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SPECIAL K-TYPES, TEMPERED CHARACTERS AND THE BEILINSON–BERNSTEIN REALIZATION

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§1. Introduction. Let $\mathscr{G}$ be a connected linear semisimple Lie group (the precise assumptions on $\mathscr{G}$ will be described in §2) and $\mathscr{K}$ be a maximal compact subgroup of $\mathscr{G}$. As an invariant attached to representations of $\mathscr{G}$, Vogan introduced in [25] a notion of "lowest" $\mathscr{K}$-types and used it to give a classification of irreducible admissible representations for $\mathscr{G}$. In this theory, roughly speaking, an irreducible representation is specified in terms of the lowest $\mathscr{K}$-types and is realized as some subquotient of a certain induced representation. In contrast to Langlands–Knapp–Zuckerman's classification ([16], [17]), Vogan's theory is technically more algebraic. Using the language of $\mathscr{D}$-module, Beilinson and Bernstein introduced in [3] a geometric theory for general modules over $\mathfrak{g}$, the complexified Lie algebra of $\mathscr{G}$, in which one can "localize" a $\mathfrak{g}$-module to a certain sheaf of $\mathscr{D}$-module on the flag variety X associated to $\mathfrak{g}$. A $\mathfrak{g}$-module can thus be realized as the space of global sections of a certain $\mathscr{D}$-module on X. From the geometric point of view, this construction is most natural and most simple. In this paper, we study $\mathscr{K}$-types of an arbitrary induced standard Harish–Chandra module (i.e., $(\mathfrak{g}, \mathscr{K})$-module) via Beilinson–Bernstein's construction. Our goal is to search for certain "special" $\mathscr{K}$-types from the geometric point of view and to put Vogan's classification into the more geometric context.

To be more precise, we fix a complexification $K \subset G$ for the pair $\mathscr{K} \subset \mathscr{G}$. K acts on the flag variety X. In the Beilinson–Bernstein theory, a quasisimple Harish–Chandra module is localized to a $(\mathscr{D}_\lambda, K)$-module on X for some dominant linear form λ on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; here $\mathscr{D}_\lambda$ is the twisted sheaf of differential operators (t.d.o. for short) on X parametrized by λ. On the other hand, to each K-orbit Q and a λ-compatible connection τ on Q, there is an associated standard module $\mathscr{I}_{Q,\tau,\lambda}$ which contains a unique irreducible submodule denoted by $\mathscr{L}_{Q,\tau,\lambda}$. We have $\Gamma(\mathscr{L}_{Q,\tau,\lambda}) \subset \Gamma(\mathscr{I}_{Q,\tau,\lambda})$. The nontrivial $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$'s exhaust all the irreducible Harish–Chandra modules. With these, we can phrase our goal more precisely as: From a geometric point of view, find a certain set of "special" K-types in $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ which will "locate" the irreducible submodule $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ (when nontrivial), and find explicit formulae for these K-types. Of course, these special K-types will be nothing but the lowest K-types in Vogan's sense.

Our results can be best explained by our methods. In the first part, we establish the following:

(a) A criterion for $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ to be nontrivial (Theorem 3.15; this is an unpublished result of Beilinson and Bernstein).

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(b) Construction of intertwining operators: for each simple root α, there is an explicit intertwining operator between standard modules associated to two α-adjacent K-orbits (with appropriate connections), which respects the transition of K-types.

(c) Duality: there are two involutions θ_1 and θ on the set of data $(Q, \tau; \lambda)$'s such that

(1) $\Gamma(\mathscr{L}_{\theta_1(Q,\tau,\lambda)})$ is the contragredient dual to $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$;
(2) $\Gamma(\mathscr{I}_{\theta(Q,\tau,\lambda)})$ has a unique irreducible quotient given by $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ (when nontrivial); θ is the Cartan involution given by K.

Intertwining functors in the context of $\mathscr{D}$-modules were first introduced by Beilinson and Bernstein in [2]. They were applied to the study of standard induced modules with different polarizations by Hecht, Miličić, Schmid, and Wolf in the second part of [12], which also contains the results of Propositions 5.9 and 5.10 concerning the intertwining operators in (b) above. The author learned the idea of using intertwining functors to study K-types from Hecht, Miličić, Schmid, and Wolf. In geometric terms, the intertwining functor with respect to the simple root α amounts to studying $\mathscr{D}_\lambda$-modules under the fibration $\pi\colon X \to X_\alpha$, the generalized flag variety consisting of parabolic subalgebras of type α. In our approach, for the purpose of studying K-types, we introduce a generalized t.d.o. $\tilde{\mathscr{D}}_\lambda$ (which is $\pi_0\mathscr{D}_\lambda$) on X_α and establish the general theory for $\tilde{\mathscr{D}}_\lambda$-modules on X_α. This constitutes the main technical part of this paper. As additional tools, we also need Harish-Chandra's regularity theorem on eigendistributions.

The second part of this paper consists of the theory of special K-types and the classification of irreducible tempered Harish-Chandra modules. Special K-types of a standard module $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ are defined as K-types consisting of sections of $\mathscr{I}_{Q,\tau,\lambda}$ on Q satisfying certain geometric restrictions: When $\mathscr{I}_{Q,\tau,\lambda}$ is interpreted as a sheaf of certain "algebraic distributions" on X supported on $\bar{Q}$, these sections do not involve normal derivatives and should grow as slowly as possible toward the boundary of Q (in other words, they should be as "flat" as possible on Q). In case Q is closed and G contains a θ-fixed Cartan subgroup, these correspond to the lowest K-types of (limit of) discrete-series representations (Schmid [23]). In case Q is open and G is split, these correspond to fine K-types of principal-series representations (Bernstein-Gelfand-Gelfand [5]). The general case is a mixture of these two.

We now come back to our goal described earlier. Using (b), we can form an intertwining operator I from $\Gamma(\mathscr{I}_{\theta(Q,\tau,\lambda)})$ to $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$. In case I is nontrivial, $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ (when nontrivial) can then be realized as the image of I. Therefore, whether a K-type in $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ actually lies in $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ can be examined via I, i.e., via SL(2) situations. The notion of special K-types fits the situation in an inborn fashion. In other words, if special K-types exist, they give the nontriviality of the map I and they survive under I more or less by definition. This comes down to the existence of special K-types and the explicit formulae for these

K-types. Here the crucial result is due to Vogan [25]. Using the Borel-Weil-Bott theorem, we get a set of explicit weights as candidates for the special K-types. The existence statement then amounts to establishing the dominance of (some of) these special weights, which can be viewed as a (complicated) combinatorial problem. In this setting, the computation needed to justify the dominance statement has been done by A. W. Knapp in [13]. It is in this sense that our goal is achieved. The deep result that the special K-types are the lowest K-types in Vogan's sense, stated in different form, is due to Vogan [25].

Our last topic is the classification of tempered Harish-Chandra modules. Using the nonvanishing criterion in (a) above and the general theory of intertwining operators (b), together with the identification of certain standard modules with the (real parabolically) induced Harish-Chandra modules (this is a special case of the more general results due to Bernstein (see [7]) and Hecht-Miličić-Schmid-Wolf [12]), we are able to give a different proof (and formulation) of the classification of irreducible tempered Harish-Chandra modules due to Knapp and Zuckerman [16]. Moreover, our method and results extend to the case when $\mathscr{G}$ is of Harish-Chandra's class. Independently, I. Mirković [21] also gave a classification of tempered characters in the context of $\mathscr{D}$-modules.

As for the organization of this paper, sections 2 and 3 recall basic results of the Beilinson-Bernstein theory. Section 4 is devoted to the study of $\tilde{\mathscr{D}}$-modules on a generalized flag variety. The construction of intertwining operators takes place in section 5. Section 6 is on the geometry of the closure of a K-orbit, which will be needed in the description of special K-types. The duality theorems are proved in section 7. The theory of special K-types is treated in section 8, and section 9 is on the classification of tempered Harish-Chandra modules. The appendix contains the proof of the criterion (a) due to Beilinson and Bernstein.

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Notations.

1. For a real (resp. complex) Lie group, say $\mathscr{G}$ (resp. G), we use the corresponding lowercase German letter with (resp. without) subscript 0, $\mathfrak{g}_0$ (resp. $\mathfrak{g}$), to denote its Lie algebra.
2. For a Lie algebra $\mathfrak{g}$, $U(\mathfrak{g})$ stands for the universal enveloping algebra of $\mathfrak{g}$ and $Z(\mathfrak{g})$ stands for the center of $U(\mathfrak{g})$.

3. Let X be a smooth variety. We denote by $\mathcal{O}_X$ the structure sheaf, by Ω_X the top exterior power of the cotangent sheaf, by $\mathcal{D}_X$ the sheaf of differential operators, and by $\Gamma(X, \cdot)$ the global section functor. When no confusion could occur, we omit the subscript X.
4. Given a morphism $\pi: Y \to X$, we denote by π_* (resp. π_0) the direct image functor in category of $\mathcal{D}$ (resp. $\mathcal{O}$) modules. If π is an embedding, we denote by $\Lambda^t\mathcal{N}_{Y|X}$ the top exterior power of the normal sheaf.

§2. Localization of $\mathfrak{g}$-modules. In this section, we recall the theory of localization of $\mathfrak{g}$-modules introduced by Beilinson and Bernstein [3]. Let G be a connected complex reductive algebraic group with Lie algebra $\mathfrak{g}$. Denote by X the flag variety. For $x \in X$, we denote by B_x ($\mathfrak{b}_x$) the corresponding Borel subgroup (subalgebra), by N_x ($\mathfrak{n}_x$) its unipotent (nilpotent) radical, and by $H_x = B_x/N_x$ ($\mathfrak{h}_x = \mathfrak{b}_x/\mathfrak{n}_x$) the Cartan subgroup (subalgebra). We denote by $R_x \subset \mathfrak{h}_x^*$ the root system of $\mathfrak{h}_x$ in $\mathfrak{g}$ and by R_x^+ the set of roots of $\mathfrak{h}_x$ in $\mathfrak{g}/\mathfrak{b}_x$. All the triples $(\mathfrak{h}_x, R_x, R_x^+)$ are canonically isomorphic. We will use an abstract triple $(\mathfrak{h}, R, R^+)$ to identify these triples. Denote by ρ the half sum of roots in R^+ and by W the Weyl group.

Consider the $\mathcal{O}_X$-module $\mathfrak{g}^0 = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$, which has a natural structure of Lie algebra. Put

$$\mathfrak{b}^0 = \{ f \in \mathfrak{g}^0 | f(x) \in \mathfrak{b}_x \quad \text{for all } x \in X \},$$

$$\mathfrak{n}^0 = \{ f \in \mathfrak{g}^0 | f(x) \in \mathfrak{n}_x \quad \text{for all } x \in X \},$$

and $\mathfrak{h}^0 = \mathfrak{b}^0/\mathfrak{n}^0$. These are $\mathcal{O}_X$ modules and $\mathfrak{h}^0 \simeq \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{h}$. Write $\tau: \mathfrak{g} \to TX$ for the action derived from G action on X. Let U^0 be the sheaf of algebras generated by $\mathcal{O}_X$ and $\mathfrak{g}$ with natural relation $[A, f] = \tau(A)f$ for $A \in \mathfrak{g}$, $f \in \mathcal{O}_X$. As $\mathcal{O}_X$-module, $U^0 \simeq \mathcal{O}_X \otimes_{\mathbb{C}} U(\mathfrak{g})$. Since G acts on $\mathfrak{n}^0$, $\mathfrak{n}^0U^0$ is two-sided ideal in U^0. Set $\mathcal{D}_{\mathfrak{h}} = U^0/\mathfrak{n}^0U^0$. The image of $\mathfrak{h}$ under the inclusion $\mathfrak{h} \to \mathfrak{h}^0 \to \mathcal{D}_{\mathfrak{h}}$ lies in the center of $\mathcal{D}_{\mathfrak{h}}$. For each $\lambda \in \mathfrak{h}^*$, set $I_\lambda = \operatorname{Ker}\{U(\mathfrak{h}) \xrightarrow{\lambda-\rho} \mathbb{C}\}$. Then $I_\lambda\mathcal{D}_{\mathfrak{h}}$ is a two-sided ideal of $\mathcal{D}_{\mathfrak{h}}$; define $\mathcal{D}_\lambda = \mathcal{D}_{\mathfrak{h}}/I_\lambda\mathcal{D}_{\mathfrak{h}}$.

LEMMA 2.1 (Beilinson–Bernstein [3], Borho–Brylinski [10]).

(a) *$i: \mathcal{O}_X \to \mathcal{D}_\lambda$ is locally isomorphic to $i: \mathcal{O}_X \to \mathcal{D}_X$.*
(b) *Let $I'_\lambda = \operatorname{Ker}\{Z(\mathfrak{g}) \to U^0 \to \mathcal{D}_\lambda\}$. Then $\lambda \to I'_\lambda$ is the Harish-Chandra homomorphism.*
(c) *The natural map $U(\mathfrak{g}) \to \mathcal{D}_\lambda$ induces an isomorphism $U(\mathfrak{g})/I'_\lambda U(\mathfrak{g}) \simeq \Gamma(X, \mathcal{D}_\lambda)$.*
(d) *$H^i(X, \mathcal{D}_\lambda) = 0$ for $i > 0$.*

A simple proof of (d) by Miličić and Taylor is in [20]. In view of (a), $\mathcal{D}_\lambda$ is called a twisted sheaf of differential operators (t.d.o. for short). Note that in case $\lambda - \rho \in \mathfrak{h}^*$ is integral, $\mathcal{D}_\lambda = \operatorname{diff}(\mathcal{O}(\lambda - \rho))$, the sheaf of differential operators in the invertible sheaf $\mathcal{O}(\lambda - \rho)$.

Let $\mathcal{M}(\mathcal{D}_\lambda)$ be the category of sheaves of $\mathcal{D}_\lambda$-modules which are quasicoherent as $\mathcal{O}_X$-modules and $\mathcal{M}(U_\lambda)$ be the category of $U_\lambda = U(\mathfrak{g})/I'_\lambda U(\mathfrak{g})$-modules, i.e., the category of $U(\mathfrak{g})$-modules with infinitesimal character given by I'_λ. We have

$$\Gamma: \mathcal{M}(\mathcal{D}_\lambda) \to \mathcal{M}(U_\lambda) \quad \text{by Lemma 2.1(c), and}$$

$$\Delta_\lambda: \mathcal{M}(U_\lambda) \to \mathcal{M}(\mathcal{D}_\lambda) \quad \text{by } V \to \Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{U_\lambda} V.$$

THEOREM 2.2 (Beilinson–Bernstein [3]).

(a) *$\operatorname{Hom}_{U_\lambda}(V, \Gamma(\mathcal{W})) = \operatorname{Hom}_{\mathcal{D}_\lambda}(\Delta_\lambda(V), \mathcal{W})$ for $V \in \mathcal{M}(U_\lambda)$ and $\mathcal{W} \in \mathcal{M}(\mathcal{D}_\lambda)$ i.e., Γ and Δ_λ are adjoint.*
(b) *If $\langle \lambda, \check{\alpha} \rangle \neq -1, -2, \ldots$ for all $\alpha \in R^+$, then Γ is exact and $V \to \Gamma\Delta_\lambda(V)$ is an isomorphism for $V \in \mathcal{M}(U_\lambda)$.*
(c) *If $\langle \lambda, \check{\alpha} \rangle \neq 0, -1, -2, \ldots$ for all $\alpha \in R^+$, then Γ is an equivalence of categories and Δ_λ is its inverse.*

The condition on λ in (b) is sometimes called dominance. For our purpose, we need a stronger version. We define a complex structure on $\mathfrak{h}^*$ by choosing the real part to be the $\mathbb{R}$-spans of roots in R^+, which decides a partial order on $\mathfrak{h}^*$. In $\mathfrak{h}^*$, λ is called *R^+-dominant* if $\lambda \geqslant 0$ in the above sense. By Lemma 2.1(b), the basic property of the Harish-Chandra homomorphism implies that $I'_\lambda = I'_{w\lambda}$ for $w \in W$. Therefore, we have

Remark 2.3. Every $\mathfrak{g}$-module with a fixed infinitesimal character can be realized as the space of global sections of some $\mathcal{D}_\lambda$-module with λ dominant.

Now we describe the general setting for this paper. We fix a connected complex reductive algebraic group G defined over $\mathbb{R}$ and consider the real Lie groups $\mathcal{G}$ in G, which has finite index in $G(\mathbb{R})$, the real points of G. This implies that the component group of any Cartan subgroup of $\mathcal{G}$ is finite and Abelian. We once and for all fix a maximal compact subgroup $\mathcal{K}$ of $\mathcal{G}$ and write $K \subset G$ for its complexification. Any finite-dimensional $\mathcal{K}$-module extends uniquely to an algebraic K-module. This is the setting used in Vogan [26]. We remark that all the results in this paper, with slight modification in the statements, hold for pairs $(\mathcal{G}, \mathcal{K})$ in the category of real reductive groups of Harish-Chandra class.

Our primary objects to study are representations of $\mathcal{G}$. Following Harish-Chandra, we consider the subcategory $\mathcal{M}(\mathfrak{g}, K)$ of $\mathcal{M}(\mathfrak{g})$ consisting of $\mathfrak{g}$-modules V with K-actions satisfying the following conditions:

(a) The morphism $\mathfrak{g} \times V \to V$ is a K-morphism.
(b) The action of $\mathfrak{g}$ and K are compatible, i.e., $\mathfrak{g}$ action restricted to $\mathfrak{k}$ coincides with the derivative of the K-action.
(c) Every vector $v \in V$ is K-finite, i.e., whose K-translate span a finite-dimensional subspace.

An object in $\mathscr{M}(\mathfrak{g}, K)$ is called a *Harish–Chandra module*. In an analogous manner, we consider the subcategory $\mathscr{M}(\mathscr{D}_\lambda, K)$ of $\mathscr{M}(\mathscr{D}_\lambda)$, consisting of $\mathscr{D}_\lambda$-modules V with K-actions such that

(a) the K-action is algebraic, i.e., V is a union of coherent $\mathscr{O}$-modules with algebraic actions of K;
(b) the action $\mathscr{D}_\lambda \otimes V \to V$ is K-equivariant; and
(c) the derivative of K-action coincides with the action of the image under $\mathfrak{k} \to TX \to \mathscr{D}_\lambda$.

We remark that K acts on X with a finite number of orbits (Wolf [28]) and the K-orbits are affinely embedded in X (this is due to Beilinson and Bernstein; see [12]).

§3. K-orbits on the flag variety. We begin with recalling the classification of irreducible $(\mathscr{D}_\lambda, K)$-modules given in [3]. Let Q be a K-orbit in X passing through $x \in X$. Then $Q = K/K_x$, where K_x is the isotropy subgroup of K at x. Write i for the inclusion $Q \hookrightarrow X$ and $\mathscr{D}_\lambda^i$ for the functorially defined t.d.o. on Q ([2]). From the definition we have

LEMMA 3.1. *Every irreducible object in $\mathscr{M}(\mathscr{D}_\lambda^i, K)$ is of the form* ind τ *for an irreducible character τ of K_x such that $d\tau = \lambda - \rho$ on* Lie(K_x).

Here ind τ is an invertible sheaf on Q induced by the character $\tau \in \hat{K}_x$. We will call such a triple $(Q, \tau; \lambda)$ a set of *Beilinson–Bernstein* data (*data* for short). It is convenient to write this as $(K \cdot x, \tau; \lambda)$, which specifies τ as a character of K_x; λ is understood as an element in $\mathfrak{h}^*$. Define $\mathscr{I}_{Q,\tau,\lambda} = i_*(\text{ind}\,\tau) \in \mathscr{M}(\mathscr{D}_\lambda, K)$. $\mathscr{I}_{Q,\tau,\lambda}$ and $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ will be referred to as *standard modules* associated to data $(Q, \tau; \lambda)$.

PROPOSITION 3.2 (Beilinson–Bernstein [3], [4]).

(a) *$\mathscr{I}_{Q,\tau,\lambda}$ contains a unique irreducible subsheaf $\mathscr{L}_{Q,\tau,\lambda}$ which is characterized by its support $\bar{Q}$ and the character τ.*
(b) *Every irreducible $(\mathscr{D}_\lambda, K)$-module arises in the above fashion.*

A triple $(Q, \tau; \lambda)$ is called *regular* if λ is dominant and $\Gamma(\mathscr{L}_{Q,\tau,\lambda}) \neq 0$. Then

PROPOSITION 3.3 (Beilinson–Bernstein [3]).

$$\{\textit{Irreducible Harish–Chandra modules with infinitesimal character } I'\} \overset{1-1}{\leftrightarrow} \{\textit{regular data with a fixed dominant } \lambda \textit{ such that } I_\lambda' = I'\}.$$

We will give a criterion, due to Beilinson and Bernstein, for a triple to be regular in terms of conditions on roots in the end of this section. Although the above results are in the context of the pair (K, X), they all remain valid in the context of pairs (N, X) with the property that the algebraic group N acts on X with a finite number of orbits. This remark will be used henceforth without further reference.

Write θ for the Cartan decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ given by the pair (G, K). The following proposition is taken from [26].

PROPOSITION 3.4 (Matsuki [19]).

(a) *Every θ-stable Cartan subalgebra of $\mathfrak{g}$ is conjugate by K to the complexification of a real θ-stable Cartan subalgebra.*
(b) *Every Borel subalgebra of $\mathfrak{g}$ contains a θ-stable Cartan subalgebra, whose K-conjugacy class is unique.*
(c) *If the Cartan subgroup of G, H is the complexification of a Cartan subgroup $\mathscr{H}$ of $\mathscr{G}$, then $(\mathscr{H} \cap \mathscr{K})/(\mathscr{H} \cap \mathscr{K})_0 \to (H \cap K)/(H \cap K)_0$ is an isomorphism.*

Therefore, from now on we may assume that, for each $x \in X$, $\mathfrak{h}_x$ in the triple $(\mathfrak{h}_x, R_x, R_x^+)$ to be θ-stable, also for H_x. Thus a character on K_x is then determined by its value on $K \cap H_x$. In the following, for an $\mathfrak{h}_x$-module A, $A_{(\mu)}$ denotes the generalized μ-eigenspace with respect to the $\mathfrak{h}_x$-action.

PROPOSITION 3.5. *For a set of data $(Q, \tau; \lambda)$, if λ is regular and dominant, then*

(a) $\bar{Q} = Cl(\{y \in X | H_i(\mathfrak{n}_y, \Gamma(\mathscr{L}_{Q,\tau,\lambda}))_{(\lambda-\rho)} \neq 0$ *for some* $i\})$, *and*
(b) *$H_x \cap K/(H_x \cap K)_0$ acts on $H_i(\mathfrak{n}_x, \Gamma(\mathscr{L}_{Q,\tau,\lambda}))_{(\lambda-\rho)}$ via the character τ for all i, $x \in Q$.*

Proof. In the following, write i_x for the inclusion of x into X and $L^p i_x^*$ for the pth left derived functor of i_x^*. By examining the geometric fiber of $\Delta_\lambda(V)$, in case λ is regular, there is a spectral sequence with E_2-term $L^p i_x^*(L^q \Delta_\lambda(v))$ converging to $H_*(\mathfrak{n}_x, V)_{(\lambda-\rho)}$ for each V in $\mathscr{M}(U_\lambda)$ (Miličić [20]). Under the assumption of the proposition, Δ_λ is exact. Therefore, for $V = \Gamma(X, \mathscr{L}_{Q,\tau,\lambda})$, we have $L^p i_x^*(\mathscr{L}_{Q,\tau,\lambda}) \Rightarrow H_{-p}(\mathfrak{n}_x, \Gamma(X, \mathscr{L}_{Q,\tau,\lambda}))_{(\lambda-\rho)}$. On the other hand, write i_x: $\{x\} \xrightarrow{i} Q \xrightarrow{i_1} (X - \partial\bar{Q})$, $\mathscr{L}$ given by $(i_1)_*(\text{ind}\,\tau)$ on $X - \partial\bar{Q}$. By Kashiwara's theorem and $Li_x^* = Li^* Li_1^*$, we have $L^p i_x^*(\mathscr{L}_{Q,\tau,\lambda}) = L^{p-k} i^*(\text{ind}\,\tau)$, where k is the codimension of Q in X. By the definition of ind τ, $H_x \cap K/(H_x \cap K)_0$ acts on $L^j i^*$ (ind τ) via τ and (b) follows. Taking into account 3.2(a), (a) follows. □

Our next task is to study the transition of a set of data under the fibration π: $X \to X_\alpha$ which, for a fixed simple root $\alpha \in R^+$, assigns to each Borel subalgebra $\mathfrak{b}_x$ the parabolic subalgebra $\mathfrak{p}_x$ of type α by adding the root vector corresponding to α. X_α thus consists of all parabolic subalgebras of type α and each fiber of π is isomorphic to $\mathbb{P}^1$.

We recall that, for a given θ-stable Cartan subalgebra $\mathfrak{h}$, a root α in $R(\mathfrak{g}, \mathfrak{h})$ is called *real* if $\theta\alpha = -\alpha$; *complex* if $\theta\alpha \neq \pm\alpha$; *compact* (resp. *noncompact*) if $\theta\alpha = \alpha$ and its corresponding root vector lies in $\mathfrak{k}$ (resp. $\mathfrak{p}$).

For each $x \in X$, write $L_x = \pi^{-1}(\pi(x))$, i.e., the fiber of π containing x. Think of α as a root in R_x^+.

LEMMA 3.6 (Vogan [26]).

(a) *If α is compact, then $L_x \subset K \cdot x$.*
(b) *If α is noncompact, then $L_x \cap (K \cdot x)$ consists of one or two points.*
(c) *If α is real, then $L_x \cap (K \cdot x) = L_x - \{y_+, y_-\}$ (y_+, y_- are two distinct points to be explained later).*
(d) *If α is complex and $\theta\alpha \in R_x^+$, then $L_x \cap (K \cdot x) = \{x\}$.*
(e) *If α is complex and $\theta\alpha \notin R_x^+$, then $L_x \cap (K \cdot x) = L_x - \{y\}$, where $\mathfrak{b}_y = s_\alpha \cdot \mathfrak{b}_x$.*

Before proceeding to the transition of a set of data $(Q = K \cdot x, \tau; \lambda)$, we remark that, for each $w \in W$, we can form a set of new data $(wQ, \tau; w\lambda)$ in the following way:

(1) $wQ = K \cdot wx$, where $\mathfrak{b}_{wx} = w\mathfrak{b}_x$, and
(2) since $H_x \cap K = H_{wx} \cap K$, we use the same character τ.

As a convention, we write τ additively. Let $H_x = TA$ be the Cartan decomposition of H_x. τ is then a character on T. The transition of data $(Q, \tau; \lambda)$ under the fibration π, in different terms, is described in chapter 8 of Vogan [27]. We examine each case separately.

Case (*c*). *α is real.*
Choose α-root vector E_α and $(-\alpha)$-root vector $E_{-\alpha}$ in $\mathfrak{g}_0$ such that $\theta E_\alpha = E_{-\alpha}$, $[E_\alpha, E_{-\alpha}] = -F_\alpha$, $[F_\alpha, E_\alpha] = 2E_\alpha$, and $[F_\alpha, E_{-\alpha}] = -2E_{-\alpha}$. Let $c_\alpha = \mathrm{Ad}(\exp(\pi i/4)(E_\alpha - E_{-\alpha}))$ be the Cayley transform (§2 of [22]). Set $Q_{\alpha+} = K \cdot y_+$, where $B_{y_+} = c_\alpha \cdot B_x$, $Z_\alpha = -i(E_\alpha + E_{-\alpha}) = c_\alpha(F_\alpha)$.

LEMMA 3.7. *$c_\alpha H_x$ has a Cartan decomposition $T_\alpha A_\alpha$ with $A_\alpha = \exp\{F \in \mathfrak{a} | \alpha(F) = 0\}$ and $T_\alpha = T \cdot \exp(\mathbb{C} \cdot Z_\alpha) = T^0 \cdot \exp(\mathbb{C} Z_\alpha)$, and $T^0 \cap \exp(\mathbb{C} \cdot Z_\alpha) = \{1, m_\alpha\}$, where $m_\alpha = \exp(\pi i Z_\alpha)$ and $T^0 = \{t \in T | \mathrm{Ad}\, t E_\alpha = E_\alpha\} = \{t \in T | \alpha(t) = 1\}$.*

This is Lemma 8.3.13 in [27].
Assume $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$. If $\tau(m_\alpha) = (-1)^{\langle \lambda, \check{\alpha} \rangle - 1}$, Lemma 3.7 allows us to define a character $\tau_{\alpha+}$ on T_α by $\tau_{\alpha+}|_{T^0} = \tau|_{T^0}$ and $\tau_{\alpha+}(\exp \pi i Z_\alpha) = (-1)^{\langle \lambda, \check{\alpha} \rangle - 1}$. Thus,

(3.8) If $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$, $\tau(m_\alpha) = (-1)^{\langle \lambda, \check{\alpha} \rangle - 1}$, then $(Q_{\alpha+}, \tau_{\alpha+}; \lambda) = (K \cdot y_+, \tau_{\alpha+}; \lambda)$ are Beilinson–Bernstein data.

If we use $c_\alpha^{-1} = \mathrm{Ad}\exp(\pi i/4)(E_{-\alpha} - E_\alpha)$ instead of c_α in the above construction, we get y_-, $Q_{\alpha-} = K \cdot y_-$, $\tau_{\alpha-}$ and

(3.8)′ If $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$, $\tau(m_\alpha) = (-1)^{\langle \lambda, \check{\alpha} \rangle - 1}$, then $(Q_{\alpha-}, \tau_{\alpha-}; \lambda) = (K \cdot y_-, \tau_{\alpha-}; \lambda)$ are Beilinson–Bernstein data.

We call α *type I* if $Q_{\alpha+} \neq Q_{\alpha-}$; *type II* if $Q_{\alpha+} = Q_{\alpha-}$. It is straightforward to check that in type II case, data in (3.8) and (3.8)′ are the same and we denote it by $(Q_\alpha, \tau_\alpha; \lambda)$.

Denote by $\underset{\sim}{\alpha}$ the root in B_{y_+} (or B_{y_-}) corresponding to α; then $\underset{\sim}{\alpha}$ is noncompact. Note that by setting $s_\alpha\tau(t) = \tau(k^{-1}tk)$, where $s_\alpha = \mathrm{Ad}\, k$ ($k \in K$), then

(3.9) $(K \cdot s_\alpha x, \tau; s_\alpha\lambda) = (K \cdot x, s_\alpha\tau; s_\alpha\lambda)$ are Beilinson–Bernstein data.

Case (*b*). *α is noncompact.*
This is dual to case (c). Similarly, we have a Cayley transform C^α. Set $Q^\alpha = K \cdot y$, where $B_y = C^\alpha \cdot B_x$. Let $C^\alpha Y_x = T^\alpha A^\alpha$ be the Cartan decomposition. We say that α is *type I* if $L_x \cap (K \cdot x)$ consists of one point, and *type II* if otherwise. Set $T_1^\alpha = \{x \in T | \alpha(x) = 1\}$.

LEMMA 3.10. (a) *If α is type I, then $T_1^\alpha = T^\alpha$.* (b) *If α is type II, then $|T^\alpha / T_1^\alpha| = 2$.*

This is Lemma 8.3.5 in [27].
If α is type I, define τ^α on T^α by $\tau|_{T^\alpha}$. Then

(3.11) $(Q^\alpha, \tau^\alpha; \lambda) = (K \cdot y, \tau^\alpha; \lambda)$ are Beilinson–Bernstein data.

If α is type II, there are two characters on T^α, τ_+^α and τ_-^α, such that $(\tau_+^\alpha \oplus \tau_-^\alpha)|_{T^\alpha} = \mathrm{Ind}_{T_1^\alpha}^{T^\alpha}(\tau|_{T^\alpha})$. Then

(3.11)′ $(Q^\alpha, \tau_\pm^\alpha; \lambda) = (K \cdot y, \tau_\pm^\alpha; \lambda)$ are Beilinson–Bernstein data.

Denote by $\tilde{\alpha}$ the root in $C^\alpha B_x$ corresponding to α; then $\tilde{\alpha}$ is a real root. Simply unraveling the definitions, we have (cf. Proposition 8.3.18 in [27])

PROPOSITION 3.12. *Assume $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$.*

1. *If α is real and the parity condition in (3.8) is satisfied, then*
 (a) *if α is type I, then $((Q_{\alpha\pm})^{\underset{\sim}{\alpha}}, (\tau_{\alpha\pm})^{\underset{\sim}{\alpha}}; \lambda) = (Q, \tau; \lambda)$;*
 (b) *if α is type II, then $(Q, \tau; \lambda)$ is either $((Q_\alpha)^{\underset{\sim}{\alpha}}, (\tau_\alpha)_+^{\underset{\sim}{\alpha}}; \lambda)$ or $((Q_\alpha)^{\underset{\sim}{\alpha}}, (\tau_\alpha)_-^{\underset{\sim}{\alpha}}; \lambda)$.*
2. *If α is noncompact, then*
 (a) *if α is type I, then $((Q^\alpha)_{\tilde{\alpha}+}, (\tau^\alpha)_{\tilde{\alpha}+}; \lambda) = (Q, \tau; \lambda)$ (under an appropriate choice of C^α);*
 (b) *if α is type II, then $((Q^\alpha)_{\tilde{\alpha}}, (\tau_\pm^\alpha)_{\tilde{\alpha}}; \lambda) = (Q, \tau; \lambda)$.*

Case (*c*) *and* (*d*). *α is complex.*
Assume $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ and set $s_\alpha\tau = \tau - (\langle \lambda, \check{\alpha} \rangle - 1)\alpha|_T$ (here α = the character of H_x acting on the α root space). Since $s_\alpha\lambda - s_\alpha\rho = \lambda - \rho - (\langle \lambda, \check{\alpha} \rangle - 1)\alpha$.

(3.13) $(s_\alpha Q, s_\alpha\tau; \lambda) = (K \cdot s_\alpha x, s_\alpha\tau; \lambda)$ are Beilinson–Bernstein data.

Note that $s_\alpha(s_\alpha\lambda) - s_\alpha\tau = \lambda - \rho + \alpha$; then

$$(3.14)\qquad (s_\alpha Q, \tau + \alpha; s_\alpha\lambda) = (K \cdot s_\alpha x, \tau + \alpha; s_\alpha\lambda) \text{ are Beilinson–Bernstein data.}$$

To conclude this section, we give a criterion, due to Beilinson and Bernstein, for a given set of data to be regular. This result is useful in studying the classification of tempered Harish-Chandra modules for $\mathscr{G}$. The proof of the following theorem will be given in the appendix.

THEOREM 3.15 (Beilinson–Bernstein). *A set of data ($Q = Kx, \tau; \lambda$) is regular if and only if the following hold:*

(a) *There is no compact simple root α with $\langle\lambda, \alpha\rangle = 0$,*
(b) *There is no complex simple root α with $\theta\alpha \notin R_x^+$, and $\langle\lambda, \alpha\rangle = 0$, and*
(c) *There is no real simple root α with $\tau(m_\alpha) = -1$ and $\langle\lambda, \alpha\rangle = 0$.*

§4. Preliminary on intertwining operators. In [2], Beilinson and Bernstein introduce certain intertwining functors in order to compare the categories $\mathscr{M}(\mathscr{D}_\lambda)$ and $\mathscr{M}(\mathscr{D}_{s_\alpha\lambda})$ for a simple root α. For the purpose of comparing standard modules and of studying K-types (later), we need a more explicit construction of intertwining operators. The basic idea is to reduce things to fibers of the fibration $\pi: X \to X_\alpha$. In other words, we want to compare the direct images of $\mathscr{M}(\mathscr{D}_\lambda)$ and $\mathscr{M}(\mathscr{D}_{s_\alpha\lambda})$ on X under π. In general, we fix a set S of simple roots and consider the variety X_s of all parabolic subalgebras of type S. Denote by π the fibration $X \to X_s$. We will introduce a generalized twisted sheaf of differential operators $\tilde{\mathscr{D}}$ on X_s (which will be nothing but $\pi_0\mathscr{D}$) and study $\tilde{\mathscr{D}}$ modules on X_s. The main result in this section is Theorem 4.14 and the Beilinson–Bernstein theory for $\tilde{\mathscr{D}}$-modules.

The construction of $\tilde{\mathscr{D}}_\lambda$ is similar to that of $\mathscr{D}_\lambda$. We only sketch the construction of $\tilde{\mathscr{D}}_\lambda$ here. The precise development would require a more careful exposition. We use the pair $(\mathfrak{p}, \mathfrak{u})$ to identify $(\mathfrak{p}_x, \mathfrak{u}_x)$ for $x \in X_s$ ($\mathfrak{u}_x$ is the nilpotent radical of $\mathfrak{p}_x$, the parabolic corresponding to x). As in the case of the flag variety, we consider the $\mathscr{O}_{X_s}$-modules $\tilde{\mathfrak{g}}^0 = \mathscr{O}_{X_s} \otimes_{\mathbf{C}} \mathfrak{g}$. Put

$$\tilde{\mathfrak{p}}^0 = \{f \in \tilde{\mathfrak{g}}^0 | f(x) \in \mathfrak{p}_x \text{ for all } x \in X_s\},$$

$$\tilde{\mathfrak{u}}^0 = \{f \in \tilde{\mathfrak{g}}^0 | f(x) \in \mathfrak{u}_x \text{ for all } x \in X_s\}, \quad \text{and}$$

$$\tilde{\mathfrak{l}}^0 = \tilde{\mathfrak{p}}^0/\tilde{\mathfrak{u}}^0.$$

These are $\mathscr{O}_{X_s}$-modules. Write $\tau: \mathfrak{g} \to TX_s$ for the action derived from the action of G on X_s. $\tilde{\mathfrak{g}}^0$ has a Lie algebra structure by putting

$$[f \otimes A, g \otimes B] = f\tau(A) \otimes B - g\tau(B) f \otimes A + fg \otimes [A, B]$$

for $f, g \in \mathscr{O}_{X_s}$, $A, B \in \mathfrak{g}$. Extending τ to $\tilde{\mathfrak{g}}^0$ by setting $\tau(f \otimes A) = f \cdot \tau(A)$, τ is then a Lie algebra homomorphism and $\tilde{\mathfrak{p}}^0$ is the kernel. Since G acts on $\tilde{\mathfrak{p}}^0$ and $\tilde{\mathfrak{u}}^0$, we have

LEMMA 4.1. *$\tilde{\mathfrak{p}}^0$ and $\tilde{\mathfrak{u}}^0$ are sheaves of ideals in $\tilde{\mathfrak{g}}^0$.*

Consider $\tilde{U}^0$ the sheaf of algebras generated by $\mathfrak{g}$ and $\mathscr{O}_{X_s}$ with natural relation $[A, f] = \tau(A)f$ for $A \in \mathfrak{g}$, $f \in \mathscr{O}_{X_s}$. $\mathscr{O}_{X_s}$-module $\tilde{U}^0$ is isomorphic to $\mathscr{O}_{X_s} \otimes_{\mathbf{C}} U(\mathfrak{g})$. Write $U(\tilde{\mathfrak{p}}^0)$, $U(\tilde{\mathfrak{u}}^0)$ and $U(\tilde{\mathfrak{l}}^0)$ for the sheaves of algebras generated by $\mathscr{O}_{X_s}$ and $\tilde{\mathfrak{p}}^0$, $\tilde{\mathfrak{u}}^0$, $\tilde{\mathfrak{l}}^0$, respectively. Let $Z(\tilde{\mathfrak{l}}^0)$ be the center of $U(\tilde{\mathfrak{l}}^0)$. In view of Lemma 4.1, $\tilde{\mathfrak{u}}^0\tilde{U}^0$ is a two-sided ideal of $\tilde{U}^0$; we can define $\tilde{\mathscr{D}}_{\mathfrak{l}} = \tilde{U}^0/\tilde{\mathfrak{u}}^0\tilde{U}^0$.

LEMMA 4.2. (a) $Z(\tilde{\mathfrak{l}}^0) = \mathscr{O}_{X_s} \otimes_{\mathbf{C}} Z(\mathfrak{l})$ ($\mathfrak{l} = \mathfrak{p}/\mathfrak{u}$).
(b) *$Z(\mathfrak{l}) \to U(\tilde{\mathfrak{l}}^0) \to \tilde{\mathscr{D}}_{\mathfrak{l}}$ has its image in the center of $\tilde{\mathscr{D}}_{\mathfrak{l}}$.*

Proof. Write $X_s \cong G/P$ for some $P = P_x$. The P-action on $\tilde{\mathfrak{l}}_x$ factors through $P/U \cong L$, but L acts trivially on $Z(\tilde{\mathfrak{l}})_x = U(\tilde{\mathfrak{l}})_x^L$. Since G acts on $Z(\tilde{\mathfrak{l}}^0)$, this gives (a). (b) follows from the facts that an element A in $Z(\mathfrak{l})$ is G-invariant. □

For each $\lambda \in \mathfrak{h}^*$, let $\tilde{I}_\lambda = \operatorname{Ker}(Z(\mathfrak{l}) \xrightarrow{\xi} U(\mathfrak{h}) \xrightarrow{\lambda-\rho} \mathbf{C})$, where ξ is the projection on $U(\mathfrak{h})$ via $U(\mathfrak{l}) = U(\mathfrak{h}) \oplus [(\mathfrak{l} \cap \mathfrak{n})U(\mathfrak{l}) + U(\mathfrak{l})(\mathfrak{l} \cap \bar{\mathfrak{n}})]$. Lemma 4.2 implies that $\tilde{I}_\lambda \cdot \tilde{\mathscr{D}}_{\mathfrak{l}}$ is a two-sided ideal of $\tilde{\mathscr{D}}_{\mathfrak{l}}$.

DEFINITION 4.3. $\tilde{\mathscr{D}}_\lambda = \tilde{\mathscr{D}}_{\mathfrak{l}}/\tilde{I}_\lambda \cdot \tilde{\mathscr{D}}_{\mathfrak{l}}$.

LEMMA 4.4. *$\tilde{\mathscr{D}}_\lambda = \tilde{\mathscr{D}}_{w\lambda}$ for $w \in W(\mathfrak{l}, \mathfrak{h})$.*

Proof. The definition of Harish-Chandra homomorphism for $(\mathfrak{l}, \mathfrak{h}, \mathfrak{l} \cap \bar{\mathfrak{n}})$ implies that $\tilde{I}_\lambda = \tilde{I}_{\lambda'}$ iff $\lambda - \rho + \rho_{\mathfrak{l}} = w \cdot (\lambda' - \rho + \rho_{\mathfrak{l}})$ for $w \in W(\mathfrak{l}, \mathfrak{h})$. Since $w(\rho - \rho_{\mathfrak{l}}) = \rho - \rho_{\mathfrak{l}}$, $\lambda = w \cdot \lambda'$. □

We denote by $\mathscr{M}(\tilde{\mathscr{D}}_\lambda)$ the category of (left) $\tilde{\mathscr{D}}_\lambda$-modules which are $\mathscr{O}_{X_s}$-quasi-coherent. On X, put

$$\mathfrak{p}^0 = \{f \in \mathfrak{g}^0 | f(x) \in \mathfrak{p}_{\pi(x)} \text{ for all } x \in X\},$$

$$\mathfrak{u}^0 = \{f \in \mathfrak{g}^0 | f(x) \in \mathfrak{u}_{\pi(x)} \text{ for all } x \in X\} \subset \mathfrak{n}^0,$$

$$\mathfrak{l}^0 = \mathfrak{p}^0/\mathfrak{u}^0, \qquad U(\mathfrak{l}^0) = \text{the algebra generated by } \mathscr{O}_X \text{ and } \mathfrak{l}^0.$$

Note that $\pi^*(\tilde{\mathfrak{g}}^0) = \mathfrak{g}^0$, $\pi^*(\tilde{\mathfrak{p}}^0) \cong \mathfrak{p}^0$, $\pi^*(\tilde{\mathfrak{u}}^0) \cong \mathfrak{u}^0$, $\pi^*(\tilde{\mathfrak{l}}^0) \cong \mathfrak{l}^0$, and $\pi^*(U(\tilde{\mathfrak{l}}^0)) \cong U(\mathfrak{l}^0)$.

PROPOSITION 4.5. *$\pi_0\mathscr{D}_\lambda \cong \tilde{\mathscr{D}}_\lambda$ as sheaves of algebras on X_s.*

Proof. As $\mathfrak{u}^0 \subset \mathfrak{n}^0$ is an ideal of $\mathfrak{g}^0$ and commutes with $\mathscr{O}_X$, we have $\pi^*(\tilde{\mathscr{D}}_{\mathfrak{l}}) = \pi^*(\tilde{U}^0/\tilde{\mathfrak{u}}^0\tilde{U}^0) \cong U^0/\mathfrak{u}^0U^0$ and the algebra structure of $U^0/\mathfrak{u}^0U^0$

extends that of $\pi^{-1}(\tilde{\mathscr{D}}_{\mathfrak{l}})$. The natural map $U^0/\mathfrak{u}^0U^0 \to U^0/\mathfrak{n}^0U^0$ gives an algebra morphism $\pi^*(\tilde{\mathscr{D}}_{\mathfrak{l}}) \to \mathscr{D}_{\mathfrak{b}}$. Since $Z(\mathfrak{l})$ commutes with $\mathscr{O}_X$, we have $\pi^*(\tilde{I}_\lambda \cdot \tilde{\mathscr{D}}_{\mathfrak{l}}) \cong \tilde{I}_\lambda \pi^*(\tilde{\mathscr{D}}_{\mathfrak{l}})$. Also, $\tilde{I}_\lambda \cdot \pi^*(\tilde{\mathscr{D}}_{\mathfrak{l}})$ goes to $I_\lambda \cdot \mathscr{D}_{\mathfrak{b}}$ under the map $U^0/\mathfrak{u}^0u^0 \to U^0/\mathfrak{n}^0U^0$. This gives rise to an algebra morphism $\pi^*(\tilde{\mathscr{D}}_\lambda) \to \mathscr{D}_\lambda$, which in turn gives $\tilde{\mathscr{D}}_\lambda \to \pi_0\mathscr{D}_\lambda$. To check that this is an isomorphism, both modules being G-equivariant, it suffices to look at their fibers. Let $x \in X_s$; write $\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{p}_x$. The map $\bar{\mathfrak{u}} \xrightarrow{\exp} \bar{U} \to G/P_x \cong X_s$ gives an affine open neighborhood, say V, of x in X_s. On V, $\tilde{U}^0 \cong \mathscr{O}_{X_s} \otimes_{\mathbb{C}} U(\bar{\mathfrak{u}}) \oplus \tilde{\mathfrak{p}}^0\tilde{U}^0$ and $\tilde{\mathscr{D}}_{\mathfrak{l}} = \tilde{U}^0/\tilde{\mathfrak{u}}^0\tilde{U}^0$ is isomorphic to $U(\tilde{\mathfrak{l}}^0) \otimes_{\mathscr{O}} (\mathscr{O} \otimes_{\mathbb{C}} U(\bar{\mathfrak{u}}))$. Set

$$(4.6) \qquad U_\lambda(\mathfrak{l}) = U(\tilde{\mathfrak{l}}^0)/\tilde{I}_\lambda U(\tilde{\mathfrak{l}}^0), \quad \text{then}$$

$$(4.6)' \qquad \tilde{\mathscr{D}}_\lambda \cong U_\lambda(\mathfrak{l}) \otimes_{\mathbb{C}} U(\bar{\mathfrak{u}}) \text{ on } V.$$

On the other hand, on $\pi^{-1}(V) \subset X$, $U^0 \cong \mathscr{O} \otimes_{\mathbb{C}} U(\bar{\mathfrak{u}}) \oplus \mathfrak{p}^0U^0$ and $\mathscr{D}_{\mathfrak{b}} = U^0/\mathfrak{n}^0U^0 \cong U(\mathfrak{l}^0)/(\mathfrak{n}^0/\mathfrak{u}^0)U(\mathfrak{l}^0) \otimes_{\mathscr{O}}(\mathscr{O} \otimes_{\mathbb{C}} U(\bar{\mathfrak{u}}))$. Set

$$(4.7) \qquad \mathscr{D}_\lambda(\mathfrak{l}) = \left[\frac{U(\mathfrak{l}^0)}{\left(\frac{\mathfrak{n}^0}{\mathfrak{u}^0}\right) \cdot U(\mathfrak{l}^0)}\right] \Bigg/ I_\lambda \cdot \left[\frac{U(\mathfrak{l}^0)}{\left(\frac{\mathfrak{n}^0}{\mathfrak{u}^0}\right) \cdot U(\mathfrak{l}^0)}\right], \quad \text{then}$$

$$(4.7)' \qquad \mathscr{D}_\lambda \cong \mathscr{D}_\lambda(\mathfrak{l}) \otimes_{\mathbb{C}} U(\bar{\mathfrak{u}}) \text{ on } \pi^{-1}(V).$$

The fiber of $\tilde{\mathscr{D}}_\lambda$ at $x = (\tilde{\mathscr{D}}_\lambda)_x = U_\lambda(\mathfrak{l}_x) \otimes_{\mathbb{C}} U(\bar{\mathfrak{u}})$. And

$$(\pi_0\mathscr{D}_\lambda)_x = [\pi_0\mathscr{D}_\lambda(\mathfrak{l})]_x \otimes_{\mathbb{C}} U(\bar{\mathfrak{u}}).$$

Since $\mathscr{D}_\lambda(\mathfrak{l})$ is an algebraic G-sheaf (i.e., a union of coherent G-sheaves) and π is proper, $[\pi_0\mathscr{D}_\lambda(\mathfrak{l})]_x = H^0(\pi^{-1}(x), \mathscr{D}_\lambda(\mathfrak{l}_x))$ (say, by Grauert's theorem; see 12.9 in [11]). Thus the isomorphism $(\tilde{\mathscr{D}}_\lambda)_x \to (\pi_0\mathscr{D}_\lambda)_x$ follows from the isomorphism $U_\lambda(\mathfrak{l}_x) \to \Gamma(\pi^{-1}(x), \mathscr{D}_\lambda(\mathfrak{l}_x))$ (Lemma 2.1). □

Now consider the diagram

$$(4.8) \qquad \begin{array}{ccc} Y & \xrightarrow{j} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y_s & \xrightarrow{i} & X_s \end{array}$$

where Y_s is a nonsingular subvariety of X and $Y = \pi^{-1}(Y_s)$, which is smooth too (π is a projection locally).

COROLLARY 4.9. (a) *$\tilde{\mathscr{D}}_\lambda$ normalizes $U_\lambda(\mathfrak{l})$ and is isomorphic to $U_\lambda(\mathfrak{l}) \otimes_{\mathscr{O}} \mathscr{D}$ locally.*

(b) $\pi_0 j^*\mathscr{D}_\lambda(\mathfrak{l}) \cong i^*\pi_0\mathscr{D}_\lambda(\mathfrak{l})$, $\pi_0 j^*\mathscr{D}_\lambda \cong i^*\pi_0\mathscr{D}_\lambda$.
(c) $\pi_0\mathscr{D}_\lambda(\mathfrak{l}) \cong U_\lambda(\mathfrak{l})$.

Proof. In view of (4.6)–(4.7) and Lemma 4.1, only the first statement of (b) needs a justification. Since $\mathscr{D}_\lambda(\mathfrak{l})$ is an algebraic G-module, we have $[R^i\pi_0\mathscr{D}_\lambda(\mathfrak{l})]_x = H^i(\pi^{-1}(x), \mathscr{D}_\lambda(\mathfrak{l}_x)) = 0$ for $i > 0$, by Lemma 2.1. $\mathscr{D}_\lambda(\mathfrak{l})$ being locally free, the base change holds. □

Denote by $\mathscr{I}_Y$ (resp. $\mathscr{I}_{Y_s}$) the ideal sheaf of Y (resp. Y_s).

Definition 4.10. (a) $\tilde{\mathscr{D}}^i_\lambda = \{A \in \tilde{\mathscr{D}}_\lambda | A\mathscr{I}_{Y_s} \subset \mathscr{I}_{Y_s} \cdot \tilde{\mathscr{D}}_\lambda\}/\mathscr{I}_{Y_s} \cdot \tilde{\mathscr{D}}_\lambda|_{Y_s}$.
(b) $\tilde{\mathscr{D}}_{X_s \leftarrow Y_s} = \Omega_{Y_s} \otimes_{\mathscr{O}_{Y_s}} i^*(\Omega^{-1}_{X_s} \otimes_{\mathscr{O}_{X_s}} \tilde{\mathscr{D}}^0_\lambda)$ ($\mathfrak{A}^0$ = the opposite algebra of $\mathfrak{A}$).

Claim 4.11. $\tilde{\mathscr{D}}_{X_s \leftarrow Y_s}$ is an $i^{-1}(\tilde{\mathscr{D}}_\lambda) - \tilde{\mathscr{D}}^i_\lambda$ bimodule.

The left $i^{-1}(\tilde{\mathscr{D}}_\lambda)$-action is evident for $\tilde{\mathscr{D}}_{X_s \leftarrow Y_s}$ has a natural right $i^{-1}(\tilde{\mathscr{D}}^0_\lambda)$-action. To describe the right $\tilde{\mathscr{D}}^i_\lambda$-action, in view of the local description in 4.9(a), we may assume $\tilde{\mathscr{D}}_\lambda = U_\lambda(\mathfrak{l}) \otimes_{\mathscr{O}} \mathscr{D}_{X_s}$ and $\tilde{\mathscr{D}}^i_\lambda = i^*(U_\lambda(\mathfrak{l})) \otimes_{i^{-1}(\mathscr{O}_{X_s})} \mathscr{D}_{Y_s}$ such that for $\xi \in TY_s$ and $A \in U_\lambda(\mathfrak{l})$, we have $[\xi, i^{-1}(A)] = i^{-1}[\xi_*, A]$, where ξ_* = the image of ξ under $TY_s \to TX_s$. In terms of these, exactly as in the usual $\mathscr{D}$-module case, $\tilde{\mathscr{D}}_{X_s \leftarrow Y_s}$ has a right $\mathscr{D}_{Y_s}$-action (cf. [2] and [20]). The extra data here, i.e., the right action of $i^{-1}(U_\lambda(\mathfrak{l}))$, is defined as right multiplication (of $i^{-1}(\tilde{\mathscr{D}}_\lambda)$) to $\tilde{\mathscr{D}}_{X_s \leftarrow Y_s}$. In other words, for $A \in U_\lambda(\mathfrak{l})$ and $\mathfrak{n} \otimes \beta \in [\Omega_{Y_s} \otimes_{\mathscr{O}} i^*(\Omega^{-1}_{X_s})] \otimes_{\mathscr{O}} i^*(\tilde{\mathscr{D}}^0_\lambda)$, $(\mathfrak{n} \otimes \beta) \cdot i^{-1}(A)$ is defined as $\mathfrak{n} \otimes [\beta \cdot i^{-1}(A)]$. Using the identity $[\xi, i^{-1}(A)] = i^{-1}[\xi_*, A]$, it is easy to check that this is well defined. □

Denote by $\mathscr{M}(\tilde{\mathscr{D}}^i_\lambda)$ the category of sheaves of (left) $\tilde{\mathscr{D}}^i_\lambda$-modules which are $\mathscr{O}_{Y_s}$-quasicoherent. Define $i_*: \mathscr{M}(\tilde{\mathscr{D}}^i_\lambda) \to \mathscr{M}(\tilde{\mathscr{D}}_\lambda)$ via $i_*(\mathscr{F}) = i_0(\tilde{\mathscr{D}}_{X_s \leftarrow Y_s} \otimes_{\tilde{\mathscr{D}}^i_\lambda} \mathscr{F})$. We remark that for $Y_s \xrightarrow{i_1} Z_s \xrightarrow{i_2} X_s$, one can define i_{1*} in the same way, and i_1, i_2 being embeddings, we have $i_* = i_{2*}i_{1*}$. Kashiwara's theorem holds in this case too:

THEOREM 4.12. *If $Y_s \xrightarrow{i_1} Z_s$ is a closed embedding, then i_{1*} defines an equivalence of the category $\mathscr{M}(\tilde{\mathscr{D}}^i_\lambda)$ and the subcategory $\mathscr{M}_{Y_s}(\tilde{\mathscr{D}}^{i_2}_\lambda) \subset \mathscr{M}(\tilde{\mathscr{D}}^{i_2}_\lambda)$ consisting of sheaves supported on Y_s.*

The proof is the same as the usual $\mathscr{D}$-module case ([4]). The inverse functor is given by $i_1^+(\mathscr{F}) = \mathrm{Hom}_{i_1^{-1}\tilde{\mathscr{D}}^{i_2}_\lambda}(\tilde{\mathscr{D}}_{Z_s \leftarrow Y_s}, i_1^{-1}(\mathscr{F}))$.

PROPOSITION 4.13. *$\pi_0\mathscr{D}^j_\lambda \cong \tilde{\mathscr{D}}^i_\lambda$ as sheaves of algebras on Y_s.*

Proof. Since π is locally a projection, we have $\pi^*\mathscr{I}_{Y_s} = \mathscr{I}_Y$. Thus $0 \to \tilde{\mathscr{D}}^i_\lambda \to \tilde{\mathscr{D}}_\lambda/\mathscr{I}_{Y_s} \cdot \mathscr{D}_\lambda|_{Y_s} = i^*\tilde{\mathscr{D}}_\lambda$ and the map $\pi^*i^*\tilde{\mathscr{D}}_\lambda \cong j^*\pi^*\tilde{\mathscr{D}}_\lambda \to j^*\mathscr{D}_\lambda$ (the second arrow is given in the proof of Proposition 4.5) gives rise to $\pi^*\tilde{\mathscr{D}}^i_\lambda \to \mathscr{D}^j_\lambda$, which gives an algebra morphism $\tilde{\mathscr{D}}^i_\lambda \to \pi_0(\mathscr{D}^j_\lambda)$. By 4.9(a), locally on Y_s, $\tilde{\mathscr{D}}^i_\lambda = i^*U_\lambda(\mathfrak{l}) \otimes_{\mathscr{O}} \mathscr{D}_{Y_s}$, and locally on $\pi^{-1}(Y_s)$, $\mathscr{D}^j_\lambda = j^*\mathscr{D}_\lambda(\mathfrak{l}) \otimes_{\mathscr{O}} \pi^*\mathscr{D}_{Y_s}$ (note that $\mathscr{D}_\lambda(\mathfrak{l})$

commutes with $\mathscr{I}_Y$). Hence, locally,

$$\pi_0(\mathscr{D}^i_\lambda) \cong \pi_0 j^*\mathscr{D}_\lambda(1) \otimes_{\mathscr{O}} \mathscr{D}_{Y_s}$$

($\mathscr{D}_{Y_s}$ *is locally free and* π *is quasicompact*)

$$\cong i^*\pi_0\mathscr{D}_\lambda(1) \otimes_{\mathscr{O}} \mathscr{D}_{Y_s} \cong \tilde{\mathscr{D}}^i_\lambda.$$

The second and the third isomorphisms follow from 4.9(b), (c). □

THEOREM 4.14. *π_0: $\mathscr{M}(\mathscr{D}_\lambda) \to \mathscr{M}(\tilde{\mathscr{D}}_\lambda)$ commutes with the direct image functors, i.e., $i_*\pi_0 \cong \pi_0 j_*$.*

Proof. Factor the diagram (4.8) into

$$\begin{tikzcd} Y \arrow[r, "j_1"] \arrow[d] & X - \partial Y \arrow[r, "j_2"] \arrow[d] & X \arrow[d] \\ Y_s \arrow[r, "i_1"] & X_s - \partial Y_s \arrow[r, "i_2"] & X_s \end{tikzcd}$$

then j_1, i_1 are closed embeddings and j_2, i_2 are open embeddings. Hence $i_* = i_{2*}i_{1*}$, $j_* = j_{2*}j_{1*}$, and $i_{2*} = i_{2\circ}$, $j_{2*} = j_{2\circ}$. It then suffices to show that $i_1^*\pi_0 \cong \pi_0 j_{1*}$. Using Kashiwara's Theorem 4.12, it suffices to check, for $\mathscr{F} \in \mathscr{M}(\mathscr{D}^j_\lambda)$, $\pi_0(\Lambda^{\mathrm{top}}\mathscr{N}_{Y|X-\partial Y} \otimes_{\mathscr{O}} \mathscr{F}) \cong \Lambda^{\mathrm{top}}\mathscr{N}_{Y_s|X_s-\partial Y_s} \otimes_{\mathscr{O}} \pi_0\mathscr{F}$ ($\mathscr{N}_{A|B}$ is the normal sheaf of A in B). Since π is locally a projection, $\pi^*(\Lambda^{\mathrm{top}}\mathscr{N}_{Y_s|X_s-\partial Y_s}) = \Lambda^{\mathrm{top}}\mathscr{N}_{Y|X-\partial Y}$. The result follows from the projection formula.

We now come to the Beilinson–Bernstein theory for $\tilde{\mathscr{D}}$ modules. Denote by $\tilde{\Gamma}$ the global section functor: $\mathscr{M}(\tilde{\mathscr{D}}_\lambda) \to \mathscr{M}(U_{o(\lambda)})$ (4.5 and 2.1).

THEOREM 4.15.
(a) *If $\langle\lambda, \check{\alpha}\rangle \neq -1, -2, \dots$ for all $\alpha \in R^+ - \langle S\rangle$, then $R^q\tilde{\Gamma} = 0$ for $q > 0$.*
(b) *If $\langle\lambda, \check{\alpha}\rangle \neq -1, -2, \dots$ for all $\alpha \in R^+ - \langle S\rangle$ and $\langle\lambda, \beta\rangle \neq 0$ for all $\beta \in R^+$, then $\tilde{\Gamma}$ is an equivalence of categories. (Here $\langle S\rangle = \{$the roots spanned by $S\}$.)*

The proof is the same as the proof of Theorem 2.2 in [3]. The assumptions in the above theorem ensure the key lemma in [3] holds in this case. Note that the left adjoint functor $\tilde{\Delta}$ to $\tilde{\Gamma}$ is given by $\tilde{\Delta}V = \tilde{\mathscr{D}}_\lambda \otimes_{U_\lambda} V$. Consider

$$\begin{tikzcd} \mathscr{M}(\mathscr{D}_\lambda) \arrow[r, "\Gamma"] \arrow[d, "\pi_0"'] & \mathscr{M}(U_\lambda) \\ \mathscr{M}(\tilde{\mathscr{D}}_\lambda) \arrow[ur, "\tilde{\Gamma}"'] & \end{tikzcd}.$$

For each $\mathscr{H} \in \mathscr{M}(\tilde{\mathscr{D}}_\lambda)$, $\pi^*\mathscr{H}$ is a $\pi^*\tilde{\mathscr{D}}_\lambda$-module. We define Δ_1: $\mathscr{M}(\tilde{\mathscr{D}}_\lambda) \to \mathscr{M}(\mathscr{D}_\lambda)$ by $\Delta_1\mathscr{H} = \mathscr{D}_\lambda \otimes_{\pi^*\tilde{\mathscr{D}}_\lambda} \pi^*\mathscr{H}$.

THEOREM 4.16. (a) *For $\mathscr{F} \in \mathscr{M}(\mathscr{D}_\lambda)$, $\mathscr{H} \in \mathscr{M}(\tilde{\mathscr{D}}_\lambda)$ we have*

$$\mathrm{Hom}_{\tilde{\mathscr{D}}_\lambda}(\mathscr{H}, \pi_0\mathscr{F}) \cong \mathrm{Hom}_{\mathscr{D}_\lambda}(\Delta_1\mathscr{H}, \mathscr{F}).\ \tilde{\Gamma}\pi_0 = \Gamma \text{ and } \Delta = \Delta_1\tilde{\Delta}.$$

(b) *If $\langle\lambda, \check{\alpha}\rangle \neq -1, -2, \dots$ for all $\alpha \in R^+$, then $R^q\pi_0 = 0$ for $q > 0$.*
(c) *If $\langle\lambda, \check{\alpha}\rangle \neq 0, -1, -2, \dots$ for all $\alpha \in R^+$, then π_0 is an equivalence of categories.*

Proof. (a) is clear from the definitions. For (b), let U be affine open in X_s. Then j: $\pi^{-1}(U) \to X$ in an affine open embedding. Since $R^q\pi_0\mathscr{F}(U) = H^q(\pi^{-1}(U), \mathscr{F}|_{\pi^{-1}(U)}) = H^q(X, j_*j^*\mathscr{F})$, (b) follows from the vanishing theorem for $\mathscr{M}(\mathscr{D}_\lambda)$ (Theorem 2.2). For (c), note that λ being regular, π_0 is faithful. Since π_0 commutes with direct sum and is exact (by (b)), π_0 is an equivalence of categories ([1]). □

§5. Intertwining operators. In this section, we will apply results of §4 to construct intertwining operators between certain standard modules.

Let $Y_1 = K \cdot x$ be a K-orbit in X. $Y_s = K \cdot \pi(x)$. Write B, P to represent x, $\pi(x) = y$, respectively. We enlarge the diagram (4.8) to

(5.1)
$$\begin{tikzcd} K \arrow[r] \arrow[dr, "q"'] & \dfrac{K}{K \cap B} \cong Y_1 \arrow[r, "j_1"] \arrow[d] & Y \arrow[r, "j"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\ & \dfrac{K}{K \cap P} \arrow[r, "\sim"] & Y_s \arrow[r, "i"] & X_s \end{tikzcd}$$

Consider the pair $(K \cap P, P/B)$. $K \cap P$ acts on $P/B \cong L/L \cap B$ with a finite number of orbits and $\mathscr{D}_\lambda(1_y)$ is a $K \cap P$-sheaf and $K \cap P$ acts on $\Gamma(\mathscr{D}_\lambda(1_y)) \cong U_\lambda(1_y)$. As before, we have the category $\mathscr{M}(U_\lambda(1_y), K \cap P)$ of $U_\lambda(1_y)$ with a compatible algebraic $K \cap P$ action (cf. §2). As in the case of $\mathscr{D}$-modules, we have

LEMMA 5.2. ind: *$\mathscr{M}(U_\lambda(1_y), K \cap P) \to \mathscr{M}(\tilde{\mathscr{D}}^i_\lambda, K)$ is an exact functor.*

Proof. The exactness of ind is evident. It remains to show that ind $E \in \mathscr{M}(\tilde{\mathscr{D}}^i_\lambda, K)$ for $E \in \mathscr{M}(U_\lambda(1_y), K \cap P)$. Since K acts on Y_s transitively, this amounts to defining a $U_\lambda(1)$-action and a compatible K-action on ind E, which is straightforward. We omit the details. □

Now consider the fiber at y of diagram (5.1), i.e.,

(5.3)
$$\begin{tikzcd} & Y_1 \arrow[r, "j_1"] & Y \\ \dfrac{K \cap P}{K \cap B} \cong & \pi^{-1}(y) \cap Y_1 \arrow[r, "\tilde{j}_1"] \arrow[u, "\tilde{i}_1"] & \pi^{-1}(y) \cong P/B \arrow[u, "i_1"'] \end{tikzcd}$$

Consider a set of data $(Y_1, \tau; \lambda)$, τ being an irreducible character of $K \cap B$. We have $\mathrm{ind}_{K \cap B}^{K \cap P} \tau \in \mathscr{M}(\mathscr{D}_\lambda(1_y)^{j_1}, K \cap P)$ and $\tilde{j}_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P}, \tau) \in \mathscr{M}(\mathscr{D}_\lambda(1_y), K \cap P)$. Hence, $\Gamma(P/B, \tilde{j}_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P} \tau)) \in \mathscr{M}(U_\lambda(1_y), K \cap P)$. By Lemma 5.1, $\mathrm{ind}\, \Gamma(P/B, \tilde{j}_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P} \tau)) \in \mathscr{M}(\tilde{\mathscr{D}}_\lambda^i, K)$.

THEOREM 5.4. *$\pi_0 j_{1*}(\mathrm{ind}\, \tau) \cong \mathrm{ind}\, \Gamma(P/B, \tilde{j}_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P} \tau))$ in $\mathscr{M}(\tilde{\mathscr{D}}_\lambda^i, K)$.*

This theorem is standard; we give a sketch of its proof. We first construct a natural morphism between these two modules in $\mathscr{M}(\tilde{\mathscr{D}}_\lambda^i, K)$. Then we show that it is an isomorphism.

LEMMA 5.5. *In the situation of (5.3), we have $i_1^* j_{1*}(\mathrm{ind}\, \tau) \cong \tilde{j}_{1*} \tilde{i}_1^*(\mathrm{ind}\, \tau)$.*

Assume this lemma first. $[j_{1*}(\mathrm{ind}\, \tau)]_{\pi^{-1}(y)} = i_1^* j_{1*}(\mathrm{ind}\, \tau) = \tilde{j}_{1*} \tilde{i}_1^*(\mathrm{ind}\, \tau) = \tilde{j}_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P} \tau)$. K acts on these restrictions:

$$k: [j_{1*}(\mathrm{ind}\, \tau)]_{\pi^{-1}(y)} \to [j_{1*}(\mathrm{ind}\, \tau)]_{\pi^{-1}(k \cdot y)}. \tag{5.6}$$

To define Φ: $\pi_0 j_{1*}(\mathrm{ind}\, \tau) \to \mathrm{ind}\, \Gamma(P/B, \tilde{j}_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P} \tau))$ in $\mathscr{M}(\tilde{\mathscr{D}}_\lambda^i, K)$, let V be open in Y_s. Note that

$$\Gamma\big(V, \mathrm{ind}\, \Gamma\big(P/B, \tilde{j}_{1*}\big(\mathrm{ind}_{K \cap B}^{K \cap P} \tau\big)\big)\big)$$
$$= \Big\{ f\colon q^{-1}(V) \to \Gamma\big(P/B, \tilde{j}_{1*}\big(\mathrm{ind}_{K \cap B}^{K \cap P} \tau\big)\big) \,\big|\, f(k \cdot p)$$
$$= \sigma(p^{-1})(f(k)) \text{ for } k \in K \cap P \Big\},$$

where $(\sigma(p) \cdot m)(z) = p(m(p^{-1}z))$ for $m \in \Gamma(P/B, j_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P} \tau))$ and $z \in P/B$. For each $k \in q^{-1}(V)$ and $\alpha \in \Gamma(V, \pi_0 j_{1*}(\mathrm{ind}\, \tau)) = \Gamma(\pi^{-1}(V), j_{1*}(\mathrm{ind}\, \tau))$, we denote by $k^*\alpha$ the pull-back of the restriction of α to $\Gamma(\pi^{-1}(k \cdot y), [j_{1*}(\mathrm{ind}\, \tau)]_{\pi^{-1}(k \cdot y)})$ via (5.6). In other words, $k^*\alpha(z) = k^{-1} \cdot (\alpha(k \cdot z))$ for $z \in \pi^{-1}(y)$. We define $\Phi\alpha(k) = k^*\alpha$. It is straightforward to check that $\Phi\alpha \in \Gamma(V, \mathrm{ind}\, \Gamma(P/B, \tilde{j}_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P} \tau)))$ and that Φ is a morphism in $\mathscr{M}(\tilde{\mathscr{D}}_\lambda^i, K)$.

To show that Φ is an isomorphism, both sides being K-equivariant, it suffices to check that isomorphism at fibers over y. Since the sheaf inside the parenthesis is an algebraic K-sheaf, and $\pi_0 j_{1*}(\mathrm{ind}\, \tau) = \pi_0 j_{1\circ}(\mathscr{D}_{Y \leftarrow Y_1} \otimes \mathrm{ind}\, \tau)$, we have

$$[\pi_0 j_{1*}(\mathrm{ind}\, \tau)]_y =$$
$$\Gamma\big(\pi^{-1}(y), \tilde{j}_{1\circ} \tilde{i}_1^*(\mathscr{D}_{Y \leftarrow Y_1} \otimes \mathrm{ind}\, \tau)\big) = \Gamma\big(\pi^{-1}(y), i_1^* j_{1\circ}(\mathscr{D}_{Y \leftarrow Y_1} \otimes \mathrm{ind}\, \tau)\big)$$

($j_{1\circ}$ is exact by (5.7)), which is

$$\Gamma\big(\pi^{-1}(y), i_1^* j_{1*}(\mathrm{ind}\, \tau)\big) = \Gamma\big(\pi^{-1}(y), \tilde{j}_{1*}\big(\mathrm{ind}_{K \cap B}^{K \cap P} \tau\big)\big)$$

by (5.5). This is the fiber of $\mathrm{ind}\, \Gamma(P/B, \tilde{j}_{1*}(\mathrm{ind}_{K \cap B}^{K \cap P} \tau))$ at y. This completes the proof of Theorem 5.4 modulo Lemma 5.5 (and 5.7). □

As for Lemma 5.5, the base change holds in derived categories ([4]). $\tilde{j}_1$ is an affine embedding; hence $\tilde{j}_{1*}$ is exact. The lemma follows from the following:

LEMMA 5.7. *j_1 is an affine embedding.*

Proof. Since $j \circ j_1$ is affine, we can take an affine open cover $(U_i; 1 \leqslant i \leqslant m)$ of X with $U_i \cap Y_1$ affine for $1 \leqslant i \leqslant n$. Therefore the open cover $(V_i = U_i \cap Y;$ $1 \leqslant i \leqslant n)$ of Y is such that $V_i \cap Y_1 = U_i \cap Y_1$ is affine for each i. Since any morphism of an affine variety is affine, this implies that Y_1 is affinely embedded in Y. □

COROLLARY 5.8. *Let $S = \{\alpha\}$, $\pi\colon X \to X_\alpha$. Then*

$$\pi_0 \mathscr{I}_{Y_1, \tau, \lambda} \cong i_*\big(\mathrm{ind}\, \Gamma\big(P/B, \tilde{j}_{1*}\big(\mathrm{ind}_{K \cap B}^{K \cap P} \tau\big)\big) \text{ in } \mathscr{M}(\tilde{\mathscr{D}}_\lambda, K)\big).$$

Proof. Recall that $\mathscr{I}_{Y_1, \tau, \lambda} = (jj_1)_*(\mathrm{ind}\, \tau)$, j and j_1 being embedding, $(jj_1)_* = j_* j_{1*}$. Thus $\pi_0 \mathscr{I}_{Y_1, \tau, \lambda} = \pi_0 j_* j_{1*}(\mathrm{ind}\, \tau) \cong i_* \pi_0 j_{1*}(\mathrm{ind}\, \tau)$ by Theorem 4.14. This corollary then follows from Theorem 5.4.

PROPOSITION 5.9. *Let $(Q, \tau; \lambda)$ be a set of Beilinson–Bernstein data. Assume α is a real simple root. In $\mathscr{M}(\mathfrak{g}, K)$, we have:*

(1) *If $\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}$, then $\Gamma(\mathscr{I}_{s_\alpha Q, \tau, s_\alpha \lambda}) \cong \Gamma(\mathscr{I}_{Q, \tau, \lambda})$.*
(2) *If $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{\geqslant 0}$ and $\tau(m_\alpha) \neq (-1)^{\langle \lambda, \check{\alpha} \rangle - 1}$ or $\langle \lambda, \alpha \rangle = 0$, then $\Gamma(\mathscr{I}_{s_\alpha Q, \tau, s_\alpha \lambda}) \cong \Gamma(\mathscr{I}_{Q, \tau, \lambda})$.*
(3) *If $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{>0}$ and $\tau(m_\alpha) = (-1)^{\langle \lambda, \check{\alpha} \rangle - 1}$ and i is affine, then the following exact sequences hold:*
 (a) *If α is type I,*

$$0 \to \Gamma\big(\mathscr{I}_{Q_{\alpha+}, \tau_{\alpha+}, \lambda}\big) \oplus \Gamma\big(\mathscr{I}_{Q_{\alpha-}, \tau_{\alpha-}, \lambda}\big) \to \tau\big(\mathscr{I}_{s_\alpha Q, \tau, s_\alpha \lambda}\big)$$
$$\to \tau\big(\mathscr{I}_{Q, \tau, \lambda}\big) \to \Gamma\big(\mathscr{I}_{Q_{\alpha+}, \tau_{\alpha+}, \lambda}\big) \oplus \Gamma\big(\mathscr{I}_{Q_{\alpha-}, \tau_{\alpha-}, \lambda}\big) \to 0.$$

 (b) *If α is type II,*

$$0 \to \Gamma\big(\mathscr{I}_{Q_\alpha, \tau_\alpha, \lambda}\big) \to \Gamma\big(\mathscr{I}_{s_\alpha Q, \tau, s_\alpha \lambda}\big) \to \Gamma\big(\mathscr{I}_{Q, \tau, \lambda}\big) \to \Gamma\big(\mathscr{I}_{Q_\alpha, \tau_\alpha, \lambda}\big) \to 0.$$

(4) *If $\langle \lambda, \alpha \rangle = 0$ and $\tau(m_\alpha) = -1$, then:*
 (a) *If α is type I, $\Gamma(\mathscr{I}_{Q, \tau, \lambda}) \cong \Gamma(\mathscr{I}_{Q_{\alpha+}, \tau_{\alpha+}, \lambda}) \oplus \Gamma(\mathscr{I}_{Q_{\alpha-}, \tau_{\alpha-}, \lambda})$.*
 (b) *If α is type II, $\Gamma(\mathscr{I}_{Q, \tau, \lambda}) \cong \Gamma(\mathscr{I}_{Q_\alpha, \tau_\alpha, \lambda})$.*

(For notations, see (3.8), (3.9).)

Proof. In view of Corollary 5.8, we should look at the corresponding modules on the fiber $\pi^{-1}(y)$, where $Y_1 = Q = K \cdot x$ and $y = \pi(x)$. Since α is real, $\pi^{-1}(y) \cap Y$ in (5.3) is dense with complement y_+, y_- (cf. §3, case (c)). Since $P/B \cong L/L \cap B$, $\theta L = L$, $P \cap K = T_\alpha U$ (3.7), and $B \cap K = T \cdot U$, where U is contained in the unipotent radical of P, we have $T^0 \subset \text{Center}(L)$, $[\mathfrak{l}_y, \mathfrak{l}_y] \cong \mathfrak{sl}(2;\mathbb{C})$, and U acts trivially on $\mathbb{P}^1$. Thus, applying SL(2) theory (see [27], for example), as $(U_\lambda(\mathfrak{l}_y), K \cap P)$-modules, the corresponding isomorphisms and exact sequences hold along the fiber $\pi^{-1}(y)$, i.e., in $\mathscr{M}(U_\lambda(\mathfrak{l}_y), K \cap P)$. By Corollary 5.8, we get (1), (2), and (4). In case i is an affine embedding, the functor Γi_* is exact, hence (3). □

PROPOSITION 5.10. *Let $(Q = K \cdot x, \tau; \lambda)$ be a set of data. Assume α is a complex simple root and $\theta\alpha \notin R_x^+$. In $\mathscr{M}(\mathfrak{g}, K)$, we have:*

(1) *If $\langle \lambda, \check{\alpha} \rangle \neq 1, 2, 3, \ldots,$ then $\Gamma(\mathscr{I}_{s_\alpha Q, \tau+\alpha, s_\alpha\lambda}) \cong \Gamma(\mathscr{I}_{Q,\tau,\lambda})$.*
(2) *If $\langle \lambda, \check{\alpha} \rangle = 1, 2, 3, \ldots,$ and i is affine, then the following is exact:*

$$0 \to \Gamma\left(\mathscr{I}_{s_\alpha Q, s_\alpha\tau, \lambda}\right) \to \Gamma\left(\mathscr{I}_{s_\alpha Q, \tau+\alpha, s_\alpha\lambda}\right) \to \Gamma\left(\mathscr{I}_{Q,\tau,\lambda}\right) \to \Gamma\left(\mathscr{I}_{s_\alpha Q, s_\alpha\tau, \lambda}\right) \to 0.$$

(For notations, see (3.13), (3.14).)

Proof. The proof is similar to that of the previous proposition. Write $K \cap B = T \cdot U$. Choose α ($\theta\alpha$ resp.) root vector E_α ($E_{\theta\alpha}$ resp.) such that $\theta E_\alpha = E_{\theta\alpha}$. Then $K \cap P = T \cdot \exp\{\mathbb{C}(E_\alpha + E_{\theta\alpha})\} \cdot U$. Thus U acts trivially on $P/B \cong \mathbb{P}^1$, and $E_\alpha + E_{\theta\alpha}$ acts in the same way as E_α acts on $L/L \cap B \cong P/B$. $\pi^{-1}(y) \cap Y_1$ in (5.3) is then an affine line. Applying the theory of highest-weight modules for SL(2), the corresponding results hold in $\mathscr{M}(U(\mathfrak{l}_y), K \cap P)$. The proposition then follows from Corollary 5.8. □

We remark that in general, unlike the situation of K-orbits in the full flag variety, the embedding i of some K-orbit in G/P is not affine. Nevertheless, the proofs above give, regardless of the affinity of i, the following maps:

For α real, I_α: $\Gamma(\mathscr{I}_{s_\alpha Q, \tau, s_\alpha\lambda}) \to \Gamma(\mathscr{I}_{Q,\tau,\lambda})$.

For α complex and $\theta\alpha \in R^+$, I_α: $\Gamma(\mathscr{I}_{s_\alpha Q, \tau+\alpha, s_\alpha\lambda}) \to \Gamma(\mathscr{I}_{Q,\tau,\lambda})$. The maps I_α will be called *α-intertwining operators.*

§6. More on K-orbits. The purpose of this section is to set up basic terminologies relating to the closure of a K-orbit. This will be used in the description of the special K-types in section 8. Let $Q = K \cdot x$ be a K-orbit.

Definition 6.1. A K-orbit Q_b is a *distinguished* (θ-stable) orbit associated to Q if $Q_b = K \cdot y$ for some y such that

(1) $\mathfrak{h}_x = \mathfrak{h}_y$,
(2) R_y^+ is θ-stable outside real roots, and
(3) the set of real roots and θ-stable roots in R_x^+ is contained in R_y^+.

Note that the condition (2) above implies that Q_b is θ-stable. Since R_y^+ is θ-stable outside real roots, each real root in R_y^+ is a sum of simple real roots in R_y^+ (cf. Lemma 8.6.2 in [27]). We may therefore replace (3) by (cf. (3.6))

(3)$'$ the set of θ-stable roots in R_x^+ is contained in R_y^+.

PROPOSITION 6.2. *For each K-orbit Q, there exists at least one distinguished (θ-stable) orbit associated to Q.*

LEMMA 6.3. *If the set of nonreal roots in R_x^+ is not θ-stable, then there is a complex $(-\theta)$-stable simple root $\beta \in R_x^+$ (i.e., $\theta\beta \notin R_x^+$).*

This is Lemma 8.6.1 in [27].

LEMMA 6.4. *Let*

$$n = \tfrac{1}{2}\#\{\alpha \in R_x^+ \mid \theta\alpha \neq -\alpha \text{ and } \theta\alpha \notin R_x^+\}$$

$$m = \tfrac{1}{2}\#\{\alpha \in R_x^+ \mid \theta\alpha \neq \alpha \text{ and } \theta\alpha \in R_x^+\}.$$

Then:

(1) *There is a sequence of roots $\beta_1, \ldots, \beta_n$ in R_x^+ satisfying, for $i = 1, \ldots, n$, β_i is a complex $(-\theta)$-stable simple root in $s_{\beta_{i-1}} \cdots s_{\beta_1} R_x^+$, such that the set of nonreal roots in $s_{\beta_n} \cdots s_{\beta_1} R_x^+$ is θ-stable.*
(2) *There is a sequence of roots $\alpha_1, \ldots, \alpha_m$ in R_x^+ satisfying, for $i = 1, \ldots, n$, α_i is a θ-stable simple root in $s_{\alpha_{i-1}} \cdots s_{\alpha_1} R_x^+$, such that the set of nonimaginary roots in $s_{\alpha_m} \cdots s_{\alpha_1} R_x^+$ is $(-\theta)$-stable.*

Proof. Replacing θ by $-\theta$, (2) follows from (1). By induction, (1) follows immediately from Lemma 6.3. □

Write $w_b = s_{\beta_n} \cdots s_{\beta_1}$ in case (1) above. Then $w_b Q$ is a distinguished orbit associated to Q. This proves Proposition 6.2.

To fix terminology, we say that $w_b Q$ is joined to Q by the sequence $\alpha_1, \ldots, \alpha_n$. Conversely, we have:

(6.5) Every distinguished orbit associated to Q is joined to Q by some sequence of roots.

For the purpose of describing the closure of a K-orbit, we first consider the case of a θ-stable orbit. Let $Q_b = K \cdot y$ be such that R_y^+ is θ-stable outside real roots. Using $S = \{\text{simple real roots in } R_y^+\}$, we have a fibration π_1: $X \to X_s$. Write $Q_r = \pi_1(Q_b)$.

(6.6) $\quad Q_r$ is a closed orbit in X_s and $\overline{Q}_b = \pi_1^{-1}(Q_r)$.

This is because the parabolic corresponding to $z = \pi_1(y)$ is θ-stable and its Levi part is split (modulo center) (cf. [19]).

The essence of the notion of distinguished orbits associated to a K-orbit lies in the following description of the closure of a K-orbit. For a simple root α and a subset S in X, denote by $P_\alpha * S$ the set $\pi^{-1}(\pi(S))$, where π is the fibration $X \to X_\alpha$.

THEOREM 6.7 (Matsuki [18], Springer [24]). *Let Q be a K-orbit and Q_b an associated distinguished orbit joined to Q by the sequence $\alpha_1, \ldots, \alpha_n$. Then*

$$\overline{Q} = P_{\alpha_1} * \cdots * P_{\alpha_n} * \overline{Q}_b.$$

Note that the closure $\overline{Q}_b$ can be described via (6.6). This theorem motivates the following consideration. Suppose $Q = K \cdot x$ and $Q_b = K \cdot y$ are in the situation of (6.1). In view of (3.6)(d), (e), and (6.6), we have:

(6.8) There is a natural K-equivariant fibration $\pi\colon Q \to Q_r$.

Denote by P the parabolic subgroup corresponding to $\pi_1(y) = z$ and by L its Levi component. Let T be the reductive part of the isotropy group K_x. Then $T_1 = T \cap L_s$ is finite ($L_s = [L, L]$) and $K \cap L/T \cong K \cap L_s/T_1$ is an open orbit in $\pi_1^{-1}(z) \cong L/T \cong L_s/T_1$. Let U be the exponential of the span of $E_\alpha + \theta E_\alpha$, for $-\alpha \in R_y^+$ and α (or $-\alpha$) is $(-\theta)$-stable complex in R_x^+ (E_α is an α-root vector). Then

(6.8)′ the fiber over z in (6.8) is canonically isomorphic to $(K \cap L_s/T_1) \times U$.

Now enter the role of $\lambda \in \mathfrak{h}^*$. Write $\lambda_+ = \frac{1}{2}(\lambda + \theta\lambda)$ and $\lambda_- = \frac{1}{2}(\lambda - \theta\lambda)$. Let $(Q = K \cdot x, \tau; \lambda)$ be a set of data with λ dominant.

Definition 6.9. An orbit $Q_b = K \cdot y$ is a *λ_+-distinguished* (θ-stable) orbit associated to $(Q, \tau; \lambda)$ if Q_b is distinguished associated to Q as in (6.1), and λ_+ (considered as element in $\mathfrak{h}_x^* = \mathfrak{h}_y^*$) is R_y^+ dominant.

PROPOSITION 6.10. *For a set of data $(Q, \tau; \lambda)$ with λ dominant, there exists at least one associated λ_+-distinguished orbit.*

Proof. Note that if $\alpha \in R_x^+$ is θ-stable, then λ_+ is α-dominant. Therefore one can find a positive root system, say R_1^+, such that λ_+ is R_1^+-dominant and R_1^+ contains all the θ-stable roots in R_x^+. Since λ_+ is θ-stable and R_1^+-dominant, the non-θ-stable complex roots in R_1^+ must be orthogonal to λ_+. Set

$$R_1^{(+)}(\lambda_+) = \{\beta \in R_1^{(+)} \mid \langle \beta, \lambda_+\rangle = 0\}$$

Applying Lemmas 6.3 and 6.4 to the positive system $R_1^+(\lambda_+)$ in $R_1(\lambda_+)$, we reach a positive root system R_y^+ satisfying (1), (2), and (3)′ in Definition 6.1 and λ_+ is R_y^+-dominant. □

For the purpose of studying K-types, we have to construct intertwining operators between standard modules associated to Q and θQ. The rest of this section is devoted to the formal construction.

We define, for $Q = K \cdot x$, $\theta Q = K \cdot \theta x$, where $\mathfrak{b}_{\theta x} = \theta(\mathfrak{b}_x)$. To build a bridge from Q to θQ, we use Lemma 6.4. In the situation of (6.4), similar to w_b, Q_b, we set $w^t = s_{\alpha_m} \cdots s_{\alpha_1}$ and $Q^t = w^t Q$.

Since the set of complex roots in $w_b R_x^+$ is θ-stable, we can find a sequence of real roots $\gamma_1, \ldots, \gamma_\ell$ in $w_b R_x^+$ satisfying, for $i = 1, \ldots, \ell$, γ_i simple in $s_{\gamma_{i-1}} \cdots s_{\gamma_1}(w_b R_x^+)$, and $s_{\gamma_\ell} \cdots s_{\gamma_1}(w_b R_x^+) = \theta(w_b R_x^+)$. Write $w_r = s_{\gamma_\ell} \cdots s_{\gamma_1}$. We have $w_r w_b R_x^+ = \theta w_b R_x^+$. In short, we have the following diagram:

(6.12)
$$\begin{tikzcd}
\theta Q^t \arrow[d, dash, "\theta\alpha_n"] & & Q^t = w^t \cdot Q \arrow[d, dash, "\alpha_n"'] \\
\vdots \arrow[d, dash, "\theta\alpha_1"] & & \vdots \arrow[d, dash, "\alpha_1"'] \\
\theta Q \arrow[d, dash, "\theta\beta_1"] & & Q \arrow[d, dash, "\beta_1"'] \\
\vdots \arrow[d, dash, "\theta\beta_n"] & & \vdots \arrow[d, dash, "\beta_n"'] \\
Q_b = w_r Q_b = \theta Q_b \arrow[r, dash, "\gamma_\ell"'] & \cdots \arrow[r, dash, "\gamma_1"'] & Q_b = w_b \cdot Q
\end{tikzcd}$$

It is clear that by applying θ to sequences $\{\alpha_i\}$ and $\{\beta_j\}$, we get similar sequences for θQ. Applying (3.14) successively, we get the Beilinson–Bernstein data $(Q_b = K \cdot (w_b x), \tau + \beta_1 + \cdots + \beta_n; w_b\lambda)$. By (3.9), $(Q_b = K \cdot (w_r w_b x), \tau + \beta_1 + \cdots \beta_n; w_r w_b \lambda)$ are also Beilinson–Bernstein data. Applying (3.14) again, we see that

$$\left(K \cdot \left(s_{\theta\beta_1} \cdots s_{\theta\beta_n} w_r w_b x\right), \tau + \beta_1 + \cdots + \beta_n - \theta\beta_n - \cdots - \theta\beta_1; s_{\theta\beta_1} \cdots s_{\theta\beta_n} w_r w_b \lambda\right)$$

are Beilinson–Bernstein data. By abuse of notation, set $\theta\lambda = s_{\theta\beta_1} \cdots s_{\theta\beta_n} w_r w_b \lambda$ (which is not given by $\theta\chi(A) = \chi(\theta(A))$ for $A \in \mathfrak{h}$). The above data is then $(K \cdot (\theta x), \tau; \theta\lambda)$ for $\beta_i = \theta\beta_i$ on T. Note that this data does not depend on the choices of sequences $\{\beta_i\}$ and $\{\gamma_j\}$. Similarly,

$$\left(Q^t = K \cdot (w^t x), \tau + \alpha_1 + \cdots + \alpha_m; w^t\lambda\right)$$

and

$$\left(\theta Q^t = K \cdot \left(s_{\theta\alpha_m} \cdots s_{\theta\alpha_1} \theta x\right), \tau + \theta\alpha_1 + \cdots + \theta\alpha_m; s_{\theta\alpha_m} \cdots s_{\theta\alpha_1} \theta\lambda\right)$$

are Beilinson–Bernstein data. Since $s_{\theta\alpha_m}\cdots s_{\theta\alpha_1} = \theta(s_{\alpha_m}\cdots s_{\alpha_1})\theta = \theta w'\theta$, we have

$$s_{\theta\alpha_m}\cdots s_{\theta\alpha_1}\theta x = \theta(w'x)$$

and

$$s_{\theta\alpha_m}\cdots s_{\theta\alpha_1}\theta\lambda = \theta w'\theta\theta w_b^{-1}\theta w_r w_b\lambda = \left(\theta w_b(w')^{-1}\theta\right)^{-1} w_r w_b (w')^{-1}\cdot w'\lambda.$$

This amounts to flipping all nonimaginary roots on $w'R^+$. By abuse of notation again, we put $s_{\theta\alpha_n}\cdots s_{\theta\alpha_1}\theta\lambda = \theta w'\lambda$. Then

$$\left(\theta Q' = K\cdot\left(s_{\alpha_m}\cdots s_{\alpha_1}\theta x\right), \tau + \theta\alpha_1 + \cdots + \theta\alpha_m; s_{\theta\alpha_m}\cdots s_{\theta\alpha_1}\theta\lambda\right)$$

$$= \left(\theta Q' = K\cdot\theta w' x, \tau + (\alpha_1 + \cdots + \alpha_m); \theta w'\lambda\right).$$

PROPOSITION 6.13. *Assume λ is dominant. We have*

$$\begin{tikzcd}
\Gamma(\mathscr{I}_{\theta Q', \tau+(\alpha_1+\cdots+\alpha_m), \theta w'\lambda}) & & & & \Gamma(\mathscr{I}_{Q', \tau+(\alpha_1+\cdots+\alpha_m), w'\lambda}) \\
\circ \arrow[u, "I_{-\theta\alpha_m}", "\iota"'] & & & & \circ \arrow[u, "\iota", "I_{-\alpha_m}"'] \\
\vdots & & & & \vdots \\
\Gamma(\mathscr{I}_{\theta Q, \tau, \theta\lambda}) \arrow[u, "I_{-\theta\alpha_1}", "\iota"'] & & & & \Gamma(\mathscr{I}_{Q, \tau, \lambda}) \arrow[u, "\iota", "I_{-\alpha_1}"'] \\
\circ \arrow[u, "I_{\theta\beta_1}", "\iota"'] & & & & \circ \arrow[u, "\iota", "I_{\beta_1}"'] \\
\vdots & & & & \vdots \\
\Gamma(\mathscr{I}_{\theta Q_b, \tau+\beta_1+\cdots\beta_n, w_r w_b\lambda}) \arrow[u, "I_{\theta\beta_n}", "\iota"'] \arrow[r, "I_{\gamma_r}"'] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r, "I_{\gamma_1}"'] & \Gamma(\mathscr{I}_{Q_b, \tau+\beta_1+\cdots+\beta_n, w_b\lambda}) \arrow[u, "\iota", "I_{\beta_n}"']
\end{tikzcd}$$

The arrow indicates the direction of intertwining map.

Proof. To avoid notation confusion, let us assume $\lambda \in \mathfrak{h}_x^*$. Using Propositions 5.9 and 5.10, it is clear that the above diagram gives the correct directions. To check isomorphisms, we use Proposition 5.10(1). λ is dominant $\Rightarrow$ $I_{-\alpha_1}, \ldots, I_{-\alpha_m}$ are isomorphisms. $w_r w_b\lambda = \theta w_b\lambda$ is then dominant with respect to $\theta w_b R_x^+ \Rightarrow I_{\theta\beta_n}, \ldots, I_{\theta\beta_1}$ are isomorphisms for $-\theta\beta_i \in \theta w_b R_x^+$. Finally, $\theta\lambda$ is dominant with respect to $\theta R_x^+ \Rightarrow I_{-\theta\alpha_1}, \ldots, I_{-\theta\alpha_m}$ are isomorphisms. □

COROLLARY 6.14 (to the proof). *If $\langle\lambda, \check{\alpha}_j\rangle \neq -1, -2, \ldots$ for $1 \leq j \leq m$, α_j as in* (6.12), *then*

$$\Gamma(\mathscr{I}_{Q,\tau,\lambda}) \cong \Gamma\left(\mathscr{I}_{Q', \tau+(\alpha_1+\cdots+\alpha_m), w'\lambda}\right).$$

§7. Duality of standard modules. In this section, we want to establish a form of duality between $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ and $\Gamma(\mathscr{I}_{\theta Q,\tau,\theta\lambda})$. We recall that, for $V \in \mathscr{M}(\mathfrak{g}, K)$, the *contragredient module* V^c to V is the space of all K-finite vectors in the algebraic dual of V. If V is admissible, then $(V^c)^c \cong V$.

THEOREM 7.1. *If $(Q = K\cdot x, \tau; \lambda)$ are regular data (hence $\Gamma(\mathscr{L}_{Q,\tau,\lambda}) \neq 0$), then $(w_G Q = K\cdot w_G x, -\tau; w_G\lambda)$ are also regular data and $\Gamma(\mathscr{L}_{w_G Q, -\tau, -w_G\lambda}) \cong \Gamma(\mathscr{L}_{Q,\tau,\lambda})^c$. Here w_G is the longest element in the Weyl group of R^+.*

The proof consists of two parts. We first treat the case that λ is regular. Then we use tensor-product techniques (Zuckerman's translation functor) to handle the singular case.

Assume now that λ is regular. Then $-w_G\lambda$ is also regular and dominant. Thus $(w_G\cdot Q, -\tau; w_G\lambda)$ are regular data. It remains to show the isomorphism. In view of Theorem 2.2, it suffices to identify $\mathscr{L}_{-w_G Q, -\tau, -w_G\lambda}$ and $\Delta_{-w_G\lambda}(\Gamma(\mathscr{L}_{Q,\tau,\lambda})^c)$. By Propositions 3.2 and 3.5, we have to show that they have the same support, and for some $x \in Q$, $H_x \cap K/(H_x \cap K)_0$ acts on $H_i(\mathfrak{n}_x, \Gamma(\mathscr{L}_{Q,\tau,\lambda})^c)_{(-w_G\lambda-\rho)}$ by $-\tau$. To check these, we use a result due to Vogan.

LEMMA 7.2.

(1) *There exists a connected component Q of $\operatorname{Aut}\mathscr{G}$ such that given a Cartan subgroup $\mathscr{H}$ of $\mathscr{G}$, there exists $\mu \in Q$ which induces the inversion $h \mapsto h^{-1}$ on $\mathscr{H}$.*

(2) *Let (π, V) be an irreducible admissible $(\mathfrak{g}, K)$-module, $\mu \in Q$ leaves $\mathscr{H}$ stable. Then $({}^{\mu}\pi, V)$ is isomorphic to the contragredient module V^c.*

For the proof, (2) is a consequence of (1) and Harish-Chandra's result on distribution characters, i.e., they are locally L^1-functions which are analytic on the set of regular elements. The proof of (1), under the assumption that $\mathscr{G}$ is connected, linear semisimple with a simply connected complexification, is given in [9] (pp. 40–41). For general connected reductive groups, such an automorphism exists on its universal covering group and the defining property for such an automorphism ensures that such automorphisms descend to the original group. In our case, since $\mathscr{G}$ sits in the real point of an algebraic reductive group G, such an automorphism always exists.

For $y \in X$, by Lemma 7.2(1), there is an inner automorphism ν of $\mathscr{G}$ such that $\sigma = \nu\circ\mu$ equals $-\mathrm{Id}$ on $\mathfrak{h}_y$. Thus $\mathfrak{h}_{\sigma(y)} = \mathfrak{h}_y$, $\mathfrak{n}_{\sigma(y)} = \sigma(\mathfrak{n}_y) = w_G\cdot\mathfrak{n}_y$. Since $\Gamma(\mathscr{L}_{Q,\tau,\lambda})^c \cong {}^{\mu}\Gamma(\mathscr{L}_{Q,\tau,\lambda}) \cong {}^{\sigma}\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ as $\mathscr{G}$-modules (here ${}^{\mu}V$ stands for

$({}^{u}\pi, V))$, we have

$$(*)\qquad H_i\left(\mathfrak{n}_{\sigma(y)}, \Gamma(\mathscr{L}_{Q,\tau,\lambda})^c\right) \cong H_i\left(\mathfrak{n}_{\sigma(y)}, {}^{\sigma}\Gamma(\mathscr{L}_{Q,\tau,\lambda})\right) \cong {}^{\sigma}\left(H_i\left(\mathfrak{n}_y, \Gamma(\mathscr{L}_{Q,\tau,\lambda})\right)\right) \text{ as } (\mathfrak{h}_y, H_y \cap K) \text{ modules.}$$

Think of $\lambda - \rho \in \mathfrak{h}_y^*$ dominant with respect to R_y^+. Then

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{h}_{\sigma(y)}}\left(H_i\left(\mathfrak{n}_{\sigma(y)}, \Gamma(\mathscr{L}_{Q,\tau,\lambda})^c\right), \mathbb{C}_{w_G(-w_G(\lambda-\rho))}\right) \\ &\cong \mathrm{Hom}_{\mathfrak{h}_{\sigma(y)}}\left({}^{\sigma}\left(H_i\left(\mathfrak{n}_y, \Gamma(\mathscr{L}_{Q,\tau,\lambda})\right)\right), \mathbb{C}_{-(\lambda-\rho)}\right) \qquad \text{by } (*) \\ &\cong \mathrm{Hom}_{\mathfrak{h}_y}\left(H_i\left(\mathfrak{n}_y, \Gamma(\mathscr{L}_{Q,\tau,\lambda})\right), \mathbb{C}_{\lambda-\rho}\right). \end{aligned}$$

Since

$$\mathrm{Hom}_{\mathfrak{h}_y}\left(H_i\left(\mathfrak{n}_y, \Gamma(\mathscr{L}_{Q,\rho,\lambda})\right), \mathbb{C}_{\lambda-\rho}\right) \begin{cases} = 0 & \text{for all } i, \quad \text{if } y \notin \overline{Q} \\ \neq 0 & \text{for some } i, \quad \text{if } y \in Q, \end{cases}$$

we have

$$\mathrm{Hom}_{\mathfrak{h}_{\sigma(y)}}\left(H_i\left(\mathfrak{n}_{\sigma(y)}, \Gamma(\mathscr{L}_{Q,\tau,\lambda})^c\right), \mathbb{C}_{-(\lambda-\rho)}\right) \begin{cases} = 0 & \text{for all } i, \\ & \text{if } \sigma(y) \notin w_G \cdot \overline{Q} = \overline{w_G \cdot Q} \\ \neq 0 & \text{for some } i, \\ & \text{if } \sigma(y) \in w_G \cdot Q. \end{cases}$$

Since $\dim(w_G \cdot Y) = \dim Y$ for any K-orbit Y (for $\theta w_G \theta = w_G$), $\mathrm{supp}(\Delta_{-w_G\cdot\lambda}\Gamma(\mathscr{L}_{Q,\tau,\lambda})^c) = \overline{w_G \cdot Q}$ by Propositions 3.2–3.5. To check the action of the component group, by Proposition 3.4(a), we may assume that H_x is the complexification of $\mathscr{H}_x$. Thus, by Proposition 3.4(c), $\mathscr{H}_x \cap \mathscr{K}/(\mathscr{H}_x \cap \mathscr{K})_0 \to H_x \cap K/(H_x \cap K)_0$ is an isomorphism. By Propositions 3.2–3.5, this group acts on $H_i(\mathfrak{n}_x, \Gamma(\mathscr{L}_{Q,\tau,\lambda}))_{(\lambda-\rho)}$ via τ. By (*) above, it acts on

$$H_i\left(\mathfrak{n}_{\sigma(x)}, \Gamma(\mathscr{L}_{Q,\tau,\lambda})^c\right)_{-(\lambda-\rho)}$$

by $-\tau$. This implies $\Gamma(\mathscr{L}_{Q,\tau,\lambda})^c \cong \Gamma(\mathscr{L}_{w_G Q, -\tau, -w_G\lambda})$ and completes the proof of Theorem 7.1 for nonsingular λ.

For general λ, choose an integral dominant weight μ such that $\lambda + \mu$ is regular dominant. Let $F_{-\mu}(F_{w_G\mu})$ be the finite-dimensional representation with lowest weight $-\mu(w_G\mu)$. Then we have $\mathscr{L}_{Q,\tau,\lambda} = \mathscr{L}_{Q,\tau+\mu,\lambda+\mu}(-\mu)$ and $\mathscr{L}_{w_G\cdot Q, -(\tau+\mu), -w_G(\lambda+\mu)}(w_G\mu) = \mathscr{L}_{w_G\cdot Q, -\tau, -w_G\lambda}$.

$$\begin{aligned} \Gamma\left(\mathscr{L}_{w_G\cdot Q, -\tau, -w_G\lambda}\right) &= \left(\Gamma\left(\mathscr{L}_{w_G\cdot Q, -(\tau+\mu), -w_G(\lambda+\mu)}\right) \otimes F_{w_G\mu}\right)_{-w_G\lambda} \\ &\cong \left(\Gamma\left(\mathscr{L}_{Q,\tau+\mu,\lambda+\mu}\right)^c \otimes F_{w_G\mu}\right)_{-w_G\lambda} \qquad (\lambda+\mu \text{ is regular}) \\ &\cong \left[\left(\Gamma\left(\mathscr{L}_{Q,\tau+\mu,\lambda+\mu}\right) \otimes F_{-\mu}\right)_{\lambda}\right]^c \\ &\cong \Gamma\left(\mathscr{L}_{Q,\tau,\lambda}\right)^c. \end{aligned}$$

The first (and last) isomorphism is Zuckerman's down functor, which is contained in the proof of Theorem 2.2 ([3] or [20]). This completes the proof of Theorem 7.1. □

Theorem 7.2. *Let $(Q, \tau; \lambda)$ be a set of regular data. Then $\Gamma(\mathscr{I}_{\theta Q,\tau,\theta\lambda})$ has a unique irreducible quotient which is $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$.*

This theorem follows from Theorem 7.1 and the following:

Proposition 7.3. *Assume λ is dominant. Then*

$$\Gamma\left(\mathscr{I}_{\theta Q,\tau,\theta\lambda}\right) \cong \Gamma\left(\mathscr{I}_{w_G Q, -\tau, -w_G\lambda}\right)^c.$$

Before proving Proposition 7.3, we list

Corollary 7.4. *Let I: $\Gamma(\mathscr{I}_{\theta Q,\tau,\theta\lambda}) \to \Gamma(\mathscr{I}_{Q,\tau,\lambda})$ be the intertwining operator given in (6.13). Assume $(Q, \tau; \lambda)$ are regular and I is nontrivial. Then $\mathrm{Im}(I) = \Gamma(\mathscr{L}_{Q,\tau,\lambda})$.*

Proof. This follows trivially from Theorem 7.2 and that $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ occurs in $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$, as a composition factor, exactly once. □

The proof of (7.3) needs some preparation. Let us first consider data ($Q = K \cdot x, \tau; \lambda$) with the property that R_x^+ is $(-\theta)$-stable outside imaginary roots (i.e., data occurring at the top in (6.12)). In this case, using $S = \{$simple imaginary roots in $w'R_x^+\}$, we can form the fibration π: $X \to X_s$. Set $y = \pi(x)$ and $\mathfrak{p} = \mathfrak{l}_y + \mathfrak{u}$ the corresponding parabolic subalgebra (similarly P, L for groups). We have

(7.5) roots in $\mathfrak{l}_y$ consist of all imaginary roots;

(7.6) roots in $\mathfrak{u}$ consist of real and $(-\theta)$-stable complex roots.

Set $\mathscr{P} = P \cap \mathscr{G}$, $\mathscr{L} = L \cap \mathscr{G}$; then $\mathscr{P}$ is a parabolic subgroup of $\mathscr{G}$ with Levi part $\mathscr{L}$, and $K \cap L$ is the complexification of $\mathscr{K} \cap \mathscr{L}$ (note that $\theta L = L$ from (7.5)). Consider the data $((K \cap L)x, \tau; \lambda)$ in the setting of $(K \cap L, \pi^{-1}(y))$ (note that $K \cap L = K \cap P$ by (7.6)). Hence $\Gamma(\mathscr{I}_{(K \cap L)x, \tau, \lambda})$ is contained in $\mathscr{M}(U_\lambda(\mathfrak{l}_y), L \cap K)$. Therefore, we can form the normalized induced Harish-Chandra module $I_{\mathscr{P}}^{\mathscr{G}}(\Gamma(\mathscr{I}_{(K \cap L)x, \tau, \lambda}))$ in $\mathscr{M}(\mathfrak{g}, K)$.

PROPOSITION 7.7. *Under the above assumptions, in* $\mathscr{M}(\mathfrak{g}, K)$,

$$\Gamma(\mathscr{I}_{Q, \tau, \lambda}) \cong I_{\mathscr{P}}^{\mathscr{G}}(\Gamma(\mathscr{I}_{(K \cap L)x, \tau, \lambda})).$$

Proof. Let i denote the inclusion of $K \cdot y$ in X_s. By Corollary 5.8, $\Gamma(\mathscr{I}_{Q, \tau, \lambda}) \cong \Gamma(i_*(\Gamma(\mathscr{I}_{(K \cap L) \cdot x, \tau, \lambda - \rho(\mathfrak{u})})))$ (this is because in the definition (4.6), there is no shift by $\rho(\mathfrak{u})$). By (7.6), i is an open embedding ([19]); therefore $\Gamma(\mathscr{I}_{Q, \tau, \lambda}) \cong \operatorname{Ind}_{K \cap L}^{K}(\Gamma(\mathscr{I}_{(K \cap L) \cdot x, \tau, \lambda - \rho(\mathfrak{u})}))$. On the other hand, we have $\mathscr{G}/\mathscr{P} \cong \mathscr{K}/\mathscr{K} \cap \mathscr{L} \subset K/K \cap L \subset X_s$ and $\mathfrak{g}_0$ acts on the modules in the proposition via vector fields on X_s. The action of $\mathfrak{p}$ on the fibers at y implies that the restriction map $\operatorname{Ind}_{K \cap L}^{K}(\Gamma(\mathscr{I}_{(K \cap L) \cdot x, \tau, \lambda - \rho(\mathfrak{u})})) \to I_{\mathscr{P}}^{\mathscr{G}}(\Gamma(\mathscr{I}_{(K \cap L) \cdot x, \tau, \lambda}))$ is a $(\mathfrak{g}, K)$-morphism. Since $K/K \cap L$ is the complexification of $\mathscr{K}/\mathscr{K} \cap \mathscr{L}$, K-types consideration shows that this is an isomorphism. □

Now we come to the proof of (7.3). Since λ is dominant, by (6.13), $\Gamma(\mathscr{I}_{\theta Q, \tau, \theta\lambda}) \cong \Gamma(\mathscr{I}_{\theta Q^t, \tau + (\alpha_1 + \cdots + \alpha_m), \theta w^t \lambda})$. The data $(\theta Q^t, \tau + (\alpha_1 + \cdots + \alpha_m), \theta w^t \lambda)$ satisfy the assumption for Proposition 7.7. Retaining the same notations as in (7.7), we have

$$\Gamma\left(\mathscr{I}_{\theta Q, \tau + (\alpha_1 + \cdots + \alpha_m), \theta w^t \lambda}\right) \cong I_{\mathscr{P}}^{\mathscr{G}}\left(\Gamma\left(\mathscr{I}_{(K \cap L) \cdot \theta w^t x, \tau + (\alpha_1 + \cdots + \alpha_m), \theta w^t \lambda}\right)\right). \tag{7.8}$$

On the other hand,

$$\Gamma\left(\mathscr{I}_{w_G \cdot Q, -\tau, -w_G \cdot \lambda}\right) \cong \Gamma\left(\mathscr{I}_{w_G \cdot w^t w_G^{-1} \cdot w_G Q, -(\tau + (\alpha_1 + \cdots + \alpha_m)), w_G w^t w_G^{-1}(-w_G \lambda)}\right)$$

($-w_G \cdot \lambda$ is dominant; we use the sequence $\{w_G(\alpha_j)\}$ to move upward, then apply Proposition 6.13). Applying (7.7) again, we have

$$\Gamma\left(\mathscr{I}_{w_G \cdot Q, \tau + (\alpha_1 + \cdots + \alpha_m), \theta w^t \lambda}\right) \cong I_{\mathscr{P}}^{\mathscr{G}}\left(\Gamma\left(\mathscr{I}_{(K \cap L) \cdot w_G w^t x, -(\tau + (\alpha_1 + \cdots + \alpha_m)), -w_G w^t \lambda}\right)\right). \tag{7.9}$$

By (6.5), the orbit $(K \cap L) \cdot \theta w^t x$ is closed in $\pi^{-1}(y)$ ([19]). Therefore the corresponding standard module is irreducible. By Theorem 7.1, we have (note that $w_G = w_L \cdot \theta$ on $w^t R_x^+$),

$$\Gamma\left(\mathscr{I}_{(K \cap L) \cdot \theta w^t x, \tau + (\alpha_1 + \cdots + \alpha_m), \theta w^t \lambda}\right) \cong \Gamma\left(\mathscr{I}_{(K \cap L) \cdot w_G w^t x, -(\tau + (\alpha_1 + \cdots + \alpha_m)), -w_G w^t \lambda}\right)'.$$

This, together with (7.8) and (7.9), proves Proposition 7.3. □

As an application of Proposition 7.7, we list a criterion for certain standard modules to vanish. For each $x \in X$, consider the set R_x^I of all imaginary roots in R_x. Then $R_x^{I,+} = R_x^I \cap R_x^+$ is a positive root system.

COROLLARY 7.10. *Assume λ is dominant. Then $\Gamma(\mathscr{I}_{Q, \tau, \lambda}) \neq 0$ iff $\langle \lambda, \alpha \rangle \neq 0$ for all compact simple roots α in $R_x^{I,+}$.*

Proof. Using 7.7, it suffices to look at the case when Q is closed and all roots are imaginary. Therefore Theorem 3.15 implies the corollary. □

§8. Special K-types. Given a standard module, say $\Gamma(\mathscr{I}_{Q, \tau, \lambda})$, we want to find a certain set of "special" K-types in it which will locate the unique irreducible submodule $\Gamma(\mathscr{L}_{Q, \tau, \lambda})$ when the data $(Q, \tau; \lambda)$ are regular. These special K-types are subtly related to the singularity of $\bar{Q}$ and the dominance of the character λ.

We first recall the notion of "fine K-types" introduced by Bernstein–Gelfand–Gelfand in [5], [6] (see also [25]). Let G be a split group and $(Q, \tau; \lambda)$ be a set of data with $Q = K \cdot x$ dense in X. K_x is then finite and $\Gamma(\mathscr{I}_{Q, \tau, \lambda}) = \operatorname{Ind}_{K_x}^{K}(\tau)$. Using notations in Lemma 3.7, $\mathfrak{k}$ has a basis $\{Z_\alpha | \alpha \in R_x^+\}$.

Definition 8.1. A K-type μ is called *fine* if for each simple root α the eigenvalue of $\mu(Z_\alpha)$ lies in the range $[-1, 1]$.

To describe this definition geometrically, we use z as coordinate on L_y $(y \in Q)$ such that $L_y \cap Q = \{z | z \neq 0, \infty\}$ with respect to the fibration $X \to X_\alpha$ (for notation, see (3.6)).

Definition 8.1′. A section $f \in \Gamma(\mathscr{I}_{Q, \tau, \lambda})$ is called *fine* on Q if for each $y \in Q$, α a simple root, the restriction of f on L_y is contained in the linear span of 1, $z^{1/2}$ and $z^{-1/2}$.

Vaguely this means that f should be as flat as possible.

PROPOSITION 8.2. *In the situation of a split group G and an open orbit Q, a fine K-type μ occurs as a subspace V_μ in $\Gamma(\mathscr{I}_{Q, \tau, \lambda})$ iff every section $f \in V_\mu$ is fine.*

This is just a restatement of definitions.

Let us remark that, for a given standard module $\mathscr{I} = \mathscr{I}_{Q, \tau, \lambda}$, there is a natural filtration defined by the degree of derivatives transversal to Q, i.e., $\mathscr{I}^0 \subset \mathscr{I}^{-1} \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathscr{I}^{-k} = \mathscr{I}$, where $\mathscr{I}^{-k}$ = subsheaf of local sections of $\mathscr{I}$ with degree of transversal derivative $\leqslant k$ $(k \in \mathbb{Z}, k \geqslant 0)$. Note that $\mathscr{I}_{Q, \tau, \lambda}^0 = \operatorname{ind} \tau \otimes \Lambda^t \mathscr{N}_{Q|X}$ on Q.

Let Q_b be a distinguished (θ-stable) orbit associated to Q (6.1). In the following we will use notations in (6.6)–(6.8). Recall that there is a K-equivariant fibration π: $Q \to Q_r$ whose fiber over $z \in Q_r$ is canonically identified with $(K \cap L_s/T_1) \times U$ and $K \cap L_s/T_1$ is an open orbit in $\pi_1^{-1}(z)$. By (6.7), we have

$\Lambda^t\mathcal{N}_{Q|X} = \pi^*(\Lambda^t\mathcal{N}_{Q_r|X_s})$. Therefore we have, on $(K \cap L_s/T_1) \times U$

$$(8.3) \quad \left[\pi_0\left(\operatorname{ind}\tau \otimes \Lambda^t\mathcal{N}_{Q|X}\right)\right]_z \cong \operatorname{Ind}_{T_1}^{K\cap L_s}\left(\tau \otimes \left[\Lambda^t\mathcal{N}_{Q_r|X_s}\right]\Big|_{T_1}\right) \otimes_{\mathbf{C}} \mathbf{C}[U].$$

Here $[\Lambda^t\mathcal{N}_{Q_r|X_s}]$ denotes the $K \cap L$-module given by the action of $K \cap L$ $(\subset K_z)$ on $[\Lambda^t\mathcal{N}_{Q_r|X_s}]_z$; the same convention will be used in similar situations. In this setting, we make

Definition 8.4. A section $f \in \Gamma(\mathcal{I}^0_{Q,\tau,\lambda}) \cong \Gamma(Q, \operatorname{ind}\tau \otimes \Lambda^t\mathcal{N}_{Q|X})$ is called *Q_b-special* if for each $z \in Q_r$ the restriction of f on $\pi^{-1}(z) = (K \cap L_s/T_1) \times U$ is constant along U and is fine on $K \cap L_s/T_1$ (Definition (8.1)′). A K-type μ occurring as a subspace V_μ in $\Gamma(\mathcal{I}^0_{Q,\tau,\lambda})$ is *Q_b-special* if each section $f \in V_\mu$ is Q_b-special.

Definition 8.5. A K-type μ which occurs as a subspace V_μ in $\Gamma(\mathcal{I}_{Q,\tau,\lambda})$ is called *special* if $V_\mu \subset \Gamma(\mathcal{I}^0_{Q,\tau,\lambda})$ and V_μ is Q_b-special for every λ_+-distinguished orbit associated to $(Q, \tau; \lambda)$.

We remark that, in view of (6.7), (6.8), and (8.1)′, a special K-type consists of sections of $\mathcal{I}^0_{Q,\tau,\lambda}$ on Q which grow as slowly as possible toward the boundary of Q (in other words, they should be as flat as possible on Q). Note that when G is split and Q is open (as in 8.1), these special K-types are just the fine K-types by (8.2). In the other extreme, i.e., when G contains a θ-fixed Cartan subgroup and Q is a closed orbit, then the special K-types are just the lowest K-types for (limit of) discrete-series representations. The general case is a mixture of these two.

PROPOSITION 8.6. *Suppose Q_b is a λ_+-distinguished orbit associated to the data $(Q = K \cdot x, \tau; \lambda)$ and assume that $\langle\lambda, \alpha\rangle \neq 0$ for all compact simple roots α in $R_x^{I,+}$, then Q_b-special K-types exist in $\Gamma(\mathcal{I}_{Q,\tau,\lambda})$.*

Note that the assumption above implies the nontriviality of $\Gamma(\mathcal{I}_{Q,\tau,\lambda})$ by (7.10) when λ is dominant. This proposition, in a different form, is due to Vogan [25]. We will translate the proposition into a statement asserting that certain weights are dominant. The necessary computations to justify the assertion on dominance have been done by A. W. Knapp [13]. For simplicity, we consider the case of connected K first, then discuss the case when K is not connected.

It suffices to produce some K-types in $\Gamma(\mathcal{I}^0_{Q,\tau,\lambda}) \cong \Gamma(Q_r, \pi_0(\operatorname{ind}\tau \otimes \Lambda^t\mathcal{N}_{Q|X}))$ satisfying the condition in (8.4). We first define a positive root system for $\mathfrak{k}$. Recall notations: $y \in Q_b$, $z = \pi(y) \in Q_r$, $K_y = T \cdot U_1$, $K_z = (L \cap K) \cdot U_1$, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Since R_y^+ is θ-stable outside real roots, we can choose a simple real root, say $\beta \in R_y^+$. Using Lemma 6.4, we can push $c_\beta R_y^+$ to a θ-stable (outside reals) positive root system. Repeating this process, we reach a certain θ-stable positive root system R_1^+, with $\mathfrak{h}_1 = \mathfrak{t}^c \oplus \mathfrak{a}'$, $\mathfrak{t}^c = \mathfrak{t} \oplus \mathfrak{t}_r$ is the Cartan subalgebra in $\mathfrak{k}$. Set $R_1^+(\mathfrak{l}, \mathfrak{h}_1) = R(\mathfrak{l}, \mathfrak{h}_1) \cap R_1^+$. Define a positive root system for $(\mathfrak{k}, \mathfrak{t}^c)$ by

setting $R^+(\mathfrak{k}, \mathfrak{t}^c)$ = the restriction of R_1^+ to $\mathfrak{t}^c$. Set

$$R^+(\mathfrak{k} \cap \mathfrak{l}, \mathfrak{t}^c) = R^+(\mathfrak{k}, \mathfrak{t}^c) \cap R(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t}^c).$$

By the construction, we have

$$(8.7) \quad \lambda_+ \text{ is } R^+(\mathfrak{k}, \mathfrak{t}^c)\text{-dominant and } R(\mathfrak{u}_1, \mathfrak{t}^c) \subset (-R^+(\mathfrak{k}, \mathfrak{t}^c)).$$

By the Borel–Weil–Bott theorem, each K-type occurring in $\Gamma(\mathcal{I}^0_{Q,\tau,\lambda}) \cong H^0(K/K_z, \operatorname{ind}(\operatorname{Ind}_T^{K\cap L}\tau) \otimes \Lambda^t\mathcal{N}_{Q_r|X_s})$ (cf. 8.3) has $R^+(\mathfrak{k}, \mathfrak{t}^c)$-highest weight $[\omega] + [\Lambda^t\mathcal{N}_{Q_r|X_s}]$, where $[\omega]$ is the highest weight of a $K \cap L$-type ω in $\operatorname{Ind}_T^{K\cap L}\tau$; and any such ω with $[\omega] + [\Lambda^t\mathcal{N}_{Q_r|X_s}]$ dominant gives rise to a K-type in $\Gamma(\mathcal{I}^0_{Q,\tau,\lambda})$. Since $\operatorname{Ind}_{T\cap L_s}^{K\cap L_s}(\tau) = \operatorname{Ind}_{T_1}^{K\cap L_s}(\tau|_{T_1})$ (cf. (6.8)), every $K \cap L_s$-type μ in $\operatorname{Ind}_{T_1}^{K\cap L_s}(\tau|_{T_1})$ gives rise to K-type in $\operatorname{Ind}_T^{K\cap L}(\tau)$, denoted by (τ, μ). In view of (8.1)′ and (8.4), a K-type in $\Gamma(\mathcal{I}^0_{Q,\tau,\lambda})$ with dominant weight $[(\tau, \mu)] + [\Lambda^t\mathcal{N}_{Q_r|X_s}]$ is Q_b-special iff μ is a fine $K \cap L_s$-type in $\operatorname{Ind}_{T_1}^{K\cap L_s}(\tau|_{T_1})$. The theorem of Bernstein–Gelfand–Gelfand in [5], [6], ensures the existence of fine $K \cap L_s$-types in $\operatorname{Ind}_{T_1}^{K\cap L_s}(\tau)$. Therefore, under the above setting.

(8.8) The set of Q_b-special K-types in $\Gamma(\mathcal{I}_{Q,\tau,\lambda})$ is in one-to-one correspondence with the set of fine $K \cap L_s$-types in $\operatorname{Ind}_{T_1}^{K\cap L_s}(\tau)$ (counted with multiplicity) such that $[(\tau, \mu)] + [\Lambda^t\mathcal{N}_{Q_r|X_s}]$ is $R^+(\mathfrak{k}, \mathfrak{t}^c)$-dominant.

Proposition 8.6 asserts that under appropriate conditions such fine $K \cap L_s$-type exists. We now translate (8.8) into a condition on weights explicitly.

Denote by Q_1 the K-orbit in X corresponding to the triple $(\mathfrak{h}_1, R_1, R_1^+)$. By the construction of R_1^+, under the fibration π_1: $X \to X_s$ in (6.6), we have

$$(8.9) \quad \begin{array}{ccccc} \pi_1^{-1}(z) & \to & \bar{Q}_b & \to & Q_r \\ & & & & \| \\ \pi_1^{-1}(z) \cap Q_1 & \to & Q_1 & \to & Q_r \end{array}.$$

Note that $\pi_1^{-1}(z) \cap Q_1$ is a $K \cap L$-orbit in $\pi_1^{-1}(z)$, which is closed. By examining the Lie algebras, we have, as characters of T^c, $[\Omega_{Q_1}] = 2\rho_K$ and $[\Omega_{\pi_1^{-1}(z)\cap Q_1}] = 2\rho_{K\cap L_s}$. Since $[\Omega_{X_s}] = -2\rho(\mu_1)$, we have $\langle[\Omega_{X_s}], Z_\alpha\rangle = \langle[\Omega_{X_s}], \check{\alpha}\rangle = 0$ for $\alpha \in R(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t}^c)$ (cf. (3.7)). Thus $[\Omega_{X_s}]$ is trivial on T_r. Moreover, since $\Lambda^t\mathcal{N}_{Q_r|X_s} = \Omega_{X_s} \otimes \Omega_{Q_r}^{-1} = \Omega_{X_s} \otimes (\Omega_{Q_1} \otimes \Omega^{-1}_{\pi_1^{-1}(z)\cap Q_1})^{-1}$, we have

$$(8.10) \quad \begin{aligned} \left[\Lambda^t\mathcal{N}_{Q_r|X_s}\right] &= \left[\Lambda^t\mathcal{N}_{Q_b|X}\right] \quad \text{on } T \\ \left[\Lambda^t\mathcal{N}_{Q_r|X_s}\right] &= 2\rho_{K\cap L_s} - 2\rho_K \quad \text{on } T_r. \end{aligned}$$

Write

$$\rho_n = \text{half sum of noncompact roots in } R_y^+$$

$$\rho_c = \text{half sum of compact roots in } R_y^+;$$

then $\rho(R_y^+) = \rho_c + \rho_n + \rho(\text{complex roots in } R_y^+)$ on $\mathfrak{t}$. Now $[\Lambda^t\mathcal{N}_{Q_r|X_s}] = 2\rho_n +$

ρ(complex roots in $R_y^+) = \rho - \rho_c + \rho_n$ on $\mathfrak{t}$. Recall that $d\tau = \lambda_+ - \rho$ on $\mathfrak{t}$. The condition on the fine $K \cap L_s$-type μ in (8.8) amounts to

$$(8.11) \qquad \left(\lambda_+ - \rho_c + \rho_n, 2\rho_{K\cap L_s} - 2\rho_K + [\mu]\right) \text{ is } R^+(\mathfrak{k}, \mathfrak{t}^c)\text{-dominant.}$$

Here (,) respects the decomposition $\mathfrak{t}^c = \mathfrak{t} \oplus \mathfrak{t}_r$.

The existence of μ in (8.11) involves a rather complicated computation on root systems. In our setting as above, this computation has been carried out by A. W. Knapp in [13],[14]. (In [13], the existence of such μ is established for nondegenerate data. It is not difficult to see that the general case can be reduced to the case with nondegenerate data, say, by using some ideas in [16]; see Knapp [15].) This completes the proof of 8.6 in case K is connected. In case K is not connected, the highest weight of a K-type does not determine the K-type. In particular, the simple statement (8.8) no longer holds. In our setting, the extension of the highest-weight theory for K has been studied by Wolf [29] and Vogan [27]. It is elementary that the existence of μ in (8.11) also provides the existence of a special K-type in this case (see Lemma 5.1.3 and Proposition 0.4.12 in [27]). □

As a corollary to the above proof, we have

PROPOSITION 8.12. *If Q_b and Q_b' are two λ_+-distinguished orbits associated to the data $(Q, \tau; \lambda)$, then the set of Q_b-special K-types coincides with that of Q_b'-special K-types.*

COROLLARY 8.13. *A K-type $V_\mu \subset \Gamma(\mathscr{I}_{Q,\tau,\lambda})$ is special iff it is Q_b-special for one λ_+-distinguished orbit Q_b associated to $(Q, \tau; \lambda)$.*

Proof of 8.12. By virtue of Lemma 6.4, we can write $Q_b = w_b Q$ and $Q_b' = w_b' Q$ for some w_b, w_b' as in (6.4). Retaining the same notations in the proof above, we have $L_s = L_s'$. By (8.7) there exists $w \in W(\mathfrak{k}, \mathfrak{t}^c)$ such that $w \cdot \lambda_+ = \lambda_+$, $w \cdot R^+(\mathfrak{k}, \mathfrak{t}) = R'^+(\mathfrak{k}, \mathfrak{t}^c)$. Since the set of imaginary roots in R_x^+ $(Q = K \cdot x)$ is the same as that of R_y^+ (and $R_y'^+$), we have $w(\lambda_+ - \rho_c + \rho_n) = \lambda_+ - \rho_c + \rho_n$. Therefore, for a fine $K \cap L_s$-type μ in $\mathrm{Ind}_{T_1}^{K\cap L_s}(\tau)$, if $(\lambda_+ - \rho_c + \rho_n, 2\rho_{K\cap L_s} - 2\rho_K + [\mu])$ is $R^+(\mathfrak{k}, \mathfrak{t}^c)$-dominant, then $w \cdot (\lambda_+ - \rho_c + \rho_n, 2\rho_{K\cap L_s} - 2\rho_K + [\mu]) = (\lambda_+ - \rho_c + \rho_n, 2\rho'_{K\cap L_s} - 2\rho_K' + w[\mu])$ is $R'^+(\mathfrak{k}, \mathfrak{t}^c)$-dominant. The result follows from (8.8) and (8.11). □

COROLLARY 8.14. *If λ is dominant and $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ is not trivial, then special K-types exist in $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$.*

Proof. This follows from (8.6), (8.13), and (6.10). □

THEOREM 8.15. *If $(Q, \tau; \lambda)$ is regular, then all the special K-types occur in $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$.*

Note that in case $(Q, \tau; \lambda)$ is regular, special K-types exist by (8.14). This theorem, stated in different form, is due to D. Vogan [25].

Proof. λ being dominant, we can complete the diagram (6.13) with Q_b an associated λ_+-distinguished orbit (cf. (6.10)). In view of (8.8), (8.11), (8.13), all the standard modules in the diagram have the same set of special K-types. By Corollary 7.4, it suffices to show that each intertwining operator I_α gives a bijection of special K-types. This holds trivially when α is a complex root for special K-types that occur in $\Gamma(\mathscr{I}^0)$ (cf. 5.10). It remains to check the case α is real. The theorem follows from the next:

LEMMA 8.16. *Assume R_x^+ is θ-stable outside real root, λ_+ is R_x^+-dominant, and $\langle \lambda, \check{\alpha} \rangle \neq -1, -2, \ldots$ for a real simple root α. Then I_α: $\Gamma(\mathscr{I}_{s_\alpha Q, \tau, s_\alpha \lambda}) \to \Gamma(\mathscr{I}_{Q,\tau,\lambda})$ gives a bijection between special K-types $(Q = K \cdot x)$.*

Proof. By Proposition 5.9, it suffices to consider the case when $\langle \lambda, \check{\alpha} \rangle = n$ is a positive integer and $\tau(m_\alpha) = (-1)^{n-1}$. It suffices to check this on a fiber. Let f be a Q-special section occurring in a special K-type. Note that in this case $Q_b = Q$ is λ_+-distinguished and f is fine on each fiber ((8.1)') by the SL(2)-theory (in the proof of 5.9) and the definition of τ_α (or $\tau_{\alpha+}, \tau_{\alpha-}$) given in (3.8). We have to show that

$$(8.17) \qquad \tau(m_\alpha) = -1 \Rightarrow \left\{\left\langle \left[\Lambda^t \mathscr{N}_{Q_{\alpha-}|X}\right], Z_\alpha \right\rangle - (n-1)\right\} < m \pm 1$$

$$< \left\{\left\langle \left[\Lambda^t \mathscr{N}_{Q_{\alpha+}|X}\right], Z_\alpha \right\rangle + (n-1)\right\}$$

$$\tau(m_\alpha) = 1 \Rightarrow \left\{\left\langle \left[\Lambda^t \mathscr{N}_{Q_{\alpha-}|X}\right], Z_\alpha \right\rangle - (n-1)\right\} < m$$

$$< \left\{\left\langle \left[\Lambda^t \mathscr{N}_{Q_{\alpha+}|X}\right], Z_\alpha \right\rangle + (n-1)\right\}.$$

Here $m = \langle [\Lambda^t \mathscr{N}_{Q|X}], Z_\alpha \rangle$. This is because the Z_α-eigenvalues of f must be of the same parity as $(\tau \otimes [\Lambda^t \mathscr{N}_{Q|X}])(m_\alpha) = \tau(m_\alpha)(-1)^m$. To check (8.17), since $[\Lambda^t \mathscr{N}_{Q_{\alpha+}|X}] = [\Lambda^t \mathscr{N}_{Q|X}] + \alpha$, we have $\langle [\Lambda^t \mathscr{N}_{Q_{\alpha+}|X}], Z_\alpha \rangle = \langle [\Lambda^t \mathscr{N}_{Q|X}], Z_\alpha \rangle + 2 = m + 2$. Similarly, $\langle [\Lambda^t \mathscr{N}_{Q_{\alpha-}|X}], Z_\alpha \rangle = m - 2$. (8.17) follows immediately from the fact that $n \geqslant 1$. The proof of the lemma is complete. □

The following result due to Vogan is useful:

THEOREM 8.18 (Vogan [25]). *Each special K-type in $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ occurs with multiplicity* 1.

By (8.8) and (8.11), this follows from the multiplicity-1 statement for fine K-types in a principal series (for split groups) (Theorem 4.3.16 in [27]).

We conclude this section with the following remarks. In Vogan [25], a K-type π occurring in a Harish-Chandra module V is called a *lowest K-type* of V if its "length" $\|\pi\|$ is minimal among all K-types in V. The length $\|\pi\|$ is defined as $|\gamma + 2\rho_K|^2$, where γ is the highest weight of π.

THEOREM 8.19 (Vogan [25]). *Let $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ be a nontrivial standard module with λ dominant. Then in $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$, the set of special K-types coincides with that of lowest K-types.*

We remark that the essence of this theorem lies in the identification of fine K-types and lowest K-types of a principal series for a split group. The hard part is to show that lowest K-types are special.

§9. Classification of tempered Harish–Chandra modules. In this section we will apply Theorem 3.15 and the theory of intertwining operators to the classification of tempered Harish–Chandra modules for $\mathscr{G}$. We do this in the context of $\mathscr{D}$-modules first, then relate it to the theory of Knapp and Zuckerman.

The standard module $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ associated to data $(Q, \tau; \lambda)$ is called *basic* if

(1) λ is dominant and $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ is nontrivial, and
(2) $\lambda_- = \frac{1}{2}(\lambda - \theta\lambda) \in \mathfrak{h}_x^*$ is purely imaginary, thus $\langle\lambda, \alpha\rangle$ is purely imaginary for all real roots α in R_x^+.

A basic standard module $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ is called *final* if, moreover, the data $(Q, \tau; \lambda)$ is regular (cf. (3.3)).

PROPOSITION 9.1. *Every basic standard module is a direct sum of certain final basic standard modules.*

Proof. Let $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ be a basic standard module associated to data $(Q, \tau; \lambda)$. We prove the proposition by induction on $\dim Q$. Note that the proposition holds trivially if Q is a closed orbit in X. For $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$, if it is final, then we are done. If it is not final, by Theorem 3.15 and definition of basic standard modules, we have that

(1) there is some simple complex root α with $\theta\alpha \notin R_x^+$ and $\langle\lambda, \alpha\rangle = 0$; or
(2) there is some simple real root α with $\tau(m_\alpha) = -1$ and $\langle\lambda, \alpha\rangle = 0$. (The case of simple compact root α with $\langle\lambda, \alpha\rangle = 0$ is eliminated by 7.10.)

In case (1), by 5.10(1), we have $\Gamma(\mathscr{I}_{Q,\tau,\lambda}) = \Gamma(\mathscr{I}_{s_\alpha Q,\tau+\alpha;\lambda})$, which is basic, too, and $\dim s_\alpha Q = \dim Q - 1$. In case (2), 5.9(4) implies that $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ is the direct sum of basic standard modules associated to data with orbits of dimension $= \dim Q - 1$. The proposition follows by induction. □

PROPOSITION 9.2. *Let $(Q, \tau; \lambda)$ be a set of regular data. Assume $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ is completely reducible. Then $\Gamma(\mathscr{I}_{Q,\tau,\lambda}) = \Gamma(\mathscr{L}_{Q,\tau,\lambda})$ is irreducible.*

This proposition follows from the following:

LEMMA 9.3. *Let $\mathscr{I} \in \mathscr{M}(\mathscr{D}_\lambda)$. Assume λ is dominant and $\mathscr{I}$ contains a unique irreducible submodule $\mathscr{L}$. Then $\Gamma(\mathscr{L})$ is contained in every nontrivial submodule of $\Gamma(\mathscr{I})$.*

Proof. Let V be a nontrivial submodule of $\Gamma(\mathscr{I})$. By 2.2(a), there is a nontrivial morphism $\Delta V \to \mathscr{I}$. Denote by $\mathscr{K}$ its kernel. Then $\mathscr{L} \hookrightarrow \Delta V/\mathscr{K}$; hence $\Gamma(\mathscr{L}) \hookrightarrow \Gamma(\Delta V/\mathscr{K})$, since λ is dominant. Since $\Gamma(\mathscr{K})$ is the kernel of $\Gamma\Delta V \to \Gamma(\mathscr{I})$, i.e., $V \to \Gamma(\mathscr{I})$, we have $\Gamma(\mathscr{K}) = 0$ and $\Gamma(\Delta V/\mathscr{K}) = \Gamma\Delta V \cong V$.

To relate the above to the theory of Knapp and Zuckerman, we begin by recalling the definition of a basic character for the real group $\mathscr{G}$ (in the setting described in the end of §2). Let $\mathscr{P}$ be a cuspidal parabolic with Levi part $\mathscr{L} = \mathscr{M}\mathscr{A}$, $\mathscr{H}$ a Cartan subgroup with Cartan decomposition $\mathscr{H} = \mathscr{T}\mathscr{A}$ such that $\mathscr{T}$ is a compact Cartan subgroup of $\mathscr{M}$ (modulo center). Fix $\lambda \in \mathfrak{h}^*$, τ a character of $\mathscr{T}$, and a positive root system $R^+(\mathfrak{m}, \mathfrak{t})$ such that

(1) $d\tau = \lambda|_{\mathfrak{t}} - \rho(R^+(\mathfrak{m}, \mathfrak{t}))$,
(2) $\lambda|_{\mathfrak{t}}$ is $R^+(\mathfrak{m}, \mathfrak{t})$-dominant, and
(3) there is no compact root α in $R^+(\mathfrak{m}, \mathfrak{t})$ with $\langle\lambda, \alpha\rangle = 0$. Then there is an associated limit of discrete series of $\mathscr{M}$, denoted by Θ_τ (cf. §1 in [16]). Let $\mathbb{C}_{\lambda_-}$ be the one-dimensional character on $\mathscr{A}$ given by $\lambda_- = \lambda|_{\mathfrak{a}} \in \mathfrak{a}^*$. Then $\Theta_\tau \otimes \mathbb{C}_{\lambda_-} \in (\mathscr{M}\mathscr{A})\hat{}$ and we can form the (normalized) induced Harish–Chandra module $I_{\mathscr{P}}^{\mathscr{G}}(\Theta_\tau \otimes \mathbb{C}_{\lambda_-})$.

Extend $R^+(\mathfrak{m}, \mathfrak{t})$ to a positive root system $R^+(\mathfrak{g}, \mathfrak{h})$ such that the nilpotent radical of $\mathfrak{p}_0$ is given by $\sum_{\alpha \in R_1^+} \mathfrak{g}_0^{-\alpha}$, where R_1^+ is the positive system for the pair $(\mathfrak{g}_0, \mathfrak{a}_0)$, determined by the restriction of $R^+(\mathfrak{g}, \mathfrak{h})$ on $\mathfrak{a}_0^*$. Then the only θ-stable roots in $R^+(\mathfrak{g}, \mathfrak{h})$ are the imaginary roots and $\rho(R^+(\mathfrak{m}, \mathfrak{t})) = \rho(R^+(\mathfrak{g}, \mathfrak{h}))|_{\mathfrak{t}}$. This positive root system corresponds to a point x in X. Thus $(Q = K \cdot x, \tau; \lambda)$ is a set of data.

THEOREM 9.4 (Bernstein; Hecht–Miličić–Schmid–Wolf).

$$\Gamma(\mathscr{I}_{Q,\tau,\lambda}) \cong I_{\mathscr{P}}^{\mathscr{G}}(\Theta_\tau \otimes \mathbb{C}_{\lambda_-}).$$

This is a special case of the more general results of Bernstein (see [7]) and Hecht–Miličić–Schmid–Wolf ([12]). We remark that in this case, in view of (7.7), it follows from the identification of limits of discrete series and standard modules associated to closed orbits for groups with θ-invariant Cartan subgroups. Using the characterization of limits of discrete series by their K-types (Schmid [23]), the identification can be deduced from the Blattner formula for such standard modules.

Recall from Knapp–Zuckerman [16] that an induced Harish–Chandra module $I_{\mathscr{P}}^{\mathscr{G}}(\Theta_\tau \otimes \mathbb{C}_{\lambda_-})$ as above is called a *basic character* if $\lambda_- = i\nu$ for some $\nu \in \mathfrak{a}_0^*$. It's called *final* if, moreover, there is no real root α with $\tau(m_\alpha) = (-1)^{\langle\rho_\alpha, \check{\alpha}\rangle}$ and $\langle\lambda_-, \alpha\rangle = 0$ (for the definition of ρ_α see page 453 in [16]).

PROPOSITION 9.5. (a) {*Nontrivial basic characters*} = {*Basic standard modules*}.
(b) {*Nontrivial final basic characters*} = {*Final basic standard modules*}.

Proof. Let $I_{\mathscr{P}}^{\mathscr{G}}(\Theta_\tau \otimes \mathbb{C}_{\lambda_-})$ be a basic character. By (9.4), it's isomorphic to $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$. By our construction of R_x^+ $(Q = K \cdot x)$, λ fails to be dominant only at complex roots. Hence, by (6.13), we can flip complex simple roots successively to some $y \in X$ such that λ is R_y^+-dominant. Then the corresponding module $\Gamma(\mathscr{I}_{K\cdot y,\tau',\lambda'})$ is basic and $\Gamma(\mathscr{I}_{K\cdot y,\tau',\lambda'}) \cong \Gamma(\mathscr{I}_{Q,\tau,\lambda}) \cong I_{\mathscr{P}}^{\mathscr{G}}(\Theta_\tau \otimes \mathbb{C}_{\lambda_-})$. Note that this process can also be reversed; thus (a) follows. In view of Proposition 5.10(1), by flipping simple complex roots if necessary, we may assume $R_y^+(\lambda) = \{\alpha \in R_y^+ \mid \langle \lambda, \alpha \rangle = 0\}$ is θ-stable outside real roots. The condition for $I_{\mathscr{P}}^{\mathscr{G}}(\Theta_\tau \otimes \mathbb{C}_{\lambda_-})$ to be final, as is easy to check from the definition, amounts to $\tau'(m_\alpha) = 1$ for α real in $R_y^+(\lambda)$, which is equivalent to $\tau'(m_\alpha) = 1$ for all simple real roots α in $R_y^+(\lambda)$ (for, in $R_y^+(\lambda)$, the real roots are spanned by simple real roots). Therefore, if $I_{\mathscr{P}}^{\mathscr{G}}(\Theta_\tau \otimes \mathbb{C}_{\lambda_-})$ is final, so is $\Gamma(\mathscr{I}_{K\cdot y,\tau',\lambda'})$. Note that this process can also be reversed, so that (b) follows. □

Since basic characters are unitary, hence completely reducible, we have, by Proposition 9.2,

COROLLARY 9.6 (Knapp–Zuckermann [16]). *Every final basic character is irreducible* (*or zero*).

We now come to the classification of tempered Harish-Chandra modules (for basics of tempered representations, see [16]).

THEOREM 9.7. {*Irreducible tempered Harish–Chandra modules*} $\overset{1-1}{\leftrightarrow}$ {*Final basic standard modules*}.

Proof. A well-known result, due to Harish-Chandra, Langlands, and Trombi, asserts that every irreducible tempered Harish–Chandra module for $\mathscr{G}$ occurs as a summand of some basic character (see [16]). This, together with (9.1), (9.6), and (3.3), implies the one-to-one correspondence in the theorem. □

In Knapp–Zuckerman [16], the classification is stated in terms of $\mathscr{G}$-conjugacy classes of nondegenerate data (Theorem 14.2 in [16]). In contrast, in (9.7) above, the redundancy is controlled by K-conjugation.

We conclude this section with the following remarks. In the classification given by Knapp and Zuckerman in [16], they use Schmid character identity (i.e., 5.10(4)) to decompose a basic character and use the theory of R-groups to provide the global control in the process of the decomposition. In the context of $\mathscr{D}$-modules, the irreducibility of a final basic character is relatively easy to see, which provides the terminal control in the process of the decomposition. In contrast, we do not need the theory of R-group (i.e., the full information about the intertwining algebra of a basic character); neither do we provide. The

advantage is that our method extends to the case when $\mathscr{G}$ is of Harish–Chandra's class (or even a large class of groups; cf. Appendix B in [12]). The very same statements of (9.6) and (9.7) hold in this context, and the only (obvious) modification needed is the condition of a set of data to be regular, which is given in Remark A.9. We remark that Ivan Mirković [21] has worked out the R-group decomposition in the context of $\mathscr{D}$-modules.

Appendix. We give a proof of Theorem 3.15 here. Note that λ is assumed to be dominant. Write Δ for the set of simple roots in R^+. Set $R^+(\lambda) = \{\alpha \in R^+ \mid \langle \lambda, \alpha \rangle = 0\}$; similarly for $\Delta(\lambda)$. Since λ is dominant, $R^+(\lambda)$ is a positive root system with $\Delta(\lambda)$ the set of its simple roots. Let $\mathscr{L}_\lambda$ be an irreducible $\mathscr{D}_\lambda$-module; then $\mathscr{L}_{\lambda+\mu} = \mathscr{O}(\mu) \otimes \mathscr{L}_\lambda$ is an irreducible $\mathscr{D}_{\lambda+\mu}$-module for any integral weight μ.

LEMMA A.1. $\Gamma(\mathscr{L}_\lambda) = 0 \Rightarrow \Gamma(\mathscr{L}_{\lambda-\mu}) = 0$ *for all dominant integral* μ.

Proof. By the key lemma in [3], we have $\mathscr{L}_{\lambda-\mu} \hookrightarrow V \otimes \mathscr{L}_\lambda$, where V is an irreducible G-module with highest weight μ. Since Γ is left exact, the result follows. □

For convenience, for a simple root α, we call $\nu \in \mathfrak{h}^*$ *α-good* if ν is dominant, $\langle \nu, \alpha \rangle = 0$ and $\langle \nu, \beta \rangle \neq 0$ for all $\beta \in R^+ - \{\alpha\}$.

LEMMA A.2. *If* $\Gamma(\mathscr{L}_\lambda) = 0$, *then there exist* $\alpha \in \Delta(\lambda)$ *and an α-good* $\nu \in \mathfrak{h}^*$ *such that* $\Gamma(\mathscr{L}_\nu) = 0$.

The proof given here is due to Beilinson and Bernstein. We remark that this lemma, in case of $(\mathfrak{g}, K)$-modules, can also be derived from Vogan's results (§7.3 in [27]). We prove this by contradiction. Let $\alpha \in \Delta(\lambda)$ and $\chi \in \mathfrak{h}^*$ with $\langle \chi, \check{\alpha} \rangle \in \mathbb{Z}_{>0}$. We consider the fibration π: $X \to X_\alpha$ and will use the notations in §4.

LEMMA A.3. *With* α, χ *as above, assume that, for every α-good* $\nu \in \mathfrak{h}^*$, $\Gamma(\mathscr{L}_\nu) \neq 0$. *Then there exists an injection* d: $\pi_0 \mathscr{L}_\chi \to \pi_0 \mathscr{L}_{s_\alpha \chi}$ *in* $\mathscr{M}(\tilde{\mathscr{D}}_\chi)$.

Assuming this lemma, we prove Lemma A.2. Denote by w the longest element in the Weyl group associated to $R^+(\lambda)$. Hence $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell}$ for $\alpha_i \in \Delta(\lambda)$. Choose an $R^+(\lambda)$-regular, dominant weight μ such that $\langle \mu, \beta \rangle = 0$ for all $\beta \in \Delta - \Delta(\lambda)$ (this can be done by using, say, the dual fundamental weights). Then $w(\lambda - \mu) = \lambda - w\mu$ is dominant-regular; hence $\Gamma(\mathscr{L}_{w(\lambda-\mu)}) \neq 0$. Assume that, for each $\alpha \in \Delta(\lambda)$ and every α-good $\nu \in \mathfrak{h}^*$, $\Gamma(\mathscr{L}_\nu) \neq 0$. Using Lemma A.2 repeatedly, we have

$$\Gamma(\mathscr{L}_{w(\lambda-\mu)}) \subset \Gamma(\mathscr{L}_{s_{\alpha_1} w(\lambda-\mu)}) \subset \cdots \subset \Gamma(\mathscr{L}_{w^{-1} w(\lambda-\mu)}).$$

Thus $\Gamma(\mathscr{L}_{\lambda-\mu}) \neq 0$. By Lemma A.1, $\Gamma(\mathscr{L}_\lambda) \neq 0$. This proves Lemma A.2.

To prove Lemma A.3, assume $\langle \chi, \check{\alpha} \rangle = n \in \mathbb{Z}_{>0}$. Let $\mu \in \mathfrak{h}^*$ be a dominant weight with $\langle \mu, \alpha \rangle = 0$. Then there is an invertible sheaf $\mathscr{O}_1(\mu)$ on X_α with

$\pi^*\mathscr{O}_1(\mu) = \mathscr{O}(\mu)$ on X. Hence $\pi_0(\mathscr{L}_{\chi+\mu}) \cong \pi_0(\mathscr{O}(\mu) \otimes \mathscr{L}_\chi) \cong \mathscr{O}_1(\mu) \otimes \pi_0(\mathscr{L}_\chi)$. Since tensoring with an invertible sheaf is exact, it is sufficient to prove Lemma A.3 for one arbitrary dominant-regular χ with $\langle \chi, \check{\alpha} \rangle = n \in \mathbb{Z}_{>0}$. To define d, note that it follows from the definition of $\mathscr{D}_\chi(1)$ that, on X, $\mathscr{O}(n\alpha)$ is the image of 1^0 in $\mathscr{D}_\chi(1)$. Thus $\mathscr{O}(n\alpha)$ acts on $\mathscr{L}_\chi$. We define

$$d_1 \colon \mathscr{L}_\chi \to \mathscr{L}_{s_\alpha \chi} = \mathscr{O}(-n\alpha) \otimes \mathscr{L}_\chi$$

via the diagram $\mathscr{L}_\chi \cong \mathscr{O}(-n\alpha) \otimes \mathscr{O}(n\alpha) \otimes \mathscr{L}_\chi \to \mathscr{O}(-n\alpha) \otimes \mathscr{L}_\chi$. The first map follows from $\mathscr{O}(n\alpha)^* = \mathscr{O}(-n\alpha)$ and the second map is the action of $\mathscr{O}(n\alpha)$ on $\mathscr{L}_\chi$. In fact, for $n = 1$, d_1 is just the exterior differentiation along fibers. It is easy to check that d_1 is a morphism as $\mathfrak{g}$-modules (but not as $\mathscr{O}_X$-modules). Denote by d the morphism $\pi_0\mathscr{L}_\chi \to \pi_0\mathscr{L}_{s_\alpha\chi}$. Then d is a morphism as $\mathscr{O}_{X_\alpha}$-modules. Hence d is a morphism in $\mathscr{M}(\tilde{\mathscr{D}}_\chi)$. Since χ is dominant-regular, $\tilde{\Gamma}$ is an equivalence of categories (Theorem 4.15), and $\tilde{\Gamma}(\pi_0\mathscr{L}_\chi) = \Gamma(\mathscr{L}_\chi)$ is an irreducible $(\mathfrak{g}, K)$-module, it suffices to show that d is nonzero, $d_y \neq 0$ on the stalks at some $y \in X_\alpha$. We can choose χ so that $\chi_1 = \chi - (n/2)\alpha$ is α-good. Hence, by assumption, $\Gamma(\mathscr{L}_{\chi_1}) \neq 0$. Therefore, $(\pi_0\mathscr{L}_{\chi_1})_y \neq 0$ for some y and

$$(\pi_0\mathscr{L}_{\chi_1})_y \otimes \left(\pi_0\mathscr{O}\left(\frac{n}{2}\alpha\right)\right)_y \to (\pi_0\mathscr{L}_\chi)_y \qquad \text{(stalks)}.$$

Note that d_y is given by the vector field E, the α-root vector in $\mathfrak{l}_y$. Since $(\pi_0\mathscr{O}((n/2)\alpha))_y$ is free over $\mathscr{O}_{X_\alpha, y}$ of dimension $n+1$ (>1) and $(\pi_0\mathscr{L}_{\chi_1})_y \neq 0$, one can find $s \in (\pi_0\mathscr{L}_{\chi_1})_y$ and $t \in (\pi_0\mathscr{O}((n/2)\alpha))_y$ such that $d_y(s \otimes t) \neq 0$. This completes the proof of Lemma A.3, hence that of Lemma A.2. □

Lemma A.2 reduces to the case when λ is a generic element on α-wall for a simple root α. It is evident that the above results hold in the category $\mathscr{M}(\mathscr{D}_\lambda, K)$, too. Let $(Q = K \cdot x, \tau; \lambda)$ be a set of data with $\lambda \in \mathfrak{h}^*$ α-good (α a simple root). As before, we have

$$\text{(A.4)} \qquad \begin{array}{ccccccc} \pi^{-1}(y) & \xrightarrow{j_3} & Y & \xrightarrow{j_2} & X - \partial Y & \xrightarrow{j_1} & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_1} \\ \pi(x) = y & \xrightarrow[i_3]{} & Y_\alpha & \xrightarrow[i_2]{} & X_\alpha - \partial Y_\alpha & \xrightarrow[i_1]{} & X_\alpha \end{array}$$

where $Y = \pi^{-1}(Y_\alpha)$, $Y_\alpha = \pi(Q) = K \cdot y$. Write $\mathscr{L}$ for $\mathscr{L}_{Q,\tau,\lambda}$, and $\mathscr{I}$ for $\mathscr{I}_{Q,\tau,\lambda}$. Since π_0 is exact (4.16(b)) and $\mathscr{L} \subset \mathscr{I}$, we have $i_2^+ i_1^* \pi_0 \mathscr{L} \hookrightarrow i_2^+ i_1^* \pi_0 \mathscr{I}$. Both sheaves being K-equivariant,

$$\text{(A.5)} \qquad i_3^* i_2^+ i_1^* \pi_0 \mathscr{L} \hookrightarrow i_3^* i_2^+ i_1^* \pi_0 \mathscr{I}.$$

On the other hand, since $\operatorname{supp} \mathscr{L} = \operatorname{supp} \mathscr{I} = \overline{Q \cap Y}$, we have $0 \neq j_2^+ j_1^* \mathscr{L} \hookrightarrow j_2^+ j_1^* \mathscr{I}$.

CLAIM 1. $j_3^* j_2^+ j^* \mathscr{L} \to j_3^* j_2^+ j_1^* \mathscr{I}$ is an inclusion.

Proof. Let $\mathscr{K}$ be the kernel of the above map. Then $\operatorname{supp} \mathscr{K} \subset \partial(\pi^{-1}(y) \cap Q)$, which is a finite set. Thus $\pi_0\mathscr{K} \neq 0$ iff $\mathscr{K} \neq 0$. Since π_0 is exact, the base change, (4.14), and (4.12) imply that $\pi_0 j_3^* j_2^+ j_1^* \mathscr{L} \to \pi_0 j_3^* j_2^+ j_1^* \mathscr{I}$ is the same as (A.5) (cf. the proof of 5.4). Hence its kernel $\pi_0\mathscr{K}$ must be 0. Therefore $\mathscr{K} = 0$. □

Write $j_4 \colon Q \hookrightarrow Y$. Then $\mathscr{I} = j_{1*} j_{2*} j_{4*}(\operatorname{ind} \tau)$. Thus $j_2^+ j_1^* \mathscr{I} \cong j_{4*}(\operatorname{ind} \tau)$. By (5.8), $j_3^* j_2^+ j_1^* \mathscr{I} = \mathscr{I}_1$, the standard module on $\pi^{-1}(y)$ associated to the data $(\pi^{-1}(y) \cap Q, \tau; \lambda - \rho(\mathfrak{u}))$ (cf. (7.7)). Let $\mathscr{L}_1$ be its unique irreducible submodule.

CLAIM 2. $\Gamma(\pi^{-1}(y), \mathscr{L}_1) \neq 0 \Rightarrow \Gamma(\mathscr{L}) \neq 0$.

Proof. By Claim 1, $\mathscr{L}_1 \subset j_3^* j_2^+ j_1^* \mathscr{L}$. If $\Gamma(\pi^{-1}(y), \mathscr{L}_1) \neq 0$, i.e., $\pi_0\mathscr{L}_1 \neq 0$, then $\pi_0 j_3^* j_2^+ j_1^* \mathscr{L} \neq 0$. Therefore $i_3^* i_2^+ i_1^* \pi_0 \mathscr{L} \neq 0$. Thus $\pi_0\mathscr{L} \neq 0$. Since λ is α-good, $\tilde{\Gamma}$ is faithful by (4.15(b)). We have $\Gamma(\mathscr{L}) = \tilde{\Gamma}(\pi_0\mathscr{L}) \neq 0$. □

LEMMA A.6. *In the above setting, $\Gamma(\pi^{-1}(y), \mathscr{L}_1) \neq 0$ iff one of the following holds*:

(a) *α is complex and $\theta\alpha \in R_x^+$.*
(b) *α is noncompact.*
(c) *α is real and $\tau(m_\alpha) \neq -1$.*

This is just a simple computation on $\mathbb{P}^1$. Therefore,

(A.7) if one of the conditions in (A.6) holds, then $\Gamma(\mathscr{L}) \neq 0$.

Conversely, we have

LEMMA A.8. *In the above setting, if none of the conditions in (A.6) holds, then* $\Gamma(\mathscr{L}) = 0$.

Proof. In view of (3.6), there are three cases:

(1) α is compact: the $\mathbb{P}^1$-computation in (A.6) shows $\Gamma(\mathscr{I}) = 0$.
(2) α is complex and $\theta\alpha \notin R_x^+$: we have $\Gamma(\mathscr{I}_{Q,\tau,\lambda}) \cong \Gamma(\mathscr{I}_{s_\alpha Q, \tau+\alpha, s_\alpha\lambda})$ by 5.10(1). Since $\langle \lambda, \alpha \rangle = 0$, $\tau + \alpha = s_\alpha\tau$ (3.14), and $s_\alpha Q \subset \partial Q$, we have $\Gamma(\mathscr{L}) = 0$.
(3) α is real and $\tau(m_\alpha) = -1$: $\Gamma(\mathscr{L}) = 0$ follows from 5.9(4) for $Q_{\alpha,+}, Q_{\alpha,-}, Q_\alpha \subset \partial Q$. □

Theorem 3.15 follows immediately from (A.2), (A.7), and (A.8).

We conclude this appendix with the following remark. In Appendix B in Hecht–Miličić–Schmid–Wolf [12], the $\mathscr{D}$-module technique is extended to the study of admissible representations for a large class of groups $\mathscr{G}$ (which contains the Harish-Chandra's class). In $\mathscr{M}(\mathscr{D}_\lambda, K, \omega)$ (see Appendix B in [12]), we have the same notion of standard modules $\mathscr{I}_{Q,\tau,\lambda}$ (and its unique irreducible submodule $\mathscr{L}_{Q,\tau,\lambda}$) for a set of Beilinson–Bernstein data $(Q = K/K_x, \tau; \lambda)$ with τ an irreducible K_x-module (may not be one-dimensional) such that $d\tau = (\lambda - \omega) \cdot \mathrm{Id}$

on $\mathfrak{t}_x$. The argument in this appendix implies that Theorem 3.15 holds in $\mathscr{M}(\mathscr{D}_\lambda, K, \omega)$.

Remark A.9. In $\mathscr{M}(\mathscr{D}_\lambda, K, \omega)$ with λ dominant, then $\Gamma(\mathscr{L}_{Q,\tau,\lambda}) \neq 0$ if and only if the conditions (a), (b) in Theorem 3.15 hold and there is no real simple root α with $\langle \lambda, \alpha \rangle = 0$, $\langle \omega, \check{\alpha} \rangle = n \in \mathbb{Z}$, and $\tau(\gamma) = (-1)^{n-1} \cdot \mathrm{Id}$ for all $\gamma \in \exp \mathbb{C} \cdot Z_\alpha$ such that under the map $\varphi\colon K \to \operatorname{Int} \mathfrak{g}$, $\varphi(\gamma) = m_\alpha$ (cf. Appendix B in [12]).

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