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1. Unipotent representations for p-adic $SO(5)$

Let $\mathbf{G}$ be the group $SO(5)$ defined over $\mathbb{Q}_p$, and let $G = \mathbf{G}(\mathbb{Q}_p)$ denote the split form. There is another form, nonsplit, G' in the same inner class with G. In the notation of [Ti], this is $G' = {}^2C_2$ (index $C_{2,1}^{(2)}$), a special unitary group of a quaternionic hermitian form in 2 variables.

The complex dual is $\check{G} = Sp(4,\mathbb{C})$, and let *triv* and *sgn* denote the trivial and the sign character respectively of the center $Z(\check{G})$ of $\check{G}$.

The DLL correspondence for unipotent representations is realized as follows:

$$\begin{aligned}
&\mathrm{Unip}(G) \leftrightarrow \{(s,e,\psi) : s \in \check{G} \text{ semisimple}, e \in \check{\mathfrak{g}} \text{ nilpotent},\ Ad(s)e = pe,\ \psi \in \widehat{A_{\check{G}}(s,e)},\ \psi|_{Z(\check{G})} = triv\}\\
&\mathrm{Unip}(G') \leftrightarrow \{(s,e,\psi) : s \in \check{G} \text{ semisimple}, e \in \check{\mathfrak{g}} \text{ nilpotent},\ Ad(s)e = pe,\ \psi \in \widehat{A_{\check{G}}(s,e)},\ \psi|_{Z(\check{G})} = sgn\}.
\end{aligned} \tag{1.0.1}$$

More precisely, via affine Hecke algebras ([Lu]):

(1) The Iwahori-spherical representations of G are parameterized by the affine Hecke algebra of type $\tilde{B}_2$ with equal parameters: $1 =>= 1 =<= 1$. Geometrically, this Hecke algebra is attached to the trivial local system on the torus of $\check{G}$.

(2) The supercuspidal unipotent representation of G (the so-called θ_{10}) is parameterized by an "empty" Hecke algebra. Geometrically, this Hecke algebra is attached to the sign local system on the principal nilpotent in $Sp(2) \times Sp(2) \subset Sp(4)$. Note that this is not a Levi, but a maximal quasi-Levi (or elliptic) subgroup. This is a general fact: the unipotent supercuspidal representations are parameterized by cuspidal local system on maximal quasi-Levi subgroups (the quasi-Levi is allowed to be $\check{G}$ itself).

(3) The unipotent representations (actually Iwahori-spherical) of G' are parameterized by a Hecke algebra which is the direct sum of two copies of the affine Hecke algebra of type $\tilde{A}_1$ with unequal parameters: $2 -^{\infty} -1$. Geometrically, these two copies are realized from the two $Sp(2)$ inside (the affine Dynkin diagram of) $Sp(4)$, with the cuspidal local system on the regular orbit of $Sp(2)$ as before.

Now, if one restricts to "real infinitesimal character", i.e., s is hyperbolic in (1.0.1), then the correspondences (1) and (3) descend to the graded Hecke algebra. The supercuspidal in (2) doesn't have real infinitesimal character, because it corresponds to a maximal quasi-Levi which is not $\check{G}$ itself, so it is the centralizer of an elliptic nontrivial semisimple element. I think this element has to be $s_e = \mathrm{diag}(i,i,-i,-i)$. (In the reduction to real infinitesimal character, this case belongs to an empty graded Hecke algebra.)

The graded Hecke algebra from (1) is the usual type C_2 one with equal parameters $2 =<= 2$. The graded Hecke algebra from (3) is one of type C_1, with "unequal" parameter 3. Here, one sees the correspondence with the geometry clearly. There are 4 nilpotent $Sp(4)$-orbits in $sp(4)$. Separate their local systems with respect to the character of the center:

$$\begin{aligned}
&triv: \ ((4), triv), ((22), triv, sgn), ((211), triv), ((1^4), triv);\\
&sgn: \ ((4), sgn), ((211), sgn).
\end{aligned} \tag{1.0.2}$$

The first set controls the geometry of the B_2 graded Hecke algebra, while the second set, that of the A_1 unequal parameter.

2. Unipotent representations of p-adic F_4

The group $\mathbf{G}$ of type F_4 is both adjoint and simply-connected. Let G denote the split group. The correspondence via affine Hecke algebras is realized in [Lu] as follows. First, there are seven supercuspidal unipotent representations of G. They are parameterized geometrically by quasi-Levis as follows:

(1) one attached to the unique cuspidal local system on $F_4(a_3)$ in F_4. This is the only one with real infinitesimal character.
(2) one attached to $C_3 \times A_1$ inside F_4. Note that $Sp(6)$ has a unique cuspidal local system suported on the nilpotent (42), and $SL(2)$ has the sign cuspidal local system supported on the principal orbit.
(3) two attached to $A_2 \times A_2$ inside F_4. Note that the group $SL(3)$ has two cuspidal local systems on the principal orbit. (In general, the cuspidal local systems in $SL(n)$ are in one to one correspondence with the relatively prime to n numbers less than n.)
(4) two attached to $A_3 \times A_1$ inside F_4. The group $SL(4)$ has two cuspidal local systems.
(5) one attached to B_4 inside F_4. The group $SO(9)$ has one cuspidal local system on the orbit (531).

Then there are the unipotent representations which appear in the induced from parahoric $P \neq G$, and a supercuspidal on the reductive quotient $\bar{P}$ of P. There is the Iwahori-spherical case, and one more case corresponding to $\bar{P} = SO(5)$ and the supercuspidal θ_{10} on $SO(5)$.

(6) The Iwahori-spherical representations are parameterized by the affine Hecke algebra of type F_4 with equal parameters $1-1-1 =>= 1-1$.
(7) The case when $\bar{P} = SO(5)$ is parameterized by the affine Hecke algebra of type $\tilde{B}_2$ with unequal parameters $3 =>= 3 =<= 1$. Geometrically, this is realized from $A_1 \times A_1$ in F_4 (and the unique cuspidal local systems on the $SL(2)$'s).

If one assumes s is hyperbolic (in (1.0.1)), then one sees in the geometry of the graded Hecke algebra of F_4 only (6) and (1).

All the semisimple elements s corresponding to (7) will have an elliptic part $s_e = \mathrm{diag}(i, i-i, -i) \in Sp(4)$ coming from θ_{10}. The question is what is the centralizer of s_e in F_4. I think it should be a group of type B_4, and then the graded Hecke algebra is the one associated to $A_1 \times A_1$ in B_4. The reason is that when one constructs the graded version of the affine Hecke algebra in (7), I think one should get type B_2 with parameters $4 =>= 3$. This one appears in [Lu1] from $sl(2)^2$ in $so(9)$.

Remark. *One can define a block partition for the category of unipotent representations for p-adic forms inner to the split adjoint form as follows: every block is* *indexed by a $\check{G}$-conjugacy class of pairs $(\check{L}, \mathcal{L})$, where $\check{L}$ is a quasi-Levi subgroup of $\check{G}$, and $\mathcal{L}$ is a cuspidal $\check{L}$-equivariant local system supported on a (necessarily distinguished) nilpotent orbit in $\mathfrak{l}$. There are two extremes: if $\check{L}$ is maximal, then the block is 1×1, and it corresponds to a supercuspidal unipotent representation in the inner class of the split group. If L is minimal, i.e., the split torus (and $\mathcal{L}$ is trivial), then we get the block of the Iwahori-spherical representations of the split p-adic group. (This would be the "big block".) As far as character formulas and Kazhdan-Lusztig polynomials go, no two such blocks interact with each other. Since for an affine Hecke algebra, one has the Harish-Chandra subquotient theorem (and the only principal series is the spherical one), this is the finest possible such block partitioning.*

References

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[Lu1] ———, *Cuspidal local systems I*
[Ti] J. Tits, *Reductive groups over local fields*, Corvallis proceedings, 1979.