KAZHDAN-LUSZTIG POLYNOMIALS

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1. Classical setting

Let G be a complex semisimple algebraic group with Lie algebra $\mathfrak{g}$, B a fixed Borel subgroup with Lie algebra $\mathfrak{b}$, $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra, $\Delta^+ \subset \Delta$ the positive roots, respectively roots, and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Let W be the Weyl group. The problem is to describe the multiplicities of the composition factors in *Verma modules*.

Let M_w be the Verma module

$$M_w = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{w\rho-\rho},$$

and L_w the unique irreducible quotient. It is known that all subquotients of M_w are of the form L_v for some $v \in W$.

Let $X = G/B$ be the *flag variety*. It has a *Bruhat decomposition*

$$X = \sqcup_{w\in W} X_w,$$

where X_w are the *Bruhat cells*. It is known that the *Schubert variety*

$$\overline{X}_w = \sqcup_{v\le w} X_v,$$

where the ordering is the *Bruhat order* in W.

Definition. *The Kazhdan-Lusztig polynomials are*

$$P_{v,w}(q) = \sum_{i\ge 0} dim \mathcal{H}^{2i} IC(\overline{X}_w) \mid_{X_v} \cdot q^i.$$

The odd cohomology vanishes, moreover $P_{v,w} = 0$ unless $v \le w$. There is an explicit algorithm for computing $P_{v,w}$.

Theorem (Kazhdan-Lusztig conjecture)**.** *The multiplicity of L_y in M_x, $x, y \in W$ is*

$$[M_x : L_y] = P_{w_0x,w_0y}(1).$$

Similar results were obtained for real reductive groups by Lusztig-Vogan (1985). If G_0 is a real form of some complex G, and $K \subset G_0$ is a maximal compact subgroup of G_0, then the geometry is in terms of $K_{\mathbb{C}}$-orbits on the complex flag variety G/B.

For a general p-adic group, the Langlands classification is still incomplete. For $GL(n)$, Zelevinsky determined a classification of irreducible representations and formulated the Kazhdan-Lusztig conjectures in that case (1981).

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Moreover, Zelevinsky proved that the KL polynomials for p-adic $GL(n)$ are a subset of the Verma modules case. For other p-adic groups, the situation is more complicated and the KL conjectures need to be refined.

2. KL conjecture for the Hecke algebra

Let $\mathbb{H}$ denote the graded Hecke algebra, which is a vector space $\mathbb{H} = \mathbb{C}W \otimes S(\mathfrak{h}^*)$ with a nontrivial commutation relation between W and $\mathfrak{h}^*$. Let us assume that $\mathbb{H}$ has *equal parameters*. It is known that the center of $\mathbb{H}$ is $\mathbb{A}^W$, where $A = S(\mathfrak{h}^*)$. Therefore, the *central characters* χ are parameterized by $\mathfrak{h}/W$.

Let $mod_\chi(\mathbb{H})$ be the category of finite dimensional modules of $\mathbb{H}$ with central character χ. The Langlands classification exists in this case, and we have standard modules X and irreducible quotients L.

Denote

$$G(\chi) = \{g \in G : Ad(g)\chi = \chi\}, \quad \mathfrak{g}_n(\chi) = \{y \in \mathfrak{g} : [\chi, y] = ny\}.$$

Theorem (Lusztig)**.** *The standard and irreducible objects in $mod_\chi(\mathbb{H})$ are parameterized by pairs $\xi = (\mathcal{O}, \mathcal{L})$, where*

1. *$\mathcal{O}$ is a $G(\chi)$-orbit on $\mathfrak{g}_2(\chi)$.*
2. *$\mathcal{L}$ is a certain $G(\chi)$-equivariant local system on $\mathcal{O}$. More precisely, choose some $e \in \mathcal{O}$. Then $\mathcal{L}$ corresponds to a representation ϕ of the component group $G(e,\chi)/G(e,\chi)^0$. The representations ϕ which are allowed are those of Springer type, that is, it must be in the restriction $G(e,\chi)/G(e,\chi)^0 \subset G(e)/G(e)^0$ of a representation which appears in the Springer correspondence.*

In this setting, the Kazhdan-Lusztig conjectures take the following form:

Theorem (Lusztig, Ginzburg)**.** *In (the Grothendieck group of) $mod_\chi(\mathbb{H})$:*

$$X_{\xi'} = \sum_{\xi} P_{\xi,\xi'}(1) \cdot L_\xi,$$

where

$$P_{\xi,\xi'}(q) = \sum_{i \geq 0} [\mathcal{L} : \mathcal{H}^{2i} IC(\overline{\mathcal{O}'}, \mathcal{L}') \mid_{\mathcal{O}}] \cdot q^i.$$

Note that, in particular, $P_{\xi,\xi} = 1$, and if $\xi \neq \xi'$, then $P_{\xi,\xi'} = 0$, unless $\mathcal{O}' \subsetneq \overline{\mathcal{O}}$.

3. Orbits

Let $Orb_n(\chi)$ denote the set of $G(\chi)$ orbits on $\mathfrak{g}_n(\chi)$. Assume that $n \in \mathbb{Z} \setminus \{0\}$. Here are some properties of $Orb_n(\chi)$:

1. $Orb_n(\chi)$ is finite.
2. For every $\mathcal{O} \in Orb_n(\chi)$, $\overline{\mathcal{O}} \setminus \mathcal{O}$ is the union of some orbits $\mathcal{O}'$ with $\dim \mathcal{O}' < \dim \mathcal{O}$.
3. There is a unique open (dense) orbit $\mathcal{O}$ in $Orb_n(\chi)$.

Example. The simplest example is when $\chi = \check{\rho}$ (the regular central character case). Let Π denote the simple roots, and fix root vectors X_α. In this case

$$\begin{aligned} \mathfrak{g}_2(\check{\rho}) &= \{y \in \mathfrak{g} : [\check{\rho}, y] = 2y\} = \oplus_{\alpha\in\Pi}\mathbb{C}\cdot X_\alpha, \\ G(\check{\rho}) &= \text{ Cartan subgroup.} \end{aligned}$$

There is a one-to-one correspondence

$$Orb_2(\check{\rho}) \leftrightarrow 2^{\Pi},$$

where to every $\Pi_P \subset \Pi$ we associate the orbit $\mathcal{O}_P = \sum_{\alpha\in\Pi_P}\mathbb{C}^*\cdot X_\alpha$. In other words, the orbits are parametrized by standard parabolic subalgebras. There are $2^{|\Pi|}$ orbits in this case, and all have smooth closure. The closure ordering is given by inclusion of subsets. Only trivial local systems appear, and therefore the KL polynomials are either 1 or 0 depending on the closure ordering.

3.1. **Parametrization.** Let e be a representative of an orbit $\mathcal{O}_e$ in $\mathfrak{g}_2(\chi)$. We recall the construction of the *Kazhdan-Lusztig parabolic* associated to e.

By a graded version of the Jacobson-Morozov triple, e can be embedded into a Lie triple $\{e, h, f\}$, such that $e \in \mathfrak{g}_2$, $h \in \mathfrak{g}_0$, and $f \in \mathfrak{g}_{-2}$. Define a gradation with respect to h as well, $\mathfrak{g}^r = \{y \in \mathfrak{g} : [h, y] = ry\}$, and set $\mathfrak{g}_t^r = \mathfrak{g}_t \cap \mathfrak{g}^r$. Then

$$\mathfrak{g} = \bigoplus_{t,r\in\mathbb{Z}} \mathfrak{g}_t^r.$$

Set

$$\mathfrak{m} = \bigoplus_{t=r}\mathfrak{g}_t^r, \quad \mathfrak{n} = \bigoplus_{t<r}\mathfrak{g}_t^r, \quad \mathfrak{p} = \mathfrak{m}\oplus\mathfrak{n}. \tag{3.1.1}$$

Properties(Lusztig):

(1) $\mathfrak{p}$ depends only on e and not on the whole triple $\{e, h, f\}$.
(2) χ is *rigid* for $\mathfrak{m}$. (By definition, this means that χ is congruent modulo $\mathfrak{z}(\mathfrak{m})$ to a middle element of a nilpotent orbit in $\mathfrak{m}$.)
(3) e is in the open $M(\chi)$-orbit in $\mathfrak{m}_2(\chi)$.
(4) $M(\chi, e) \subset G(\chi, e)$ ($\subset P$) induces an isomorphism of the component groups.
(5) The $P(\chi)$-orbit of e in $\mathfrak{p}_2$ is open, dense in $\mathfrak{p}_2$.

An immediate corollary of (5) is a dimension formula for $\mathcal{O}_e = G(\chi)\cdot e$.

Proposition. $\dim\mathcal{O}_e = \dim\mathfrak{p}_2 - \dim\mathfrak{p}_0 + \dim\mathfrak{g}_0$.

Definition. *A parabolic subgroup P (or subalgebra $\mathfrak{p}$) is called good for χ if it satifies conditions (3.1.1) and (2) above.*

Let $\mathcal{P}_2(\chi)$ denote the set of good parabolic subgroups for χ.

Theorem (Lusztig)**.** *There is a bijection between $Orb_2(\chi)$ and $G(\chi)$-conjugacy classes in $\mathcal{P}_2(\chi)$.*

Proof. The inverse map is given as follows. Let $P = MN$ be a good parabolic for χ. Then there exists s a middle element of a Lie triple in $\mathfrak{m}$, such that $\chi \equiv s \pmod{\mathfrak{z}(\mathfrak{m})}$. Moreover, the decomposition (3.1.1) must hold with respect to χ and s. Let $G' \subset G(\chi)$ be the reductive subgroup whose Lie algebra is $\mathfrak{g}_0^0$. Then G' acts on $\mathfrak{g}_2^2(\chi)$ and there is a unique open orbit of this action. Let $\mathcal{O}$ be the unique $G(\chi)$-orbit on $\mathfrak{g}_2$ containing it. The inverse map associates $\mathcal{O}$ to P.

□

3.2. **An application: reducibility.** The category $mod_\chi(\mathbb{H})$ has also a "classical" Langlands classification (Evens). For every standard parabolic $P = MN$, one can define a subalgebra $\mathbb{H}_M \subset \mathbb{H}_P \subset \mathbb{H}$, and form induced standard modules $X(P, \sigma, \nu)$, where σ is a *tempered* module of $\mathbb{H}_M$, and ν is a dominant character. The tempered modules are defined by a Casselman criterion in terms of the weights under the action of the abelian part of the Hecke algebra. In the geometric classification, σ is tempered for $\mathbb{H}_M$, if χ is rigid for M and the corresponding $M(\chi)$-orbit on $\mathfrak{m}_2(\chi)$ is the open orbit.

Lemma. *Let σ be a tempered module for $\mathbb{H}_M$ parametrized by (the open orbit of $M(\chi)$ on $\mathfrak{m}_2(\chi)$) and a representation ψ of the component group $A_M(\chi, e)$. Then the $G(\chi)$-orbit on $\mathfrak{g}_2(\chi)$ of the Langlands quotient $L(P, \sigma, \nu)$ is parametrized by P and ψ viewed now as a representation of the component group $A_G(\chi, e)$.*

Proposition. *The reducibility points of the standard module $X(P, \sigma, \nu)$ (with ν strictly dominant) are necessarily a subset of the zeros of the rational function*

$$\prod_{\alpha \in \Delta(\mathfrak{n}^-)} \frac{2 - \langle \alpha, \chi \rangle}{\langle \alpha, \chi \rangle}.$$

If in addition σ is generic (parametrized by the trivial local system), then the reducibility points are precisely the set of zeros.

The first assertion follows immediately from the dimension formula for $\mathcal{O}_P$. The second one follows again by using in addition a result of M. Reeder that the generic modules are parametrized by the trivial local system on the open orbit.

3.3. **Combinatorial parametrization.** In type A, an explicit combinatorial parametrization of the orbits and irreducible modules (there are only trivial local systems in that case) comes down to *Zelevinsky's multisegment* (or *strings*): every orbit is described by a set of segments, each segment being a sequence of numbers increasing by 2, such that the support of the strings is χ. The closed orbit corresponds to all segments being singletons, while the open orbit corresponds to the unique *nested* multisegment of support χ. The closure ordering can be described combinatorially.

For other classical types, we developed a combinatorial parametrization which takes into account the local systems as well. Roughly, every local system corresponds to a set of segments and a Young tableau. Incidently, this answers a conjecture of Opdam and Slooten on the combinatorial parametrization of tempered modules when the Hecke algebra is of geometric type.

4. Polynomials

Lusztig's algorithm (2006) computes polynomials

$$c_{\xi,\xi'}(v) = \sum_j [\mathcal{L}' : \mathcal{H}^i(\overline{\mathcal{O}}, \mathcal{L}) \mid_{\mathcal{O}'}] v^{\dim \mathcal{O} - \dim \mathcal{O}' - j}.$$

The relation with KL polynomials is given by the formula

$$c_{\xi,\xi'}(v) = v^{\dim \mathcal{O} - dim \mathcal{O}'} \cdot P_{\xi,\xi'}\left(\frac{1}{v^2}\right).$$

The algorithm construct *four* bases $\mathcal{U}_\pm$ and $\mathcal{Z}_\pm$ inside a space $\mathcal{K}(\chi)/Rad$:

- the $\mathbb{Q}(v)$-vector space $\mathcal{K}(\chi)$ has a basis indexed by the cosets $W/W(\chi)$ and
- it has a symmetric bilinear form whose radical is *Rad*.
- the matrix of polynomials is given by the change of basis from $\mathcal{Z}$ to $\mathcal{U}$.
- the elements of the bases $\mathcal{U}$ and $\mathcal{Z}$ are indexed by the orbits and local systems in $Orb_2(\chi)$.
- to construct $\mathcal{U}$ and $\mathcal{Z}$ one uses induction from KL parabolics and the Fourier-Deligne transform (bijection between simple perverse sheaves on $\mathfrak{g}_2$ and simple perverse sheaves on $\mathfrak{g}_{-2}$. This is why one needs 4 bases! As a byproduct, the algorithm computes the Fourier-Deligne transform, and therefore (by the result of Evens-Mirković, the Iwahori-Matsumoto involution). As a side remark, in type A, the FDT is the Pyasetsky map.
- one important step which makes the algorithm work is the fact that if $\#Orb_2(\chi) > 1$, then the FDT of the open orbit in $\mathfrak{g}_2(\chi)$ is *not* the open orbit in $\mathfrak{g}_{-2}(\chi)$. In representation theoretic terms, this is the statement that the Iwahori-Matsumoto involution of a tempered module cannot be tempered (unless the tempered module is the full spherical principal series).

4.1. **An example in $\mathfrak{s}p(6)$.** The central character is $\chi = (3,1,1)$, the middle element of the nilpotent $(4,2)$. There are 10 orbits of $G(\chi)$ on $\mathfrak{P}_2$. We list the parametrization of these orbits, the dimensions, the corresponding Levi subalgebras and the basis elements $\mathcal{Z}_-$ and $\mathcal{U}_-$. The bases $\mathcal{Z}_+$ and $\mathcal{U}_+$ are obtained by multiplication by w_0.

We encode the cosets $W/W(\chi)$ by the of W action on $(3,1,1)$.

s	Levi $\mathfrak{l}_s$	Dim	$\mathcal{Z}_-$
$(0,0,0)$	$\pm\{\epsilon_2-\epsilon_3\}$	0	$\frac{1}{v+v^{-1}}[3,1,1]$
$(1,-1,0)$	$\pm\{\epsilon_1-\epsilon_2\}$	2	$[1,3,1]+v[3,1,1]$
$(0,0,1)$	$\pm\{2\epsilon_3\}$	2	$[3,1,-1]+v[3,1,1]$
$(1,-1,1)$	$\pm\{\epsilon_1-\epsilon_2,2\epsilon_3\}$	3	$[1,3,-1]+v[1,3,1]+v[3,1,-1]+v^2[3,1,1]$
$(0,1,1)$	$\pm\{\epsilon_2\pm\epsilon_3,2\epsilon_2,2\epsilon_3\}$	3	$\frac{1}{v+v^{-1}}[3,-1,-1]-v[3,1,-1]-\frac{v}{v+v^{-1}}[3,1,1]$
$(0,1,1)$		3	$[3,-1,1]+\frac{1}{v+v^{-1}}[3,-1,-1]-\frac{v}{v+v^{-1}}[3,1,1]$
$(2,0,2)$	$\pm\{\epsilon_1-\epsilon_2,\epsilon_2+\epsilon_3,\epsilon_1+\epsilon_3\}$	4	$[-1,1,3]+v[1,3,-1]+v[3,-1,1]+v^2[3,1,-1]$
$(3,1,0)$	$\pm\{\epsilon_1\pm\epsilon_2,2\epsilon_2,2\epsilon_1\}$	4	$[1,-3,-1]+v[1,1,3]+v[1,3,-1]+v^2[1,3,1]$
$(3,1,1)$	Δ	5	$\frac{1}{v+v^{-1}}[-3,-1,-1]-v[-1,1,3]-v[1,-3,-1]$ $-\frac{v^2}{v+v^{-1}}[3,-1,-1]-v^2[1,1,3]-v^2[1,3,-1]$ $-v^2[3,-1,1]-v^3[1,3,1]$
$(3,1,1)$		5	$[-3,-1,1]+\frac{1}{v+v^{-1}}[-3,-1,-1]+v[-1,1,3]+v[1,3,1]$ $-\frac{v^2}{v+v^{-1}}[3,-1,-1]+v^2[1,3,-1]+v^2[3,1,1]+v^3[3,1,-1]$

s	$\mathcal{U}_-$
$(0,0,0)$	$\frac{1}{v+v^{-1}}[3,1,1]$
$(1,-1,0)$	$[1,3,1]+\frac{1}{v+v^{-1}}[3,1,1]$
$(0,0,1)$	$[3,1,-1]+\frac{1}{v+v^{-1}}[3,1,1]$
$(1,-1,1)$	$[1,3,-1]$
$(0,1,1)$	$\frac{1}{v+v^{-1}}[3,-1,-1]$
$(0,1,1)$	$[3,-1,1]+\frac{1}{v+v^{-1}}[3,-1,-1]$
$(2,0,2)$	$[-1,1,3]$
$(3,1,0)$	$[1,-3,-1]+v[1,1,3]+v^2[1,3,-1]+\frac{v^2}{v+v^{-1}}[1,3,1]$
$(3,1,1)$	$\frac{1}{v+v^{-1}}[-3,-1,-1]$
$(3,1,1)$	$[-3,-1,1]+\frac{1}{v+v^{-1}}[-3,-1,-1]$

The basis elements satisfy the conditions:

(1) $(\xi:\xi')=0$, when $\xi,\xi'\in\mathcal{Z}$ correspond to distinct orbits.
(2) $(\xi:\xi')=1+v\mathbb{Z}[v]$, when $\xi,\xi'\in\mathcal{Z}$ correspond to the same orbit.
(3) the change of basis matrix is upper triangular (see below).
(4) $\overline{\mu}-\mu$ is an element of the radical of the bilinear form.
(5) $\mathcal{Z}$ (similarly $\mathcal{U}$) is a basis for $\mathcal{K}/Rad$.
(6) The elements in $\mathcal{U}_+$ which give the two tempered basis elements in $\mathcal{U}_-$ correspond to the orbits $(0,0,0)$ and $(0,0,1)$.

The Lusztig polynomials $c_{\xi,\xi'}$ (coming from the change of basis matrix from $\mathcal{Z}$ to $\mathcal{U}$) are:

$$\left(\begin{array}{cccccccc|cc}
1 & v^2 & v^2 & (v+v^3) & v^3 & v & (v^2+v^4) & v^4 & v^5 & v^3 \\
0 & 1 & 0 & v & 0 & 0 & v^2 & v^2 & v^3 & v \\
0 & 0 & 1 & v & v & 0 & v^2 & v^2 & v^3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & v & v & v^2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & v & 0 & v^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & v & 0 & 0 & v^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & v & v \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

Remark. (Jantzen conjecture) It is natural to conjecture that the polynomials $c_{\xi,\xi'}(v)$ give precisely the levels of the Jantzen filtration.

Note that this would imply the *standard injectivity conjecture* in this case: the generic subquotient must be a submodule of the standard module.

It is known that the generic subquotient is not the *unique* submodule. One can see Tadić's example in the above table (row 7): the induced from the Steinberg on the Siegel parabolic of $SO(7)$ has two submodules at the above χ. (In the Hecke algebra, this is the induced from the Steinberg on a type A_2 subalgebra of the type C_3 Hecke algebra).