Tempered modules in exotic Deligne-Langlands correspondence

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Abstract

The main purpose of this paper is to identify the tempered modules for the affine Hecke algebra of type $C_n^{(1)}$ with arbitrary, non-root of unity, unequal parameters, in the exotic Deligne-Langlands correspondence in the sense of [Ka08a]. This classification has certain applications to the Weyl group structure of the tempered Hecke algebra modules.

Introduction

The main purpose of this paper is to study the tempered modules for the affine Hecke algebra $\mathbb{H}$ of type $C_n^{(1)}$(Definition 1.1) using the exotic nilpotent cones defined in [Ka08a]. Here $\mathbb{H}$ is an algebra over $\mathcal{A} = \mathbb{C}[\mathbf{q}_0^{\pm 1}, \mathbf{q}_0^{\pm 1}, \mathbf{q}_0^{\pm 1}]$, where $\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}$ are three indeterminate parameters. This is the most general affine Hecke algebra with unequal parameters of classical type.

The main case we deal in this paper is the affine Hecke algebra $\mathbb{H}_{n,m}$ of type $B_n^{(1)}$ with unequal parameters (see §1.3). The algebra $\mathbb{H}$ specializes to $\mathbb{H}_{n,m}$ by the specialization $(\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}) \mapsto (-t^{m/2}, t^{m/2}, t)$ with $t \in \mathbb{C}^\times$ and $m > 0$. These algebras, for special values of m, appear as convolution algebras in the theory of p-adic groups. For example, if $m = 1$ or $m = 1/2$ (and an appropriate specialization of t), they correspond to Iwahori-Hecke algebras for split p-adic $SO(2n+1)$ or $PSp(2n)$, respectively. More generally, when $m \in \mathbb{Z} + \epsilon$, where $\epsilon \in \{0, 1/2, 1/4\}$, these algebras appear from representations of p-adic groups with unipotent cuspidal support, in the sense of [Lu95a].

Set $G = Sp(2n, \mathbb{C})$, let T denote its diagonal torus, and let $W = N_G(T)/T$. The algebra $\mathbb{H}$ has a large abelian subalgebra $\mathcal{A} \otimes R(T)$, and the tempered and discrete series $\mathbb{H}$-modules are defined by the Casselman criterion for the generalized $R(T)$-weights (Definition 1.7). By a result of Bernstein and Lusztig, the center of $\mathbb{H}$ is $\mathcal{A} \otimes R(T)^W$. Therefore the central characters of irreducible $\mathbb{H}$-modules are of the form $a = (s, q_0, q_1, q_2)$, where $s \in T$. The $\mathbb{H}$-action on a module with central character a factors through a finite-dimensional algebra $\mathbb{H}_a$. Lusztig [Lu89] have shown that $\mathbb{H}_a$ contains W. We say that the central

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character a is positive real if $-q_0, q_1, q_2 \in \mathbb{R}_{>1}$, and s is hyperbolic (see Definition 1.5).

We are interested in the identification of tempered $\mathbb{H}$-modules for non-root of unity parameters. Via Lusztig's reduction ([Lu89]), this can essentially be reduced to the determination of the tempered modules and discrete series with positive real central character for the case of $\mathbb{H}_{n,m}$.

For the algebra $\mathbb{H}_{n,m}$, we say that the parameter $m > 0$ is *generic* if $m \notin \{1/2, 1, 3/2, 2, \ldots, n-1/2\}$. By [OS08] (also [Lu02] for $m \in \mathbb{Z} + 1/4$), we know that the central characters which allow discrete series in the positive real generic range are in one-to-one correspondence with partitions σ of n (see Definition 3.1).

Example 0.1 ($n = 2$ case). There are two central character which afford discrete series corresponding to $\sigma_1 = (2)$ and $\sigma_2 = (1^2)$. The first discrete series is always the Steinberg module (corresponding to the sgn representation of the affine Weyl group), but the dimension and the W-module structure of the second discrete series depend on m.

We call $s_\sigma \in T$ which afford discrete series distinsguished.

Our main result is the description of parameters corresponding to discrete series $\mathbb{H}_{n,m}$-module within the framework of the exotic Deligne-Langlands correspondence (eDL for short). To be more precise, let us recall it briefly (c.f. §1.3 or [Ka08a]). Let $V_1 = \mathbb{C}^{2n}$ be the vector representation of G, and $V_2 = \wedge^2 V_1$. Set $\mathbb{V} = V_1^{\oplus 2} \oplus V_2$. This is a representation of $\mathcal{G} = G \times (\mathbb{C}^\times)^3$, where G acts diagonally on (V_1, V_1, V_2) and $(c_1, c_2, c_3) \in (\mathbb{C}^\times)^3$ acts by multiplication by $(c_1^{-1}, c_2^{-1}, c_3^{-1})$. The eDL correspondence is stated as a one-to-one correspondence

$$G(a) \backslash \mathbb{V}^a \longleftrightarrow \mathsf{Irrep}\mathbb{H}_a, \quad X \mapsto L_{(a,X)}, \tag{0.1}$$

where $G(a)$ is the centralizer of s in G and $\mathbb{V}^a$ is the set of a-fixed points in $\mathbb{V}$.

There is a combinatorial parameterization of the left hand side in terms of marked partitions (see §1.4). The right hand side is the set of irreducible $\mathbb{H}$-modules with central character a. The irreducible module $L_{(a,X)}$ is obtained as a quotient of a standard geometric module $M_{(a,X)}$. We call (a, X) the eDL parameter of $L_{(a,X)}$. There are some remarks to be made at this point:

1. There are no local systems in this picture: the isotropy group of every X is connected;

2. There exists an exotic Springer correspondence ([Ka08a]). Moreover, the homology groups of all (classical) Springer fibers of $Sp(2n)$ and $SO(2n+1)$ can be realized via the homology of suitable exotic Springer fibers (c.f. Corollary 1.23);

3. For the Hecke algebras which are known to appear in the representation theory of p-adic groups, the Deligne-Langlands-Lusztig correspondence (DLL for short) was established in [KL87, Lu89, Lu95b, Lu02]. The connection between the eDL and DLL correspondences is non-trivial. In particular, the "lowest W-types" of a fixed irreducible module differ between the eDL correspondence and the Langlands classification;

4. In the Langlands classification, we can specify a discrete series by its "lowest W-type". However, our "eDL lowest W-type" does not single out

discrete series. Hence, we need the full eDL parameter in order to specify discrete series.

The third phenomenon gives a restriction on the W-character of discrete series which seems far from trivial (c.f. §4.6).

In §3.2, we give an effective, simple algorithm which produces from a distinguished central character s_σ, an element $X_{\mathsf{out}(\sigma)}$ in $\mathbb{V}^a$.

Theorem 0.2 (=Theorem 3.5)**.** *Let σ be a partition of n. Let $a = (\vec{q}, s_\sigma)$ be the real positive generic central character, where $\vec{q} = (-t^{m/2}, t^{m/2}, t)$, and s_σ is the distinguished semisimple element corresponding to σ. Then $(a, X_{\mathsf{out}(\sigma)})$ is the eDL parameter of the unique discrete series $\mathbb{H}_a$-module.*

We also give a geometric characterization of $\mathsf{out}(\sigma)$ in §3.6, as the minimal orbit with respect to certain conditions.

More generally, one can describe the tempered spectrum of $\mathbb{H}_{n,m}$ with real positive generic parameter as follows.

Theorem 0.3 (=Theorem 4.7 and Corollary 4.18)**.** *If $n_1 + n_2 = n$, we set $\mathbb{H}^S_m := \mathbb{H}^{\mathsf{A}}_{n_1,m} \times \mathbb{H}_{n_2,m} \subset \mathbb{H}_{n,m}$, where $\mathbb{H}^{\mathsf{A}}_{n_1,m}$ is an affine Hecke algebra of $GL(n_1)$.*

(i) Let L_1^{A} and L_2 be a tempered $\mathbb{H}^{\mathsf{A}}_{n_1,m}$-module and a discrete series $\mathbb{H}_{n_2,m}$-module with real positive generic parameter, respectively. Then,

$$L := \mathsf{Ind}^{\mathbb{H}_{n,m}}_{\mathbb{H}^S_m} (L_1^{\mathsf{A}} \boxtimes L_2)$$

is a tempered $\mathbb{H}_{n,m}$-module.

(ii) Every irreducible tempered $\mathbb{H}_{n,m}$-module with real positive generic central character can be realized in a unique way as in (i).

The classification of tempered modules for the Hecke algebras of type A is well-known from [Ze80] and [KL87].

Example 0.4 (Table of discrete series for $n = 2$). Let $\{\epsilon_1 - \epsilon_2, \epsilon_1, \epsilon_2\}$ denote the positive roots (of $SO(5, \mathbb{C})$). Let v_β denote a T-eigenvector of weight β in $\mathbb{V}$. Assume the notation for W-representations in Remark 1.24. We have:

m	$(0, 1/2)$	$(1/2, 1)$	$(1, \infty)$
$X_{\mathsf{out}(\sigma_1)}$	$v_{\epsilon_1-\epsilon_2} + v_{\epsilon_2}$	$v_{\epsilon_1-\epsilon_2} + v_{\epsilon_2}$	$v_{\epsilon_1-\epsilon_2} + v_{\epsilon_2}$
$X_{\mathsf{out}(\sigma_2)}$	$v_{\epsilon_1-\epsilon_2}$	$v_{\epsilon_1-\epsilon_2} + v_{\epsilon_1}$	v_{ϵ_2}
$\mathsf{ds}(\mathsf{out}(\sigma_2))\vert_W$	$(11) \times (0)$	$(1) \times (1) + (0) \times (11)$	$(0) \times (2)$

To transfer our description from generic parameters to special parameters, we prove a continuity result of tempered modules, which is an algebraic analogue of a result of [OS08].

Theorem 0.5 (=Corollary 2.18)**.** *Let $a^t = a \exp(\gamma t)$ be a one-parameter family of positive real central characters depending on $t \in \mathbb{R}_{\geq 0}$ by*

$$\gamma \in \mathfrak{t} \oplus \{0\} \oplus \mathbb{R}^2_{\geq 0} \subset \mathrm{Lie}(T \times (\mathbb{C}^\times)^3).$$

Let $X_t \in \mathbb{V}^{a^t}$, $t \geq 0$, be a family of exotic nilpotent element corresponding to the same marked partition τ. We assume that a^t is generic for all $t \in (-\epsilon, \epsilon) \setminus \{0\}$ for some small $\epsilon > 0$.

(i) The module $L_{(a^0,X_0)}$ *is an irreducible quotient of the both of the two limit modules* $\lim_{t\to 0^\pm} L_{(a^t,X_t)}$.

(ii) The module $L_{(a^0,X_0)}$ *is tempered if* $L_{(a^t,X_t)}$ *is a tempered module in at least one of the regions* $0 < t < \epsilon$ *or* $-\epsilon < t < 0$.

(iii) The module $L_{(a^0,X_0)}$ *is a discrete series if* $L_{(a^t,X_t)}$ *are discrete series for* $t \in (-\epsilon, \epsilon) \setminus \{0\}$.

Remark 0.6. In general, the limit tempered module $\lim_{t\to 0} L_{(a^t,X_t)}$ is reducible. For example, let us consider $\mathbb{H}_{n,m}$ with $n = 2$, and $1/2 < m < 1$. There is one tempered $\mathbb{H}_{2,m}$-module L_m with its W_2-structure $(0)\times(11)+(0)\times(2)+(1)\times(1)$. We have

$$\lim_{m\to 1^-} L_m \cong U_1 \oplus U_2, \quad U_1|_W \cong (0)\times(11) + (1)\times(1), \text{ and } U_2|_W \cong (0)\times(2),$$

where U_1, U_2 are tempered modules of $\mathbb{H}_{2,1}$ (the affine Hecke algebra with equal parameters of type B_2). In the usual DLL correspondence, the tempered modules U_1 and U_2 are parameterized by the same nilpotent adjoint orbit in $\mathfrak{so}(5)$, i.e., they are in the same L-packet.

The result of [OS08] guarantees that every tempered module arises as $L_{(a^0,X_0)}$ via Theorem 0.5. This completes the description of tempered modules in the eDL correspondence. It may be worth mentioning that the classification of [OS08] in the case of $\mathbb{H}_{n,m}$ is basically the same as a conjecture of Slooten [Sl06]. It is also closely related to the classification of the tempered and discrete series $\mathbb{H}_{n,m}$-modules by [KL87, Lu89, Lu95b, Lu02]. We provide a direct link between [Sl06] and (the $Spin(\ell)$-case of) [Lu89] in §4.3 for the sake of completeness.

In addition, we include certain applications of this "exotic" classification for the W-structure of tempered modules in §4.

The organization of this paper is as follows: In §1, we fix notation and recall the basic results. Some of the material (like Corollary 1.23) is new in the sense it was not included in [Ka08a]. In §2, we present various technical lemmas needed in the sequel. Among them, the deepest result is Theorem 2.16, which is essential for the reduction step in the proof of Theorem 0.2. In §3, we formulate and prove our main result, Theorem 0.2. Namely, after recalling some preliminary facts in §3.1, we present our main algorithm $\sigma \to \mathsf{out}(\sigma)$ and state Theorem 0.2 in §3.2. We reduce the proof to the case when σ is "rigid" in §3.3. We analyze the weight distribution of certain special discrete series in §3.4. Then, we use the induction theorem repeatedly to prove that the module $L_{(a_\sigma,X_{\mathsf{out}(\sigma)})}$ must be a discrete series whenever σ is rigid. We also give an alternative characterization of $\mathsf{out}(\sigma)$ in §3.6. The last section §4 concerns various applications: we characterize those discrete series $L_{(a_\sigma,X_{\mathsf{out}(\sigma)})}$ which contain sgn, and analyze their deformations in §4.1. We prove Theorem 0.3 in §4.2. We explain the relation between the view-points of Lusztig and Slooten-Opdam-Solleveld in §4.3. We deduce the W-independence of tempered modules in §4.4, and characterize the tempered $\mathbb{H}_{n,m}$-modules which are irreducible as W-modules. We finish this paper by presenting several constraints on the W-structure of tempered $\mathbb{H}_{n,m}$-module coming from the comparison of the eDL correspondence and the Langlands correspondence of [Lu02].

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1 Preliminaries

1.1 Affine Hecke algebra $\mathbb{H}$

In this paper, G will denote the group $Sp(2n, \mathbb{C})$. Fix a Borel subgroup B, and a maximal torus T in B, and let $W = N_G(T)/T$ be the Weyl group. We denote the character lattice of T by $X^*(T)$. Fix a root system R of (G,T) with positive roots R^+ given by B, and simple roots Π. In coordinates, the roots are

$$R^+ = \{\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{2\epsilon_i\} \subset \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\} = R. \tag{1.1}$$

For every $S \subset \Pi$, we let $P_S = L_S U_S$ denote the unique parabolic subgroup containining B and generated by S.

We set $\mathcal{A}_{\mathbb{Z}} := \mathbb{Z}[\mathbf{q}_0^{\pm 1}, \mathbf{q}_1^{\pm 1}, \mathbf{q}_2^{\pm 1}]$ and $\mathcal{A} := \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}_{\mathbb{Z}} = \mathbb{C}[\mathbf{q}_0^{\pm 1}, \mathbf{q}_1^{\pm 1}, \mathbf{q}_2^{\pm 1}]$.

Definition 1.1 (Hecke algebras of type $C_n^{(1)}$)**.** A Hecke algebra of type $C_n^{(1)}$ with three parameters is an associative algebra $\mathbb{H}_n = \mathbb{H}$ over $\mathcal{A}$ generated by $\{T_i\}_{i=1}^n$ and $\{e^\lambda\}_{\lambda \in X^*(T)}$ subject to the following relations:

(Toric relations) For each $\lambda, \mu \in X^*(T)$, we have $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$ (and $e^0 = 1$);

(The Hecke relations) We have

$$(T_i + 1)(T_i - \mathbf{q}_2) = 0 \ (1 \le i < n) \text{ and } (T_n + 1)(T_n + \mathbf{q}_0\mathbf{q}_1) = 0; \tag{1.2}$$

(The braid relations) We have

$$\begin{aligned} &T_i T_j = T_j T_i \ (\text{if } |i-j| > 1), \\ &(T_n T_{n-1})^2 = (T_{n-1} T_n)^2, \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \ (\text{if } 1 \le i < n-1); \end{aligned} \tag{1.3}$$

(The Bernstein-Lusztig relations) For each $\lambda \in X^*(T)$, we have

$$T_i e^\lambda - e^{s_i \lambda} T_i = \begin{cases} (1 - \mathbf{q}_2) \frac{e^\lambda - e^{s_i\lambda}}{e^{\alpha_i} - 1} & (i \neq n) \\ \frac{(1+\mathbf{q}_0\mathbf{q}_1) - (\mathbf{q}_0 + \mathbf{q}_1)e^{\epsilon_n}}{e^{\alpha_n} - 1}(e^\lambda - e^{s_n\lambda}) & (i = n) \end{cases}. \tag{1.4}$$

Definition 1.2 (Parabolic subalgebras of $\mathbb{H}$)**.** Let $S \subset \Pi$ be a subset. We define $\mathbb{H}^S$ to be the $\mathcal{A}$-subalgebra of $\mathbb{H}$ generated by $\{T_i; \alpha_i \in S\}$ and $\{e^\lambda; \lambda \in X^*(T)\}$. If $S = \Pi - \{n\}$, then we may denote $\mathbb{H}^S = \mathbb{H}_n^S$ by $\mathbb{H}^{\mathsf{A}} = \mathbb{H}_n^{\mathsf{A}}$.

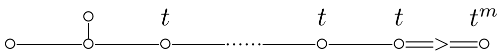
Remark 1.3. **1)** The standard choice of parameters (t_0, t_1, t_n) is: $t_1^2 = \mathbf{q}_2$, $t_n^2 = -\mathbf{q}_0\mathbf{q}_1$, and $t_n(t_0 - t_0^{-1}) = (\mathbf{q}_0 + \mathbf{q}_1)$. This yields

$$T_n e^\lambda - e^{s_n\lambda} T_n = \frac{1 - t_n^2 - t_n(t_0 - t_0^{-1})e^{\epsilon_n}}{e^{2\epsilon_n} - 1}(e^\lambda - e^{s_n\lambda});$$

2) An extended Hecke algebra of type $B_n^{(1)}$ with two-parameters is obtained by requiring $\mathbf{q}_0 + \mathbf{q}_1 = 0$. An equal parameter extended Hecke algebra of type $B_n^{(1)}$

is obtained by requiring $\mathbf{q}_0 + \mathbf{q}_1 = 0$ and $\mathbf{q}_1^2 = \mathbf{q}_2$. An equal parameter Hecke algebra of type $C_n^{(1)}$ is obtained by requiring $\mathbf{q}_2 = -\mathbf{q}_0\mathbf{q}_1$ and $(1+\mathbf{q}_0)(1+\mathbf{q}_1) = 0$;

3) The Hecke algebra with unequal parameters of type $B_n^{(1)}$ with the parameters



is obtained as

$$\mathbb{H}_{n,m} := \mathbb{H}/(\mathbf{q}_0 + t^{m/2}, \mathbf{q}_1 - t^{m/2}, \mathbf{q}_2 - t).$$

Set $\mathcal{G} = G \times (\mathbb{C}^\times)^3$. We recall the following well-known description of the center of $\mathbb{H}$.

Theorem 1.4 (Bernstein-Lusztig [Lu89])**.** *The center of the Hecke algebra* $\mathbb{H}$ *is*

$$Z(\mathbb{H}) \cong \mathcal{A}[e^\lambda; \lambda \in X^*(T)]^W \cong \mathcal{A} \otimes_{R(\mathbb{C}^\times)^3} R(\mathcal{G}). \tag{1.5}$$

A semisimple element $a \in \mathcal{G}$ is a triple $a = (s, \vec{q}) = (s, q_0, q_1, q_2)$, for some $s \in G$ is semisimple, and $\vec{q} \in (\mathbb{C}^\times)^3$. In view of the previous theorem, each semisimple $a \in \mathcal{G}$ defines a character $\mathbb{C}_a$ of $Z(\mathbb{H})$ by taking traces of elements of $R(\mathcal{G})$ at a. We can define therefore the specialized Hecke algebras

$$\mathbb{H}_a := \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H} \text{ and } \mathbb{H}_a^S := \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H}^S, \tag{1.6}$$

where $S \subset \Pi$.

A $\mathbb{H}_a$-module M is said to be a $\mathbb{H}$-module with central character a or s. By Theorem 1.4, having central character s is equivalent to having central character $v \cdot s$ for each $v \in W$. (Here $v \cdot s$ is the action induced by the adjoint action of $N_G(T)$ on T inside G.)

We fix a maximal subset $\mathbb{S} \subset \mathbb{R}_{>0}$ with the following properties:

- We have $q_0^{\pm 1} \notin \mathbb{S}$ and $q_1 \in \mathbb{S}$;
- $\mathbb{S}$ is stable by $q_2^{\mathbb{Z}}$-action;
- We have $\{\xi, \xi^{-1}\} \not\subset \mathbb{S}$ for all $\xi \in \mathbb{R}_{>0}$.

We recall next the geometric construction of irreducible $\mathbb{H}_a$-modules from [Ka08a], under the assumption in the following definition.

Definition 1.5. A semisimple element $a = (s, \vec{q}) = (s, q_0, q_1, q_2)$ in $\mathcal{G}$ is called positive real if $-q_0, q_1, q_2 \in \mathbb{R}_{>1}$ and s has only positive real eigenvalue on V_1. A positive real semisimple element $a = (s, \vec{q}) = (s, q_0, q_1, q_2)$ in $\mathcal{G}$ is called generic if it satisfies:

$$q_1^2 \neq q_2^m \quad (m \neq -2n+1, -2n+2, \cdots, 2n-1). \tag{1.7}$$

The set of semisimple positive real generic elements of $\mathcal{G}$ will be denoted by $\mathcal{G}_0$. We call $a = (s, \vec{q}) \in \mathcal{G}_0$ to be $\mathbb{S}$-positive if we have $\langle s, \epsilon_i \rangle \in \mathbb{S}$ for each $i = 1, \ldots, n$. The set of $\mathbb{S}$-positive elements of $\mathcal{G}_0$ is denoted by $\mathcal{G}_0^+$.

Example 1.6. In the case of the Hecke algebra $\mathbb{H}_{n,m}$ of type $B_n^{(1)}$ (as is defined in Example 1.3 **3)**), if we assume $t \in \mathbb{R}$ and $m \in \mathbb{Z}$, then the condition (1.7) turns into $m \notin \{-n-1/2, -n, \ldots, n-1/2\}$.

We have $R(T) \subset \mathbb{H}$. Thus, we can consider the set of $R(T)$-weights of V for a finite dimensional $\mathbb{H}_a$-module V. We denote it by $\Psi(V) \subset T$. It is well known $\Psi(V) \subset W \cdot s$ whenever $a = (s, \vec{q})$.

Definition 1.7. An $\mathbb{H}_a$-module V is called tempered, if for all $\chi \in \Psi(V)$, one has

$$\langle \chi, \epsilon_1 + \cdots + \epsilon_j \rangle \leq 1, \text{ for all } 1 \leq j \leq n. \tag{1.8}$$

The module V is called a discrete series, if the inequalities in (1.8) are strict.

1.2 Reduction to positive real character

We briefly recall Lusztig's reduction to positive real central character. The original reference is [Lu89] (see also §4 in [Lu02]), and there is a complete treatment relative to the tempered spectrum in [OS08]. In this subsection alone, we denote the Hecke algebra defined in 1.1 by $\mathbb{H}_{R',X'}^{\lambda_0,\lambda_1,\lambda_n}$, where $X' = X^*(T')$, T' is the torus of type B_n (dual to T), R' denotes the roots of type B_n (coroots for R of type C_n), and $t_i = t^{\lambda_i}$ (as in Remark 1.3), where t acts by some element in $\mathbb{R}_{>0}$. Let us also denote by $\mathsf{Irrep}_{W\cdot s,\nu_0}\mathbb{H}_{R',X'}^{\lambda_0,\lambda_1,\lambda_n}$, the irreducible modules on which t acts by ν_0, and the central character is $s \in T'$. (The emphasis on W in the notation will be justified by the reduction procedure.)

Every $s \in T'$ has a unique decomposition $s = s_e \cdot s_h$, into an elliptic part, and a hyperbolic part: $s_e \in S^1 \otimes_{\mathbb{Z}} X_*(T')$, $s_h \in \mathbb{R}_{>0} \otimes_{\mathbb{Z}} X_*(T')$. Note that s_h is the $\mathbb{R}_{>0}$-part, in the sense of Definition 1.5.

Fix a central character $s = s_e \cdot s_h$. Define

$$R'_{s_e} = \left\{ \alpha \in R' : \begin{array}{ll} \alpha(s_e) = 1, & \text{if } \alpha \notin \{\pm\epsilon_n\} \\ \alpha(s_e) = \pm 1, & \text{if } \alpha \in \{\pm\epsilon_n\} \end{array} \right\}. \tag{1.9}$$

Then $R'_{s_e} \subset R'$ is a root subsystem. Let $W_{s_e} \subset W$ denote the subgroup generated by the reflections in the roots of R'_{s_e}.

Definition 1.8. Let $\underline{R}' \subset \underline{E}'$, $\check{\underline{R}}' \subset \check{\underline{E}}'$ be a root system in the usual sense, and denote by $\underline{E}'_{\mathbb{C}}$ and $\check{\underline{E}}'_{\mathbb{C}}$ the complexifications. Let $\Pi' \subset R'$ be a set of simple roots, and W the Coxeter group. Let μ be a W-invariant function on Π. Define the graded (or degenerate) affine Hecke algebra $\overline{\mathbb{H}}^{\mu}_{\underline{R}',\underline{E}'}$ ([Lu89]) to be the associative $\mathbb{C}[r]$-algebra with unity generated by $\{t_w : w \in W\}$, $\omega \in \underline{E}'_{\mathbb{C}}$ subject to the relations:

$$\begin{aligned}
&t_w t'_w = t_{ww'}, \text{ for all } w, w' \in W;\\
&\omega\omega' = \omega'\omega, \text{ for all } \omega, \omega' \in \underline{E}'_{\mathbb{C}};\\
&\omega t_{s_\alpha} - t_{s_\alpha} s_\alpha(\omega) = r\mu(\alpha)\langle\check{\alpha}, \omega\rangle, \text{ for all } \alpha \in \Pi', \omega \in \underline{E}'_{\mathbb{C}}.
\end{aligned}$$

Remark 1.9. The center of $\overline{\mathbb{H}}^{\mu}_{\underline{R}',\underline{E}'}$ is $\mathbb{C}[r] \otimes S(\underline{E}'_{\mathbb{C}})^W$, where $S(\ )$ denotes the symmetric algebra ([Lu89]). Therefore, the central characters of irreducible modules, on which r acts by a certain number, are given by W-conjugacy classes of elements in $\check{\underline{E}}'_{\mathbb{C}}$. Denote by $\mathsf{Irrep}_{W\cdot\underline{s},r_0}\overline{\mathbb{H}}^{\mu}_{\underline{R}',\underline{E}'}$ the class of irreducible modules with central character $\underline{s} \in \check{\underline{E}}'_{\mathbb{C}}$, and on which r acts by r_0.

Fix r_0 such that $e^{r_0} = \nu_0$. Recall that every hyperbolic element $s_h \in T'$ has a unique logarithm $\log s_h \in \mathfrak{t}'$.

Theorem 1.10 ([Lu89]). *There are natural one-to-one correspondences:*

$$\mathsf{Irrep}_{W\cdot s,\nu_0}\mathbb{H}^{\lambda_0,\lambda_1,\lambda_n}_{R',X'} \cong \mathsf{Irrep}_{W_{s_e}\cdot s,\nu_0}\mathbb{H}^{\lambda_0,\lambda_1,\lambda_n}_{R'_{s_e},X'} \cong \mathsf{Irrep}_{W_{s_e}\cdot \log s_h,r_0}\overline{\mathbb{H}}^{\mu_1,\mu_n}_{R'_{s_e},\mathfrak{t}'^*},$$

where $\mu_1 = \lambda_1$, $\mu_n = \lambda_n + \epsilon_n(s_e)\lambda_0$.

These correspondences follow from an isomorphism of certain completions of these Hecke algebras. By applying the theorem again for $\mathsf{Irrep}_{W_{s_e}\cdot s_h,\nu_0}\mathbb{H}^{\mu_n/2,\mu_1,\mu_n/2}_{R'_{s_e},X'}$, we see that:

Corollary 1.11. *There is a natural one-to-one correspondence*

$$\mathsf{Irrep}_{W\cdot s,\nu_0}\mathbb{H}^{\lambda_0,\lambda_1,\lambda_n}_{R',X'} \cong \mathsf{Irrep}_{W'_{s_e}\cdot s_h,\nu_0}\mathbb{H}^{\mu_n/2,\mu_1,\mu_n/2}_{R'_{s_e},X'},$$

where $\mu_1 = \lambda_1$, $\mu_n = \lambda_n + \epsilon_n(s_e)\lambda_0$.

We should mention that, for a general affine Hecke algebra (not necessarily of type $C_n^{(1)}$), the same correspondence as Theorem 1.10 holds, but the affine graded algebra $\overline{\mathbb{H}}$ needs to be replaced with an extension of it by a group of diagram automorphisms.

Corollary 1.12. *In the correspondence of Corollary 1.11, tempered modules and discrete series modules correspond, respectively.*

This is easily seen from the isomorphism of algebra completions we mentioned above. See for example §2 (particularly 2.28) of [OS08].

1.3 Irreducible $\mathbb{H}$-modules

We use the notation of §1.1. In addition, we introduce the following notation. For a group H, an element $h \in H$, and a H-variety $\mathcal{X}$ we denote by $\mathcal{X}^h$ and $\mathcal{X}^H$, the subvariety of h-fixed and H-fixed points in $\mathcal{X}$, respectively. For $x \in \mathcal{X}$, we define $\mathsf{Stab}_H x := \{h \in H; hx = x\}$.

Let $V_1 = \mathbb{C}^{2n}$ denote the vector representation of G. Set $V_2 = \wedge^2 V_1$ and $\mathbb{V} = V_1^{\oplus 2} \oplus V_2$. Then $\mathbb{V}$ is a representation of $\mathcal{G}$ as follows: G acts diagonally, and an element $(c_1, c_2, c_3) \in (\mathbb{C}^\times)^3$ acts on (V_1, V_1, V_2) via multiplication by $(c_1^{-1}, c_2^{-1}, c_3^{-1})$.

For each weight $0 \neq \beta \in X^*(T)$, we set v_β to be a non-zero T-eigenvector of $\{0\} \oplus V_1 \oplus V_2 \subset \mathbb{V}$, which is unique up to scalar.

We denote by $\mathbb{V}^+$ the sum of B-positive T-weight spaces in $\mathbb{V}$. For each $S \subset \Pi$, we will denote by $\mathbb{V}_S$ the sum of T-weight spaces for the weights in the $\mathbb{Q}$-span of S. We define the collapsing map (an analogue of the moment map)

$$\mu : F := G \times^B \mathbb{V}^+ \longrightarrow \mathbb{V}, \quad \mu(g, v^+) = g \cdot v^+,\ g \in G,\ v^+ \in \mathbb{V}^+, \tag{1.10}$$

and denote the image of μ by $\mathfrak{N}$. We call it the exotic nilpotent cone ([Ka08a]). For each $a = (s, \vec{q})$, we denote by μ^a the restriction of μ to the a-fixed points of F. We denote by $G(s)$ or $G(a)$ the centralizer $Z_G(s)$. This is a connected (reductive) subgroup, since G is simply connected. Its action on $\mathfrak{N}^a$ has finitely many orbits. We will describe this in more detail in §1.4.

Let $\mathsf{pr}_B : F \to G/B$ be the projection $\mathsf{pr}_B(g, v^+) = gB$. We define

$$\mathcal{E}^a_X := \mathsf{pr}_B(\mu^{-1}(X)^a) \subset G/B, \tag{1.11}$$

and call it an exotic Springer fiber.

Definition 1.13 (Standard module)**.** Let $a \in \mathcal{G}_0$ and let $X \in \mathfrak{N}^a$. The total Borel-Moore homology space

$$M_{(a,X)} := \bigoplus_{m \geq 0} H_m(\mathcal{E}_X^a, \mathbb{C}) \tag{1.12}$$

admits a structure of finite dimensional $\mathbb{H}_a$-module. We call this module a standard module. Fix $S \subset \Pi$. If $a \in L_S \times (\mathbb{C}^\times)^3$ and $X \in \mathbb{V}_S$, then

$$M_{(a,X)}^S := \bigoplus_{m \geq 0} H_m(\mathcal{E}_X^a \cap P_S/B, \mathbb{C}) \tag{1.13}$$

admits a $\mathbb{H}^S$-module structure.
If $S = \Pi - \{n\}$, then we may denote $M_{(a,X)}^S$ by $M_{(a,X)}^{\mathsf{A}}$.

Let $\mathbb{V}^S$ be the unique T-equivariant splitting of the map $\mathbb{V}^+ \longrightarrow \mathbb{V}^+/(\mathbb{V}^+ \cap \mathbb{V}_S)$. If $X \in \mathbb{V}_S$, then we have necessarily $\mathfrak{u}_S X \subset \mathbb{V}^S$. The induction theorem is the following:

Theorem 1.14 ([Ka08a] Theorem 7.4)**.** *Let $S \subset \Pi$. Let $a = (s, \vec{q}) \in \mathcal{G}_0$ and let $X \in \mathfrak{N}^a$. Assume $s \in L_S$ and $X \in (\mathfrak{N} \cap \mathbb{V}_S)$. If we have*

$$(\mathbb{V}^S)^a \subset \mathfrak{u}_S X, \tag{1.14}$$

then we have an isomorphism

$$\mathsf{Ind}_{\mathbb{H}^S}^{\mathbb{H}} M_{(a,X)}^S \cong M_{(a,X)}$$

as $\mathbb{H}$-modules. □

Fix the semisimple element $a_0 = (1, -1, 1, 1) \in \mathcal{G}$. The following result is an exotic version of the well-known Springer correspondence.

Theorem 1.15 ([Ka08a] Theorem 8.3)**.** *Let $X \in \mathfrak{N}^{a_0}$. Then, the space*

$$L_X := H_{2d_X}(\mathcal{E}_X^{a_0}, \mathbb{C}), \text{ where } d_X := \dim \mathcal{E}_X^{a_0} \tag{1.15}$$

admits a structure of irreducible W-module.

Moreover the map $X \mapsto L_X$ defines a one-to-one correspondence between the set of orbits $G \backslash \mathfrak{N}^{a_0}$ and $\mathsf{Irrep}\ W$.

In this correspondence, if X is in the open dense $G(a_0)$-orbit of $\mathfrak{N}^{a_0}$ then L_X is the sgn W-representation. If $X = 0$, then L_0 is the triv W-representation.

In the following proposition, $Z = F \times_{\mathfrak{N}} F$ denotes the exotic Steinberg variety, and $H_\bullet^A(\bullet)$ is equivariant (Borel-Moore) homology with respect to the group A.

Proposition 1.16 ([Ka08a] Theorem 9.2)**.** *Let $a = (s, q_0, q_1, q_2) \in \mathcal{G}$ be a real positive parameter. Let $\underline{a} := (\log s, r_1, r_2) \in \mathfrak{t} \oplus \mathbb{R}^2$ be an element such that*

$$a = (\exp(\log s), q_0, e^{r_1}, e^{r_2}).$$

Let A be a connected subtorus of $\mathcal{G}$ which contains $(s, 1, q_1, q_2)$, and let $\mathfrak{a}$ be its Lie algebra. If $X \in \mathfrak{N}^A$, then $H_\bullet^A(Z^{a_0})$ acquires a structure of a $\mathbb{C}[\mathfrak{a}]$-algebra which we denote by $\mathbb{H}_a^+$. We have:

1. *The quotient of $\mathbb{H}_a^+$ by the ideal generated by functions of $\mathbb{C}[\mathfrak{a}]$ which are zero along $\underline{a}$ is isomorphic to $\mathbb{H}_a$;*

2. *We have a natural inclusion $\mathbb{C}[W] \hookrightarrow \mathbb{H}_a^+$.*

Moreover, we have

$$\mathbb{C}[\mathfrak{a}] \otimes H_\bullet(\mathcal{E}_X) \cong H_\bullet^A(\mathcal{E}_X) \text{ for each } X \in \mathfrak{N}^A$$

as a compatible $(\mathbb{C}[W], \mathbb{C}[\mathfrak{a}])$-module, where W acts on $\mathbb{C}[\mathfrak{a}]$ trivially. □

Corollary 1.17. *Keep the setting of Proposition 1.16. We have*

$$M_{(a,X)} \cong \bigoplus_{m \geq 0} H_m(\mathcal{E}_X^{a_0}, \mathbb{C}). \tag{1.16}$$

as $\mathbb{C}[W]$-modules. □

In (1.16), the irreducible W-module L_X appears exactly once in the decomposition of $M_{(a,X)}$. So there is a unique irreducible subquotient, denoted $L_{(a,X)}$ which contains L_X.

Theorem 1.18 ([Ka08a] Theorem 10.2)**.** *Let $a = (s, \vec{q}) \in \mathcal{G}$, be a positive real element so that $\mathfrak{N}^a \subset \mathfrak{N}^{a_0}$. Then, we have a one-to-one correspondence*

$$G(a) \backslash \mathbb{V}^a \leftrightarrow \mathsf{Irrep}\ \mathbb{H}_a, \qquad X \mapsto L_{(a,X)}. \tag{1.17}$$

The module $L_{(a,X)}$ is a $\mathbb{H}_a$-module quotient of $M_{(a,X)}$. Moreover, if $L_{(a,Y)}$ appears in $M_{(a,X)}$, then we have $X \in \overline{G(a)Y}$.

We need to emphasize that, unlike the case of [KL87], there are no nontrivial local systems appearing in the parameterization of $\mathsf{Irrep}\ \mathbb{H}_a$ in Theorem 1.18 for $a \in \mathcal{G}_0$ (see also Corollary 1.33).

Corollary 1.19. *If $a \in \mathcal{G}$ is a positive real element, the set of $G(a)$-orbits of $\mathfrak{N}^a$ is finite. In particular, there exists a unique dense open $G(a)$-orbit $\mathcal{O}_0^a$ of $\mathfrak{N}^a$.*

It is useful to remark that $L_{X_0} \cong \mathsf{sgn}$ ($X_0 \in \mathcal{O}_0^{a_0}$) appears with multiplicity one in every standard module $M_{(a,X)}$.

From Theorem 1.18, together with the DLL correspondence of type A, we deduce:

Theorem 1.20. *Keep the same setting as Theorem 1.18. Let $S \subset \Pi$ and assume $s \in L_S$. Then, we have a one-to-one correspondence*

$$L(s) \backslash \mathbb{V}_S^a \leftrightarrow \mathsf{Irrep}\ \mathbb{H}_a^S, \qquad X \mapsto L_{(a,X)}^S. \tag{1.18}$$

The module $L_{(a,X)}^S$ is a $\mathbb{H}_a^S$-module quotient of $M_{(a,X)}^S$. Moreover, if $L_{(a,Y)}^S$ appears in $M_{(a,X)}^S$, then we have $X \in \overline{L_S(a)Y}$.

Convention 1.21. Assume that $a \in T \times (\mathbb{C}^\times)^3$. If $S = \Pi - \{n\}$, then we set $L_{(a,X)}^{\mathsf{A}} = L_{(a,X)}^S$. Let ${}^{\mathsf{t}}a = (w_0^{\mathsf{A}} \cdot s^{-1}, \vec{q})$, where $w_0^{\mathsf{A}} \in \mathfrak{S}_n \subset W$ is the longest element. Since $X \in \mathfrak{gl}(n)$, we have ${}^{\mathsf{t}}aX = X$. As a consequence, the $\mathbb{H}^{\mathsf{A}}$-modules

$${}^{\mathsf{t}}M_{(a,X)}^{\mathsf{A}} := M_{({}^{\mathsf{t}}a,X)}^{\mathsf{A}} \text{ and } {}^{\mathsf{t}}L_{(a,X)}^{\mathsf{A}} := L_{({}^{\mathsf{t}}a,X)}^{\mathsf{A}}$$

are well-defined.

The following results are not explicitly stated in [Ka08a], but are immediate consequences.

Theorem 1.22. *Let $a \in \mathcal{G}_0$. Let L be an irreducible $\mathbb{H}_a$-module which contains* sgn *as its W-module constituent. Then, we have necessarily $L \cong L_{(a,X)} = M_{(a,X)}$ for $X \in \mathcal{O}_0^a$.*

Proof. Taking account into the fact that the stabilizer of X in G is connected and $\mathfrak{N}^a$ is a vector space (i.e. a smooth algebraic variety), the assertion follows by exactly the same argument as in [EM97]. $\square$

Corollary 1.23. *Let $G' = Sp(2n, \mathbb{C})$ or $SO(2n+1, \mathbb{C})$ and let $\mathfrak{g}'$ denote its Lie algebra. Let $Y \in \mathfrak{g}'$ be a nilpotent element. Let A_Y be the component group of the stabilizer of G'-action on Y. Let $\mathcal{B}_Y$ be the Springer fiber of Y. Then, there exists $X \in \mathfrak{N}^{a_0}$ such that*

$$H_\bullet(\mathcal{B}_Y)^{A_Y} \cong H_\bullet(\mathcal{E}_X)$$

as W-modules (without grading).

Proof. Let $\mathbb{H}^{G'}$ denote the one-parameter affine Hecke algebra coming from $T^*(G/B)$ in the sense of the DLL correspondence. Let $G'_{>0}$ denote the set of points of G' which acts on the natural representation with only positive real eigenvalues. The Bala-Carter theory implies that there exists a semi-simple element $a_Y \in G'_{>0} \times \mathbb{R}_{>0}$ so that $G'(a_Y)Y$ defines a open dense subset of ${\mathfrak{g}'}^{a_Y}$. By an argument of Lusztig [Lu95b], we deduce that $H_\bullet(\mathcal{B}_Y^{s_Y})^{A_Y} \cong H_\bullet(\mathcal{B}_Y)^{A_Y}$ acquires a structure of irreducible $\mathbb{H}^G$-module. Here we know that $H_0(\mathcal{B}_Y)^{A_Y} = H_0(\mathcal{B}_Y) = \mathsf{sgn}$ as $\mathbb{C}[W]$-modules. It is easy to verify that we have some positive real element $a \in \mathcal{G}$ such that $\mathbb{H}^{G'}_{a_Y} \cong \mathbb{H}_a$, where $\mathbb{H}^{G'}_{a_Y}$ is the specialized Hecke algebra of $\mathbb{H}^{G'}$ defined in a parallel fashion to (1.6). Let $X \in \mathcal{O}_0^a$. Then, the module $M_{(a,X)} = L_{(a,X)}$ is the unique $\mathbb{H}_a$-module which contains sgn as W-modules. This forces

$$M_{(a,X)} \cong H_\bullet(\mathcal{B}_Y)^{A_Y}$$

as W-modules (without grading). Therefore, Corollary 1.16 implies the result. $\square$

Remark 1.24. Before presenting an example of the correspondence in Corollary 1.23, let us recall the parameterization of $\mathsf{Irrep}\ W_n$ in terms of bipartitions. Recall that $W_n \cong \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. Let $\xi = (\underbrace{\mathsf{triv}, \ldots, \mathsf{triv}}_{k}, \underbrace{\mathsf{sgn}, \ldots, \mathsf{sgn}}_{n-k})$ be a character of $(\mathbb{Z}/2\mathbb{Z})^n$, and let $\mathfrak{S}_k \times \mathfrak{S}_{n-k} = \mathsf{Stab}_{\mathfrak{S}_n}(\xi)$. The representations of symmetric groups are parameterized by partitions. Let α be a partition of k and β be a partition of $n-k$, and (α), (β) the corresponding representations of $\mathfrak{S}_k$ and $\mathfrak{S}_{n-k}$, respectively. We denote by $(\alpha) \times (\beta)$ (and call it a bipartition) the irreducible representation of W_n obtained by induction from $(\alpha) \boxtimes (\beta) \boxtimes \xi$ on $\mathfrak{S}_k \times \mathfrak{S}_{n-k} \times (\mathbb{Z}/2\mathbb{Z})^n$. All elements of $\mathsf{Irrep}\ W_n$ are obtained by this procedure, hence the one-to-one correspondence with bipartitions of n.

Example 1.25. We explain 1.23 in the case $n = 3$. There are 7 nilpotent adjoint orbits for $SO(7, \mathbb{C})$, and 8 for $Sp(6, \mathbb{C})$. There are 10 exotic orbits (the same as the number of irreducible representations for W_3). Let us denote the representatives of these 10 orbits as follows:

X_1	$v_{\epsilon_1-\epsilon_2}+v_{\epsilon_2-\epsilon_3}+v_{\epsilon_3}$	$(0)\times(1^3)$
X_2	$v_{\epsilon_1-\epsilon_2}+v_{\epsilon_2-\epsilon_3}+v_{\epsilon_2}$	$(1)\times(11)+(0)\times(1^3)$
X_3	$v_{\epsilon_1-\epsilon_2}+v_{\epsilon_2-\epsilon_3}+v_{\epsilon_1}$	$(11)\times(1)+(1)\times(11)+(0)\times(1^3)$
X_4	$v_{\epsilon_1-\epsilon_2}+v_{\epsilon_2}$	$\mathsf{Ind}_{W_2}^{W_3}((0)\times(11))$
X_5	$v_{\epsilon_1-\epsilon_2}+v_{\epsilon_1}$	$\mathsf{Ind}_{W_2}^{W_3}((1)\times(1)+(0)\times(11))$
X_6	$v_{\epsilon_1-\epsilon_2}+v_{\epsilon_2-\epsilon_3}$	$\mathsf{Ind}_{W(A_2)}^{W_3}(\mathsf{sgn})$
X_7	$v_{\epsilon_1-\epsilon_2}+v_{\epsilon_3}$	$\mathsf{Ind}_{W_1\times W(A_1)}^{W_3}((0)\times(1)\boxtimes\mathsf{sgn})$
X_8	v_{ϵ_1}	$\mathsf{Ind}_{W_1}^{W_3}((0)\times(1))$
X_9	$v_{\epsilon_1-\epsilon_2}$	$\mathsf{Ind}_{W(A_1)}^{W_3}(\mathsf{sgn})$
X_{10}	0	$\mathsf{Ind}_{\{1\}}^{W_3}((0))$

The last column gives the W_3-structure of $H_\bullet(\mathcal{E}_X)$ in every case. With this notation, the correspondences from 1.23 are as follows:

$SO(7)$	(7)	(511)	(331)	(322)	(31^4)	(221^3)	(1^7)
	X_1	X_2	X_3	X_7	X_5	X_9	X_{10}

$Sp(6)$	(6)	(42)	(411)	(33)	(222)	(2211)	(21^4)	(1^6)
	X_1	X_2	X_4	X_6	X_7	X_5	X_8	X_{10}

The notation for the parameterization for classical nilpotent orbits is as [Ca85].

1.4 A parameterization of exotic orbits $G(a)\backslash\mathfrak{N}^a$

We recall the combinatorial parameterization of $G(a)$-orbits in $\mathfrak{N}^a$ following [Ka08a]. Fix $a=(s,\vec{q})\in\mathcal{G}_0^+$.

Definition 1.26 (Marked partition)**.** A segment adapted to a is a subset $I\subset[1,n]$ such that for every $i\in I$, we have either

- there exists no $j\in I$ such that $\langle s,\epsilon_i\rangle>\langle s,\epsilon_j\rangle$;
- there exists a unique $j\in I$ such that $\langle s,\epsilon_i\rangle=q_2\langle s,\epsilon_j\rangle$.

A marked partition adapted to a is a pair $(\{I_m\}_{m=1}^k,\delta)$ where

1. $I_1\sqcup I_2\sqcup\cdots\sqcup I_k=[1,n]$ is a division of the set of integers $[1,n]$ into a union of segments;
2. $\delta:[1,n]\to\{0,1\}$ such that $\delta(l)=0$ whenever $\langle s,\epsilon_l\rangle\neq q_1$.

We refer $\{I_m\}_{m=1}^k$ as the support of $(\{I_m\}_{m=1}^k,\delta)$.

Let us denote by $\mathsf{MP}(a)$ the set of marked partitions adapted to a.

Proposition 1.27 ([Ka08a])**.** *The map $\Upsilon:\ \mathsf{MP}(a)\to G(a)\backslash\mathfrak{N}^a$, given by*

$$(\{I_m\}_{m=1}^k,\delta)\mapsto\sum_{m=1}^k v_{I_m}+\sum_{i=1}^n\delta(i)v_{\epsilon_i},\quad\textit{where } v_{I_m}=\sum_{i,j\in I_m,\langle s,\epsilon_i\rangle=q_2\langle s,\epsilon_j\rangle}v_{\epsilon_i-\epsilon_j},\tag{1.19}$$

is a surjection.

For each $\tau \in \mathsf{MP}(a)$, we put $v_\tau := \Upsilon(\tau)$ and $\mathcal{O}_\tau := G(a)v_\tau$.

In order to describe the fibers of the map Υ, we define first a partial order on the set of subsets of $[1,n]$. Let $I, I' \subset [1,n]$. Set $\underline{I} = \{\langle s, \epsilon_i \rangle : i \in I\}$, and similarly for $\underline{J}$. Then we define

$$I \lhd I' \iff \min \underline{I} \le \min \underline{I'} \le \max \underline{I} \le \max \underline{J}. \tag{1.20}$$

If $I \lhd I'$, we say that I is dominated by I'.

If $(\{I_m\}_{m=1}^k, \delta) \in \mathsf{MP}(a)$ is given, we define $(\{I_m\}_{m=1}^k, \widetilde{\delta})$ by modifying δ as follows. If some i such that $\langle s, \epsilon_i \rangle = q_1$ belongs to an I_m which is dominated by a marked $I_{m'}$ (i.e., $\delta(I_{m'}) = \{0,1\}$), then we set $\widetilde{\delta}(i) = 1$ (i.e., we mark I_m as well).

A permutation $w \in \mathfrak{S}_n$ is said to be adapted to a if we have

$$\langle s, \epsilon_i \rangle = \left\langle s, \epsilon_{w(i)} \right\rangle$$

for $i = 1, \ldots, n$. Let $\mathfrak{S}_n^a$ denote the subgroup of $\mathfrak{S}_n$ formed by permutations adapted to a.

It is straight-forward that $\mathfrak{S}_n^a$ acts on $\mathsf{MP}(a)$ by applying w to $(\{I_m\}_{m=1}^k, \delta) \in \mathsf{MP}(a)$ as $(\{I_m\}_{m=1}^k, \delta) \mapsto (\{w(I_m)\}_{m=1}^k, w^*\delta)$, where $(w^*\delta)(i) = \delta(w(i))$.

Proposition 1.28 ([Ka08a]). *Let Υ be the map defined in Proposition 1.27. Then $\Upsilon((\{I_m\}_{m=1}^k, \delta)) = \Upsilon((\{I'_m\}_{m=1}^{k'}, \delta'))$ if and only if $\{I_m\}_{m=1}^k = \{I'_m\}_{m=1}^{k'}$ and $\widetilde{\delta} = \widetilde{\delta}'$ up to $\mathfrak{S}_n^a$-action.*

The marked partition corresponding to the open $G(a)$-orbit in $\mathfrak{N}^a$ is obtained by extracting the longest possible I_1 subject to $av_I = v_I$, then the longest possible I_2 from $[1,n] \setminus I_1$ subject to the same condition etc. Then we mark all I_j such that $q_1 \in \underline{I_j}$.

Let $\mathsf{MP}_0(a)$ be the set of marked partitions with trivial markings. (I.e. $\tau = (\mathbf{I}, \delta) \in \mathsf{MP}(a)$ with $\delta \equiv 0$.) The following result is a re-interpretation of the closure relations of type A quiver orbits with uniform orientation.

Theorem 1.29 ([AD80],[Ze81]). *Let $\tau = (\mathbf{I}, 0) \in \mathsf{MP}_0(a)$. Let $\tau' = (\mathbf{I}', 0) \in \mathsf{MP}_0(a)$ be obtained from τ by the following procedure:*

($\spadesuit$) *For some two segments $I_l, I_m \in \mathbf{I}$ such that*

$$\min \underline{I_l} < \min \underline{I_m} \le \max \underline{I_l} < \max \underline{I_m},$$

we define $\mathbf{I}'$ to be the set of segments obtained from $\mathbf{I}$ by replacing $\{I_k, I_l\}$ with $\{I^+, I^-\}$, where $I^\pm$ are segments such that $I^+ \sqcup I^- = I_l \sqcup I_m$, $\underline{I^+} = \underline{I_l} \cup \underline{I_m}$, and $\underline{I^-} = \underline{I_l} \cap \underline{I_m}$.

Then, we have $\mathcal{O}_\tau \subset \overline{\mathcal{O}_{\tau'}}$. Moreover, every $G(a)$-orbit which is larger than $\mathcal{O}_\tau$ and parameterized by $\mathsf{MP}_0(a)$ is obtained by a successive application of the procedure ($\spadesuit$).

Convention 1.30. For each $\tau \in \mathsf{MP}(a)$, we sometimes denote $L_{(a,v_\tau)}$ by $L_{(a,\tau)}$ or just L_τ when the central character is clear. We may use similar notation like $M_{(a,\tau)}$ or M_τ.

Each of $\tau = (\{I_m\}, \delta) \in \mathsf{MP}_0(a)$ defines a representation R_τ of type A-quiver corresponding to the multisegments $\{\underline{I_m}; m \geq 0\}$ in the sense of Zelevinsky. (Here we identify $G(a) \circlearrowright V_2^{(s,q_2)}$ with the representation space of type A-quiver of appropriate dimension vector.) In particular, R_τ is a direct sum of indecomposable modules R_{I_m} corresponding to the segment I_m (or rather $\underline{I_m}$).

Lemma 1.31. *Let $\tau = (\{I_m\}, \delta) \in \mathsf{MP}_0(a)$. We have a non-zero map $R_{I_m} \to R_{I_{m'}}$ (as modules of type A-quivers) if and only if $I_m \lhd I_{m'}$. Moreover, such a non-zero map is unique up to scalar.*

Proof. Straight-forward. $\square$

Theorem 1.32 (c.f. Brion [Br08] Proposition 2.29)**.** *Let $a \in \mathcal{G}_0^+$ and let $\tau = (\mathbf{I}, 0) \in \mathsf{MP}_0(a)$. The group of automorphisms of R_τ as type A-quiver representation is isomorphic to $\mathsf{Stab}_{G(a)} v_\tau$.* $\square$

Corollary 1.33. *Keep the setting of Theorem 1.32. Let r_τ be the number of segments of $\mathbf{I}$, and let u_τ be the set of distinct pairs of segments I, I' in $\mathbf{I}$ such that $I \lhd I$. Then, $\mathsf{Stab}_{G(a)} v_\tau$ is a connected algebraic group of rank r_τ and dimension $(r_\tau + u_\tau)$.*

Proof. Taking account into Theorem 1.32, the assertion is a straight-forward corollary of Lemma 1.31. $\square$

2 Some weight calculations

2.1 Varieties corresponding to weight spaces

In this section, we use the language of perverse sheaves (corresponding to middle perversity) on complex algebraic varieties. Some of the standard references for this theory are Beilinson-Bernstein-Deligne [BBD], Kashiwara-Schapiro [KS90], Gelfand-Manin [GM94], and Hotta-Tanisaki [HT08].

For a variety $\mathcal{X}$, we denote by $\underline{\mathbb{C}}$ the constant sheaf (shifted by $\dim \mathcal{X}$). For a locally closed subset $\mathcal{O} \subset \mathcal{X}$, we have an embedding $j_{\mathcal{O}} : \mathcal{O} \to \mathcal{X}$. We have a constant sheaf $(j_{\mathcal{O}})_! \underline{\mathbb{C}}$ obtained by extending the constant sheaf on $\mathcal{O}$ by zero to $\mathcal{X}$. We have an intermediate extension object $\mathsf{IC}(\mathcal{O})$ which admits a morphism $(j_{\mathcal{O}})_! \underline{\mathbb{C}} \to \mathsf{IC}(\mathcal{O})$.

Let us present one technical lemma, which is a straight-forward consequence of the general theory. We will need this result later, in the proof of Theorem 2.16.

Lemma 2.1. *Let $\mathcal{X}$ be a smooth variety with a projective morphism $f : \mathcal{X} \to \mathcal{Y}$. Let $\mathcal{U} \subset \mathcal{Y}$ be a Zariski open subset. Let f' be the restriction of f to $f^{-1}(\mathcal{U})$. If $\mathsf{IC}(\mathcal{O})|_{\mathcal{U}}$ is a simple perverse sheaf on $\mathcal{U}$ which appears as a direct factor of $f'_* \underline{\mathbb{C}}$ (up to degree shift), then $\mathsf{IC}(\mathcal{O})$ appears in $f_* \underline{\mathbb{C}}$ (up to degree shift).*

Proof. Both maps f, f' are projective over the base. The Beilinson-Bernstein-Deligne-Gabber decomposition theorem asserts that both $f_* \underline{\mathbb{C}}$ and $f'_* \underline{\mathbb{C}}$ decompose into direct sums of simple perverse sheaves on $\mathcal{Y}$ or $\mathcal{U}$, respectively. Since $\mathcal{U}$ is open in $\mathcal{Y}$, the intersection $\mathcal{O} \cap \mathcal{U}$ is open in $\mathcal{O}$ if $\mathcal{O} \cap \mathcal{U} \neq \emptyset$. Since a simple perverse sheaf can be written as an extension of a local system on a smooth locally closed subset, we have $\mathsf{IC}(\mathcal{O}) = \mathsf{IC}(\mathcal{O} \cap \mathcal{U})$ whenever $\mathcal{O} \cap \mathcal{U} \neq \emptyset$. The complex

$\mathsf{IC}(\mathcal{O} \cap \mathcal{U})$ is the unique simple object (in the category of perverse sheaves on $\mathcal{Y}$) supported on $\overline{\mathcal{O}}$ such that $\mathsf{IC}(\mathcal{O} \cap \mathcal{U})|_{\mathcal{O} \cap \mathcal{U}}$ is the constant sheaf. Therefore, we conclude that the simple constituent $\mathcal{L}$ of $f_* \underline{\mathbb{C}}$ such that $\mathcal{L}|_{\mathcal{U}} \cong \mathsf{IC}(\mathcal{O})|_{\mathcal{U}}$ must be $\mathsf{IC}(\mathcal{O})$ itself as desired. $\square$

For every $w \in W$, let $\dot{w}$ denote a representative in $N_G(T)$. We put ${}^w\mathbb{V}^+ := \dot{w}^{-1}\mathbb{V}^+$ and ${}^w\mathbb{V}(a) := \mathbb{V}^a \cap {}^w\mathbb{V}^+$. We denote $(\mathrm{Ad}(\dot{w}^{-1})B)(s)$ by ${}^wB(s)$. It is clear that these definitions do not depend on the choice of $\dot{w}$. Recall the restriction of the collapsing map $\mu^a : F^a \to \mathfrak{N}^a$. Set

$$F^a_w = G(s) \times^{{}^wB(s)} {}^w\mathbb{V}(a). \tag{2.1}$$

Let W_s be the reflection subgroup of W corresponding to the subroot system of R defined by the roots α such that $\langle s, \alpha \rangle = 1$. Following Lemma 3.6 in [Ka08a], we have a decomposition

$$F^a = \sqcup_{w \in W/W_s} F^a_w. \tag{2.2}$$

Denote by μ^a_w the restriction of μ^a to a piece F^a_w, where w is a representative in W/W_s.

For each $y = w \cdot s^{-1} \in W \cdot s^{-1}$, let $\mathcal{E}^a_X[y]$ denote the preimage of of $X \in \mathfrak{N}^a$ under μ^a_w, projected to G/B:

$$\mathcal{E}^a_X[y] = \{g\dot{w}^{-1}B;\ gs = sg, X \in g^w\mathbb{V}(a)\} \tag{2.3}$$

Notice that replacing w by ww' ($w' \in W_s$) in this construction gives the same variety, hence $\mathcal{E}^a_X[y]$ only depends on $w \in W/W_s$.

Proposition 2.2. *Let $\tau \in \mathsf{MP}(a)$. For $w \in W/W_s$, the $y = (w \cdot s^{-1})$-weight space of the standard module $M_{(a,v_\tau)}$ is $H_\bullet(\mathcal{E}^a_{v_\tau}[y])$. In particular, y is a weight of $M_{(a,v_\tau)}$ if and only if $(\mu^a_w)_*\underline{\mathbb{C}}$ contains $\mathsf{IC}(\mathcal{O}_\tau)$.*

Proof. See Chriss-Ginzburg [CG97] §8.1. $\square$

An important consequence is that we can characterize certain weights of $L_{(a,v_\tau)}$.

Corollary 2.3. *If $\mathcal{O}_\tau$ meets ${}^w\mathbb{V}(a)$ in a dense open subset, then $w \cdot s^{-1}$ is a weight of $L_{(a,v_\tau)}$.*

Proof. We have $(\mu^a_w)^{-1}(X) \neq \emptyset$ if and only if $X \in \overline{\mathcal{O}_\tau}$. Hence, we have $\dim H_\bullet((\mu^a_w)^{-1}(X)) \neq 0$ when $X \in \mathcal{O}_\tau$. It follows that there exist a simple constituent of $(\mu^a_w)_*\underline{\mathbb{C}}$ in $D^b(\mathfrak{N}^a)$ which has support contained in $\overline{\mathcal{O}_\tau}$. By the BBD-Gabber theorem and [Ka08a] 4.10, we have $\mathsf{IC}(\mathcal{O}_\tau)$ as a direct summand of $(\mu^a_w)_*\underline{\mathbb{C}}$ (up to degree shift). Now Ginzburg theory (see [CG97] §8.7) implies the result. $\square$

2.2 Special weights

Fix $a \in \mathcal{G}^+_0$ and let $\tau = (\mathbf{I}, \delta) \in \mathsf{MP}(a)$.

Construction 2.4. We divide $\mathbf{I}$ into four sets $D^1_+, D^2_+, D^1_-, D^2_-$ as follows:

- If $\max \underline{I} < q_1$, we put $I \in \mathbf{I}$ into D^2_+;
- If $\min \underline{I} > q_1$, we put $I \in \mathbf{I}$ into D^2_-;

Note that all segments I in $D_+^2 \cup D_-^2$ are unmarked, since $q_1 \notin \underline{I}$. Now we consider only segments in $\mathbf{I} \setminus (D_+^2 \cup D_-^2)$.

- If there exists some I' such that $\delta(I') = \{0,1\}$ (i.e., I' is marked) and $I \lhd I'$, then we put I into D_+^1;
- If we have $\delta(I) = \{0\}$ and there exists no I' such that $\delta(I') = \{0,1\}$ and $I \lhd I'$, then we put I into D_-^1

We denote $D_+ := (D_+^1 \cup D_+^2)$ and $D_- := (D_-^1 \cup D_-^2)$.

Notice that $D_+ \cup D_-$ exhausts $\mathbf{I}$. One sees immediately that in this construction, all segments I in D_+ are marked ($\delta(I) = \{0,1\}$) whenever $q_1 \in \underline{I}$. We change the marking of τ so that every $I \in D_+$, with $q_1 \in \underline{I}$, is marked. By Proposition 1.28, this procedure does not change the $G(s)$-orbit of v_τ. We introduce a second partial ordering on $\mathbf{I}$, weaker than $\lhd$:

$$I \prec I' \iff \min \underline{I} \leq \min \underline{I}'. \tag{2.4}$$

The following proposition is our main criterion for finding some special weight of each simple $\mathbb{H}_a$-modules. The notation $w(j)$ refers to the usual action of $W(B_n)$ by permutations and sign changes on $[-n, n]$.

Proposition 2.5. *Let τ be a marked partition as above. Assume that we have $\langle s, \epsilon_i \rangle > \langle s, \epsilon_j \rangle$ for every $i < j$. Assume that $w \in W$ satisfies the following conditions:*

- *Let $I \in D_+$. Then, we have $w(j) > 0$ for all $j \in I$. Moreover, we have $w(i) < w(j)$ for each $i, j \in I$ such that $\langle s, \epsilon_i \rangle > \langle s, \epsilon_j \rangle$;*
- *Let $I \in D_-$. Then, we have $w(j) < 0$ for all $j \in I$. Moreover, we have $w(i) < w(j)$ for each $i, j \in I$ such that $\langle s, \epsilon_i \rangle < \langle s, \epsilon_j \rangle$;*
- *If $I, I' \in D_+$ and $I \prec I'$, then we have*
$$w(j) < w(j') \text{ for every } (j, j') \in I \times I'; \tag{2.5}$$
- *If $I, I' \in D_-$ and $I \prec I'$, then we have*
$$w(j) > w(j') \text{ for every } (j, j') \in I \times I'; \tag{2.6}$$
- *If $I, I' \in \mathbf{I}$ and $\min \underline{I} = \min \underline{I'}$, then we have either*
$$w(j) > w(j') \text{ for every } (j, j') \in I \times I', \text{ or} \tag{2.7}$$
$$w(j) < w(j') \text{ for every } (j, j') \in I \times I' \tag{2.8}$$

Then $\mathcal{O}_\tau$ meets ${}^w\mathbb{V}(a)$ densely. In particular, we have $w \cdot s^{-1} \in \Psi(L_{(a, v_\tau)})$.

Proof. Now the first two conditions implies $\dot{w} v_I \in \mathbb{V}^+$, for all $I \in \mathbf{I}$, and $\dot{w} v_{\epsilon_i} \in \mathbb{V}^+$ (if $\delta(i) > 0$). Therefore, we deduce $v_\tau \in {}^w\mathbb{V}^+$, which implies $v_\tau \in {}^w\mathbb{V}(a)$.

For each ordered pair $(l, m) \in \mathbb{Z}^2$, we define

$$\mathfrak{p}_\tau^{l,m} := \bigoplus_{l \in I_i, m \in I_j; (\star)} (\mathfrak{g}(s) \cap \mathfrak{g}[\epsilon_i - \epsilon_j]),$$

where $(\star)$ denotes the condition $\epsilon_{w(i)} - \epsilon_{w(j)} \in R^+$, and $\mathfrak{g}[\epsilon_i - \epsilon_j]$ are the weight spaces. The condition $(\star)$ is also rephrased as:

$(\star)_1$ If $w(i)w(j) > 0$, then we have $w(i) < w(j)$;

$(\star)_2$ If $w(i) < 0$, then we have $w(j) < 0$.

It is straight-forward to see that $\mathfrak{p}_\tau^{l,m}$ is an abelian subalgebra of $\mathfrak{g}$.

Since $\{I_m\}_m$ exhaust $[1,n]$, condition $(\star)$ implies

$$\mathfrak{p}_\tau := \mathfrak{t} \oplus \bigoplus_{l,m} \mathfrak{p}_\tau^{l,m} = (\dot{w}^{-1}\mathfrak{b}) \cap \mathfrak{g}(s).$$

Hence, the Lie algebra $\mathfrak{p}_\tau$ preserves ${}^w\mathbb{V}^+$. Since $\mathfrak{p}_\tau \subset \mathfrak{g}(s)$, it preserves $\mathbb{V}^a$. Thus, $\mathfrak{p}_\tau$ acts on ${}^w\mathbb{V}(a)$. Moreover, we can replace $\mathfrak{p}_\tau$ with the connected algebraic subgroup $P_\tau \subset G(s)$ with $\mathrm{Lie} P_\tau = \mathfrak{p}_\tau$ to deduce that P_τ acts on ${}^w\mathbb{V}(a)$. We wish to prove that $P_\tau v_\tau$ is dense in ${}^w\mathbb{V}(a)$. We will be able to deduce this from the following claim, which is proved by computations.

Claim A. $\mathfrak{p}_\tau v_\tau = {}^w\mathbb{V}(a)$.

Proof. Since $\mathfrak{p}_\tau^{l,m}$ is a direct sum of T-weight spaces, we deduce that $(\mathfrak{t} \oplus \mathfrak{p}_\tau^{l,m})$ is again a Lie subalgebra of $\mathfrak{g}(s)$. We set $\mathfrak{t}^m := \bigoplus_{i \in I_m} \mathbb{C}\epsilon_i$, where $\epsilon_i \in \mathfrak{t}^*$ is identified with the dual basis $\epsilon_i \in \mathfrak{t}$ by the pairing $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. We have

$$\mathfrak{t}^m v_\tau = \mathfrak{t}^m (v_{I_m} + \sum_{i \in I_m} \delta(i) v_{\epsilon_i}) = \bigoplus_{i,j \in I_m; \langle s, \epsilon_i \rangle = q_2 \langle s, \epsilon_j \rangle} \mathbb{C} v_{\epsilon_i - \epsilon_j} \oplus \bigoplus_{i \in I_m; \delta(i) = 1} \mathbb{C} v_{\epsilon_i}$$

by a simple calculation. (Here we used the fact the the weights appearing in the RHS are linearly independent.)

By the first two conditions on w, the signs of the entries in $w(I_l)$ and $w(I_m)$ are constant on each segment. We calculate $\mathfrak{p}_\tau^{l,m} v_\tau$ in each of the four possible cases of signs.

Case 1) $(w(I_l), w(I_m) > 0)$ This means $I_l, I_m \in D_+$. We have either $0 < w(I_m) < w(I_l)$ or $0 < w(I_l) < w(I_m)$. If we have $0 < w(I_m) < w(I_l)$, then we have $\epsilon_i - \epsilon_j \notin w^{-1}R^+$ for every $i \in I_l, j \in I_m$. Therefore, $\mathfrak{p}_\tau^{i,j} = \{0\}$ in this case.

Now we assume $0 < w(I_l) < w(I_m)$. We have $\min \underline{I_m} \ge \min \underline{I_l}$ by assumption. We have $\epsilon_i - \epsilon_j \in \Psi(\mathfrak{p}_\tau^{l,m})$ if and only if $i \in I_l, j \in I_m$ and $\langle s, \epsilon_i - \epsilon_j \rangle = 1$. By the definition of segments, we deduce that

$$\mathfrak{p}_\tau^{l,m} = \bigoplus_{b \in \underline{I_l} \cap \underline{I_m}} \mathfrak{g}[\epsilon_{i_b} - \epsilon_{j_b}],$$

where $i_b \in I_l, j_b \in I_m$ satisfies $\langle s, \epsilon_{i_b} \rangle = b = \langle s, \epsilon_{j_b} \rangle$. By explicit computation, we have

$$\mathfrak{g}[\epsilon_{i_b} - \epsilon_{j_b}] v_\tau = \mathbb{C}(v_{\epsilon_{i_{q_2 b}} - \epsilon_{j_b}} + v_{\epsilon_{i_b} - \epsilon_{j_{q_2^{-1} b}}}).$$

Let b^- be the minimal element of $\underline{I_l} \cap \underline{I_m}$. Then, the number $j_{q_2^{-1} b^-}$ does not exist. It follows that

$$\mathfrak{p}_\tau^{l,m} v_\tau = \sum_{b \in \underline{I_l} \cap \underline{I_m}} \mathfrak{g}[\epsilon_{i_b} - \epsilon_{j_b}] v_{2,\tau} = \sum_{b \in \underline{I_l} \cap \underline{I_m}} \mathbb{V}[\epsilon_{i_b} - \epsilon_{j_{q_2^{-1} b}}],$$

where $v_{2,\tau}$ be the V_2-part of v_τ.

Here both I_l and I_m are marked. Hence, we conclude that

$$(\mathfrak{t}^l \oplus \mathfrak{t}^m \oplus \mathfrak{p}_\tau^{l,m})v_\tau = \sum_{b\in \underline{I_l}\cap \underline{I_m}} \mathfrak{g}[\epsilon_{i_b} - \epsilon_{j_b}]v_\tau = \mathfrak{t}^l v_\tau \oplus \mathfrak{t}^m v_\tau \oplus V,$$

where

$$V = \begin{cases} \sum_{b\in \underline{I_l}\cap \underline{I_m}} \mathbb{V}[\epsilon_{i_b} - \epsilon_{j_{q_2^{-1}b}}] & (w(I_m) > w(I_l)) \\ \{0\} & (w(I_m) < w(I_l)) \end{cases}.$$

Case 2) $(w(I_l), w(I_m) < 0)$ This means $I_l, I_m \in D_-$. This case is exactly the same as **Case 1** if we uniformly multiply -1 to every weights. Therefore, we conclude that

$$\mathfrak{p}_\tau^{l,m} = \begin{cases} \bigoplus_{b\in \underline{I_l}\cap \underline{I_m}} \mathfrak{g}[\epsilon_{i_b} - \epsilon_{j_b}] & (w(I_l) < w(I_m)) \\ 0 & (w(I_l) > w(I_m)) \end{cases},$$

and

$$(\mathfrak{t}^l \oplus \mathfrak{t}^m \oplus \mathfrak{p}_\tau^{l,m})v_\tau = \sum_{b\in \underline{I_l}\cap \underline{I_m}} \mathfrak{g}[\epsilon_{i_b} - \epsilon_{j_b}]v_\tau = \mathfrak{t}^l v_\tau \oplus \mathfrak{t}^m v_\tau \oplus V,$$

where

$$V = \begin{cases} \sum_{b\in \underline{I_l}\cap \underline{I_m}} \mathbb{V}[\epsilon_{i_b} - \epsilon_{j_{q_2^{-1}b}}] & (w(I_m) < w(I_l)) \\ \{0\} & (w(I_m) > w(I_l)) \end{cases}.$$

Case 3) $(w(I_l) < 0, w(I_m) > 0)$ This means $I_l \in D_-$, $I_m \in D_+$. We have $\epsilon_i - \epsilon_j \notin w^{-1}R^+$ when $i \in I_l$ and $j \in I_m$. It follows that $\mathfrak{p}_\tau^{l,m} = 0$. Therefore we have

$$(\mathfrak{t}^l \oplus \mathfrak{t}^m \oplus \mathfrak{p}_\tau^{l,m})v_\tau = \mathfrak{t}^l v_\tau \oplus \mathfrak{t}^m v_\tau.$$

Case 4) $(w(I_l) > 0, w(I_m) < 0)$ This means $I_l \in D_+$, $I_m \in D_-$. We have $\epsilon_i - \epsilon_j \in w^{-1}R^+$ when $i \in I_l$ and $j \in I_m$. By a similar argument as in **Case 1**, we deduce that

$$\mathfrak{p}_\tau^{l,m} = \bigoplus_{b\in \underline{I_l}\cap \underline{I_m}} \mathfrak{g}[\epsilon_{i_b} - \epsilon_{j_b}],$$

where $i_b \in I_l, j_b \in I_m$ satisfies $\langle s, \epsilon_{i_b}\rangle = b = \langle s, \epsilon_{j_b}\rangle$. By assumption, we have $I_l \succ I_m$ only if $\underline{I_l} \subsetneq \underline{I_m}$. If $\min \underline{I_l} \le \min \underline{I_m}$, then we have $j_{q_2^{-1}b^-} = \emptyset$ for $b^- = \min(\underline{I_l} \cap \underline{I_m})$. If $\min \underline{I_l} \ge \min \underline{I_m}$, then we have $i_{q_2 b^+} = \emptyset$ for $b^+ = \max(\underline{I_l} \cap \underline{I_m})$.

The segment I_l is marked while I_m is not. In particular, the vector $\sum_{i\in I_l\cup I_m} \delta(i)v_{\epsilon_i}$ is annihilated by $\mathfrak{p}_\tau^{l,m}$.

Therefore, by a similar argument as in **Case 1**, we have

$$(\mathfrak{t}^l \oplus \mathfrak{t}^m \oplus \mathfrak{p}_\tau^{l,m})v_\tau = \mathfrak{t}^l v_\tau \oplus \mathfrak{t}^m v_\tau \oplus V,$$

where

$$V = \begin{cases} \sum_{b\in \underline{I_l}\cap \underline{I_m}} \mathbb{V}[\epsilon_{i_b} - \epsilon_{j_{q_2^{-1}b}}] & (\min \underline{I_l} \le \min \underline{I_m}) \\ \sum_{b\in \underline{I_l}\cap \underline{I_m}} \mathbb{V}[\epsilon_{i_{q_2 b}} - \epsilon_{j_b}] & (\min \underline{I_l} \ge \min \underline{I_m}) \end{cases}.$$

We can rephrase the conclusion of the above case-by-case calculations as follows:

- $\mathfrak{t}^l v_\tau$ is a sum of T-weight spaces of ${}^w\mathbb{V}(a)$ of weight ϵ_i or $\epsilon_i - \epsilon_j$ such that $i, j \in I_l$;

- $\mathfrak{p}_\tau^{l,m} v_\tau$ is a sum of T-weight spaces of ${}^w\mathbb{V}(a)$ of weight $\epsilon_i - \epsilon_j$ such that $i \in I_l$ and $j \in I_m$.

From this, we deduce that

$$\mathfrak{p}_\tau v_\tau = \mathfrak{t} \oplus \sum_{l,m} \mathfrak{p}^{l,m} v_\tau = {}^w\mathbb{V}(a).$$

$\square$

We have a natural identification $\mathfrak{p}_\tau v_\tau = T_{v_\tau}(P_\tau v_\tau)$ (the RHS must be read as the tangent space of $P_\tau v_\tau$ at v_τ). We deduce that

$$\dim P_\tau v_\tau = \dim {}^w\mathbb{V}(a).$$

Since $P_\tau v_\tau \subset {}^w\mathbb{V}(a)$, this forces $\overline{P_\tau v_\tau} = {}^w\mathbb{V}(a)$, which implies the result. $\square$

Convention 2.6. Let $a \in \mathcal{G}_0^+$. A function $c : [1,n] \to [1,n]$ is called a c-function adapted to a if

$$c(i) < c(j) \text{ if } \langle s, \epsilon_i \rangle > \langle s, \epsilon_j \rangle .$$

Using this notion, it is convenient to rephrase Proposition 2.5 as follows:

Proposition 2.7. *Let $\tau \in \mathsf{MP}(a)$. Fix a c-function c adapted to a. Assume that $w \in W$ satisfies the following properties:*

- *Let $I \in D_+$. Then, we have $w(c(j)) > 0$ for all $j \in I$. Moreover, we have $w(c(i)) < w(c(j))$ for each $i, j \in I$ such that $c(i) < c(j)$;*
- *Let $I \in D_-$. Then, we have $w(c(j)) < 0$ for all $j \in I$. Moreover, we have $w(c(i)) < w(c(j))$ for each $i, j \in I$ such that $c(i) > c(j)$;*
- *If $I, I' \in D_+$ and $I \prec I'$, then we have*
$$w(c(j)) < w(c(j')) \text{ for every } (j, j') \in I \times I'; \tag{2.9}$$
- *If $I, I' \in D_-$ and $I \prec I'$, then we have*
$$w(c(j)) > w(c(j')) \text{ for every } (j, j') \in I \times I'; \tag{2.10}$$
- *If $I, I' \in \tau$ and $\min \underline{I} = \min \underline{I'}$, then we have either*
$$w(c(j)) > w(c(j')) \text{ for every } (j, j') \in I \times I', \text{ or} \tag{2.11}$$
$$w(c(j)) < w(c(j')) \text{ for every } (j, j') \in I \times I' \tag{2.12}$$

If $y \in T$ is such that $\left\langle y, \epsilon_{c(i)} \right\rangle = \left\langle w \cdot s^{-1}, \epsilon_i \right\rangle$ for every $1 \le i \le n$, then $y \in \Psi(L_{(a, v_\tau)})$. $\square$

Proposition 2.8. *Let $\tau = (\mathbf{I}, \delta) \in \mathsf{MP}(a)$. Let $w \in W$. If ${}^w\mathbb{V}(a)$ meets $\mathcal{O}_\tau$, then there exists $v \in \mathfrak{S}_n$ which satisfies:*

$(\clubsuit)_w$ *For every $i, j \in I_m \in \mathbf{I}$ such that $\langle \epsilon_i, s \rangle \in q_2^{\mathbb{Z}_{>0}} \langle \epsilon_j, s \rangle$, we have either*

$$wv(i) < wv(j) \text{ or } wv(i) > 0 > wv(j).$$

Proof. The space ${}^{w}\mathbb{V}(a)$ is stable under the action of ${}^{w}B(s)$. It follows that the space $G(s)^{w}\mathbb{V}(a)$ is a closed subset of $\mathfrak{N}^{a}$. In particular, a $G(s)$-orbit $\mathcal{O}_\tau \subset \mathfrak{N}^a$ meets ${}^{w}\mathbb{V}(a)$ if and only if we have $\mathcal{O}_\tau \subset \overline{\mathcal{O}}$, where $\mathcal{O}$ is the open dense $G(s)$-orbit of $G(s)^{w}\mathbb{V}(a)$.

Condition $(\clubsuit)_w$ is independent of the marking. Let $\tau' := (\mathbf{I}, \delta_0)$, where $\delta_0 \equiv 0$. We have $\mathcal{O}_{\tau'} \subset \overline{\mathcal{O}_\tau}$. Hence, it suffices to verify the assumption in the case $\tau = \tau'$. We put $T_\tau := \mathsf{Stab}_T v_\tau$. It is easy to verify that T_τ is a connected torus of rank $\#\mathbf{I}$ (the number of segments of τ). Since we have $\langle s, \epsilon_i \rangle \neq \langle s, \epsilon_j \rangle$ for each $i, j \in I_m \in \mathbf{I}$, it follows that $Z_{G(s)}(T_\tau) = T$ and $T v_\tau = \mathcal{O}_\tau^{T_\tau}$. The set of ${}^{v}T_\tau$-fixed points of

$$F_w^a := G(s) \times^{{}^{w}B(s)} {}^{w}\mathbb{V}(a) \xrightarrow{\pi_w^a} G(s)/B$$

is concentrated on the fiber of $(G(s)/B)^T$ for every $w \in W$ and $v \in W_s$. The image of μ_w^a contains all of $\mathcal{O}_\tau$ if and only if $\mathcal{O}_\tau \subset \mathfrak{N}^a$ meets ${}^{w}\mathbb{V}(a)$. Therefore, ${}^{w}\mathbb{V}(a)$ meets $\mathcal{O}_\tau$ if and only if

$$\dim \left({}^{w}\mathbb{V}(a)\right)^{{}^{v'}T_\tau} = \dim T v_\tau \text{ for some } v' \in W_s.$$

Since $a \in \mathcal{G}_0^+$, we have $\mathbb{V}^a \subset \mathfrak{gl}(n) \oplus \mathbb{C}^n$. In particular, having ${}^{v'}T_\tau$-fixed points is equivalent to having ${}^{v}T_\tau$-fixed points for $v \in \mathfrak{S}_n^a$. Now $(\clubsuit)_w$ is equivalent to the fact that $\dot{v} v_{I_m} \in {}^{w}\mathbb{V}(a)$ for each $I_m \in \mathbf{I}$. $\square$

Corollary 2.9. *Keep the setting of Proposition 2.8. We have $w \cdot s^{-1} \in \Psi(L_{(a, v_\tau)})$ only if there exists some $v \in \mathfrak{S}_n$ satisfying $(\clubsuit)_w$.*

Proof. By Proposition 2.2, the assumption implies that $\mathsf{IC}(\mathcal{O}_\tau)$ is a direct summand of $(\mu_w^a)_* \underline{\mathbb{C}}$ (up to shift). This implies $\mathrm{Supp}\, \mathsf{IC}(\mathcal{O}_\tau) = \overline{\mathcal{O}_\tau} \subset \mathrm{Im} \mu_w^a$. This happens only if ${}^{w}\mathbb{V}^a$ meets $\mathcal{O}_\tau$, which implies the result by Proposition 2.8. $\square$

2.3 Nested component decomposition

An important ingredient of the determination of tempered modules is a reduction theorem. This is similar to Zelevinsky's nested segments, but there are important differences in our setting.

Let $\tau = (\mathbf{I}, \delta)$, $\mathbf{I} = \{I_m\}_{m=1}^k$, be a marked partition adapted to $a = (s, \vec{q}) \in \mathcal{G}_0^+$. Assume that $\mathbf{I}$ can be split into a disjoint union of two collections of segments $\mathbf{I}^1$ and $\mathbf{I}^2$ with the property

$$\begin{aligned} &\underline{I} \sqsubset \underline{I}', \quad \text{for every } I \in \mathbf{I}^1, I' \in \mathbf{I}^2, \text{ or} \\ &\underline{I}' \sqsubset \underline{I}, \quad \text{for every } I \in \mathbf{I}^1, I' \in \mathbf{I}^2, \end{aligned} \tag{2.13}$$

where $\underline{I} \sqsubset \underline{J}$ means

$$\min \underline{J} < \min \underline{I} \text{ and } \max \underline{I} < \max \underline{J}.$$

Let n_1 and n_2 be the sums of cardinalities of segments of $\mathbf{I}^1$ and $\mathbf{I}^2$, respectively. By applying an appropriate permutation, we assume that

- $\mathbf{I}^1$ and $\mathbf{I}^2$ are divisions of $[1, n_1]$ and $(n_1, n]$, respectively;
- If $1 \leq i < j \leq n_1$ or $n_1 < i < j \leq n$, then $\langle s, \epsilon_i \rangle > \langle s, \epsilon_j \rangle$

Then we can regard $\tau^1 = (\mathbf{I}^1, \delta|_{\mathbf{I}^1})$ and $\tau^2 = (\mathbf{I}^2, \delta|_{\mathbf{I}^2})$ as marked partitions for $G_1 = Sp(2n_1)$ and $G_2 = Sp(2n_2)$ respectively, where $Sp(2n_1) \times Sp(2n_2)$ is embedded diagonally in $Sp(2n)$.

The marked partition τ parameterizes a $G(s)$-orbit $\mathcal{O}_\tau$ on $\mathfrak{N}^a$. We define semisimple elements $s_1 \in Sp(2n_1)$, $s_2 \in Sp(2n_2)$ to be the projections of s onto the $Sp(2n_1)$, $Sp(2n_2)$ factors, respectively. We set $a_1 := (s_1, \vec{q})$, and $a_2 := (s_2, \vec{q})$. The marked partitions τ^1 and τ^2 define orbits $\mathcal{O}_{\tau^1}$, $\mathcal{O}_{\tau^2}$, of $G_1 = G_1(s_1)$, $G_2 = G_2(s_2)$ respectively on the corresponding exotic nilcones.

Lemma 2.10. *Every $G(s)$-orbit $\mathcal{O}$ of $\mathfrak{N}^a$ which contains $\mathcal{O}_\tau$ as its closure is written as $G(s)(\mathcal{O}_1 \times \mathcal{O}_2)$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are exotic nilpotent orbits of G_1 and G_2, respectively. In addition, the marked partitions corresponding to $\mathcal{O}_1$ and $\mathcal{O}_2$ are nested in the sense of* (2.13).

Proof. The condition (2.13) is independent of markings. Thus, it suffices to prove the assertion for all orbits with no markings. In the algorithm of Theorem 1.29, it is straight-forward to see we cannot choose $I_m \in \mathbf{I}^1$ and $I_l \in \mathbf{I}^2$. Therefore, the above procedure preserves $\bigsqcup_{I_m \in \mathbf{I}^1} I_m$ and $\bigsqcup_{I_m \in \mathbf{I}^2} I_m$, respectively. Moreover, it preserves the nestedness of the modified $\mathbf{I}^1$ and $\mathbf{I}^2$, which implies the result. $\square$

Let $\mathbb{V}_{(1)}$ and $\mathbb{V}_{(2)}$ be the exotic representations of G_1 and G_2, respectively. We set

$$\mathcal{O}_\tau^\uparrow := \bigcup_{\mathcal{O}_\tau \subset \overline{\mathcal{O}}} \mathcal{O},$$

where $\mathcal{O}$ runs over all $G(s)$-orbits of $\mathfrak{N}^a$. This is a $G(s)$-stable open subset of $\mathfrak{N}^a$. Similarly, for $i = 1, 2$, we define $\mathcal{O}_{\tau^i}^\uparrow$ to be the union of $G_i(s_i)$-orbits of $\mathfrak{N}^a \cap \mathbb{V}_{(i)}$ which contain $\mathcal{O}_{\tau^i}$ in their closure.

Corollary 2.11. *Keep the setting of Lemma 2.10. We have*

$$G(s)(\mathcal{O}_{\tau^1}^\uparrow \times \mathcal{O}_{\tau^2}^\uparrow) = \mathcal{O}_\tau^\uparrow.$$

Lemma 2.12. *For each $x = x_1 \times x_2 \in \mathcal{O}_{\tau^1}^\uparrow \times \mathcal{O}_{\tau^2}^\uparrow \subset \mathcal{O}_\tau^\uparrow$, we have*

$$\mathsf{Stab}_{G(s)} x = \mathsf{Stab}_{G_1(s_1)} x_1 \times \mathsf{Stab}_{G_2(s_2)} x_2.$$

Proof. If suffices to prove the result for $\tau = (\mathbf{I}, \delta)$ with $\delta \equiv 0$. In that case, the assertion follows by Lemma 1.31 and Theorem 1.32. $\square$

Corollary 2.13. *Keep the setting of Lemma 2.12. Then, we have*

$$G(s) \times^{(G_1(s_1) \times G_2(s_2))} \left(\mathcal{O}_{\tau^1}^\uparrow \times \mathcal{O}_{\tau^2}^\uparrow \right) \xrightarrow{\cong} \mathcal{O}_\tau^\uparrow.$$

Let M and L be two $\mathbb{H}_a$-modules with L simple. Let $[M : L]$ denote the multiplicity of L in M in the Grothendieck group of $\mathbb{H}-\mathsf{mod}$.

Corollary 2.14. *Keep the setting of Lemma 2.12. Then, for every $y = y_1 \times y_2 \in \mathcal{O}_{\tau^1}^\uparrow \times \mathcal{O}_{\tau^2}^\uparrow$, we have*

$$[M_{(a,x)} : L_{(a,y)}] = [M_{(a_1,x_1)} : L_{(a_1,y_1)}][M_{(a_2,x_2)} : L_{(a_2,y_2)}].$$

Proof. By the Ginzburg theory, the coefficient $[M_{(a_*,x_*)} : L_{(a_*,y_*)}]$ is interpreted as $\dim H^\bullet_{G_*(a_*)x_*}(\mathsf{IC}(G_*(a_*)y_*))$, where $* = \emptyset, 1, 2$ in a uniform fashion. We have

$$\begin{aligned}\dim H^\bullet_{G(a)x}(\mathsf{IC}(G(a)y)) &= \dim H^\bullet_{(G_1(a_1)\times G_2(a_2))x}(\mathsf{IC}((G_1(a_1)\times G_2(a_2))y))\\ &= (\dim H^\bullet_{G_1(a_1)x_1}(\mathsf{IC}(G_1(a_1)y_1)))(\dim H^\bullet_{G_2(a_2)x_2}(\mathsf{IC}(G_2(a_2)y_2))).\end{aligned}$$

This implies the assertion. □

Proposition 2.15. *Let P be a parabolic subgroup G with Levi decomposition $P = LU$. We assume that $T \subset L$ and $L(s) = G_1(s_1) \times G_2(s_2)$. For each $w \in W_{\mathsf{B}_{n_1}} \times W_{\mathsf{B}_{n_2}}$ such that $U(s) \subset {}^wB$, we have:*

$$\begin{tikzcd} G(s) \times^{L(s)} \left(L(s) \times^{({}^wB\cap L)(s)} ({}^w\mathbb{V}(a) \cap (\mathcal{O}^{\uparrow}_{\tau^1} \times \mathcal{O}^{\uparrow}_{\tau^2}))\right) \arrow[r, equal] \arrow[d] & G(s) \times^{({}^wB)(s)} ({}^w\mathbb{V}(a) \cap \mathcal{O}^{\uparrow}_{\tau}) \arrow[d, "\nu^a_w"] \\ G(s) \times^{L(s)} \left(\mathcal{O}^{\uparrow}_{\tau^1} \times \mathcal{O}^{\uparrow}_{\tau^2}\right) \arrow[r, equal] & \mathcal{O}^{\uparrow}_{\tau} \end{tikzcd}$$

Here the vertical arrows are natural action maps and the bottom equality is from Corollary 2.13.

Proof. We have $({}^wB)(s) = ({}^wB \cap L(s))({}^wB \cap U(s))$. It follows that the natural action map

$$\mathsf{a} : ({}^wB \cap U(s)) \times ({}^w\mathbb{V}(a) \cap (\mathcal{O}^{\uparrow}_{\tau^1} \times \mathcal{O}^{\uparrow}_{\tau^2})) \longrightarrow {}^w\mathbb{V}(a) \cap \mathcal{O}^{\uparrow}_{\tau} \qquad (2.14)$$

is an inclusion.

Claim B. *Let $\eta \in \mathsf{MP}(a)$ be such that $\mathcal{O}_\tau \subset \overline{\mathcal{O}_\eta}$. We have ${}^w\mathbb{V}(a) \cap \mathcal{O}_\eta \neq \emptyset$ if and only if ${}^w\mathbb{V}(a) \cap (\mathcal{O}_{\eta^1} \times \mathcal{O}_{\eta^2}) \neq \emptyset$.*

Proof. The proof is by contradiction. Set $\eta = (\mathbf{J}, \delta') = (\{J'_m\}_m, \delta')$ and let $\mathbf{J} = \mathbf{J}^1 \sqcup \mathbf{J}^2$ be the nested component decomposition of η determined by that of τ. The set of maximal and minimal values of J_m are determined by that of $\mathbf{I}$ by Theorem 1.29.

We first argue by replacing η with $\eta' = (\mathbf{J}, \delta'')$, where $\delta'' \equiv 0$. If ${}^w\mathbb{V}(a) \cap (\mathcal{O}_{\eta^1} \times \mathcal{O}_{\eta^2}) \neq \emptyset$ and ${}^w\mathbb{V}(a) \cap \mathcal{O}_\eta = \emptyset$, then (2.13) claims that there exists a non-trivial sequence of weights of ${}^w\mathbb{V}(a)$ of the form:

$$w^{-1}(\epsilon_{i_1} - \epsilon_{i_2}), w^{-1}(\epsilon_{i_2} - \epsilon_{i_3}), \ldots, w^{-1}(\epsilon_{i_k} - \epsilon_{i_{k+1}}), \cdots, w^{-1}(\epsilon_{i_{l-1}} - \epsilon_{i_l}),$$

where $0 < i_1 < i_2 < \cdots < i_l$ and either **a)** $i_1, i_l \in [1, n_1]$ and $i_k \in (n_1, n]$ or **b)** $i_1, i_l \in (n_1, n]$ and $i_k \in [1, n_1]$ for some k. However, both cases are impossible to achieve. Therefore, we get a contradiction and the result follows when $\delta' \equiv 0$.

Recall that ${}^w\mathbb{V}^+$ is $({}^wB)(s)$-stable and $U(s)$ is $L(s)$-stable. It follows that the map a is isomorphism when $\delta' \equiv 0$. Since $V_1^{(s,q_1)} \subset \mathbb{V}_{(1)} \oplus \mathbb{V}_{(2)}$, and the vectors corresponding to the nontrivial marking δ' are in $V_1^{(s,q_1)}$, we can rearrange a point in ${}^w\mathbb{V}(a) \cap \mathcal{O}_\eta$ to give a point in ${}^w\mathbb{V}(a) \cap (\mathcal{O}_{\eta^1} \times \mathcal{O}_{\eta^2})$, even if $\delta' \neq 0$. This completes the proof. □

We return to the proof of Proposition 2.15. Set $\mathbb{V}^{++}$ to be the unique $({}^w B)(s)$-equivariant splitting map of ${}^w\mathbb{V}^+/({}^w\mathbb{V}(a) \cap \mathbb{V}_{(1)} \oplus \mathbb{V}_{(2)})$ to ${}^w\mathbb{V}^+$. Thanks to Claim B and Corollary 2.13, the composition of the following $(({}^w B)(s)$-equivariant) quotient map and its T-equivariant splitting

$$ {}^w\mathbb{V}(a) \to {}^w\mathbb{V}(a)/\mathbb{V}^{++} \hookrightarrow {}^w\mathbb{V}(a) $$

does not change the orbits in $\mathcal{O}_\tau^\uparrow$. In particular, the map a is an isomorphism.

Inducing both sides of (2.14) up to $G(s)$, we conclude the result. □

Theorem 2.16. *For each $y_1 \in \Psi(L_{\tau^1})$ and $y_2 \in \Psi(L_{\tau^2})$, we have $(y_1 \times y_2) \in \Psi(L_\tau)$.*

Proof. Write $y_1 = w_1 \cdot s_1^{-1}$ and $y_2 = w_2 \cdot s_2^{-1}$. We assume the situation of Proposition 2.15. By Proposition 2.2, we need to prove that $(\mu_w^a)_* \underline{\mathbb{C}}$ contains $\mathsf{IC}(\mathcal{O}_\tau)$ provided that $(\mu_{w_i}^{a_i})_* \underline{\mathbb{C}}$ contains $\mathsf{IC}(\mathcal{O}_{\tau^i})$, $i = 1, 2$. Since the orbits we concerned with are contained in $\mathcal{O}_\tau^\uparrow$, we can replace μ_w^a by ν_w^a and consider the corresponding problem by Lemma 2.1. By Proposition 2.15, $(\nu_w^a)_* \underline{\mathbb{C}}$ must contain the inflation of $\mathsf{IC}(\mathcal{O}_{\tau^1}) \boxtimes \mathsf{IC}(\mathcal{O}_{\tau^2})$ from $(\mathcal{O}_{\tau^1}^\uparrow \times \mathcal{O}_{\tau^2}^\uparrow)$ to $\mathcal{O}_\tau^\uparrow$, which is nothing but $\mathsf{IC}(\mathcal{O}_\tau)$. This verifies the assertion. □

2.4 Specialization of parameters

Proposition 2.17. *Let R be a $\mathbb{C}[t]$-algebra of finite rank. Let M be a R-module which is free as a $\mathbb{C}[t]$-module. Assume that we have a R-submodule $N \subset M$ whose localization $\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}[t]} N$ is a free $\mathbb{C}[t^{\pm 1}]$-module. Then, there exists a R-submodule $N' \subset M$ such that*

$$ \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}[t]} N = \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}[t]} N' \subset \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}[t]} M $$

and the quotient M/N' is free over $\mathbb{C}[t]$.

Proof. Let $h_1, \ldots, h_k \in N$ be a collection of elements of N which gives a free basis of $\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}[t]} N$. If $h_1(0), \ldots, h_k(0) \in \mathbb{C} \otimes_{\mathbb{C}[t]} N$ is linearly dependent, then we multiply t^{-1} to a $\mathbb{C}$-linear combination f of $h_1, \ldots, h_k$ such that $f(0) = 0$. By replacing some of h_l by f and repeating the above procedure for as long as possible, we obtain a collection of elements $h_1', \ldots, h_k'$ of M such that $h_1'(0), \ldots, h_k'(0) \in \mathbb{C} \otimes_{\mathbb{C}[t]} M$ are linearly independent and $t^{n_1} h_1', \ldots, t^{n_k} h_k' \in N$ $(n_1, \ldots, n_k \geq 0)$ define a $\mathbb{C}[t]$-basis of N. It is straight-forward to see $\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}[t]} N$ is a R-module. It follows that

$$ M \cap \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}[t]} N \subset \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}[t]} M $$

is a R-submodule of M, which has $h_1', \ldots h_k'$ as $\mathbb{C}[t]$-basis. This gives the desired N'. □

Corollary 2.18. *Let $a^t = a \exp(\gamma t)$ be a one-parameter family in $\mathcal{G}_0^+$ depending on $t \in \mathbb{R}$ by*

$$ \gamma \in \mathfrak{t} \oplus \{0\} \oplus \mathbb{R}^2_{\geq 0} \subset \mathrm{Lie}(T \times (\mathbb{C}^\times)^3). $$

Let τ be a marked partition adapted to each of a^t. We assume that a^t is generic except for finitely many values. Then, we have

$$ \Psi(L_{(a^0, v_\tau)}) \subset \lim_{t \to 0} \Psi(L_{(a^t, v_\tau)}). $$

In particular, the module $L_{(a^0,v_\tau)}$ is tempered if $L_{(a^t,v_\tau)}$ defines a tempered module in (at least) one of the region

$$\epsilon > t > 0 \text{ or } -\epsilon < t < 0 \text{ for some positive number } \epsilon \ll 1.$$

Proof. Let $\underline{a^t} \in \mathfrak{t} \oplus \mathbb{R}^2$ be the element defined from a^t via the statement of Proposition 1.16. We have $\underline{a^t} = \underline{a} + t\gamma$. We choose A in Proposition 1.16 so that $a^t \in A$. Let $\ell \subset \mathfrak{a}$ denote the line $\{\underline{a^t}\}_{t\in\mathbb{R}}$. We have the corresponding surjection $\mathbb{C}[\mathfrak{a}] \to \mathbb{C}[t]$. Therefore, we have a family of $\mathbb{H}_t := \mathbb{C}[t] \otimes_{\mathbb{C}[\mathfrak{a}]} \mathbb{H}_a^+$-module $M_t := \mathbb{C}[t] \otimes_{\mathbb{C}[\mathfrak{a}]} H_\bullet^A(\mathcal{E}_{v_\tau})$. We apply Proposition 2.17 to the family of maximal $\mathbb{H}_{a^t}$-submodules ($t \neq 0$) of $M_{(a^t,v_\tau)}$ for which the corresponding quotients are $L_{(a^t,v_\tau)}$. Every such $\mathbb{H}_{a^t}$-submodule extends to a $\mathbb{H}_t$-submodule $N \subset M_t$ whose quotient specializes to $L_{(a^t,v_\tau)}$ unless $t = 0$. Since a finite-dimensional W-module is rigid under flat deformation, it follows that the W-module structure of N must be constant along $t \in \mathbb{R}$. Therefore, $\mathbb{C} \otimes_{\mathbb{C}[t]} M_t/N$ contains a non-trivial $\mathbb{H}_a$-module which contains L_{v_τ} (as W-modules). This must be $L_{(a^0,v_\tau)}$. Since M_t is an algebraic family of $\mathbb{H}_t$-modules, we have

$$\Psi(L_{(a^0,v_\tau)}) \subset \lim_{t'\to 0} \Psi(\mathbb{C}_{t'} \otimes_{\mathbb{C}[t]} M_t/N) = \lim_{t'\to 0} \Psi(L_{(a^{t'},v_\tau)}),$$

where $\mathbb{C}_{t'}$ is the quotient of $\mathbb{C}[t]$ by the ideal $(t - t')$. The rest of the assertions are clear. $\square$

Corollary 2.19. *With the notation from corollary 2.18, assume that $L_{(a^t,v_\tau)}$ is a discrete series for $t \in (-\epsilon, \epsilon) \setminus \{0\}$. Then $L_{(a^0,v_\tau)}$ is a discrete series.*

Proof. By 2.18, $\Psi(L_{(a^0,v_\tau)}) \subset \lim_{t\to 0} \Psi(L_{(a^t,v_\tau)})$. Let $w \cdot a^t$ be a one-parameter family of weights, $w \in W$, such that $w \cdot a^t \in \Psi(L_{(a^t,v_\tau)})$ and $w \cdot a^0 \in \Psi(L_{(a^0,v_\tau)})$. By the discrete series condition, $\langle w \cdot a^t, \epsilon_1 + \cdots + \epsilon_j\rangle < 1$, for all $1 \le j \le n$, and for all $t \in (-\epsilon, \epsilon) \setminus \{0\}$. Since $\langle w \cdot a^t, \epsilon_1 + \cdots + \epsilon_j\rangle$ is continuous and linear in t, it follows that $\langle w \cdot a^0, \epsilon_1 + \cdots + \epsilon_j\rangle < 1$ (for every j) as well. $\square$

3 Parameters corresponding to discrete series

Recall that for any finite dimensional $\mathbb{H}_a$-module V, we denote by $\Psi(V) \subset T$ the set of its $R(T)$-weights.

3.1 Distinguished marked partitions

We restrict now to the case of the specialized affine Hecke algebra of type $B_n^{(1)}$ with $\vec{q} = (-t^{m/2}, t^{m/2}, t)$, $t \in \mathbb{R}_{>1}, m \in \mathbb{R}_{>0}$, and we assume the genericity condition, i.e., $m \notin \{0, 1, 2, \ldots, 2n-1\}$.

Let $a = (s, \vec{q}) \in \mathcal{G}_0^+$ be given.

Definition 3.1. We say that a (or s) is distinguished if the dense $G(a)$-orbit on $\mathfrak{N}^a$ is parameterized by a marked partition $(\{I_m\}_{m=1}^k, \delta)$ which satisfies:

1. $\max \underline{I}_1 > \max \underline{I}_2 > \cdots > \max \underline{I}_k$;
2. $\min \underline{I}_1 < \min \underline{I}_2 < \cdots < \min \underline{I}_k$;
3. $\delta(I_m) = \{0, 1\}$, for all m (which in particular means $q_1 \in I_m$ for all m).

We call such a marked partition distinguished as well.

Note that the distinguished marked partitions are in one to one correspondence with partitions of n by a "folding" procedure: for every $J \in \{I_m\}_{m=1}^k$, define $\underline{\#}J$ to be the number of elements in $\underline{J}$ strictly smaller than q_1, and $\overline{\#}J$ to be the number of elements in $\underline{J}$ greater than or equal to q_1. If $\mathsf{mp}(\sigma)$ is a distinguished marked partition, then one can build a left-justified nondecreasing partition (tableau) σ of n, as follows: put $\overline{\#}I_1$ boxes on the first row and $\underline{\#}I_1$ boxes on the first column below the first row (so the I_1 looks bent like a hook), then add $\overline{\#}I_2$ boxes on the second row and $\underline{\#}I_2$ on the second column, below the second row etc. Remark that, in the end, the diagonal of the tableau σ has boxes exactly corresponding to the markings of $\mathsf{mp}(\sigma)$ (see figure 3.1).

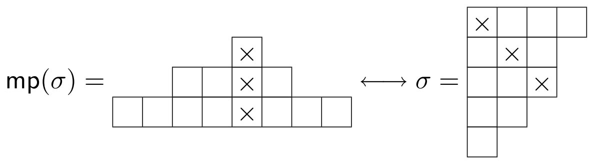


Figure 1: The correspondence $\mathsf{mp}(\sigma) \leftrightarrow \sigma$, for $\sigma = (4, 3, 3, 2, 1)$.

Theorem 3.2 ([Lu02],[OS08]). *Assume $s \in T$, and $a = (s, \vec{q})$ is as above. Then there exists a discrete series module with central character s if and only if s is distinguished in the sense of Definition 3.1.*

In particular, a distinguished semisimple a (or s) corresponds to a partition σ of n. We write a_σ and s_σ to emphasize this dependence. Note that, by §1.4, the marked partition $\mathsf{mp}(\sigma)$ above parameterizes the open $G(a_\sigma)$-orbit in $\mathfrak{N}^{a_\sigma}$. The goal of this section is to identify which $G(a_\sigma)$-orbit in $\mathfrak{N}^{a_\sigma}$ parameterizes the discrete series $\mathbb{H}_{a_\sigma}$-module under Theorem 1.18. By Propositions 1.27 and 1.28, we need to describe a marked partition, denoted $\mathsf{ds}(\sigma)$ or $\mathsf{ds}(s_\sigma)$, which gives a representative of the orbit via the map Υ.

3.2 Algorithm

We start with a distinguished marked partition $\mathsf{mp}(\sigma)$ corresponding to a partition σ of n as in §3.1, and let s_σ denote the corresponding semisimple element. We put integer coordinates (i, j) in the boxes of σ such that the boxes on the first row have coordinates: $(1, -1)$, $(2, -1)$, $(3, -1)$ etc., the boxes on the second row: $(1, -2)$, $(2, -2)$, $(3, -2)$ etc., the numbering starting from the left. Note that the boxes of the diagonal have coordinates $(i, -i)$.

We define a function on the boxes of σ, which we call an e-function. For a box (i, j), we set

$$e(i, j) = q_1 q_2^{i+j}. \tag{3.1}$$

Given σ, the following algorithm gives a marked partition $\mathsf{out}(\sigma)$ which turns out to parameterize the discrete series with central character s_σ (i.e., $\mathsf{out}(\sigma) = \mathsf{ds}(\sigma)$).

Algorithm 3.3. 1. Set $\ell = 0$, $\sigma_{(\ell)} = \sigma$, $L^+ = L^- = \emptyset$. (L^+ and L^- will be collections of subsets of σ.)

2. Find the unique $(i,j) \in \sigma_{(\ell)}$ such that $e(i,j)$ or $e(i,j)^{-1}$ attains the maximum in the set $\{e(i,j)^{\pm 1} : (i,j) \in \sigma_{(\ell)}\}$.

   (a) If the maximum is at $e(i,j)$, append the set (horizontal strip) $\{(i-k,j) \in \sigma_{(\ell)} : k \geq 0\}$ to L^+.

   (b) If the maximum is at $e(i,j)^{-1}$, append the set (vertical strip) $\{(i,j+k) \in \sigma_{(\ell)} : k \geq 0\}$ to L^-.

   Remove the horizontal or vertical string as above from $\sigma_{(\ell)}$ and call the resulting partition $\sigma_{(\ell+1)}$. If $\sigma_{(\ell+1)} \neq \emptyset$, increase ℓ to $\ell+1$ and go back to the beginning of step 2.

3. Set $L = \emptyset$. (This will be a collection of sets.) For every $-n \leq k \leq n$, form

$$L_k^+ = \{I \in L^+ : \min_{(i,j)\in I} e(i,j) = q_1 q_2^k\}, \quad L_k^- = \{I \in L^- : \max_{(i,j)\in I} e(i,j) = q_1 q_2^{k-1}\}. \tag{3.2}$$

   For every k, order the elements in L_k^+, respectively L_k^- decreasingly with respect to their cardinality: $I_{k,1}^+, \dots, I_{k,m_1}^+$ and $I_{k,1}^-, \dots, I_{k,m_2}^-$. By adding empty sets at the tail of the appropriate sequence, we may assume $m_1 = m_2$. Then for $j = 1, \dots, m_1$, form the segment $I_{k,j}^+ \sqcup I_{k,j}^-$, and append it to L. (Notice that $I_{k,j}^+ \sqcup I_{k,j}^-$ is a segment since we have started from $\mathsf{mp}(\sigma)$ instead to σ itself.)

   Then L is the collection of segments in the marked partition $\mathsf{out}(\sigma)$. We specify the marking δ next.

4. Define a temporary marking δ' first. For every $I \in L$, let $e(I)$ denote the set of $e(i,j)$ for $(i,j) \in I$. Recall that I could be marked only if $q_1 \in e(I)$, and if so, the marking could only be on the box (i,j) with $e(i,j) = q_1$. Set

$$\delta'(I) = \begin{cases} 1, \text{ if } q_1 \in e(I) \text{ and } \displaystyle\prod_{(i,j)\in I} e(i,j) > 1, \\ 0, \text{ otherwise.} \end{cases} \tag{3.3}$$

   We refine δ' to δ by removing the marking of any segment I which is dominated by marked segment I'.

Remark 3.4. 1. The hypothesis that $a = (s, \vec{q})$ is generic is essential for the uniqueness of the box (i,j) realizing the maximum in Step 2 of the algorithm.

2. The first two steps of the algorithm are identical with the algorithm conjectured by Slooten ([Sl06]) for a generalized Springer correspondence for the graded Hecke algebra of type B_n with generic unequal labels. We will see that this algorithm is equivalent with the one described by Lusztig-Spaltenstein ([LS85]) for the graded Hecke algebra of type B_n with (representative) generic unequal labels constructed from cuspidal local systems in Spin groups. We explain this in more detail in section 4.3.

3. To clarify the algorithm, we offer an example. Consider $n = 14$, and the partition $\sigma = (4,3,3,2,2)$, assuming that $q_2^2 < q_1 < q_2^{\frac{5}{2}}$, see figure 3 (in the figure an entry k in the box means the e-value is $q_1 q_2^k$).

0	1	2	3
-1	0	1	
-2	-1	0	
-3	-2		
-4			

Figure 2: Partition $\sigma = (4, 3, 3, 2, 1)$ and $q_2^2 < q_1 < q_2^{\frac{5}{2}}$

We find $L^+ = \{(q_1, q_1q_2, q_1q_2^2, q_1q_2^3), (q_1q_2^{-1}, q_1, q_1q_2), (q_1q_2^{-2}, q_1q_2^{-1}, q_1), (q_1q^{-2})\}$ and $L^- = \{(q_1q_2^{-4}, q_1q_2^{-3})\}$. We separate the segments based on where they begin or end: $L_0^+ = \{(q_1, q_1q_2, q_1q_2^2, q_1q_2^3)\}$, $L_{-1}^+ = \{(q_1q_2^{-1}, q_1, q_1q_2)\}$, and $L_{-2}^+ = \{(q_1q_2^{-2}, q_1q_2^{-1}, q_1), (q_1q_2^{-2})\}$, respectively and $L_{-2}^- = \{(q_1q_2^{-4}, q_1q_2^{-3})\}$. Next we may combine the segment in L_{-2}^+ with the longest segment in L_{-2}^-. The resulting marked partition τ has support $\mathbf{I}$ given by the segments (with e-values) $I_1 = (q_1, q_1q_2, q_1q_2^2, q_1q_2^3)$, $I_2 = (q_1q_2^{-1}, q_1, q_1q_2)$, and $I_3 = (q_1q_2^{-4}, q_1q_2^{-3}, q_1q_2^{-2}, q_1q_2^{-1}, q_1)$, $I_4 = (q_1q_2^{-2})$. According to the algorithm, we temporarily mark the first three segments at q_1, but then since $I_3 \lhd I_2 \lhd I_1$, we remove the markings on I_2 and I_3. In conclusion, the output of the algorithm is the marked partition $\mathsf{out}(\sigma)$ (see figure 3) with support given by $\{I_1, I_2, I_3, I_4\}$ and a single marking on I_1. (This marked partition is in the same orbit with the one where all three I_1, I_2, I_3 are marked.)

0	1	2	3	
-1	0	1		
-4	-3	-2	-1	0
-2				

or, after aligning the rows,

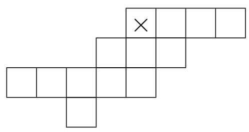
$\mathsf{out}(\sigma) =$ 

Figure 3: Output of Algorithm 3.3 when $\sigma = (4, 3, 3, 2, 1)$ and $q_2^2 < q_1 < q_2^{\frac{5}{2}}$

The main result of section 3 is next.

Theorem 3.5. *Let σ be a partition of n, and let a_σ, s_σ be the semisimple elements constructed from σ in §3.1. The discrete series $\mathbb{H}_{a_\sigma}$-module (with central character s_σ) is $L_{(a_\sigma, \Upsilon(\mathsf{out}(\sigma)))}$, where $\mathsf{out}(\sigma)$ is the marked partition constructed in Algorithm 3.3. In other words, $\mathsf{out}(\sigma) = \mathsf{ds}(\sigma)$.*

The proof will be broken up into parts in the next sections.

Example 3.6. **(One hook partitions.)** Before proving Theorem 3.5, let us present an example which illustrates the algorithm. Assume σ is a partition given by a single hook, i.e., $\sigma = (M, \underbrace{1, \ldots, 1}_{n-M})$, for some $1 \leq M \leq n$. This means that the semisimple element is $s_\sigma = (q_1 q_2^{M-1}, q_1 q_2^{M-2}, \dots, q_1, \dots, q_1 q_2^{M-n})$. In this case,

$$\mathfrak{N}^{a_\sigma} = \bigoplus_{i=1}^{n} \mathbb{C} v_{\epsilon_i - \epsilon_{i+1}} \oplus \mathbb{C} v_{\epsilon_M}, \quad \text{and } G(s_\sigma) = (\mathbb{C}^\times)^n \text{ (i.e., the maximal torus)}, \tag{3.4}$$

so there are 2^n orbits in $G(s_\sigma)\backslash\mathfrak{N}^{a_\sigma}$, each orbit corresponding to a subset $\mathcal{S}$ of $\{v_{\epsilon_i-\epsilon_{i+1}},\ 1 \le i < n, v_{\epsilon_M}\}$. To $\mathcal{S}$, there corresponds a marked partition $\tau_{\mathcal{S}} = (\mathbf{I}, \delta)$, as follows: for every maximal string of consecutive weight vectors $\{v_{\epsilon_i-\epsilon_{i+1}}, \dots, v_{\epsilon_{i+t-1}-\epsilon_{i+t}}\}$ in S, we attach a segment $I = [i, i+1, \dots, i+t-1] \in \mathbf{I}$, of length t with e-values $(q_1 q_2^{M-i}, \dots, q_1 q_2^{M-i-t+1})$. In addition, we mark at q_1, and write $\delta = 1$, if $v_{\epsilon_M} \in \mathcal{S}$, and we don't mark, and write $\delta = 0$, otherwise.

We remark that, as a consequence of the results about weights, one sees that the any $R(T)$-weight for an irreducible $\mathbb{H}_{a_\sigma}$-module with central character s_σ, i.e., parameterized by a marked partition of the form $\tau_{\mathcal{S}}$, is one-dimensional.

By applying the algorithm explicitly, as a corollary of Theorem 3.5, we find that the discrete series $\mathsf{ds}(\sigma)$ is parameterized by the marked partition $\tau = (\mathbf{I}, \delta)$ as follows:

(a) if $q_1 q_2^{M-n} > 1$, then τ has $\mathbf{I} = \{(1, 2, \dots, M), (M+1), (M+2), \dots, (n)\}$ and $\delta = 1$;

(b) if $q_1 q_2^{M-n} < 1$, and

  (b1) $(q_1 q_2^{M-n})^{-1} > q_1 q_2^{M-1}$, then τ has $\mathbf{I} = \{(1, 2, \dots, n)\}$ and $\delta = 0$;

  (b2) $q_1 q_2^{-1} < (q_1 q_2^{M-n})^{-1} < q_1 q_2^{M-1}$, then τ has $\mathbf{I} = \{(1, 2, \dots n)\}$ and $\delta = 1$;

  (b3) there exists $k > M$ with the property that $q_1 q_2^{k-1} < (q_1 q_2^{M-n})^{-1} < q_1 q_2^{k}$, then τ had $\mathbf{I} = \{(1, 2, \dots, M), (M+1), \dots, (k-1), (k, k+1, \dots, n)\}$ and $\delta = 1$.

3.3 Reduction

We begin with two consequences of the discussion about weights in §2 which allow us to make some reductions in the proof of Theorem 3.5.

Recall that we defined a notion of nested component decomposition in §2.3. Let τ be a marked partition for $Sp(2n)$ obtained by nested induction from the marked partitions τ_1 and τ_2 of $Sp(2n_1)$ and $Sp(2n_2)$, respectively.

The experience of $GL(n)$ would suggest that the irreducible module $L_{(a,v_\tau)}$ it is (at least) a subquotient of the induced module

$$\mathrm{Ind}_{\mathbb{H}(B_{n_1}) \times \mathbb{H}(B_{n_2})}^{\mathbb{H}(B_n)} (L_{(a_1, v_{\tau_1})} \boxtimes L_{(a_2, v_{\tau_2})}). \tag{3.5}$$

This however is false. The first example where this can be seen is for $n = 4$, $n_1 = 1$, and $n_2 = 3$, and parameter $m = 1/4$. One considers $\tau = \mathsf{mp}((31))$. (The notation (31) means the partition $3 + 1 = 4$.) In this case, $L_{(a_1, v_{\tau_1})}$ is the one-dimensional Steinberg module, and $L_{(a_2, v_{\tau_2})}$ is a four dimensional discrete series, and so the induced in (3.5) is 16-dimensional. But the irreducible $L_{(a,v_\tau)}$ (which turns out to be a discrete series as well) is 22-dimensional.

But as a consequence of Theorem 2.16, we can relate the temperedness of $L_{(a,v_\tau)}$ to that of $L_{(a_1, v_{\tau_1})}$ and $L_{(a_2, v_{\tau_2})}$.

Corollary 3.7. *The irreducible module $L_{(a,v_\tau)}$ is tempered only if both $L_{(a_1,v_{\tau_1})}$ and $L_{(a_2,v_{\tau_2})}$ are tempered.*

Proof. This is clear from Theorem 2.16 by applying the temperedness condition on weights (1.8). □

From Corollary 3.7, we will be able to reduce the verification of tempered modules to the case when the marked partition τ is rigid, i.e., it does not admit a nested component decomposition.

Example 3.8. In the following I_ℓ will denote the ℓ-th segment extracted by the Algorithm 3.3 from $\sigma_{(\ell)}$. This is put in L^+ or in L^-, depending if it is a horizontal or a vertical strip, respectively. Assume the algorithm runs $I_1 \in L^+$ and $I_2 \in L^-$, or $I_1 \in L^+$ and $I_2 \in L^-$. Set $p = \#I_1 + \#I_2$. Then $\sigma' := \sigma_{(3)}$ is a partition of $n - p$ which gives a distinguished semisimple element $s_{\sigma'}$ of $Sp(2n-2p)$ for $\mathbb{H}_{a'}$, where $a' = (s_{\sigma'}, \vec{q})$. (The point is that it is the same $\vec{q}$ as for σ.) Let τ_o denote the marked partition having a single segment $I_o := I_1 \cup I_2$ which is marked if $\prod_{j \in I_o} e(j) > 1$ and unmarked otherwise. This is a marked partition for $Sp(2p)$. Then the algorithm implies that $\mathsf{out}(\sigma)$ admits a nested component decomposition into τ_o and $\mathsf{out}(\sigma')$. Moreover, τ_o parameterizes a discrete series module for $Sp(2p)$, and so $\mathsf{out}(\sigma)$ is tempered only if $\mathsf{out}(\sigma')$ is tempered.

The criterion in Proposition 2.5 gives the following restriction on the form of a marked partition with all special weights satisfies (1.8). The notation is as in §2.2.

Corollary 3.9. *Let τ be a marked partition adapted to a_σ. Assume that every weight of $L_{(a_\sigma,v_\tau)}$ satisfies the tempered condition (1.8). Then, the following conditions hold:*

1. *We have $D_-^2 = \emptyset$;*
2. *Let I be a minimal element of D_+ with respect to $\prec$. Then, we have $\prod_{j \in I} e(j) > 1$;*
3. *Let I be a minimal element of D_- with respect to $\prec$. Then, we have $\prod_{j \in I} e(j) < 1$.*

Proof. We put $L_\tau = L_{(a_\sigma,v_\tau)}$.

Assume that $D_-^2 \neq \emptyset$ to deduce contradiction. We have $w_1 \in W$ which satisfies the assumptions of Proposition 2.5 and $w_1(I_m) = \{-1, -2, \cdots, -k\}$ for some $I_m \in D_-^2$ and $1 \le k \le n$. It follows that

$$\left\langle \epsilon_1, w_1 \cdot s^{-1} \right\rangle = b > 1$$

for some $b \in \underline{I_m}$. Since $w_1 \cdot s^{-1} \in \Psi(L_\tau)$, we have contradiction, which implies 1).

Let I be a minimal element of D_+ with respect to $\prec$. We have $w_2 \in W$ which satisfies the assumptions of Proposition 2.5 and $w_2(I) = \{1, 2, \cdots, k\}$ for some $1 \le k \le n$. We have

$$\left\langle \epsilon_1 + \cdots + \epsilon_k, w_2 \cdot s^{-1} \right\rangle = \prod_{j \in I} e(j)^{-1} < 1$$

in order that $w_2 \cdot s^{-1} \in \Psi(L_\tau)$ defines a tempered weight.

Let I be a minimal element of D_- with respect to $\prec$. We have $w_3 \in W$ which satisfies the assumptions of Proposition 2.5 and $w_3(I) = \{-1, -2, \cdots, -k\}$ for some $1 \le k \le n$. We have

$$\left\langle \epsilon_1 + \cdots + \epsilon_k, w_3 \cdot s^{-1} \right\rangle = \prod_{j \in I} e(j) < 1$$

in order that $w_3 \cdot s^{-1} \in \Psi(L_\tau)$ defines a tempered weight. $\square$

In order to use a reduction with respect to nested component decomposition, we need to explain how/when the nested components appear by looking at the distinguished central character $s = s_\sigma$.

Let us denote by $K_1, K_2, \dots$ the hooks of σ, starting from the outside, and we think of them as carrying the corresponding e values as in Algorithm 3.3. There are two cases, completely analogous: $\prod_{j \in K_1} e(j) > 1$ or $\prod_{j \in K_1} e(j) < 1$. Let us assume we are in the first case, and let $k_0 \ge 1$ be such that

$$\prod_{j \in K_1} e(j) > 1, \dots, \prod_{j \in K_{k_0}} e(j) > 1, \max \underline{K_{k_0+1}} < (\min \underline{K_{k_0}})^{-1}. \tag{3.6}$$

Set $m = \#K_1 + \cdots + \#K_{k_0}$. Let us denote by σ' the partition of m given by the hooks $K_1, \dots, K_{k_0}$, and by σ'' the partition of $n - m$ remaining from σ after removing the first k_0 hooks. Then σ' and σ'' give rise to distinguished semisimple elements in $\mathbb{H}(C_m)$ and $\mathbb{H}(C_{n-m})$, respectively, for the same vector $\vec{q}$ as σ.

Remark 3.10. The output $\mathsf{out}(\sigma)$ of Algorithm 3.3 admits a nested component decomposition into $\mathsf{out}(\sigma')$ and $\mathsf{out}(\sigma'')$, where $\sigma, \sigma', \sigma''$ are as in the previous paragraph.

Next we show that this decomposition is actually forced upon a marked partition τ adapted to a_σ, which carries a tempered module.

Proposition 3.11 ((+)-reduction)**.** *Assume the following conditions:*

1. *The pair (σ, s_σ) satisfies (3.6);*
2. *Both $L_{(a_{\sigma'}, \mathsf{out}(\sigma'))}$ and $L_{(a_{\sigma''}, \mathsf{out}(\sigma''))}$ are tempered modules.*

If $\tau = (\mathbf{I}, \delta)$ is a marked partition such that $L_{(a_\sigma, v_\tau)}$ is tempered, then we have $\tau = \mathsf{out}(\sigma') \times \mathsf{out}(\sigma'')$.

Proof. We set $e_i^- := \min e(K_i)$ and set $e_i^+ := \max e(K_i)$ for $i \ge 0$. (We define $e_1^+, \dots, e_{k_0+1}^+, e_{k_0+2}^+, \dots$)

Notice that if there exists $\mathbf{I}' = \{I_1, \dots, I_k\} \subset \mathbf{I}$ such that

$(\#)_1$ $(\mathbf{I}', \delta_{\mathbf{I}'})$ is a marked partition adapted to $a_{\sigma'}$ defined as above;

$(\#)_2$ Each $I_m \in \mathbf{I}'$ satisfies $\min \underline{I_m} \in \{e_i^-; 1 \le i \le k_0\}$ and $\max \underline{I_m} \in \{e_i^+; 1 \le i \le k_0\}$, then $\tau = (\mathbf{I}', \delta|_{\mathbf{I}'}) \times (\mathbf{I} \backslash \mathbf{I}', \delta|_{(\mathbf{I} \backslash \mathbf{I}')})$ defines a nested component decomposition of τ. By the second assumption, Theorem 3.2, and Theorem 2.16, we deduce that $\tau = \mathsf{out}(\sigma') \times \mathsf{out}(\sigma'')$ is the only possible marked partition which can be tempered in $\mathsf{MP}(a_{\sigma'}) \times \mathsf{MP}(a_{\sigma''})$. Therefore, it suffices to verify condition $(\#) = (\#)_1 \wedge (\#)_2$ by assuming $L_{(a_\sigma, v_\tau)}$ is tempered.

The proof is by contradiction. Let $i_0 \le k_0$ be the smallest integer such that there exists a segment $I \in \mathbf{I}$ with $\min e(I) = e^-_{i_0}$ and $\max e(I) \notin \{e^+_i; 1 \le i \le k_0\}$. Then we have two possibilities:

- $(\max e(I) < (e^-_{i_0})^{-1})$ We have $\prod_{\Box \in I} e(\Box) < 1$. By the induction assumption, I is the minimal segment (with respect to $\succ$) which has the product of its e-values < 1. Choose a c-function $c : \cup_m I_m \to [1, n]$. Fix $w \in W$ so that $w(c(I)) = \{1, 2, \ldots, \# I\}$, $v_\tau \in {}^w\mathbb{V}(a)$, and w satisfies the conditions of Proposition 2.5 for $\tau^I := (\mathbf{I} - \{I\}, \delta|_{(\mathbf{I} - \{I\})})$.

Claim C. *Assume that either I is marked or there exists $I' \in \mathbf{I}$ such that I' is marked and $I \lhd I'$. (In particular, $I \in D_+$ for τ.) Then, the orbit $\mathcal{O}_\tau$ meets ${}^w\mathbb{V}(a)$ in a dense open part.*

Proof. Since $a \in \mathcal{G}_0^+$, we deduce that $\mathbb{V}^a[\epsilon_i + \epsilon_j] = \{0\} = \mathfrak{g}(s)[\epsilon_i + \epsilon_j]$ for each pair $1 \le i, j \le n$. Let $\mathfrak{u}_m = \bigoplus_{(i,j) \in I \times I_m} \mathfrak{g}(s)[\epsilon_i - \epsilon_j]$ be the direct sum of weight spaces such that $\langle \epsilon_i, s \rangle = \langle \epsilon_j, s \rangle$. We put $\mathbf{V}_m := \bigoplus_{i \in I; j \in I_m} \mathbb{V}^a[\epsilon_i - \epsilon_j]$. By a similar argument as in the proof of Proposition 2.5, it suffices to show that $\mathbf{V}_m \subset \mathfrak{u}_m v_\tau$ for each $I \neq I_m \in \mathbf{I}$. If $I_m \prec I$, then we automatically have $\underline{I} \sqsubset \underline{I_m}$. Otherwise, we have $I \prec I_m$. In both cases, we can verify $\mathbf{V}_m \subset \mathfrak{u}_m v_\tau$ directly. $\Box$

We return to the proof of Proposition 3.11. If the assumption of Claim C holds, then we have $\left\langle \varpi_{\# I}, w \cdot s^{-1} \right\rangle > 1$. By Corollary 2.3, it follows that $L_{(a, v_\tau)}$ is not tempered under the assumption of Claim C. If the assumption of Claim C fails, then I is the minimal element of D_-. By the pigeon hole principle and (3.6), there exists some segment $J \in \mathbf{I}$, $J \in D_-$, such that one of the following conditions holds: **1)** $\min \underline{J} > q_1$, **2)** $\max \underline{J} = e^+_i$ for some $1 \le i \le k_0$, and the product of its e-values is positive. Here **1)** is equivalent to $J \in D^2_-$, and so it is excluded by Corollary 3.9 1). We assume case **2)**. We replace J if necessary to assume that $\max \underline{J}$ is maximal among all the segments with the property **2)** inside D_-. Notice that Corollary 3.9 1) implies that J is uniquely determined. Fix $w' \in W$ so that $w'(c(J)) = \{-1, -2, \ldots, -\# J\}$, $v_\tau \in {}^{w'}\mathbb{V}(a)$, and w' satisfies the conditions of Proposition 2.5 for $\tau^J := (\mathbf{I} - \{J\}, \delta|_{(\mathbf{I} - \{J\})})$.

Claim D. *The orbit $\mathcal{O}_\tau$ meets ${}^{w'}\mathbb{V}(a)$ in a dense open part.*

Proof. Since $a \in \mathcal{G}_0^+$, we deduce that $\mathbb{V}^a[\epsilon_i + \epsilon_j] = \{0\} = \mathfrak{g}(s)[\epsilon_i + \epsilon_j]$ for each pair $1 \le i, j \le n$. Let $\mathfrak{u}_m = \bigoplus_{(i,j) \in J \times I_m} \mathfrak{g}[\epsilon_i - \epsilon_j]$ be the direct sum of weight spaces such that $\langle \epsilon_i, s \rangle = \langle \epsilon_j, s \rangle$. We put $\mathbf{V}_m := \bigoplus_{i \in J; j \in I_m} \mathbb{V}^a[\epsilon_i - \epsilon_j]$. By a similar argument as in the proof of Proposition 2.5, it suffices to show that $\mathbf{V}_m \subset \mathfrak{u}_m v_\tau$ for each $J \neq I_m \in D_-$. If suffices to verify that $J \not\lhd I_m$, meaning that the following situation does not occur:

$$\min \underline{J} \le \min \underline{I_m} \le \max \underline{J} \le \max \underline{I_m}.$$

This follows from the maximality assumption on $\max \underline{J}$. $\Box$

We return to the proof of Proposition 3.11. If the assumption of Claim D holds, then we have $\langle \varpi_{\#J}, w' \cdot s^{-1} \rangle > 1$. As a consequence, τ cannot be tempered in this case.

- $(\max e(I) > (e^-_{i_0})^{-1})$ We have $\max e(I) \neq e^+_j$ for all j. By (3.6), we have $\max e(I) > e^+_{k_0+1}$. There is some segment $J \in \tau$ such that $\min e(J) = q_2 \max e(I)$. This means $\min e(J) > q_1$. In particular, we have $J \in D^-_2$. This contradicts Corollary 3.9 1). Therefore, L_τ cannot be a tempered module.

By the above case-by-case analysis, we verified (#) as desired. $\square$

We also have the similar results for the case when $\prod_{j\in K_1} e(j) < 1.$. We state them without proofs, since the proofs are completely analogous.

Proposition 3.12 ((−)-reduction)**.** *Assume that (σ, s_σ) satisfies*

$$\prod_{j\in K_1} e(j) < 1, \ldots, \prod_{j\in K_{k_0}} e(j) < 1, (\min \underline{K_{k_0+1}})^{-1} < \max \underline{K_{k_0}}. \tag{3.7}$$

Let σ' and σ'' be partitions defined completely parallel to the case (3.6). Let $\tau \in \mathsf{MP}(a_\sigma)$ be a marked partition such that $L_{(a_\sigma, v_\tau)}$ is tempered. If both of $\mathsf{out}(\sigma')$ and $\mathsf{out}(\sigma'')$ define tempered modules, then we have $\tau = \mathsf{out}(\sigma') \times \mathsf{out}(\sigma'')$.

Proof. Since the proof is analogous to Proposition 3.11, we omit the details. $\square$

By applying these reductions repeatedly, we see that it remains to prove that the output of the Algorithm 3.3 is a tempered module in the case when (σ, s_σ) satisfies $\prod_{j\in K_k} e(j) > 1$ for all k, or $\prod_{j\in K_k} e(j) < 1$, for all k.

3.4 A particular case: (±)-ladders

Recall that $\sigma = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ is a distinguished partition, and that $a_\sigma = (s_\sigma, \vec{q})$ is the corresponding semisimple element.

We begin with a particular instance of Algorithm 3.3: the cases when the algorithm produces $L^- = \emptyset$ or $L^+ = \emptyset$.

Definition 3.13 ((±)-ladder)**.** A positive ladder corresponding to a_σ is a marked partition $\tau = (\mathbf{I}, \delta)$ adapted to a_σ which satisfies the following conditions:

1. We have $\mathbf{I} = \{I_1, I_2, \ldots\}$ such that
$$e(I_i) = \{q_1 q_2^{1-i}, \ldots, q_1 q_2^{\lambda_i - i}\};$$
2. We have $\delta(\square) = 1$ if $e(\square) = q_1$ and $\square \in I_1$, and $\delta(\square) = 0$ otherwise.

A negative ladder corresponding to a_σ is a marked partition $\tau = (\mathbf{I}, \delta)$ adapted to a_σ which satisfies the following conditions:

1. We have $\mathbf{I} = \{I_1, I_2, \ldots\}$ such that
$$e(I_i) = \{q_1 q_2^{i-\lambda_i}, \ldots, q_1 q_2^{i-1}\};$$
2. We have $\delta \equiv 0$.

For every distinguished σ there are unique $(\pm)$-ladders: the positive ladder has the collection of segments $\mathbf{I}$ as the rows of σ, and every one containing q_1 is marked, while the negative ladder has the collection of segments $\mathbf{I}$ as the columns of σ, and has no marking. It is clear that a $(\pm)$-ladder is rigid.

Recall that in general, the weights $\Psi(L_\tau)$ are a subset of $W \cdot s_\sigma^{-1}$. If τ is a $(\pm)$-ladder, then the weigths have a particular form:

Proposition 3.14. *1. Assume that τ is the positive ladder for σ. Then $\Psi(L_\tau) \subset \mathfrak{S}_n \cdot s_\sigma^{-1}$.*

2. Assume that τ is the negative ladder for σ. Then $\Psi(L_\tau) \subset \mathfrak{S}_n \cdot s_\sigma$.

Proof. The proofs of the two assertions are completely analogous, therefore we only present the details when τ is the positive ladder. The proof is by induction on k, the number of rows of σ, or equivalently, the number of segments J_i in the support of τ.

In the base case, $k = 1$, the orbit corresponding to τ is G-regular in $\mathfrak{N} = \mathfrak{N}_1$, and the corresponding module is the Steinberg module. It follows that $\Psi(L_\tau) = \{s_\sigma^{-1}\}$, which proves the assertion in this case.

Assume the result holds for all σ' with less than k rows, and assume σ has k rows. Choose a c-function for σ (see 2.6). We want to show that for every weight $w^{-1} \cdot s_\sigma^{-1}$, we have $w^{-1}i > 0$ for $1 \leq i \leq n$ (which implies that $w \in \mathfrak{S}_n$), or, equivalently, that $w^{-1}c(\boxed{j}) > 0$ for every box $\boxed{j}$ of τ.

Let τ_1 and τ_2 be the positive ladder partitions corresponding to the first $k-1$ rows, respectively last row, of σ, and let s_1, s_2 be the corresponding semisimple elements. We form the $\mathbb{H}^{\mathsf{A}}_{\lambda_k}$-module (one dimensional) $L^{\mathsf{A}}_{\lambda_k} = M^{\mathsf{A}}_{\lambda_k}$ corresponding to (s_2, q_2, v_{τ_2}), and let M_{τ_1} be the stardard module of $\mathbb{H}_{n-\lambda_k}$. The induction theorem 1.14 applies, and we have

$$M_\tau \cong \mathsf{Ind}^{\mathbb{H}_n}_{\mathbb{H}^{\mathsf{A}}_{\lambda_k} \times \mathbb{H}_{n-\lambda_k}} ({}^{\mathsf{t}}L^{\mathsf{A}}_{\tau_2} \boxtimes M_{\tau_1}). \tag{3.8}$$

(The notation is as in Convention 1.21.) For every coset representative w of $W_n/\mathfrak{S}_{\lambda_k} \times W_{n-\lambda_k}$, similarly to 2.5, we analyze the homology $H_\bullet(\mathcal{E}^{a_\sigma}_{v_\tau}[w \cdot s_\sigma^{-1}])$ to see if the $w \cdot s_\sigma^{-1}$-weight space is nonempty. By the induction hypothesis, we have $wc(\boxed{j}) > 0$ for all $\boxed{j} \in \tau_1$. It remains to show that the same holds for all $\boxed{j} \in \tau_2$.

Notice that the minimal e-value e_{min} in τ is attained by an element $\boxed{\mathsf{min}}$ of τ_2. Recall that this makes $c(\boxed{\mathsf{min}}) = n$. There are at most two elements in τ which have e-value equal to $q_2 e_{\mathsf{min}}$: one in τ_1, denoted $\boxed{1}$, and, if τ_2 is not a singleton, one in τ_2, denoted $\boxed{2}$.

If $wc(\boxed{\mathsf{min}}) > 0$, then in order to have $v_{\tau_2} \in {}^w\mathbb{V}(a)$, one must have $wc(\boxed{j}) > 0$, for all $j \in \tau_2$.

If $wc(\boxed{\mathsf{min}}) < 0$, then we need $v_\alpha \in {}^w\mathbb{V}(a)$ for $\alpha = \epsilon_{n-j} - \epsilon_n$, where we set $c(\boxed{j}) = n - j$ for $j = 1, 2$ (the first case follows by the induction hypothesis, and the second appears if τ_2 is not a singleton). But this implies that in order for

$$H_\bullet(\mathcal{E}^{a_\sigma}_{v_\tau}[w \cdot s_\sigma^{-1}]) = H_\bullet((\mu^{a_\sigma}_w)^{-1}(v_\tau))$$

to carry L_τ, the orbit $\mathcal{O}_\tau$ must meet $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}^2) \subset \mathfrak{N}^a$ if τ_2 is not a singleton, respectively $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \subset \mathfrak{N}^a$ if τ_2 is a singleton, in its open dense part. But since $\boxed{\mathsf{min}}$ and $\boxed{1}$ are not in the same segment of τ, this is not the case for τ. ☐

Corollary 3.15. *1. Assume Algorithm 3.3 produces $L^- = \emptyset$ for σ. Then the output of the algorithm $\mathsf{out}(\sigma)$, which is the positive ladder, is a discrete series. In particular, this is the case when $q_1 > q_2^{n-1} > 1$.*

2. Assume Algorithm 3.3 produces $L^+ = \emptyset$ for σ. Then the output of the algorithm $\mathsf{out}(\sigma)$, which is the negative ladder, is a discrete series. In particular, this is the case when $q_1 < q_2^{n-1} < 1$.

Proof. Assume $L^- = \emptyset$, so that $\mathsf{out}(\sigma)$ is the positive ladder τ. Then any weight $w \cdot s_\sigma^{-1}$ of L_τ is given by a permutation of the entries of s_σ^{-1} by Proposition 3.14.

For each $w \cdot s_\sigma^{-1} \in \Psi(L_\tau)$, we have a sequence of integers $(i_1^m, \ldots, i_{\#I_m}^m)$ (associated to each $I_m \in \mathbf{I}$) so that

$$\left\langle s_\sigma, \epsilon_{i_k^m} \right\rangle = q_2 \left\langle s_\sigma, \epsilon_{i_{k+1}^m} \right\rangle \text{ and } w(i_k^m) < w(i_{k+1}^m) \qquad (1 \leq k < \#I)$$

by Proposition 2.8. It follows that $\langle w \cdot s_\sigma^{-1}, \epsilon_1 + \cdots + \epsilon_j \rangle < 1$, for all $1 \leq j \leq n$. The case $L^+ = \emptyset$ is analogous. ☐

3.5 Proof of the main theorem

We continue with the proof of Theorem 3.5. Recall that σ is a partition of n, with associated semisimple element s_σ. We wish to prove that $\mathsf{out}(\sigma)$ is tempered, or equivalently $\mathsf{out}(\sigma) = \mathsf{ds}(\sigma)$. Assume that in the first two steps of 3.3, the segments produced are $L^+ \sqcup L^- = \{I_1, \ldots, I_N\}$. Section 3.4 proves the claim when either $L^+ = \emptyset$ or $L^- = \emptyset$. We may now assume that both L^+ and L^- are nonempty. Firstly, we give a condition for σ such that $\mathsf{ds}(\sigma)$ (and also $\mathsf{out}(\sigma)$) admits a nested component decomposition.

Proposition 3.16. *We retain the previous notation. Assume that there exists a positive integer M, with $M < N/2$ such that*

$$M = \#\{I_1, \ldots, I_{2M}\} \cap L^+ = \#\{I_1, \ldots, I_{2M}\} \cap L^-.$$

Let s_1, s_2 be the semisimple elements coming from $\sqcup_{k=1}^{2M} e(I_k)$ and $\sqcup_{k=2M+1}^{N} e(I_k)$, respectively. Then $\mathsf{ds}(s_\sigma) = \mathsf{ds}(s_1) \times \mathsf{ds}(s_2)$.

Proof. It suffices to prove the result when $M > 0$ is the smallest number which satisfies the assumption of the claim. Let $K_1, K_2, \ldots$ be the set of extremal hooks of σ taken from outside. Then, we have $K_i = I_{m_i^+} \cup I_{m_i^-}$ for some $I_{m_i^+} \in L^+$ and $I_{m_i^-} \in L^-$ for some $1 \leq m_i^+, m_i^- \leq 2M$ whenever $1 \leq i \leq M$. This implies that either the condition (3.6) or (3.7) holds. Then, we can apply Proposition 3.11 or Proposition 3.12 to deduce the result. ☐

We now prove the result in two particular cases, to which the general case will be reduced.

Proposition 3.17. *Assume that the algorithm runs as*

$$I_1, \ldots, I_r \in L^+, I_{r+1}, \ldots, I_{r+t} \in L^-,$$

for some $0 \leq t \leq r$.

1. *If $w \cdot s_\sigma^{-1}$ is a weight of $L_{\mathsf{out}(\sigma)}$, then $wc(\boxed{x}) > 0$ for all $\boxed{x} \in \tau$ such that $e(\boxed{x}) \geq q_1$.*

2. *$L_{\mathsf{out}(\sigma)}$ is a tempered module.*

Proof. The proof is by induction. For every $0 \leq u \leq t$, define the marked partitions $\tau_1^{(u)}$ and $\tau_2^{(u)}$ as follows: the support of $\tau_1^{(u)}$ is $\{I_{r+u+1}\}$ and it is unmarked, while the support of $\tau_2^{(u)}$ is $\{I_1, \ldots, I_{r-u}, I_{r-u+1} \sqcup I_{r+u}, \ldots, I_r \sqcup I_{r+1}\}$, and every $\boxed{x}$ such that $e(\boxed{x}) = q_1$ is marked. Note that $e(\boxed{y}) < q_1$ for all $\boxed{y} \in \tau_1^{(u)}$. For $u = 0$, $\tau_1^{(0)}$ has support $\{I_{r+1}\}$, while $\tau_2^{(0)}$ has support L^+, and in fact it is a positive ladder. By 3.14, we have $\tau_2^{(0)} = \mathsf{out}(\sigma')$ for some smaller partition σ' and $L_{\tau_2^{(0)}}$ is a tempered module which satisfies condition 1). Notice that $\tau_1^{(t)} = \emptyset$, and $\tau_2^{(t)} = \mathsf{out}(\sigma)$.

We proceed by induction on u to prove that $L_{\tau_2^{(u)}}$ is tempered, and condition 1) is satisfied. As just mentioned, this holds for $u = 0$. Let $u > 0$ be fixed, and assume the theorem holds for all smaller $u' \leq u$, and we will prove it for $u + 1$.

Let $n_1^{(u)}$ and $n_2^{(u)}$ be the sizes of $\tau_1^{(u)}$ and $\tau_2^{(u)}$, respectively. Let $P \supset B$ be the parabolic subgroup with Levi factor $GL(n_1^{(u)}) \times Sp(2n_2^{(u)}) \subset Sp(n_2^{(u+1)})$. Define $W_P = N_P(T)/T \subset W$. We regard $v_{\tau_1^{(u)}}$ as a regular nilpotent Jordan normal form of $\mathfrak{gl}_{n_1^{(u)}}$ and $v_{\tau_2^{(u)}}$ as a normal form (see Proposition 1.27) of an exotic representation of $Sp(2n_2^{(u)})$. We have $M^{\mathsf{A}}_{\tau_1^{(u)}} = L^{\mathsf{A}}_{\tau_1^{(u)}}$ as (one-dimensional) modules for $\mathbb{H}^{\mathsf{A}}_{n_1}$.

Claim E. *We have $\mathcal{O}_{\tau_1^{(u)} \times \tau_2^{(u)}} \subset \overline{\mathcal{O}_{\tau_2^{(u+1)}}}$ and*

$$\dim \mathcal{O}_{\tau_1^{(u)} \times \tau_2^{(u)}} + 1 = \dim \mathcal{O}_{\tau_2^{(u+1)}}.$$

Proof. We set $I^\star := I_{r+u+1} \cup I_{r-u}$. The segment $J := I_{r+u+1}$ satisfies $J \sqsubset I$ or $\underline{J} \cap \underline{I} = \emptyset$ for each $I \in \tau_2^{(u)}$. We have $I_{r-u} \lhd I$ or $I_{r-u} \rhd I$ for $I \in \tau_2^{(u)}$ if and only if $I^\star \lhd I$ or $I^\star \rhd I$, respectively. It follows that $u_{\tau_2^{(u+1)}} = u_{\tau_2^{(u)}} + u_{\tau_1^{(u)}}$. (The definition of u_τ is as in Corollary 1.33.) Using Corollary 1.33, we conclude the dimension estimate. The existence of closure relation is straight-forward since we have an attracting map from $v_{\tau_2^{(u+1)}}$ to $v_{\tau_1^{(u)}} + v_{\tau_2^{(u)}}$ defined as the scalar multiplication of the T-component of $v_{\tau_2^{(u+1)}}$ which does not appear in $v_{\tau_1^{(u)}} + v_{\tau_2^{(u)}}$. $\square$

We return to the proof of Proposition 3.17. Notice that both of $\mathcal{O}_{\tau_1^{(u)} \times \tau_2^{(u)}}$ and $\mathcal{O}_{\tau_2^{(u+1)}}$ are open subsets of vector bundles over their projections to $V_2^{(s,q_2)}$ with equal dimensional fibers. It follows that the regularity of the orbit closure $\mathcal{O}_{\tau_1^{(u)} \times \tau_2^{(u)}} \subset \overline{\mathcal{O}_{\tau_2^{(u+1)}}}$ is equivalent to the regularity of the corresponding orbit closure in $V_2^{(s,q_2)}$. We identify $V_2^{(s,q_2)}$ with some type A-quiver representation space. By the Abeasis-Del Fra-Kraft theorem [ADK81], $\overline{\mathcal{O}_{\tau_2^{(u+1)}}}$ is normal along $\mathcal{O}_{\tau_1^{(u)} \times \tau_2^{(u)}}$ since its projection to $V_2^{(s,q_2)}$ is so. Since normality implies regularity in codimension one, it follows that $\dim H^\bullet_{\mathcal{O}_{\tau_1^{(u)} \times \tau_2^{(u)}}}(\mathsf{IC}(\mathcal{O}_{\tau_2^{(u+1)}})) = 1$. Hence, the

Ginzburg theory implies

$$[M_{\tau_1^{(u)}\times\tau_2^{(u)}} : L_{\tau_2^{(u+1)}}] = [\mathrm{Ind}_{\mathbb{H}_P}^{\mathbb{H}}({}^{\mathsf{t}}M^{\mathsf{A}}_{\tau_1^{(u)}} \boxtimes M_{\tau_2^{(u)}}) : L_{\tau_2^{(u+1)}}] = 1 > 0. \tag{3.9}$$

In other words, $L_{\tau_2^{(u+1)}}$ is an $\mathbb{H}$-subquotient of $M_{\tau_1^{(u)}\times\tau_2^{(u)}}$.

In particular, every $R(T)$-weight $s_0 \in \Psi(L_{\tau_2^{(u+1)}})$ can be written as $s_0 = w \cdot s^{-1}$ by $w \in W$, which is a minimal coset representative of W/W_P. Such w is written as $w = w_1(v_1 \times 1)$, where $w_1 \in \mathfrak{S}_n$, $v_1 \in (\mathbb{Z}/2\mathbb{Z})^{n_1^{(u)}} \subset W_{\mathsf{B}_{n_1^{(u)}}}$, and $1 \in W_{\mathsf{B}_{n_2^{(u)}}}$. Therefore, we have

$$s = w_1 \cdot (s_1 \times s_2),$$

where $s_1 \in (\mathbb{Z}/2\mathbb{Z})^{n_1^{(u)}}\Psi(M^{\mathsf{A}}_{\tau_1^{(u)}}) = \Psi(L^{\mathsf{A}}_{\tau_1^{(u)}})$, $s_2 \in \Psi(M_{\tau_2^{(u)}})$, and $w_1 \in \mathfrak{S}_n$.

Taking into account the fact that we have no $\boxed{x} \in \tau_1^{(u)}$ such that $e(\boxed{x}) \geq q_1$, we deduce

$$wc(\boxed{x}) > 0 \text{ if } e(\boxed{x}) \geq q_1. \tag{3.10}$$

Let $v \cdot s_\sigma^{-1}$ be a $R(T)$-weight of $L_{\tau_2^{(u+1)}}$. Applying Proposition 2.8 and (3.10), we deduce that for each marked segment $(\{I\}, \delta|_{\{I\}})$, we have either

- For each $i < j \in c(I)$, we have $0 < v(i) < v(j)$;
- There exists a unique $k_I \in I$ such that $e(k_I) < q_1$ and for all $i, j \in c(I)$ we have:

  $$0 < v(i) < v(j) \text{ if } i < j < c(k_I), v(c(k_I)) \leq v(i) < v(j) < 0 \text{ if } c(k_I) < i < j$$

Let $\varpi_l^I := \sum_{i \in c(I); |v(i)| < l} \epsilon_{|v(i)|}$. Then, we have

$$\left\langle v \cdot s_\sigma^{-1}, \varpi_l^I \right\rangle < 1.$$

This implies that $L_{\tau_2^{(u+1)}}$ must be tempered. Therefore, the induction proceeds and we obtained the result. □

Proposition 3.17 has the following counterpart, with the analogous proof.

Proposition 3.18. *Assume that the algorithm runs as*

$$I_1, \ldots, I_r \in L^-, I_{r+1}, \ldots, I_{r+t} \in L^+,$$

for some $0 \leq t \leq r$.

1. *If* $w \cdot s_\sigma^{-1}$ *is a weight of* $L_{\mathsf{out}(\sigma)}$, *then* $wc(\boxed{x}) < 0$ *for all* $\boxed{x} \in \mathsf{out}(\sigma)$ *such that* $e(\boxed{x}) \leq q_1$.
2. $L_{\mathsf{out}}(\sigma)$ *is a tempered module.*

Proof. The proof (of Proposition 3.17) works by changing the definition of $\tau_2^{(u)}$ so that the support is

$$I_1, \ldots, I_{r-u}, (I_{r-u+1} \cup I_{r+u}), \ldots, (I_r \cup I_{r+1}),$$

and set the support of $\tau_1^{(u)}$ to be I_{r+u+1}. □

Theorem 3.19 (also Theorem 3.5)**.** *The output* $\mathsf{out}(\sigma)$ *of Algorithm 3.3 defines a tempered module.*

Proof. There are two cases, depending if the Algorithm 3.3 begins with $I_1 \in L^+$ or $I_1 \in L^-$. We present the case $I_1 \in L^+$, the other situation being analogous. We fix some c-function of $\mathsf{out}(\sigma)$. Denote $\mathsf{out}(\sigma) = (\mathbf{I}, \delta)$. Assume that the algorithm runs as

$$I_1, \dots, I_r \in L^+, I_{r+1}, \dots, I_{r+t} \in L^-, \emptyset \neq I_{r+t+1} \in L^+, \dots,$$

for some $0 < t < r$. (Note that 3.16 deals in particular with the case $t \geq r$, while 3.17 considered the situation $I_{r+t+1} = \emptyset$.) From step 3 of 3.3, we see that the first t segments in $\mathbf{I}$ (with respect to $\prec$) are $I_r \sqcup I_{r+1}, I_{r-1} \sqcup I_{r+2}, \dots, I_{r-t+1} \sqcup I_{r+t}$.

Set $\tau_1 = (\{I_r \sqcup I_{r+1}\}, \delta|_{\{I_r \sqcup I_{r+1}\}})$, and let τ_2 be the marked partition obtained from $\mathsf{out}(\sigma)$ by removing the segment $I_r \sqcup I_{r+1}$. Denote by n_1, n_2, their sizes, $n_1 + n_2 = n$. Let $\mathbb{H}_P$ be the Hecke algebra for $GL(n_1) \times Sp(2n_2)$. It is clear that $I_r \sqcup I_{r+1}$ attains the minimal e-value. Hence, Theorem 1.14 is applicable and $L_{\mathsf{out}(\sigma)}$ is a subquotient of $\mathrm{Ind}_{\mathbb{H}_P}^{\mathbb{H}}(L_{\tau_1} \boxtimes L_{\tau_2})$.

In terms of the partitions, I_r and I_{r+1} correspond some row part and column part of the partition obtained by extracting $I_1, \dots, I_{r-1}$ from σ. So if σ_2 is the partition obtained from removing these two pieces, then we have $\tau_2 = \mathsf{out}(\sigma_2)$. By induction, we may assume that τ_2 is tempered. But notice that τ_1 is not a tempered module of $GL(n_1)$ in general, so we need to check that L_{τ_1}'s contribution to $\Psi(L_{\mathsf{out}(\sigma)})$ satisfies the temperedness condition.

Define the subset $I^* \subset I_r \sqcup I_{r+1}$ by

$$\boxed{x} \in I^* \text{ if } e(\boxed{x}) > e(\boxed{y})^{-1}, \text{ for every } \boxed{y} \in I_{r+1}.$$

(Since $I_r \in L^+$ is picked in the algorithm before $I_{r+1} \in L^-$, we have $I^* \neq \emptyset$.) One can see easily that if $\boxed{x} \in I^*$, then $e(\boxed{x}) > e(\boxed{y})$ (equivalently, $c(\boxed{x}) < c(\boxed{y})$), for every $\boxed{y} \in I_{r+t+1}$. Actually, this relation holds for every $\boxed{y} \in \sqcup_{k \geq 1} I_{r+t+k}$, since $I_{r+t+1} \in L^+$, and so the maximal e-value in $\sqcup_{k \geq 1} I_{r+t+k}$ is in I_{r+t+1}. Moreover, we have $e(\boxed{x}) > q_1$ for every $\boxed{x} \in I^*$.

Taking account into Corollary 2.9, it is sufficient to prove that

$$wc(I^*) > 0, \text{ for every } w \cdot s_\sigma^{-1} \in \Psi(L_{\mathsf{out}(\sigma)}). \tag{3.11}$$

If we think of w, as always, as acting by permutations of $[1, n]$ and sign changes, then in order to have $wc(\boxed{x}) < 0$ for some $\boxed{x} \in I^*$, it is sufficient to look at $\boxed{x}$ with $e(\boxed{x}) \geq \min \underline{I^*}$. This means that one can ignore the part of $\mathsf{out}(\sigma)$ coming from $I_{r+t+1}, \dots$. In other words, (3.11) is equivalent with

$$wc(I^*) > 0, \text{ for every } w \cdot s_\sigma^{-1} \in \Psi(L_{\tau^+}), \tag{3.12}$$

where τ^+ is the marked subpartition of $\mathsf{out}(\sigma)$ obtained by neglecting I_k for $k > r + t$. Now the assumption of Proposition 3.17 is satisfied. Proposition 3.17 1) implies that L_{τ^+} is tempered, while 2) implies that (3.12) and (3.11) hold. $\square$

3.6 A characterization of $\mathsf{ds}(\sigma)$

We finish this section with certain combinatorial properties that the output $\mathsf{out}(\sigma)$ must satisfy. By (repeated applications of) Propositions 3.11 and 3.12,

$\mathsf{out}(\sigma)$ acquires a nested component decomposition whenever the products of e-values in the hooks of σ are not uniformly greater than 1 or not uniformly less than 1. By Lemma 2.10, the same is true for every other $G(s_\sigma)$-orbit which contains $G(s_\sigma) \cdot \mathsf{out}(\sigma)$ in its closure. Let us refer to this decomposition here as the "σ-hook nested components" decomposition. The hooks of σ which contribute to a given σ-hook nested component have the products of e-values uniformly greater than 1, in which case we call the component positive, or uniformly less than 1, in which case we call it negative. As an application of Theorem 3.5 and Algorithm 3.3, we obtain a combinatorial characterization of $\mathsf{out}(\sigma) = \mathsf{ds}(\sigma)$. Consider the following properties for a marked partition $\tau = (\mathbf{J}', \delta')$:

(p1) for every $i \geq 0$, there are at most i segments in $\mathbf{J}'$ with all e-values greater than $q_1 q_2^{-i+1}$.

(n1) for every $i \geq 0$, there are at most i segments in $\mathbf{J}'$ with all e-values less than $q_1 q_2^{i-1}$.

(p2) for every segment $J' \in \mathbf{J}'$, we have $\prod_{j' \in J'} e(j') > 1$.

(n2) for every segment $J' \in \mathbf{J}'$, we have $\prod_{j' \in J'} e(j') < 1$.

(p3) for every $J' \in \mathbf{J}'$, if $q_1 \in e(J')$, then $\delta(J') = 1$.

Corollary 3.20. *The $G(s_\sigma)$-orbit $\mathsf{out}(\sigma) = \mathsf{ds}(\sigma)$ is minimal among all $G(\sigma)$-orbits τ admitting the σ-hook nested decomposition and satisfying the properties:*

1. *(p1), (p2), (p3) on every positive σ-hook nested component.*
2. *(n1), (n2) on every negative σ-hook nested component.*

Proof. It is sufficient to check the claim when σ has only one hook nested component. The case when σ consists of a single hook is easily verified directly (see Example 3.6).

More generally, let us assume that the second step of the algorithm runs as $I_1, I_2, \ldots$. There are two situations with respect to I_1: either there exists $k \geq 2$ such that I_k combines with I_1 in the third step of the algorithm, or if not, then I_1 appears in the support of $\mathsf{out}(\sigma) = (\mathbf{J}, \delta)$ by itself.

• In the first case, we must be in the setting of Proposition 3.16, and therefore, the proof may reduced further to the setting when the algorithm for σ runs as $I_1, I_2, \ldots, I_{2M}$ for some $M \geq 1$, and such that $\#\{I_1, \ldots, I_{2M}\} \cap L^+ = M = \#\{I_1, \ldots, I_{2M}\} \cap L^-$. Then $\mathsf{out}(\sigma)$ has exactly M segments in its support all of the form $I_j \sqcup I_{j'}$ (see figure 3.6).

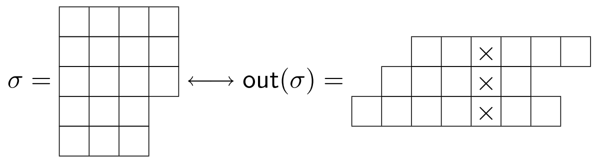


Figure 4: Output of Algorithm 3.3 when $\sigma = (4, 4, 4, 3, 3)$ and $q_2 < q_1 < q_2^{\frac{3}{2}}$

From the algorithm we see that every segment $J \in \mathbf{J}$ in $\mathsf{out}(\sigma) = (\mathbf{J}, \delta)$ contains q_1 as an e-value and $\delta(J) = 1$ (the product of e-values of J being greater than 1). We claim that there is no τ in the closure of $\mathsf{out}(\sigma)$ which can satisfy the required conditions. Let J, J' be two segments in $\mathbf{J}$, and we assume that $J \prec J'$. Since $e(J) \cap e(J') \neq \emptyset$ (because q_1 is in the intersection), there are only two cases: either $J \lhd J'$, or else $J' \sqsubset J$. If $J \lhd J'$, by the closure relations of section 1.4, we see that J, J' cannot combine to give a smaller orbit. Assume $J' \sqsubset J$. Then from step 2 of the algorithm one sees that necessarily $(\min J)^{-1} > \max J'$. If they combine two give a smaller orbit τ, then τ must contain J_1, J_1' such that $e(J_1) = \{\min e(J), q_2 \min e(J), \ldots, \max e(J')\}$, and $e(J_1') = \{\min e(J'), q_2 \min e(J'), \ldots, \max e(J)\}$. But then condition (p2) fails for J_1. By a similar argument, one may also see that if a single segment $J \in \mathbf{J}$ is broken into two pieces such (p1) holds, then the smaller segment with respect to $\prec$ has to fail (p2).

• In the second case, I_1 forms a segment in $\mathbf{J}$ by itself. If $\tau = (\mathbf{J}', \delta')$ is in the closure of $\mathsf{out}(\sigma)$ and satisfies the required assumptions, then we see that $I_1 \in \mathbf{J}'$. (This is because of the conditions (p1,2), the segment I_1 cannot be broken into two pieces to yield such a τ, and it is also clear that if it is combined with some other segment, the resulting marked partition would not be in the closure of $\mathsf{out}(\sigma)$. So one can ignore the segment I_1 from consideration. This amounts to analyzing a smaller partition σ' which is obtained from σ by removing the first row and replacing m by $m-1$, in the positive nested component case, or by removing the first column and replacing m by $m+1$ in the negative nested component case. Then one proceeds by induction. □

4 Applications of the classification

We present some consequences of the classification to the structure of discrete series. Recall that $\mathbb{H}_{n,m}$ denotes the affine Hecke algebra of type B_n with parameter $\vec{q} = (-t^{m/2}, t^{m/2}, t)$, $m > 0$. The generic values of m are all positive real numbers *not* in the set $\{0, 1/2, 1, 3/2, 2, \ldots, n - 1/2\}$.

4.1 Discrete series and deformations

One immediate corollary of the algorithm is the classification of discrete series which contain the sgn W-representation, including for nongeneric values m. (At generic values of m, the inequalities in Corollary 4.1 are all strict.)

Corollary 4.1. *Let σ be a partition of n. The discrete series $\mathsf{ds}(\sigma)$ contains the sgn W-representation if and only if $\mathsf{ds}(\sigma)$ parameterizes the open $G(a)$-orbit in $\mathfrak{N}^a$. In the notation of §3.1, the condition is that $\mathsf{mp}(\sigma) = \mathsf{ds}(\sigma)$, which happens if and only if their support $\{I_m\}_{m=1}^k$ satisfy:*

$$\max e(I_1) \geq (\min e(I_1))^{-1} \geq \max e(I_2) \geq (\min e(I_2))^{-1} \geq \ldots. \tag{4.1}$$

Proof. The condition (4.1) implies that we have $I_k \sqsubset I_l$ for every $k > l$. Hence, we cannot apply the algorithm of Theorem 1.29. It follows that $\mathsf{ds}(\sigma)$ defines the dense open orbit when projected to $V_2^{(s,q_2)}$-part. Moreover, we have $\prod_{i \in I_m} e(i) > 1$ for every m. This implies that the temporary marking δ' in Algorithm 3.3 is uniformly marked. Therefore, we deduce that $\mathsf{mp}(\sigma) = \mathsf{ds}(\sigma)$ as desired. □

Recall that the discrete series for $\mathbb{H}_{n,m}$ are in one-to-one correspondence $\sigma \leftrightarrow \mathsf{ds}(\sigma)$ with partitions σ of n. To every σ and m, one attached a semisimple element $s_{\sigma,m}$. The following corollary describes the properties of this family of $\mathbb{H}_{n,m}$-modules, as m varies in appropriate intervals.

Corollary 4.2. *Let $k < m < k + 1/2$ for some $k \in \{0, 1/2, 1, \ldots, n - 1/2\}$. Let $\mathsf{ds}(\sigma)$ be the parameter of discrete series of $\mathbb{H}_{n,m}$ attached to σ. Let a_m be a family of semi-simple element attached of σ with $\vec{q} = (-t^{m/2}, t^{m/2}, t)$. Assume that $L_{(a_m, v_{\mathsf{ds}(\sigma)})}$ contains sgn as $\mathbb{C}[W]$-modules. Then, the modules in the family $\{L_{(a_m, v_{\mathsf{ds}(\sigma)})}\}_m$ in the region $k \leq m \leq k + 1/2$:*

1. *have the same dimension;*
2. *are all simple tempered modules;*
3. *are all isomorphic as W-representations.*

Proof. Taking account into Corollary 2.18 and Theorem 1.22, it suffices to prove that $v_{\mathsf{ds}(\sigma)}$ defines an open dense orbit of $\mathfrak{N}^{a_m}$ when $m = k, k + 1/2$. By the description of the orbit structure of $\mathfrak{N}^{a}$ in [Ka08a] 1.17, we deduce that we obtain no new orbits by the specialization process. Hence, the condition (4.1) with $>$ replaced by $\geq$ is already enough to guarantee that $\mathcal{O}^{a_m}_{\mathsf{ds}(\sigma)} \subset \mathfrak{N}^{a_m}$ is dense open for every $k \leq m \leq k + 1/2$. □

Notice that when $k < m < k + 1/2$, every $L_{(a_m, v_{\mathsf{ds}(\sigma)})}$ is in fact a discrete series, but at the endpoints of the interval, $m \in \{k, k + 1/2\}$ they could be just tempered. On the other hand, Corollary 2.19 effectively says that if $L_{(a_m, v_{\mathsf{ds}(\sigma)})}$ is a discrete series in the interval $k < m < k + 1/2$, but also in the interval $k - 1/2 < m < k$, then it is a discrete series at $m = k$. This gives a combinatorial condition on σ, viewed as a tableau for m, as follows. The idea is due to [Sl06], 5.3, 5.17.

Definition 4.3. Assume k is a critical value, and let $\{k\}$ denote the fractional part of k (which is 0 or 1/2). The extremities of σ at k is the set $E(\sigma, k)$ defined by the procedure: put in $E(\sigma, k)$ the maximal entry in every row above or on the $t^{\{k\}}$-diagonal, and also the inverse of the minimal entry in every column below or on the $t^{-\{k\}}$-diagonal. One allows repetitions in this set, if they exist.

Corollary 4.4. *Assume k is a critical value. If the family $\{L_{(a_m, v_{\mathsf{ds}(\sigma)})}\}$ consists of discrete series in the interval $k - 1/2 < m < k + 1/2$, then the set $E(\sigma, k)$ does not have repetitions.*

Proof. In combinatorial terms, the condition that $\{L_{(a_m, v_{\mathsf{ds}(\sigma)})}\}$ consists of discrete series in the interval $k - 1/2 < m < k + 1/2$ means that the output of the Algorithm 3.3 is the same for σ when $k - 1/2 < m < k$ or $k < m < k + 1/2$. This is equivalent to the fact that step 2 of the algorithm is the same in these two intervals, which implies that step 2 of the algorithm is well-defined at $m = k$ as well. From this, it is easy to see that $E(\sigma, k)$ must not allow repetitions. □

4.2 Tempered modules in generic parameters

Let us assume that m is generic. Let $\mathbb{H}^{\mathsf{A}}_{n,m}$ be the subalgebra of $\mathbb{H}_{n,m}$ generated by $R(T)$ and $T_1, \ldots, T_{n-1}$. This is an affine Hecke algebra for $GL(n)$.

Theorem 4.5 ([KL87], [Ze80]). *The set of tempered modules with positive real central character of $\mathbb{H}^{\mathsf{A}}_{n,m}$ is in one-to-one correspondence with the set of partitions of n. Fix a partition $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ of n. The corresponding tempered module $L^{\mathsf{A}}_{(a,v_\tau)}$ is*

1. *Form a sequence $\mathbf{I} = \{I_k\}$ of subsets of $[1,n]$ by setting $I_1 = [1,\sigma_1], \ldots, I_k = [\sigma_1 + \cdots + \sigma_{k-1} + 1, \sigma_1 + \cdots + \sigma_k] \ldots$.*
2. *Set the trivial labeling $\delta \equiv 0$ and form $\tau := (\mathbf{I}, \delta)$;*
3. *Form a semi-simple element $s \in T$ such that τ is adapted to $a = (s, \vec{q})$ and*
$$\{\langle \epsilon_i, s\rangle\, ; i \in I_k\} = \{\langle \epsilon_i, s\rangle^{-1}\, ; i \in I_k\} \text{ for each } k.$$

More precisely, the set of tempered modules with positive real central character for type A_{n-1} are in one to one correspondence with nilpotent adjoint orbits of type A_{n-1}, and this is in turn are parameterized by partitions of n. If σ is a partition as above, then one forms the (block upper triangular) parabolic subgroup P with Levi subgroup $GL(\sigma_1)\times\cdots\times GL(\sigma_\ell)$. Let $\mathbb{H}_P = \mathbb{H}^{\mathsf{A}}_{\sigma_1,m}\times\cdots\times\mathbb{H}^{\mathsf{A}}_{\sigma_\ell,m}$ be the Hecke subalgebra of $\mathbb{H}^{\mathsf{A}}_{n,m}$ corresponding to P. Let St_{σ_j} denote the Steinberg $\mathbb{H}^{\mathsf{A}}_{\sigma_j,m}$-module. The induced module $\mathsf{Ind}_{\mathbb{H}_P}^{\mathbb{H}^{\mathsf{A}}_{n,m}}(\mathsf{St}_{\sigma_1}\boxtimes\cdots\boxtimes\mathsf{St}_{\sigma_\ell})$ is irreducible and tempered, and this is L_{a,v_τ}, in the notation of 4.5. The element s corresponds to the middle element of the nilpotent orbit parameterized by σ. Explicitly, $s = \exp(\frac{\sigma_1-1}{2}, \ldots, -\frac{\sigma_1-1}{2}, \ldots, \frac{\sigma_\ell-1}{2}, \ldots, -\frac{\sigma_\ell-1}{2})$.

When we have $S_0 = \Pi - \{\alpha_{n_1}\}$, then we have $L_{S_0} \cong GL(n_1) \times Sp(2n_2)$. We have $\mathfrak{N} \cap \mathbb{V}_{S_0} \subset \mathfrak{gl}(n_1) \oplus \mathbb{V}_{(2)}$ (as L_{S_0}-varieties), where $\mathbb{V}_{(2)}$ is the exotic representation of $Sp(2n_2)$.

Theorem 4.6. *Let $S_0 = \Pi - \{\alpha_{n_1}\}$. Let $a = (s, \vec{q}) \in \mathcal{G}_0^+ \cap L_{S_0}$. We define s_1, s_2 to be the projections of s to $GL(n_1)$ and $Sp(2n_2)$, respectively. Fix $X \in \mathfrak{N}^a$ and a decomposition $X = X_1 \oplus X_2 \in \mathfrak{gl}(n_1) \oplus \mathbb{V}_{(2)}$. We assume*

1. *$L^{\mathsf{A}}_{(s_1,q_2,X_1)}$ and $L_{(s_2,\vec{q},X_2)}$ define irreducible modules of $\mathbb{H}^{\mathsf{A}}_{n_1,m}$ and $\mathbb{H}_{n_2,m}$, respectively;*
2. *We have $\{\langle \epsilon_i, s\rangle\, ; 1 \le i \le n_1\} \subset q_2^{\frac{1}{2}\mathbb{Z}}$ and $\{\langle \epsilon_i, s\rangle\, ; n_1 < i \le n\} \subset q_1 q_2^{\mathbb{Z}}$.*

Then, we have an isomorphism

$$\mathsf{Ind}^{\mathbb{H}}_{\mathbb{H}^{S_0}}(L^{\mathsf{A}}_{(s_1,q_2,X_1)} \boxtimes L_{(s_2,\vec{q},X_2)}) \cong L_{(a,X)} \tag{4.2}$$

as $\mathbb{H}$-modules.

Proof. The assumption (1.7) implies that $q_2^{\frac{1}{2}\mathbb{Z}} \cap q_1 q_2^{\mathbb{Z}} = \emptyset$. If follows that $(\mathbb{V}^{S_0})^a = \{0\}$, hence the induction theorem is applicable. Since we have $L_{S_0}(s) = G(s)$ and $\mathfrak{N}^a = \mathfrak{N}^a \cap (\mathfrak{gl}(n_1) \oplus \mathbb{V}_{(2)})$, it follows that the isomorphism classes of irreducible $\mathbb{H}_a$-modules and irreducible $\mathbb{H}_a^{S_0}$-modules are in one-to-one correspondence through the identification of parameters. For $X \in \mathcal{O}_0^a$, the Ginzburg theory implies that both $M^{S_0}_{(a,X)}$ and $M_{(a,X)}$ are irreducible modules of $\mathbb{H}_a^{S_0}$ and $\mathbb{H}_a$, respectively. Hence, the assertion holds in this case. We prove the assertion by induction on the closure relation of orbits. Let $\mathcal{O} \subset \mathfrak{N}^a$ be a $G(s)$-orbit. Assume that (4.2) holds for all $\mathbb{H}_a^{S_0}$-module such that the orbit closure of the corresponding $G(s)$-orbit contains $\mathcal{O}$. Let $X \in \mathcal{O}$. By the induction theorem, we have

$$\mathsf{Ind}_{\mathbb{H}^{S_0}}^{\mathbb{H}} M_{(a,X)}^{S_0} \cong M_{(a,X)}.$$

Moreover, the Ginzburg theory asserts that the multiplicity of each irreducible module inside $M_{(a,X)}^{S_0}$ and $M_{(a,X)}$ (as $\mathbb{H}^{S_0}$ and $\mathbb{H}$-modules, respectively) is the same, under the correspondence between irreducibles of $\mathbb{H}_{S_0}$ and $\mathbb{H}$. Hence we deduce

$$\mathsf{Ind}_{\mathbb{H}^{S_0}}^{\mathbb{H}} L_{(a,X)}^{S_0} \cong L_{(a,X)},$$

where $L_{(a,X)}^{S_0}$ is the unique $\mathbb{H}_{S_0}$-module corresponding to $\mathcal{O}$. This is nothing but (4.2). Hence, the induction proceeds and we conclude the result. □

Theorem 4.7. *Let $S_0 = \Pi - \{\alpha_{n_1}\}$. Let $a = (s, \vec{q}) \in \mathcal{G}_0^+ \cap L_{S_0}$. We define s_1, s_2 to be the projections of s to $GL(n_1)$ and $Sp(2n_2)$, respectively. Fix $X \in \mathfrak{N}^a$ and a decomposition $X = X_1 \oplus X_2 \in \mathfrak{gl}(n_1) \oplus \mathbb{V}_{(2)}$. We assume*

1. *$L_{(s_1,q_2,X_1)}^{\mathsf{A}}$ is a tempered module of $\mathbb{H}_{n_1,m}^{\mathsf{A}}$;*
2. *$L_{(s_2,\vec{q},X_2)}$ is a discrete series of $\mathbb{H}_{n_2,m}$.*

Then, $L_{(a,X)}$ is a tempered $\mathbb{H}$-module.

Proof. The proof of the induction theorem (see [Ka08a] §7 or [KL87] §7) claims that we have an isomorphism

$$H_\bullet(\mu^{-1}(X)^a) \cong \bigoplus_w H_\bullet(\mu^{-1}(X)^a \cap P_{S_0}\dot{w}^{-1}B/B) \text{ as vector spaces,} \tag{4.3}$$

where $w \in W/(\mathfrak{S}_{n_1} \times W_{\mathsf{B}_{n_2}})$ denotes its minimal length representative in W. Here we have $(P_{S_0}\dot{w}^{-1}B/B)^s \cong G(s)/B$, which implies that (4.3) is in fact a direct sum decomposition as $\mathbb{H}^{S_0}$-modules (up to semi-simplification). It follows that weight $y = w \cdot s^{-1} \in \Psi(L_{(a,X)})$ is written as $v \cdot ((w_1 \cdot s_1^{-1}) \times s_2')$, where $v \in \mathfrak{S}_n/(\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$, $w_1 \in W_{\mathsf{B}_{n_1}}$ so that the condition $(\clubsuit)_w$ of Proposition 2.8 is satisfied, and $s_2' \in \Psi(L_{(s_2,\vec{q},X_2)})$. (Notice that $(s_1, \vec{q}) \notin \mathcal{G}_0^+$, which means that Proposition 2.8 does not apply as is. However, the same argument still applies thanks to Corollary 2.18.) Let us take $I \subset [1, n_1]$ so that $I = \{i_1, \ldots, i_k\}$ satisfies $i_1 < i_2 < \cdots < i_k$, $\langle \epsilon_{i_l}, s \rangle = q_2 \langle \epsilon_{i_{l+1}}, s \rangle$ and $\langle \epsilon_{i_l}, s \rangle = \langle \epsilon_{i_{k+1-l}}, s \rangle^{-1}$ holds for each $1 \le l \le k$. Then, the condition $v_I \in {}^{w_1}\mathbb{V}(a)$ implies that

$$\langle \omega_p', s^{-1} \rangle \le 1, \text{ where } \omega_p' = \sum_{|w_1(i_l)|<p} \epsilon_{|w_1(i_l)|} \text{ for each } p.$$

This implies that

$$\langle \omega_i, \chi \rangle \le 1 \text{ for every } \chi \in \Psi(L_{(a,X)}) \text{ and every } 1 \le i \le n,$$

which implies that $L_{(a,X)}$ is tempered as desired. □

4.3 Cuspidal local systems in Spin groups

In this section, we explain the constructions and algorithms of Lusztig and Slooten and the relation with our setting.

Let $\overline{\mathbb{H}}_{n,m}$ be the affine graded Hecke algebra of type B_n with unequal parameters (see Definition 1.8), with parameters normalized as:

$$\overset{1}{\circ}\text{---}\overset{1}{\circ}\text{---}\overset{1}{\circ}\text{---}\cdots\text{---}\overset{1}{\circ}\text{---}\overset{1}{\circ}\Rightarrow\overset{m,}{\circ} \tag{4.4}$$

where $4m \equiv 1$ or $3 \bmod 4$.

Define X_ℓ to be the set of nilpotent orbits in $so(\ell)$ parameterized by partitions containing odd parts with multiplicity one, and even parts with even multiplicity. For every nilpotent orbit $\mathcal{O} \subset so(\ell)$ given by the partition $(a_1, \ldots, a_s)$ define the defect of $\mathcal{O}$

$$d(\mathcal{O}) := \sum_{i=1}^{s} d(a_i), \text{ where } d(a_i) = \begin{cases} 1, & \text{if } a_i \equiv 1 \bmod 4 \\ 0, & \text{if } a_i \equiv 0, 2 \bmod 4 \\ -1, & \text{if } a_i \equiv 3 \bmod 4. \end{cases} \tag{4.5}$$

For every $d \in \mathbb{Z}$, set $X_{\ell,d}$ to be the set of elements in X_ℓ of defect equal to d. Then one has

$$X_\ell = \bigcup_{d\in\mathbb{Z}, 4|(\ell-d)} X_{\ell,d}. \tag{4.6}$$

The generalized Springer correspondence ([Lu85]) for the cuspidal local systems in $Spin(\ell)$ which do not factor through $SO(\ell)$ takes the following combinatorial form.

Theorem 4.8 ([Lu85, LS85])**.** *There is a one to one correspondence*

$$X_{\ell,d} \longleftrightarrow \mathsf{Irrep} W_n, \text{ where } n = \frac{\ell - d(2d-1)}{4}.$$

Remark 4.9.
1. It is not hard to see using a generating functions argument ([Lu85]) that the two sets in Theorem 4.8 have the same cardinality. Moreover, a slight modification of that argument shows that the number of distinguished orbits in $X_{\ell,d}$ equals $P(n)$, the number of partitions of n.
2. In the generalized Springer correspondence in this setting, there is a unique local system on each orbit in $X_{\ell,d}$ which enters, and this is why the correspondence can be regarded as one between orbits and Weyl group representations.
3. The relation between $\overline{\mathbb{H}}_{n,m}$ and $X_{\ell,d}$ is given by

$$4n + d(2d-1) = \ell, \; d = -d(4m)[m + 1/4]. \tag{4.7}$$

The left to right map in Theorem 4.8 is given by an explicit algorithm which we recall now. We use the notation for $\mathsf{Irrep}\ W_n$ from Remark 1.24.

Algorithm 4.10 ([LS85]). Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of ℓ of defect d. Here, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$. We will produce inductively a bipartition $\rho(\lambda)$ of $n = \frac{\ell - d(2d-1)}{4}$, which parameterizes an element of $\mathsf{Irrep} W_n$. Define the (smaller) partition μ as follows:

(i) if λ_m is odd, then set $\mu = (\lambda_1, \dots, \lambda_{m-1})$;

(ii) if λ_m is even, then set $\mu = (\lambda_1, \dots, \lambda_{m-2})$.

By induction $\rho(\mu)$ is known, say it is of the form $\rho(\mu) = (\gamma) \times (\delta)$ for some bipartition $(\gamma), (\delta)$.

(a) $d(\lambda_m) = 0$ (λ is even). Set $r = [(\lambda_m + 2)/4] - d(\mu)$ and $s = [\lambda_m/4] + d(\mu)$. (Note that $r + s = \frac{\lambda_m}{2}$.)

  (a1) If $d(\mu) > 0$, set $\rho(\lambda) = (\gamma, r) \times (\delta, s)$.

  (a2) If $d(\mu) \leq 0$, set $\rho(\lambda) = (\gamma, s) \times (\delta, r)$.

(b) $d(\lambda_m) = 1$ ($\lambda_m \equiv 1 \bmod 4$). Set $r = \frac{\lambda_m - 1}{4} - d(\mu)$.

  (b1) If $d(\mu) > 0$, set $\rho(\lambda) = (\gamma, r) \times (\delta)$.

  (b2) If $d(\mu) = 0$, set $\rho(\lambda) = (\delta, r) \times (\gamma)$.

  (b3) If $d(\mu) < 0$, set $\rho(\lambda) = (\gamma) \times (\delta, r)$.

(c) $d(\lambda_m) = -1$ ($\lambda_m \equiv 3 \bmod 4$). Set $r = \frac{\lambda_m - 3}{4} + d(\mu)$.

  (c1) If $d(\mu) > 1$, set $\rho(\lambda) = (\gamma) \times (\delta, r)$.

  (c2) If $d(\mu) = 1$, set $\rho(\lambda) = (\delta, r) \times (\gamma)$.

  (c3) If $d(\mu) < 1$, set $\rho(\lambda) = (\gamma, r) \times (\delta)$.

Theorem 4.11 ([Lu02]). *The tempered modules of $\overline{\mathbb{H}}_{n,m}$ with positive real central character are parameterized by the orbits in $X_{\ell,d}$. The discrete series of $\overline{\mathbb{H}}_{n,m}$ with positive real central character are parameterized by the distinguished orbits in $X_{\ell,d}$. In particular, there are $\#P(n)$ discrete series.*

In [Sl06], a conjecture relating discrete series of $\overline{\mathbb{H}}_{n,m}$, partitions of n, and Weyl group representations (a Springer correspondence) was proposed. We explain this next.

Partitions of n to distinguished orbits. Let σ be a partition of n. We think of σ as left justified Young tableau, with the length of rows decreasing, same as in §3.2. Fill out the boxes of σ starting at the left upper corner with m and increase by one to the right, and decrease by one down. In this way, all the boxes on the diagonal have the entry m. Recall that $m \equiv 1$ or $3 \bmod 4$. Let s_σ denote the collection of the absolute values of the entries of σ (with multiplicities), ordered nonincreasingly. We think of s_σ as being a central character for $\overline{\mathbb{H}}_{n,m}$.

To s_σ, we build a distinguished nilpotent orbit $\mathcal{O}_\sigma$ in $X_{\ell,d}$, $\ell = 4n + d(2d - 1)$, as follows (we are thinking of $\mathcal{O}_\sigma$ as a partition of ℓ with defect d). Let $\{m\} = m - [m]$ denote the fractional part of m. This is either $1/4$ or $3/4$. Start with the cuspidal part $\lambda_c = \{4m - 2, 4m - 6, \dots, 4 - 4\{m\}\}$. This is of the form $\{3, 7, 11, \dots\}$ or $\{1, 5, 9, \dots\}$, depending if $\{m\} = 1/4$ or $3/4$, respectively. When $m = 1/4$, we have $\lambda_c = \emptyset$. Note that the defect of λ_c is d, and the sum of entries in λ_c is $2(m + 1/4)(m - 1/4) = d(2d - 1)$. Set $\lambda = \lambda_c$. For every hook in σ, we will modify λ so that the defect remains the same, and the sum of entries increases by the four times the length of the hook. Assume there are h hooks,

$h \geq 1$. For every hook j starting from the exterior, denote by q_j the entry in the right extremity, and by q'_j the bottom extremity. Note that

$$q_1 > q_2 > \cdots > q_h \geq m \geq q'_h > q'_{h-1} > \cdots > q'_1, \tag{4.8}$$

and that the length of the hook is $q_j - q'_j + 1$. Starting with the most interior hook, for every hook j, there are two cases:

1. if $q'_j \leq 1/4$, add to λ, $4q_j + 2$ and $-4q'_j + 2$ (they have opposite defect);
2. if $q'_j > 1/4$, add to λ, $4q_j + 2$, and remove $4q'_j - 2$ (they have same defect).

The end partition λ is $\mathcal{O}_\sigma$. We summarize the obvious properties of this construction.

Claim F. *The above procedure $\sigma \mapsto \mathcal{O}_\sigma$ is well-defined, and gives a distinguished orbit in $X_{\ell,d}$. Moreover, two different partitions give different elements of $X_{\ell,d}$.*

Example 4.12. Let us consider the example $n = 13$, $m = 9/4$, and the partition $\sigma = (4, 3, 3, 2, 1)$. Then $d = -2$ and $\ell = 62$. We view the partition as:

$\frac{9}{4}$	$\frac{13}{4}$	$\frac{17}{4}$	$\frac{21}{4}$
$\frac{5}{4}$	$\frac{9}{4}$	$\frac{13}{4}$	
$\frac{1}{4}$	$\frac{5}{4}$	$\frac{9}{4}$	
$-\frac{3}{4}$	$\frac{1}{4}$		
$-\frac{7}{4}$			

Figure 5: Partition $(4, 3, 3, 2, 1)$ for $\overline{\mathbb{H}}_{13,\frac{9}{4}}$.

By the algorithm, we start with $\lambda_c = (3, 7)$. There are three hooks. The most interior hook has $q_3 = q'_3 = 9/4$, so we add 11 and remove 7, and get $\lambda = (3, 11)$. Next $q_2 = 13/4$, $q'_2 = 1/4$, so we add 15 and 1, and get $\lambda = (1, 3, 11, 15)$. Finally, $q_1 = 21/4$ and $q'_1 = -7/4$, so we add 23 and 9. Therefore $\mathcal{O}_\sigma = (1, 3, 9, 11, 15, 23)$, which is in $X_{62,-2}$.

Behind the reasoning for this algorithm is the fact that the middle element of the nilpotent $\mathcal{O}_\sigma$ is obtained from the central character s_σ and the middle element for the cuspidal part.

Partitions, distinguished orbits, and W-representations. Let us recall the conjecture of [Sl06], and show that it is equivalent to the [LS85] algorithm presented above.

Algorithm 4.13. Start with σ a partition of n viewed as before. We form a bipartition $S_m(\sigma) = (\gamma) \times (\delta)$ of n as follows. Begin by setting $\gamma = \delta = \emptyset$. Then find the largest in absolute value entry in σ. (This is necessarily one of the extremities of the first hook.) Remove all the boxes to the left of it in the same row (including it), or all boxes above it in the same column (including it). Let x be the number of boxes removed. If they were in the same row, append x to γ, if they were in the same column, append x to δ. Repeat the process until there are no boxes left, or until there is a single box left. In the latter case, if the entry in the single box left is positive, append 1 to γ, if it is negative, append 1 to δ.

Example 4.14. In Example 4.12, first we remove the boxes to the left of 21/4, append 4 to γ, then the boxes to the left of 13/4, append 3 to γ, then the boxes to the left of 9/4, append 3 to γ, then the remaining boxes above $-7/4$, append 2 to δ, finally, the remaining box 1/4, so append 1 to γ. So the bipartition $S_{9/4}(4,3,3,2,1)$ equals $(4,3,3,1)\times(2)$.

Claim G. *For every σ a partition of n, and $4m \equiv 1,3$ mod 4, the W_n-representations (or rather, bipartitions of n) $S_m(\sigma)$ and $\rho(\mathcal{O}_\sigma)$ coincide.*

In other words, the algorithm for the Springer correspondence of Lusztig-Spaltenstein coincides with the algorithm of Slooten in the case when $4m \equiv 1,3$ mod 4.

Proof. We prove this statement by induction on n, the size of the Young tableau. The base $n=2$ is straightforward. Let σ be a partition of $n>2$ viewed as a Young tableau, and $\mathcal{O}_\sigma = (\lambda_1 < \cdots < \lambda_k)$ the orbit constructed before. Since this is a distinguished orbit, only cases (b) and (c) of the Algorithm 4.10 enter. The largest entry in absolute value max is given by one of the extremities q_1 or $-q_1'$ (if $q_1'<0$) of the first hook. It corresponds to λ_k: $\lambda_k = 4\mathsf{max}+2$. There are two cases.

a) Assume $\mathsf{max} = q_1$. Then $d(\lambda_k) = -d(4m)$. Let r' be the number of boxes on the first row in σ, so $r' = q_1 - m + 1$. In 3.3, one forms $r = [\frac{\lambda_k}{4}] - d(\lambda_k)(d(\mathcal{O}_\sigma) - d(\lambda_k)) = [q_1+\frac{1}{2}] + d(4m)(-d(4m))([m+\frac{1}{4}]-1) = [r'+m-1+\frac{1}{2}] - [m+\frac{1}{4}]+1 = r' + [m+\frac{1}{2}] - [m+\frac{1}{4}] = r'$. So $r = r'$. In 4.13, r' is placed in the left side of the bipartition. We check that in 4.10, r is also placed in the left side of the bipartition. There are two subcases: if $d(4m)=1$, then $d(\lambda_k)=-1$, and so, in 4.10 (c), $d(\mu) = -[m+\frac{1}{4}]+1 \le 1$, so we are in the cases (c2,c3); if $d(4m)=-1$, so $m \ge \frac{3}{4}$, $d(\lambda_k)=1$, and so, in 4.10 (b), $d(\mu) = [m+\frac{1}{4}]-1 \ge 0$, so we are in cases (b1,b2).

Now let τ be the Young tableau obtained after removing the first row. The entry in the left upper corner is $m-1$, which is positive unless $m = 1/4, 3/4$. If $m>1$, we can regard τ as a partition of $n-r$ and with $m-1$ instead of m. It is immediate that the correponding $\mathcal{O}_\tau$ is the same as μ in 4.10. By induction if $Sl(\tau) = (\gamma)\times(\delta)$, then $\rho(\mu) = (\gamma)\times(\delta)$, and we are done.

So consider the cases $m = 1/4$ or $m = 3/4$. Then $m-1<0$. Let $\bar{\tau}$ be the Young tableau which is obtained from τ by first taking the transpose tableau and then multiplying all the entries by (-1). The left upper corner of $\bar{\tau}$ has entry $1-m>0$, so we can regard $\bar{\tau}$ as a partition of $n-r$ for $1-m$ (not m), and associate $\mathcal{O}_{\bar{\tau}}$. Note that if $Sl(\tau) = (\delta)\times(\gamma)$, then $Sl(\bar{\tau}) = (\gamma)\times(\delta)$. The only observation left to make is that $\mathcal{O}_{\bar{\tau}} = \mu$, where $\mu = (\lambda_1,\ldots,\lambda_{k-1})$. This follows easily from the algorithm for $\mathcal{O}_\sigma$ and $\mathcal{O}_{\bar{\tau}}$. b) Assume $\mathsf{max} = -q_1' > 0$. Then $d(\lambda_k) = d(4m)$. By the same argument as in case a), one shows that r', the number of boxes in the first column, equals r from 4.10. In 4.13, r' is placed in the right side of the bipartition. We check that in 4.10, r is also placed in the right side of the bipartition. There are two subcases: if $d(4m)=-1$, so that $m \ge \frac{3}{4}$, then $d(\lambda_k)=-1$, and so, in 4.10 (c), $d(\mu) = [m+\frac{1}{4}]+1 > 1$, so we are in the cases (c1); if $d(4m)=1$, $d(\lambda_k)=1$, and so, in 4.10 (b), $d(\mu) = -[m+\frac{1}{4}]-1 < 0$, so we are in case (b3).

If τ is the Young tableau obtained after removing the first column, the entry in the upper left corner is $m+1>0$, so we can regard τ as a partition for $n-r$ and with $m+1$ instead of m. It is immediate that the correponding $\mathcal{O}_\tau$ is the same as μ in 4.10. By induction if $Sl(\tau)=(\gamma)\times(\delta)$, then $\rho(\mu)=(\gamma)\times(\delta)$, and this concludes this case.

□

Since we showed these algorithms yield the same W-representation, let us denote it by $\rho(\sigma)$. There are two particular cases worth mentioning.

Assume that (σ,m) are such that the Springer correspondence algorithm picks only rows, and so $\rho(\sigma)=(a_1,\dots,a_k)\times\emptyset$, or only columns, and so $\rho(\sigma)=\emptyset\times(b_1,\dots,b_k)$. These are the cases we refered as positive, respectively negative, ladders, and so by the previous results, the discrete series representation with central character s_σ is irreducible as an W-module, and equals $\rho(\sigma)\otimes\mathsf{sgn}$. (The tensoring with sgn) is a normalization, so that the Steinberg module is sgn as a W-representation.

The second particular case is that for discrete series which contain the sgn W-representation. By the previous results, if the hook extremities of σ are $q_1>q_2>\dots>q_h\geq m\geq q'_h>q'_{h-1}>\dots>q'_1$, then in order for s_σ to contain the sgn, they must satisfy:

$$q_1>-q'_1>q_2>-q'_2>\dots>q_h>-q'_h.$$

This means that $\rho(\sigma)$ is obtained as follows: remove the first row, then the first column in the remaining tableau, then remove the first row remaining, then the first column etc.

We remark that in these two cases, the W-representation attach to σ by the exotic Springer correspondence coincides with the W-representation attached to σ by the algorithms of Lusztig-Spaltenstein and Slooten.

4.4 W-independence of tempered modules

Using the geometric realization and results of Lusztig for the graded Hecke algebras arising from cuspidal local systems, one was able to prove in [Ci08] a certain independence result for tempered modules with positive real central characters. This is a generalization of the similar result of Barbasch-Moy for Hecke algebras with equal parameters, and it is a Hecke algebra analogue of Vogan's lowest K-types.

Retain the notation from §4.3. We formulate this result in the setting of the graded Hecke algebra $\overline{\mathbb{H}}_{n,m}$ from §4.3, with $4m\equiv 1,3 \pmod 4$. For every tempered module π with positive real central character, which by Theorem 4.11, corresponds to an orbit $\mathcal{O}_\pi\in X_{\ell,d}$, let $\rho(\pi)$ be the generalized Springer correspondence W-representation attached to $\mathcal{O}_\pi$. The following result follows from the fact that any other W-type appearing in the restriction $\pi|_W$ is attached in the generalized Springer correspondence to an orbit larger than $\mathcal{O}_\pi$ in the closure ordering.

Proposition 4.15 (cf. [Ci08]). *1. There is a bijection $\pi\mapsto\rho(\pi)$ between tempered modules $\overline{\mathbb{H}}_{n,m}$, $4m\equiv 1,3$ (mod 4) with positive real central character and* $\mathsf{Irrep}\ W$.

2. *The set of positive real tempered $\overline{\mathbb{H}}$-modules viewed in $R(W)$ is linear independent. Moreover, in the ordering coming from the closure ordering in $X_{\ell,d}$, the change of basis matrix to* $\mathsf{Irrep}\ W$ *is uni-triangular.*

Theorem 1.18 allows us to extend this result to all generic real positive parameters $\vec{q} = (-t^{m/2}, t^{m/2}, t)$, for the affine Hecke algebra of type $B_n^{(1)}$. (Here we use implicitly the correspondence between the affine Hecke algebra and the graded Hecke algebra for positive real central characters.)

Corollary 4.16. *The set of tempered $\mathbb{H}_a$-modules for generic positive real a is W-independent in $R(W)$. Moreover, in the ordering coming from the generalized Springer correspondence, the change of basis matrix to* $\mathsf{Irrep}W$ *is uni-triangular.*

Proof. Let m be in the open interval $(\frac{k}{2}, \frac{k+1}{2})$, for some integer $k \geq 0$ and a be the corresponding generic parameter. Let $\mathsf{MP}(m)$ be the set of (exotic) marked partitions which parameterizes the set of tempered $\mathbb{H}_a$-modules. Set $m_0 = \frac{2k+1}{2}$, and fix $\tau \in \mathsf{MP}(m_0)$. The results in this paper imply that the set $\mathsf{MP}(m)$ is the same for all $m \in (\frac{k}{2}, \frac{k+1}{2})$. Moreover, if we denote by $\mathsf{temp}_m(\tau)$ the tempered module parameterized by τ at the parameter m, then

$$\mathsf{temp}_m(\tau) \cong \mathsf{temp}_{m'}(\tau) \text{ as } W\text{-modules}, \tag{4.9}$$

for any $m, m' \in (\frac{k}{2}, \frac{k+1}{2})$. In particular, $\mathsf{temp}_m(\tau) \equiv \mathsf{temp}_{m_0}(\tau)$, for all $\tau \in \mathsf{MP}(m) = \mathsf{MP}(m_0)$. Then the claim follows from Proposition 4.15. □

Remark 4.17. One can ask naturally if a similar uni-triangular correspondence as in Corollary 4.16 holds if one considers instead the exotic Springer correspondence (see [Ka08b] for an explicit algorithm). This is not the case however: in general, the map assigning to a tempered $\mathbb{H}_s$-module its exotic Springer representation is not one-to-one, as one can see in the example $n = 4$, and $0 < m < 1/2$ for the partitions of n, $\sigma_1 = (2+1+1)$ and $\sigma_2 = (1+1+1+1)$. The exotic Springer map assigns the W_4-representation $(1^4) \times (0)$ to both $\mathsf{ds}(\sigma_1)$ and $\mathsf{ds}(\sigma_2)$, while the generalized Springer map assigns $(1^3) \times (1)$ to $\mathsf{ds}(\sigma_1)$ and $(1^4) \times (0)$ to $\mathsf{ds}(\sigma_2)$.

In particular, the "lowest W-type" correspondence of Corollary 4.16 shows that the construction of Theorem 4.7 exhausts all tempered modules in the real positive generic range.

Corollary 4.18. *Every tempered $\mathbb{H}_a$-module for generic positive real a is obtained by induction as in Theorem 4.7.*

Proof. It is sufficient to show that Theorem 4.7 produces $\#\mathsf{Irrep}W_n$ distinct tempered modules. Let $\mathcal{P}(k)$ denote the number of partitions of k, and $\mathcal{P}_2(k)$ denote the number of bipartitions of k, i.e., $\mathcal{P}_2(k) = \#\mathsf{Irrep}W_k$. For every $1 \leq n_1 \leq n$, a tempered $\mathbb{H}_a$-module is constructed from a tempered $GL(n_1)$ module and a discrete series of $Sp(2n_2)$, where $n_2 = n - n_1$. There are $\mathcal{P}(n_1)$ tempered modules of $GL(n_1)$ and $\mathcal{P}(n_2)$ discrete series of $Sp(2n_2)$. Therefore we get $\sum_{n_1=1}^{n} \mathcal{P}(n_1)\mathcal{P}(n-n_1) = \mathcal{P}_2(n)$ tempered $\mathbb{H}_a$-modules. These are all distinct $\mathbb{H}_a$-modules since they are nonisomorphic as W_n-modules.

□

4.5 One W-type discrete series

We show that the only tempered $\mathbb{H}_{n,m}$-modules (with real positive generic parameter) which are irreducible as W-modules are the $(\pm)$-ladder representations (see §3.4). Any tempered module which is not a discrete series is obtained by parabolic unitary induction from a discrete series module of a proper parabolic Hecke subalgebra. Therefore, no such module could be W-irreducible, so we can restrict to the case of discrete series, and we can restrict to the equivalent setting of $\overline{\mathbb{H}}_{n,m}$-modules.

Let $\overline{\mathbb{H}}_n^{\mathsf{A}}$ be the one-parameter graded Hecke algebra for $GL(n)$, viewed as a subalgebra of $\overline{\mathbb{H}}_{n,m}$. We have that $\overline{\mathbb{H}}_n^{\mathsf{A}}$ is generated by $\{\epsilon_1, \ldots, \epsilon_n\}$ and $\{t_{i,i+1} : 1 \leq i \leq n-1\}$, where t_{ij} denotes the generator in corresponding to the reflection $s_{\epsilon_i - \epsilon_j}$. The following lemma is well-known and easy to prove by direct computation.

Lemma 4.19. *There is a surjective algebra map $\phi : \overline{\mathbb{H}}_{A_{n-1}} \mapsto \mathbb{C}\mathfrak{S}_n$, given on generators by*

$$\begin{aligned} \phi(t_{i,i+1}) &= s_{i,i+1}, \\ \phi(\epsilon_j) &= s_{j,j+1} + s_{j,j+2} + \cdots + s_{j,n}. \end{aligned} \tag{4.10}$$

Note that ϕ allows us to lift any irreducible $\mathfrak{S}_n$-representation to an irreducible $\overline{\mathbb{H}}_n^{\mathsf{A}}$-module. For σ a partition of n, let $\phi^*(\sigma)$ denote the irreducible $\overline{\mathbb{H}}_{A_{n-1}}$-module obtained in this way from lifting $\sigma \otimes \mathsf{sgn}$.

A simple modification of ϕ lifts any irreducible $\mathfrak{S}_n$-representation to an irreducible $\overline{\mathbb{H}}_{n,m}$-modules. The following statement can be viewed as a particular case of the construction in [BM99].

Lemma 4.20. *Let $\eta \in \{+1, -1\}$ be given and let σ be a fixed partition of n. The assigment*

$$\begin{aligned} t_{i,i+1} &\mapsto \phi^*(\sigma)(t_{i,i+1}), & 1 \leq i \leq n-1, \\ \epsilon_i &\mapsto \eta m \mathsf{Id} + \phi^*(\sigma)(\epsilon_i), & 1 \leq i \leq n-1, \\ t_n &\mapsto \eta \mathsf{Id}, \\ \epsilon_n &\mapsto \eta m \mathsf{Id}, \end{aligned} \tag{4.11}$$

gives an irreducible $\overline{\mathbb{H}}_{n,m}$-module, $\pi(\sigma, \eta)$.

Proof. By Lemma 4.19, we only need to check that the Hecke relations

$$\begin{aligned} t_n \cdot \epsilon_n &= -\epsilon_n t_n + 2m, \\ t_{n-1} \cdot \epsilon_n &= \epsilon_{n-1} t_{n-1} - 1, \end{aligned} \tag{4.12}$$

are satisfied for this assignment. This is straighforward. □

Note that $\pi(\sigma, +)$ equals $(\sigma) \times (\emptyset) \otimes \mathsf{sgn}$ as a $W(B_n)$-representations, while $\pi(\sigma, -)$ equal $(\emptyset) \times (\sigma) \otimes \mathsf{sgn}$, in the bipartition notation of $W(B_n)$-representations from §4.3. We show that these are precisely the $(\pm)$-ladder representations from §3.4.

Proposition 4.21. *Let s_σ be a distinguished central character and assume that the $\mathsf{ds}(s_\sigma)$ is a $(\pm)$-ladder. Then $\mathsf{ds}(s_\sigma)$ is irreducible as a W-representation.*

Proof. Let us assume that $\mathsf{ds}(s_\sigma)$ is a positive ladder. The proof for the other case is analogous. We wish to show that $\mathsf{ds}(s_\sigma) = \pi(\sigma, +)$. A direct proof of this fact would be to compute the central character of $\pi(\sigma, +)$ and show that it is s_σ. We give an indirect proof. In the bijection of Corollary 4.16, $\mathsf{ds}(s_\sigma)$ (contains and) corresponds to the $W(B_n)$-representation $\rho(\mathsf{ds}(s_\sigma)) = (\sigma) \times (\emptyset) \otimes \mathsf{sgn}$. By the Lusztig classification [Lu95b], $\pi(\sigma, +)$ is the unique irreducible quotient of a standard module $M = M_{a_\sigma, \mathcal{O}_\sigma}$. There is a continuous deformation of $s_\sigma \longrightarrow s_0$, and $M \longrightarrow M_0$ such that M_0 is a tempered module at the semisimple element $a_0 = (s_0, q_1, q_2)$, and $M_0|_W = M|_W$. Moreover, the tempered module M_0 must contain $\rho(\mathsf{ds}(s_\sigma))$, and in Proposition 4.15, $\rho(M_0) = \rho(\mathsf{ds}(s_\sigma))$. But this implies that $M_0 = \mathsf{ds}(s_\sigma)$. Since this is a discrete series, we then have $\pi(\sigma, +) = M = M_0 = \mathsf{ds}(s_\sigma)$.

□

Remark 4.22. Proposition 4.21 gives in fact all discrete series which are irreducible W-representations. To see this, recall that in [BM99], one determined which W-representations can be extended to hermitian graded Hecke algebra modules. When the Hecke algebra is $\overline{\mathbb{H}}_{n,m}$, the only cases are W-representations of the form $(\gamma) \times (\emptyset)$, $(\emptyset) \times (\delta)$, or $(\underbrace{d, \ldots, d}_{k}) \times (\underbrace{f, \ldots, f}_{l})$, when $k - d = l - f + m$.

Note that, using Algorithm 4.13, it is immediate that there is no discrete series $\mathsf{ds}(\sigma)$ such that $\rho(\mathsf{ds}(\sigma)) = (\underbrace{d, \ldots, d}_{k}) \times (\underbrace{f, \ldots, f}_{l})$, for $k > 1$ and $l > 1$. We check the case $k = l = 1$, $d > 0, f > 0$. In order to have $\rho(\mathsf{ds}(\sigma)) = (d) \times (f)$, by Algorithm 4.13, we must have σ a one hook partition, with the largest two entry values at the two extremities of the hook. If the largest entry is the the right extremity of the hook, then by 4.1, $\mathsf{ds}(\sigma)$ also contains the sgn W-representation, so it is not W-irreducible. So it remains the case when the largest entry is in the bottom extremity of the hook. In that case, by Example 3.6, the exotic Springer correspondence attaches to $\mathsf{ds}(\sigma)$, the $W(B_n)$-representation given by the bipartition $(\emptyset) \times (n)$. So again $\mathsf{ds}(\sigma)$ contains at least two W-types.

4.6 Closure relation of orbits

Fix $\vec{q} = (-t^{m/2}, t^{m/2}, t)$ for $t \in \mathbb{R}_{>1}$ and $m \in \frac{1}{4}\mathbb{Z} \backslash \frac{1}{2}\mathbb{Z}$. Let σ be a partition of n. Attached to $\rho(\sigma)$ and $\mathsf{mp}(\sigma)$, we have irreducible W-modules L_σ and E_σ, respectively. For an irreducible W-module K, we denote by $\mathbb{O}(K)$ the nilpotent orbit of $Spin(\ell)$ corresponding to K via (the inverse of) a generalized Springer correspondence (c.f. §4.3). Let $\mathcal{C}$ be the pair $(\mathbb{O}, \mathcal{L})$ of $Spin(\ell)$-orbit of $\mathfrak{so}(\ell)$ and the local system which contribute to the generalized Springer correspondence. Let $\mathcal{O}(K) \subset \mathfrak{N}^{a_0}$ be the G-orbit corresponding to K via Theorem 1.15. Let $\mathsf{pr} : \mathbb{V} \to V_2$ be the G-equivariant projection map.

Theorem 4.23. *Under the above setting:*

1. *We have* $[\mathsf{ds}(\sigma) : L_\sigma] = 1 = [\mathsf{ds}(\sigma) : E_\sigma]$ *as* W*-modules;*
2. *For each irreducibe* W*-submodule* K *of* $\mathsf{ds}(\sigma)$*, we have* $\mathcal{O}(L_\sigma) \subset \overline{\mathcal{O}(K)}$ *and* $\mathbb{O}(E_\sigma) \subset \overline{\mathbb{O}(K)}$*. In particular, we have* $\mathcal{O}(L_\sigma) \subset \overline{\mathcal{O}(E_\sigma)}$ *and* $\mathbb{O}(E_\sigma) \subset \overline{\mathbb{O}(L_\sigma)}$*;*

3. *Let E be an irreducible W-module such that* $\mathsf{pr}(\mathcal{O}(E)) = \mathsf{pr}(\mathcal{O}(E_\sigma))$. *Then, we have* $[\mathsf{ds}(\sigma) : E] = 0$ *as W-modules.*

Proof. Recall that the both of the constructions of Lusztig [Lu95b] and [Ka08a] depend on the realization of $\mathbb{H}_{a_\sigma}$ in terms of the self-extention algebras of certain complexes. Let $\mathsf{IC}(\mathbb{O}(K), \mathcal{L})$ be the minimal extension of $\mathcal{L}$. In Lusztig's case, we have (by [Lu95b]):

i) There exists $\mathbf{a}_\sigma \in Spin(\ell) \times \mathbb{C}^\times$. Let us denote by $\mathcal{C}_\sigma$ the set of connected components of $(\mathbb{O}^{\mathbf{a}_\sigma}, \mathcal{L}|_{\mathbb{O}^{\mathbf{a}_\sigma}})$;

ii) Define $G^L := Z_{Spin(\ell)\times\mathbb{C}^\times}(\mathbf{a}_\sigma)$. Then, each element of $\mathcal{C}_\sigma$ is a single G^L-orbit with a local system. The set $\mathcal{C}_\sigma$ is in bijection with $\mathsf{Irrep}\mathbb{H}_{a_\sigma}$;

iii) For each $(\mathbb{O}, \mathcal{L}|_{\mathbb{O}}) \in \mathcal{C}_\sigma$, we define $\mathbb{O}^\sim := Spin(\ell)\mathbb{O} \subset \mathfrak{so}(\ell)$. It defines a W-representation $\rho(\mathbb{O})$ via a genralized Springer correspondence;

iv) The standard module $M(\mathbb{O})$ contains an irreducible W-module K with multiplicity (as W-modules) equal to $\dim H^\bullet_{\mathbb{O}^\sim}(\mathsf{IC}(\mathbb{O}(K), \mathcal{L}))$;

v) The standard module $M(\mathbb{O})$ has a unique simple quotient $L(\mathbb{O})$ and we have $[M(\mathbb{O}) : L(\mathbb{O}')] = \dim H^\bullet_{\mathbb{O}}(\mathsf{IC}(\mathbb{O}', \mathcal{L}|_{\mathbb{O}'}))$.

Now the assertion $[\mathsf{ds}(\sigma) : L_\sigma] = 1$ follows by the combination of **iv)** and **v)**. The assertion $[\mathsf{ds}(\sigma) : E_\sigma] = 1$ follows by the construction of $L_{(a,X)}$ and Theorem 1.18. We have $\mathbb{O}(E_\sigma) \subset \overline{\mathbb{O}(L_\sigma)}$ by the combination of **iv)** and **v)**. We have $\mathcal{O}(L_\sigma) \subset \overline{\mathcal{O}(E_\sigma)}$ via an analogous statement for eDL correspondences (see [Ka08a] §8 for example). This proves **1)** and **2)**. We prove **3)**. Let $L^\sim$ and L denote a simple $\mathbb{H}_{a_\sigma}$-module and a simple W-module corresponding to a $G(a_\sigma)$-orbit $\mathcal{O}_{\mathsf{mp}(\sigma)} \subset \mathfrak{N}^{a_\sigma}$. Notice that $\mathsf{pr}(\mathcal{O}(E)) = \mathsf{pr}(\mathcal{O}(E_\sigma))$ implies either $\overline{\mathcal{O}(E)} \cap \mathcal{O}(E_\sigma) = \emptyset$ or $\mathcal{O}(E)$ is a (open dense subset of a) vector bundle over $\mathcal{O}(E_\sigma)$. By **3)**, it suffices to consider the latter case. By an analogue of **iii)** for $\mathfrak{N}$, we see that $[M_{(a_\sigma,X)}, E] = 1$ as W-modules for each $X \in \mathcal{O}_{\mathsf{mp}(\sigma)}$. By an analogue of **iv)** for $\mathfrak{N}$ implies $[M_{(a_\sigma,X)}, L_{(a_\sigma,Y)}] = 1$ as $\mathbb{H}_{a_\sigma}$-module for each $Y \in \mathfrak{N}$ such that $GY = \mathcal{O}(E)$ and $X \in \overline{G(a_\sigma)Y}$. The existence of such Y is straight-forward. Therefore **1)**, applied to $L_{(a_\sigma,Y)}$, implies the result. □

Remark 4.24. To use Theorem 4.23, one needs to know the Weyl group representation attached to each orbit and the closure relations between orbits. These are contained in [Ka08b] and Achar-Henderson [AH08], respectively.

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