$$

The isomorphism in ① is (5.6) and it is a map in $\mathscr{HC}_{q,\mathrm{Weil}}$. But now $\tilde{U}_\alpha X$ is perverse and $\mathscr{L} L(\tilde{U}_\alpha J_q X)$ is concentrated in negative (perverse-cohomology) degrees. Therefore ② gives a map in

$$\mathrm{Hom}_{\mathscr{HC}_{q,\mathrm{Weil}}}(\mathscr{L}^0 L \tilde{U}_\alpha J_q X, \tilde{U}_\alpha X) \approx \mathrm{Hom}_{\mathscr{O}'_{q,\mathrm{Weil}}}(\tilde{U}_\alpha J_q X, J_q \tilde{U}_\alpha X);$$

hence, ② gives a map $\tilde{U}_\alpha J_q X \to J_q \tilde{U}_\alpha X$ in $\mathscr{O}'_{q,\mathrm{Weil}}$. We leave to the reader to verify that this is d. Q.E.D.

Proof of (5.5). It is enough to show that for each $s \in S$, $\mathbb{J}$ commutes with $T_s + 1$. Take $\delta \in \mathscr{D}_K$ such that $s = s_\alpha \notin \tau(\delta)$, the τ-invariant of the Harish-Chandra module attached to δ. Then (5.18) and (5.19) imply

$$(T_s + 1)\mathbb{J}(C_\delta) = \mathbb{J}((T_s + 1) C_\delta),$$

because $\tilde{U}_\alpha$ corresponds to $-(T_s + 1)$ in the Grothendieck group. If $s \in \tau(\delta)$, then $T_s C_\delta = q C_\delta$ [17] and $\mathbb{J}(T_s C_\delta) = \mathbb{J}(q C_\delta) = q \mathbb{J}(C_\delta)$. We now recall that if $s \in \tau(X)$, all the irreducible constituents of $J_q X$ have s in their τ-invariant. Therefore $T_s \mathbb{J}(C_\delta) = q \mathbb{J}(C_\delta)$. Q.E.D.

6. Proof of main theorem

Let $\mathscr{D}_K^0 \subset \mathscr{D}_K$ be as in Sect. 1; the set of parameters δ corresponding to the open K_q-orbit in $\mathscr{B}_q$. The set $\mathscr{D}_K^0$ parametrizes Hecht's largest-growth Harish-Chandra modules in $\mathscr{HC}_F$ [12].

(6.1) *Definition.* A function $A\colon \mathscr{D}_K^0 \to \mathbb{Z}[B]$ is called *realizable* if there exists a map of Hecke algebra modules

$$\mathbb{J}_A\colon \mathscr{M}'_K \to \mathscr{M}'_N$$

such that for each $\delta \in \mathscr{D}_K^0$, if we set $\mathbb{J}_A(\hat{C}_\delta) = \sum_{w \in W} a_w(\delta)\hat{C}_w$. Then $a_{w_0}(\delta) = A(\delta)$ with w_0 the longest element in W. We additionally ask that for any $\sigma \in \mathscr{D}_K$, $a_w(\sigma) \neq 0$ imply $\ell(w) \leqq \ell(\sigma)$.

(6.2) *Example.* Let $A\colon \mathscr{D}_K^0 \to \mathbb{Z}[B]$ associate to each δ the number $1 \in \mathbb{Z}[B]$. In [8] it is shown that A is realizable. We will denote this function by A_{trivial}.

The following lemma is a very slight variation of the uniqueness theorem of [8].

(6.3) **Lemma.** *Let A be a realizable map. Then the Hecke-module map $\mathbb{J}_A$ satisfying definition* (6.1) *is unique and computable in terms of A. If the image of A lies inside $\mathbb{Z}[q^{1/2}, q^{-1/2}] \hookrightarrow \mathbb{Z}[B]$, then $\mathbb{J}_A$ induces a Hecke-module map $\mathbb{J}_A$: $\mathscr{M}_K \to \mathscr{M}_N$.*

Brief sketch of proof. Given $\delta \in \mathscr{D}_K$, one must give an algorithm in terms of A to compute $\mathbb{J}_A(\hat{C}_\delta) = \sum_{w \in W} a_w(\delta)\hat{C}_w$. We show that each $a_w(\delta)$ is computable in terms of A by decreasing induction on the length $\ell(w)$. Applying operators of the form $T_s + 1$, $s \in S$, it is possible to take any given $\hat{C}_w$ to $\hat{C}_{w_0}$ in a way that the only other terms $\hat{C}_{w'}$ that are taken to $\hat{C}_{w_0}$ have length bigger than $\ell(w)$. Then one uses that $\mathbb{J}_A$ coincides with A for largest growth terms $\hat{C}_\delta$, $\delta \in \mathscr{D}_K^0$. For details see [8] and example (1.15). The second assertion in (6.3) also follows from this proof. Q.E.D.

Computability of A

Recall the duality operation D on the Hecke-modules $\mathscr{M}_K$, induced by Verdier duality. By [17] there is an algorithm for computing $D\delta$, for $\delta \in \mathscr{D}_K$, in the basis $\mathscr{D}_K$ of $\mathscr{M}_K$. We have

(6.4) **Proposition.** *Let A be the realizable map associated to $\mathbb{J}$ of* (5.5). *Then A can be computed from the equation*

$$(-1)^{\ell(w_0)} T_{w_0} D\Big(\sum_{\delta \in \mathscr{D}_K^0} \delta\Big) = q^{-\ell(w_0)}\Big(\sum_{\delta \in \mathscr{D}_K^0} A(\delta)^{-1}\delta\Big)$$

(6.6) *Example.* Consider $G_{\mathbb{R}} = \mathrm{SL}_2\mathbb{R}$ as in (1.15). We have

$$D\delta_+ = q^{-1} C_{\delta_+} - \delta_h - \delta_a$$

$$D\delta_- = q^{-1}\delta_-.$$

Therefore

$$\begin{aligned} T_s D(\delta_+ + \delta_-) &= C_{\delta_+} - \delta_a - \delta_h - q^{-1}\delta_- \\ &= -q^{-1}(-q\delta_+ + \delta_-) \end{aligned}$$

and $A(\delta_+)=-q^{-1}$, $A(\delta)=1$. This simply shifts the weights in the filtration obtained in (1.15) in a harmless way.

Let $M_w \in 0b\mathcal{O}'_F$ be the Verma module attached to $w\in W\approx\mathcal{D}_N$. If i^w: $\mathcal{O}_w \hookrightarrow \mathcal{B}_q$ is the inclusion of the corresponding N_q-orbit, and 1_w is a trivial sheaf on $\mathcal{O}_w$, then $M'_w = i^w_! 1_w$ is the perverse sheaf associated to M_w by (2.2). In the context of Weil sheaves we will assume 1_w has the trivial Frobenius action

(6.7) **Lemma.** *Let $X(\delta)$ be the irreducible object in $\mathcal{HC}_q$ corresponding to $\delta\in\mathcal{D}^0_K$. Then $X(\delta)$ occurs exactly once in $\mathcal{L}^0 L_q(M'_{w_0})$ as a quotient.*

Proof. This is a consequence of (2.16) and the following Frobenius reciprocity law

$$\mathrm{Hom}_R(LM_{w_0}, \bar{\pi}(\delta)) = \mathrm{Hom}_{\mathfrak{h}}(\bar{\pi}(\delta)/\mathfrak{n}\,\bar{\pi}(\delta), \mathbb{C}_\lambda)$$

where $w_0\lambda$ is the highest weight of M_{w_0}. Q.E.D.

(6.8) **Corollary.** *In the setting of* (6.7), *assume that $X(\delta)$ occurs with weight $q^{n/2}$, $n\in\mathbb{Z}$ in $\mathcal{L}^0 L_q(M'_{w_0})$. Then $A(\delta)=(-1)^{\ell(w_0)}q^{-n/2}$.*

Proof. We have

$$\mathrm{Hom}_{\mathcal{HC}_{q,\mathrm{Weil}}}(\mathcal{L}^0 L_q(M'_{w_0}), L^{n/2}\otimes X(\delta)) = \mathrm{Hom}_{\mathcal{O}_{q,\mathrm{Weil}}}(M'_{w_0}, L^{n/2}\otimes J_q X(\delta)).$$

Therefore, we obtain a non-trivial map $M'_{w_0}\to L^{n/2}\otimes J_q X(\delta)$ implying (6.8) (we use the fact that the finite dimensional module occurs exactly once in $J(\bar{\pi}(\delta))$. Q.E.D.

Recall the two Hecke-module maps $\mathbb{L}_{\mathrm{I}}$, $\mathbb{L}_{\mathrm{II}}$ of Sect. 5. Since $\mathcal{L}^i L_q(M'_{w_0})$ can be non-zero for $i\neq 0$, one cannot obtain, in general, the weight of $X(\delta)$ in $\mathcal{L}^0 L_q(M'_{w_0})$ by computing $\mathbb{L}_{\mathrm{I}}(T_{w_0})$. Using (2.17) and that $\mathcal{L}^i L_{\mathrm{II},q}(M'_{w_0})=0$, $i\neq 0$ [22], it suffices to compute $\mathbb{L}_{\mathrm{II}}(T_{w_0}) = T_{w_0}\mathbb{L}_{\mathrm{II}}(T_e)$. Here we are considering the operator T_w as imbedded in $\mathcal{M}_N$ (see Sect. 2).

Computation of $\mathbb{L}_{\mathrm{II}}(T_e)$

Let $\{x_0\}$ be the 0-dimensional, N_q-orbit in $\mathcal{B}_q$. Then (by our choice of N_q), the K_q-orbit $K_q\cdot x_0$ is the open orbit in $\mathcal{B}_q$. We have maps f: $K_q/T^0_q\to\mathcal{B}_q$ and h: $K_q/T_q\to\mathcal{B}_q$ given by $kT^0_q\to k\cdot x_0$ and $kT_q\to k\cdot x_0$. Let $s=\dim(B\cap K/T)$.

(6.9) **Lemma.** *Let $1_{K_q/T^0_q}$ be the trivial sheaf on K_q/T^0_q. The perverse sheaf $\mathcal{L}^{-s}L_{\mathrm{II},q}(M'_e)\otimes L^{\ell(w_0)/2}$ can be computed as*

$$f_*(1_{K_q/T^0_q}[\ell(w_0)]) = \bigoplus_{\delta\in\mathcal{D}^0_K} h_*(\delta)[\ell(w_0)].$$

Proof. Define categories $\mathcal{A}(K_q)_{T^0_q,\mathrm{Weil}}$, $\mathcal{A}(K_q)_{\mathrm{Weil}}$ in analogy of $\mathcal{A}_{T^0,\mathrm{Weil}}$, $\mathcal{A}_{\mathrm{Weil}}$ of Sect. 2, with $\{x_0\}$ in the place of $\mathcal{B}_q$. In this case all the objects consist of constant sheaves. We obtain a commutative diagram

(6.10)
$$\begin{tikzcd}
\mathcal{A}_{T^0,\mathrm{Weil}} \arrow[d, "\xi"'] & \mathcal{A}(K_q)_{T^0,\mathrm{Weil}} \arrow[l, "i_*"'] \arrow[d, "\xi"] \\
\mathcal{A}_{\mathrm{Weil}} & \mathcal{A}(K_q)_{\mathrm{Weil}} \arrow[l, "i_*"']
\end{tikzcd}$$

where i is the inclusion $K_q \times \{x_0\} \hookrightarrow K_q \times \mathscr{B}_q$ or $K_q/T_q^0 \times \{x_0\} \hookrightarrow K_q \times_{T_q^0} \mathscr{B}_q$. Recall that $L^{\ell(w_0)/2} \otimes \mathscr{L}^0 L_{\mathrm{II},q}(M'_e) = \mathrm{act}'_* \xi p^*(M'_e)[\ell(w_0)]$ and

$$p^*(M'_e) = 1_{K_q} \boxtimes M'_e = i_*(1_{K_q} \boxtimes 1_{\{x_0\}})[\ell(w_0)],$$

since M'_e is a skyscraper sheaf at $\{x_0\}$. We obtain

$$\begin{aligned} L^{\ell(w_0)/2} \otimes \mathscr{L}^0_{\mathrm{II},q}(M'_e) &= \mathrm{act}'_* \xi i_*(1_{K_q} \boxtimes 1_{\{x_0\}})[\ell(w_0)] \\ &= \mathrm{act}'_* i_*(1_{K_q/T_q^0} \boxtimes 1_{\{x_0\}})[\ell(w_0)] \quad \text{(by (6.10))}. \end{aligned}$$

On the other hand $\mathrm{act}' \circ i = f$, therefore

$$L^{\ell(w_0)/2} \otimes \mathscr{L}^0 L_{\mathrm{II},q}(M'_e) = f_*(1_{K_q/T_q^0})[\ell(w_0)].$$

If we denote by g the projection $K_q/T_q^0 \to K_q/T_q$, $f = h \circ g$ with g being a proper map. We obtain

$$f_*(1_{K_q/T_q^0}) = h_* g_*(1_{K_q/T_q^0}) = \bigoplus_{\delta \in \mathscr{D}^0_{\mathbf{K}}} h_*(\delta).$$

This proves the lemma in the quasi-split case; the general case is a slight modification. Q.E.D.

Proof of (6.4). From (6.8) we obtain that

$$\mathscr{L}^{-s} L_{\mathrm{II},q}(M'_e) = \sum_{\delta \in \mathscr{D}^0_{\mathbf{K}}} h_*(\delta)[\ell(w_0)] \otimes L^{-\frac{\ell(w_0)}{2}} = \bigoplus D h_!(\delta[\ell(w_0)] \otimes L^{-\frac{\ell(w_0)}{2}}).$$

This becomes in the Grothendieck group the identity

$$\mathbb{L}_{\mathrm{II}}(T_e) = (-1)^{\ell(w_0)} D\Big(\sum_{\delta \in \mathscr{D}^0_{\mathbf{K}}} \delta^{-\frac{\ell(w_0)}{2}}\Big).$$

Therefore

$$\begin{aligned} \mathbb{L}_{\mathrm{II}}(T_{w_0}) &= T_{w_0} \mathbb{L}_{\mathrm{II}}(T_e) \\ &= (-1)^{\ell(w_0)} T_{w_0} D\Big(\sum_{\delta \in \mathscr{D}^0_{\mathbf{K}}} \delta^{-\frac{\ell(w_0)}{2}}\Big) \\ &= \sum_{\delta \in \mathscr{D}^0_{\mathbf{K}}} A(\delta)^{-1} q^{-\frac{\ell(w_0)}{2}} \delta \end{aligned}$$

and we obtain (6.4). Q.E.D.

(6.11) **Corollary.** *The Frobenius action on $J_q X$ is mixed whenever $X \in 0b\mathscr{H}\mathscr{C}_{q,\mathrm{mix}}$. In particular, $J_q X$ has an increasing filtration*

$$E_{-p} \subset E_{-p+1} \subset \ldots \subset E_p = J_q X$$

whose subquotients E_{j+1}/E_j are pure of weight $q^{j+1/2}$ (and therefore semisimple in $\mathscr{H}\mathscr{C}_q$).

Proof. If X is mixed, $J_q X$ is mixed because we have shown that the image of the function A lies inside $\mathbb{Z}[q^{1/2}, q^{-1/2}] \hookrightarrow \mathbb{Z}[B]$ and by (6.3). The filtration statement is O. Gabber's result [2]. Q.E.D.

This concludes the proof of (1.11) and (1.4).

(6.12) *Remark.* The methods employed in this last section, namely the fact that $\mathbb{L}_{\mathrm{II}}$ is a Hecke algebra map can be used to obtain weight filtrations for principal series which are not necessarily standard modules and leads to their indecomposability. This will be pursued elsewhere; [25].

Appendix to section 2

We give a sketch of proof of some results of Bernstein (2.16) which were announced in [4]. The categories that are used are categories of D-modules. The notation f_*, f^*, etc. where f is some algebraic map refers to operations on D-modules that correspond to f_*, f^* via the (covariant) Riemann-Hilbert correspondence. Let M be in $\mathcal{O}'_F$ and $\mathcal{M}$ its localization $D_F \underset{R}{\otimes} M$. For simplicity assume that F is the trivial module. It is enough to prove that the global sections functor $\Gamma_{\mathscr{B}}$ on $\mathscr{B}$, applied to $\mathrm{act}_* p^*(\mathcal{M})$ can be identified with $R(K) \overset{\mathscr{L}}{\underset{\mathfrak{k}}{\otimes}} M = \mathscr{L} L_{\mathrm{I}}(M)$. Similarly for $\mathrm{act}'_* \circ \xi \circ p^*(\mathcal{M})$, whose global sections should correspond to $R(K) \overset{\mathscr{L}}{\underset{\mathfrak{k}, T^0}{\otimes}} M$, when M is in $\mathcal{O}'_{F,T^0}$. We deal only with the case of $\mathscr{L} L_{\mathrm{I}}$ and leave $\mathscr{L} L_{\mathrm{II}}$ to the reader. Taking a free R-resolution of M, corresponding to a free $D_{\mathscr{B}}$-resolution of $\mathcal{M}$, by localization it suffices to compute $\Gamma_{\mathscr{B}} \mathrm{act}_* p^*(D_{\mathscr{B}})$. Recall that

(A.1) (a) $p^*(D_{\mathscr{B}}) = \mathcal{O}_K \boxtimes D_{\mathscr{B}}$

(b) $\mathrm{act}_* [p^* D_{\mathscr{B}}] = \mathrm{act} \cdot \left[D_{\mathscr{B} \leftarrow K \times \mathscr{B}} \overset{\mathscr{L}}{\underset{D_{K \times B}}{\otimes}} (\mathcal{O}_K \boxtimes D_{\mathscr{B}}) \right].$

Here $f.$ denotes the direct image as sheaves of vector spaces. We are implicitly using that act has an affine fiber (isomorphic to K) so that it is unnecessary to compute derived functors of act.

(A.2) *Remark.* The commutative diagram

$$\begin{tikzcd} K \times \mathscr{B} \arrow[r, "\varphi^{-1}"] \arrow[d, "p"'] & K \times \mathscr{B} \arrow[dl, "\mathrm{act}"] \\ \mathscr{B} & \end{tikzcd}$$

with $\varphi(k,x) = (k, k \cdot x)$, shows that, up to a change of coordinates given by φ, act is simply a projection. Therefore $\mathscr{L}^0 \mathrm{act}_*$ can be obtained by dividing by vector fields along the fibers $\{(k, k^{-1}x)\} \simeq S_x$. We now make this precise.

By taking a projective resolution of $D_{\mathscr{B} \leftarrow K \times \mathscr{B}}$ as a right $D_{K \times \mathscr{B}}$-module, it is possible to compute $\mathrm{act}_* p^*(\mathcal{M})$ in (A.1). This can be done using a relative de Rham complex. We obtain

(A.3) $$\mathrm{act}_* p^*(\mathcal{M}) = p \cdot \mathrm{Hom}_{\mathbb{C}}(\Lambda \mathfrak{k}, \mathcal{O}_K \boxtimes D_{\mathscr{B}})[\dim K]$$

and (A.3) is the Lie algebra cohomology complex obtained by considering $\mathcal{O}_K \boxtimes D_{\mathscr{B}}$ as a $\mathfrak{k}$-module, where $\xi \in \mathfrak{k}$ acts by

(A.4) $$\xi(f \otimes d) = \xi f \otimes d + f \otimes b(\xi) d, \qquad \xi f = -f \cdot \xi$$

Here $b\colon \mathfrak{U}(\mathfrak{k}) \to R$ is the projection to R, identifying ξ with a global vector field (since R in turn can be identified with global section in $D_{\mathscr{B}}$). This gives as global sections

(A.5) $$\mathscr{R}\Gamma_{\mathscr{B}} \mathrm{act}_* p^*(D_{\mathscr{B}}) \approx R(K) \underset{\mathfrak{k}}{\otimes} R$$

and $\mathscr{L} L_{\mathrm{I}}(M) \simeq R(K) \overset{\mathscr{L}}{\underset{\mathfrak{k}}{\otimes}} M$ can be identified with $\mathrm{act}_* p^*(\mathcal{M})$ for any $M \in \mathrm{Ob}\,\mathcal{O}'_F$. As noted in (A.2), in (A.5) we have divided by vector fields along the fibers $S_x = \{(k, k^{-1}x) : k \in K\}$. We now take care of the R-action.

R-actions

Note that (A.5) simply means that we have divided by vector fields along the fibers of act. In the case of p_* (p the projection to the second coordinate), dividing by vector fields along K is "harmless" because $D_{\mathscr{B}}$ commutes with D_K in $D_{K\times\mathscr{B}}$. Therefore the $D_{\mathscr{B}}$-action is evident in the quotient. For the case of act the same is true up to a change of coordinates given by φ^{-1} as in (A.2). Let

$$\text{(A.6)}\qquad \begin{aligned} Q&=\mathfrak{U}(\mathfrak{k})\otimes R(K)\otimes R\\ Q'&=1\otimes R(K)\otimes R. \end{aligned}$$

Then Q corresponds to the global sections of $D_{K\times\mathscr{B}}$. We now define an automorphism of Q (as an algebra).

Let $\underset{\sim}{a}\colon Q\to Q$ be given by $\underset{\sim}{a}(1\otimes f\otimes 1)=1\otimes f\otimes 1$, $f\in R(K)$ and $\underset{\sim}{a}(1\otimes 1\otimes r)=\sum 1\otimes f_i\otimes r_i$, where $r\in R$ and for all $k\in K$, $\sum f_i(k)\,r_i=\mathrm{Ad}(k^{-1})\,r$. Using that $\mathrm{Ad}(k)(r_1 r_2)=\mathrm{Ad}(k)\,r_1\,\mathrm{Ad}(k)\,r_2$, we obtain that $\underset{\sim}{a}$ is a map of algebras. Define

$$\text{(A.7)}\qquad R^a\overset{\text{def}}{=}\underset{\sim}{a}(R)\hookrightarrow 1\otimes R(K)\underset{\mathbb{C}}{\otimes}R.$$

The algebra map $\underset{\sim}{a}$ extends to $\tilde{a}\colon Q\to Q$ by specifying that for $\xi\in\mathfrak{k}$, $\tilde{a}(\xi\otimes 1\otimes 1)=\xi\otimes 1\otimes 1+1\otimes 1\otimes b(\xi)$.

(A.8) *Remark.* The algebra map $\tilde{a}$ is induced by the derivative of $\varphi^{-1}\colon K\times\mathscr{B}\to K\times\mathscr{B}$ acting on vector fields. Moreover, $\tilde{a}(\mathfrak{k}\otimes 1\otimes 1)$ and R^a commute.

Note that $R(K)\underset{\mathbb{C}}{\otimes}M$ is a Q-module. By restriction it becomes an $\tilde{a}(\mathfrak{k})$ module and an R^a-module. Also

$$\text{(A.9)}\qquad R(K)\underset{\mathfrak{k}}{\otimes}M=R(K)\underset{\mathbb{C}}{\otimes}M/\tilde{a}(\mathfrak{k})[R(K)\underset{\mathbb{C}}{\otimes}M]$$

and since R^a commutes with $\tilde{a}(\mathfrak{k})$, $R(K)\underset{\mathfrak{k}}{\otimes}M$ is an R^a-module. If we identify $R^a\approx R$ we obtain the R-module action in (A.5).

The case of $\mathscr{L}L_{\mathrm{II}}$ is handled in a similar way. Roughly speaking, the reason why we will get $\mathscr{L}L_{\mathrm{II}}$ instead of $\mathscr{L}L_{\mathrm{I}}$ is that the fiber of $K\underset{T^0}{\times}\mathscr{B}\to\mathscr{B}$ is K/T and in (A.3) one needs to use $\Lambda(\mathfrak{k}/\mathfrak{t})$ leading to $\mathscr{L}L_{\mathrm{II}}(M)=H^*(\mathfrak{k},\mathfrak{t},R(K)\otimes M)$.

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