

WHITTAKER LIMITS OF DIFFERENCE SPHERICAL FUNCTIONS

IVAN CHEREDNIK [†]

CONTENTS

0. Introduction	3
0.1. Results and applications	3
0.2. Growth estimates	4
0.3. Our approach	5
0.4. Difference spherical functions	6
0.5. The setting of the paper	7
1. Double Hecke algebra	8
1.1. Affine Weyl group	9
1.2. The length on $\widehat{\mathbf{W}}$	11
1.3. Reduction modulo $\mathbf{W}$	11
1.4. More notations	13
1.5. Main definition	14
2. Polynomial representation	15
2.1. Macdonald polynomials	16
2.2. Symmetric polynomials	17
2.3. Using intertwiners	18
2.4. Spherical polynomials	20
2.5. The limit $\mathbf{t} \rightarrow \mathbf{0}$	21
3. Spherical and Whittaker functions	22
3.1. Gauss-type integrals	23
3.2. Global spherical function	24
3.3. Global Whittaker function	25
3.4. One-dimensional theory	28
3.5. Whittaker function for $\mathbf{A}_1$	30
3.6. The case $ \mathbf{q} > 1$	31
4. Harmonic analysis topics	32

[†] August 12, 2008 Partially supported by NSF grant DMS-0800642.

4.1.	Growth estimates	33
4.2.	Exact asymptotic formulas	35
4.3.	When $\mathbf{k} \rightarrow \infty$	38
4.4.	Jackson integrals	41
4.5.	The special case $\boldsymbol{\xi} = -\boldsymbol{\rho}_{\mathbf{k}}$	42
4.6.	Taking the limit	43
	References	47

.

0. INTRODUCTION

The main aim of this paper is to introduce *global* q -Whittaker functions as the limit $t \rightarrow 0$ of the (renormalized) generalized symmetric spherical functions constructed in [C5] for arbitrary reduced root systems (see [Sto] in the $C^\vee C$ -case). This work is inspired by [GLO1] and [GLO2], though our approach is different. For instance, we obtain a q -version of the classical Shintani-Casselman-Shalika formula [Shi, CS] via the q -Mehta-Macdonald integral in the Jackson setting. The Shintani-type formulas (in the case of GL_n) play an important role in [GLO1, GLO2], but the q -Gauss integrals are not considered there as well as globally-defined q -Whittaker functions. We use these formulas to obtain a q, t -generalization of the Harish-Chandra asymptotic formula for the classical spherical function.

0.1. Results and applications. The key observation is that the definition of the symmetric q, t -spherical functions from [C5] is compatible with taking the Whittaker limit and results in globally-defined q -Whittaker functions. The definition from [C5] is based on the q -Mehta-Macdonald integrals calculated there for the constant term functional, i.e., in the setting of Laurent series. In this paper, we mainly treat the spherical functions as *global* ones, analytic or meromorphic.

The q -Whittaker functions are solutions of the q -Toda eigenvalue problem and are expected to have important applications in mathematics and physics, including the Langlands program. Concerning the latter and relations to the affine flag varieties, see, for instance, [GiL, BF, Ion2]. The q -Shintani-Casselman-Shalika formula gives a (relatively simple) example of the Langlands correspondence. The affine Toda lattice provides another link; it is (presumably) dual to the q -Toda lattice in the sense of [KL]).

It is important to note that the coefficients of the expansion of our Whittaker function are essentially polynomials in terms of q with *positive integral* coefficients. It can be verified using the intertwining operators or via the relation to the Demazure characters. This fact is of obvious importance for the “categorization” of the q -Whittaker function and its geometric applications. The “ q -integrality” has no known counterpart in the general q, t -theory (with a reservation concerning the *stable* GL -case); one of the parameters, q or t , has to be eliminated or expressed in terms of the remaining one. However, the q, t -spherical

functions are of more fundamental nature and more convenient to deal with in many other aspects.

The cornerstone of their theory is the *duality* based on the DAHA-Fourier transform; see [C8] and [C9]. It is missing in the theory of q -Whittaker functions and for many other degenerations, including the Harish-Chandra theory ($q \rightarrow 1$), the p -adic limit ($q \rightarrow \infty$) and q -Hermite polynomials ($t \rightarrow 0$). A specific problem with the Whittaker limiting procedure among other degenerations, is that it destroys the W -invariance; it gives another reason for treating Whittaker functions as limits of the spherical functions rather than for creating their intrinsic theory. On the other hand, the Whittaker functions satisfy quite a few identities that cannot be obtained from the q, t -theory. These identities, the q -integrality of the coefficients and various applications obviously make the q -Whittaker functions an important independent direction, requiring developing specific methods.

At the end of the paper, we outline the approach to the global spherical and Whittaker functions via the harmonic analysis. Our formulas for these functions are actually equivalent to certain fundamental properties of the corresponding integral transforms in the space of Laurent polynomials multiplied by the Gaussian. The latter space is the simplest and the most natural choice here, but the same functions can serve other algebraic and analytic situations. This direction seems very promising. For instance, the existence of the q -Whittaker limit of the global spherical function appears a boundary case of the *growth estimates* for the q, t -spherical function with respect to x together with k (a new feature of the q, t -theory).

0.2. Growth estimates. Provided that $\Re(x), \Re(\lambda)$ are inside the positive Weyl chamber $\mathfrak{C}_+$ (the exact choice of $\mathfrak{C}$ is not important but the walls must be avoided), the global spherical function for $0 < q < 1$ approaches asymptotically in the limit of large $\Re(x)$

$$|W| \text{ CT } \Theta(\rho_k) \frac{\Theta(x + \lambda - \rho_k)}{\Theta(x)\Theta(\lambda)} \prod_{\alpha \in R_+} \frac{\Gamma_q(\lambda_\alpha^\vee)}{\Gamma_q(\lambda_\alpha^\vee + k_\alpha)}$$

for the theta-series Θ and q -Gamma function associated with a given root system R ; $t = q^k$, $\rho_k = k\rho$ in the simply-laced case, CT is the constant term of the celebrated Macdonald function. No inequalities for (fixed) k are necessary but (rational negative) k where the polynomial representation of DAHA becomes non-semisimple must be excluded.

Up to a periodic function, the x -dependence of this function is $q^{(x, \rho_k - \lambda)}$, so this theorem is an *exact* q, t -analog of the Harish-Chandra formula [HC] describing the asymptotic behavior of the classical spherical function in terms of the c -function (the case of complex Lie groups). In the Whittaker limit, ρ_k is removed and $\Re(x)$ must be taken from $-\mathfrak{C}_+$ (the Whittaker function it is not W -invariant with respect to x).

It is one of the major results of this paper, which seems a beginning of fruitful analytic q -theory.

0.3. Our approach. It is different from that of [GLO1, GLO2] (and we deal with arbitrary reduced root systems). The technique of the Gaussians is the key to introduce the *global* q -Whittaker function and prove the Shintani-type formulas. The q -Whittaker function is mainly treated in [GLO1, GLO2] as a *discrete* function on the weight lattice for GL_n satisfying the q -Toda system of *difference* equations.

The space of all solutions is, generally, $|W|$ -dimensional over the field of periodic functions, playing the role of constants in the difference theory; upon the restriction to the weight lattice it is $|W|$ -dimensional over $\mathbb{C}$. Choosing the “right” Whittaker function in this space requires certain growth conditions; using the W -symmetric dependence on the spectral parameters gives another approach. There is no intrinsic definition of the q -Whittaker function so far, but our formula and the growth conditions we establish clarify what can be expected. First, only positive powers appear in its Laurent series expansion (after dropping the Gaussians). Second, our x -asymptotic formula for the q -Whittaker function inside the negative Weyl chamber is sufficient to fix it uniquely.

We note that in the *differential* setting, the spherical and Whittaker functions can be uniquely determined from the eigenvalue problem (subject to the W -invariance for the spherical function and certain growth conditions in the Whittaker case). It simplifies the starting definitions. However the difference theory is more universal and, remarkably, has important algebraic and analytic advantages. The self-duality of the DAHA-Fourier transform and the technique of the Gaussians are the key; these are special features of the q, t -setting and are mainly absent in the trigonometric-differential and p -adic cases. In this respect, the q, t -theory is somewhat similar to the rational-differential theory of (multi-variable) Bessel functions.

0.4. Difference spherical functions. The *global* nonsymmetric and symmetric q, t -spherical functions were defined in [C5] and then in [Sto] (the $C^\vee C$ -case) as the reproducing kernels of the Fourier transform of the standard polynomial representation twisted by the Gaussian. In this approach, the spherical function is determined uniquely (the Macdonald eigenvalue problem fixes it only up to periodic factors). Using the Gaussians, among other things, provides the global convergence. These construction appeared compatible with the Whittaker limit.

The Gaussians play the key role in our approach to the Shintani-Casselman-Shalika formula. In the q, t -setting, it becomes the Mehta-Macdonald formula in the Jackson case from [C5], where a special vector, $-\rho_k$, is taken as the origin of the Jackson summation.

Developing this direction, we conclude the paper with the Jackson-Gauss integrals for the *global* spherical and Whittaker functions; such formulas were given only for Macdonald polynomials in [C5]. These formulas seem an important step toward systematic difference harmonic analysis, although the case of the real integration is still beyond the existing theory. Now, with the q, t -Harish-Chandra asymptotic formulas from this paper, it seems that there are no obstacles for developing the real integration theory generalizing the classical “non-compact” case.

Conceptually, as it was observed in [GLO2], the q -variant of the Shintani-Casselman-Shalika formula is nothing but the duality formula for the Macdonald polynomials from [C3] considered upon the limit $t \rightarrow 0$. However, establishing exact relations is, generally, a subtle problem. The Shintani-type formulas play the major role in the paper, including the growth estimates.

This interpretation gives evidence that the DAHA-Fourier transform is connected with the (local quantum) geometric Langlands correspondence. Generally, the DAHA-localization functor, which includes the modular transformation $q \mapsto q'$, is expected to play its role in the quantum geometric Langlands correspondence; the DAHA-Fourier transform is likely to be one of its ingredients.

We note that DAHA leads to a theory that is a priori more general than the one needed for the (local) quantum Langlands correspondence because it contains an extra parameter t . However, there is growing evidence that the general q, t -DAHA appear in the Langlands program. It makes important the exact relations between the q, t -spherical functions and q -Whittaker ones (which are already a part of the Langlands

program). We expect this paper to trigger interesting new developments.

It is worth mentioning that the approach to spherical functions via the Fourier transform significantly depends on the choice of the corresponding representation of the double affine Hecke algebra. Technically, the choice of this space influences only the normalization; spherical functions are defined up to periodic factors. However, the analytic properties of the q, t -spherical function, exact factors in the Shintani-type formulas and many other similar features do depend on the considered representation (equivalently, the choice of the normalization).

For instance, if the Gaussian is interpreted as a theta-function, then the corresponding spherical function is *meromorphic* but not analytic. Treating the Gaussian as $q^{x^2/2}$ (not as a Laurent series), i.e., using a somewhat different analytic setting, leads to the q, t -spherical functions analytic everywhere, but not single-valued in terms of q^x . If the Gaussians are omitted in this definition, i.e., the DAHA-Fourier transform acts from the polynomial representation to the space of delta-functions, then the corresponding spherical function will become a generalized function. The problem of its convergence and meromorphic continuation (upon a proper renormalization) is, generally, much more involved than for the *global functions*.

0.5. The setting of the paper. Only the symmetric theory will be considered in this work; the (truly) nonsymmetric q -Whittaker function can be defined as certain limits of the *nonsymmetric* global spherical function, but the construction becomes more involved and will be a subject of the next work(s). Nevertheless, we begin the paper with the account of the nonsymmetric Macdonald polynomials including their (straight) degeneration as $t \rightarrow 0$, which is closely related to the Demazure characters of irreducible affine Lie algebras; see [San, Ion1]. We mainly need the formulas in terms of the intertwining operators to justify some of our claims and estimates; the intertwiners can be naturally defined only in the nonsymmetric theory.

We mention that the Macdonald symmetric polynomials considered under the limit $t \rightarrow 0$ generalize the classical q -Hermite polynomials, so the main result of the paper is in establishing the formula for the q -Whittaker function in terms of *multi-variable q -Hermite polynomials*. Presumably, when the theory of nonsymmetric Whittaker limit is completed (it is beyond this paper), the *nonsymmetric q -Whittaker*

function will be a *generating function for all Demazure characters*, not only the ones for anti-dominant weights. Such interpretation can clarify the role of Whittaker functions in the Kac-Moody theory and may have connections to [GiL] and to questions and conjectures from [BF].

There are two possible setups in the DAHA theory for the non-simply-laced root systems, which correspond to two possible choices of the affine extension. In this paper, we introduce the affine root system using $\alpha_0 = [-\vartheta, 1]$ in terms of the maximal *short* root ϑ . The conjugation by the Gaussian and the Fourier transform preserve the double affine Hecke algebra in this setup. By the way, it is exactly the case where a relation to the Demazure characters can be established according to [Ion1], Theorem 1.

The case of the “standard” affine root system with $\alpha_0 = [-\theta, 1]$ for the maximal *long* root θ is analogous, although the Fourier transform acts from the double affine Hecke algebra to its dual in the B, C -cases. This is the setting that is expected to be related to the geometric Langlands correspondence. Technically, the switch to the “standard” DAHA can be achieved by changing only one formula in this paper, namely, that for the action of T_0 in the polynomial representation. This change influences the relations of T_0 with the X -operators (indexed by the weights). The Y -operators become labeled by the coweights for such choice of T_0 ; they are labeled by the weights in this paper.

In this paper, the difference between these two DAHA-theories is not crucial. Generally, quite a few formulas become more “symmetric” when the standard affine extension is considered (cf. [C9]), but the main constructions are, generally, more transparent in our setting.

Acknowledgements. The author is thankful to D. Kazhdan for alerting me to the works of Gerasimov et. al and for our various conversations on the Whittaker functions and the Langlands correspondence. I indebted to D. Gaitsgory for the discussion of the quantum geometric Langlands duality. Special thanks go to A. Gerasimov for his explanations of the results of [GLO1, GLO2], which influenced this paper a great deal.

1. DOUBLE HECKE ALGEBRA

Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type $A, B, \dots, F, G$ with respect to a euclidean form (z, z') on $\mathbb{R}^n \ni z, z'$, W the *Weyl group* generated by the reflections s_α , R_+ the set of positive roots ($R_- = -R_+$)

corresponding to fixed simple roots $\alpha_1, \dots, \alpha_n$, Γ the Dynkin diagram with $\{\alpha_i, 1 \leq i \leq n\}$ as the vertices. Respectively,

$$R^\vee = \{\alpha^\vee = 2\alpha/(\alpha, \alpha)\}.$$

The root lattice and the weight lattice are:

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for the simple coroots $\alpha_i^\vee$. Replacing $\mathbb{Z}$ by $\mathbb{Z}_\pm = \{m \in \mathbb{Z}, \pm m \geq 0\}$ we obtain $Q_\pm, P_\pm$. Here and further see [B].

The form will be normalized by the condition $(\alpha, \alpha) = 2$ for the *short* roots in this paper. Thus,

$$\nu_\alpha \stackrel{\text{def}}{=} (\alpha, \alpha)/2 \text{ can be either } 1, \text{ or } \{1, 2\}, \text{ or } \{1, 3\}.$$

This normalization leads to the inclusions $Q \subset Q^\vee, P \subset P^\vee$, where $P^\vee$ is defined to be generated by the fundamental coweights $\{\omega_i^\vee\}$ dual to $\{\alpha_i\}$.

We set $\nu_i = \nu_{\alpha_i}$, $\nu_R = \{\nu_\alpha, \alpha \in R\}$ and

$$(1.1) \quad \rho_\nu \stackrel{\text{def}}{=} (1/2) \sum_{\nu_\alpha = \nu} \alpha = \sum_{\nu_i = \nu} \omega_i, \text{ where } \alpha \in R_+, \nu \in \nu_R.$$

Note that $(\rho_\nu, \alpha_i^\vee) = 1$ as $\nu_i = \nu$.

1.1. Affine Weyl group. The vectors $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R, j \in \mathbb{Z}$ form the *affine root system* $\tilde{R} \supset R$ ($z \in \mathbb{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \stackrel{\text{def}}{=} [-\vartheta, 1]$ to the simple roots for the *maximal short root* $\vartheta \in R_+$. It is also the *maximal positive coroot* because of the choice of normalization.

The corresponding set $\tilde{R}_+$ of positive roots equals $R_+ \cup \{[\alpha, \nu_\alpha j], \alpha \in R, j > 0\}$. Indeed, any positive affine root $[\alpha, \nu_\alpha j]$ is a linear combinations with non-negative integral coefficients of $\{\alpha_i, 0 \leq i \leq n\}$.

We complete the Dynkin diagram Γ of R by α_0 (by $-\vartheta$, to be more exact); it is called *affine Dynkin diagram* $\tilde{\Gamma}$. One can obtain it from the completed Dynkin diagram from [B] for the *dual system* $R^\vee$ by reversing all arrows.

The set of the indices of the images of α_0 by all the automorphisms of $\tilde{\Gamma}$ will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O' = \{r \in O, r \neq 0\}$. The elements ω_r for $r \in O'$ are the so-called minuscule weights: $(\omega_r, \alpha^\vee) \leq 1$ for $\alpha \in R_+$.

Given $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}$, $b \in P$, let

$$(1.2) \quad s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee) \tilde{\alpha}, \quad b'(\tilde{z}) = [z, \zeta - (z, b)]$$

for $\tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1}$.

The *affine Weyl group* $\widetilde{W}$ is generated by all $s_{\tilde{\alpha}}$ (we write $\widetilde{W} = \langle s_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{R}_+ \rangle$). One can take the simple reflections $s_i = s_{\alpha_i}$ ($0 \leq i \leq n$) as its generators and introduce the corresponding notion of the length. This group is the semidirect product $W \ltimes Q'$ of its subgroups $W = \langle s_\alpha, \alpha \in R_+ \rangle$ and $Q' = \{a', a \in Q\}$, where

$$(1.3) \quad \alpha' = s_\alpha s_{[\alpha, \nu_\alpha]} = s_{[-\alpha, \nu_\alpha]} s_\alpha \quad \text{for } \alpha \in R.$$

The *extended Weyl group* $\widehat{W}$ generated by W and P' (instead of Q') is isomorphic to $W \ltimes P'$:

$$(1.4) \quad (wb')([z, \zeta]) = [w(z), \zeta - (z, b)] \quad \text{for } w \in W, b \in B.$$

From now on, b and b' , P and P' will be identified.

Given $b \in P_+$, let w_0^b be the longest element in the subgroup $W_0^b \subset W$ of the elements preserving b . This subgroup is generated by simple reflections. We set

$$(1.5) \quad u_b = w_0 w_0^b \in W, \quad \pi_b = b(u_b)^{-1} \in \widehat{W}, \quad u_i = u_{\omega_i}, \pi_i = \pi_{\omega_i},$$

where w_0 is the longest element in W , $1 \leq i \leq n$.

The elements $\pi_r \stackrel{\text{def}}{=} \pi_{\omega_r}$, $r \in O'$ and $\pi_0 = \text{id}$ leave $\tilde{\Gamma}$ invariant and form a group denoted by Π , which is isomorphic to P/Q by the natural projection $\{\omega_r \mapsto \pi_r\}$. As to $\{u_r\}$, they preserve the set $\{-\vartheta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (u_r)^{-1}(-\vartheta)$ distinguish the indices $r \in O'$. Moreover,

$$(1.6) \quad \widehat{W} = \Pi \ltimes \widetilde{W}, \quad \text{where } \pi_r s_i \pi_r^{-1} = s_j \text{ if } \pi_r(\alpha_i) = \alpha_j, \quad 0 \leq j \leq n.$$

We will need the following *affine action* of $\widehat{W}$ on $z \in \mathbb{R}^n$:

$$(1.7) \quad (wb)((z)) = w(b + z), \quad w \in W, b \in P, \\ s_{\tilde{\alpha}}((z)) = z - ((z, \alpha^\vee) + j)\alpha, \quad \tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}.$$

For instance, $(bw)((0)) = b$ for any $w \in W$. The relation to the above action is given in terms of the *affine pairing* $([z, l], z' + d) \stackrel{\text{def}}{=} (z, z') + l$:

$$(1.8) \quad (\widehat{w}([z, l]), \widehat{w}([z', l]) + d) = ([z, l], z' + d) \quad \text{for } \widehat{w} \in \widehat{W},$$

where we treat d formally.

1.2. The length on $\widehat{W}$. Setting $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$, $\pi_r \in \Pi$, $\widetilde{w} \in \widetilde{W}$, the length $l(\widehat{w})$ is by definition the length of the reduced decomposition $\widetilde{w} = s_{i_1} \dots s_{i_2} s_{i_1}$ in terms of the simple reflections s_i , $0 \leq i \leq n$. The number of s_i in this decomposition such that $\nu_i = \nu$ is denoted by $l_\nu(\widehat{w})$.

The *length* can be also defined as the cardinality $|\lambda(\widehat{w})|$ of the λ -set of $\widehat{w}$:

$$(1.9) \quad \lambda(\widehat{w}) \stackrel{\text{def}}{=} \widetilde{R}_+ \cap \widehat{w}^{-1}(\widetilde{R}_-) = \{\tilde{\alpha} \in \widetilde{R}_+, \widehat{w}(\tilde{\alpha}) \in \widetilde{R}_-\}, \quad \widehat{w} \in \widehat{W}.$$

Respectively,

$$(1.10) \quad \lambda(\widehat{w}) = \cup_\nu \lambda_\nu(\widehat{w}), \quad \lambda_\nu(\widehat{w}) \stackrel{\text{def}}{=} \{\tilde{\alpha} \in \lambda(\widehat{w}), \nu(\tilde{\alpha}) = \nu\}.$$

The coincidence with the previous definition is based on the equivalence of the *length equality*

$$(1.11) \quad (a) \quad l_\nu(\widehat{w}\widehat{u}) = l_\nu(\widehat{w}) + l_\nu(\widehat{u}) \quad \text{for } \widehat{w}, \widehat{u} \in \widehat{W}$$

and the *cocycle relation*

$$(1.12) \quad (b) \quad \lambda_\nu(\widehat{w}\widehat{u}) = \lambda_\nu(\widehat{u}) \cup \widehat{u}^{-1}(\lambda_\nu(\widehat{w})),$$

which, in its turn, is equivalent to the *positivity condition*

$$(1.13) \quad (c) \quad \widehat{u}^{-1}(\lambda_\nu(\widehat{w})) \subset \widetilde{R}_+$$

and is also equivalent to the *embedding condition*

$$(1.14) \quad (d) \quad \lambda_\nu(\widehat{u}) \subset \lambda_\nu(\widehat{w}).$$

See, e.g., [C4, C8] and also [B, Hu]. Applying (1.12) to the reduced decomposition $\widehat{w} = \pi_r s_{i_1} \dots s_{i_2} s_{i_1}$,

$$(1.15) \quad \begin{aligned} \lambda(\widehat{w}) = \{ & \tilde{\alpha}^l = \widetilde{w}^{-1} s_{i_l}(\alpha_{i_l}), \dots, \tilde{\alpha}^3 = s_{i_1} s_{i_2}(\alpha_{i_3}), \\ & \tilde{\alpha}^2 = s_{i_1}(\alpha_{i_2}), \tilde{\alpha}^1 = \alpha_{i_1} \}. \end{aligned}$$

1.3. Reduction modulo W . It generalizes the construction of the elements π_b for $b \in P_+$; see [C4] or [C8].

Proposition 1.1. *Given $b \in P$, there exists a unique decomposition $b = \pi_b u_b$, $u_b \in W$ satisfying one of the following equivalent conditions:*

- (i) $l(\pi_b) + l(u_b) = l(b)$ and $l(u_b)$ is the greatest possible,
- (ii) $\lambda(\pi_b) \cap R = \emptyset$.

The latter condition implies that $l(\pi_b) + l(w) = l(\pi_b w)$ for any $w \in W$. Besides, the relation $u_b(b) \stackrel{\text{def}}{=} b_- \in P_- = -P_+$ holds, which, in its turn, determines u_b uniquely if one of the following equivalent conditions is imposed:

- (iii) $l(u_b)$ is the smallest possible,
- (iv) if $\alpha \in \lambda(u_b)$ then $(\alpha, b) \neq 0$.

□

Condition (ii) readily gives a complete description of the set $\pi_P = \{\pi_b, b \in P\}$, namely, only $[\alpha < 0, \nu_\alpha j > 0]$ can appear in $\lambda(\pi_b)$.

Explicitly,

$$(1.16) \quad \lambda(b) = \{\tilde{\alpha} > 0, (b, \alpha^\vee) > j \geq 0 \text{ if } \alpha \in R_+, \\ (b, \alpha^\vee) \geq j > 0 \text{ if } \alpha \in R_-\},$$

$$(1.17) \quad \lambda(\pi_b) = \{\tilde{\alpha} > 0, \alpha \in R_-, (b_-, \alpha^\vee) > j > 0 \text{ if } u_b^{-1}(\alpha) \in R_+, \\ (b_-, \alpha^\vee) \geq j > 0 \text{ if } u_b^{-1}(\alpha) \in R_-\},$$

For instance, $l(b) = l(b_-) = -2(\rho^\vee, b_-)$ for $2\rho^\vee = \sum_{\alpha > 0} \alpha^\vee$.

The element $b_- = u_b(b)$ is a unique element from P_- that belongs to the orbit $W(b)$. Thus the equality $c_- = b_-$ means that b, c belong to the same orbit. We will also use $b_+ \stackrel{\text{def}}{=} w_0(b_-)$, a unique element in $W(b) \cap P_+$. In terms of π_b ,

$$u_b \pi_b = b_-, \pi_b u_b = b_+.$$

Note that $l(\pi_b w) = l(\pi_b) + l(w)$ for all $b \in P, w \in W$. For instance,

$$(1.18) \quad l(b_- w) = l(b_-) + l(w), \quad l(w b_+) = l(b_+) + l(w), \\ l(u_b \pi_b w) = l(u_b) + l(\pi_b) + l(w) \text{ for } b \in P, w \in W.$$

Partial ordering on P . It is necessary in the theory of nonsymmetric polynomials. See [Op, M3]. This ordering was also used in [C2] in the process of calculating the coefficients of Y -operators. The definition is as follows:

$$(1.19) \quad b \leq c, c \geq b \text{ for } b, c \in P \text{ if } c - b \in Q_+,$$

$$(1.20) \quad b \preceq c, c \succeq b \text{ if } b_- < c_- \text{ or } \{b_- = c_- \text{ and } b \leq c\}.$$

Recall that $b_- = c_-$ means that b, c belong to the same W -orbit. We write $<, >, \prec, \succ$ respectively if $b \neq c$.

The following sets

$$(1.21) \quad \begin{aligned} \sigma(b) &\stackrel{\text{def}}{=} \{c \in P, c \succeq b\}, \quad \sigma_*(b) \stackrel{\text{def}}{=} \{c \in P, c \succ b\}, \\ \sigma_-(b) &\stackrel{\text{def}}{=} \sigma(b_-), \quad \sigma_+(b) \stackrel{\text{def}}{=} \sigma_*(b_+) = \{c \in P, c_- > b_-\}. \end{aligned}$$

are convex. By *convex*, we mean that if $c, d = c + r\alpha \in \sigma$ for $\alpha \in R_+, r \in \mathbb{Z}_+$, then

$$(1.22) \quad \{c, c + \alpha, \dots, c + (r-1)\alpha, d\} \subset \sigma.$$

1.4. More notations. By m , we denote the least natural number such that $(P, P) = (1/m)\mathbb{Z}$. Thus $m = 2$ for D_{2k} , $m = 1$ for B_{2k} and C_k , otherwise $m = |\Pi|$.

We will need to include the case $t = 0$ in our definition, which requires minor deviations from the definitions of [C8], [C4] and other author's papers. Namely, we multiply all T_i there by $t_i^{1/2}$ and change the formulas correspondingly.

The double affine Hecke algebra depends on the parameters $q, t_\nu, \nu \in \{\nu_\alpha\}$. It will be defined over the ring

$$\mathbb{Q}_{q,t} \stackrel{\text{def}}{=} \mathbb{Q}[q^{\pm 1/m}, t_\nu]$$

formed by polynomials in terms of $q^{\pm 1/m}$ and $\{t_\nu\}$. We set

$$(1.23) \quad \begin{aligned} t_{\tilde{\alpha}} &= t_\alpha = t_{\nu_\alpha}, \quad t_i = t_{\alpha_i}, \quad q_{\tilde{\alpha}} = q^{\nu_\alpha}, \quad q_i = q^{\nu_{\alpha_i}}, \\ \text{where } \tilde{\alpha} &= [\alpha, \nu_{\alpha j}] \in \tilde{R}, \quad 0 \leq i \leq n. \end{aligned}$$

It will be convenient to use the parameters $\{k_\nu\}$ together with $\{t_\nu\}$, setting

$$t_\alpha = t_\nu = q_\alpha^{k_\nu} \quad \text{for } \nu = \nu_\alpha, \quad \text{and } \rho_k = (1/2) \sum_{\alpha > 0} k_\alpha \alpha.$$

Note that $(\rho_k, \alpha_i^\vee) = k_i = k_{\alpha_i} = ((\rho_k)^\vee, \alpha_i)$ for $i > 0$; $(\rho_k)^\vee \stackrel{\text{def}}{=} \sum k_\nu (\rho_\nu)^\vee$. Using that $w_0(\rho_k) = -\rho_k$, we obtain that $(\rho_k, -w_0(b)) = (\rho_k, b)$. For instance, $(\rho_k, b_+) = -(\rho_k, b_-)$, where $b_+ \stackrel{\text{def}}{=} w_0(b_-)$ (see above).

By $q^{(\rho_k, \alpha)}$, we mean $\prod_{\nu \in \nu_R} t_\nu^{((\rho_\nu)^\vee, \alpha)}$; here $\alpha \in R$, $(\rho_\nu)^\vee = \rho_\nu / \nu$, and this product contains only *integral* powers of t_{sht} and t_{lng} .

For pairwise commutative $X_1, \dots, X_n$,

$$(1.24) \quad X_{\tilde{b}} = \prod_{i=1}^n X_i^{l_i} q^j \text{ if } \tilde{b} = [b, j], \quad \widehat{w}(X_{\tilde{b}}) = X_{\widehat{w}(\tilde{b})}.$$

where $b = \sum_{i=1}^n l_i \omega_i \in P$, $j \in \frac{1}{m}\mathbb{Z}$, $\widehat{w} \in \widehat{W}$.

For instance, $X_0 \stackrel{\text{def}}{=} X_{\alpha_0} = qX_{\vartheta}^{-1}$.

We set $(\tilde{b}, \tilde{c}) = (b, c)$ ignoring the affine extensions in this pairing.

1.5. Main definition. We note that π_r^{-1} is π_{r^*} and u_r^{-1} is u_{r^*} for $r^* \in O$, $u_r = \pi_r^{-1}\omega_r$. The reflection $*$ is induced by an involution of the nonaffine Dynkin diagram Γ .

Definition 1.2. *The double affine Hecke algebra $\mathcal{H}$ is generated over $\mathbb{Q}_{q,t}$ by the elements $\{T_i, 0 \leq i \leq n\}$, pairwise commutative $\{X_b, b \in P\}$ satisfying (1.24), and the group Π , where the following relations are imposed:*

- (o) $(T_i - t_i)(T_i + 1) = 0$, $0 \leq i \leq n$;
- (i) $T_i T_j T_i \dots = T_j T_i T_j \dots$, m_{ij} factors on each side;
- (ii) $\pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j$;
- (iii) $T_i X_b = X_b X_{\alpha_i}^{-1} \{t_i T_i^{-1}\}$ if $(b, \alpha_i^\vee) = 1$, $0 \leq i \leq n$;
- (iv) $T_i X_b = X_b T_i$ if $(b, \alpha_i^\vee) = 0$ for $0 \leq i \leq n$;
- (v) $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{r^*}, b)}$, $r \in O'$.

Here and further the brackets $\{\cdot\}$ will be used to show explicitly the elements that belong to $\mathcal{H}$. One can rewrite (iii, iv) as in [L]:

$$(1.25) \quad T_i X_b - X_{s_i(b)} T_i = (t_i - 1) \frac{X_{s_i(b)} - X_b}{X_{\alpha_i} - 1}, \quad 0 \leq i \leq n.$$

Given $\tilde{w} \in \widetilde{W}$, $r \in O$, the product

$$(1.26) \quad T_{\pi_r \tilde{w}} \stackrel{\text{def}}{=} \pi_r \prod_{k=1}^l T_{i_k}, \quad \text{where } \tilde{w} = \prod_{k=1}^l s_{i_k}, l = l(\tilde{w}),$$

does not depend on the choice of the reduced decomposition (because $\{T\}$ satisfy the same “braid” relations as $\{s\}$ do). Moreover,

$$(1.27) \quad T_{\widehat{v}} T_{\widehat{w}} = T_{\widehat{v}\widehat{w}} \text{ whenever } l(\widehat{v}\widehat{w}) = l(\widehat{v}) + l(\widehat{w}) \text{ for } \widehat{v}, \widehat{w} \in \widehat{W}.$$

In particular, we arrive at the pairwise commutative elements:

$$(1.28) \quad Y_b = q^{(b_+ - b, \rho_k)} \prod_{i=1}^n Y_i^{l_i} \quad \text{if } b = \sum_{i=1}^n l_i \omega_i \in P, \quad Y_i \stackrel{\text{def}}{=} T_{\omega_i}, b \in P.$$

The factors here are needed to make them from $\mathcal{H}$; b_+ is a unique element in $W(b) \cap P_+$. Note that $Y_b Y_{-b} = q^{-2(b_+, \rho_k)}$.

Generally, if we replace s_i by T_i or T_i^{-1} in any reduced decomposition of $\widehat{w} \in \widehat{W}$, then such product belongs to $\mathcal{H}$ upon the multiplication by the product of t_i corresponding to the terms T_i^{-1} .

The relations dual to (iii,iv) hold (for $i > 0$ only):

$$(1.29) \quad \begin{aligned} \{t_i T_i^{-1}\} Y_b &= Y_{s_i(b)} T_i \quad \text{if } (b, \alpha_i^\vee) = 1, \\ T_i Y_b &= Y_b T_i \quad \text{if } (b, \alpha_i^\vee) = 0, \quad 1 \leq i \leq n. \end{aligned}$$

The counterpart of (1.25) is as follows:

$$(1.30) \quad T_i Y_b - Y_{s_i(b)} T_i = (t_i - 1) \frac{Y_b - Y_{s_i(b)}}{1 - q^{-(\theta', \rho_k)} Y_{-\alpha_i}}, \quad 1 \leq i \leq n,$$

where $\theta' = \theta, \vartheta$ respectively for long, short α_i (it is the only root in the intersection $W(\alpha_i) \cap P_+$).

Here and below we use that given $b \in P$, replacing all $T_i^{\pm 1}$ by $t_i^{\pm 1}$ in the product of (1.28) for Y_b results in the t -power $q^{2(\rho_k, b)} = \prod_{\nu} t_{\nu}^{2((\rho_{\nu})^\vee, b)}$.

Also, Y_b for any b can be represented as the product $\pi_r T_{i_l}^{\pm 1} \cdots T_{i_1}^{\pm 1}$ for a given reduced decomposition $b = \pi_r s_{i_l} \cdots s_{i_1}$ and proper choice of $\{\pm\}$. Here $l = l(b) = 2(\rho^\vee, b_+)$. Only positive powers T_i^{+1} will appear in this product when $b \in P_+$. The total number of the terms $T_i^{\pm 1}$ with $\nu_i = \nu$ in this product equals $2((\rho_{\nu})^\vee, b_+)$.

2. POLYNOMIAL REPRESENTATION

From now on, we will switch from $\mathcal{H}$ to its *intermediate subalgebra* $\mathcal{H}^b \subset \mathcal{H}$ with P replaced by a lattice B between Q and P (see [C7]). Respectively, Π is changed to the preimage Π^b of B/Q in Π . Generally, there can be two different lattices B_X and B_Y for X and Y . We consider only $B_X = B = B_Y$ in the paper; respectively, $a, b \in B$ in X_a, Y_b .

We also set $\widehat{W}^b = B \cdot W \subset \widehat{W}$, and replace m by the least $\tilde{m} \in \mathbb{N}$ such that $\tilde{m}(B, B) \subset \mathbb{Z}$ in the definition of the $\mathbb{Q}_{q,t}$.

Note that $\mathcal{H}^b$ and the polynomial representations (and their rational and trigonometric degenerations) are actually defined over $\mathbb{Z}$ extended by the parameters of DAHA. However the field $\mathbb{Q}_{q,t}$ will be sufficient in this paper.

The *Demazure-Lusztig operators* are as follows:

$$(2.1) \quad T_i = t_i s_i + (t_i - 1)(X_{\alpha_i} - 1)^{-1}(s_i - 1), \quad 0 \leq i \leq n;$$

they obviously preserve $\mathbb{Q}[q, t_\nu][X_b]$. We note that only the formula for T_0 involves q :

$$(2.2) \quad T_0 = t_0 s_0 + (t_0 - 1)(X_0 - 1)^{-1}(s_0 - 1), \text{ where} \\ X_0 = qX_\vartheta^{-1}, \quad s_0(X_b) = X_b X_\vartheta^{-(b, \vartheta)} q^{(b, \vartheta)}, \quad \alpha_0 = [-\vartheta, 1].$$

The map sending T_j to the corresponding operator from (2.1), X_b to X_b (see (1.24)) and $\pi_r \mapsto \pi_r$ induces a $\mathbb{Q}_{q,t}$ -linear homomorphism from $\mathcal{H}^b$ to the algebra of linear endomorphisms of $\mathbb{Q}_{q,t}[X]$. This $\mathcal{H}^b$ -module is faithful and remains faithful when q, t take any nonzero complex values assuming that q is not a root of unity. It will be called the *polynomial representation*; the notation is

$$\mathcal{V} \stackrel{\text{def}}{=} \mathbb{Q}_{q,t}[X_b] = \mathbb{Q}_{q,t}[X_b, b \in B].$$

The images of the Y_b are called the *difference-trigonometric Dunkl operators*.

The polynomial representation is the $\mathcal{H}^b$ -module induced from the one-dimensional representation $T_i \mapsto t_i$, $Y_b \mapsto q^{2(\rho_k, b)}$ of the affine Hecke subalgebra $\mathcal{H}_Y^b = \langle T_i, Y_b \rangle$.

2.1. Macdonald polynomials. There are two equivalent definitions of the *nonsymmetric Macdonald polynomials*, denoted by $E_b(X) = E_b^{(k)}$ for $b \in B$; they belong to $\mathbb{Q}(q, t)[X_a, a \in B]$. The first is based on the truncated theta function due to Macdonald:

$$(2.3) \quad \mu = \mu^{(k)} = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \frac{(1 - X_\alpha q_\alpha^j)(1 - X_\alpha^{-1} q_\alpha^{j+1})}{(1 - X_\alpha t_\alpha q_\alpha^j)(1 - X_\alpha^{-1} t_\alpha q_\alpha^{j+1})}.$$

We will mainly consider μ as a Laurent series with the coefficients in the ring $\mathbb{Q}[t_\nu][[q_\nu]]$ for $\nu \in \nu_R = \{\nu_{\text{sht}}, \nu_{\text{lng}}\}$. The constant term of a

Laurent series $f(X)$ will be denoted by $\langle f \rangle$. Then

$$(2.4) \quad \langle \mu \rangle = \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \frac{(1 - q^{(\rho_k, \alpha) + j \nu_\alpha})^2}{(1 - t_\alpha q^{(\rho_k, \alpha) + j \nu_\alpha})(1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + j \nu_\alpha})}.$$

Recall that $q^{(z, \alpha)} = q_\alpha^{(z, \alpha^\vee)}$, $t_\alpha = q_\alpha^{k_\alpha}$. This equality is equivalent to the Macdonald constant term conjecture proved in complete generality in [C2].

Let $\mu_\circ \stackrel{\text{def}}{=} \mu / \langle \mu \rangle$. The coefficients of the Laurent series $\mu_\circ$ are from the field of rationals $\mathbb{Q}(q, t) \stackrel{\text{def}}{=} \mathbb{Q}(q_\nu, t_\nu)$, where $\nu \in \nu_R$.

The polynomials E_b are uniquely determined from the relations

$$(2.5) \quad E_b - X_b \in \oplus_{c \succ b} \mathbb{Q}(q, t) X_c, \quad \langle E_b X_c^{-1} \mu_\circ \rangle = 0 \quad \text{for } B \ni c \succ b.$$

for generic q, t and form a basis in $\mathbb{Q}(q, t)[X_b]$.

This definition is due to Macdonald (for $k_{\text{sht}} = k_{\text{lng}} \in \mathbb{Z}_+$), who extended the construction from [Op]. The general (reduced) case was considered in [C4].

Another approach is based on the Y -operators. We continue using the same notation X, Y, T for these operators acting in the polynomial representation. Let $X_a(q^b) = q^{(a, b)}$ as $a, b \in P$.

Proposition 2.1. *The polynomials $\{E_b, b \in B\}$ are unique (up to proportionality) eigenfunctions of the operators Y_a ($a \in P$) acting in $\mathbb{Q}_{q, t}[X]$:*

$$(2.6) \quad Y_a(E_b) = q^{(a_+, \rho_k) - (a, b_\sharp)} E_b \quad \text{for } b_\sharp \stackrel{\text{def}}{=} b - u_b^{-1}(\rho_k),$$

$u_b = \pi_b^{-1} b$ is from Proposition 1.1, $b_\sharp = \pi_b((- \rho_k))$.

□

The coefficients of the Macdonald polynomials are rational functions in terms of q_ν, t_ν .

2.2. Symmetric polynomials. Following Proposition 2.1, the *symmetric Macdonald polynomials* $P_b = P_b^{(k)}$ can be introduced as eigenfunctions of the W -invariant *difference operators*

$$(2.7) \quad L_{a_+} = \text{Red}_W \left(\sum_{a' \in W(a_+)} Y_{a'} \right) \quad \text{for } a_+ \in B_+,$$

where Red_W is the restriction to the space $\mathcal{V}^W$ of W -invariants of $\mathcal{V}$. Explicitly,

$$(2.8) \quad \begin{aligned} L_{a_+}(P_{b_-}) &= q^{(a_+, \rho_k)} \left(\sum_{a' \in W(a_+)} q^{-(a', b_- - \rho_k)} \right) P_{b_-}, \quad b_- \in B_-, \\ P_{b_-} &= \sum_{b \in W(b_-)} X_b \pmod{\oplus_{c_- \succ b_-} \mathbb{Q}(q, t) X_c}. \end{aligned}$$

These polynomials were introduced in [M2, M1]. They were used for the first time in Kadell's unpublished work (classical root systems). In the case of A_1 , they are due to Rogers.

The connection between E and P is as follows

$$(2.9) \quad \begin{aligned} P_{b_-} &= \mathbf{P}_{b_+} E_{b_+}, \quad b_- \in B_-, \quad b_+ = w_0(b_-), \\ \mathbf{P}_{b_+} &\stackrel{\text{def}}{=} \sum_{c \in W(b_+)} T_{w_c}, \quad \text{where} \end{aligned}$$

$w_c \in W$ is the element of the least length such that $c = w_c(b_+)$. Taking the complete t -symmetrization $\mathbf{P}$ here (with the summation over all w), one obtains P_{b_-} up to proportionality. See [Op, M3, C4].

There are two different kinds of inner products in $\mathcal{V}$ from [C8] and other works. In the symmetric setting, they essentially coincide. We will need here only the inner products of the symmetric polynomials P_b for $b = b_-$:

$$(2.10) \quad \begin{aligned} &\langle P_b(X) P_c(X^{-1}) \mu_o \rangle \\ &= \delta_{bc} \prod_{\alpha > 0} \prod_{j=0}^{-(\alpha^\vee, b)-1} \left(\frac{(1 - q_\alpha^{j+1} t_\alpha^{-1} X_\alpha(q^{\rho_k}))(1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k}))}{(1 - q_\alpha^j X_\alpha(q^{\rho_k}))(1 - q_\alpha^{j+1} X_\alpha(q^{\rho_k}))} \right). \end{aligned}$$

2.3. Using intertwiners. The following map can be uniquely extended to an automorphism of $\mathcal{H}^b$ where proper fractional powers of q are added (see [C1], [C4], [C7]):

$$(2.11) \quad \begin{aligned} \tau_+ : X_b &\mapsto X_b, \quad \pi_r \mapsto q^{-(\omega_r, \omega_r)} X_r \pi_r, \quad Y_r \mapsto X_r Y_r q^{-\frac{(\omega_r, \omega_r)}{2}}, \\ \tau_+ : T_0 &\mapsto X_0^{-1} \{t_0 T_0^{-1}\}, \quad Y_\vartheta \mapsto X_0^{-1} \{t_0 T_0^{-1}\} T_{s_\vartheta}. \end{aligned}$$

This automorphism fixes T_i ($i \geq 1$), t_ν , q and fractional powers of q .

The Y -intertwiners serve as creation operators in the theory of non-symmetric Macdonald polynomials. Following [C6, C8], let

$$(2.12) \quad \Psi_i^c = \tau_+(T_i) + (t_i - 1)(X_{\alpha_i}(q^{c^\sharp}) - 1)^{-1}, 0 \leq i \leq n.$$

We will use the pairing from (1.8) and the affine action $\widehat{w}((c))$ from (1.7).

Theorem 2.2. *Given $c \in B$, $0 \leq i \leq n$ such that $(\alpha_i, c + d) > 0$,*

$$(2.13) \quad q^{(c,c)/2-(b,b)/2} E_b = \Psi_i^c(E_c) \text{ for } b = s_i((c)).$$

If $(\alpha_i, c + d) = 0$, then

$$(2.14) \quad \tau_+(T_i)(E_c) = t_i E_c, \quad 0 \leq i \leq n,$$

which results in the relations $s_i(E_c) = E_c$ as $i > 0$. For $b = \pi_r((c))$, where the indices r are from O' ,

$$(2.15) \quad q^{(c,c)/2-(b,b)/2} E_b = \tau_+(\pi_r)(E_c) = X_{\omega_r} q^{-(\omega_r, \omega_r)/2} \pi_r(E_c).$$

Also $\tau_+(\pi_r)(E_c) \neq E_c$ for $\pi_r \neq id$, since $\pi_r((c)) \neq c$ for any $c \in B$.

□

If $(\alpha_i, c) > 0$ and $i > 0$, then the set $\lambda(\pi_b)$ is obtained from $\lambda(\pi_c)$ by adding $[\alpha, (c_-, \alpha)]$ for $\alpha = u_c(\alpha_i) \in R_-$ and $(c_-, \alpha^\vee) = (c, \alpha_i^\vee) > 0$. When $i = 0$ and $(\alpha_0, c + d) = -(c, \vartheta) + 1 > 0$, then the root $[\alpha, (c_-, \alpha) + 1]$ is added to $\lambda(\pi_c)$ for $\alpha = u_c(-\vartheta) = \alpha^\vee \in R_-$ and $(c_-, \alpha) = -(c, \vartheta) \geq 0$.

In each of these two cases, $(\alpha_i, u_c^{-1}(\rho)) = (\alpha, \rho) < 0$ and the powers of t_ν in

$$(2.16) \quad X_{\alpha_i}(q^{c^\sharp}) = q^{(\alpha_i, c - u_c^{-1}(\rho_k) + d)} = q^{(a_i, c + d)} \prod_{\nu} t_\nu^{-(\alpha, (\rho_\nu)^\vee)}$$

are from $\mathbb{Z}_+$ with that of t_i strictly positive.

Due to Theorem 2.2 (see also [C6], Corollary 5.3), the polynomial E_b exists if

$$\prod_{[\alpha, j] \in \lambda'(\pi_b)} (1 - q_\alpha^j X_\alpha(q^{\rho_k})) \neq 0.$$

If $b \in B_-$ and the latter inequality holds for $b_+ = w_0(b) \in B_+$, then the symmetric polynomials P_b is well defined.

2.4. Spherical polynomials. The following renormalization of the E -polynomials is of major importance in the Fourier analysis (see [C4]):

$$(2.17) \quad \mathcal{E}_b \stackrel{\text{def}}{=} E_b(X)(E_b(q^{-\rho_k}))^{-1}, \quad \text{where } b \in B,$$

$$E_b(q^{-\rho_k}) = q^{(\rho_k, b_-)} \prod_{[\alpha, j] \in \lambda'(\pi_b)} \left(\frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \right).$$

This definition requires the t -localization.

We call them nonsymmetric *spherical polynomials*. Formula (2.17) is the Macdonald *evaluation conjecture* in the nonsymmetric variant from [C4]. See [C3] for the symmetric evaluation conjecture.

The following *duality formula* holds for $b, c \in B$:

$$(2.18) \quad \mathcal{E}_b(q^{c_\sharp}) = \mathcal{E}_c(q^{b_\sharp}), \quad b_\sharp = b - u_b^{-1}(\rho_k),$$

which is the main justification of the definition of $\mathcal{E}_b$.

Given $b \in B$, the polynomial $\mathcal{E}_b$ is well defined for $q, t \in \mathbb{C}^*$ if

$$(2.19) \quad \prod_{[\alpha, j] \in \lambda'(\pi_b)} (1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})) \neq 0.$$

In the symmetric setting,

$$(2.20) \quad \mathcal{P}_b \stackrel{\text{def}}{=} P_b(X)(P_b(q^{-\rho_k}))^{-1} \quad \text{where } b \in B_-,$$

$$P_b(q^{-\rho_k}) = P_b(q^{\rho_k}) = q^{(\rho_k, b_-)} \prod_{\alpha > 0} \prod_{j=0}^{-(\alpha^\vee, b)-1} \left(\frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \right).$$

The symmetric duality reads as follows:

$$(2.21) \quad \mathcal{P}_b(q^{c-\rho_k}) = \mathcal{P}_c(q^{b-\rho_k}), \quad \text{for } b, c \in B_-.$$

The norm formula becomes entirely conceptual:

$$(2.22) \quad (\langle \mathcal{P}_b(X) \mathcal{P}_b(X^{-1}) \mu_\circ \rangle)^{-1} = \sum_{a \in W(b)} \mu(\pi_a) \mu(\text{id})^{-1},$$

$$\text{where } \mu(\widehat{w}) \stackrel{\text{def}}{=} \mu(\widehat{w}((q^{-\rho_k}))) \quad \text{for } \widehat{w} \in \widehat{W}.$$

It is a direct corollary of the fact that the Fourier transform sends the $\mathcal{P}$ -polynomials to the delta-functions; see [C8].

2.5. The limit $t \rightarrow 0$. Let $\overline{\mathcal{H}}^b$ by the reduction of $\mathcal{H}^b$ as $t_\nu = 0$ for $\nu \in \nu_R$. It can be called the *nil-DAHA* or the *crystal DAHA*. The polynomials $\overline{E}_b, \overline{P}_{b_-}$ are well defined and linearly generate $\overline{\mathcal{V}}$ and $\overline{\mathcal{V}}^W$ correspondingly; $\overline{\mathcal{V}} = \mathbb{Q}_q[X_b, b \in B]$.

Theorem 2.2 holds under this specialization and gives quite a constructive approach to the $\overline{E}$ -polynomials. The intertwiners Ψ_i^c from (2.12) that appear in the formulas for $\overline{E}_b$ are all in the form $\tau_+(T_i) + 1$ in this limit. It is directly connected with the fact that $T'_i = T_i + 1$ satisfy the same homogeneous Coxeter relations as $\{T_i, 0 \leq i \leq n\}$ do, a special feature of the nil-DAHA. It results from the theory of intertwiners and, of course, can be checked directly too.

The action of π_r on $\{T'_i\}$ by conjugation obviously remains unchanged. Thus relations (i,ii) from Definition 1.2 hold and, given $\widehat{w} \in \widehat{W}$, the element $T'_w = \pi_r T'_{i_1} \cdots T'_{i_l}$ does not depend on the choice of the reduced decomposition $\widehat{w} = \pi_r s_{i_1} \cdots s_{i_l}$. For instance, operators $\Pi'_i \stackrel{\text{def}}{=} \tau_+(T'_{-\omega_i})$ for $i = 1, \dots, n$ are pairwise commutative and, importantly, W -invariant. One has: $\Pi'_b = \prod_{i=1}^n (\Pi'_i)^{n_i}$ for $B_- \ni b = -\sum n_i \omega_i$.

The W -invariance of $\{\Pi_b, b \in B_-\}$ simplifies significantly the relation of the $\overline{E}$ -polynomials to the $\overline{P}$ -polynomials:

$$(2.23) \quad \overline{P}_b = \overline{E}_b \text{ for } b = b_- \in B_-.$$

In more detail, we have the following explicit proposition.

Proposition 2.3. (i) In the representation $\overline{\mathcal{V}}$ of $\overline{\mathcal{H}}^b$, the polynomial $\tau_+(T'_w)(1)$ equals $q^{r_b} \overline{E}_b$ for $\widehat{w} = \pi_b, b \in B, r_b \in \mathbb{Q}$.
(ii) In the symmetric case,

$$(2.24) \quad \Pi'_b(1) = q^{r_b} \overline{P}_b \text{ for } b \in B_-, r_b \in \mathbb{Q},$$

where Π'_i can be replaced by their restrictions $\text{Red}_W(\Pi'_i)$ to $\overline{\mathcal{V}}^W$, which are pairwise commutative W -invariant difference operators. $\square$

We note that only positive powers of q appear in the coefficients of $\overline{E}_b$. As $q \rightarrow 0$, the polynomials $\overline{E}_{b_-}$ become the classical finite dimensional Lie characters, which can be seen from (2.28) (upon the symmetrization).

For the affine root systems considered in this paper (with α_0 in terms of the maximal *short* root ϑ), the connection was established between

the polynomials $E_b(t \rightarrow \infty)$ and the Demazure characters of the corresponding irreducible affine Lie algebras. See [San] and, especially, [Ion1], Theorem 1. Paper [Ion1] is based on the technique of intertwiners (from [KS] in the GL_n -case and [C6] for arbitrary reduced root systems). We will not discuss this direction in this paper.

There is a relation between the limit $t \rightarrow 0$ used here and the one $t \rightarrow \infty$. It goes through the general formula

$$(2.25) \quad E_b^* = \prod_{\nu \in \nu_R} t_\nu^{l_\nu(u_b) - l_\nu(w_0)} T_{w_0}(E_{\varsigma(b)}), \quad \text{where} \\ X^* = X^{-1}, q^* = q^{-1}, t^* = t^{-1}, \varsigma(b) = -w_0(b),$$

form [C8] and other author's works. This connection is especially simple for the symmetric polynomials: $P_b(X)^* = P_b(X^{-1})$ as $b = b_-$, i.e., $\overline{P}_b = P_b(t \rightarrow 0) = P_{\varsigma(b)}(t \rightarrow \infty)$. We use that $P_b(X^{-1}) = P_{\varsigma(b)}(X)$.

Concerning the orthogonality of $\overline{P}$, the denominator of the μ -function from (2.26) vanishes in the limit:

$$(2.26) \quad \overline{\mu} = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} (1 - X_\alpha q_\alpha^j) (1 - X_\alpha^{-1} q_\alpha^{j+1}).$$

The constant term formula becomes a well-known identity:

$$(2.27) \quad \langle \overline{\mu} \rangle = \prod_{i=1}^n \prod_{j=1}^{\infty} \frac{1}{1 - q_i^j}, \quad \text{where } q_i = q^{\nu_i}.$$

For $b, c \in B_-$, the norm formula from (2.10) reads as:

$$(2.28) \quad \langle \overline{P}_b(X) \overline{P}_c(X^{-1}) \overline{\mu}_o \rangle = \delta_{bc} \prod_{i=1}^n \prod_{j=1}^{-(\alpha_i^\vee, b)} (1 - q_i^j).$$

3. SPHERICAL AND WHITTAKER FUNCTIONS

We will begin with the identities involving the Gaussians, which are essentially from [C5]; then their limits $t \rightarrow 0$ will be considered.

The second part of this section is devoted to the Whittaker limit of the q, t -spherical function from [C5], which results in a formula for the q -Whittaker function in terms of the $\overline{P}$ -polynomials.

We note that the *Whittaker limit* is a general procedure that can be applied to any solutions of the Macdonald eigenvalue problem (and its various degenerations and generalizations).

3.1. Gauss-type integrals. By the *Gaussians* $\tilde{\gamma}$ we mean

$$(3.1) \quad \tilde{\gamma}^{\oplus} = \sum_{b \in B} q^{-(b,b)/2} X_b, \quad \tilde{\gamma}^{\ominus} = \sum_{b \in B} q^{(b,b)/2} X_b.$$

The multiplication by $\tilde{\gamma}^{\ominus}$ preserves the space of Laurent series with coefficients in $\mathbb{Q}[t][[q^{\frac{1}{2\tilde{m}}}]]$, where $\tilde{m}(B, B) = \mathbb{Z}$ is from the definition of $\mathbb{Q}_{q,t}$. Respectively, the coefficients must be taken from $\mathbb{Q}[t][[q^{-\frac{1}{2\tilde{m}}}]]$ when the Gaussian $\tilde{\gamma}^{\oplus}$ is taken.

We will also use the *real Gaussians* defined as

$$(3.2) \quad \gamma^{\pm 1} = q^{\pm x^2/2}, \quad \text{where } X_b \stackrel{\text{def}}{=} q^{x_b}, x_b = (x, b), x^2 = \sum_i x_{\alpha_i} x_{\omega_i^\vee}.$$

Note that considering $\tilde{\gamma}^{\oplus, \ominus}$ as holomorphic functions (provided that $|q| > 1$ and, respectively, $|q| < 1$) the functions $\tilde{\gamma}^{\oplus}/\gamma$ and $\tilde{\gamma}^{\ominus}\gamma$ are B -periodic in terms of x .

The *q-Mehta-Macdonald identity* from [C5]

$$(3.3) \quad \langle \tilde{\gamma}^{\ominus} \mu_o \rangle = \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - t_{\alpha}^{-1} q_{\alpha}^{(\rho_k, \alpha^\vee) + j}}{1 - q_{\alpha}^{(\rho_k, \alpha^\vee) + j}} \right)$$

provides the normalization constant for the q -Gauss integrals

$$(3.4) \quad \begin{aligned} & \langle P_b(X) P_c(X) \tilde{\gamma}^{\ominus} \mu_o \rangle \\ &= q^{\frac{(b,b) + (c,c)}{2} - (b+c, \rho_k)} P_c(q^{b-\rho_k}) P_b(q^{-\rho_k}) \langle \tilde{\gamma}^{\ominus} \mu_o \rangle, \end{aligned}$$

where $b, c \in B_-$. Obviously, it implies the duality formula (2.21). Formula (3.4) can be naturally extended to the E -polynomials (the proof even becomes simpler), but we do not need it in this paper.

There are counterparts of (3.4) for $\tilde{\gamma}^+$ (with $|q| > 1$ if it is treated as an analytic function), and for the Jackson summation taken instead of the constant term functional. See [C5, C8]. The considerations from this paper can be readily extended to these cases.

Taking the limit. Let us tend $t \rightarrow 0$ in (3.4). The definition of the P -polynomials implies that

$$(3.5) \quad \lim_{t \rightarrow 0} q^{-(c, \rho_k)} P_c(q^{z-\rho_k}) = q^{(c_+, z)} \quad \text{for } c \in B_-, c_+ = w_o(c).$$

Note that $-(c, \rho_k) = (c_+, \rho_k)$. For instance, it matches the evaluation formula in (2.20): $\lim_{t \rightarrow 0} q^{-(c, \rho_k)} P_c(q^{-\rho_k}) = 1$.

We come to the following formulas ($c \in B_-$):

$$(3.6) \quad \langle \tilde{\gamma}^\ominus \bar{\mu}_o \rangle = \prod_{i=1}^n \prod_{j=1}^\infty (1 - q_i^j),$$

$$(3.7) \quad \langle \bar{P}_b(X) \bar{P}_c(X) \tilde{\gamma}^\ominus \bar{\mu}_o \rangle = q^{\frac{(b,b)+(c,c)}{2}} X_{c_+}(q^b) \langle \tilde{\gamma}^\ominus \bar{\mu}_o \rangle.$$

Here $X_{c_+}(q^b) = q^{(c_+, b)} = q^{(c, b_+)} = X_{b_+}(q^c)$.

3.2. Global spherical function. One of the main advantages of the technique of Gaussians is a possibility to introduce the spherical function as a reproducing kernel of the Fourier transform from $\mathcal{V}\gamma^{-1}$, the polynomial representation multiplied by the Gaussian γ^{-1} , to the $\mathcal{H}^b$ -module $\mathcal{V}\gamma$. We will need only the symmetric case here. We assume that $|q| < 1$, which makes the considerations “naturally” compatible with the limit $t \rightarrow 0$. In this setting, the construction below is directly related to the identities (3.4) (respectively, (3.7) in the limit).

We note that if the whole polynomial representation is considered, then the corresponding anti-involutions of $\mathcal{H}^b$, generally, require the t -localizations. Correspondingly, the definition of the Whittaker limit of the nonsymmetric counterpart of formula (3.9) below (see [C5]) becomes more subtle.

We will use the notation $\tilde{\gamma}_\lambda$ and γ_λ for the Gaussians defined for another set of variables Λ completely analogous to X ($\tilde{\gamma}_x, \gamma_x$ are old $\tilde{\gamma}, \gamma$). Thus, $\tilde{\gamma}_\lambda = \tilde{\gamma}(q^\lambda)$ and $\gamma_\lambda = \gamma(q^\lambda) = q^{\lambda^2/2}$. We will also use

$$(3.8) \quad \langle \gamma \rangle_{\rho_k} \stackrel{\text{def}}{=} \sum_{a \in B} q^{\frac{(\rho_k + a, \rho_k + a)}{2}} = \tilde{\gamma}^\ominus(q^{\rho_k}) q^{\frac{(\rho_k, \rho_k)}{2}}.$$

Theorem 3.1. *Provided that $|q| < 1$, the function Ψ from the relation*

$$(3.9) \quad \begin{aligned} & \tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus \mathfrak{P}^\circ(X, \Lambda) / \tilde{\gamma}_x^\ominus(q^{\rho_k}) \\ &= \Psi(X, \Lambda; q, t) \stackrel{\text{def}}{=} \sum_{b \in B_-} q^{\frac{(b,b)}{2} - (\rho_k, b)} \frac{P_b(X) P_b(\Lambda^{-1})}{\langle P_b(X) P_b(X^{-1}) \mu_o \rangle} \end{aligned}$$

is a well-defined Laurent series. It is an analytic function for all X, Λ and for any choice of t_ν assuming that all P -polynomials exist (the conditions $|t_\nu| < 1$ are sufficient).

The function $\mathfrak{P}^\circ(X, \Lambda)$ defined via (3.9) is meromorphic for all X, Λ and analytic apart from the zeros of $\tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus$. Replacing $\tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus$ by $\gamma_x^{-1} \gamma_\lambda^{-1}$ in this definition, the corresponding function will be denoted simply by $\mathfrak{P}(X, \Lambda)$; it becomes totally analytic but not a (single-valued) function in terms of X_b, Λ_b .

Both functions, $\mathfrak{P}^\circ(X, \Lambda)$ and $\mathfrak{P}(X, \Lambda)$, are $X \leftrightarrow \Lambda$ -symmetric, W -invariant with respect to X and Λ and satisfy the following extension of the eigenvalue problem from (2.8):

$$(3.10) \quad L_{a_+}(\mathfrak{P}(X, \Lambda)) = q^{(a_+, \rho_k)} \left(\sum_{a' \in W(a_+)} \Lambda_{a'}^{-1} \right) \mathfrak{P}(X, \Lambda).$$

□

We note that $P_b(X^{-1})P_b(\Lambda) = P_{\varsigma b}(X)P_{\varsigma b}(\Lambda^{-1})$ in (3.9); recall that $\varsigma(x) = -w_0(x)$ and $P_{\varsigma b}(X) = P_b(X^{-1})$. Applying ς to the summation index b does not change the result. Thus:

$$\mathfrak{P}^\circ(X, \Lambda) = \mathfrak{P}^\circ(\Lambda, X) = \mathfrak{P}^\circ(\varsigma(X), \varsigma(\Lambda)).$$

The following can be used for an *abstract* (i.e., without an explicit formula) definition of the function $\mathfrak{P}^\circ(X, \Lambda)$. It goes through the spherical polynomials $\{\mathcal{P}_c = P_c/P_c(q^{-\rho_k}), c \in B_-\}$ with a common coefficient of proportionality:

$$(3.11) \quad \mathfrak{P}^\circ(X, q^{c-\rho_k}) = \frac{P_c(X)}{P_c(q^{-\rho_k})} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q_\alpha^{(\rho_k, \alpha^\vee) + j}}{1 - t_\alpha^{-1} q_\alpha^{(\rho_k, \alpha^\vee) + j}} \right).$$

Here we substitute $\lambda = c_\sharp, \Lambda = q^{c_\sharp}$ in the left-hand side of (3.10) and divide it by the Gaussian $\tilde{\gamma}_x^\ominus$. This formula can be considered as a q, t -generalization of the Shintani-Casselman-Shalika formula from [Shi, CS]. Its limit as $t \rightarrow 0$ will be discussed in the next section.

3.3. Global Whittaker function. We are now in a position to define the *global q -Whittaker function* $\tilde{\mathfrak{P}}_x^\circ(X, \Lambda)$ from the relation

$$(3.12) \quad \tilde{\gamma}_x^\ominus \tilde{\mathfrak{P}}_x^\circ(X) \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} \frac{\tilde{\gamma}^\ominus(q^{x-\rho_k})}{\tilde{\gamma}^\ominus(q^{\rho_k})} \mathfrak{P}^\circ(q^{-\rho_k} X).$$

Here we always assume that $t_\nu \rightarrow 0$ for all ν . The function $\tilde{\mathfrak{P}}_x$ is defined for γ^{-1} instead of $\tilde{\gamma}^\ominus$:

$$(3.13) \quad \tilde{\mathfrak{P}}_x(X, \Lambda) \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} q^{(x, \rho_k)} \mathfrak{P}(q^{-\rho_k} X, \Lambda).$$

More explicitly, provided that $|q| < 1$,

$$(3.14) \quad \tilde{\mathfrak{P}}_x(X) \stackrel{\text{def}}{=} \gamma_x \lim_{k \rightarrow \infty} q^{\frac{(\rho_k, \rho_k)}{2}} \left(\gamma_x^{-1} \mathfrak{P} \right) (q^{x - \rho_k}) = \\ \lim_{k \rightarrow \infty} q^{\frac{(\rho_k, \rho_k)}{2}} q^{-\frac{(x - \rho_k, x - \rho_k)}{2}} \mathfrak{P}(q^{x - \rho_k}) = \lim_{k \rightarrow \infty} q^{(x, \rho_k)} \mathfrak{P}(q^{x - \rho_k}).$$

In this definition, Λ remains untouched, so the limit is a W -invariant function with respect to Λ . As a matter of fact, the key fact we need is the existence of the limit

$$(3.15) \quad \lim_{k \rightarrow \infty} \Psi(q^{x - \rho_k}, \Lambda; q, t) = \tilde{\Psi}(X, \Lambda; q)$$

for $\Psi(X, \Lambda; q, t)$ from (3.9). Let us calculate the Whittaker $\tilde{\Psi}$ in full detail. It is essentially *a generating function for the $\overline{P}$ -polynomials*; see Proposition 2.3.

Theorem 3.2. (i) *Provided that $|q| < 1$, the Whittaker function $\tilde{\mathfrak{P}}_x^\circ$ is given by the formula*

$$(3.16) \quad \tilde{\mathfrak{P}}_x^\circ(X, \Lambda) \tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus \\ = \tilde{\Psi}(X, \Lambda; q) \stackrel{\text{def}}{=} \sum_{b \in B_-} q^{\frac{(b, b)}{2}} \frac{X_{b_+} \overline{P}_b(\Lambda^{-1})}{\prod_{i=1}^n \prod_{j=1}^{-(\alpha_i^\vee, b)} (1 - q_i^j)},$$

where the power series in the right-hand side is well defined coefficient-wise and converges everywhere; see (3.5) and (2.28). The formula for $\tilde{\mathfrak{P}}_x$ is with $\gamma_x^{-1} \gamma_\lambda^{-1}$ instead of $\tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus$ and with the same summation in the right-hand side.

(ii) The ratio of the functions $\tilde{\mathfrak{P}}_x^\circ(X, \Lambda)$, $\tilde{\mathfrak{P}}_x(X, \Lambda)$ is B -periodic with respect to X and Λ . The dependence on Λ is governed by (3.10) for the limits $\overline{L}_{a_+}^\Lambda$ of the operators L_{a_+} as $t \rightarrow 0$ upon $X \mapsto \Lambda$:

$$(3.17) \quad \overline{L}_{a_+}^\Lambda (\tilde{\mathfrak{P}}_x(X, \Lambda)) = X_{a_-}^{-1} \tilde{\mathfrak{P}}_x(X, \Lambda), \quad X_{a_-}^{-1} = X_{-w_0(a_+)}.$$

In terms of X , these functions satisfy the q -Toda system of difference equations:

$$(3.18) \quad \tilde{L}_{a_+}(\tilde{\mathfrak{P}}(X, \Lambda)) = \left(\sum_{a' \in W(a_+)} \Lambda_{a'}^{-1} \right) \tilde{\mathfrak{P}}(X, \Lambda),$$

$$(3.19) \quad \tilde{L}_{a_+} \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} q^{-(a_+, \rho_k)} \left(q^{(x, \rho_k)} (\Gamma_{\rho_k}^{-1} L_{a_+} \Gamma_{\rho_k}) q^{-(x, \rho_k)} \right), \\ \Gamma_b(F(X)) = F(q^b X), \quad \Gamma_b X_a = q^{(b, a)} X_a \Gamma_b \quad \text{for } b \in \mathbb{C}^n.$$

Here the difference operators L_{a_+} ($a_+ \in B_+$) from (2.8) are conjugated by the translation $\Gamma_{-\rho_k}$ (it is $(\rho_k)'$ in the notation from (1.2)) and then by the operator of multiplication by $q^{(x, \rho_k)}$. $\square$

We note that $X_{b_+} \overline{P}_b(\Lambda^{-1})$ in the summation for $\tilde{\Psi}$ can be replaced by $X_b^{-1} \overline{P}_b(\Lambda)$. Recall that, generally, $P_b(X^{-1}) = P_{\varsigma(b)}(X)$ and $b \mapsto \varsigma(b) = -w_0(b)$ does not change the coefficients in the summation from (3.16).

Concerning the notation, one can use $\tilde{\mathfrak{P}}_\lambda^\circ$, $\tilde{\mathfrak{P}}_\lambda$ for the Whittaker limits with Λ, λ instead of X, x , but we do not need these functions in the paper. We put x in $\tilde{\mathfrak{P}}_x$ (not always) only to emphasize that the Whittaker limit makes the dependence on X and Λ asymmetric.

The construction of the Toda operators in terms of the Macdonald operators (and their various degenerations) is essentially due to Inozemtsev and Etingof. The paper [Et] contains a systematic consideration of various Whittaker functions and related objects. This paper is mainly about GL_n , but our (3.19) is quite analogous to the limiting procedure there, as was expected in Remark 1 at the end of [Et].

We remark that our q -Toda operators are “dual” to those from [Et, GLO1] (the translation operators must be replaced by their inverses), which is connected with our choice of the limit $t \rightarrow 0$ versus $t \rightarrow \infty$ in these papers. It will be discussed below in greater detail.

Theorem 3.3. *Continuing the previous theorem, let $X = q^c$ for $c \in B_-$. Then the Shintani-type identity holds:*

$$(3.20) \quad \tilde{\gamma}^\ominus(1) \tilde{\mathfrak{P}}^\circ(q^c, \Lambda) = \overline{P}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \left(\frac{1}{1 - q_i^j} \right),$$

where $\tilde{\gamma}^\ominus(1) = \sum_{b \in B} q^{b^2/2}$. More explicitly,

$$(3.21) \quad \begin{aligned} & \sum_{b \in B_-} \frac{q^{(c-b, c-b)/2} \overline{P}_b(\Lambda)}{\prod_{i=1}^n \prod_{j=1}^{(\alpha_i^\vee, b_+)} (1 - q_i^j)} \\ &= \tilde{\gamma}^\ominus(\Lambda) \overline{P}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \left(\frac{1}{1 - q_i^j} \right). \end{aligned}$$

Proof. Due to (3.12),

$$(3.22) \quad \tilde{\gamma}^\ominus(q^c) \tilde{\mathfrak{P}}_x^\circ(q^c, \Lambda) = \lim_{t \rightarrow 0} \frac{\tilde{\gamma}^\ominus(q^{c-\rho_k})}{\tilde{\gamma}^\ominus(q^{-\rho_k})} \left(\mathfrak{P}^\circ(q^{c-\rho_k}, \Lambda) \right).$$

Applying the identity (3.11) for X transposed with Λ (the duality) inside $\left(\cdot \right)$,

$$(3.23) \quad \tilde{\gamma}^\ominus(q^c) \tilde{\mathfrak{P}}_x^\circ(q^c, \Lambda) = \lim_{t \rightarrow 0} \frac{\tilde{\gamma}^\ominus(q^{c-\rho_k})}{\tilde{\gamma}^\ominus(q^{-\rho_k})} \left(\frac{P_c(\Lambda)}{P_c(q^{-\rho_k})} \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1 - q_i^j} \right).$$

Recall that

$$(3.24) \quad \langle \gamma \rangle_{\rho_k} = \langle \gamma \rangle_{c-\rho_k} = \tilde{\gamma}^\ominus(q^{c-\rho_k}) q^{\frac{(c-\rho_k, c-\rho_k)}{2}},$$

where we use that c is from B ; see (3.8). Hence,

$$(3.25) \quad \frac{\tilde{\gamma}^\ominus(q^{c-\rho_k})}{\tilde{\gamma}^\ominus(q^{-\rho_k})} = q^{(c, \rho_k) - c^2/2},$$

Moving $q^{(c, \rho_k)}$ from (3.25) to the denominator and combining it with $P_c(q^{-\rho_k})$, we apply (3.5):

$$\lim_{t \rightarrow 0} q^{-(c, \rho_k)} P_c(q^{-\rho_k}) = 1.$$

Finally, we move $q^{-c^2/2}$ from (3.25) to the left-hand side of (3.23) and observe that $q^{c^2/2} \tilde{\gamma}^\ominus(q^c)$ does not depend on c , so it equals $\tilde{\gamma}^\ominus(1)$. $\square$

We note that by making $q = 0$ in (3.21), we arrive at the trivial identity $\overline{P}_b(\Lambda; q = 0) = \overline{P}_b(\Lambda; q = 0)$, where $\overline{P}_b(\Lambda; q = 0)$ is the classical character for the dominant weight $w_0(b)$.

The p -adic limit $q \rightarrow 0$ (in this setting) transforms (3.11) to the classical Shintani-Casselman-Shalika formulas. See [C8] concerning the p -adic degeneration of the DAHA theory (the limit $q \rightarrow \infty$ is considered there).

3.4. One-dimensional theory. We will begin with the explicit formula for the $\overline{P}$ -polynomials in the case of A_1 . The formulas for the Rogers polynomials are well known as well as for their limits as $t \rightarrow 0$. Such limits are the q -Hermite polynomials introduced by Szegő and considered in many works; see, e.g., [ASI]. Let us reestablish the formulas we need for these polynomials using the (nonsymmetric) intertwining operators.

Let $\alpha = \alpha_i = \vartheta$, $s = s_1$, $\omega = \omega_1 = \rho$; so $\alpha = 2\omega$ and the standard invariant form is $(n\omega, m\omega) = nm/2$. Similarly,

$$X = X_\omega = q^x, \quad X(q^{n\omega}) = q^{n/2}, \quad \Gamma(F(X)) = F(q^{1/2}X),$$

i.e., $x(n\omega) = n/2$, $\Gamma(x) = x + 1/2$, $\Gamma X = q^{1/2}X\Gamma$.

We will also use $\pi \stackrel{\text{def}}{=} s\Gamma : X \mapsto q^{1/2}X^{-1}$; then $\pi^2 = \text{id}$ and $Y = Y_\omega = \pi T$ in DAHA of type A_1 . Concerning the Gaussians, $(x, x) = x_\alpha x_\omega = 2x^2$ and $\gamma = q^{(x, x)/2} = q^{x^2}$; note that $\gamma(q^{n\omega}) = q^{n^2(\omega, \omega)/2} = q^{n^2/4}$. Also, $x - \rho_k = \Gamma^{-k}(x) = x - k/2$ and $q^{(x, \rho_k)} = q^{xk}$ in the formulas for the Whittaker limit.

We set $t = t_0 = q^k$ for $k \in \mathbb{C}$,

$$\overline{P}_n = \overline{P}_{-n\omega} = \overline{E}_{-n\omega}.$$

Then $\overline{P}_1 = 1$, $\overline{P}_2 = X + X^{-1}$,

$$\begin{aligned} \overline{P}_2 &= X^2 + X^{-2} + 1 + q, \quad \overline{P}_3 = X^3 + X^{-3} + \frac{1 - q^3}{1 - q}(X + X^{-1}), \\ (3.26) \quad \overline{P}_4 &= X^4 + X^{-4} + \frac{1 - q^4}{1 - q}(X^2 + X^{-2}) + \frac{(1 - q^4)(1 - q^3)}{(1 - q)(1 - q^2)}. \end{aligned}$$

Generally, for the monomial symmetric functions $M_0 = 1$, $M_n = X^n + X^{-n}$ as $n > 1$,

$$(3.27) \quad \overline{P}_n = M_n + \sum_{j=1}^{[n/2]} \frac{(1 - q^n) \cdots (1 - q^{n-j+1})}{(1 - q) \cdots (1 - q^j)} M_{n-2j}.$$

The norm formulas from (2.26), (2.27), (2.28) read as follows:

$$(3.28) \quad \langle \overline{P}_m(X) \overline{P}_n(X) \overline{\mu}_o \rangle = \delta_{mn} \prod_{j=1}^n (1 - q^j),$$

where $m, n = 0, 1, \dots$, $\overline{\mu}_o = \overline{\mu} / \langle \overline{\mu}_o \rangle$ for the classical theta-function

$$(3.29) \quad \overline{\mu} = \prod_{j=0}^{\infty} (1 - X^2 q^j)(1 - X^{-2} q^{j+1}), \quad \langle \overline{\mu} \rangle = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}.$$

Due to Theorem 2.2, the composition $\mathcal{R} = (1 + T)X\pi$ is the raising operator for the $\overline{P}$ -polynomials. Namely, upon the restriction, Red, to the symmetric polynomials:

$$(3.30) \quad q^{\frac{n}{2}} R(\overline{P}_n) = \overline{P}_{n+1}, \quad \text{where } R = \text{Red}(\mathcal{R}) = \frac{X^2 \Gamma^{-1} - X^{-2} \Gamma}{X - X^{-1}}.$$

This readily gives (3.27).

The L -operators. We will begin with the formula for the q -Toda operator from (3.19):

$$(3.31) \quad \tilde{L} = \lim_{t \rightarrow 0} (q^{kx} \Gamma_k^{-1} L \Gamma_k q^{-kx}) = (1 - X^{-2})\Gamma + \Gamma^{-1}$$

for $L = L_\omega = Y + tY^{-1}$, where $\Gamma_k X = t^{k/2} X \Gamma_k$. We will also use

$$(3.32) \quad \tilde{L}_\gamma = \gamma^{-1} \tilde{L} \gamma = q^{1/4} (X\Gamma + X^{-1}(\Gamma^{-1} - \Gamma)) \quad \text{for } \gamma = q^{x^2}.$$

Similarly for the straight specialization of L at $t = 0$,

$$(3.33) \quad \begin{aligned} \overline{L} &= \lim_{t \rightarrow 0} L = (1 - X^2)\Gamma + (1 - X^{-2})\Gamma^{-1}, \\ \overline{L}_\gamma &= \gamma^{-1} \overline{L} \gamma = -q^{1/4} (X - X^{-1})^{-1} (\Gamma - \Gamma^{-1}). \end{aligned}$$

This operator is proportional to the so-called Askey-Wilson divided difference operator. The latter serves as the *shift operator* in the theory of Rogers' polynomials (with any t) and the basic hypergeometric function. See [AI] and also [C8], Chapter 2.

Its defining property is the relation

$$(3.34) \quad \overline{L}_\gamma(\overline{P}_n) = -q^{1/4} (q^n - q^{-n}) \overline{P}_{n-1}, \quad n = 1, 2, \dots$$

Let us give a convenient reference concerning (3.30), (3.34): [OS], formulas (20-25).

3.5. Whittaker function for A_1 . Provided that $|q| < 1$, we can now introduce the Whittaker function $\tilde{\mathfrak{P}}_x^\circ$ from the relation:

$$(3.35) \quad \tilde{\mathfrak{P}}_x^\circ(X, \Lambda) \tilde{\gamma}_x^\ominus \tilde{\gamma}_\lambda^\ominus = \tilde{\Psi}(X, \Lambda) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} q^{\frac{n^2}{4}} \frac{X^n \overline{P}_n(\Lambda)}{\prod_{j=1}^n (1 - q^j)},$$

where $\tilde{\gamma}_x^\ominus = \sum_{j=-\infty}^{\infty} q^{j^2/4} X^j$ ($\tilde{\gamma}_\lambda^\ominus$ is defined in terms of λ).

The function $\tilde{\Psi}(X, \Lambda)$ is actually the so-called generating function for q -Hermite polynomials; see, e.g., [Sus] formulas (26), (27) and the references there. Its interpretation as a q -Whittaker function (upon the multiplication by the Gaussians) does not seem to have been noticed, although the difference equation for $\tilde{\Psi}(X, \Lambda)$ was certainly known (formula (19) *ibid.*).

The power series $\tilde{\Psi}(X, \Lambda)$ converges everywhere. The Λ -dependence (see (3.17)) readily follows from (3.34):

$$(3.36) \quad \overline{L}_\gamma^\Lambda(\tilde{\Psi}(X, \Lambda)) = X \tilde{\Psi}(X, \Lambda), \quad \overline{L}_\gamma^\Lambda = \overline{L}_\gamma(X \mapsto \Lambda).$$

In terms of X , the function $\tilde{\Psi}$ satisfies the γ -*twisted* q -Toda equation, which reads as follows:

$$(3.37) \quad \tilde{L}_\gamma(\tilde{\Psi}(X, \Lambda)) = (\Lambda + \Lambda^{-1}) \tilde{\Psi}(X, \Lambda).$$

Shintani-type formula. Let us consider Theorem 3.3 in the A_1 -case; we plug in $X = q^{-n/2}$ for $n = 0, 1, \dots$. Then

$$(3.38) \quad q^{n^2/4} \tilde{\Psi}(q^{-n/2}, \Lambda) = \tilde{\gamma}^\ominus(\Lambda) \overline{P}_n(\Lambda) \prod_{j=1}^{\infty} \left(\frac{1}{1 - q^j} \right).$$

Recall that $\tilde{\gamma}^\ominus(q^\lambda) = \sum_{j \in \mathbb{Z}} q^{j\lambda + j^2/4}$. Here the left-hand side and the right-hand side coincide as Laurent series or as analytic functions.

This formula becomes a trivial identity for $q = 0$, i.e., in the case of the classical characters

$$\overline{P}_n(X; q = 0) = \frac{X^{n+1} - X^{-n-1}}{X - X^{-1}}.$$

3.6. The case $|q| > 1$. Generally, the Whittaker-type limiting procedure as $t \rightarrow \infty$ is naturally connected with the theory at $|q| > 1$ and can lead to new formulas. However, in the symmetric setting of this paper, there is a direct connection between the Whittaker functions defined as $|q| < 1, t \rightarrow 0$ and $|q| > 1, t \rightarrow \infty$, which we are going to discuss now.

We follow [C5] and use $\tilde{\gamma}^\oplus$ instead of $\tilde{\gamma}^\ominus$ and γ instead of γ^{-1} . In the nonsymmetric setting, the corresponding global spherical function is really different from that for $|q| < 1$. However, there exists a simple connection in the symmetric case.

The q, t -definition we need is as follows (cf. (3.9)):

$$(3.39) \quad \tilde{\gamma}_x^\oplus \tilde{\gamma}_\lambda^\oplus \mathfrak{P}_\star^\circ / \tilde{\gamma}^\oplus(q^{\rho_k}) = \sum_{b \in B_-} q^{-\frac{(b,b)}{2} + (\rho_k, b)} \frac{P_b(X) P_b(\Lambda)}{\langle P_b(X) P_b(X^{-1}) \mu_\circ \rangle},$$

where $q > 1$ and $\mathfrak{P}_\star^\circ$ satisfies the claims of Theorem 3.1. The Whittaker limiting procedure requires here taking $t \rightarrow \infty$ for ensuring the convergence.

The formulas are:

$$(3.40) \quad \tilde{\gamma}_\lambda^\oplus \tilde{\gamma}_x^\oplus \tilde{\mathfrak{P}}_\star^\circ = \tilde{\gamma}_\lambda^\oplus \lim_{t \rightarrow \infty} \frac{\tilde{\gamma}_x^\oplus(q^{x-\rho_k})}{\tilde{\gamma}_x^\oplus(q^{\rho_k})} \mathfrak{P}_\star^\circ(q^{-\rho_k} X)$$

$$(3.41) \quad \stackrel{\text{def}}{=} \tilde{\Psi}_\star(X, \Lambda; q) = \sum_{b \in B_-} q^{-\frac{(b,b)}{2}} \frac{X_b P_b(\Lambda; q, t \rightarrow \infty)}{\prod_{i=1}^n \prod_{j=1}^{-(\alpha_i^\vee, b)} (1 - q_i^{-j})}.$$

Cf. (3.12) and (3.16). Here

$$(3.42) \quad \lim_{t \rightarrow \infty} P_b(\Lambda; q, t) = \lim_{t^{-1} \rightarrow 0} P_b(\Lambda^{-1}; q^{-1}, t^{-1}) = \overline{P}_b(\Lambda^{-1}; q^{-1}).$$

Therefore $\tilde{\Psi}_\star(X, \Lambda; q)$ simply *coincides* with $\tilde{\Psi}(X^{-1}, \Lambda; q^{-1})$ in the notation from (3.16).

We conclude that $\tilde{\mathfrak{P}}_\star^\circ$ satisfies the eigenvalue problem

$$(3.43) \quad \tilde{L}_{a_+}^\star(\tilde{\mathfrak{P}}_\star(X, \Lambda)) = \left(\sum_{a' \in W(a_+)} \Lambda_{a'}^{-1} \right) \tilde{\mathfrak{P}}_\star(X, \Lambda),$$

$$(3.44) \quad \tilde{L}_{a_+}^\star \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} q^{-(a_+, \rho_k)} \left(q^{-(x, \rho_k)} (\Gamma_{\rho_k}^{-1} L_{a_+} \Gamma_{\rho_k}) q^{(x, \rho_k)} \right),$$

where $\Gamma_b(F(X)) = F(q^b X)$. Compare with (3.19); the conjugation by $q^{(x, \rho_k)}$ there is replaced by the conjugation by $q^{-(x, \rho_k)}$. Thus the operators $L_{a_+}^\star(X, q^{-1})$ generalize those considered in [Et, GLO2]. For instance, in the one-dimensional case in the notation from (3.31):

$$(3.45) \quad \begin{aligned} \widetilde{L}^\star &= \lim_{t \rightarrow \infty} (q^{-kx} \Gamma_k^{-1} L \Gamma_k q^{kx}) \\ &= \lim_{t \rightarrow \infty} t^{-1/2} \left(\frac{t(t^{-1} X^2) - 1}{t^{-1} X^2 - 1} (t^{1/2} \Gamma) + \frac{t(t X^{-2}) - 1}{t X^{-2} - 1} (t^{-1/2} \Gamma^{-1}) \right) \\ &= (1 - X^2) \Gamma + \Gamma^{-1}. \end{aligned}$$

4. HARMONIC ANALYSIS TOPICS

The real integration or Jackson integration is, generally, necessary when the Gaussian γ^{-1} in the constructions above is replaced by γ . A typical example is as follows. Let us consider the DAHA-Fourier transform in terms of the constant term functional (or using the imaginary integration) in the space of Laurent polynomials multiplied by γ^{-1} . Then the inverse transform will involve the Jackson (or real) integration and the proper choice of the Gaussian is γ instead of γ^{-1} .

Such “switch” of the Gaussians is necessary algebraically due to the properties of the involution of DAHA that governs the Fourier transform. Correspondingly, the contour of integration, real or imaginary, must ensure the convergence, i.e., its choice is of analytic nature. The direction is real for γ and imaginary for γ^{-1} . It is of course for $|q| < 1$; if $|q| > 1$ then it must be the other way round. Generally, especially, in the absence of the Gaussians (for instance, in the Harish-Chandra theory), the directions, real or imaginary, are selected to match the growth estimates for the spherical function, used as kernels of the corresponding transforms.

We establish such estimates in the *real direction*. The theory appears surprisingly “precise”, although the results of the paper are far from being complete. Only the first term of the asymptotic expansion is obtained. We note that in our setting, the global spherical function is periodic in the imaginary direction, so the imaginary growth estimates are irrelevant. We stick to the Jackson integration, which is actually very similar to the “classical” case of real integration; the estimates we obtain serve both theories.

4.1. Growth estimates. It is possible to evaluate the growth of the global q, t -spherical function $\mathfrak{P}(X, \Lambda; q, t)$ from Theorem 3.1 in the real directions. Let $0 < q < 1$, $t_\nu = q_\nu^{k_\nu}$ (or, simply, $t = q^k$) for $k_\nu \in \mathbb{C}$ provided the existence of all spherical symmetric polynomials $\{\mathcal{P}_{b_-}\}$, equivalently, provided that the polynomial representation is semisimple and the radical of the evaluation pairing vanishes (see [C9]). The assumption $\Re k_\nu > -1/h_\nu$ for the *Coxeter numbers* $h_\nu = 1 + (\rho, (\theta')^\vee)$, where $\theta' = \theta, \vartheta$ for $\nu = \nu_{\text{lng}}, \nu_{\text{sht}}$, is sufficient (but not necessary).

For $x \in \mathbb{C}^n$, let $x_+ \stackrel{\text{def}}{=} u(x)$ where $u(\Re(x))$ is a unique vector belonging to the closure $\overline{\mathfrak{C}}_+ = \sum_{i=1}^n \mathbb{R}_+ \omega_i$ of the standard positive nonaffine Weyl chamber $\mathfrak{C}_+ = \sum_{i=1}^n \mathbb{R}_{>0} \omega_i$.

Given a p -sequence of vectors $\mathbf{x}' = \{x'_1, \dots, x'_p\} \subset \mathbb{R}^n$ and a p -sequence of positive integers $\mathbf{n} = \{n_1, \dots, n_p\}$, we use the dot-notation $\mathbf{n} \cdot \mathbf{x}'$ for $\sum_{j=1}^p n_j x'_j$.

Theorem 4.1. (i) For arbitrary $x, \lambda \in \mathbb{C}^n, k_\nu \in \mathbb{C}$, we set:

$$(4.1) \quad \mathfrak{P}_+^\circ(x, \lambda; q, k) \stackrel{\text{def}}{=} \frac{\tilde{\gamma}^\circ(q^x) \tilde{\gamma}^\circ(q^\lambda)}{\tilde{\gamma}^\circ(q^{x_+ + \lambda_+ - \rho_k}) \tilde{\gamma}^\circ(q^{\rho_k})} \mathfrak{P}^\circ(q^x, q^\lambda; q, q^k).$$

Given a real p -sequence $\mathbf{x}'$, let the components of $\mathbf{n}$ tend to $+\infty$ in any way provided that $(\mathbf{n} \cdot \mathbf{x}')_+ \in \mathfrak{C}_+$. Then the limit

$$(4.2) \quad \lim_{\mathbf{n} \rightarrow \infty} \mathfrak{P}_\dagger^\circ(x + \mathbf{n} \cdot \mathbf{x}', \lambda; q, k)$$

exists if $\Re(\lambda)_+ \in \mathfrak{C}_+$; moreover, it depends only on λ and is nonzero for all such λ . Here we choose x to ensure that $\tilde{\gamma}^\ominus(q^{(x+\mathbf{n} \cdot \mathbf{x}')_+ + (\lambda)_+ - \rho_k}) \neq 0$ for any $\mathbf{n}$.

Under the same constraints, consider $\mathfrak{P}_\dagger^\circ(x + \mathbf{n} \cdot \mathbf{x}', \lambda + \mathbf{n} \cdot \boldsymbol{\lambda}'; q, k)$ for a real p -sequence $\boldsymbol{\lambda}'$ satisfying $(\mathbf{n} \cdot \boldsymbol{\lambda}')_+ \in \mathfrak{C}_+$. Then the limit exists too and is an absolute nonzero constant depending only on q, k .

(ii) In the case of the Whittaker function $\tilde{\mathfrak{P}}$, we remove k from the formulas and replace x_+ by $-x$:

$$(4.3) \quad \tilde{\mathfrak{P}}_\dagger^\circ(x, \lambda; q) \stackrel{\text{def}}{=} \frac{\tilde{\gamma}^\ominus(q^x) \tilde{\gamma}^\ominus(q^\lambda)}{\tilde{\gamma}^\ominus(q^{\lambda_+ - x})} \tilde{\mathfrak{P}}^\circ(q^x, q^\lambda; q).$$

Provided that $\mathbf{n} \cdot \mathbf{x}' \in -\mathfrak{C}_+$ (it was not needed in the q, t -case), the claims from (i) hold true for

$$(4.4) \quad \lim_{\mathbf{n} \rightarrow \infty} \tilde{\mathfrak{P}}_\dagger^\circ(x + \mathbf{n} \cdot \mathbf{x}', \lambda + \mathbf{n} \cdot \boldsymbol{\lambda}'; q).$$

Here $\tilde{\gamma}^\ominus(q^{\{\cdot\}})$ is nonzero at $(\lambda + \mathbf{n} \cdot \boldsymbol{\lambda}')_+ - (x + \mathbf{n} \cdot \mathbf{x}')$; we continue to assume that $(\mathbf{n} \cdot \mathbf{x}')_+ \in \mathfrak{C}_+$ and, respectively, either $\Re(\lambda)_+ \in \mathfrak{C}_+$ as $\boldsymbol{\lambda}' = 0$ or $(\mathbf{n} \cdot \boldsymbol{\lambda}')_+ \in \mathfrak{C}_+$. $\square$

The justification of (i) involves the analysis of the corresponding difference equations for $\mathfrak{P}$ in the limit of large x and/or large λ , but we use the explicit formulas too. We note that the *asymptotic* difference equations provide the asymptotic limit (the factor in the definition of $\mathfrak{P}_\dagger^\circ$ from (4.3)) only up to a periodic function. So we need to use that both, $\mathfrak{P}^\circ$ and $\mathfrak{P}_\dagger^\circ$, are meromorphic. Part (ii) is obtained as a limit of (i).

Taking $\mathbf{x}'$ and $\boldsymbol{\lambda}'$ real vectors is, actually, insignificant in the theorem. Since $\mathfrak{P}^\circ$ and $\mathfrak{P}_\dagger^\circ$ are $2\pi i \log(q) P^\vee$ -periodic in the imaginary direction, it suffices to impose the conditions from (i,ii) for their real parts only.

Due to the claim that the limits do not depend on the particular way the integers $\{n_i\}$ approach the infinity, the estimates can be calculated *precisely* using the Shintani-type formulas. Moreover, this approach provides the best justification of (i) if one involves here that $\mathfrak{P}^\circ$ is a unique (in its class) solution of the spherical eigenvalue problem that

“goes through” the $\mathcal{P}$ -polynomials in the sense of Shintani-type formulas.

4.2. Exact asymptotic formulas. Let us obtain them in the (most important) case $\lambda' = 0$. The key ingredient is the *inverse* of the positive half of the μ -function, a direct q, t -counterpart of the celebrated the Harish-Chandra c -function [HC]:

$$(4.5) \quad \sigma(X; q, t) = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \frac{1 - t_{\alpha} X_{\alpha} q_{\alpha}^j}{1 - X_{\alpha} q_{\alpha}^j}.$$

Corollary 4.2. (i) *Provided the conditions of part (i) of the theorem as $\lambda' = 0$, including $\Re(\lambda)_+ \in \mathfrak{C}_+$,*

$$(4.6) \quad \lim_{\mathbf{n} \rightarrow \infty} \mathfrak{P}_{\dagger}^{\circ}(x + \mathbf{n} \cdot \mathbf{x}', \lambda; q, k) = \varrho(q, t) \langle \mu \rangle \sigma(q^{\lambda^+}; q, t)$$

$$\text{for } \varrho(q, t) \stackrel{\text{def}}{=} \langle \mu \rangle \prod_{\alpha > 0} \frac{1 - q^{(\rho_k, \alpha)}}{1 - t_{\alpha} q^{(\rho_k, \alpha)}}$$

$$(4.7) \quad = \prod_{\alpha > 0} \prod_{j=1}^{\infty} \frac{(1 - q^{(\rho_k, \alpha) + (j-1)\nu_{\alpha}})(1 - q^{(\rho_k, \alpha) + j\nu_{\alpha}})}{(1 - t_{\alpha} q^{(\rho_k, \alpha) + (j-1)\nu_{\alpha}})(1 - t_{\alpha}^{-1} q^{(\rho_k, \alpha) + j\nu_{\alpha}})},$$

where $\langle \mu \rangle$ is the constant term of μ from (2.4); $\varrho = \langle \sum_{w \in W} w(\mu) \rangle$.

(ii) *Correspondingly, imposing $\mathbf{n} \cdot \mathbf{x}' \in -\mathfrak{C}_+$ and the other conditions in the Whittaker case,*

$$(4.8) \quad \lim_{\mathbf{n} \rightarrow \infty} \tilde{\mathfrak{P}}_{\dagger}^{\circ}(x + \mathbf{n} \cdot \mathbf{x}', \lambda; q) = \langle \bar{\mu} \rangle \sigma(q^{\lambda^+}; q, 0)$$

$$= \prod_{i=1}^n \prod_{j=0}^{\infty} \frac{1}{(1 - q_i^{j+1})(1 - q_i^{(\lambda_+, \alpha_i^{\vee}) + j})}.$$

In contrast to this formula, assuming that $(\mathbf{n} \cdot \lambda')_+ \in \mathfrak{C}_+$, the λ -limit does not depend on x :

$$(4.9) \quad \lim_{\mathbf{n} \rightarrow \infty} \tilde{\mathfrak{P}}_{\dagger}(x, \lambda + \mathbf{n} \cdot \lambda'; q) = \langle \bar{\mu} \rangle = \prod_{i=1}^n \prod_{j=1}^{\infty} \frac{1}{1 - q_i^j}.$$

The limit remains the same if we substitute $x \mapsto x + \mathbf{n} \cdot \mathbf{x}'$ in (4.9) for $\mathbf{x}'$ such that $\Re((\mathbf{n} \cdot \lambda')_+ - \mathbf{n} \cdot \mathbf{x}') \in \mathfrak{C}_+$.

Proof. It suffices to calculate

$$(4.10) \quad \lim_{c_+ \rightarrow \infty} \mathfrak{P}_{\dagger}^{\circ}(c - \rho_k, \lambda; q, k), \quad \text{where } c \in B_-,$$

and by $c_+ \rightarrow \infty$, we mean that $(\alpha_i, c_+) \rightarrow \infty$ for all $i = 1, \dots, n$. Recall that $c_+ = w_0(c)$, where c is always from B_- in this calculation.

Using the definition and formula (3.11),

$$(4.11) \quad \mathfrak{P}_+^\circ(c - \rho_k, \lambda; q, k) = \frac{\tilde{\gamma}^\ominus(q^{c-\rho_k})\tilde{\gamma}^\ominus(q^\lambda)}{\tilde{\gamma}^\ominus(q^{\lambda_+-c})\tilde{\gamma}^\ominus(q^{\rho_k})}$$

$$(4.12) \quad \times \frac{P_c(q^\lambda)}{P_c(q^{-\rho_k})} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q_\alpha^{(\rho_k, \alpha^\vee)+j}}{1 - t_\alpha^{-1} q_\alpha^{(\rho_k, \alpha^\vee)+j}} \right).$$

The special value $P_c(q^{-\rho_k})$ is given by (2.20); it is the exponent $q^{(\rho_k, c)}$ times the product term, which will be combined (in the limit of large c_+) with the product from (4.12). The result is exactly $\varrho(q, t)$, the constant term of the symmetrization of μ from [M1, M2, C2].

We note that $\langle \mu \rangle$ was obtained in this calculation without any reference to its “true” meaning as the constant term of μ . It is interesting but not very much surprising; in [C8] the norm-formula for Macdonald polynomials (including the constant term formula) was actually deduced from the evaluation formula. Something similar occurs here.

Since $c \in B$ (actually $c \in B_-$), we can remove it from the theta-functions $\tilde{\gamma}^\ominus(q^{c-\rho_k})$ and $\tilde{\gamma}^\ominus(q^{\lambda_+-c})$, the multipliers are the same as for the Gaussians $q^{-(c-\rho_k)^2/2}$ and $q^{-(\lambda_+-c)^2/2}$. It gives:

$$(4.13) \quad \frac{\tilde{\gamma}^\ominus(q^{c-\rho_k})\tilde{\gamma}^\ominus(q^\lambda)}{\tilde{\gamma}^\ominus(q^{\lambda_+-c})\tilde{\gamma}^\ominus(q^{\rho_k})} = q^{(c, \rho_k - \lambda_+)}.$$

The factor $q^{(c, \rho_k)}$ will cancel the same term from $P_c(q^{-\rho_k})$ (in the denominator). The remaining part of (i) is taking the limit

$$\lim_{c_+ \rightarrow \infty} q^{-(c, \lambda_+)} P_c(q^\lambda),$$

which is a subject of the following lemma.

Lemma 4.3. *Provided that $|q| < 1$ and $\Re(x_+) \in \mathfrak{C}_+$,*

$$\lim_{c_+ \rightarrow \infty} q^{-(c, \lambda_+)} P_c(q^x) = \sigma(q^{x_+}; q, t).$$

Proof. In the multiplicative notations, $q^{(c, x_+)} = q^{(w^{-1}(c), x)} = X_{w(c)}$ for $w(\Re(x)) \in \mathfrak{C}_+$, i.e., this monomial is from the leading symmetric monomial function of the $P_c(X)$. Its coefficient is 1 by construction. One can assume here that $w = 1$ due to the W -invariance of P_c . Then $X_c^{-1} P_c$ will be a power series in terms of X_{α_i} for $i = 1, \dots, n$.

Calculating the corresponding difference equations (in the limit of large c_+) is the most direct way to identify its expansion with $\sigma(X)$. It suffices to use the leading terms of the L -operators serving the symmetric Macdonald polynomials calculated in [C2], Proposition 3.4. Then we observe that $\sigma(X)$ is a solution of this system of equations. It gives the required since both are power series in terms of X_{α_i} with the constant term 1. It is also possible to involve the inner product from the theory of Macdonald polynomials. $\square$

The lemma gives (4.6). The Whittaker variants from (ii) are its straightforward limits; the condition $\Re(x) \in -\mathfrak{C}_+$ must be imposed in (4.8) and no such conditions are necessary in (4.9). A direct calculation of these two limits via the Shintani-type formula (3.20) is possible as well. $\square$

The lemma is known for the Askey-Wilson polynomials [Is] ([FZ] contains a comprehensive discussion of the A_1 -case). See [vD] and references therein for a counterpart of this lemma in the case of Koornwinder polynomials (the root system $C^\vee C_n$). We note that the Laurent expansion of the rank one μ -function is very explicit, and Lemma 4.3 is straightforward. As for Corollary 4.2, its one-dimensional versions (for the basic hypergeometric function or similar) are likely to be known.

The Harish-Chandra formula. The corollary is an exact generalization of the Harish-Chandra fundamental asymptotic formula for the classical spherical functions. Indeed, in terms of $\mathfrak{P}^\circ$, asymptotically,

$$(4.14) \quad \lim_{\mathbf{n} \rightarrow \infty} \mathfrak{P}^\circ(x + \mathbf{n} \cdot \mathbf{x}', \lambda; q, k) \\ = \varrho(q, t) \frac{\tilde{\gamma}^\ominus(q^{x_+ + \lambda_+ - \rho_k}) \tilde{\gamma}^\ominus(q^{\rho_k})}{\tilde{\gamma}^\ominus(q^{x_+}) \tilde{\gamma}^\ominus(q^{\lambda_+})} \sigma(q^{\lambda_+}; q, t).$$

Up to a simple W -invariant and B -periodic factor $C(x, \lambda)$, depending of course on q, k (it is $\mathbb{Z}$ -periodic in terms of k), we can switch to $\mathfrak{P}$ here, replacing all $\tilde{\gamma}^\ominus(q^x)$ by $\gamma^{-1}(q^x) = q^{-x^2/2}$. It gives that in the limit of large $\Re(x)_+ \in \mathfrak{C}_+$,

$$(4.15) \quad \mathfrak{P}^\circ(x, \lambda; q, k) \sim C(x, \lambda) \varrho(q, t) q^{-(x_+, \lambda_+ - \rho_k) + (\lambda_+, \rho_k)} \sigma(q^{\lambda_+}; q, t).$$

Corollary 4.4. *We continue to assume that all spherical polynomial $\{\mathcal{P}_{b_-}\}$ exist; for instance, the conditions $k_\nu \notin -1/h_\nu - \mathbb{Q}_+$ for the Coxeter numbers h_ν of R are sufficient. Provided that $\Re(\lambda)_+ \in \mathfrak{C}_+$, the global spherical function $\mathfrak{P}^\circ(x, \lambda; q, k)$ is bounded in terms of x as*

$\mathfrak{C}_+ \ni \Re(x)_+ \rightarrow \infty$ if and only if

$$0 < (\Re(\lambda)_+, \alpha_i^\vee) \leq \Re(k_i) \text{ for } i = 1, \dots, n, \text{ which implies } \Re(k_\nu) > 0.$$

If $\Re(\lambda)_+ \in \overline{\mathfrak{C}}_+$ is allowed, then $\mathfrak{P}_+^\circ(x, \lambda; q, k)$ asymptotically approaches a polynomial in terms of $\{x_i\}$ of degree no greater than n , the rank of the root system. $\square$

The dependence of x in the right-hand side of (4.15) is as in the Harish-Chandra formula [HC]. The corresponding degeneration of $\mathcal{H}$ (and all related objects) is the procedure $q \rightarrow 1$, where we set $X_b = e^{-z_b}$ and z_b, λ_b, k are considered the basic new variables upon the degeneration. We take $-z_b$ here because the base q is smaller than 1. The limit of the right-hand side of (4.15) can be readily controlled using the functional equation for the theta-function $\tilde{\gamma}^\ominus$. Up to some renormalization, it becomes (for large $\Re(z_+)$):

$$\text{Const} \prod_{i=1}^n Z_i^{(\alpha_i^\vee, \lambda) - k_i} \prod_{\alpha \in R_+} \frac{\Gamma(\lambda_\alpha^\vee)}{\Gamma(\lambda_\alpha^\vee + k_\alpha)} \text{ for } Z = e^z, z = z_+, \lambda = \lambda_+.$$

The factor $q^{-(\lambda_+, \rho_k)}$, which ensures the $X \leftrightarrow \Lambda$ -duality of the q, t -formula, vanishes in the limit; the duality collapses under the degeneration to the Harish-Chandra theory.

Respectively, Corollary 4.4 is a q, t -version of the description of the bounded spherical functions from [HJ].

Technically, (4.15) matches the growth estimates for *complex* Lie groups because real Lie group result in the terms like $\Gamma(\lambda_\alpha^\vee/2)$ in this formula, which is not the case.

The discussion of the limit $q \rightarrow 1$ in more detail, including managing the global spherical functions, i.e., the left-hand side of (4.15), will be continued elsewhere. It should be possible to reprove the classical estimates using the q, t -theory.

Claim (i) of the theorem can be naturally extended toward including the Whittaker limit as follows.

4.3. When $k \rightarrow \infty$. Let us reformulate (4.17) entirely in terms of the function Ψ from (3.9). Namely, provided the conditions from Theorem 4.5, (i), the limit of the function

$$(4.16) \quad \Psi_+(x, \lambda; q, k) \stackrel{\text{def}}{=} \left(\tilde{\gamma}^\ominus(q^{x_+ + \lambda_+ - \rho_k}) \right)^{-1} \Psi(X, \Lambda; q, t)$$

exists. Similarly, $\tilde{\Psi}_{\dagger}(x, \lambda; q) \stackrel{\text{def}}{=} (\tilde{\gamma}^{\ominus}(q^{\lambda+ -x}))^{-1} \tilde{\Psi}(X, \Lambda; q)$. The Whittaker limit becomes simply:

$$\lim_{k \rightarrow \infty} \Psi(q^{x-\rho_k}, q^{\lambda}; q, q^k) = \tilde{\Psi}(q^x, q^{\lambda}; q).$$

See (3.15).

Given real $k'_{\nu} \geq 0$, let us replace k by $k + n'k'$ for $n' \in \mathbb{N}$ in (4.2) and analyze the limit

$$(4.17) \quad \lim_{\{\mathbf{n}, n'\} \rightarrow \infty} \Psi_{\dagger}(x + \mathbf{n} \cdot \mathbf{x}', \lambda + \mathbf{n} \cdot \boldsymbol{\lambda}'; q, k + n'k').$$

In the non-simply-laced case, n' can be treated as a 2-vector $\{n'_{\text{sh}}, n'_{\text{lg}}\}$ and $n' \cdot k'$ considered instead of $n'k'$; then both components are supposed to approach infinity (in this paper).

Theorem 4.5. *We represent $k'_{\nu} = u_{\nu} + v_{\nu}$ for non-negative real u_{ν}, v_{ν} and pick the directions $\mathbf{x}', \boldsymbol{\lambda}'$ such that*

$$(4.18) \quad \begin{aligned} (a) \quad & (\mathbf{n} \cdot \mathbf{x}')_+ - n'\rho_u \in \mathfrak{C}_+ \ni \Re(\lambda)_+ \text{ when } \boldsymbol{\lambda}' = 0 \text{ or} \\ (b) \quad & (\mathbf{n} \cdot \mathbf{x}')_+ - n'\rho_u \in \mathfrak{C}_+ \ni (\mathbf{n} \cdot \boldsymbol{\lambda}')_+ - n'\rho_v \text{ when } \boldsymbol{\lambda}' \neq 0 \end{aligned}$$

for all $\mathbf{n}, n'$. Then the limit (4.17) exists subject to conditions from part (i) of Theorem 4.1, including the strict positivity requirement $\Re(\lambda)_+ \in \mathfrak{C}_+$. It does not depend on x in case (a) and is a x, λ -constant under (b). If $k' > 0$ then the limit does not depend on k too, i.e., depends only on x for (a) and is an absolute constant for (b). $\square$

Theorem 4.1 corresponds to the case $k' = 0$; then the limit does depend on k . The rule here is that the limit does not depend on the vectors x, λ or k involved in the limit, provided that the corresponding directions and the values of the vectors which are fixed are generic.

The Whittaker limiting procedure can be treated as an *extreme case* of the theorem as follows. Let $k = n'k'$ assuming that $k' > 0$ and $\mathbf{n} = \{n'\}$. We take $\lambda' = 0, x' = -\rho_{k'}$. Then the limit (4.17) still exists but now it depends on x (and depends on λ too because we set $\lambda' = 0$). Explicitly,

$$\Psi_{\dagger}(x, \lambda; q, k) = (\tilde{\gamma}^{\ominus}(q^{\lambda+ -x}))^{-1} \Psi(q^{x+n'x'}, q^{\lambda}; q, q^{n'k'}),$$

since $(x + n'x')_+ = \rho_k - x$ for sufficiently large n' . Actually, we do not need $\Psi_{\dagger}$ here; the correction factor $(\tilde{\gamma}^{\ominus}(q^{\lambda+ -x}))^{-1}$ does not depend on n' . We arrive at the procedure from (3.12).

We believe that the following calculation is clarifying. Let us take *generic extreme* x' and λ' in (4.18):

$$x' = \rho_u, \lambda' = \rho_v, \text{ so } x'_+ + \lambda'_+ - \rho'_k = 0.$$

Similar to the Whittaker case, we do not need $\Psi_\dagger$ here. Assuming that all u_ν and v_ν are *nonzero*,

$$(4.19) \quad \begin{aligned} & \lim_{n' \rightarrow \infty} \Psi(x + n' x', \lambda + n' \lambda'; q, k + n' k') \\ &= \sum_{b \in B_+} q^{\frac{(b,b)}{2}} \frac{X_b \Lambda_b^\varsigma}{\prod_{i=1}^n \prod_{j=1}^{(\alpha_i^\vee, b)} (1 - q_i^j)} \end{aligned}$$

for $\Lambda^\varsigma = w_0(\Lambda^{-1})$. Thus, we obtain a non-constant dependence on x and λ here, but the output is (one of the variants of) the multi-variable q -exponential function, i.e., significantly simpler than the Whittaker one.

A Whittaker variant of this calculation is actually an extreme case of formula (4.9). It is:

$$(4.20) \quad \begin{aligned} & \lim_{\mathbf{n} \rightarrow \infty} \tilde{\Psi}(x + (\mathbf{n} \cdot \mathbf{y}')_+, \lambda + \mathbf{n} \cdot \mathbf{y}'; q) \\ &= \sum_{b \in B_+} q^{\frac{(b,b)}{2}} \frac{q^{x_b - \lambda_b}}{\prod_{i=1}^n \prod_{j=1}^{(\alpha_i^\vee, b)} (1 - q_i^j)}, \end{aligned}$$

where we use the same $\mathbf{y}'$ for x and λ (but in somewhat different way), assuming that $\Re(\mathbf{n} \cdot \mathbf{y}')_+ \in \mathfrak{C}_+$. Note the sign of $(\mathbf{n} \cdot \mathbf{y}')_+$; the growth estimates for the q -Whittaker functions considered above required taking the direction from the negative Weyl chamber. The proof is simple; we only need to know the leading coefficient of $\overline{P}_b$ is 1.

Discussion. The theorems guarantee exponential growth (to be exact, no greater) of the function $\mathfrak{P}^\circ$ including the boundaries of the domains in the theorems.

In more detail, the Gaussian-type corrections used in the definitions of $\mathfrak{P}$ -functions and the corresponding Ψ -functions are not sufficient to ensure the existence of the limits on the boundary of the domains considered in Theorem 4.1 and 4.5. Even if they are sufficient for the convergence (as in the Whittaker case), then the limits can depend on the initial x, λ . For instance, when $(\mathbf{n} \cdot \mathbf{x}')_+, (\mathbf{n} \cdot \boldsymbol{\lambda}')_+$ belong to faces of the Weyl chamber $\overline{\mathfrak{C}}_+$, the limits are expected to be connected with the spherical (and Whittaker) functions for *subsystems* of R .

The role of the condition $\Re(\lambda)_+ \in \mathfrak{C}_+$ as $\lambda' = 0$ is also important and not clarified in full. As it was claimed, if $\Re(\lambda)_+ \notin \mathfrak{C}_+$ then (4.16), generally, diverges, but the growth is polynomial.

A description of such and similar *extreme* situations and the corresponding *asymptotic* systems of difference equations is a natural challenge.

Numerical experiments in the rank one case confirm that the convergence condition (4.18) is sharp. It is not clear what happens if x' is taken non-proportional to $\rho_{k'}$ (especially in the non-simply-laced case when $k = \{k_{\text{sht}}, k_{\text{lng}}\}$). Generally, for any $x' \in \mathfrak{C}_+$, the convergence of $\Psi(x + n'x', \lambda; q, k + n'k')_{\dagger}$ is granted for $0 \leq k' < k_o$, where $k_o = k_o(x') > 0$. What is the formula for $k_o(x')$ and for which x' the limit exists at such extreme $k_o(x')$?

4.4. Jackson integrals. We are going to integrate the product of two global spherical functions for the μ -measure twisted by the plus-Gaussian. The previous section guarantees that the growth of this function in real directions is no greater than exponential. Due to the presence of the Gaussian, this is sufficient to ensure the convergence of the Jackson summations in the theorem below. This theorem is not from [C5], but its proof is based on the same technique (see also [C8]).

Let us fix $\xi \in \mathbb{C}^n$ and define the Jackson summation as follows:

$$\langle f \rangle_{\xi} \stackrel{\text{def}}{=} |W|^{-1} \sum_{w \in W, b \in B} f(q^{w(\xi)+b}), \quad \text{where } w(\xi) + b = (bw)((\xi)).$$

Here the affine action of $\widehat{W}$ from (1.7) is used; f can be any function well defined at the set $\{q^{w(\xi)+b}\}$. Recall that the notation $\langle f \rangle$ was used for the constant term of a Laurent series f . We continue to assume that $|q| < 1$.

As above, $X_{\alpha}(q^{\xi}) = q^{(\alpha, \xi)}$, $\gamma(q^z) = q^{(z, z)/2}$, $(z, z) = \sum_{i=1}^n z_i z_{\alpha_i}$, say, $(z, z)/2 = z_1^2 - z_1 z_2 + z_2^2$ for A_2 . For instance,

$$\langle \gamma \rangle_{\xi} = \sum_{a \in B} q^{(\xi + a, \xi + a)/2} = \widetilde{\gamma}^{\ominus}(q^{\xi}) q^{(\xi, \xi)/2}, \quad \gamma \widetilde{\gamma}^{\ominus} = \sum_{a \in B} a((\gamma)).$$

We will constantly use that $\langle \gamma \rangle_{\xi}$ is periodic with respect to the substitutions $\xi \mapsto \xi + b, b \in B$.

Let us normalize the μ -function at q^ξ : $\mu_\bullet \stackrel{\text{def}}{=} \mu/\mu(q^\xi)$. Explicitly, using the sets $\lambda(bw) = \tilde{R}_+ \cap (bw)^{-1}(-\tilde{R}_+)$,

$$(4.21) \quad \mu_\bullet(q^{w(\xi)+b}) = \prod_{[\alpha, \nu_{\alpha j}] \in \lambda(bw)} \left(\frac{t_\alpha^{-1/2} - t_\alpha^{1/2} q^{(\alpha, \xi) + \nu_{\alpha j}}}{t_\alpha^{1/2} - t_\alpha^{-1/2} q^{(\alpha, \xi) + \nu_{\alpha j}}} \right).$$

Theorem 4.6. *For arbitrary weights $\Lambda = q^\lambda, \Lambda' = q^{\lambda'}$,*

$$(4.22) \quad \begin{aligned} & (\tilde{\gamma}^\ominus(q^{\rho_k}))^2 \langle \mathfrak{P}^\circ(X, \Lambda) \mathfrak{P}^\circ(X^{-1}, \Lambda') \gamma \mu_\bullet \rangle_\xi \\ &= \langle \gamma \mu_\bullet \rangle_\xi \tilde{\gamma}_\lambda^\ominus \tilde{\gamma}_{\lambda'}^\ominus \mathfrak{P}^\circ(\Lambda, \Lambda') \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q_\alpha^{(\rho_k, \alpha^\vee) + j}}{1 - t_\alpha^{-1} q_\alpha^{(\rho_k, \alpha^\vee) + j}} \right), \\ & \langle \gamma \mu_\bullet \rangle_\xi = \mu(q^\xi; t^{-1}) |W|^{-1} \langle \gamma \rangle_\xi \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \left(\frac{1 - t_\alpha^{-1} q_\alpha^{-(\rho_k, \alpha^\vee) + j}}{1 - q_\alpha^{-(\rho_k, \alpha^\vee) + j}} \right), \end{aligned}$$

where by $\mu(q^\xi; t^{-1})$ we mean the μ -function evaluated at $X = q^\xi$ with all t_α replaced by t_α^{-1} . $\square$

In these formulas t_ν are arbitrary provided the existence of all $\{\mathcal{P}_b\}$. The products are considered as the limits if $k_\nu \in \mathbb{Z}_+ \setminus \{0\}$. The normalization factor is obtained by taking $\Lambda = q^{-\rho_k}, \Lambda' = q^{-\rho_k}$. Indeed,

$$(4.23) \quad \begin{aligned} \mathfrak{P}^\circ(X, q^{-\rho_k}) &= \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q_\alpha^{(\rho_k, \alpha^\vee) + j}}{1 - t_\alpha^{-1} q_\alpha^{(\rho_k, \alpha^\vee) + j}} \right) \mathfrak{P}^\circ(q^{\rho_k}, q^{-\rho_k}) \text{ and} \\ & \langle \mathfrak{P}^\circ(X, q^{-\rho_k}) \mathfrak{P}^\circ(X, q^{-\rho_k}) \gamma \mu_\bullet \rangle_\xi \\ &= \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q_\alpha^{(\rho_k, \alpha^\vee) + j}}{1 - t_\alpha^{-1} q_\alpha^{(\rho_k, \alpha^\vee) + j}} \right) \mathfrak{P}^\circ(q^{\rho_k}, q^{-\rho_k}) \langle \gamma \mu_\bullet \rangle_\xi \end{aligned}$$

due to formula (3.11).

4.5. The special case $\xi = -\rho_k$. The theory of Jackson-Gauss integrals is essentially algebraic, similar to that for the constant term functional. Analytically, we need only the exponential growth of $\mathfrak{P}$ in real directions; (4.2) is more than sufficient. The growth estimates can be equally used in the theory based on the *real integration* instead of the Jackson summation. This theory is a q -generalization of the so-called non-compact case in the harmonic analysis on the symmetric spaces. Formulas like (4.22) hold in such theory but the corresponding factors of proportionality (generally, periodic functions in terms of X

and Λ) are not calculated so far with a reservation about the A_1 -case (see [C8]).

There is a special case when (4.22) becomes a straightforward *algebraic* exercise; it occurs for $\xi = -\rho_k$ taken as the starting point of the Jackson summation. In this case, $\mu_\bullet(q^{w(\xi)+b})$ is nonzero if and only if $bw = \pi_b = bu_b^{-1}$, i.e., at $b_\# = \pi_b((- \rho_k)) = b - u_b^{-1}(\rho_k)$ in the notations from Proposition 1.1. One has:

$$(4.24) \quad \mu_\bullet(q^{b_\#}) = q^{2(b_-, \rho_k)} \prod_{\nu} t_{\nu}^{l_{\nu}(u_b)} \prod_{[\alpha, j] \in \lambda'(\pi_b)} \left(\frac{1 - t_{\alpha} q_{\alpha}^{(\alpha^{\vee}, \rho_k) + j}}{1 - t_{\alpha}^{-1} q_{\alpha}^{(\alpha^{\vee}, \rho_k) + j}} \right),$$

where $\lambda'(\pi_b) = \{ [\alpha, j] \mid [-\alpha, \nu_{\alpha} j] \in \lambda(\pi_b) \}$. Then $\langle \gamma \rangle_{\xi} = \langle \gamma \rangle_{\rho_k}$ and

$$(4.25) \quad \langle \gamma \mu_{\bullet} \rangle_{-\rho_k} = |W|^{-1} \langle \gamma \rangle_{\rho_k} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q_{\alpha}^{(\rho_k, \alpha^{\vee}) + j}}{1 - t_{\alpha}^{-1} q_{\alpha}^{(\rho_k, \alpha^{\vee}) + j}} \right).$$

Formula (4.23) reads as follows:

$$(4.26) \quad \begin{aligned} & |W| (\tilde{\gamma}^{\ominus}(q^{\rho_k}))^2 \langle \mathfrak{P}^{\circ}(X, \Lambda) \mathfrak{P}^{\circ}(X^{-1}, \Lambda') \gamma \mu_{\bullet} \rangle_{-\rho_k} \\ &= \langle \gamma \rangle_{\rho_k} \tilde{\gamma}_{\lambda}^{\ominus} \tilde{\gamma}_{\lambda'}^{\ominus} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q_{\alpha}^{(\rho_k, \alpha^{\vee}) + j}}{1 - t_{\alpha}^{-1} q_{\alpha}^{(\rho_k, \alpha^{\vee}) + j}} \right)^2 \mathfrak{P}^{\circ}(\Lambda, \Lambda'). \end{aligned}$$

It is important to note that (4.26) is not a *new* identity. It formally results from the definition of $\mathfrak{P}^{\circ}$ and the duality of the P -polynomials if the Shintani-type relations from (2.22) are supposed to be known. Thus, we can say that the special case trivializes the Jackson integrals.

Actually, relations (2.22) were deduced in [C5] from the general ξ -theory of Jackson integration, so the *special case* is analogical to the normalization condition in the theory of spherical functions only with some reservations.

4.6. Taking the limit. Let us interpret the identity (4.26) upon the Whittaker limit. The Jackson summation will be now over B : $\langle f \rangle_{\diamond} \stackrel{\text{def}}{=} \sum_{b \in B} f(q^b)$; notice that there is no $|W|$ -factor versus the previous definition. For instance, $\langle \gamma \rangle_{\diamond} = \tilde{\gamma}^{\ominus}(1)$. The corresponding $\bar{\mu}$ -measure is nonzero only on B_+ :

$$(4.27) \quad \bar{\mu}_{\diamond}(q^{b_+}) = \prod_{i=1}^n \prod_{j=1}^{(\alpha_i^{\vee}, b_+)} (1 - q_i^j)^{-1}.$$

We come to the following “reformulation” of the definition of $\tilde{\mathfrak{P}}^\circ$:

$$\begin{aligned}
 & \langle \gamma \rangle_\diamond \langle q^{(x,\lambda)} \tilde{\mathfrak{P}}^\circ(X^{-1}, \Lambda') \gamma \bar{\mu}_\diamond \rangle_\diamond \\
 (4.28) \quad &= \tilde{\gamma}_\lambda^\ominus \tilde{\gamma}_{\lambda'}^\ominus \prod_{i=1}^n \prod_{j=1}^\infty \left(\frac{1}{1 - q_i^j} \right) \tilde{\mathfrak{P}}^\circ(\Lambda, \Lambda').
 \end{aligned}$$

Here the Shintani-type formulas were employed.

It is instructional to obtain (4.28) as a Whittaker-type limit of (4.26). We suggest the following way.

First, let us make k a positive *integer*; to be exact, $k = N$ for $N = \{N_\nu \in \mathbb{N}\}$. Then $\rho_N \in P_+$ and, for instance, $\langle \gamma \rangle_{\lambda + \rho_N} = \langle \gamma \rangle_\lambda$, which will be used constantly. Second, let us renormalize the μ -measure (4.24):

$$\begin{aligned}
 & \tilde{\mu}_\bullet(q^{b_\#}) = q^{-2(b_-, \rho_k)} \mu_\bullet(q^{b_\#}) \\
 (4.29) \quad &= \prod_\nu t_\nu^{l_\nu(u_b)} \prod_{[\alpha, j] \in \lambda'(\pi_b)} \left(\frac{1 - t_\alpha q_\alpha^{(\alpha^\vee, \rho_k) + j}}{1 - t_\alpha^{-1} q_\alpha^{(\alpha^\vee, \rho_k) + j}} \right).
 \end{aligned}$$

The limit of $\tilde{\mu}_\bullet(q^{b_\#})$ as $N \rightarrow \infty$ exists for any $b \in B$ and is nonzero only for $b = b_-$. Namely,

$$\lim_{N \rightarrow \infty} \tilde{\mu}_\bullet(q^{b_-}) = \bar{\mu}_\diamond(q^{b_+}).$$

Third, we will use the following property of the *spherical* polynomials:

$$(4.30) \quad \lim_{N \rightarrow \infty} \mathcal{P}_b(q^{\lambda - \rho_N}) = q^{(\lambda, b_+)} \quad \text{for } b = b_-,$$

which is a reformulation of (3.5).

Forth, we observe that the condition $\rho_N \in B$ guarantees that

$$\langle \gamma \rangle_{x - \rho_N} = \sum_b q^{(b + x - \rho_N, b + x - \rho_N)/2} = \langle \gamma \rangle_x.$$

for any x . Therefore the limiting procedure for obtaining $\tilde{\mathfrak{P}}^\circ$ from $\mathfrak{P}^\circ$ from (3.12) *coincides* with that for $\tilde{\mathfrak{P}}$ from (3.13):

$$(4.31) \quad \tilde{\mathfrak{P}}^\circ(X, \Lambda) = \lim_{N \rightarrow \infty} q^{(x, \rho_N)} \mathfrak{P}^\circ(q^{-\rho_N} X, \Lambda).$$

Replacing now λ by $\lambda - \rho_N$ in (4.26), one obtains:

$$(4.32) \quad \langle \gamma \rangle_{\diamond}^{-1} (\tilde{\gamma}^{\ominus}(q^{\rho_N}))^2 \langle \mathfrak{P}^{\circ}(X, q^{\lambda - \rho_N}) \mathfrak{P}^{\circ}(X^{-1}, \Lambda') \gamma \mu_{\bullet} \rangle_{-\rho_N} \\ = \{q^{-\frac{(\lambda - \rho_N)^2}{2}} \langle \gamma \rangle_{\lambda}\} \tilde{\gamma}_{\lambda'}^{\ominus} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q_{\alpha}^{(\rho_N, \alpha^{\vee}) + j}}{1 - t_{\alpha}^{-1} q_{\alpha}^{(\rho_N, \alpha^{\vee}) + j}} \right)^2 \mathfrak{P}^{\circ}(q^{-(\lambda - \rho_N)}, \Lambda').$$

Here $|W|$ is not present due to our definition of the Jackson summation in the Whittaker case.

We can restrict ourselves only with $b = b_-$, since the other b appear in (4.29) with strictly positive t -factors $\prod_{\nu} t_{\nu}^{l_{\nu}(u_b)}$. Then the left-hand side of (4.32) *modulo higher powers of t* is as follows: $LHS \bmod(t) =$

$$Q \sum_{b \in B_-} q^{\frac{(b_-)^2}{2}} \tilde{\mu}_{\bullet}(b_-) \mathfrak{P}^{\circ}(q^{b_- - \rho_N}, q^{\lambda - \rho_N}) \{q^{(b_-, \rho_N)} \mathfrak{P}^{\circ}(q^{-b_+ - \rho_N}, \Lambda')\} \\ = Q \sum_{b \in B_-} q^{\frac{(b_-)^2}{2}} \tilde{\mu}_{\bullet}(b_-) \{ \mathcal{P}(q^{\lambda - \rho_N}) \Pi \} \{q^{(-b_+, \rho_N)} \mathfrak{P}^{\circ}(q^{-b_+ - \rho_N}, \Lambda')\} \\ \text{for } Q \stackrel{\text{def}}{=} \{q^{\rho_N^2/2} \tilde{\gamma}^{\ominus}(q^{\rho_N}) / \langle \gamma \rangle_{\rho_N}\} \tilde{\gamma}^{\ominus}(q^{\rho_N}) = \tilde{\gamma}^{\ominus}(q^{\rho_N}) \\ \text{and } \Pi \stackrel{\text{def}}{=} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \frac{1 - q_{\alpha}^{(\rho_N, \alpha^{\vee}) + j}}{1 - t_{\alpha}^{-1} q_{\alpha}^{(\rho_N, \alpha^{\vee}) + j}}.$$

Transforming correspondingly the right-hand side of (4.32), one arrives at

$$RHS = q^{-\frac{\rho_N^2}{2}} \{q^{-\frac{\lambda^2}{2}} \langle \gamma \rangle_{\lambda}\} \tilde{\gamma}_{\lambda'}^{\ominus} \Pi^2 \{q^{(\lambda, \rho_N)} \mathfrak{P}^{\circ}(q^{\lambda - \rho_N}, \Lambda')\}.$$

The term $q^{-\rho_N^2/2}$ can be moved to the LHS and combined with Q , namely,

$$q^{\rho_N^2/2} \tilde{\gamma}^{\ominus}(q^{\rho_N}) = \langle \gamma \rangle_{\rho_N}.$$

One Π can be reduced in the $LHS \bmod(t)$ and the RHS.

Then we use (4.30) for $\mathcal{P}(q^{\lambda - \rho_N})$ and the definition of the Whittaker limit (4.31) for $\mathfrak{P}^{\circ}(q^{-b_+ - \rho_N}, \Lambda')$ and for $\mathfrak{P}^{\circ}(q^{\lambda - \rho_N}, \Lambda')$. Replacing (back) $q^{-\frac{\lambda^2}{2}} \langle \gamma \rangle_{\lambda}$ by $\tilde{\gamma}^{\ominus}(\lambda)$ and changing the summation set in the LHS from B_- to B_+ , we eventually obtain (4.28).

This calculation is expected to be a sample for the general ξ -Jackson integration theory in the Whittaker case (presumably, for the real integration too); it will be discussed elsewhere. We note that the term $q^{(x, \lambda)}$ in the integrand of (4.28) can be naturally combined with $\gamma = q^{x^2/2}$

and “eliminated” upon the change of variables $x + \lambda \mapsto x$. However this substitution will change the summation set from B_+ to $\lambda + B_+$, i.e., the general Jackson summation (with an arbitrary starting vector) naturally emerges even in the special case under consideration.

The extreme case. There is no “natural” way to eliminate Λ, Λ' from (4.28) by evaluating this formula at certain special points. Generally, it may lead to interesting identities that contain only q . However, one can try to take $\lambda \sim \lambda'$ and tend them to infinity for $\Lambda = q^\lambda, \Lambda' = q^{\lambda'}$. Let us perform this calculation in detail.

We will use (4.9):

$$(4.33) \quad \lim_{\mathbf{n} \rightarrow \infty} \tilde{\Psi}(\lambda + (\mathbf{n} \cdot \mathbf{y}')_+, \lambda' + \mathbf{n} \cdot \mathbf{y}'; q) \\ = \sum_{b \in B_+} q^{\frac{(b,b)}{2}} \frac{q^{\lambda_b - \lambda'_b}}{\prod_{i=1}^n \prod_{j=1}^{(\alpha_i^\vee, b)} (1 - q_i^j)},$$

where $(\mathbf{n} \cdot \mathbf{y}')_+ \in \mathfrak{C}_+$. In this limit, formula (4.28) reads as:

$$(4.34) \quad \langle \gamma \rangle_\diamond \sum_{b \in B_+} q^{(b, \lambda)} \frac{\tilde{\gamma}^\ominus(q^{\lambda'_+ + b})}{\tilde{\gamma}^\ominus(q^{\lambda'_+}) \tilde{\gamma}^\ominus(q^b)} \gamma(q^b) \bar{\mu}_\diamond(q^b) \\ = \sum_{b \in B_+} q^{\frac{(b,b)}{2}} \frac{q^{\lambda_b - \lambda'_b}}{\prod_{i=1}^n \prod_{j=1}^{(\alpha_i^\vee, b)} (1 - q_i^j)},$$

where we canceled out $\langle \bar{\mu} \rangle = \prod_{i=1}^n \prod_{j=1}^\infty (1 - q_i^j)^{-1}$ in both sides. Moving b from the arguments of $\tilde{\gamma}^\ominus$ and using that $\langle \gamma \rangle_\diamond = \tilde{\gamma}^\ominus(1)$, we come to an *identical equality*. No new formulas appear in this way.

Discussion. We think that the growth estimates and formula (4.28) show great potential of the q -theory of Whittaker functions in harmonic analysis. For instance, an immediate interpretation of (4.28) is the fact that the global q -Whittaker function multiplied by the Gaussian is essentially *invariant* with respect to the q -Fourier-Jackson transform for the measure $\bar{\mu}_\diamond$ from (4.27), which is very much standard in the theory of q -functions.

This paper seems a convincing demonstration of the key role of Shintani-type formulas in the algebraic and analytic theory of spherical and Whittaker functions. Even a q -generalization of the Harish-Chandra asymptotic formula, which, naturally, is a cornerstone of the analytic theory, can be deduced from these formulas. This is different

from the differential setting and makes the q -theory significantly more algebraic than the classical harmonic analysis on the symmetric spaces.

We would like to mention that global spherical and Whittaker functions have properties similar to celebrated Ramanujan's mock theta functions, including the theory at $|q| = 1$. For instance, certain (but not direct) counterparts of Maas forms can be introduced following the same lines. To be more exact, the natural objects are Maas-type *theta functions*; they are not holomorphic in terms of x, λ but satisfy the modular equation with respect to q .

Also, the q, t -theory has connections with the mock theta functions at level of formulas. It is not very surprising because the *basic* hypergeometric function is known to be related to (some) mock functions. Our global spherical functions are its multi-variable generalizations.

REFERENCES

- [ASI] W.A. Al-Salam, and M.E.H. Ismail, *q-Beta integrals and q-Hermite polynomials*, Pacific Journal of Mathematics, 135:2 (1988), 209–221. [28](#)
- [AI] R. Askey, and M.E.H. Ismail, *A generalization of ultraspherical polynomials*, in *Studies in Pure Mathematics*, Ed. P. Erdős, Birkhäuser, Boston (1983), 55–78. [30](#)
- [B] N. Bourbaki, *Groupes et algèbres de Lie*, Ch. 4–6, Hermann, Paris (1969) [9](#), [11](#)
- [BF] A. Braverman, and M. Finkelberg, *Finite-difference quantum Toda lattice via equivariant K-theory*, Transformation Groups 10 (2005), 363–386. [3](#), [8](#)
- [CS] W. Casselman, and J. Shalika, *The unramified principal series of p-adic groups, II. The Whittaker function*, Comp. Math. 41 (1980), 207–231. [3](#), [25](#)
- [C1] —, I. Cherednik, *Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators*, IMRN 9 (1992), 171–180. [18](#)
- [C2] —, *Double affine Hecke algebras and Macdonald's conjectures*, Annals of Mathematics 141 (1995), 191–216. [12](#), [17](#), [36](#), [37](#)
- [C3] —, *Macdonald's evaluation conjectures and difference Fourier transform*, Inventiones Math. 122 (1995), 119–145. [6](#), [20](#)
- [C4] —, *Nonsymmetric Macdonald polynomials*, IMRN 10 (1995), 483–515. [11](#), [13](#), [17](#), [18](#), [20](#)
- [C5] —, *Difference Macdonald-Mehta conjecture*, IMRN 10 (1997), 449–467. [3](#), [6](#), [22](#), [23](#), [24](#), [31](#), [41](#), [43](#)

- [C6] — , *Intertwining operators of double affine Hecke algebras*, Selecta Math. New ser. 3 (1997), 459–495 [19](#), [22](#)
- [C7] — , *Double affine Hecke algebras and difference Fourier transforms*, Inventiones Math. 152 (2003), 213–303. [15](#), [18](#)
- [C8] — , *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, Cambridge, 2006. [4](#), [11](#), [13](#), [18](#), [19](#), [20](#), [22](#), [23](#), [28](#), [30](#), [36](#), [41](#), [43](#)
- [C9] — , *Non-semisimple Macdonald polynomials*, Preprint arXiv:0709.1742 [math.QA]; Selecta Mathematica (2008). [4](#), [8](#), [33](#)
- [vD] J.F. van Diejen, *An asymptotic formula for the Koornwinder polynomials*, Journal of Computational and Applied Mathematics, 178 (2005), 465–471. [37](#)
- [Et] P. Etingof, *Whittaker functions on quantum groups and q -deformed Toda operators*, AMS Transl. Ser. 2, 194, 9–25, AMS, Providence, Rhode Island, 1999. [27](#), [32](#)
- [FZ] P. Freund, and A. Zabrodin, *Z_n -Baxter models and quantum symmetric spaces*, Physics Letters B 284 (1992), 283–288. [37](#)
- [GLO1] A. Gerasimov, and D. Lebedev, and S. Oblazin, *On q -deformed $\mathfrak{gl}_{l+1}$ -Whittaker functions, I*, Preprint arXiv: 0803.0145 [math.RT]. [3](#), [5](#), [8](#), [27](#)
- [GLO2] *On q -deformed $\mathfrak{gl}_{l+1}$ -Whittaker functions, III*, Preprint arXiv: 0805.3754 [math.RT]. [3](#), [5](#), [6](#), [8](#), [32](#)
- [GiL] A. Givental, and Y.-P. Lee, *Quantum K -theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Inventiones Math., 151 (2003), 193–219. [3](#), [8](#)
- [HC] Harish-Chandra, *Discrete series for semisimple Lie groups, II.*, Acta Mathematica 116 (1963), 1–111. [5](#), [35](#), [38](#)
- [HJ] S. Helgason, and K. Johnson, *The bounded spherical functions on symmetric spaces*, Advances Math. 3 (1969), 586–593. [38](#)
- [Hu] J. Humphreys, *Reflection groups and Coxeter Groups*, Cambridge University Press (1990). [11](#)
- [Ion1] B. Ion, *Nonsymmetric Macdonald polynomials and Demazure characters*, Duke Mathematical Journal 116:2 (2003), 299–318. [7](#), [8](#), [22](#)
- [Ion2] B. Ion, *A weight multiplicity formula for Demazure modules*, Preprint arXiv:math/0409218 [math.RT]. [3](#)
- [Is] M.E.H. Ismail, *Asymptotics of the Askey-Wilson and q -Jacobi polynomials*, SIAM J. Math. Anal. 17 (1986), 1475–1482. [37](#)
- [KL] D. Kazhdan, and G. Lusztig, *Tensor structures arising from affine Lie algebras. III*, J. of AMS 7 (1994), 335–381. [3](#)
- [KS] F. Knop, and S. Sahi, *A recursion and a combinatorial formula for Jack polynomials*, Inventiones Math., 128:1 (1997), 9–22. [22](#)
- [L] G. Lusztig, *Affine Hecke algebras and their graded version*, J. of the AMS 2:3 (1989), 599–635. [14](#)
- [M1] I. Macdonald, *Orthogonal polynomials associated with root systems*, Preprint (1988). [18](#), [36](#)

- [M2] — , *A new class of symmetric functions*, Publ I.R.M.A., Strasbourg, Actes 20-e Seminaire Lotharingen, (1988), 131–171 . [18](#), [36](#)
- [M3] — , *Affine Hecke algebras and orthogonal polynomials*, Séminaire Bourbaki 47:797 (1995), 01–18. [12](#), [18](#)
- [OS] S. Odakea, and R. Sasaki, *q-oscillator from the q-Hermite Polynomial*, Preprint arXiv:0710.2209v2 [hep-th]. [30](#)
- [Op] E. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. 175 (1995), 75–121. [12](#), [17](#), [18](#)
- [San] Y. Sanderson, *On the Connection Between Macdonald Polynomials and Demazure Characters*, J. of Algebraic Combinatorics, 11 (2000), 269–275. [7](#), [22](#)
- [Shi] T. Shintani, *On an explicit formula for class 1 Whittaker functions on GL_n over p-adic fields*, Proc. Japan Acad. 52 (1976), 180–182. [3](#), [25](#)
- [Sto] J. Stokman, *Difference Fourier transforms for nonreduced root systems*, Preprint arXiv:math/0111221 [math.QA]. [3](#), [6](#)
- [Sus] S. Suslov, *Another addition theorem for the q-exponential function*, J. Phys. A: Math. Gen. 33: 41 (2000) L375-L380. [30](#)

(I. Cherednik) DEPARTMENT OF MATHEMATICS, UNC CHAPEL HILL, NORTH CAROLINA 27599, USA, CHERED@EMAIL.UNC.EDU