ON THE UNITARY DUAL OF HECKE ALGEBRAS WITH UNEQUAL PARAMETERS

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Abstract. We determine the unitary dual of the geometric Hecke algebras with unequal parameters which appear in [Lu6]'s classification of unipotent representations for *exceptional* p-adic groups. The largest such algebra is of type F_4. Via the [BM1]-[BM2] correspondences applied to this setting, this is equivalent to the identification of the unitary unipotent (in the sense of [Lu6]) representations of the p-adic group.

Contents

1. Introduction

In this paper, we extend to the setting of (geometric) Hecke algebras $\mathbb{H}$ with *unequal parameters* the methods of [BC2] for computing signatures of Hermitian forms on Hecke algebras modules via comparison with signatures of Hermitian modules of spherical modules for certain Hecke algebras attached to centralizers of nilpotent orbits. As an application, we determine the unitary dual of the Hecke algebras of types G_2 and F_4, with unequal parameters, which appear in [Lu6], in the classification of unipotent representations of p-adic groups.

We give an outline of the paper. In section 2.1-2.6, we present the necessary definitions and background on the representation theory of $\mathbb{H}$, assuming the parameters are arbitrary. In particular, we recall the Langlands classification (actually, the reduction to tempered modules) as in [Ev], the unitarity of tempered modules, as it follows from [Op1], and the results about the Hermitian forms and intertwining operators from [BM3]. In section 2.7, we present a reduction to *real* central ("infinitesimal") character for the unitary

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dual problem, which is the complete analogue of the result for real reductive groups, as in [Kn], XVI.4. This is, of course, well-known to the specialists.

From section 3 on, we restrict to the case of *geometric* Hecke algebras, *i.e.* those for which there exists a geometric classification, as in [Lu2]-[Lu4]. We recall the relevant results about the classification of simple $\mathbb{H}$-modules, and the generalized Springer correspondence ([Lu5]). An important consequence is that the set of irreducible tempered $\mathbb{H}$-modules with real central character are linearly independent in the Grothendieck group of W. Therefore the methods of [BM1],[BM2] hold, and the correspondence with unitary (unipotent) representations of p-adic groups can be verified.

In section 4, we analyze Hermitian forms and intertwining operators for simple modules. We restrict, as we may, to modules with real central character. A particular case is that of spherical modules. In this case, we extend a theorem from [BC1] which says that the signature of the Hermitian form is constant over the faces of the arrangement of positive root hyperplanes. As an easy application of these methods, we present the spherical unitary dual for type G_2 with arbitrary unequal parameters. (In fact, if we factor in the *Iwahori-Matsumoto involution*, definition (4.2.1), and the tempered modules, one obtains all the unitary modules with real central character of G_2.)

Section 5 presents a more interesting application. We determine the unitary dual of the Hecke algebra (of type F_4) constructed in [Lu2] from a cuspidal local system on the principal nilpotent orbit in the Levi $(3A_1)''$ of E_7. Besides the equal parameter case, this is, essentially, the only Hecke algebra of type F_4 which appears in the classification of unipotent representations of p-adic groups from [Lu6] (the equal parameter $\mathbb{H}(F_4)$ was treated in [Ci]). Although the methods employed are mostly uniform, details need to be checked case by case for each nilpotent orbit in E_7 in the classification. The answer is summarized in theorems 5.1 and 5.4. Since the Iwahori-Matsumoto involution, IM, preserves unitary module, we verify (in section 5.3) that our answer is closed under IM. In order to use IM, we compute need to compute the decompositions of standard modules and the W-structure of unitary modules.

2. Preliminaries

2.1. Let $\mathfrak{h}$ be a finite dimensional vector space, $R \subset \mathfrak{h}^*$ a root system, with $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots, $\check{R} \subset \mathfrak{h}$ the set of coroots, and W the Weyl group. Let $c : R \to \mathbb{Z}_{>0}$ be a function such that $c_\alpha = c_\beta$, whenever α and β are W-conjugate. As a vector space,

$$\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{C}[r] \otimes \mathbb{A}, \tag{2.1.1}$$

where $\mathbb{A}$ is the symmetric algebra over $\mathfrak{h}^*$. The generators are $t_w \in \mathbb{C}[W]$, $w \in W$ and $\omega \in \mathfrak{h}^*$. The relations between the generators are:

$$\begin{aligned} &t_w t'_w = t_{ww'}, && \text{for all } w, w' \in W; \\ &t_s^2 = 1, && \text{for any simple reflection } s \in W; \\ &t_s \omega = s(\omega) t_s + r c_\alpha \omega(\check{\alpha}), && \text{for simple reflections } s = s_\alpha. \end{aligned} \tag{2.1.2}$$

Assume the root system R is irreducible. If R is simply-laced, the function c must be constant. In the two root lengths case, we will denote the function c by the pair $(c_l, c_s) \in \mathbb{Z}^2_{>0}$, where c_l, c_s specify the values of c on the long and short roots, respectively (in this order). Let $k \in \mathbb{Z}_{>0}$ be the ratio $k = (\alpha_l, \alpha_l)/(\alpha_s, \alpha_s)$ for some long root α_l and short root α_s. Then it is straightforward to see that $\mathbb{H}(R, (c_1, c_2)) \cong \mathbb{H}(R, (kc_2, c_1))$. For example, $\mathbb{H}(F_4, (1, 2)) \cong \mathbb{H}(F_4, (4, 1))$.

2.2. By [Lu1], the center of $\mathbb{H}$ is $\mathbb{C}[r] \otimes \mathbb{A}^W$.

On any simple (finite dimensional) $\mathbb{H}$-module, the center of $\mathbb{H}$ acts by character, which we will call a *central character*. The central characters correspond to W-conjugacy classes of semisimple elements $(r_0, s) \in \mathbb{C} \oplus \mathfrak{h}$.

Definition. *A central character (r_0, s) is called real if $(r_0, s) \in \mathbb{R} \oplus \mathfrak{h}_{\mathbb{R}}$.*

2.3. $\mathbb{H}$ has a $*$- operation given on generators as follows (as in [BM2], section 5):

$$\begin{aligned} &t_w^* = t_{w^{-1}},\ w \in W;\ r^* = r; \\ &\omega^* = -\overline{\omega} + r \sum_{\alpha \in R^+} c_\alpha \overline{\omega}(\check{\alpha}) t_{s_\alpha},\ \omega \in \mathfrak{h}^*. \end{aligned} \tag{2.3.1}$$

The $\mathbb{H}$-module V is called *Hermitian* if it admits a Hermitian form $\langle\ ,\ \rangle$ such that

$$\langle x \cdot v_1, v_2 \rangle = \langle v_1, x^* \cdot v_2 \rangle,\ \text{for all } v_1, v_2 \in V,\ x \in \mathbb{H}. \tag{2.3.2}$$

It is called *unitary*, if in addition the Hermitian form is positive definite.

2.4. We present the *Langlands classification* for $\mathbb{H}$ as in [Ev].

If V is a (finite dimensional) simple $\mathbb{H}$-module, $\mathbb{A}$ induces a generalized weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}} V_\lambda. \tag{2.4.1}$$

Call λ a *weight* if $V_\lambda \neq 0$.

Definition. *The irreducible module σ is called tempered if $\omega_i(\lambda) \leq 0$, for all weights $\lambda \in \mathfrak{h}$ of σ and all fundamental weights $\omega_i \in \mathfrak{h}^*$. If σ is tempered and $\omega_i(\lambda) < 0$, for all λ, ω_i as above, σ is called a discrete series.*

For every $\Pi_M \subset \Pi$, define $R_M \subset R$ to be the set of roots generated by Π_M, $\check R_M \subset \check R$ the corresponding set of coroots, and $W(M) \subset W$ the corresponding Weyl subgroup. Let $\mathbb{H}_M$ be the Hecke algebra attached to $(\mathfrak{h}, R_M)$. It can be regarded naturally as a subalgebra of $\mathbb{H}$.

Define $\mathfrak{t} = \{\nu \in \mathfrak{h} : \langle\alpha,\nu\rangle = 0, \text{ for all } \alpha \in \Pi_M\}$ and $\mathfrak{t}^* = \{\lambda \in \mathfrak{h}^* : \langle\lambda,\check\alpha\rangle = 0, \text{ for all } \alpha \in \Pi_M\}$. Then $\mathbb{H}_M$ decomposes as

$$\mathbb{H}_M = \mathbb{H}_{M_0} \otimes S(\mathfrak{t}^*),$$

where $\mathbb{H}_{M_0}$ is the Hecke algebra attached to $(\mathbb{C}\langle\Pi_M\rangle, R_M)$.

We will denote by $I(M,U)$ the induced module $I(M,U) = \mathbb{H} \otimes_{\mathbb{H}_M} U$.

Theorem ([Ev])**.**

(1) *Every irreducible $\mathbb{H}$-module is a quotient of a standard induced module $X(M,\sigma,\nu) = I(M,\sigma\otimes\mathbb{C}_\nu)$, where σ is a tempered module for $\mathbb{H}_{M_0}$, and $\nu \in \mathfrak{t}^+ = \{\nu \in \mathfrak{t} : Re(\alpha(\nu)) > 0, \text{ for all } \alpha \in \Pi \setminus \Pi_M\}$.*
(2) *Assume the notation from (1). Then $X(M,\sigma,\nu)$ has a unique irreducible quotient, denoted by $L(M,\sigma,\nu)$.*
(3) *If $L(M,\sigma,\nu) \cong L(M',\sigma',\nu')$, then $M = M'$, $\sigma \cong \sigma'$ as $\mathbb{H}_{M_0}$-modules, and $\nu = \nu'$.*

We will call a triple (M,σ,ν) as in theorem 2.4 a *Langlands parameter*.

2.5. We need to recall some results about the affine Hecke algebra as in [Op1], and their implications for $\mathbb{H}$. Fix $q > 1$. The *affine Hecke algebra* $\mathcal{H}$ is the $\mathbb{C}$-algebra with basis $\{T_w : w \in W_{\text{aff}}\}$ (W_{aff} is the affine Weyl group) which satisfies the relations:

(1) $T_wT_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$.
(2) $(T_{s_\alpha}+1)(T_{s_\alpha} - q^{c_\alpha}) = 0$, for all simple affine reflections s_α.

The $*$-operation for $\mathcal{H}$ is given by $T_w^* = T_{w^{-1}}$. The algebra $\mathcal{H}$ has a *trace*, $\tau : \mathcal{H} \to \mathbb{C}$, given by

$$\tau(T_w) = \delta_{w,1}. \tag{2.5.1}$$

It is easy to check that τ defines an inner product on $\mathcal{H}$ via

$$(x,y) := \tau(x^*y),\ x,y \in \mathcal{H}, \tag{2.5.2}$$

and the basis $\{T_w : w \in W_{\text{aff}}\}$ is orthogonal with respect to $(\ ,\ )$. Note that τ is easy to define using the Iwahori-Matsumoto presentation, but not so if one uses instead the Bernstein-Lusztig presentation, $\mathcal{H} = \mathcal{H}_W \otimes \mathcal{A}$. (See [Op2] for details.) Similarly to section 2.4, one defines *tempered* and *discrete series* modules for $\mathcal{H}$ using the weights of the abelian subalgebra $\mathcal{A}$ (Casselman's criteria).

Definition. *Let $\mathfrak{H}$ be the Hilbert space completion of $\mathcal{H}$ with respect to $(\ ,\ )$.*

Proposition ([Op1],2.22)**.** *A finite dimensional representation σ of $\mathcal{H}$ is a discrete series if and only if it is a subrepresentation of $\mathfrak{H}$. In particular, σ is a unitary $*$-representation of $\mathcal{H}$.*

A standard argument then implies the unitarity of tempered $\mathcal{H}$-modules as well. We can transfer this results to the graded Hecke algebra, using theorem 9.3 in [Lu1] and theorem 4.3 [BM2].

In conclusion, all tempered modules of $\mathbb{H}$ are unitary.

2.6. Given a module V, let V^h denote the Hermitian dual.

Every element $x \in \mathbb{H}$ can be written uniquely as $x = \sum_{w\in W/W(M)} t_w x_w$, with $x_w \in \mathbb{H}_M$. Let $\epsilon_M : \mathbb{H} \to \mathbb{H}_M$ be the map defined by $\epsilon_M(x) = x_1$, that is, the component of the identity element $1 \in W$. In the particular case $\Pi_M = \emptyset$, we will denote the map by $\epsilon : \mathbb{H} \to \mathbb{A}$.

Lemma ([BM3], 1.4)**.** *If U is module for $\mathbb{H}_M$, and $\langle\ ,\ \rangle_M$ denotes the Hermitian pairing with U^h, then the Hermitian dual of $I(M,U)$ is $I(M,U^h)$, and the Hermitian pairing is given by*

$$\langle t_x \otimes v_x, t_y \otimes v_y\rangle_h = \langle \epsilon_M(t_y^* t_x)v_x, v_y\rangle_M,\ x, y \in W/W(M),\ v_x, v_y \in U.$$

Applying this result to a Langlands parameter (M,σ,ν) as in section 2.4, we find that the Hermitian dual of $X(M,\sigma,\nu)$ is $I(M,\sigma\otimes\mathbb{C}_{-\overline{\nu}})$.

Let w_0 denote the longest Weyl group element in W, and let $W(w_0M)$ be the subgroup of W generated by the reflections in w_0R_M. Let w_m denote a shortest element in the double coset $W(w_0M)w_0W(M)$. Then $w_m\Pi_M$ is a subset of Π, which we denote by Π_{w_mM}.

Proposition ([BM3], 1.5)**.** *The Hermitian dual of the irreducible Langlands quotient $L(M,\sigma,\nu)$ is $L(w_mM, w_m\sigma, -w_m\overline{\nu})$. In particular, $L(M,\sigma,\nu)$ is Hermitian if and only if there exists $w \in W$ such that*

$$wM = M, \quad w\sigma \cong \sigma \quad \textit{and} \quad w\nu = -\overline{\nu}.$$

If this is the case, we will denote by $a_w : w\sigma \to \sigma$.

Let $w = s_1 \dots s_k$ be a reduced decomposition of w. For each simple root α, define

$$r_{s_\alpha} = t_{s_\alpha}\alpha - c_\alpha; \quad r_w = r_{s_{\alpha_1}} \dots r_{s_{\alpha_k}}. \tag{2.6.1}$$

Lemma 1.6 in [BM3] (based on proposition 5.2 in [Lu1]) proves that r_w does not depend on the reduced expression of w.

Assume $L(M,\sigma,\nu)$ is Hermitian. Define

$$\mathcal{A}(M,\sigma,\nu) : X(M,\sigma,\nu) \to I(M,\sigma\otimes\mathbb{C}_{-\overline{\nu}}),\ x\otimes(v\otimes 1_\nu) \mapsto xr_w\otimes(a_w(v)\otimes 1_{-\overline{\nu}}). \tag{2.6.2}$$

One can verify, as in section 1.6 of [BM3], that this is an intertwining operator. The image of $\mathcal{A}(M,\sigma,\nu)$ is the Langlands quotient $L(M,\sigma,\nu)$.

2.7. Let (M,σ,ν) be a Langlands parameter as in section 2.4, and $\nu = Re\ \nu + \sqrt{-1}Im\ \nu$ with $Im\ \nu \neq 0$. Set

$$R_{M_1} = \{\alpha \in R : \langle Im\ \nu, \alpha\rangle = 0, \quad R_{N_1} = \{\alpha \in R : \langle Im\ \nu, \alpha\rangle > 0\}. \tag{2.7.1}$$

Clearly, $\Pi_M \subset R_{M_1}$. Moreover, R_{M_1} is a root subsystem of R. Set $R^+_{M_1} = R_{M_1} \cap R^+$, and let Δ_{M_1} denote the set of corresponding simple roots. Note that $\Pi_M \subset \Delta_{M_1}$, but Δ_{M_1} need not be a subset of Π. But $\mathbb{H}_M$ is naturally a subalgebra of $\mathbb{H}_{M_1}$.

Assume $L(M,\sigma,\nu)$ is Hermitian and let $w \in W$ be as in proposition 2.6. From $w\nu = -\overline{\nu}$, it follows that $w \in W(M_1)$.

The triple $(M, \sigma, Re\ \nu)$ is a Langlands triple for $\mathbb{H}_{M_1}$, so it makes sense to consider $\sigma_1 = L_{M_1}(M,\sigma, Re\ \nu)$. Moreover, σ_1 is Hermitian in $\mathbb{H}_{M_1}$.

Proposition. *With the notation as above,*

$$I(M_1, (\sigma_1 \otimes \mathbb{C}_{Im\ \nu})) \cong L(M,\sigma,\nu).$$

Proof. It is known that there exists $w' \in W$ (of shortest length) which maps Δ_{M_1} onto a subset $\Pi_{M'}$ of Π. This defines an isomorphism $a_{w'} : \mathbb{H}_{M_1} \to \mathbb{H}_{M'}$, and, similarly to section 2.6, an intertwining operator $\mathcal{A}(M_1 : M', \mathcal{V}, \nu')$. Setting $\mathcal{V} = X_{M_1}(M,\sigma, Re\ \nu)$ and $\nu' = Im\ \nu$, we find that this operator is invertible.

$$X(M,\sigma,\nu) \cong I(M_1, X_{M_1}(M,\sigma, Re\ \nu) \otimes \mathbb{C}_{Im\ \nu}). \tag{2.7.2}$$

Now, $X(M,\sigma,\nu)$ maps onto $L(M,\sigma,\nu)$ via the operator $\mathcal{A}(M,\sigma,\nu)$ in (2.6.2). On the other hand, $X_{M_1}(M,\sigma, Re\,\nu)$ maps onto σ_1 by the operator $\mathcal{A}_{M_1}(M,\sigma, Re\,\nu)$, and by induction (which is an exact functor) we find a map from the right hand side of (2.7.2) onto $I(M_1, (\sigma_1 \otimes \mathbb{C}_{Im\ \nu})$ whose kernel is $\mathbb{H} \otimes_{\mathbb{H}_{M_1}}$ $(\ker \mathcal{A}_{M_1}(M,\sigma, Re\,\nu)) \cong \ker \mathcal{A}(M',\sigma,\nu)$, which is identical to $\ker \mathcal{A}(M,\sigma,\nu)$. ∎

Corollary. *Assuming the previous notation, $L(M,\sigma,\nu)$ is unitary if and only if $L_{M_1}(M,\sigma, Re\ \nu)$ is unitary.*

Proof. This follows immediately from proposition 2.7, and the fact that if χ is a purely imaginary character and $I(M, U \otimes \mathbb{C}_\chi)$ is irreducible, then $I(M, U \otimes \mathbb{C}_\chi)$ is unitary for $\mathbb{H}$ if and only if U is unitary for $\mathbb{H}_M$. ∎

3. Geometric Hecke algebras

We will restrict to the case of *geometric Hecke algebras*, as in [Lu2]-[Lu4]. We review some facts about these algebras and their geometric classification.

3.1. For an algebraic group $\mathbf{G}$ and a $\mathbf{G}$-variety X, let $H^\bullet_{\mathbf{G}}(X) = H^\bullet_{\mathbf{G}}(X, \mathbb{C})$, respectively $H_\bullet^{\mathbf{G}}(X) = H_\bullet^{\mathbf{G}}(X,\mathbb{C})$ denote the equivariant cohomology, respectively homology (as in section 1 of [Lu2]). The component group of $\mathbf{G}$, $A(\mathbf{G}) = \mathbf{G}/\mathbf{G}^0$ acts naturally on $H^\bullet_{\mathbf{G}^0}(X)$ and $H_\bullet^{\mathbf{G}^0}(X)$. The cup product defines a structure of graded $H^\bullet_{\mathbf{G}}(X)$-module on $H_\bullet^{\mathbf{G}}(X)$. If pt is a point of X, one uses the notation $H^\bullet_{\mathbf{G}} = H^\bullet_{\mathbf{G}}(\{pt\})$, respectively $H_\bullet^{\mathbf{G}} = H_\bullet^{\mathbf{G}}(\{pt\})$. There is a $\mathbb{C}$-algebra homomorphism $H^\bullet_{\mathbf{G}} \to H^\bullet_{\mathbf{G}}(X)$ induced by the map $X \to \{pt\}$, and therefore $H^\bullet_{\mathbf{G}}$ and $H_\bullet^{\mathbf{G}}$ can both' be considered as $H^\bullet_{\mathbf{G}}$-modules.

For a subset $\mathcal{S}$ of $\mathbf{G}$, or $\mathfrak{g}$, let $Z_{\mathbf{G}}(\mathcal{S})$, $N_{\mathbf{G}}(\mathcal{S})$ denote the centralizer, respectively the normalizer of S in $\mathbf{G}$.

Let G be a reductive connected complex algebraic group, with Lie algebra $\mathfrak{g}$. Let $P = LN$ denote a parabolic subgroup, with $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ the corresponding Lie algebras, such that $\mathfrak{l}$ admits an irreducible L-equivariant cuspidal local system (as in [Lu2],[Lu5]) $\mathcal{L}$ on a nilpotent L-orbit $\mathcal{C} \subset \mathfrak{l}$. The classification of cuspidal local systems can be found in [Lu5]. In particular, $W = N(L)/L$ is a Coxeter group.

Let H be the center of L with Lie algebra $\mathfrak{h}$, and let R be the set of nonzero weights α for the ad-action of $\mathfrak{h}$ on $\mathfrak{g}$, and $R^+ \subset R$ the set of weights for which the corresponding weight space $\mathfrak{g}_\alpha \subset \mathfrak{n}$. For each parabolic $P_j = L_j N_j$, $j = 1, n$, such that $P \subset P_j$ maximally and $L \subset L_j$, let $R_j^+ = \{\alpha \in R^+ : \alpha(\mathfrak{z}(\mathfrak{l}_j)) = 0\}$, where $\mathfrak{z}(\mathfrak{l}_j)$ denotes the center of $\mathfrak{l}_j$. It is shown in [Lu2] that each R_j^+ contains a unique α_j such that $\alpha_j \notin 2R$.

Let $Z_G(\mathcal{C})$ denote the centralizer in G of a Lie triple for $\mathcal{C}$, and $\mathfrak{z}(\mathcal{C})$ its Lie algebra.

Proposition ([Lu2]).

(a) R is a (possibly non-reduced) root system in $\mathfrak{h}^*$, with simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. Moreover, W is the corresponding Weyl group.

(b) H is a maximal torus in $Z^0 = Z^0_G(\mathcal{C})$.

(c) W is isomorphic to $W(Z^0_G(\mathcal{C})) = N_{Z^0}(H)/H$.

(d) The set of roots in $\mathfrak{z}(\mathcal{C})$ with respect to $\mathfrak{h}$ is exactly the set of non-multipliable roots in R.

For each $j = 1, n$, let $c_j \geq 2$ be such that

$$ad(e)^{c_j-2} : \mathfrak{l}_j \cap \mathfrak{n} \to \mathfrak{l}_j \cap \mathfrak{n} \neq 0, \text{ and } ad(e)^{c_j-1} : \mathfrak{l}_j \cap \mathfrak{n} \to \mathfrak{l}_j \cap \mathfrak{n} = 0. \quad (3.1.1)$$

By proposition 2.12 in [Lu2], $c_i = c_j$ whenever α_i and α_j are W-conjugate. Therefore, one can define a Hecke algebra $\mathbb{H}$ as in (2.1.1),(2.1.2). In view of the proposition above, one can think of this algebra as essentially a graded Hecke algebra with unequal parameters for the centralizer $\mathfrak{z}(\mathcal{C})$. The explicit algebras which may appear are listed in 2.13 of [Lu2]. The more familiar case of Hecke algebras with equal parameters arise when one takes P to be a Borel subgroup, and $\mathcal{C}$ and $\mathcal{L}$ to be trivial.

The geometric realization of $\mathbb{H}$ is obtained as follows. Consider the varieties

$$\begin{aligned} \dot{\mathfrak{g}}_N &= \{(x, gP) \in \mathfrak{g} \times G/P |\ Ad(g^{-1})x \in \mathfrak{n}\}, \quad (3.1.2) \\ \ddot{\mathfrak{g}}_N &= \{(x, gP, g'P) \in \mathfrak{g} \times G/P \times G/P |\ (x, gP), (x, g'P) \in \dot{\mathfrak{g}}_N\}. \end{aligned}$$

Clearly $\ddot{\mathfrak{g}}_N \subset \dot{\mathfrak{g}}_N \times \dot{\mathfrak{g}}_N$. The local system $\mathcal{L}$ gives a local systems $\dot{\mathcal{L}}$ on $\dot{\mathfrak{g}}_N$ and $\ddot{\mathcal{L}}$ on $\ddot{\mathfrak{g}}_N$.

The group $G \times \mathbb{C}^*$ acts on $\mathfrak{g}$ by $(g_1, \lambda) \cdot x = \lambda^{-2} Ad(g_1)x$, for every $x \in \mathfrak{g}, g_1 \in G, \lambda \in \mathbb{C}^*$. In [Lu2], the vector space $H^{G\times\mathbb{C}^*}_\bullet(\ddot{\mathfrak{g}}_N, \ddot{\mathcal{L}})$ is endowed with left and right actions of W and $S(\mathfrak{h}^* \oplus \mathbb{C}) \cong H^\bullet_{G\times\mathbb{C}^*}(\dot{\mathfrak{g}}_N)$, and it is proved (theorem 6.3 and corollary 6.4) that

$$H^{G\times\mathbb{C}^*}_\bullet(\ddot{\mathfrak{g}}_N, \mathbb{C}) \cong \mathbb{H}, \text{ as } \mathbb{H}\text{-bimodules}. \quad (3.1.3)$$

3.2. Now we recall the construction of standard modules for $\mathbb{H}$. Fix a nilpotent element e in $\mathfrak{g}$, and let $\mathcal{P}_e$ be the variety

$$\mathcal{P}_e = \{gP \in G/P : Ad(g^{-1})e \in \mathcal{C} + \mathfrak{n}\}. \tag{3.2.1}$$

The centralizer $Z_{G\times\mathbb{C}^*}(e)$ acts on $\mathcal{P}_e$ by $(g_1,\lambda).gP = (g_1g)P$.

[Lu2] constructs actions of W and $S(\mathfrak{h}^*\oplus\mathbb{C})$ on $H_\bullet^{Z^0_{G\times\mathbb{C}^*}(e)}(\mathcal{P}_e,\dot{\mathcal{L}})$, and proves that these are compatible with the relations between the generators of $\mathbb{H}$, therefore obtaining a module of $\mathbb{H}$ (theorem 8.13). The component group $A_{G\times\mathbb{C}^*}(e)$ acts on $H_\bullet^{Z^0_{G\times\mathbb{C}^*}(e)}(\mathcal{P}_e,\dot{\mathcal{L}})$, and commutes with the $\mathbb{H}$-action (8.5).

Consider the variety $\mathcal{V}$ of semisimple $Z^0_{G\times\mathbb{C}^*}(e)$-orbits on the Lie algebra $\mathfrak{z}_{G\times\mathbb{C}^*}(e) = \{(x,r_0)\in\mathfrak{g}\oplus\mathbb{C} : [x,e]=2r_0e\}$ of $Z_{G\times\mathbb{C}^*}(e)$. The affine variety $\mathcal{V}$ has $H^\bullet_{Z^0_{G\times\mathbb{C}^*}(e)}$ as the coordinate ring. Define the $\mathbb{H}$-modules

$$X(s,r_0,e) = \mathbb{C}_{(s,r_0)} \otimes_{H^\bullet_{Z^0_{G\times\mathbb{C}^*}(e)}} H_\bullet^{Z^0_{G\times\mathbb{C}^*}(e)}(\mathcal{P}_e,\dot{\mathcal{L}}), \tag{3.2.2}$$

where $\mathbb{C}_{(s,r_0)}$ denotes the $H^\bullet_{Z^0_{G\times\mathbb{C}^*}(e)}$-module given by the evaluation at $(s,r_0)\in\mathcal{V}$, $H^\bullet_{Z^0_{G\times\mathbb{C}^*}(e)}\to\mathbb{C}$.

Let $A_{G\times\mathbb{C}^*}(e,s,r_0)$ denote the stabilizer of (s,r_0) in $A_{G\times\mathbb{C}^*}(e)$. For each $\psi\in\widehat{A}_{G\times\mathbb{C}^*}(e,s,r_0)$, define

$$X(s,r_0,e,\psi) = \mathrm{Hom}_{A_{G\times\mathbb{C}^*}(e,s,r_0)}[\psi : X(s,r_0,e)]. \tag{3.2.3}$$

In particular, when $(s,r_0)=\mathbf{0}$, we have (7.2, 8.9 in [Lu2] and 10.12(d) in [Lu3])

$$X(\mathbf{0},e)\cong H_\bullet^{\{1\}}(\mathcal{P}_e,\dot{\mathcal{L}}) \text{ as } W\times A_G(e)\text{-modules}. \tag{3.2.4}$$

Let $\widehat{A}^0_G(e)$ and $\widehat{A}^0_G(e,s)$ denote the set of representations ψ which appear in the $A_G(e)$-module $H_\bullet^{\{1\}}(\mathcal{P}_e,\dot{\mathcal{L}})$, respectively in the restriction of this module to $A_G(e,s)$.

Theorem ([Lu2],[Lu3])**.** *Assume $r_0\neq 0$.*
(a)([Lu2],8.10) $X(s,r_0,e,\psi)\neq 0$ if and only if $\psi\in\widehat{A}^0_G(e,s,r_0)$.
(b)([Lu2],8.15) Any simple $\mathbb{H}$-module on which r acts by r_0 is a quotient $\overline{X}(s,r_0,e,\psi)$ of an $X(s,r_0,e,\psi)$, where $\psi\in\widehat{A}^0_G(e,s)$.
(c)([Lu3],8.18) The set of isomorphism classes of simple $\mathbb{H}$-modules with central character (s,r_0) is in 1-1 correspondence with the set

$$\mathcal{M}_{s,r_0} = Z_G(s)\text{-conjugacy classes on } \{(e,\psi) : [s,e]=2r_0e,\ \psi\in\widehat{A}^0_G(e,s)\}. \tag{3.2.5}$$

3.3. Fix a semisimple class $(s,r_0)\in\mathcal{V}$. The following theorem is immediately implied by the results in [Lu3], 10.5-10.7.

Theorem ([Lu3])**.** *If a composition factor Y of the standard module $X(s,r_0,e,\psi)$ is parametrized by the class $(e',\psi')\in\mathcal{M}_{s,r_0}$, then necessarily $\mathcal{O}\subset\overline{\mathcal{O}'}$, where*

$\mathcal{O}$, $\mathcal{O}'$ are the $Z_G(s)$-orbits of e, respectively e' in $\mathcal{M}_{s,r_0}$. Moreover, $\mathcal{O} = \mathcal{O}'$ if and only if $Y = \overline{X}(s, r_0, e, \psi)$.

This is the geometric equivalent of the classical results for real and p-adic groups concerning the minimality of the Langlands parameter in a standard module (see [BW], IV.4.13 and XI.2.13).

Finally, an immediate corollary of theorem 3.3 (see 8.17 in [Lu2], equivalently 10.9 in [Lu3]) is that, if $\mathcal{O}$ is the unique open $Z_G(s)$-orbit in $\mathcal{M}_{s,r_0}$, then $X(s, r_0, e, \psi)$, $\psi \in \widehat{A}^0_G(e, s)$ is simple.

3.4. An important role in the determination of the unitary dual will be played by the W-structure of standard and simple modules.

The continuation argument in 10.13 in [Lu3] in conjunction with (3.2.4) shows that

$$X(s, r_0, e)|_W \cong H^{\{1\}}_\bullet(\mathcal{P}_e, \dot{\mathcal{L}}), \text{ as } W \times A_G(s, e)\text{-representations.} \tag{3.4.1}$$

(By definition, $H^{\{1\}}_j(X, \mathcal{L}) = H^{2dim(X)-j}_c(X, \mathcal{L}^*)^*$, where $H^\bullet_c$ denotes the cohomology with compact support, while * denotes the dual vector space or local system.)

Let $\mathcal{N}_G$ denote the set G-conjugacy classes of pairs $\{(e, \phi) : e \in \mathfrak{g}_N, \phi \in \widehat{A}_G(e)\}$. The *generalized Springer correspondence* ([Lu5]) gives, in particular, an injection

$$\Phi_{\mathcal{C},\mathcal{L}} : \widehat{W} \hookrightarrow \mathcal{N}_G. \tag{3.4.2}$$

(Recall that $W = N_G(L)/L$.) For the classical Springer correspondence (P a Borel subgroup).

More precisely, let (e, ϕ) be a G-conjugacy class $\mathcal{N}_G$. Set $\mathcal{O}_e = G \cdot e$, and let ε_ϕ be the local system on $\mathcal{O}_e$ corresponding to ϕ. Then [Lu5] (6.2,6.3) attaches to (e, ϕ) a unique G-conjugacy class $(L', \mathcal{C}', \mathcal{L}')$, where $L' \subset P'$ is a Levi subgroup, $\mathcal{C}'$ a nilpotent L'-orbit in $\mathfrak{l}'$, and $\mathcal{L}'$ is a local system on $\mathcal{C}'$, such that:

$$\begin{aligned} &(a)\ H_c^{dim(\mathcal{O}_e)-dim(\mathcal{C}')}(\mathcal{O}_e \cap (\mathcal{C}' + \mathfrak{n}'), \varepsilon_\phi) \neq 0; \\ &(b)\ P' \text{ is minimal with respect to } (a). \end{aligned} \tag{3.4.3}$$

The local system $\mathcal{L}'$ on $\mathcal{C}'$ is constructed from ε_ϕ (see [Lu5], 6.2, for the precise definition). (Note that in fact, (3.4.3) gives a definition of cuspidal: one could define $\mathcal{L}'$ to be a cuspidal local system for G if in (3.4.3), $P' = G$.) It is shown in [Lu5] that all $\mathcal{L}'$ appearing in this way must be cuspidal for the corresponding L'.

If we denote by $\mathcal{M}_{\mathcal{C},\mathcal{L}}$ the subset of $\mathcal{N}_G$ attached to $(L, \mathcal{C}, \mathcal{L})$ by (3.4.3), the generalized Springer correspondence (3.4.2) can be reformulated as a bijection $\mathcal{M}_{\mathcal{C},\mathcal{L}} \leftrightarrow \widehat{W}$. For $(e, \phi) \in \mathcal{M}_{\mathcal{C},\mathcal{L}}$, there corresponds an irreducible W-representation, which we will denote by $\mu(\mathcal{O}_e, \phi)$, constructed in $\mathrm{Hom}_{A(e)}[\phi : H^\bullet_c(\mathcal{P}_e, \dot{\mathcal{L}})]$.

With respect to the closure ordering of nilpotent orbits, the smallest orbit, $\mathcal{O}_{min}$, appearing in $\mathrm{Im}\Phi_{\mathcal{C},\mathcal{L}}$ is $G \cdot \mathcal{C}$, and the largest orbit, $\mathcal{O}_{max}$, is the

Lusztig-Spaltenstein induced orbit $\mathrm{Ind}_L^G(\mathcal{C})$. Moreover, they both appear exactly once.

The correspondence is normalized such that

$$\mu(\mathcal{O}_{min}, \phi_{min}) = trivial \text{ and } \mu(\mathcal{O}_{max}, \phi_{max}) = sgn.$$

3.5. The previous discussion gives a classification of simple $\mathbb{H}$-modules via *lowest W-types.*

Fix $\psi \in \widehat{A}_G(s,e)$. If $\phi \in \widehat{A}^0_G(e)$ and

$$\mathrm{Hom}_W[(\mu(\mathcal{O}_e, \phi) : X(s, r_0, e, \psi)] \neq 0, \tag{3.5.1}$$

then we will call $\mu(\mathcal{O}, \phi)$ a *lowest W-type* for $X(s, r_0, e, \psi)$.

Proposition. *The simple module $\overline{X}(s, r_0, e, \psi)$ is the unique composition factor of $X(s, r_0, e, \psi)$ which contains the lowest W-types $\sigma(\mathcal{O}_e, \phi)$ with multiplicity $[\phi \mid_{A_G(s,e)} : \psi]$.*

Proof. It follows directly from theorem 3.3 and the discussion in section 3.4, particularly, equation (3.4.1). □

3.6. If $s \in \mathfrak{g}$ is semisimple, let $s = s_{hyp} + s_{ell}$ denote its decomposition into hyperbolic and elliptic parts.

Theorem ([Lu4]). *A simple $\mathbb{H}$-module $\overline{X}(s, r_0, e, \psi)$ is tempered if and only if $\{s_{hyp}, e\}$ can be embedded into a Lie triple of $\mathcal{O}_e$. In this case $\overline{X}(s, r_0, e, \psi) = X(s, r_0, e, \psi)$.*

If in addition, $\mathcal{O}_e$ is a distinguished nilpotent orbit, then $X(s, r_0, e, \psi)$ is a discrete series.

Note that, if (s, e, ϕ) is the geometric parameter of a real ($s_{ell} = 0$) tempered simple $\mathbb{H}$-module, $A_G(s, e) = A_G(e)$, and $X(s, r_0, e, \phi)$ has a unique lowest W-type, namely $\mu(\mathcal{O}_e, \phi)$. In addition, $\mu(\mathcal{O}_e, \phi)$ has multiplicity one.

This gives a bijection between $\widehat{W}$ and simple tempered $\mathbb{H}$-modules with real central character. In particular, we have the following result.

Corollary. *The set of simple tempered $\mathbb{H}$-modules with real central character is linearly independent in the Grothendieck group of W.*

4. Hermitian forms

We retain the notation from the previous sections. The graded Hecke algebra $\mathbb{H}$ will be assumed geometric and all parameters and central characters are assumed to be real. We will fix $r_0 = 1/2$ and drop it from the notation. (It is sufficient to determine the unitary dual of $\mathbb{H}$ for one particular value $r_0 \neq 0$.)

4.1. Let (M,σ,ν), ν real, be a Hermitian Langlands parameter as in section 2.6. We constructed the intertwining operator

$$\mathcal{A}(M,\sigma,\nu): X(M,\sigma,\nu) \to I(M,\sigma\otimes\mathbb{C}_{-\nu}),\ x\otimes(v\otimes 1_\nu) \mapsto xr_w\otimes(a_w(v)\otimes 1_{-\nu}), \tag{4.1.1}$$

where w is such that $wM=M$, $a_w: w\sigma \xrightarrow{\cong} \sigma$, and $w\nu=-\nu$.

By the results in the previous section, $X(M,\sigma,\nu)$ contains a special set of W-representations, the lowest W-types. It is an empirical fact that every $L(M,\sigma,\nu)$ contains one lowest W-type μ_0 with multiplicity 1. (This is verified case by case for the exceptional Hecke algebra, for those of classical types this is automatic from the fact that the component groups $A_G(e)$ of nilpotent elements e are always abelian.)

As a $\mathbb{C}[W]$-module,

$$I(M,\sigma\otimes\mathbb{C}_\nu)\mid_W = \mathbb{C}[W]\otimes_{\mathbb{C}[W(M)]}(\sigma\mid_W). \tag{4.1.2}$$

For any W-type (μ,V_μ), $\mathcal{A}(M,\sigma,\nu)$ induces an operator

$$r_\mu(M,\sigma,\nu):\ \mathrm{Hom}_W(\mu,\mathbb{C}[W]\otimes_{\mathbb{C}[W(M)]}\sigma) \to \mathrm{Hom}_W(\mu,\mathbb{C}[W]\otimes_{\mathbb{C}[W(M)]}\sigma). \tag{4.1.3}$$

By Frobenius reciprocity,

$$\mathrm{Hom}_W(\mu,\mathbb{C}[W]\otimes_{\mathbb{C}[W(M)]}\sigma) \cong \mathrm{Hom}_{W(M)}(\mu,\sigma). \tag{4.1.4}$$

In conclusion, $\mathcal{A}(M,\sigma,\nu)$ gives rise to operators

$$r_\mu(M,\sigma,\nu):\ \mathrm{Hom}_{\mathbb{C}[W(M)]}(\mu,\sigma) \to \mathrm{Hom}_{\mathbb{C}[W(M)]}(\mu,\sigma). \tag{4.1.5}$$

The operator $r_{\mu_0}(M,\sigma,\nu)$ is a scalar, and we normalize the intertwining operator $\mathcal{A}(M,\sigma,\nu)$ so that this scalar is 1.

Recall the map $\epsilon:\mathbb{H}\to\mathbb{A}$ (section 2.6). We denote by $\epsilon(x)(\nu)$, the evaluation of an element $\epsilon(x)\in\mathbb{A}=S(\mathfrak{h}^*)$ at $\nu\in\mathfrak{h}$.

Theorem ([BM3])**.** *Let (M,σ,ν) be a Hermitian Langlands parameter.*

1. *The map $\mathcal{A}(M,\sigma,\nu)$ is an intertwining operator.*
2. *The image of the operator $\mathcal{A}(M,\sigma,\nu)$ is $L(M,\sigma,\nu)$ and the Hermitian form on $L(M,\sigma,\nu)$ is given by:*

$$\begin{aligned}\langle t_x\otimes(v_x\otimes 1_\nu), t_y\otimes(v_y\otimes 1_\nu)\rangle &= \langle t_x\otimes(v_x\otimes 1_\nu), t_yr_{w_m}\otimes(a_m(v_y)\otimes 1_{-\nu})\rangle_h,\\ &= \langle \epsilon(t_y^*t_xr_{w_m})(\nu)a_m(v_x), v_y\rangle_M.\end{aligned}$$

The discussion in this section can be summarized in the following corollary.

Corollary. *A Langlands parameter (M,σ,ν), ν real, is unitary if and only if the following two conditions are satisfied:*

1. *$w_mM=M$, $w_mV\cong V$, $w_m\nu=-\nu$;*
2. *the normalized operators $r_\mu(M,\sigma,\nu)$ are positive semidefinite for all $\mu\in\widehat{W}$, such that* $\mathrm{Hom}_{W(M)}[\mu:V]\neq 0$.

One of the main tools for showing $\mathbb{H}$-modules are *not* unitary is to compute the signature of certain $r_\mu(M,\sigma,\nu)$.

4.2. We consider now the particular case of spherical modules.

Definition. *A $\mathbb{H}$-module V is called spherical if* $\mathrm{Hom}_W[V, triv] \neq 0$. *It is called generic if* $\mathrm{Hom}_W[V, sgn] \neq 0$.

In the case of Hecke algebra with equal parameters, these definitions are motivated by the correspondence with the representations of p-adic groups. (In that case, spherical, respectively generic, $\mathbb{H}$-modules correspond to representations of the p-adic group having fixed vectors under a maximal compact open subgroup, respectively admitting Whittaker models.)

The Iwahori-Matsumoto involution, IM, defined by

$$\begin{aligned} IM(t_w) &:= (-1)^{l(w)} t_w, \\ IM(\omega) &:= -\omega, \quad \omega \in \mathfrak{h}^*, \end{aligned} \tag{4.2.1}$$

interchanges spherical and generic modules. From (2.3.1), it follows that IM commutes with the $*$-operation on $\mathbb{H}$. From (2.3.2), it follows immediately then that IM preserves Hermitian and unitary modules.

Let $X(\nu) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu$, $\nu \in \mathfrak{h}_{\mathbb{R}}$, be a principal series module. Clearly, $X(\nu) \cong \mathbb{C}[W]$, as W-modules, and in particular, $X(\nu)$ has a unique subquotient containing the $triv$ representation of W. Denote this by $L(\nu)$. Let $\mathfrak{h}_{\mathbb{R}}^+$ denote the dominant parameters in $\mathfrak{h}_{\mathbb{R}}$. The Langlands classification and the considerents about intertwining operators presented in the previous sections are summarized as follows.

Proposition. *Assume $\nu \in \mathfrak{h}_{\mathbb{R}}^+$.*
(1) The principal series $X(\nu)$ has a unique irreducible quotient, $L(\nu)$, which is spherical. Moreover, every spherical $\mathbb{H}$-module appears in this way.
(2) A spherical module $L(\nu)$ is Hermitian if and only if $w_0\nu = -\nu$.
(3) If $w_0\nu = -\nu$, the image of the intertwining operator

$$A(\nu) : X(\nu) \to X(-\nu), \ A(\nu)(x \otimes 1_\nu) = x r_{w_0} \otimes 1_{-\nu},$$

normalized so that it is $+1$ on the spherical vector, is $L(\nu)$.

In particular, $X(\nu)$ is reducible if and only if $\langle \alpha, \nu \rangle = c_\alpha$, for some $\alpha \in R^+$. For every $\mu \in \widehat{W}$, let

$$r_\mu(\nu) : \mu^* \to \mu^* \tag{4.2.2}$$

be the operator defined by (4.1.5). Let $w_0 = s_1 \cdots s_k$ be a reduced decomposition of w_0 ($k = |R^+|$), where $s_j = s_{\alpha_j}$, $\alpha_j \in \Pi$. The operator $r_\mu(\nu)$ has a decomposition

$$r_\mu(\nu) = r_{\mu,s_1}(s_2 \cdots s_k\nu) \cdot r_{\mu,s_2}(s_3 \cdots s_k\nu) \cdots r_{\mu,s_k}(\nu), \tag{4.2.3}$$

where each $r_{\mu,s_j}(\nu')$ is determined by the equation

$$r_{\mu,s_j}(\nu') = \begin{cases} 1, & \text{on the } (+1)\text{-eigenspace of } s_j \text{ on } \mu^* \\ \frac{c_{\alpha_j} - \nu'}{c_{\alpha_j} + \nu'}, & \text{on the } (-1)\text{-eigenspace of } s_j \text{ on } \mu^*. \end{cases} \tag{4.2.4}$$

4.3. As in corollary 4.1, a spherical Hermitian module $L(\nu)$ is unitary if and only if $r_\mu(\nu)$ are positive semidefinite for all $\mu \in \widehat{W}$.

Definition. *Define the* 0*-complementary series to be the set* $\{\nu \in \mathfrak{h}_{\mathbb{R}}^+ : X(\nu) = L(\nu)$ *unitary*$\}$.

The 0-complementary series when $\mathbb{H}$ is of type B_n/C_n, with arbitrary unequal parameters, were determined in [BC1]. For type G_2, using the machinery presented in the previous section, in particular the signatures of $r_\mu(\nu)$, it is an easy calculation. We record the result, without proof, next. We use the simple roots

$$\alpha_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \text{ and } \alpha_2 = (-1, 1, 0).$$

The Hecke algebra $\mathbb{H}(G_2)$ has $c_{\alpha_1} = 1$ and $c_{\alpha_2} = c > 0$.

Proposition. *Let* $\nu = (\nu_1, \nu_1+\nu_2, -2\nu_1-\nu_2)$, $\nu_1 \geq 0$, $\nu_2 \geq 0$, *be a dominant (spherical) parameter for* $\mathbb{H}(G_2)$. *There are nine cases depending on the parameter* c. *The* 0*-complementary series are as follows.*
(1) $0 < c < 1 : \{3\nu_1 + 2\nu_2 < c\} \cup \{3\nu_1 + \nu_2 > c, \nu_2 < c, 2\nu_1 + \nu_2 < 1\}$.
(2) $c = 1 : \{3\nu_1 + 2\nu_2 < 1\} \cup \{3\nu_1 + \nu_2 > 1, 2\nu_1 + \nu_2 < 1\}$.
(3) $1 < c < \frac{3}{2} : \{3\nu_1 + 2\nu_2 < c\} \cup \{3\nu_1 + \nu_2 > c, 2\nu_1 + \nu_2 < 1\}$.
(4) $c = \frac{3}{2} : \{3\nu_1 + 2\nu_2 < \frac{3}{2}\}$.
(5) $\frac{3}{2} < c < 2 : \{3\nu_1 + 2\nu_2 < c, 2\nu_1 + \nu_2 < 1\}$.
(6) $c = 2 : \{2\nu_1 + \nu_2 < 1\}$.
(7) $2 < c < 3 : \{2\nu_1 + \nu_2 < 1\} \cup \{\nu_1 + \nu_2 > 1, 3\nu_1 + 2\nu_2 < c\}$.
(8) $c = 3 : \{2\nu_1 + \nu_2 < 1\} \cup \{\nu_1 + \nu_2 > 1, 3\nu_1 + 2\nu_2 < 3\}$.
(9) $c > 3 : \{2\nu_1 + \nu_2 < 1\} \cup \{\nu_1 + \nu_2 > 1, \nu_1 < 1, 3\nu_1 + 2\nu_2 < c\}$.

In addition, the only other unitary spherical representations are the endpoints of the 0*-complementary series, and the trivial representation, for which* $(\nu_1, \nu_2) = (1, c)$.

In this notation, the Hecke algebras of type G_2 which appear geometrically in [Lu6] are the cases $c = 1$ (equal parameters) and $c = 9$. The latter corresponds to a cuspidal local system on the nilpotent $2A_2$ in E_6.

4.4. Recall the arrangement of hyperplanes $\langle \alpha, \nu \rangle = c_\alpha$ in the dominant Weyl chamber. A *face* $\mathcal{F}$ in $\mathfrak{h}_{\mathbb{R}}^+$ is a set determined by the roots $\alpha \in R^+$ being $= 0$, $> c_\alpha$, $= c_\alpha$, or $< c_\alpha$. It is natural to expect that the spherical operators $r_\mu(\nu)$ have constant signature as ν varies over a face $\mathcal{F}$. For the Hecke algebra with equal parameters (*i.e.* $c_\alpha = 1$, for all α), this is theorem 2.4 in [BC1]. We prove a generalization here.

Theorem. *Let* $\mathbb{H}$ *be a geometric Hecke algebra. The multiplicities of the* W*-types in the spherical irreducible module* $L(\nu)$ *are constant as* ν *ranges over a face* $\mathcal{F}$.

Proof. Similar to [BC1]. Details to follow. □

5. Geometric Hecke algebra of type F_4

We present the classification of the unitary dual for the Hecke algebra $\mathbb{H}$ with unequal parameters of type F_4, which appears geometrically in the classification of [Lu6]. It has labels 1 on the long roots, and 2 on the short roots. We use the following choice for the roots:

$$\alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4, \quad \alpha_2 = 2\epsilon_4, \quad \alpha_3 = \epsilon_3 - \epsilon_4, \quad \alpha_4 = \epsilon_2 - \epsilon_3.$$

This algebra is attached to the cuspidal local system on the nilpotent orbit $(3A_1)''$ in E_7. The simple modules are parametrized therefore by a subset of the nilpotent orbits in E_7, larger in the closure ordering than $(3A_1)''$. We list these nilpotent orbits $\mathcal{O}$, each with the corresponding infinitesimal character in $\mathbb{H}$, the W-types coming from the generalized Springer correspondence (LWT), and the centralizer $\mathfrak{z}(\mathcal{O})$ in E_7. The generalized Springer correspondence for this case was computed in [Sp]. The closure ordering for complex nilpotent orbits in E_7 can be found in [Ca]. (We mention that there is a typographical error in [Ca], the nilpotent orbits $D_6(a_2)$ and $D_5(a_1)+A_1$ are comparable, see figure 2.)

Table 1: Nilpotent orbits for $(3A_1)''$ in E_7

$\mathcal{O}$	**Central character**	$\mathfrak{z}(\mathcal{O})$	**LWT**
$(3A_1)''$	$(\nu_1,\nu_2,\nu_3,\nu_4)$	F_4	1_1
$4A_1$	$(\nu_1,\nu_2,\nu_3,\frac{1}{2})$	C_3	2_3
A_2+3A_1	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_1(2,1,1,0)+\nu_2(1,1,0,0)$	G_2	1_3
$(A_3+A_1)''$	$(0,0,1,-1)+\nu_1(\frac{1}{2},\frac{1}{2},0,0)+\nu_2(\frac{1}{2},-\frac{1}{2},0,0)$ $+\nu_3(0,0,\frac{1}{2},\frac{1}{2})$	B_3	4_2
A_3+2A_1	$(\nu_1,1+\nu_2,-1+\nu_2,\frac{1}{2})$	$2A_1$	8_3
$D_4(a_1)+A_1$	$(\nu_1,\nu_2,\frac{3}{2},\frac{1}{2})$	$2A_1$	$9_1,2_1$
$A_3+A_2+A_1$	$(\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{1}{2})+\nu(2,1,1,0)$	A_1	4_4
D_4+A_1	$(\nu_1,\nu_2,\frac{5}{2},\frac{1}{2})$	B_2	9_3
$(A_5)''$	$(\nu_2+\frac{3\nu_1}{2},2+\frac{\nu_1}{2},\frac{\nu_1}{2},-2+\frac{\nu_1}{2})$	G_2	8_1
$D_5(a_1)+A_1$	$(\frac{3}{2}+\nu,-\frac{3}{2}+\nu,\frac{3}{2},\frac{1}{2})$	A_1	4_1
A_5+A_1	$(\frac{1}{4},\frac{7}{4},-\frac{1}{4},-\frac{9}{4})+\nu(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	A_1	6_1
$D_6(a_2)$	$(\nu,\frac{5}{2},\frac{3}{2},\frac{1}{2})$	A_1	16_1
$E_7(a_5)$	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	1	$12_1,6_2$
D_5+A_1	$(2+\nu,-2+\nu,\frac{5}{2},\frac{1}{2})$	A_1	2_2
$D_6(a_1)$	$(\nu,\frac{7}{2},\frac{3}{2},\frac{1}{2})$	A_1	9_2
$E_7(a_4)$	$(\frac{7}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	1	$4_3,8_2$
D_6	$(\nu,\frac{9}{2},\frac{5}{2},\frac{1}{2})$	A_1	9_4
$E_7(a_3)$	$(\frac{9}{2},\frac{5}{2},\frac{1}{2},\frac{1}{2})$	1	$8_4,1_2$
$E_7(a_2)$	$(\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$	1	4_5
$E_7(a_1)$	$(\frac{13}{2},\frac{7}{2},\frac{3}{2},\frac{1}{2})$	1	2_4
E_7	$(\frac{17}{2},\frac{9}{2},\frac{5}{2},\frac{1}{2})$	1	1_4

5.1. Results. Let (M, σ, ν) be a (Hermitian) Langlands parameter, where σ is a limit of discrete series for $\mathbb{H}_M$ (or rather $\mathbb{H}_{M_0}$, notation as in section 2.4). We construct a Hecke algebra $\mathbb{H}_\sigma$ with possibly unequal parameters, with Weyl group W_σ.

Proposition. *With the notation above, there exists a set of W-types $\{\mu_0, \mu_1, \dots, \mu_l\}$ in $X(M, \sigma, \nu)$ such that, if the corresponding W_σ-types are $\{\rho(\mu_0), \rho(\mu_1), \dots, \rho(\mu_l)\}$, then*

$$r_\mu(M, \sigma, \nu) = r_{\rho(\mu)}(\nu), \text{ for all } \mu \in \{\mu_0, \dots, \mu_l\},$$

where $r_\mu(M, \sigma, \nu)$ is the operator in $\mathbb{H}$ (defined as in (4.1.5)), and $r_{\rho(\mu)}$ is the spherical operator in $\mathbb{H}_\sigma$.

Proof. Details to follow. □

The spherical unitary dual is listed in table 3. These are the unitary modules parametrized by $\mathcal{O}_{min}$. For the rest of the nilpotents, the unitary sets are presented next. For every nilpotent $\mathcal{O}$ in table 1, with $\{e, h, f\} \subset \mathcal{O}$ a Lie triple, and $\phi \in \widehat{A}_G(e)^0$, let

$$\mathcal{U}(\mathcal{O}, \phi) = \{(\nu, \psi) : s = \frac{h}{2} + \nu, \phi \in \widehat{A}_G(s, e)^0, \mathrm{Hom}_{A(s,e)}[\psi : \phi] \neq 0, \overline{X}(s, e, \psi) \text{ is unitary}\} \tag{5.1.1}$$

denote the set of unitary parameters associated to $(\mathcal{O}, \phi)$.

Theorem. *With the notation as before (particularly table 1 and (5.1.1)), the unitary (real) parameters for $\mathbb{H}(F_4, (1, 2))$ are:*

$\mathcal{O}$	$\mu(\mathcal{O}, \phi)$	**Unitary** ν
E_7	1_4	*D.S.*
$E_7(a_1)$	2_4	*D.S.*
$E_7(a_2)$	4_5	*D.S.*
$E_7(a_3)$	8_4	*D.S.*
$E_7(a_3)$	1_2	*D.S.*
D_6	9_4	$\mathcal{SU}(A_1, (1))$
$E_7(a_4)$	4_3	*D.S.*
	2_2	*D.S.*
$D_6(a_1)$	9_2	$\mathcal{SU}(A_1, (1))$
$D_5 + A_1$	8_2	$\mathcal{SU}(A_1, (1))$
$E_7(a_5)$	12_1	*D.S.*
	6_2	*D.S.*
$D_6(a_2)$	16_1	$\mathcal{SU}(A_1, (1))$
$A_5 + A_1$	6_1	$\mathcal{SU}(A_1, (1))$
$D_5(a_1) + A_1$	4_1	$0 \leq \nu \leq 1$
$(A_5)''$	8_1	$\mathcal{SU}(G_2, (1, 1))$
$D_4 + A_1$	9_3	$\mathcal{SU}(B_2, (1, \frac{3}{2}))$
$A_3A_2A_1$	4_4	$\mathcal{SU}(A_1, (2))$

Table 2 – continued from previous page

$D_4(a_1)+A_1$	9_1 2_1	$\{0\le\nu_2<\nu_1\le 3-\nu_2,\ \nu_1\le\frac{5}{2}\}\cup\{0\le\nu_2=\nu_1\le\frac{3}{2}\}$ $\{0\le\nu_2<\nu_1\le 3-\nu_2,\ \nu_1\le\frac{5}{2}\}\cup\{0\le\nu_2=\nu_1\le\frac{5}{2}\}$
A_3+2A_1	8_3	$\{0\le\nu_1\le\frac{1}{2},\ \nu_1+2\nu_2\le\frac{3}{2}\}\cup$ $\{0\le\nu_1\le\frac{1}{2},\ 0\le\nu_2\le\frac{3}{2},\ 2\nu_2-\nu_1\ge\frac{3}{2}\}$
$(A_3+A_1)''$	4_2	$\mathcal{SU}(B_3,(1,1))\cup\{(1+\nu,\nu,-1+\nu):1<\nu<2\}$
A_2+3A_1	1_3	$\mathcal{SU}(G_2,(2,1))\cup\{\nu_1+\nu_2=1,\ 0<\nu_2<1\}\cup$ $\{\nu_1=1,\ 0\le\nu_2\le 1\}\cup\{3\nu_1+\nu_2=2,\ \frac{1}{2}<\nu_2\le 2\}$
$4A_1$	2_3	$\{\nu_1+\nu_2+\nu_3\le\frac{3}{2}\}\cup\{\nu_1+\nu_2-\nu_3\ge\frac{3}{2},\nu_1<\frac{3}{2},\nu_1+\nu_2<2\}$ $\cup\{(\nu_1,\nu_2,\frac{3}{2}):\nu_1+\nu_2\le 3,\nu_1\le\frac{5}{2}\}$ $\cup\{(\nu_1,1+\nu_2,-1+\nu_2):0\le\nu_1\le\frac{1}{2},0\le\nu_2\le\frac{1}{2}\}$ $\cup\{(\nu_1,1+\nu_2,-1+\nu_2):\nu_1+2\nu_2=\frac{3}{2},\frac{1}{2}\le\nu_1\le\frac{3}{2}\}$ $\cup\{(\nu,\frac{3}{2},\frac{1}{2}):0\le\nu\le\frac{5}{2}\}\cup\{(1+\nu,-1+\nu,\frac{3}{2}):0\le\nu<\frac{3}{2}\}$ $\cup\{(2+\nu,\nu,-2+\nu):0\le\nu<\frac{1}{2}\}\cup\{(\nu,\frac{7}{2},\frac{3}{2}):0\le\nu<\frac{1}{2}\}$ $\cup\{(\frac{5}{2},\frac{3}{2},\frac{1}{2})\}\cup\{(\frac{7}{2},\frac{3}{2},\frac{1}{2})\}\cup\{(\frac{13}{2},\frac{7}{2},\frac{3}{2})\}$
$(3A_1)''$	1_1	$\mathcal{SU}(F_4,(1,2))$, *see table 3*

We will present, in detail, in the next two sections, the calculations giving the unitary dual of $\mathbb{H}(F_4,(1,2))$. The discussion is organized by nilpotent orbits.

For the distinguished orbits, there is nothing to do. They parametrize discrete series, and the lowest W-types and infinitesimal characters can be read from tables 1 and 2.

5.2. Maximal parabolic cases. First we treat the cases of those nilpotent orbits which, in the Bala-Carter classification, correspond to Levi subalgebras of maximal parabolic subalgebras. In terms of the (classical) Langlands classification, they correspond to the induced modules from discrete series on Levi components of maximal parabolic subalgebras.
$\mathbf{D_6}$: The infinitesimal character is $(\nu,\frac{9}{2},\frac{5}{2},\frac{1}{2})$, the lowest W-type is 9_4, and $\mathfrak{z}=A_1$. The standard module is induced from the Steinberg representation on the Hecke subalgebra of type C_3. The W-structure is

$$X(D_6)|_W = 9_4+8_4+4_5+2_4+1_4.$$

We calculate the intertwining operators for arbitrary parameter c. In that case, the infinitesimal character is $(\nu,2c+\frac{1}{2},c+\frac{1}{2},\frac{1}{2})$.

9_4 1;

8_4 $\frac{c-\frac{3}{2}-\nu}{c-\frac{3}{2}+\nu}$;

4_5 $\frac{c+\frac{3}{2}-\nu}{c+\frac{3}{2}+\nu}$;

2_4 $\frac{(c-\frac{3}{2}-\nu)(3c-\frac{1}{2}-\nu)}{(c-\frac{3}{2}+\nu)(3c-\frac{1}{2}+\nu)}$;

1_4 $\frac{(c+\frac{3}{2}-\nu)(3c+\frac{5}{2}-\nu)}{(c+\frac{3}{2}+\nu)(3c+\frac{5}{2}+\nu)}$.

In particular, when $c = 2$, the calculation implies that $\overline{X}(D_6, \nu)$ is unitary if and only if $0 \leq \nu \leq \frac{1}{2}$.

$\mathbf{D_6(a_1)}$: The infinitesimal character is $(\nu, \frac{7}{2}, \frac{3}{2}, \frac{1}{2})$, the lowest W-type is 9_2, and $\mathfrak{z} = A_1$. In terms of Langlands classification, the standard module is induced from a one-dimensional discrete series in C_3. The W-structure is

$$X(D_6(a_1))|_W = \mathrm{Ind}_{W(C_3)}^{W(F_4)}(111 \times 0) = 9_2 + 4_3 + 8_4 + 1_2 + 2_4.$$

The intertwining operators in this case give:

9_2 1;

4_3 $\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$;

8_4 $\frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu}$;

1_2 $\frac{(\frac{1}{2}-\nu)(\frac{7}{2}-\nu)}{(\frac{1}{2}+\nu)(\frac{7}{2}+\nu)}$;

2_4 $\frac{(\frac{7}{2}-\nu)(\frac{13}{2}-\nu)}{(\frac{7}{2}+\nu)(\frac{13}{2}+\nu)}$.

This indicates that $\overline{X}(D_6(a_1), \nu)$ is unitary if and only if $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$, the standard module decomposes as

$$X(D_6(a_1), \frac{1}{2}) = \overline{X}(D_6(a_1), \frac{1}{2}) + X(E_7(a_4), \phi_1)$$
$$\overline{X}(D_6(a_1), \frac{1}{2}) = 9_2 + 8_4 + 2_4.$$

$\mathbf{D_5 + A_1}$: The infinitesimal character is $(2+\nu, -2+\nu, \frac{5}{2}, \frac{1}{2})$, the lowest W-type is 8_2, and $\mathfrak{z} = A_1$. The standard module is induced from the Steinberg representation on the Hecke subalgebra of type B_3. The W-structure is

$$X(D_5 + A_1)|_W = 8_2 + 2_2 + 9_4 + 4_5 + 1_4.$$

We calculate the intertwining operators for arbitrary parameter c. In that case, the infinitesimal character is $(\frac{c}{2} + 1 + \nu, -\frac{c}{2} - 1 + \nu, c + \frac{1}{2}, \frac{1}{2})$.

8_2 $+1$;

2_2 $\frac{\frac{c-3}{2}+\nu}{\frac{c-3}{2}-\nu}$;

9_4 $\frac{\frac{3c-1}{2}-\nu}{\frac{3c-1}{2}+\nu}$;

4_5 $\frac{(\frac{3c-1}{2}-\nu)(\frac{3c+1}{2}-\nu)}{(\frac{3c-1}{2}+\nu)(\frac{3c+1}{2}+\nu)}$;

1_4 $\frac{(\frac{3c-1}{2}-\nu)(\frac{3c+1}{2}-\nu(\frac{5c+3}{2}-\nu))}{(\frac{3c-1}{2}+\nu)(\frac{3c+1}{2}+\nu)(\frac{5c+3}{2}+\nu)}$.

In particular, if $c = 2$, $\overline{X}(D_5 + A_1, \nu)$ is unitary if and only if $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$, the standard module decomposes as

$$X(D_5 + A_1, \frac{1}{2}) = \overline{X}(D_5 + A_1, \frac{1}{2}) + X(E_7(a_4), \phi_2)$$
$$\overline{X}(D_5 + A_1, \frac{1}{2})|_W = 8_2 + 9_4 + 4_5 + 1_4.$$

$\mathbf{D_6(a_2)}$: The infinitesimal character is $(\nu, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$, the lowest W-type is 16_1, and $\mathfrak{z} = A_1$. In terms of Langlands classification, the standard module is induced from a four dimensional discrete series on C_3. The W-structure is

$$\begin{aligned} X(D_6(a_2))|_W &= \mathrm{Ind}_{W(C_3)}^{W(F_4)}(1 \times 11 + 0 \times 111) \\ &= 16_1 + 2_4 + 9_2 + 2 \cdot 9_4 + 2 \cdot 8_4 + 12_1 + 8_2 + 2 \cdot 4_5 + 6_2 + 1_4. \end{aligned}$$

The intertwining operators are (for the W-types with multiplicity greater than 1, we only give the determinant):

16_1	$+1;$
12_1	$\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu};$
6_2	$\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu};$
2_4	$\frac{(\frac{5}{2}-\nu)(\frac{9}{2}-\nu)}{(\frac{5}{2}+\nu)(\frac{9}{2}+\nu)};$
8_2	$\frac{(\frac{1}{2}-\nu)(\frac{5}{2}-\nu)}{(\frac{1}{2}+\nu)(\frac{5}{2}+\nu)};$
1_4	$\frac{(\frac{1}{2}-\nu)(\frac{5}{2}-\nu)(\frac{9}{2}-\nu)(\frac{11}{2}-\nu)}{(\frac{1}{2}+\nu)(\frac{5}{2}+\nu)(\frac{9}{2}+\nu)(\frac{11}{2}+\nu)};$
9_2	$\frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu}$
8_4	$\mathrm{Det}=\frac{(\frac{1}{2}-\nu)(\frac{5}{2}-\nu)(\frac{9}{2}-\nu)}{(\frac{1}{2}+\nu)(\frac{5}{2}+\nu)(\frac{9}{2}+\nu)};$
4_5	$\mathrm{Det}=\frac{(\frac{1}{2}-\nu)^2(\frac{5}{2}-\nu)(\frac{9}{2}-\nu)(\frac{11}{2}-\nu)}{(\frac{1}{2}+\nu)^2(\frac{5}{2}+\nu)(\frac{9}{2}+\nu)(\frac{11}{2}+\nu)}.$

It follows that $\overline{X}(D_6(a_2), \nu)$ is unitary if and only if $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$, the standard module decomposes as

$$X(D_6(a_2), \frac{1}{2}) = \overline{X}(D_6(a_2), \frac{1}{2}) + X(E_7(a_5), \phi_1) + X(E_7(a_5), \phi_2)$$
$$\overline{X}(D_6(a_2), \frac{1}{2})|_W = 16_1 + 2_4 + 9_2 + 8_4 + 9_4.$$

$\mathbf{A_5 + A_1}$: The infinitesimal character is $(\frac{1}{4}, \frac{7}{4}, -\frac{1}{4}, -\frac{9}{4}) + \nu(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the lowest W-type is 6_1, and $\mathfrak{z} = A_1$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on $\widetilde{A}_2 + A_1$.

$$\begin{aligned} X(A_5 + A_1)|_W &= \mathrm{Ind}_{W(\widetilde{A}_2 + A_1)}^{W(F_4)}((111) \otimes (11)) \\ &= 6_1 + 16_1 + 2 \cdot 8_4 + 8_2 + 4_3 + 4_5 + 12_1 + 9_2 + 2 \cdot 9_4 + 2_4 + 1_4. \end{aligned}$$

We compute the intertwining operator for arbitrary parameter c. In that case, the infinitesimal character is $(\frac{1}{4}, c - \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4} - c) + \nu(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

6_1 $1;$
16_1 $\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu};$
8_2 $\frac{(\frac{1}{2}-\nu)^2(\frac{3}{2}-\nu)}{(\frac{1}{2}+\nu)^2(\frac{3}{2}+\nu)};$
12_1 $\frac{(\frac{1}{2}-\nu)^2}{(\frac{1}{2}+\nu)^2};$
9_2 $\frac{(\frac{1}{2}-\nu)(2c-\frac{3}{2}-\nu)}{(\frac{1}{2}+\nu)(2c-\frac{3}{2}+\nu)};$
4_3 $\frac{2c-\frac{3}{2}-\nu}{2c-\frac{3}{2}+\nu};$
2_4 $\frac{(\frac{1}{2}-\nu)(c+\frac{1}{2}-\nu)(2c-\frac{3}{2}-\nu)(2c+\frac{1}{2}-\nu)}{(\frac{1}{2}+\nu)(c+\frac{1}{2}+\nu)(2c-\frac{3}{2}+\nu)(2c+\frac{1}{2}+\nu)};$
4_5 $\frac{(\frac{1}{2}-\nu)^2(\frac{3}{2}-\nu)(c+\frac{1}{2}-\nu)(2c+\frac{1}{2}-\nu)}{(\frac{1}{2}+\nu)^2(\frac{3}{2}+\nu)(c+\frac{1}{2}+\nu)(2c+\frac{1}{2}+\nu)};$
1_4 $\frac{(\frac{1}{2}-\nu)^2(\frac{3}{2}-\nu)(c+\frac{1}{2}-\nu)(2c+\frac{1}{2}-\nu)(2c+\frac{3}{2}-\nu)}{(\frac{1}{2}+\nu)^2(\frac{3}{2}-\nu)(c+\frac{1}{2}+\nu)(2c+\frac{1}{2}+\nu)(2c+\frac{3}{2}+\nu)};$
8_4 $\text{Det}=\frac{(\frac{1}{2}-\nu)^3(c+\frac{1}{2}-\nu)(2c-\frac{3}{2}-\nu)(2c+\frac{1}{2}-\nu)}{(\frac{1}{2}+\nu)^3(c+\frac{1}{2}+\nu)(2c-\frac{3}{2}+\nu)(2c+\frac{1}{2}+\nu)};$
9_4 $\text{Det}=\frac{(\frac{1}{2}-\nu)^3(\frac{3}{2}-\nu)(c+\frac{1}{2}-\nu)(2c+\frac{1}{2}-\nu)}{(\frac{1}{2}+\nu)^3(\frac{3}{2}+\nu)(c+\frac{1}{2}+\nu)(2c+\frac{1}{2}+\nu)}.$

It follows that in the case $c=2$, $\overline{X}(A_5+A_1,\nu)$ is unitary if and only if $0\le\nu\le\frac{1}{2}$. At $\nu=\frac{1}{2}$, the standard module decomposes as

$$X(A_5A_1,\frac{1}{2})=\overline{X}(A_5A_1,\frac{1}{2})+\overline{X}(D_6(a_2),\frac{1}{2})+X(E_7(a_5),\phi_1)$$
$$\overline{X}(A_5A_1,\frac{1}{2})=6_1+4_3.$$

$\mathbf{D_5(a_1)+A_1}$: The infinitesimal character is $(\frac{3}{2}+\nu,-\frac{3}{2}+\nu,\frac{3}{2},\frac{1}{2})$, the lowest W-type is 4_1, and $\mathfrak{z}=A_1$. In terms of Langlands classification, the standard module is induced from a two-dimensional discrete series on B_3. The W-structure is

$$X(D_5(a_1)+A_1))|_W=\text{Ind}_{W(B_3)}^{W(F_4)}(0\times 12)=4_1+16_1+2_4+9_4+9_2+8_4.$$

The intertwining operators are:
4_1 $1;$
16_1 $\frac{1-\nu}{1+\nu};$
9_2 $\frac{(1-\nu)(2-\nu)}{(1+\nu)(2+\nu)};$
9_4 $\frac{(1-\nu)(4-\nu)}{(1+\nu)(4+\nu)};$
8_4 $\frac{(1-\nu)(2-\nu)(4-\nu)}{(1+\nu)(2+\nu)(4+\nu)};$
2_4 $\frac{(1-\nu)(2-\nu)(4-\nu)(5-\nu)}{(1+\nu)(2+\nu)(4+\nu)(5+\nu)}.$

It follows that $\overline{X}(D_5(a_1)+A_1)$ is unitary if and only if $0\le\nu\le 1$. At $\nu=1$, the standard module decomposes as

$$X(D_5(a_1)+A_1,1)=\overline{X}(D_5(a_1)+A_1,1)+\overline{X}(D_6(a_2),\frac{1}{2})$$
$$\overline{X}(D_5(a_1)+A_1,1)|_W=4_1.$$

$\mathbf{A_3+A_2+A_1}$: The infinitesimal character is $(\frac12,\frac12,-\frac32,\frac12)+\nu(2,1,1,0)$, the lowest W-type is 4_4, and $\mathfrak{z}=A_1$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on $A_2+\widetilde{A}_1$. The W-structure is:

$$\begin{aligned}X(A_3+A_2+A_1)|_W &= \mathrm{Ind}_{W(A_2+\widetilde{A}_1)}^{W(F_4)}((111)\otimes(11))\\ &= 4_4+16_1+8_4+2\cdot 8_2+4_5+12_1+6_1+9_3+2\cdot 9_4+2_2+1_4.\end{aligned}$$

We compute the intertwining operators for general parameter c. In that case the infinitesimal character is $(\frac12,\frac{c-1}{2},-\frac{c+1}{2},\frac12)+\nu(2,1,1,0)$.

4_4 1;

9_3 $\frac{\frac{c}{2}-\nu}{\frac{c}{2}+\nu}$;

6_1 $\frac{\frac{3c-2}{2}-\nu}{\frac{3c-2}{2}+\nu}$;

16_1 $\frac{(\frac{c}{2}-\nu)(\frac{3c-2}{2}-\nu)}{(\frac{c}{2}+\nu)(\frac{3c-2}{2}+\nu)}$;

12_1 $\frac{(\frac{c}{2}-\nu)^2(\frac{3c-2}{2}-\nu)}{(\frac{c}{2}+\nu)^2(\frac{3c-2}{2}+\nu)}$;

8_4 $\frac{(\frac{c}{2}-\nu)^2(\frac{3c-2}{2}-\nu)(\frac{3c}{2}-\nu)}{(\frac{c}{2}+\nu)^2(\frac{3c-2}{2}+\nu)(\frac{3c}{2}+\nu)}$;

2_2 $\frac{(\frac{c}{2}-\nu)(\frac{c+1}{2}-\nu)(\frac{c+2}{2}-\nu)}{(\frac{c}{2}+\nu)(\frac{c+1}{2}+\nu)(\frac{c+2}{2}+\nu)}$;

4_5 $\frac{(\frac{c}{2}-\nu)^2(\frac{c+1}{2}-\nu)(\frac{c+2}{2}-\nu)(\frac{3c-2}{2}-\nu)(\frac{3c}{2}-\nu)}{(\frac{c}{2}+\nu)^2(\frac{c+1}{2}+\nu)(\frac{c+2}{2}+\nu)(\frac{3c-2}{2}+\nu)(\frac{3c}{2}+\nu)}$;

1_4 $\frac{(\frac{c}{2}-\nu)^2(\frac{c+1}{2}-\nu)(\frac{c+2}{2}-\nu)(\frac{3c-2}{2}-\nu)(\frac{3c}{2}-\nu)(\frac{3c+2}{2}-\nu)}{(\frac{c}{2}+\nu)^2(\frac{c+1}{2}+\nu)(\frac{c+2}{2}+\nu)(\frac{3c-2}{2}+\nu)(\frac{3c}{2}+\nu)(\frac{3c+2}{2}+\nu)}$;

8_2 Det$=\frac{(\frac{c}{2}-\nu)^3(\frac{c+1}{2}-\nu)(\frac{c+2}{2}-\nu)(\frac{3c-2}{2}-\nu)}{(\frac{c}{2}+\nu)^3(\frac{c+1}{2}+\nu)(\frac{c+2}{2}+\nu)(\frac{3c-2}{2}+\nu)}$;

9_4 Det$=\frac{(\frac{c}{2}-\nu)^3(\frac{c+1}{2}-\nu)(\frac{c+2}{2}-\nu)(\frac{3c-2}{2}-\nu)^2(\frac{3c}{2}-\nu)}{(\frac{c}{2}+\nu)^3(\frac{c+1}{2}+\nu)(\frac{c+2}{2}+\nu)(\frac{3c-2}{2}+\nu)^2(\frac{3c}{2}+\nu)}$.

In the case $c=2$, it follows that $\overline{X}(A_3+A_2+A_1,\nu)$ is unitary if and only if $0\le\nu\le1$. At $\nu=1$, the standard module decomposes as

$$\begin{aligned}&X(A_3A_2A_1,1)=\overline{X}(A_3A_2A_1,1)+\overline{X}(D_4A_1,(\frac32,\frac12))+X(E_7(a_5),\phi_1)\\ &\overline{X}(A_3A_2A_1,1)|_W=4_4+6_1.\end{aligned}$$

Note that $\overline{X}(A_3A_2A_1,1)$ is the IM-dual of $\overline{X}(A_5A_1,\frac12)$.

5.3. Matching of intertwining operators.

$(\mathbf{A_5})''$: The infinitesimal character is $(\nu_2+\frac{3\nu_1}{2},2+\frac{\nu_2}{2},\frac{\nu_1}{2},-2+\frac{\nu_1}{2})$, the lowest W-type is 8_1, and $\mathfrak{z}=G_2$. In the Langlands classification, the standard module is induced from the Steinberg representation on $\widetilde{A}_2$. The W-structure is

$$X((A_5)'')|_W=\mathrm{Ind}_{W(\widetilde{A}_2)}^{W(F_4)}((111)).$$

The restrictions of the nearby W-types are as follows:

Nilpotent	$(A_5)''$	A_5A_1	$D_6(a_2)$	$E_7(a_5)$	$E_7(a_5)$
W-type	8_1	6_1	16_1	12_1	6_2
Multiplicity	1	1	2	2	1
$\widetilde{A}_2 \subset \widetilde{A}_2 + A_1$	(2)	(11)	$(2),(11)$	$(2),(11)$	(2)
$\widetilde{A}_2 \subset C_3$	11×1	11×1	$11 \times 1, 1 \times 11$	$11 \times 1, 1 \times 11$	1×11
$W(\mathfrak{z}) = W(G_2)$	$(1,0)$	$(1,3)''$	$(2,2)$	$(2,1)$	$(1,3)'$

We need the intertwining operators for the induced module from the Steinberg representation on A_2 in the Hecke algebra of type C_3, with parameters 2–2⇐1. The operators are

$$11 \times 1 : 1, \quad 1 \times 11 : \frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}, \quad 111 \times 0 : \frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}.$$

Therefore, the matching of operators is with the Hecke algebra of type G_2 with equal parameters, $\mathbb{H}(G_2)$. The nearby W-types match all the relevant $W(G_2)$-types, so the unitary set $\mathcal{U}(A_5'')$ is included in the spherical unitary dual of $\mathbb{H}(G_2)$. The modules of $\mathbb{H}(G_2)$ are parametrized by nilpotent orbits in the Lie algebra of type G_2. We analyze the composition series and unitarity of $X(A_5'')$ by cases corresponding to these nilpotent orbits.

(1) : In $\mathbb{H}(G_2)$, this is the spherical complementary series

$$\{3\nu_1 + 2\nu_2 < 1\} \cup \{3\nu_1 + \nu_2 > 1 > 2\nu_1 + \nu_2\}. \tag{5.3.1}$$

The standard module $X(A_5'')$ is irreducible in these regions, and therefore unitary (being unitarily induced and unitary when $\nu_2 = 0$).

A_1 : The parameters are of the form $(\nu_1, \nu_2) = (-\frac{1}{2} + \nu, 1)$. They are unitary for $0 \le \nu < \frac{1}{2}$, being endpoints of the complementary series (5.3.1). The decomposition of the standard module is

$$\begin{aligned} X(A_5'', (1, -\frac{1}{2} + \nu)) &= \overline{X}(A_5'', (1, -\frac{1}{2} + \nu)) + X(A_5 + A_1, \nu) \\ \overline{X}(A_5'', (1, -\frac{1}{2} + \nu))|_W &= 8_1 + 16_1 + 12_1 + 6_2 + 2 \cdot 8_4 + 4_3 + 4_5 + 2 \cdot 9_2 \\ &\quad + 9_4 + 2_4 + 1_2. \end{aligned}$$

$\widetilde{A}_1$: The parameters are of the form $(\nu_1, \nu_2) = (1, -\frac{3}{2} + \nu, 1)$. They are unitary for $0 \le \nu < \frac{1}{2}$, being endpoints of the complementary series (5.3.1). The decomposition of the standard module is

$$\begin{aligned} X(A_5'', (-\frac{3}{2} + \nu, 1)) &= \overline{X}(A_5'', (-\frac{3}{2} + \nu, 1)) + X(D_6(a_2), \nu) \\ \overline{X}(A_5'', (-\frac{3}{2} + \nu, 1))|_W &= 8_1 + 6_1 + 16_1 + 12_1 + 2 \cdot 8_4 + 2 \cdot 4_3 + 2 \cdot 9_2 \\ &\quad + 9_4 + 2_4 + 1_2. \end{aligned}$$

$G_2(a_1)$: This is the parameter $(\nu_1, \nu_2) = (0, 1)$. It is unitary, being an endpoint of the complementary series (5.3.1). The decomposition of the standard module is

$$X(A_5'',(0,1)) = \overline{X}(A_5'',(0,1)) + \overline{X}(A_5A_1,\frac{1}{2}) + \overline{X}(D_6(a_2),\frac{1}{2})$$
$$+ X(E_7(a_5),\phi_1) + X(E_7(a_5),\phi_2)$$
$$\overline{X}(A_5'',(0,1))|_W = 8_1 + 12_1 + 8_4 + 9_2 + 1_2.$$

G_2 : This is the parameter $(\nu_1,\nu_2) = (1,1)$. In $\mathbb{H}(G_2)$, it is isolated. We compute the W-structure of the standard module. In addition to the operators given by the nearby W-types, we need to check the operator on 9_2.

$$X(A_5'',(1,1)) = \overline{X}(A_5'',(1,1)) + \overline{X}(A_5A_1,\frac{3}{2}) + \overline{X}(D_6(a_2),\frac{5}{2})$$
$$+ \overline{X}(D_5A_1,\frac{1}{2}) + \overline{X}(D_6(a_1),\frac{1}{2}) + X(E_7(a_4),\phi_2)$$
$$\overline{X}(A_5'',(1,1))|_W = 8_1 + 9_2.$$

Moreover, the operator on 9_2 is positive at $(1,1)$, so this point is unitary.

In conclusion, the unitary parameter set $\mathcal{U}(A_5'')$ is identical with the spherical unitary dual $\mathcal{SU}(\mathbb{H}(G_2))$.

$\mathbf{D_4 + A_1}$: The infinitesimal character is $(\nu_1,\nu_2,\frac{5}{2},\frac{1}{2})$, the lowest W-type is 9_3, and $\mathfrak{z} = B_2$. In Langlands classification, the standard module is induced from the Steinberg representation on C_2. The W-structure is

$$X(D_4 + A_1)|_W = \mathrm{Ind}_{W(C_2)}^{W(F_4)}(0 \times 11).$$

The restrictions of the nearby W-types are as follows:

Nilpotent	$D_4 + A_1$	$D_5(a_1)A_1$	$D_6(a_2)$	$E_7(a_5)$	$D_6(a_1)$
W-type	9_3	4_1	16_1	$12_1, 6_2$	9_2
Multiplicity	1	1	2	1, 1	1
$B_2 \subset B_3$	1×11	0×12	$0 \times 12, 1 \times 11$	1×11	0×12
$C_2 \subset C_3$	0×12	0×12	$0 \times 12, 1 \times 11$	1×11	1×11
$W(\mathfrak{z}) = W(B_2)$	2×0	11×0	1×1	0×2	0×11

We need the intertwining operators for the induced modules from the Steinberg representation of $C_2 = B_2$ in the Hecke algebras of types B_3, and C_3, with parameters 1–1⇒2, respectively 2–2⇐1.

In B_3 (1–1⇒2), the infinitesimal character is $(\nu,3,2)$, and the standard module is the induced from the Steinberg representation on B_2. The operators are

$$1 \times 11 : 1, \quad 0 \times 12 : \frac{1-\nu}{1+\nu}, \quad 0 \times 111 : \frac{4-\nu}{4+\nu}.$$

In C_3 (2–2⇐1), the infinitesimal character is $(\nu,\frac{5}{2},\frac{1}{2})$, and the standard module is the induced from the Steinberg representation on C_2. The operators are

$$0 \times 12 : 1, \quad 1 \times 11 : \frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}, \quad 0 \times 111 : \frac{(\frac{3}{2}-\nu)(\frac{9}{2}-\nu)}{(\frac{3}{2}+\nu)(\frac{9}{2}+\nu)}.$$

Therefore, the matching of operators is with the Hecke algebra of type B_2 with parameters $1 \Rightarrow 3/2$. The nearby W-types match all the $W(B_2)$-representations of this B_2, therefore, the unitary parameter set $\mathcal{U}(D_4 + A_1)$ is included in the spherical unitary dual of $\mathbb{H}(B_2, 1, 3/2)$. The modules of $\mathbb{H}(B_2, 1, 3/2)$ are parametrized by a cuspidal local system on $\mathfrak{sp}(2) \oplus \mathbb{C}^2 \subset \mathfrak{sp}(6)$. Specifically the nilpotent orbits of $\mathfrak{sp}(6)$ which enter in the parametrization are $(6), (42), (411), (222), (21^4)$. We analyze the composition series and unitarity of $X(D_4 + A_1)$ by cases corresponding to these nilpotent orbits.

(21^4) : In $H(B_2, 1, 3/2)$, this is the spherical complementary series

$$\{\nu_1 + \nu_2 < 1\} \cup \{\nu_1 - \nu_2 > 1, \nu_1 < \frac{3}{2}\}. \tag{5.3.2}$$

The standard module $X(D_4 + A_1)$ is irreducible in these regions, and therefore unitary (being unitarily induced and unitary when $\nu_2 = 0$).

(222) : The parameters are of the form $(\nu_1, \nu_2) = (\frac{1}{2} + \nu, -\frac{1}{2} + \nu)$. They are unitary for $0 \le \nu < \frac{3}{2}$, being endpoints of the complementary series (5.3.2). The standard module decomposes as follows

$$X(D_4A_1, (\frac{1}{2} + \nu, -\frac{1}{2} + \nu)) = \overline{X}(D_4A_1, (\frac{1}{2} + \nu, -\frac{1}{2} + \nu) + X(D_5(a_1)A_1, \nu)$$

$$\overline{X}(D_4A_1, (\frac{1}{2} + \nu, -\frac{1}{2} + \nu)|_W = 9_3 + 16_1 + 12_1 + 2 \cdot 9_4 + 8_4 + 2 \cdot 8_2 + 6_2 + 2 \cdot 4_5 + 2_2 + 1_4.$$

(411) : The parameters are $(\nu_1, \nu_2) = (\nu, \frac{3}{2})$. They are unitary for $0 \le \nu \le \frac{1}{2}$, being endpoints of the complementary series (5.3.2). The standard module decomposes as follows

$$X(D_4A_1, (\nu, \frac{3}{2})) = \overline{X}(D_4A_1, (\nu, \frac{3}{2})) + X(D_6(a_2), \nu)$$

$$\overline{X}(D_4A_1, (\nu, \frac{3}{2}))|_W = 9_3 + 16_1 + 8_2 + 4_1 + 2_2 + 9_4.$$

(42) : The parameters are $(\nu_1, \nu_2) = (\frac{3}{2}, \frac{1}{2})$. This is an endpoint of the complementary series (5.3.2), therefore unitary. The decomposition of the standard module is

$$X(D_4A_1, (\frac{3}{2}, \frac{1}{2})) = \overline{X}(D_4A_1, (\frac{3}{2}, \frac{1}{2})) + \overline{X}(D_5(a_1)A_1, 1) + \overline{X}(D_6(a_2), \frac{1}{2}) + X(E_7(a_5), \phi_1) + X(E_7(a_5), \phi_2)$$

$$\overline{X}(D_4A_1, (\frac{3}{2}, \frac{1}{2}))|_W = 9_3 + 16_1 + 8_2 + 9_4 + 2_2.$$

(6) : The parameters are $(\nu_1, \nu_2) = (\frac{5}{2}, \frac{3}{2})$. This point is isolated. From the W-structure of $\overline{X}(D_4A_1, (\nu, \frac{3}{2}))$, we see that the only W-types which can be present in $\overline{X}(D_4A_1, (\frac{5}{2}, \frac{3}{2}))$ are $9_3, 16_1, 8_2, 4_1, 2_2, 9_4$. By virtue of the matching with operators in $\mathbb{H}(B_2, 1, 3/2)$, we know 16_1 and 4_1 cannot be present. We compute the operators on the remaining ones and find that the W-structure is

$$\overline{X}(D_4A_1,(\frac{5}{2},\frac{3}{2}))=9_3+8_2.$$

Moreover, the operator on 8_2 is positive, so $\overline{X}(D_4A_1,(\frac{5}{2},\frac{3}{2}))$ is unitary. Note also that $\overline{X}(D_4A_1,(\frac{5}{2},\frac{3}{2}))$ is the IM-dual of $\overline{X}(A_5'',(1,1))$.

In conclusion, the unitary set $\mathcal{U}(D_4{+}A_1)$ is identical with $\mathcal{SU}(\mathbb{H}(B_2,1,3/2)$.

$\mathbf{D_4(a_1)+A_1}$: The infinitesimal character is $(\nu_1,\nu_2,\frac{3}{2},\frac{1}{2})$, the lowest W-types are 9_1 and 2_1, and $\mathfrak{z}=2A_1$. The two lowest W-types are separate when $\nu_1=\nu_2$. If $\nu_1\neq\nu_2$, then the (Langlands classification) standard module is induced from a one dimensional discrete series on B_2 (1⇒2). As a W-module

$$X(D_4(a_1)A_1,(\nu_1,\nu_2))|_W=\mathrm{Ind}_{W(B_2)}^{W(F_4)}(0\times 2),\quad \text{if } \nu_1\neq\nu_2.$$

When $\nu_1=\nu_2=\nu$, there are two standard modules, corresponding to the two lowest W-types, induced from two tempered representations of B_3 (1–1⇒2). As W-modules,

$$\begin{aligned}X(D_4(a_1)A_1,(\nu,\nu),\phi_1)|_W&=\mathrm{Ind}_{W(B_3)}^{W(F_4)}(1\times 2+0\times 12)=9_1+8_1+4_3+6_1\\&\quad+2\cdot 9_2+2\cdot 16_1+2\cdot 8_4+9_4+12_1+4_1+2_4.\\X(D_4(a_1)A_1,(\nu,\nu),\phi_2)|_W&=\mathrm{Ind}_{W(B_3)}^{W(F_4)}(0\times 3)=2_1+8_1+4_3+9_2+1_2.\end{aligned}$$

The restrictions of the nearby W-types are:

Nilpotent	$D_4(a_1)A_1$	$D_4(a_1)A_1$	A_5''	$A_3A_2A_1$	$D_5(a_1)A_1$
W-type	9_1	2_1	8_1	4_3	4_1
Multiplicity	$1+0$	$0+1$	$1+1$	$1+1$	$1+0$
$B_2\subset B_3$	1×2	0×3	$1\times 2, 0\times 3$	$1\times 2, 0\times 3$	0×12
$C_2\subset C_3$	12×0	12×0	$12\times 0, 11\times 1$	$11\times 1, 111\times 0$	0×12
$W(B_2)=W(\mathfrak{z})\rtimes\mathbb{Z}/2$	2×0	11×0	1×1	1×1	11×0

The matching of the intertwining operators is with the Hecke algebra $\mathbb{H}(B_2,0,5/2)$. More specifically,

$9_1:\ 1;$

$2_1:\ 1;$

$$8_1:\ \begin{pmatrix}\frac{\frac{5}{2}-\nu_1}{\frac{5}{2}-\nu_2} & 0\\ 0 & \frac{\frac{5}{2}-\nu_2}{\frac{5}{2}+\nu_2}\end{pmatrix}.$$

For the calculations, we need some intertwining operators on modules in the Hecke algebras of type B_3 and C_3.

In $\mathbb{H}(B_3,1,2)$, the infinitesimal character is $(\nu,2,1)$. There are two lowest W-types 1×2 and 0×3, which are separate at $\nu=0$ only. When $\nu>0$, the operators are

$$1\times 2:1,\quad 0\times 3:1,\quad 0\times 12:\frac{3-\nu}{3+\nu}.$$

In $\mathbb{H}(C_3,2,1)$, the infinitesimal character is $(\nu,\frac{3}{2},\frac{1}{2})$. The lowest W-type is 12×0. The operators are

$$12\times 0:1,\quad 11\times 1:\frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu},\quad 111\times 0:\frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu}.$$

From this calculations, it follows that the reducibility lines for $X(D_4(a_1),A_1,(\nu_1,\nu_2))$, when $\nu_1\neq\nu_2$, are $\nu_1\pm\nu_2=3$, $\nu_i=\frac{5}{2},\frac{7}{2}$, $i=1,2$. The operator on 8_1 shows that the unitary set $\mathcal{U}(D_4(a_1)A_1)$ is included in $0\le\nu_2\le\nu\le\frac{5}{2}$. However, the line $\nu_1+\nu_2=3$ cuts this region. The restrictions in the table above imply that the operator on 4_1 is

$$4_1:\frac{(3-(\nu_1+\nu_2))(3-(\nu_1-\nu_2))}{(3+(\nu_1+\nu_2))(3+(\nu_1-\nu_2))}.$$

So for $\nu_1\neq\nu_2$, $X(D_4(a_1)A_1,(\nu_1,\nu_2))$ is unitary if and only if

$$\{\nu_1+\nu_2<3,\ \nu_1<\frac{5}{2}\}.$$

We also note that $\overline{X}(D_4(a_1)A_1,(\nu,\frac{5}{2}))$, $0\le\nu<\frac{1}{2}$ is the IM-dual of $\overline{X}(D_4A_1,(\nu,\frac{3}{2}))$, and $\overline{X}(D_4(a_1)A_1,(\frac{5}{2},\frac{1}{2}))$ is the IM-dual of $\overline{X}(D_4A_1,(\frac{3}{2},\frac{1}{2}))$. On $\nu_1+\nu_2=3$, we write the parameters as $(\frac{3}{2}+\nu,-\frac{3}{2}+\nu)$ (which are unitary for $0\le\nu\le 1$). The W-structure here is

$$X(D_4(a_1)A_1,(\frac{3}{2}+\nu,-\frac{3}{2}+\nu))=\overline{X}(D_4(a_1)A_1,(\frac{3}{2}+\nu,-\frac{3}{2}+\nu))+X(D_5(a_1)A_1,\nu)$$

$$\overline{X}(D_4(a_1)A_1,(\frac{3}{2}+\nu,-\frac{3}{2}+\nu))|_W=9_1+2_1+2\cdot 9_4+16_1+8_4+2\cdot 8_1 +2\cdot 4_3+12_1+6_1+1_2.$$

It remains to analyze the case $\nu_1=\nu_2=\nu$. The calculation on 8_1 can be used to conclude that both Langlands quotients $\overline{X}(D_4(a_1)A_1,(\nu,\nu))$ fail to be unitary for $\nu>\frac{5}{2}$. Moreover, the calculation for 4_1 implies that $\overline{X}(D_4(a_1)A_1,(\nu,\nu),\phi_1)$ is not unitary for $\nu>\frac{3}{2}$.

For ϕ_1, the first possible reducibility point is $\nu=\frac{3}{2}$ (this is seen by looking at the W-types in $X(D_4(a_1)A_1,(\nu,\nu),\phi_1)$ and their corresponding nilpotent orbits). So it is unitary for $0\le\nu\le\frac{3}{2}$. The decomposition at $\nu=\frac{3}{2}$ is

$$X(D_4(a_1)A_1,(\frac{3}{2},\frac{3}{2}),\phi_1)=\overline{X}(D_4(a_1)A_1,(\frac{3}{2},\frac{3}{2}),\phi_1)+X(D_5(a_1)A_1,0)$$

$$X(D_4(a_1)A_1,(\frac{3}{2},\frac{3}{2}),\phi_1)|_W=9_1+9_2+16_1+8_4+8_1+12_1+6_1+4_3.$$

For ϕ_2, the first possible reducibility point is $\nu=\frac{5}{2}$, so this is unitary if and only if $0\le\nu\le\frac{5}{2}$. Note that $\overline{X}(D_4(a_1)A_1,(\frac{5}{2},\frac{5}{2}),\phi_2)$ is the IM-dual of $X(E_7(a_4),\phi_1)$. As a W-representation, it is just 2_1.

In conclusion, the unitary set $\mathcal{U}(D_4(a_1)A_1)$ is given by:

(1) $\{\nu_1+\nu_2\le 3,\ \nu_1\le \frac{5}{2}\}$, if $\nu_1\neq\nu_2$.
(2) $0\le\nu\le\frac{3}{2}$, if $\nu_1=\nu_2=\nu$ for the ϕ_1-factor.
(3) $0\le\nu\le\frac{5}{2}$, if $\nu_1=\nu_2=\nu$ for the ϕ_2-factor.

$\mathbf{A_3+2A_1}$: The infinitesimal character is $(\nu_1,1+\nu_2,-1+\nu_2,\frac{1}{2})$, the lowest W-type 8_3, and $\mathfrak{z}=2A_1$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on the Hecke algebra of type $A_1+\widetilde{A}_1$. As a W-module,

$$X(A_3+2A_1)|_W=\mathrm{Ind}_{W(A_1+\widetilde{A}_1)}^{W(F_4)}((11)\otimes(11)).$$

The restrictions of nearby W-types are:

Nilpotent	A_3+2A_1	$D_4(a_1)A_1$	$A_3A_2A_1$	A_5''	D_4A_1	$D_5(a_1)A_1$
W-type	8_3	9_1	4_4	8_1	9_3	4_1
Multiplicity	1	1	1	1	2	1
$C_1\times A_1\subset C_3$	1×2	1×2	1×2	11×1	$1\times 2, 0\times 12$	0×12
$B_1\times A_1\subset B_3$	11×1	1×2	11×1	1×2	$1\times 2, 2\times 1$	0×12
$A_1\subset A_2$	(21)	(21)	(1^3)	(21)	$(21),(1^3)$	(21)
$\widetilde{A}_1\subset\widetilde{A}_2$	(21)	(21)	(1^3)	(21)	$(21),(21)$	(21)

We need the following calculations from the Hecke algebras of type B_3 and C_3.

In $\mathbb{H}(C_3,2,1)$, the infinitesimal character is $(1+\nu,-1+\nu,\frac{1}{2})$. The standard module is induced from the Steinberg module on $C_1\times A_1$, and it is reducible at $\nu=\frac{3}{2},\frac{5}{2},\frac{7}{2}$. The relevant operators are:

$$1\times 2:1,\quad 11\times 1:\frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu},\quad 0\times 12:\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}.$$

In $\mathbb{H}(B_3,1,2)$, the infinitesimal character is $(\frac{1}{2}+\nu,-\frac{1}{2}+\nu,2)$. The standard module is induced from the Steinberg module on $B_1\times A_1$, and it is reducible at $\nu=\frac{1}{2},\frac{5}{2},\frac{7}{2}$. The relevant operators are:

$$11\times 1:1,\quad 1\times 2:\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu},\quad 1\times 11:\frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu}.$$

The reducibility lines for $X(A_3+2A_1,(\nu_1,\nu_2))$ are $\nu_1=\frac{1}{2},\frac{5}{2},\frac{7}{2}$, $\nu_2=\frac{3}{2},\frac{5}{2},\frac{7}{2}$, and $\pm\nu_1\pm2\nu_2=\frac{3}{2}$ and $\pm\nu_1\pm\nu_2=3$. (The last two types come from the restrictions of the operators to $A_1\subset A_2$ and $\widetilde{A}_1\subset\widetilde{A}_2$.)

From the restrictions we that the operators are as follows:

$8_3:$ 1;

$9_1:$ $\frac{\frac12-\nu_1}{\frac12+\nu_1}$;

$4_1:$ $\frac{(\frac12-\nu_1)(\frac52-\nu_1)(\frac32-\nu_2)}{(\frac12+\nu_1)(\frac52+\nu_1)(\frac32+\nu_2)}$;

$8_1:$ $\frac{(\frac12-\nu_1)(\frac52-\nu_2)}{(\frac12+\nu_1)(\frac52+\nu_2)}$;

$4_4:$ $\frac{(\frac32-(\nu_1+2\nu_2))(\frac32-(\nu_1-2\nu_2)}{(\frac32+(\nu_1+2\nu_2))(\frac32+(\nu_1-2\nu_2)}$;

$9_3:$ Det$=\frac{(\frac32-\nu_2)(\frac52-\nu_1)(\frac32-(\nu_1+2\nu_2))(\frac32-(\nu_1-2\nu_2)}{(\frac32+\nu_2)(\frac52+\nu_1)(\frac32+(\nu_1+2\nu_2))(\frac32+(\nu_1-2\nu_2)}$.

In this case, it is natural to try to match the unitary set $\mathcal{U}(A_3+2A_1)$ with the spherical unitary dual of A_1+A_1 with parameters 1 and 3. This is just the set $\{0\le\nu_1\le\frac12,\ 0\le\nu_2\le\frac32\}$. The operators above imply that actually the unitary set is

$$\mathcal{U}(A_3{+}2A_1)=\{0\le\nu_1\le\frac12,\nu_1{+}2\nu_2\le\frac32\}\cup\{0\le\nu_1\le\frac12,0\le\nu_2\le\frac32,2\nu_2{-}\nu_1\ge\frac32\}.$$

We also record the relevant decompositions and W-structure of the standard module. On $\nu_2=\frac32$, $0\le\nu_1<\frac12$, the decomposition is

$$X(A_3+2A_1,(\nu,\frac32))=\overline{X}(A_3+2A_1,(\nu,\frac32))+X(D_4+A_1,(\nu,\frac12).$$

Moreover, $\overline{X}(A_3+2A_1,(\nu,\frac32))$ is self IM-dual.

On $\nu_1=\frac12$, $0\le\nu_2<\frac12$ and $1<\nu_2<\frac32$, the decomposition is

$$X(A_3+2A_1,(\frac12,\nu))=\overline{X}(A_3+2A_1,(\frac12,\nu))+X(D_4(a_1)+A_1,(\nu,\nu),\phi_1).$$

The IM-dual of $\overline{X}(A_3+2A_1,(\frac12,\nu))$ is a spherical module.

On $2\nu_2\pm\nu_1=\frac32$, we write the parameter as $(\nu_1,\nu_2)=(-\frac12+2\nu,\frac12+\nu)$. Then it is unitary for $0<\nu<\frac12$. Here, the decomposition is

$$X(A_3+2A_1,(-\frac12+2\nu,\frac12+\nu))=\overline{X}(A_3+2A_1,(-\frac12+2\nu,\frac12+\nu))+X(A_3+A_2+A_1,\nu). \tag{5.3.3}$$

The IM-dual of $\overline{X}(A_3+2A_1,(-\frac12+2\nu,\frac12+\nu))$ is parametrized by $4A_1$.

At $(\frac12,\frac12)$, the decomposition is

$$X(A_3+2A_1,(\frac12,\frac12))=\overline{X}(A_3+2A_1,(\frac12,\frac12))+X(D_4(a_1)A_1,(\frac12,\frac12),\phi_1)+X(A_3A_2A_1,0)$$

$$\overline{X}(A_3+2A_1,(\frac12,\frac12))|_W=8_3+16_1+8_2+12_1+6_2+9_3+9_4+4_5.$$

The similar decomposition (and identical W-structure) holds at $(1,1)$.

Finally, at the point $(\frac{1}{2},\frac{3}{2})$, the decomposition is

$$X(A_3+2A_1,(\frac{1}{2},\frac{3}{2}))=\overline{X}(A_3+2A_1,(\frac{1}{2},\frac{3}{2}))+\overline{X}(D_4(a_1)A_1,(\frac{3}{2},\frac{3}{2}),\phi_1)$$
$$+\overline{X}(D_4A_1,(\frac{1}{2},\frac{1}{2}))+X(D_5(a_1)A_1,0)$$
$$\overline{X}(A_3+2A_1,(\frac{1}{2},\frac{3}{2}))|_W=8_3+16_1+8_2+4_4+12_1+6_1+9_3+9_4.$$

Note that $\overline{X}(A_3+2A_1,(\frac{1}{2},\frac{3}{2}))$ is the IM-dual of $\overline{X}(D_4(a_1)A_1,(\frac{3}{2},\frac{3}{2}),\phi_1)$.

$(\mathbf{A_3+A_1})''$: The infinitesimal character is $(\frac{\nu_1+\nu_2}{2},\frac{\nu_1-\nu_2}{2},1+\frac{\nu_2}{2},-1+\frac{\nu_2}{2})$, the lowest W-type is 4_2, and centralizer B_3. In the Langlands classification, the standard module is induced from the Steinberg representation on $\widetilde{A}_1$, so as a W-module, it is $X((A_3+A_1)'')|_W=\mathrm{Ind}_{W(\widetilde{A}_1)}^{W(F_4)}((11))$.

The restrictions of nearby W-types are:

Nilpotent	$(A_3A_1)''$	A_32A_1	$D4(a_1)A_1$	$D_4(a_1)A_1$	A_5''
W-types	4_2	8_3	9_1	2_1	8_1
$\widetilde{A}_1\subset\widetilde{A}_2$	(21)	$2\cdot(21)$	$3\cdot(21)$	(21)	$3\cdot(21),(1^3)$
A_1	(2)	$(2),(11)$	$2\cdot(2),(11)$	(2)	$3\cdot(2),(11)$
$A_1\subset C_2$	1×1	$2(1\times1)$	$2(1\times1),11\times0$	11×0	$2(1\times1),2(11\times0)$
$W(\mathfrak{z})$	3×0	12×0	2×1	0×3	$3\times0+1\times2$

We need a calculation from the Hecke algebra of type C_2, $2\Leftarrow1$. The infinitesimal character is $(1+\nu,-1+\nu)$, and the standard module is induced from the Steinberg representation on $\widetilde{A}_1$. Then the reducibility points are $\nu=\frac{1}{2},\frac{3}{2}$, and the operators are

$$1\times1:1,\quad 11\times0:\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}.$$

It follows that the reducibility hyperplanes for $X((A_3+A_1)'',(\nu_1,\nu_2))$ are $\nu_i=1,3$ (from $A_1\subset C_2$), $\nu_i\pm\nu_j=1$ (from $\widetilde{A}_1\subset\widetilde{A}_1+A_1$), and $\nu_1\pm\nu_2\pm\nu_3=6$ (from $\widetilde{A}_1\subset\widetilde{A}_2$).

The matching of operators is with the spherical operators for $\mathbb{H}(B_3)$, the Hecke algebra of type B_3 with equal parameters. From the tables, we see that we can match the operators on all relevant $W(B_3)$-types, except 1×2. (The operator on 8_1 fails to match it because of the restriction to $\widetilde{A}_1\subset\widetilde{A}_2$.) The representation 1×2 in B_3 is the only one which rules out the interval $\{(1+\nu,\nu,-1+\nu):1<\nu<2\}$ in the spherical dual. It follows that the unitary set $\mathcal{U}((A_3A_1)'')$ is included in the union $\mathcal{SU}(\mathbb{H}(B_3))\cup\{(1+\nu,\nu,-1+\nu):1<\nu<2\}$. We also note that the extra hyperplanes of reducibility of $X((A_3+A_1)'')$ which do not correspond to B_3, intersect this union only in the point $(3,2,1)$. We analyze the parameters in $\mathcal{U}((A_3A_1)'')$ partitioned by the nilpotent orbits in type $\mathfrak{so}(7)$ (that is, in the same way we parametrize the spherical dual of $\mathbb{H}(B_3)$).

(1^7) : The parameters are (ν_1, ν_2, ν_3); the spherical complementary series for $\mathbb{H}(B_3)$ is

$$\{\nu_1 + \nu_2 < 1\} \cup \{nu_1 + \nu_3 > 1, \nu_2 + \nu_3 < 1, \nu_1 < 1\}. \tag{5.3.4}$$

In these regions, $X((A_3 + A_1)'')$ is irreducible, and it is unitarily induced and unitary for $\nu_3 = 0$, so it is unitary in (5.3.4).

(221^3) : The parameters are $(\frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_1, 2\nu_2)$, unitary for $\{0 \le \nu_1 < \frac{1}{2}, 0 \le \nu_2 < \frac{1}{2}\}$, being endpoints of the complementary series (5.3.4). The decomposition of the standard module in this region is

$$\begin{aligned} X((A_3A_1)'', \frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_1, 2\nu_2)) &= \overline{X}((A_3A_1)'', \frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_1, 2\nu_2)) \\ &+ X(A_3 + 2A_1, (\nu_1, \nu_2)). \end{aligned}$$

The IM-dual of $\overline{X}((A_3A_1)'', \frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_1, 2\nu_2))$ is parametrized by $4A_1$.

(31^4) : The parameters are $(\nu_1 + \nu_2, \nu_1 - \nu_2, 1)$, unitary for $\{\nu_1 < \frac{1}{2}\}$, being endpoints of (5.3.4). The decomposition of the standard module is

$$\begin{aligned} X((A_3A_1)'', (\nu_1 + \nu_2, \nu_1 - \nu_2, 1)) &= \overline{X}((A_3A_1)'', (\nu_1 + \nu_2, \nu_1 - \nu_2, 1)) \\ &+ X(D_4(a_1)A_1, (\nu_1, \nu_2)). \end{aligned}$$

The IM-dual of $\overline{X}((A_3A_1)'', (\nu_1 + \nu_2, \nu_1 - \nu_2, 1))$ is a spherical module.

(322) : The parameters are $(\frac{1}{2} + \nu, -\frac{1}{2} + \nu, 1)$, unitary for $0 \le \nu < \frac{1}{2}$ (again endpoints of (5.3.4). The standard module decomposes as

$$\begin{aligned} X((A_3A_1)'', (\frac{1}{2} + \nu, -\frac{1}{2} + \nu, 1)) &= \overline{X}((A_3A_1)'', (\frac{1}{2} + \nu, -\frac{1}{2} + \nu, 1)) \\ &+ X(A_3 + 2A_1, (\nu, \frac{1}{2}) + X(D_4(a_1)A_1, (\nu, \frac{1}{2})) \end{aligned}$$

Moreover, $\overline{X}((A_3A_1)'', (\frac{1}{2} + \nu, -\frac{1}{2} + \nu, 1))$ is self IM-dual.

(331) : The parameters are $(1 + \nu, \nu, -1 + \nu)$. At $\nu = 0$, we have a unitary module, endpoint of (5.3.4). (Note also that $\overline{X}((A_3A_1)'', (1, 1, 0))$ is the IM-dual of $\overline{X}(A_3 + 2A_1, (\frac{1}{2}, \frac{1}{2}))$.) In $\mathbb{H}(B_3)$, this is the only such parameter which is unitary. In F_4 however, by the observations preceding this analysis, we need to also consider the segment $1 < \nu < 2$. In this interval, the decomposition is

$$\begin{aligned} X((A_3 + A_1)'', (1 + \nu, \nu, -1 + \nu)) &= \overline{X}((A_3 + A_1)'', (1 + \nu, \nu, -1 + \nu)) \\ &+ 2 \cdot \overline{X}(A_3 + 2A_1, (-\frac{1}{2} + \nu, \frac{1}{2} + \frac{\nu}{2})) + X(A_3A_2A_1, \frac{\nu}{2}) \\ \overline{X}((A_3 + A_1)'', (1 + \nu, \nu, -1 + \nu))|_W &= 4_2 + 9_1 + 2_1 + 6_2 + 2 \cdot 8_1 + 4_3 \\ &+ 2 \cdot 9_2 + 12_1 + 16_1 + 8_4 + 1_2. \end{aligned}$$

This segment is isolated, so to prove that it is unitary, we compute explicitly the signatures on all the W-types which appear in the restriction of $\overline{X}((A_3 + A_1)'', (1 + \nu, \nu, -1 + \nu))|_W$. The IM-dual of $\overline{X}((A_3 + A_1)'', (1 + \nu, \nu, -1 + \nu))$ is a module parametrized by $A_2 + 3A_1$, and our calculations will be confirmed by the unitarity of those duals.

(511) : The parameters are $(\nu, 2, 1)$, which are not unitary for $\nu > 0$. At $\nu = 0$, $\overline{X}((A_3A_1)'', (2,1,0))$ is the IM-dual of $\overline{X}(A_3 + 2A_1, (\frac{1}{2}, 1))$, and therefore it is unitary.

(7) : The parameter is $(3, 2, 1)$. Then $\overline{X}((A_3+A_1)'', (3,2,1))$ is the IM-dual of $X(E_7(a_5), \phi_2)$, and therefore it is unitary.

In conclusion, $\mathcal{U}((A_3 + A_1)'') = \mathcal{SU}(\mathbb{H}(B_3)) \sqcup \{(1+\nu, \nu, -1+\nu) : 1 < \nu < 2\}$.

$\mathbf{A_2 + 3A_1}$: The infinitesimal character is $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu_1(2,1,1,0) + \nu_2(1,1,0,0)$, the lowest W-type is 1_3, and $\mathfrak{z} = G_2$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on A_2, and as a W-module, $X(A_2 + 3A_1) = \mathrm{Ind}_{W(A_2)}^{W(F_4)}((111))$.

The restrictions of nearby W-types are:

Nilpotent	A_23A_1	A_32A_1	$A_3A_2A_1$	D_4A_1	A_5A_1
W-type	1_3	8_3	4_4	9_3	6_1
Multiplicity	1	1	2	3	1
$A_2 \subset B_3$	111×0	11×1	111×0 11×1	$111 \times 0, 1 \times 11$ 11×1	11×1
$A_2 \subset A_2 + \widetilde{A}_1$	(2)	(2)	$(2), (11)$	$2 \cdot (2), (11)$	(11)
$W(\mathfrak{z})$	$(1,0)$	$(1,3)'$	$(2,1)$	$(1,0) + (2,2)$	$(1,6)$

We need a calculation in the Hecke algebra $\mathbb{H}(B_3, 1, 2)$. The infinitesimal character is $(\nu+1, \nu, -1+\nu)$, and the standard module is induced from the Steinberg representation on A_2. The reducibility points are $1, 2, 3$, and the relevant operators are

$$111 \times 0 : 1, \quad 11 \times 1 : \frac{1-\nu}{1+\nu}, \quad 1 \times 11 : \frac{(1-\nu)(2-\nu)}{(1+\nu)(2+\nu)}.$$

Then the reducibility lines for $X(A_2 + 3A_1, (\nu_1, \nu_2))$ are $\nu_1, \nu_1 + \nu_2, 2\nu_1 + \nu_2 = 1, 2, 3$, and $\nu_2, 3\nu_1 + \nu_2, 3\nu_1 + 2\nu_2 = 2$. The matching of operators is with $\mathbb{H}(G_2, 1, 2)$, the Hecke algebra of type G_2: 1<≡2. From the tables, it follows that we are able to match only the operators on $(1,0)$, $(1,3)'$, and $(2,1)$ in G_2. We will also use the operator on 9_3 (which is the closets one to match $(2,2)$). The spherical unitary dual of $\mathbb{H}(G_2, 1, 2)$ consists of the closed region $\{2\nu_1 + \nu_2 \leq 1\}$, and the isolated points $(\frac{1}{2}, \frac{1}{2})$ and $(1, 2)$. The matched operators are sufficient to conclude that, when irreducible, $X(A_2 + 3A_1, (\nu_1, \nu_2))$ is unitary only in the region $2\nu_1 + \nu_2 < 1$, but on the lines we need more information.

On the line $\nu_1 = -1 + \nu, \nu_2 = 2$, the decomposition is

$$X(A_2 + 3A_1, (-1+\nu, 2)) = \overline{X}(A_2 + 3A_1, (-1+\nu, 2)) + X(A_3 + A_2 + A_1, \nu)$$

$$\overline{X}(A_2 + 3A_1, (-1+\nu, 2))|_W = 1_3 + 8_3 + 4_4 + 6_2 + 16_1 + 2 \cdot 8_2 + 2 \cdot 9_3 + 12_1 + 9_4 + 2_2 + 4_5.$$

To simplify notation, set $X_1(\nu) = X(A_2+3A_1,(-1+\nu,2))$. The matched operators give

$$8_3 : -\frac{(\frac{1}{2}-\nu)(2-\nu)}{(\frac{1}{2}+\nu)(2+\nu)}, \quad 4_4 : \frac{2-\nu}{2+\nu},$$

showing that $X_1(\nu)$ can only be unitary for $\frac{1}{2} \le \nu \le 2$. The operator on 9_3 has two nonzero eigenvalues with product $-\frac{(\frac{1}{2}-\nu)(1-\nu)(2-\nu)^2(3-\nu)}{(\frac{1}{2}+\nu)(1+\nu)(2+\nu)^2(3+\nu)}$, so the only segment which can be unitary is $\frac{1}{2} \le \nu \le 1$. At $\nu = \frac{1}{2}$, $X_1(\nu)$ decomposes further:

$$X_1(\frac{1}{2}) = \overline{X}(A_2+3A_1,(\frac{1}{2},\frac{1}{2})) + \overline{X}(A_3+2A_1,(\frac{1}{2},\frac{1}{2}))$$
$$\overline{X}(A_2+3A_1,(\frac{1}{2},\frac{1}{2}))|_W = 1_3+4_4+8_2+9_3+2_2.$$

Since $X(A_2+3A_1,(\frac{1}{2},\frac{1}{2}))$ is the IM-dual of $X(D_4(a_1)A_1,\phi_2)$ (which is tempered), it must be unitary. Also the other factor is unitary, because $\overline{X}(A_3+2A_1,(\frac{1}{2},\frac{1}{2}))$ is. Since 9_2 appears in both factors, and the operator on it is positive for $\frac{1}{2} < \nu < 1$, it follows that $X_1(\nu)$ is unitary for $\frac{1}{2} < \nu < 1$. Note also that $X_1(\nu)$ is the IM-dual of $\overline{X}((A_3+A_1)'',1+2\nu,2\nu,-1+2\nu)$, which confirms its unitarity.

On the line $\nu_1 = 1, \nu_2 = -\frac{3}{2}+\nu$, the decomposition is

$$X(A_2+3A_1,(1,-\frac{3}{2}+\nu)) = \overline{X}(A_2+3A_1,(1,-\frac{3}{2}+\nu)) + \overline{X}(A_3+2A_1,(\frac{1}{2},\nu))$$
$$\overline{X}(A_2+3A_1,(1,-\frac{3}{2}+\nu))|_W = 1_3+4_4+8_2+9_3+2_2. \tag{5.3.5}$$

Denote $X_2(\nu) = \overline{X}(A_2+3A_1,(1,-\frac{3}{2}+\nu))$. The intertwining operators calculations give us

$$4_4 : \frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu}, \quad 9_3 : \frac{(\frac{5}{2}-\nu)(\frac{7}{2}-\nu)}{(\frac{5}{2}+\nu)(\frac{7}{2}+\nu)}.$$

So the only interval on which $X_2(\nu)$ can be unitary is $0 \le \nu \le \frac{5}{2}$. But on this interval, $X_2(\nu)$ is the IM-dual of $X(D_4(a_1)A_1,(\nu,\nu),\phi_2)$, so it is in fact unitary. Note that at the endpoint, $\nu = \frac{5}{2}$, $\overline{X}(A_2+3A_1,(1,1))$ is the IM-dual of the discrete series $X(E_7(a_4),\phi_2)$.

Finally, at the point $(1,2)$, $\overline{X}(A_2+3A_1,(1,2))$ is the IM-dual of the discrete series $X(E_7(a_3),\phi_2)$.

In conclusion, the unitary set $\mathcal{U}(A_2+3A_1)$ is the one pictured in the figure 1. It is strictly larger than the spherical unitary dual $\mathcal{SU}(\mathbb{H}(G_2,1,2))$.

$\mathbf{4A_1}$: The parameters are $(\nu_1,\nu_2,\nu_3,\frac{1}{2})$, the lowest W-type is 2_3, and $\mathfrak{z} = C_3$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on A_1, and as a W-representation, $X(4A_1) = \mathrm{Ind}_{W(A_1)}^{W(F_4)}((11))$.

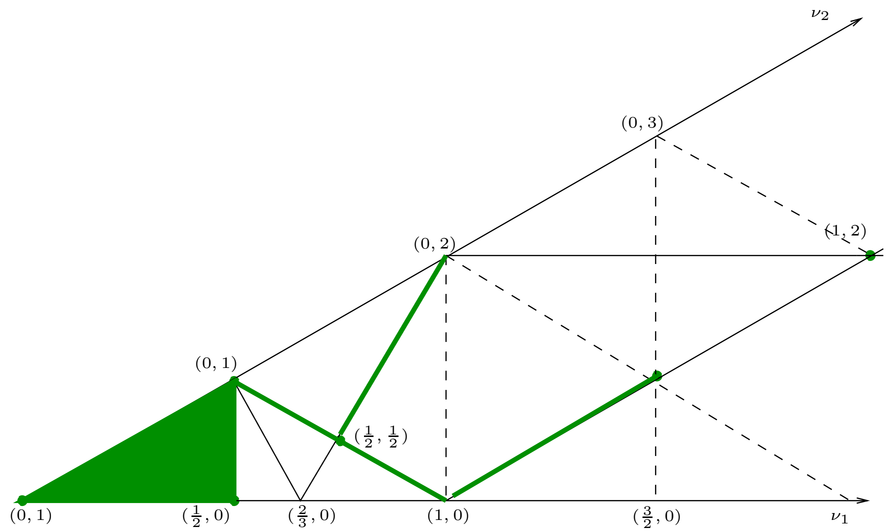

FIGURE 1. Unitary parameters and reducibility lines for A_2+3A_1

The restrictions of the nearby W-types are:

Nilpotent	$4A_1$	$(A_3A_1)''$	A_23A_1	A_32A_1	$D_4(a_1)A_1$	$A_3A_2A_1$
W-type	2_3	4_2	1_3	8_3	9_1	4_4
Multiplicity	1	1	1	4	3	3
$C_1\subset C_2$	0×2	1×1	0×2	$2(0\times 2)$	0×2	$2(0\times 2)$
				$2(1\times 1)$	$2(1\times 1)$	1×1
$A_1\subset A_2$	(21)	(21)	(1^3)	$3(21),(1^3)$	$3(21)$	$(21),2(1^3)$
$\widetilde{A}_1$						
$W(\mathfrak{z})$	3×0	0×3	3×0	$0\times 3+2\times 1$	1×2	2×1

We need one calculation in the Hecke algebra $\mathbb{H}(C_2,2,1)$ of type C_2, $2\Leftarrow 1$. The standard module is induced from the Steinberg representation on C_1, and it has infinitesimal character $(\nu,\frac{1}{2})$. The operators are

$$0\times 2:1,\quad 1\times 1:\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu},\quad 0\times 11:\frac{(\frac{3}{2}-\nu)(\frac{5}{2}-\nu)}{(\frac{3}{2}+\nu)(\frac{5}{2}+\nu)}.$$

The matching of intertwining operators is with the spherical operators for $\mathbb{H}(C_3,2,3)$, the Hecke algebra of type C_3, 2–2$\Leftarrow$3. The only W-types which give matchings are

2_3 with 3×0;
4_2 with 0×3;
9_1 with 1×2;
8_1 with 0×12.

The last one is irrelevant for the calculation. To get an inclusion of $\mathcal{U}(4A_1)$ into $\mathcal{SU}(\mathbb{H}(C_3,2,3))$, we would have needed to find matchings for the $W(C_3)$-types 2×1 and 12×0 as well. The $W(F_4)$-type 8_3 fails to match 2×1 because of its restriction to $A_1 \subset A_2$, but we will have to use it in the calculation nevertheless. We will also use the operators on 1_3 and 4_1.

The hyperplanes of reducibility for $X(4A_1),(\nu_1,\nu_2,\nu_3))$ are: $\nu_i \pm \nu_j = 2$, $\nu_i = \frac{3}{2}$ (as in $\mathbb{H}(C_3, 2, 3)$), but also $\nu_i = \frac{5}{2}$, and $\nu_1 \pm \nu_2 \pm \nu_3 = \frac{3}{2}$. Especially the second extra family of hyperplanes of reducibility will affect the unitarity in an essential way.

The Hecke algebra $\mathbb{H}(C_3, 2, 3)$ arises geometrically from a cuspidal local system on $SL(3)^3$ in $Spin(13)$; its dual is parametrized by a subset of the nilpotent orbits in type B_6.. We organize our analysis of the unitary set $\mathcal{U}(4A_1)$ by infinitesimal characters corresponding to these orbits.

(2^61) : In $\mathbb{H}(C_3, 2, 3)$, these are parameters (ν_1,ν_2,ν_3) in the complement of the hyperplanes $\nu_i \pm \nu_j = 2$, $\nu_i = \frac{3}{2}$. In $\mathbb{H}(C_3, 2, 3)$ the unitary regions are:

(i) $\{\nu_1 < \frac{3}{2}, \nu_1 + \nu_2 < 2\}$.

(ii) $\{\nu_1 < \frac{3}{2}, \nu_1 + \nu_3 > 2, \nu_2 + \nu_3\}$.

But the operators on the matched $W(C_3)$-types 3×0, 0×3, 1×2 (equivalently, on the $W(F_4)$-operators 2_3, 4_2, respectively 9_1) are also positive in the region

(iii) $\{\nu_3 < \frac{3}{2}, \nu_2 > \frac{3}{2}, \nu_1 + \nu_3 > 2, \nu_1 - \nu_3 < 2, \nu_2 + \nu_3 < 2\}$.

In F_4, $X(4A_1)$ becomes reducible also on the hyperplanes $\nu_1 \pm \nu_2 \pm \nu_3 = \frac{3}{2}$, some of which cut the regions (i)–(iii). More specifically, region (i) is cut by the hyperplanes $\nu_1 + \nu_2 + \nu_3 = \frac{3}{2}$ and $\nu_1 + \nu_2 - \nu_3 = \frac{3}{2}$, and region (ii) is cut by $\nu_1 + \nu_2 - \nu_3 = \frac{3}{2}$. We use the operators on 1_2 and 8_4 and these are indefinite in all the resuting (sub)regions except

$$\{\nu_1 + \nu_2 + \nu_3 \leq \frac{3}{2}\} \cup \{\nu_1 + \nu_2 - \nu_3 \geq \frac{3}{2}, \nu_1 < \frac{3}{2}, \nu_1 + \nu_2 < 2\}. \tag{5.3.6}$$

In these two remaining regions, one can deform ν_3 to 0, and $X(4A_1)$ stays irreducible, and for $\nu_3 = 0$, it is unitarily induced and unitary. So the parameters in (5.3.6) are unitary. On the hyperplanes $\nu_1 + \nu_2 \pm \nu_3 = \frac{3}{2}$, in (5.3.6), the decomposition is

$$X(4A_1) = \overline{X}(4A_1) + X(A_2 + 3A_1),$$

and the factor $\overline{X}(4A_1)$ is self IM-dual.

(52^4) : The parameter is $(\nu_1, \nu_2, \frac{3}{2})$. In $\mathbb{H}(C_3, 2, 3)$, the corresponding parameters are unitary for $\{\nu_1 < \frac{1}{2}\} \cup \{\nu_2 > \frac{1}{2}, \nu_1 + \nu < 2\}$. However, out of the matched operators, the only nontrivial one is on 9_1, and it is $\frac{(\frac{7}{2}-\nu_1)(\frac{7}{2}-\nu_2)}{(\frac{7}{2}+\nu_1)(\frac{7}{2}+\nu_2)}$. We also need the operators on 1_3 and 4_1:

$$1_3 : \begin{cases} \frac{(3-(\nu_1-\nu_2))(3-(\nu_1+\nu_2))}{(3+(\nu_1-\nu_2))(3+(\nu_1+\nu_2))}, & \nu_1 \neq \nu_2 \\ 0, & \nu_1 = \nu_2 \end{cases}, \quad 4_1 : \frac{(\frac{5}{2}-\nu_1)(\frac{5}{2}-\nu_2)(\frac{7}{2}-\nu_1)(\frac{7}{2}-\nu_2)}{(\frac{5}{2}+\nu_1)(\frac{5}{2}+\nu_2)(\frac{7}{2}+\nu_1)(\frac{7}{2}+\nu_2)}.$$

From these calculations, it follows that for $\nu_1 \neq \nu_2$, the only unitary parameters are in the region $\{\nu_1 + \nu_2 < 3, \nu_1 < \frac{5}{2}\}$ (and its closure). In this region, the decomposition of the standard module is

$$X(4A_1,(\nu_1,\nu_2,\frac{3}{2}))=\overline{X}(4A_1,(\nu_1,\nu_2,\frac{3}{2}))+\overline{X}((A_3A_1)'',(\nu_1+\nu_2,\nu_1-\nu_2,1))$$
$$+X(D_4(a_1)A_1,(\nu_1,\nu_2)).$$

Moreover, $\overline{X}(4A_1,(\nu_1,\nu_2,\frac{3}{2}))$ is the IM-dual of $X(D_4(a_1)+A_1)$, so it is indeed unitary.

When $\nu_1=\nu_2=\nu$, the decomposition is

$$X(4A_1,(\nu,\nu,\frac{3}{2}))=\overline{X}(4A_1,(\nu,\nu,\frac{3}{2}))+\overline{X}((A_3A_1)'',(2\nu,1,0))$$
$$\overline{X}(A_2+3A_1,(1,-\frac{3}{2}+\nu))+X(D_4(a_1)A_1,(\nu,\nu)).$$

The factor $\overline{X}(4A_1,(\nu,\nu,\frac{3}{2}))$ is the IM-dual of $X(D_4(a_1)A_1,(\nu,\nu),phi_1)$, so it is unitary if and only if $0\le\nu\le\frac{3}{2}$.

(44221) : The parameters are $(\nu_1,1+\nu_2,-1+\nu_2)$. In $\mathbb{H}(C_3,2,3)$, they are unitary in the region $\{\nu_1<\frac{3}{2},\nu_2<\frac{1}{2}\}$. In F_4, the unitary set will be different. Generically, here the standard module decomposes as

$$X(4A_1,(\nu_1,1+\nu_2,-1+\nu_2))=\overline{X}(4A_1,(\nu_1,1+\nu_2,-1+\nu_2))+X(A_3+2A_1,(\nu_1,\nu_2)),$$

and $\overline{X}(4A_1)$ is the IM-dual of $\overline{X}((A_3+A_1)'',(\frac{1}{2}+\nu_1,-\frac{1}{2}+\nu_1,2\nu_2))$, which is unitary for $\{0\le\nu_1<\frac{1}{2},0\le\nu_2<\frac{1}{2}\}$.

Similar calculations as in the previous case ((52^4)) show that the parameters $(\nu_1,1+\nu_2,-1+\nu_2)$ can be unitary only for $\{0\le\nu_1<\frac{1}{2},0\le\nu_2<\frac{1}{2}\}$ (by the remark in the previous paragraph, $\overline{X}(4A_1,(\nu_1,1+\nu_2,-1+\nu_2))$ has to be unitary in this region), and on the segment $\nu_1+2\nu_2=\frac{3}{2}$, for $\frac{1}{2}\le\nu_1\le\frac{3}{2}$. For this, we use, in addition to the matched W-types, the operators on $1_3,4_1,8_3$.

It remains to analyze the segment $\nu_1+2\nu_2=\frac{3}{2}$. We rewrite the parameters as $\nu_1=\frac{1}{2}+2\nu,\nu_2=\frac{1}{2}-\nu$. Then the factor $\overline{X}(4A_1,(\frac{1}{2}+2\nu,\frac{1}{2}+\nu,-\frac{3}{2}+\nu))$ is the IM-dual of $\overline{X}(A_3+2A_1,(-\frac{1}{2}+\nu,\frac{1}{2}+\nu))$, which is unitary for $0\le\nu<\frac{1}{2}$, which is precisely the segment we were looking at. Therefore, this segment is also unitary for $\overline{X}(4A_1)$.

(53221) : The infinitesimal character is $(\nu,\frac{3}{2},\frac{1}{2})$. In $\mathbb{H}(C_3,2,3)$, the corresponding spherical module is unitary for $0\le\nu<\frac{3}{2}$.

In F_4, the only nonzero matched operator is on 9_1, and it is $\frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu}$. The operator on 1_3 is $\frac{(\frac{5}{2}-\nu)(\frac{7}{2}-\nu)}{(\frac{5}{2}+\nu)(\frac{7}{2}+\nu)}$. So it remains to check the segment $0\le\nu\le\frac{5}{2}$. For $0\le\nu\le\frac{5}{2}$, $\overline{X}(4A_1,(\nu,\frac{3}{2},\frac{1}{2}))$ is the IM-dual of $X(D_4(a_1)A_1,(\nu,\frac{1}{2}))$, so it is indeed unitary.

(544) : The infinitesimal character is $(1+\nu,-1+\nu,\frac{3}{2})$. In $\mathbb{H}(C_3,2,3)$, the corresponding spherical module is unitary for $0\le\nu<\frac{3}{2}$. In F_4, the matched operator on 9_4 gives $\frac{(\frac{5}{2}-\nu)(\frac{9}{2}-\nu)}{(\frac{5}{2}+\nu)(\frac{9}{2}+\nu)}$. The operator on 1_2 is $\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}$, which implies the only unitary parameters can be in $0\le\nu\le\frac{3}{2}$. For $0\le\nu<\frac{3}{2}$,

$\overline{X}(4A_1,(1+\nu,-1+\nu,\frac{3}{2}))$ is the IM-dual of $X(D_4(a_1)A_1,(1+\nu,-1+\nu))$, so it is unitary.

(661) : The infinitesimal character is $(2+\nu,\nu,-2+\nu)$. In $\mathbb{H}(C_3,2,3)$, the corresponding spherical module is unitary for $0\le\nu<\frac{1}{2}$. In F_4, the matched operators give $4_5:\frac{(\frac{1}{2}+\nu)(\frac{3}{2}-\nu)(\frac{7}{2}-\nu)}{(\frac{1}{2}-\nu)(\frac{3}{2}+\nu)(\frac{7}{2}+\nu)}$, respectively $9_4:\frac{(\frac{3}{2}-\nu)(\frac{7}{2}-\nu)}{(\frac{3}{2}+\nu)(\frac{7}{2}+\nu)}$. For $0\le\nu<\frac{1}{2}$, the factor $\overline{X}(4A_1,(2+\nu,\nu,-2+\nu))$ is the IM-dual of $X((A_5)'',(1,-\frac{1}{2}+\nu))$, therefore it is unitary.

(751) : The infinitesimal character is $(\frac{5}{2},\frac{3}{2},\frac{1}{2})$, which is unitary in $\mathbb{H}(C_3,2,3)$. In F_4, $\overline{X}(4A_1,(\frac{5}{2},\frac{3}{2},\frac{1}{2}))$ is unitary as well, being the IM-dual of $\overline{X}(D_6(a_1),\frac{1}{2})$.

(922) : The infinitesimal character is $(\nu,\frac{7}{2},\frac{3}{2})$, which in $\mathbb{H}(C_3,2,3)$ is unitary for $0\le\nu<\frac{1}{2}$. In F_4, the matched operators are 0, but the operators on 1_2 and 8_4 are $\frac{(\frac{1}{2}+\nu)(\frac{13}{2}-\nu)}{(\frac{1}{2}-\nu)(\frac{13}{2}+\nu)}$, respectively $\frac{\frac{13}{2}-\nu}{\frac{13}{2}+\nu}$.

For $0\le\nu<\frac{1}{2}$, the factor $\overline{X}(4A_1,(\nu,\frac{7}{2},\frac{3}{2}))$ is the IM-dual of $X(D_6(a_1),\nu)$, and therefore unitary. But also the point $\nu=\frac{13}{2}$ is unitary, since $\overline{X}(4A_1,(\frac{13}{2},\frac{7}{2},\frac{3}{2}))$ is the IM-dual of $X(E_7(a_1))$. (This point has no correspondent in $\mathbb{H}(C_3,2,3)$.)

(931) : The infinitesimal character is $(\frac{7}{2},\frac{3}{2},\frac{1}{2})$, which is unitary in $\mathbb{H}(C_3,2,3)$. It is also unitary in F_4, as $\overline{X}(4A_1,(\frac{7}{2},\frac{3}{2},\frac{1}{2}))$ is the IM-dual of $\overline{X}(D_6(a_1),\frac{1}{2})$.

(13) : The infinitesimal character is $(\frac{11}{2},\frac{7}{2},\frac{3}{2})$. In $\mathbb{H}(C_3,2,3)$ the corresponding spherical module is the trivial representation. But $\overline{X}(4A_1,(\frac{11}{2},\frac{7}{2},\frac{3}{2}))$ is not unitary as seen from the operators for (922) at $\nu=\frac{11}{2}$.

5.4. Spherical modules. The spherical modules are parameterized by the nilpotent orbit $(3A_1)$". If a spherical module does not contain the sign W-representation, via IM its unitarity was already determined in the previous section. We record those results next.

Table 3: Spherical unitary modules for $\mathbb{H}(F_4,(1,2))$

Nilpotent	**Central character**	**Unitary**
E_7	$(\frac{17}{2},\frac{9}{2},\frac{5}{2},\frac{1}{2})$	
$E_7(a_2)$	$(\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$	
D_6	$(\nu,\frac{9}{2},\frac{5}{2},\frac{1}{2})$	$\{0\le\nu\le\frac{1}{2}\}$
D_5+A_1	$(2+\nu,-2+\nu,\frac{5}{2},\frac{1}{2})$	$\{0\le\nu\le\frac{1}{2}\}$
$E_7(a_5)$	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	
$D_6(a_2)$	$(\nu,\frac{5}{2},\frac{3}{2},\frac{1}{2})$	$\{0\le\nu<\frac{1}{2}\}$
A_5+A_1	$(\frac{1}{4},\frac{7}{4},-\frac{1}{4},-\frac{9}{4})+\nu(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\{0\le\nu<\frac{1}{2}\}$
(A_5)"	$(\nu_2+\frac{3\nu_1}{2}+2+\frac{\nu_1}{2},\frac{\nu_1}{2},-2+\frac{\nu_1}{2})$	$\{3\nu_1+2\nu_2<1\}$ $\{2\nu_1+\nu_2<1<3\nu_1+\nu_2\}$
D_4+A_1	$(\nu_1,\nu_2,\frac{5}{2},\frac{1}{2})$ $(\frac{1}{2}+\nu,-\frac{1}{2}+\nu,\frac{5}{2},\frac{1}{2})$	$\{\nu_1+\nu_2<1\}$ $\{\nu_1-\nu_2>1,\nu_1<\frac{3}{2}\}$ $\{0\le\nu<\frac{3}{2}\}$
$A_3A_2A_1$	$(\frac{1}{2},\frac{1}{2},-\frac{3}{2},\frac{1}{2})+\nu(2,1,1,0)$	$\{0\le\nu<1\}$
A_3+2A_1	$(\nu_1,1+\nu_2,-1+\nu_2,\frac{1}{2})$	$\{0\le\nu_1<\frac{1}{2},\nu_1+2\nu_2<\frac{3}{2}\}$ $\{0\le\nu_1<\frac{1}{2},0\le\nu_2<\frac{3}{2},2\nu_2-\nu_1>\frac{3}{2}\}$

Table 3 – continued from previous page

	$(1+\nu,-1+\nu,\frac{1}{2},\frac{1}{2})$	$\{0\le\nu_1<\frac{1}{2}\}\cup\{1<\nu<\frac{3}{2}\}$
(A_3+A_1)"	$(\frac{\nu_1+\nu_2}{2},\frac{\nu_1-\nu_2}{2},1+\frac{\nu_3}{2},-1+\frac{\nu_3}{2})$	$\{0\le\nu_3\le\nu_2\le\nu_1<1-\nu_2\}$ $\{0\le\nu_3\le\nu_2<1-\nu_3<\nu_1<1\}$
	$(\nu_1,\nu_2,\frac{3}{2},\frac{1}{2})$	$\{0\le\nu_2\le\nu_1<\frac{1}{2}\}$
A_2+3A_1	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})+\nu_1(2,1,1,0)$ $+\nu_2(1,1,0,0)$	$\{2\nu_1+\nu_2<1\}$
$4A_1$	$(\nu_1,\nu_2,\nu_3,\frac{1}{2})$	$\{\nu_1+\nu_2+\nu_3<\frac{3}{2}\}$ $\{\nu_1+\nu_2-\nu_3>\frac{3}{2},\nu_1<\frac{3}{2},\nu_1+\nu_2<2\}$

It remains to determine the unitarity of the irreducible spherical principal series.

5.5. The 0-complementary series. In this section we determine the unitary irreducible spherical series $X(\nu)$. The parameter $\nu=(\nu_1,\nu_2,\nu_3,\nu_4)$ is assumed in the dominant Weyl chamber $\mathcal{C}$. This is partitioned by the hyperplanes

$$\langle\alpha_l,\nu\rangle=1,\ \alpha_l \text{ long root and } \langle\alpha_s,\nu\rangle=c,\alpha_s \text{ short root.} \tag{5.5.1}$$

We assume first that $c>1$ is arbitrary. The principal series $X(\nu)$ is reducible precisely when ν is on one of the hyperplanes in (5.5.1). If $\mathcal{F}$ is an open connected component of the complement of (5.5.1) in $\mathcal{C}$ (we call $\mathcal{F}$ a *region*), then all the intertwining operators $r_\sigma(\nu)$, $\sigma\in\widehat{W}$, are invertible, and therefore have constant signature in $\mathcal{F}$. We say that the region $\mathcal{F}$ is unitary if $X(\nu)$ is unitary for all (equivalently, any) $\nu\in\mathcal{F}$. The walls of any region $\mathcal{F}$ are of the form (5.5.1), or of the form $\langle\alpha,\nu\rangle=0$, for $\alpha\in\Pi$.

Proposition. *Consider the half-space $K_s=\{\nu:\langle\epsilon_1+\epsilon_2,\nu\rangle<c\}$. The unitary regions $\mathcal{F}$ in $\mathcal{C}\cap K$ are:*

($\mathcal{F}_1$) $\{2\nu_1<1,\ \nu_1+\nu_2<c\}$;
($\mathcal{F}_2$) $\{\nu_1+\nu_2+\nu_3+\nu_4>1,\ \nu_1+\nu_2+\nu_3-\nu_4<1,\ \nu_1+\nu_2<c\}$;
($\mathcal{F}_3$) $\{\nu_1+\nu_2-\nu_3+\nu_4>1,\ \nu_1-\nu_2+\nu_3+\nu_4<1,\ \nu_1+\nu_2-\nu_3-\nu_4<1,\ \nu_1+\nu_2<c\}$;
($\mathcal{F}_4$) $\{\nu_1-\nu_2+\nu_3+\nu_4<1,\ 2\nu_2>1,\ \nu_1+\nu_2<c\}$;
($\mathcal{F}_5$) $\{2\nu_2>1,\ \nu_1-\nu_2+\nu_3-\nu_4>1,\ 2\nu_3<1,\ \nu_1-\nu_2-\nu_3+\nu_4<1,\ \nu_1+\nu_2<c\}$;
($\mathcal{F}_6$) $\{2\nu_2>1,\ \nu_1-\nu_2-\nu_3-\nu_4>1,\ 2\nu_3<1,\ \nu_1+\nu_2<c\}$;
($\mathcal{F}_7$) $\{\nu_1-\nu_2-\nu_3-\nu_4>1,\ 2\nu_4>1,\ \nu_1+\nu_2<c\}$.

Proof. The proof is a case by case analysis, which we sketch here.

Note that $\epsilon_1+\epsilon_2$ is the highest short root of F_4. The condition that $\mathcal{F}$ be in the half-space K means that the walls of $\mathcal{F}$ can only be of the form $\langle\alpha_l,\nu\rangle=1$, $\langle\alpha,\nu\rangle=0$ (or $\nu_1+\nu_2=c$). From the partial order relation among the (long) roots, we see that there are 19 such regions. A case by case analysis gives that each of these regions has a wall of the form $\langle\alpha,\nu\rangle=0$.

If $\mathcal{F}$ has a wall of the form $\langle \alpha, \nu \rangle = 0$, then on this wall, $X(\nu)$ is unitarily induced irreducibly from a principal series $X_M(\nu')$ for a Hecke subalgebra H_M, with M Levi of type B_3 or C_3. In [BC1], theorem 3.4 and 3.6, the unitary irreducible spherical principal series for the Hecke algebras of type B_n/C_n with arbitrary unequal parameters are determined, so in particular we know the unitarity of $X_M(\nu')$, and therefore of $X(\nu)$.

□

In [BC1], one was able to prove by hand that, in type B_n, if the highest short root is greater than c, then no region $\mathcal{F}$ can be unitary. This is false in F_4, as seen in the following example.

Example. *The region* $\mathcal{F}_8 = \{\nu_1 - \nu_2 - \nu_3 - \nu_4 > 1,\ 2\nu_4 > 1,\ \nu_1 + \nu_3 > c,\ \nu_1 + \nu_4 < c\}$ *is unitary.*

Proof. The region $\mathcal{F}_8$ has a wall given by $\langle \epsilon_2 - \epsilon_3, \nu \rangle = 0$. We deform the parameter ν to this wall, *i.e.* $\nu_2 = \nu_3$. The corresponding module $X(\nu)$ is unitarily induced irreducible from the principal series $X_{B_3}(\nu')$, where $\nu' = (\nu_1', \nu_2', \nu_3')$ satisfy $\nu_1' - \nu_2' > 1$, $\nu_2' - \nu_3' > 1$, and $\nu_1' < c$. These parameters ν' are unitary in B_3, *cf.* [BC1], theorem 3.6. □

In order to show that the regions $\mathcal{F}_1 - \mathcal{F}_8$ are the *only* unitary regions, we determined by brute force computer calculations the signatures of $r_\sigma(\nu)$, with $\sigma \in \{1_1, 4_2, 9_1\}$, and $\nu \in \mathcal{F}$, for $\mathcal{F}$ any region in $\mathcal{C}$.

Corollary. *In the case of the geometric Hecke algebra* $\mathbb{H}(F_4, (1, 2))$*, the* 0-*complementary series is given by the regions* $\mathcal{F}_1 - \mathcal{F}_5$ *from proposition 5.5.*

(When $c = 2$, the regions $\mathcal{F}_6 - \mathcal{F}_8$, in the notation as before, become empty.)

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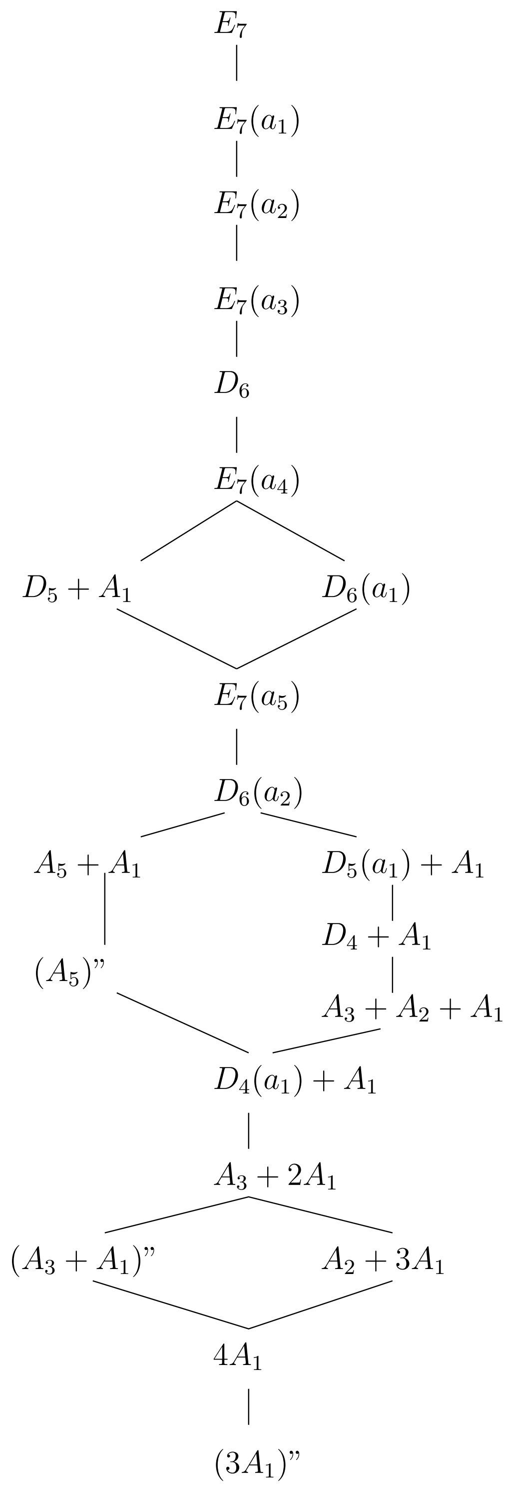

FIGURE 2. Nilpotent orbits parametrizing $(3A_1)'' \subset E_7$