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Strings for $\mathfrak{sp}(\mathbf{2n})$

This is a possible Zelevinsky-type parametrization of irreducible modules for the Hecke algebra of type C_n with equal parameters. Equivalently, this is a parametrization of the orbits and the local systems (coming from the Springer correspondence) in Lusztig's classification for $sp(2n)$.

Every irreducible module is parametrized by a multisegment

$$\kappa = (\gamma_1, \gamma_2, \dots, \gamma_k, \Lambda, -\gamma_k, \dots, -\gamma_2, -\gamma_1),$$

subject to the following conditions:

1) a *gl*-string γ_i is a segment, the entries are decreasing by 1.

2) the *gl*-strings are ordered: if $\gamma_i \neq \gamma_{i+1}$, then the first entry in γ_i is greater than the first entry in γ_{i+1}, or else $length(\gamma_i) > length(\gamma_{i+1})$.

3) the unique *sp*-string Λ, if nonempty, say of length $2m$, $m \leq n$, corresponds to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0)$ of m, as follows: $\Lambda = \Lambda' \sqcup (-\Lambda')$, where

$$\Lambda' = \left(\underbrace{\frac{1}{2}, \frac{3}{2}, \dots, \lambda_1 - \frac{1}{2}}_{\lambda_1}, \underbrace{-\frac{1}{2}, \frac{1}{2}, \dots, \lambda_2 - \frac{3}{2}}_{\lambda_2}, \dots, \underbrace{-\ell + \frac{3}{2}, -\ell + \frac{1}{2}, \dots, \lambda_\ell - \ell + \frac{1}{2}}_{\lambda_\ell} \right).$$

For example, if $m = 4$, there are 5 partitions of 4, and the corresponding strings are as follows:

Partition λ	Diagram	Λ
4	$\begin{array}{\|c\|c\|c\|c\|}\hline \frac12 & \frac32 & \frac52 & \frac72 \\ \hline\end{array}$	$(\frac72, \frac52, \frac32, \frac12; -\frac12, -\frac32, -\frac52, -\frac72)$
$3+1$	$\begin{array}{\|c\|c\|c\|}\hline \frac12 & \frac32 & \frac52 \\ \hline -\frac12 & & \\ \hline\end{array}$	$(\frac52, \frac32, \frac12, -\frac12; \frac12, -\frac12, -\frac32, -\frac52)$
$2+2$	$\begin{array}{\|c\|c\|}\hline \frac12 & \frac32 \\ \hline -\frac12 & \frac12 \\ \hline\end{array}$	$(\frac32, \frac12, \frac12, -\frac12; \frac12, -\frac12, -\frac12, -\frac32)$
$2+1+1$	$\begin{array}{\|c\|c\|}\hline \frac12 & \frac32 \\ \hline -\frac12 & \\ \hline -\frac32 & \\ \hline\end{array}$	$(\frac32, \frac12, -\frac12, -\frac32; \frac32, \frac12, -\frac12, -\frac32)$
$1+1+1+1$	$\begin{array}{\|c\|}\hline \frac12 \\ \hline -\frac12 \\ \hline -\frac32 \\ \hline -\frac52 \\ \hline\end{array}$	$(\frac12, -\frac12, -\frac32, -\frac52; \frac52, \frac32, \frac12, -\frac12)$

If κ is a multisegment as above, the union of entries in all the segments of κ, with multiplicities, is called the *support* of κ, and denoted $|\kappa|$. (So the length of $|\kappa|$ is $2n$.)

Conjecture 1. Let χ be a fixed dominant semisimple (diagonal) element in $sp(2n)$.

1) The irreducible modules with central character χ are parameterized by all the possible multisegments κ with support $|\kappa| = \chi$. In other words, this is the parametrization of all local systems for the equal parameter case supported by the orbits of $G(\chi)$ on $\mathfrak{g}_2(\chi)$.

2) Two strings κ_1 and κ_2 correspond to the same orbit on $\mathfrak{g}_2(\chi)$, if one is obtained from the other by a transformations of the form:
(i) a permutation of the entries in the *sp*-string.
(ii) one *gl*-string is identical to its negative, and the two are concatenated to the *sp*-string. For example: $((\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})) \mapsto (\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, -\frac{1}{2})$, the *sp*-string in this case corresponding to the partition $(1+1)$ in $sp(4)$.

3) The closure ordering should be as in Zelevinsky, except when the relation involves the *sp*-string Λ, both a γ_j and $-\gamma_j$ must be used.

If one is interested in a parametrization of the orbits only, then a variant of the previous construction should suffice:

Conjecture 2. The orbits of $G(\chi)$ on $\mathfrak{g}_2(\chi)$ are parametrized by multisegments κ of the same form as before, but now Λ has the property that it is obtained from a partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ of m, as follows: $\Lambda = \Lambda' \sqcup (-\Lambda')$, where

$$\Lambda' = \left(\underbrace{\frac{1}{2}, \frac{3}{2}, \ldots, \lambda_1 - \frac{1}{2}}_{\lambda_1}, \underbrace{\frac{1}{2}, \frac{3}{2}, \ldots, \lambda_2 - \frac{1}{2}}_{\lambda_2}, \ldots, \underbrace{\frac{1}{2}, \frac{3}{2}, \ldots, \lambda_\ell - \frac{1}{2}}_{\lambda_\ell} \right).$$

For example, if $m = 4$, there are only 2 partitions of 4 with distinct parts, and the corresponding strings are as follows:

Partition λ	Diagram	Λ
4	$\begin{array}{|c|c|c|c|}\hline \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ \hline \end{array}$	$(\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2})$
$3+1$	$\begin{array}{|c|c|c|}\hline \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \hline \frac{1}{2} & & \\ \hline \end{array}$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2})$

Example. For the central character $\chi = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$ in $sp(6)$, the parametrization is the following:
1) $(\frac{3}{2})\ (\frac{1}{2})\ (\frac{1}{2})\ (-\frac{1}{2})\ (-\frac{1}{2})\ (-\frac{3}{2})$.
2) $(\frac{3}{2}, \frac{1}{2})\ (\frac{1}{2})\ (-\frac{1}{2})\ (-\frac{1}{2}, -\frac{3}{2})$.

3) $(\frac{3}{2})\ (\frac{1}{2})\ (\frac{1}{2},-\frac{1}{2})\ (-\frac{1}{2})\ (-\frac{3}{2})$.
4) $(\frac{3}{2},\frac{1}{2})\ (\frac{1}{2},-\frac{1}{2})\ (-\frac{1}{2},-\frac{3}{2})$.
5) $(\frac{3}{2})\ (\frac{1}{2},-\frac{1}{2})\ (\frac{1}{2},-\frac{1}{2})\ (-\frac{3}{2})$.
6) $(\frac{3}{2})\ (\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2})\ (-\frac{3}{2})$.
7) $(\frac{3}{2},\frac{1}{2},-\frac{1}{2})\ (\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$.
8) $(\frac{1}{2})\ (\frac{3}{2},\frac{1}{2};-\frac{1}{2},-\frac{3}{2})\ (-\frac{1}{2})$.
9) $(\frac{3}{2},\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$.
10) $(\frac{1}{2},-\frac{1}{2},-\frac{3}{2};\frac{3}{2},\frac{1}{2},-\frac{1}{2})$.

In this example, 5 and 6, respectively 9 and 10 belong to the same orbit. For the record, the IM involution in this case pairs (1,9), (3,10), (2,5), (4,7) and (6,8).