All about n!

by Bill Casselman

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1. Introduction

$$
\begin{aligned}
n! &= 1 \cdot 2 \cdot 3 \cdot \dots \cdot n \\
0! &= 1 \\
n! &= n \cdot (n-1)!
\end{aligned}
$$

$$
\begin{aligned}
0! &= 1 \\
1! &= 1 \\
2! &= 2 \\
3! &= 6 \\
4! &= 24 \\
5! &= 120 \\
6! &= 720 \\
7! &= 5040 \\
8! &= 40320 \\
9! &= 362880 \\
10! &= 3628800 \\
11! &= 39916800 \\
12! &= 479001600 \\
13! &= 6227020800 \\
14! &= 87178291200 \\
15! &= 1307674368000 \\
16! &= 20922789888000 \\
17! &= 355687428096000 \\
18! &= 6402373705728000
\end{aligned}
$$

19! = 121645100408832000
20! = 2432902008176640000
21! = 51090942171709440000
22! = 1124000727777607680000
23! = 25852016738884976640000
24! = 620448401733239439360000
25! = 15511210043330985984000000
26! = 403291461126605635584000000
27! = 10888869450418352160768000000
28! = 304888344611713860501504000000
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34! = 295232799039604140847618609643520000000
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36! = 371993326789901217467999448150835200000000

1000! = 4023872600770937735437024339230039857193748642107146325437999104299385123986290205920442084 8
696940480047998861019719605863166687299480855890132382966994459099742450408707375991882362 77
271887325197795059509952761208749754624970436014182780946464962910563938874378864873371191 81
045825783647849977012476632889835955735432513185323958463075557409114262417474349347553428 64
657661166779739666882029120737914385371958824980812686783837455973174613608537953452422158 65
932019280908782973084313928444032812315586110369768013573042161687476096758713483120254785 89
320767169132448426236131412508780208000261683151027341827977704784635868170164365024153691 39
828126481021309276124489635992870511496497541990934222156683257208082133318611681155361583 65
469840467089756029009505376164758477284218896796462449451607653534081989013854424879849599 53
319101723355556602139450399736280750137837615307127761926849034352625200015888535147331611 70
210396817592151090778801939317811419454525722386554146106289218796022383897147608850627686 29
671466746975629112340824392081601537808898939645182632436716167621791689097799119037540312 74
622289988005195444414282012187361745992642956581746628302955570299024324153181617210465832 03
678690611726015878352075151628422554026517048330422614397428693306169089796848259012545832 71
682264580665267699586526822728070757813918581788896522081643483448259932660433676601769996 12
831860788386150279465955131156552036093988180612138558600301435694527224206344631797460594 68
257310379008402443243846565724501440282188525247093519062092902313649327349756551395872055 96
542287497740114133469627154228458623773875382304838656889764619273838149001407673104466402 59
899490222221765904339901886018566526485061799702356193897017860040811889729918311021171229 84
590164192106888438712185564612496079872290851929681937238864261483965738229112312502418664 93
531439701374285319266498753372189406942814341185201580141233448280150513996942901534830776 44
569099073152433278288269864602789864321139083506217095002597389863554277196742822248757586 76
575234422020757363056949882508796892816275384886339690995982628095612145099487170124451646 12
603790293091208890869420285106401821543994571568059418727489980942547421735824010636774045 95
741785160829230135358081840096996372524230560855903700624271243416909004153690105933983835 77
7939410970027753472000000000000000000000000000000000000000000000000000000000000000000000000 0
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2. Where is it used?

Permutations

$$n! = \text{number of arrangements of } n \text{ objects}$$

Binomial theorem

$$\begin{aligned}(x+h)^n &= \sum_0^n \binom{n}{k} x^{n-k} h^k \\ &= x^n + n x^{n-1} h + \frac{n(n-1)}{2!} x^{n-2} h^2 + \cdots\end{aligned}$$

$$\text{since } \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots\big(n-(k-1)\big)}{k!}$$

Binomial distribution

If an event has probability p, the probability of k of these in n independent trials is

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Exponential function

$$\begin{aligned}e^x &= \lim_{\to\infty} \left(1 + \frac{x}{n}\right)^n \\ &= \lim 1 + n\left(\frac{x}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2 + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\end{aligned}$$

Taylor series

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + \cdots$$

or

$$\begin{aligned}\tau_h f &= \left(I + hD + \frac{h^2 D^2}{2!} + \cdots\right) f \\ &= e^{hD} f\end{aligned}$$

3. Estimating n! for large n

On your calculator or in a spreadsheet, you'll get something like this:

$$\begin{aligned}
9! &= 362880.0\\
10! &= 3628800.0\\
11! &= 3.99168E7\\
12! &= 4.790016E8\\
&\ldots\\
69! &= 1.71122245E99\\
70! &= 1.1978572E100\\
&\ldots\\
169! &= 4.2690680090047027E304\\
170! &= 7.257415615307994E306\\
171! &= \texttt{Infinity}
\end{aligned}$$

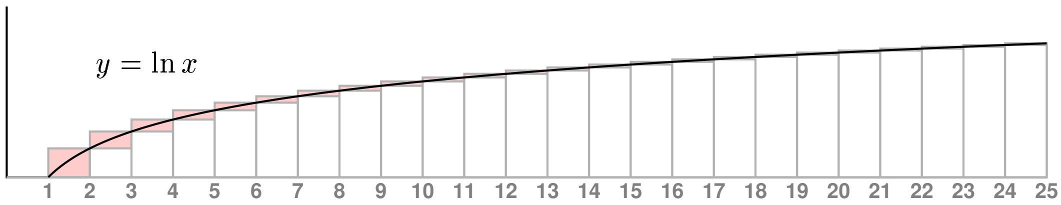
What's going on? *Hint:* $\log_2 170! = 1019.3695$.

$$\begin{aligned}
\log_{10} n! &= \log_{10} 1 + \log_{10} 2 + \cdots + \log_{10} n\\
\log_{10} x &= \ln x / \ln 10
\end{aligned}$$

So

$$\begin{aligned}
\log_{10} 171! &= 309.0937781052\\
171! &= 10^{0.0937781052} 10^{309}\\
&= 1.24101807E309
\end{aligned}$$

$$\begin{aligned}
\log_{10} 1000! &= 2567.60464422214\\
1000! &= 10^{0.60464422214} 10^{2567}\\
&= 4.0238726008E2567
\end{aligned}$$



$$
\begin{aligned}
\int_1^n \ln(x)\,dx &\sim \sum_1^{n-1} \frac{\ln(i)+\ln(i+1)}{2} \\
&= \sum_1^n \ln(i) - \frac{\ln(1)+\ln(n)}{2} \\
\ln n! &\sim \int_1^n \ln(x)\,dx + \frac{\ln(1)+\ln(n)}{2} \\
&= [x\ln x - x]_1^n + \frac{\ln(1)+\ln(n)}{2} \\
&= (n+1/2)\ln n - n + C\,.
\end{aligned}
$$

Stirling's formula:

$$n! \sim n^n e^{-n} e^C \sqrt{n}$$

What is the constant term e^C?

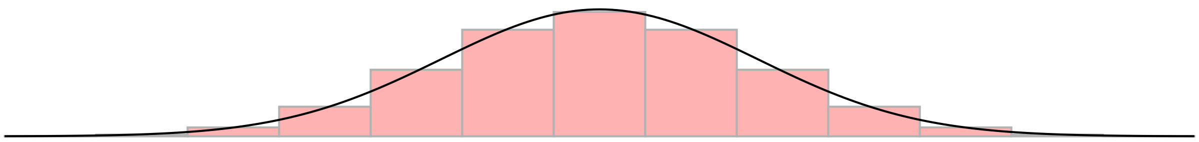
The binomial distribution

$$\binom{n}{k} p^k (1-p)^{n-k}$$

is approximately the normal distribution

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\mu = np$ **is the** *mean*, $\sigma = \sqrt{np(1-p)}$ **is the** *standard deviation*



For $n = 2m$ **and** $p = 1/2$**,** $\sigma = \sqrt{(1/2)(1/2)2m} = \sqrt{m/2}$**, so**

$$\begin{aligned}\frac{(2m)!}{(m!)^2}\frac{1}{2^{2m}} &\sim \frac{1}{2^{2m}}\frac{\sqrt{2m}(2m)^{2m}e^{-2m}e^C}{(\sqrt{m}\,m^m e^{-m}e^C)^2} = \frac{2^{2m}m^{2m}e^{-2m}}{2^{2m}m^{2m}e^{-2m}}\frac{\sqrt{2m}e^C}{me^{2C}} \\ &= \frac{\sqrt{2}}{e^C\sqrt{m}} \\ &\sim \frac{1}{\sqrt{2\pi}\sqrt{m/2}} = \frac{1}{\sqrt{\pi m}}\end{aligned}$$

so

$$e^C = \sqrt{2\pi}, \quad C = \ln\sqrt{2\pi}\,.$$

leading to the guess:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} \quad (\textbf{Stirling's formula})\,.$$

4. How accurate is Stirling's formula?

n	$n!$	**Stirling's**	**relative error = (t-a)/t**
1	1.0	0.922137	0.07786299
2	2.0	1.9190043	0.040497825
3	6.0	5.83621	0.027298402
4	24.0	23.506176	0.020576036
5	120.0	118.019165	0.016506935
6	720.0	710.0782	0.013780299
7	5040.0	4980.396	0.011826224
8	40320.0	39902.395	0.010357256
9	362880.0	359536.88	0.009212762
10	3628800.0	3598695.5	0.008295961
11	$3.99168E7$	$3.9615624E7$	0.0075450675
12	$4.790016E8$	$4.75687488E8$	0.0069187945
13	$6.2270208E9$	$6.1872394E9$	0.0063885003
14	$8.7178289E10$	$8.6661005E10$	0.005933696
15	$1.30767441E12$	$1.30043072E12$	0.0055393344
16	$2.09227906E13$	$2.08141141E13$	0.0051941196

n	$n!$	Stirling's	relative error
1	1.0	0.922137	0.07786299
2	2.0	1.9190043	0.040497825
4	24.0	23.506176	0.020576036
8	40320.0	39902.395	0.010357256
16	$2.09227906E13$	$2.08141141E13$	0.0051941196
32	$2.6313083E35$	$2.6244652E35$	0.0026006943
64	$1.26886932E89$	$1.26721823E89$	0.0013012254
128	$3.85620482E215$	$3.8536951E215$	0.000650828

This leads to a more precise guess:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}}$$

with a relative error roughly proportional to $1/n$*.*

We conjecture:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}}$$

with a relative error roughly proportional to $1/n$*.* **Since**

$$\ln S(1+c/n)/S = \ln(1+c/n) \sim c/n$$

this is equivalent to the conjecture that

$$\ln n! = (n+1/2)\ln n - n + \ln\sqrt{2\pi} + O(1/n)\,.$$

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5. Proving Stirling's formula

Theorem.

$$\ln n! = (n-1/2)\ln n - n + \ln\sqrt{2\pi} + E_n$$

where $|E_n| \leq 3/24n$*.*

Let $f(x)$ be an arbitrary smooth function. Integrate by parts:

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^1 f(x)\cdot 1\cdot\,dx \\ &= [f(x)(x+c)]_0^1 - \int_0^1 f'(x)(x+c)\,dx \\ &= \big((1+c)f(1) - c\,f(0)\big) - \int_0^1 (x+c)\,dx \\ &= \frac{f(1)+f(0)}{2} - \int_0^1 f'(x)\beta_1(x)\,dx\end{aligned}$$

if $c = -1/2$, hence

$$\beta_1(x) = x - 1/2\,.$$

$$\int_0^1 f(x)\,dx = \frac{f(1)+f(0)}{2} - \int_0^1 f'(x)\beta_1(x)\,dx$$

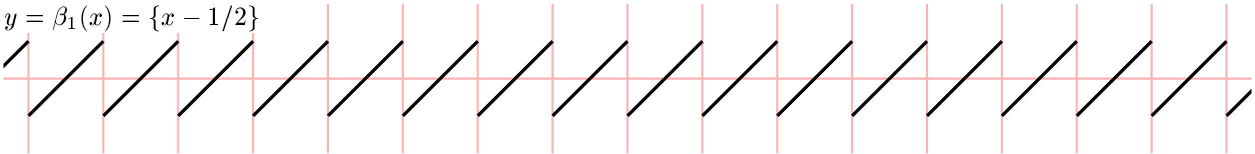
where

$$\beta_1(x) = x - 1/2\,.$$

Hence if a, b are integers

$$\int_a^b f(x)\,dx = \sum_a^b f(i) - \frac{f(b)+f(a)}{2} - \int_a^b f'(x)\beta_1(x)\,dx$$

where $\beta_1(x) = x - 1/2$ extended periodically from $[0,1]$ to $(-\infty,\infty)$ (Trapezoid rule with exact error formula).



We can invert this and then continue:

$$\begin{aligned}\sum_a^b f(i) &= \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \int_a^b f'(x)\beta_1(x)\,dx \\ &= \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \big[f'(x)\beta_2(x)\big]_a^b \\ &\qquad\qquad - \int_a^b f''(x)\beta_2(x)\,dx\end{aligned}$$

if

$$\beta_2'(x) = \beta_1(x) = \{x - 1/2\}$$
$$\beta_2 = \left\{\frac{x^2}{2} - \frac{x}{2} + C\right\}$$

where we'll set $C = 1/12$ **for reasons to appear later.**

This is

$$\begin{aligned}\sum_a^b f(i) &= \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \big[f'(x)\beta_2(x)\big]_a^b \\ &\qquad\qquad - \int_a^b f''(x)\beta_2(x)\,dx \\ &= \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \beta_2(0)\big(f'(b) - f'(a)\big) \\ &\qquad - \int_a^b f''(x)\beta_2(x)\,dx\,.\end{aligned}$$

since

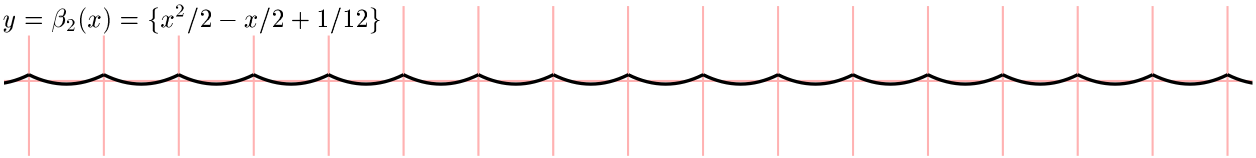
$$\beta_2(1) - \beta_2(0) = \int_0^1 \beta_2'(x)\,dx = \int_0^1 \beta_1(x)\,dx = 0$$

Lemma.

$$\sum_a^b f(i) = \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \frac{1}{12}\big(f'(b) - f'(a)\big) - \int_a^b f''(x)\beta_2(x)\,dx$$

where

$$\beta_2(x) = \left\{\frac{x^2}{2} - \frac{x}{2} + \frac{1}{12}\right\}.$$



We apply this to the case we want:

$$\begin{aligned}
\ln n! &= \sum_1^n \ln i \\
&= \int_1^n \ln x\,dx + \frac{\ln n}{2} + \frac{1}{12n} - \frac{1}{12} + \int_1^n \frac{\beta_2(x)}{x^2}\,dx \\
&= (n+1/2)\ln n - n + \frac{11}{12} + \int_1^\infty \frac{\beta_2(x)}{x^2}\,dx \\
&\qquad + \frac{1}{12n} + \int_n^\infty \frac{\beta_2(x)}{x^2}\,dx \\
&= (n+1/2)\ln n - n + \left(\frac{11}{12} + \int_1^\infty \frac{\beta_2(x)}{x^2}\,dx\right) \quad \big(\text{i.e. } e^C\big) \\
&\qquad + \frac{1}{12n} + E \quad \big(|E| \le 1/24n\big).
\end{aligned}$$

We can keep on going.

$$\sum_a^b f(i) = \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \beta_2(0)\big(f'(b) - f'(a)\big) - \int_a^b f''(x)\beta_2(x)\,dx$$
$$= \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \beta_2(0)\big(f'(b) - f'(a)\big) - \beta_3(0)\big(f''(b) - f''(a)\big) + \int_a^b f'''(x)\beta_3(x)\,dx$$

as long as

$$\beta_3'(x) = \beta_2(x)$$
$$\beta_3(1) - \beta_3(0) = \int_0^1 \beta_2(x)\,dx = 0 \quad (\textbf{motivating } \beta_2(0) = 1/12)$$

Theorem. (Euler-Maclaurin formula)

$$f(a) + \cdots + f(b) = \int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} + \sum_{k=2}^{n} (-1)^k \beta_k(0)\big(f^{(k)}(b) - f^{(k)}(a)\big) + (-1)^{n+1}\int_a^b f^{(n)}(x)\beta_n(x)\,dx$$

where

$$\beta_1(x) = \{x - 1/2\}$$
$$\beta_n'(x) = \beta_{n-1}(x)$$
$$\int_0^1 \beta_n(x)\,dx = 0 \quad (n \geq 2)\,.$$

$$\beta_1(x) = x - 1/2$$
$$\beta_2(x) = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12}$$
$$\beta_3(x) = \frac{x^3}{3!} - \frac{x^2}{4} + \frac{x}{12}$$
$$\beta_4(x) = \frac{x^4}{4!} - \frac{x^3}{12} + \frac{x^2}{24} + \frac{4}{5}$$
$$\cdots$$
$$\beta_n(x) = \frac{x^n}{n!} + \cdots$$

7. Bernoulli polynomials and numbers

It's easier to work with monic polynomials:

$$\mathbf{B}_n(x) = n!\,\beta_n(x)$$

$$\mathbf{B}_0 = 1$$
$$\mathbf{B}_1 = x - {}^1/_2$$
$$\mathbf{B}_2 = x^2 - x + {}^1/_6$$
$$\mathbf{B}_3 = x^3 - {}^3/_2\,x^2 + {}^1/_2\,x$$
$$\mathbf{B}_4 = x^4 - 2x^3 + x^2 - {}^1/_{30}$$
$$\mathbf{B}_5 = x^5 - {}^5/_4\,x^4 + {}^5/_3\,x^3 - {}^1/_6\,x$$
$$\mathbf{B}_6 = x^6 - 3x^5 + {}^5/_2\,x^4 - {}^1/_2\,x^2 + {}^1/_{42}$$
$$\cdots$$

$$\mathbf{B}_n'(x) = n\,\mathbf{B}_{n-1}(x), \int_0^1 \mathbf{B}_n(x)\,dx = 0 \quad (n \geq 2)$$

$$\mathbf{B}_n = \sum_0^n \binom{n}{k} B_{n-k} x^k \quad \textbf{where } B_n = \mathbf{B}_n(0)\,.$$

$$\begin{aligned}
B_0 &= 1 \\
B_1 &= -1/2 \\
B_2 &= 1/6 \\
B_4 &= -1/30 \\
B_6 &= 1/42 \\
B_8 &= -1/30 \\
B_{2m+1} &= 0 \quad (m \geq 1)
\end{aligned}$$

since $\mathbf{B}_n(1-x) = (-1)^n \mathbf{B}_n(x)$**,** $\mathbf{B}_n(1) = \mathbf{B}_n(0)$ **for** $n \geq 2$**.**

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k$$

$$\begin{aligned}
B_{10} &= 5/66 \\
B_{12} &= -691/2730 \\
B_{14} &= 7/6 \\
B_{16} &= -3617/510 \\
B_{18} &= 43867/798 \\
B_{20} &= -174611/330 \\
B_{22} &= 854513/138 \\
B_{24} &= -236364091/2730 \\
B_{26} &= 8553103/6 \\
B_{28} &= -23749461029/870 \\
B_{30} &= 8615841276005/14322 \\
B_{32} &= -7709321041217/510 \\
B_{34} &= 2577687858367/6 \\
B_{36} &= -26315271553053477373/1919190
\end{aligned}$$

$$
\begin{aligned}
B_{2n} &\sim \frac{2(2n)!}{(2\pi)^{2n}} \\
&\sim 4\sqrt{\pi m}\left(\frac{m}{e\pi}\right)^{2m} \\
\frac{B_{2n}}{(2n)!} &\sim \frac{2}{(2\pi)^{2n}}\,.
\end{aligned}
$$

8. Stirling's series

We can now write the Euler-Maclaurin formula as

$$
\begin{aligned}
f(a) + \cdots + f(b) = \int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} \\
+ \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}\big(f^{(2k-1)}(b) - f^{(2k-1)}(a)\big) \\
+ \int_a^b f^{(2n+1)}(x)\mathbf{B}_{2n+1}(x)\,dx\,.
\end{aligned}
$$

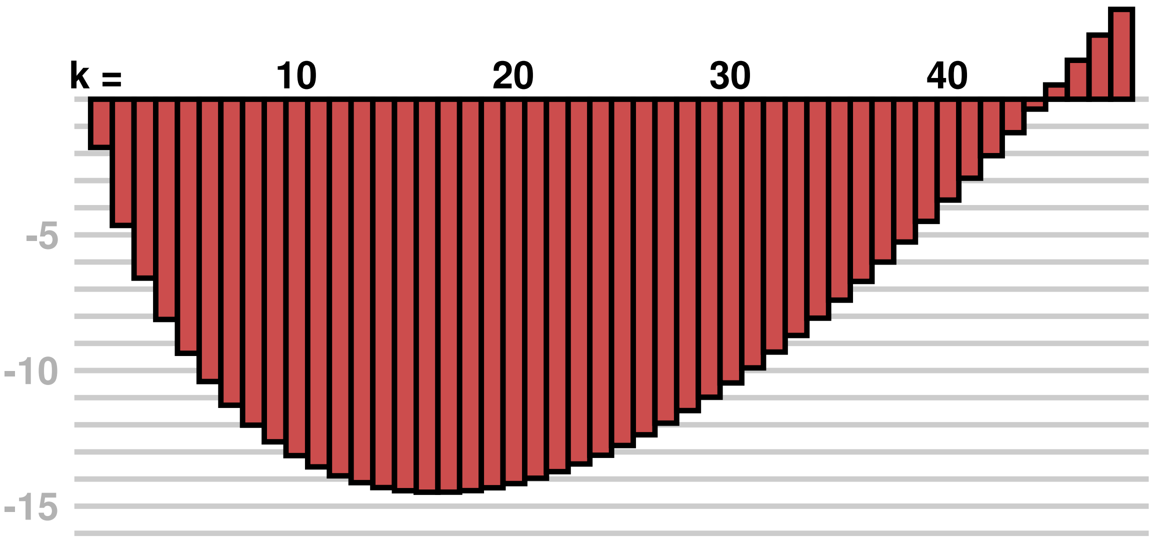
$$
\begin{aligned}
\ln n! &= (n+1/2)\ln n - n + \sqrt{2\pi} + \frac{1}{12n} + \frac{1}{2}\int_n^\infty \frac{B_2(x)}{x^2}\,dx \\
&= (n+1/2)\ln n - n + \sqrt{2\pi} + \frac{B_2}{1\cdot 2\cdot n} + \frac{B_4}{3\cdot 4\cdot n^3} \\
&\quad + \frac{B_6}{5\cdot 6\cdot n^5} + \frac{B_8}{7\cdot 8\cdot n^7} + \frac{B_{10}}{9\cdot 10\cdot n^9} + \cdots \\
&\quad + \frac{1}{2n+1}\int_n^\infty \frac{B_{2n+1}(x)}{x^{2n+1}}\,dx
\end{aligned}
$$

which suggests using the full infinite series.

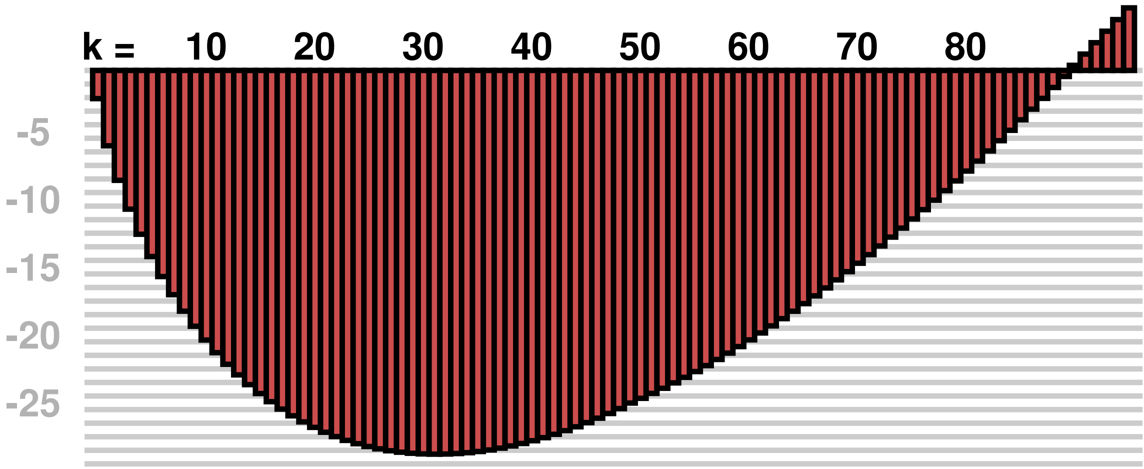
The error in truncating at a given term is roughly the size of the first term left out. For n = 5, it is possible to get 15 decimal accuracy:

k	$B_{2k}/(2k)(2k-1)n^{2k-1}$	sum
2	0.016666668	4.78751371825889
4	-2.2222222E-5	4.787491496036668
6	2.5396827E-7	4.787491750004922
8	-7.619048E-9	4.787491742385875
10	4.3097642E-10	4.787491742816851
12	-3.927095E-11	4.78749174277758
14	5.251282E-12	4.787491742782831
16	-9.683158E-13	4.787491742781863
18	2.3546348E-13	4.787491742782098
20	-7.300355E-14	4.787491742782025
22	2.8107843E-14	4.787491742782054
24	-1.3157388E-14	4.7874917427820405
26	7.358833E-15	4.787491742782048
28	-4.846437E-15	4.787491742782043
30	3.7123133E-15	4.787491742782047
32	-3.2723833E-15	4.787491742782043
34	3.2890925E-15	4.787491742782047
36	-3.739118E-15	4.787491742782043
38	4.773534E-15	4.787491742782048
40	-6.800261E-15	4.7874917427820405
42	1.07485635E-14	4.787491742782051
44	-1.8753564E-14	4.7874917427820325
46	3.595052E-14	4.787491742782069
48	-7.540077E-14	4.787491742781993
50	1.7235154E-13	4.787491742782166
52	-4.278401E-13	4.787491742781738
54	1.1496225E-12	4.787491742782888
56	-3.3336894E-12	4.787491742779554
58	1.04034194E-11	4.787491742789957
60	-3.484811E-11	4.787491742755109
62	1.2499216E-10	4.787491742880102

64	-4.789658E-10	4.787491742401135
66	1.9567048E-9	4.78749174435784
68	-8.505167E-9	4.787491735852673
70	3.9261494E-8	4.787491775114167
72	-1.9213842E-7	4.787491582975752
74	9.951884E-7	4.787492578164144
76	-5.4470347E-6	4.787487131129519
78	3.14583E-5	4.787518589429264
80	-1.9143477E-4	4.787327154655804
82	0.0012258523	4.788553006999118
84	-0.008249683	4.78030332349266
86	0.058276694	4.838580019365776
88	-0.4316309	4.406949095247427
90	3.3482258	7.755174840430623
92	-27.17362	-19.41844542742654
94	230.5032	211.0847638576286
96	-2041.6816	-1830.5969223379595
98	18866.143	17035.5462597014
100	-181710.98	-164675.4421449314



log (base 10) of 2k-th term in Stirling's series for n = 5



log (base 10) of 2k-th term in Stirling’s series for n = 10

9. What I haven’t told you

$$\sum_{1}^{n} i^k$$

$$\Gamma(s)$$

$$\zeta(s)$$

$$\zeta(2n), \quad \zeta(1-2n)$$

$$\frac{x}{e^x - 1}, \quad \frac{1}{e^D - 1}$$

$$\gamma$$

Pick’s Theorem

Lanczos’ formula

and much, much more.

Let $S(n) = 1 + \cdots n$. As we know, $S(n) = n(n+1)/2$. This is actually a function defined for all values of n, satisfying $S(x+1) = S(x) + (x+1)$ for all x. In the early 18th century, many mathematicians wondered if there were some similar *interpolation* of the function $n!$. It was Euler (born just about exactly 300 years ago) who found the good solution of this problem. Define

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt$$

This is defined for $s > 0$, and $\Gamma(s) = (n-1)!$, and integration by parts shows that $\Gamma(s+1) = s\Gamma(s)$. Thus $\Gamma(s+1)$ is the interpolation we want. This function occurs almost everywhere—for example in the formula for the volumes of higher-dimensional spheres, in number theory, and in the theory of nuclear particle scattering.

If $P(x)$ is a polynomial of degree n, then $P^{(n+1)} \equiv 0$, so the Euler-Maclaurin formula is an exact equation:

$$\sum_1^n P(n) = \int_1^n P(x)\, dx + \frac{P(1) + P(n)}{2} + \sum_{2k-1 \le n} \frac{B_{2k}}{(2k)!} \left(P^{(2k-1)}(n) - P^{(2k-1)}(1) \right).$$

The zeta function is defined by the formula

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s}\,.$$

The integral test shows that it converges for $s > 1$. There are maany connections between this function and the Euler-Maclaurin formula. For one thing, the most accurate formula used to calculate $\zeta(s)$ numerically is a version of Euler-Maclaurin. Another connection is more mysterious:

$$\zeta(2n) = \frac{2\pi)^{2n} |B_{2n}|}{2(2n)!}\,.$$

For example

$$\zeta(2) = \pi^2/6, \quad \zeta(4) = \pi^4/90\,.$$

Since $\zeta(2n)$ converges rapidly to 1 as $n \to \infty$, this implies the earlier estimate for $|B_{2n}|$.

$$\frac{x}{e^x - 1} = \sum_0^\infty \frac{B_n}{n!}\, x^n$$

is the Taylor series of $x/(e^x - 1)$. This follows from the Euler-Maclaurin formula applied to e^{tx} as a function of t.

Applied to $f(x) = 1/x$, the Euler-Maclaurin formula gives very accurate estimates for the harmonic sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

There are Euler-Maclaurin formulas in higher dimensions, but in thi sdomain there are more mysteries than final results. In fact this is an extremely active area of research. The analogue of an interval with integral endpoints is a polygon whose vertices have integral coordinates. Pick's Theorem is the simplest generalization of Euler-Maclaurin to higher dimensions. It tells you the number of points with integral coordinates inside such a polygon in terms of its area and the lengths of its sides, thus relating the integral of $f(x) \equiv 1$ over the polygon with the sum over its interior integral points and its boundary. There is apparently no entirely satisfactory analogue in dimensions more than three.

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10. References

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The End