Functors for Matching Kazhdan-Lusztig polynomials for $GL(n, \mathbb{F})$

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- Define the functor for $GL(n, \mathbb{R})$.

References

T. Arakawa, T. Suzuki, A. Tsuchiya: *Degenerate double affine Hecke algebras...*, Proceedings Kyoto, 1997.

T. Arakawa, T. Suzuki: *Duality between* $sl_n(\mathbb{C})$ *and the degenerate affine Hecke algebra*, J. Algebra 209, 1998.

T. Suzuki: *Rogawski's conjecture...*, Represent. Theory 2, 1998.

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Set $\mathfrak{g} = gl(n, \mathbb{C})$ with decomposition $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$, and Borel subalgebras $\mathfrak{b}^+, \mathfrak{b}^-$. Then $\Delta(\mathfrak{g}, \mathfrak{h})$, $\Delta^+(\mathfrak{g}, \mathfrak{h})$, $\Pi(\mathfrak{g}, \mathfrak{h})$, ρ are as usual.

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$\mathcal{O}(\mathfrak{g})_\mu$ = the full subcategory of modules with the same infinitesimal character as the Verma module $M(\mu)$.

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$\mathbb{H}$ is a degeneration of the Iwahori-Hecke algebra $\mathcal{IH}$ of $GL(n, \mathbb{Q}_p)$ (Lusztig, Drinfeld).

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Lu Assume λ is hyperbolic. Then the categories $Rep(\mathcal{IH})_\lambda$ and $Rep(\mathbb{H})_\lambda$ are equivalent.

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Moreover $Y \in \mathcal{O}(\mathfrak{g})_\mu$, $\mu \in \mathfrak{h}^*$, the weights of $H_0(\mathfrak{n}^-, Y)$ are all of the form

$$w \circ \mu := w(\mu + \rho) - \rho, \text{ for some } w \in W.$$

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Note that for $1 \leq i < j \leq n$, $\Omega_{i,j}$ is the natural permutation of the V_n-factors.

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So one sees that the functor F_λ can actually be defined as

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Then, $F_\lambda(M(\mu)) \cong \mathbb{C}[W_n/(W_{\ell_1} \times \cdots \times W_{\ell_n})]$, as a W-representation.

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for every multiset $\underline{\chi} = (\chi_1, \ldots, \chi_n)$ of central characters.

Every χ_i can be thought as a *segment* $(a_i, a_i + 1, \ldots, b_i)$, with $b_i - a_i + 1 = \ell_i$.

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- If $\underline{\chi}$ is *nested* (in the sense of Zelevinski), then $X(\underline{\chi})$ has a unique simple quotient $\overline{X}(\underline{\chi})$. Call such $X(\underline{\chi})$ a *standard module*.

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In addition, a nondegenerate $\mathfrak{g}$-form on $L(\mu)$ induces a nondegenerate $\mathbb{H}$-form on $F_\lambda(L(\mu))$.

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4. If $\lambda - \mu \in P(V_n^{\otimes n})$, then $F_\lambda(M(\mu)) = X(\underline{\chi}(\lambda, \mu))$.
5. From (3) and (4), it follows that $F_\lambda(L(\mu)) = \overline{X}(\underline{\chi}(\lambda, \mu))$.

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• (Then (3) follows by a calculation similar to that for category $\mathcal{O}$.)

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$B^+ = H^s N^{s,+}$ is a Borel subgroup and $B^- = H^s N^{s,-}$ the opposite;

$\Delta^+(\mathfrak{g}, \mathfrak{h}^s)$ are the positive roots with respect to B^+;

$\mathfrak{h}^s$, $\mathfrak{n}^{s,-}$ etc. are the complexified Lie algebras.

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In this setting (Casselman):

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A: Final limit standard modules (Vogan) with unique simple submodules.

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Every irreducible module is such a submodule for some P, δ.

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Make a *choice* of positive roots $\Delta(\mathfrak{g}, \mathfrak{h})$, such that

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In $GL(n, \mathbb{R})$, $\dim LKT^+(\delta) = 1$.

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Then the rest of the machinery can be applied.

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- We know the matching fails for exceptional groups, in F_4 for example. Is there a fix (or a good explanation)?