'ed in proof. Recently Viehweg succeeded to prove $C_{3,1}$. Hence, in the above dim $V=3$, $k(V)\leq 0$; the exceptional cases do not occur.

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Jacquet Modules for Real Reductive Groups

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In the past few years work of several people has established a precise relationship between the behaviour at infinity of matrix coefficients of admissible representations—of both real and p-adic reductive groups—and embeddings into representations induced from parabolic subgroups, as well as the relationship of both to characters. Historically, these ideas were already implicit in Harish-Chandra's theory of the "constant term", and one might consider the new results as an algebraic elaboration of this theory. One crucial observation was made by Jacquet **[10]** concerning the p-adic case, and indeed this case has had a strong influence on the development of the other. For p-adic groups the relationships established are fundamental and perhaps indispensable to the whole theory of admissible representations (see **[1]**, **[3]**, **[8]**). For real groups they seem to me somewhat more technical, and I must admit that it is not clear to me how important they will turn out to be in the long run. At any rate their present interest lies mainly in the light they shed on the nature and significance of the series expansions of matrix coefficients, a notoriously obscure matter up to now. They also offer a new approach to old results on the algebra of representations; have been applied to a problem in cohomology of arithmetic groups; have been used to formulate and prove some striking analytical properties of intertwining operators among principal series.

The p-adic case is simpler than the other in both statements and proofs, and if only for that reason I would have liked to say something about it here. It was not possible because of limitations on time and space but I would like to call attention — as Harish-Chandra is fond of doing—to the continuing suggestive parallels in the two theories.

Most of what I will say should appear soon in joint papers with Dragan Milicic and Nolan Wallach. It represents to some extent independent work on their part and to some extent joint work with them. I also include some recent and not yet published results of Henryk Hecht and Wilfried Schmid.

1. Let

$G = \boldsymbol{R}$-valued points on a Zariski-connected reductive group defined over $\boldsymbol{R}$,
K = a maximal compact subgroup of G,
P = a minimal parabolic subgroup of G,
N = the unipotent radical of P,
M = a reductive component of P stable under the Cartan involution associated to K,
A = the (topological) neutral component of the maximal split torus of G contained in M.

Let $\mathfrak{g}$, $\mathfrak{k}$, etc. be their complexified Lie algebras.

I call a *Harish-Chandra module* of the pair $(\mathfrak{g}, K)$ a representation (π, V) of $\mathfrak{g}$ and K simultaneously on the same space such that (1) the representation of K is the direct sum of continuous finite-dimensional irreducibles; (2) the representation of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$ agrees with the differential of the representation of K; (3) for all $X \in \mathfrak{g}$, $k \in K$, $\pi(\mathrm{Ad}(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$; (4) it is $Z(\mathfrak{g})$-finite—i.e. every $v \in V$ is contained in a finite-dimensional subspace stable under the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$. It is well known that condition (4) is superfluous if the representation has finite length and that K-representations occur with finite multiplicity if it is finitely generated.

If (π, V) is a continuous representation of G (on a reasonable topological vector space) then $\mathfrak{g}$ and K act on the dense subspace of smooth K-finite vectors. Frequently—for example if π is unitary and irreducible—this yields a Harish-Chandra module. In fact, almost all interest in such modules arises in the intimate relationship with G-spaces. It is only recently, however, that one knows:

Lemma (Prischepionok). *Every finitely generated Harish-Chandra module is the canonical representation on the K-finite vectors in some smooth representation of G.*

As will be seen, one may even choose this to be on a nuclear Fréchet space.

There are available two rather different proofs of this—see **[5]** and **[15]**. Prischepionok even constructs a canonical extension to G.

Of course when π is irreducible the Lemma follows from an old and famous result of Harish-Chandra (which is proved nicely in **[13]**).

If (π, V) is a finitely generated Harish-Chandra module, its contragredient is the natural representation on the K-finite linear functionals on V. It is again a Harish-Chandra module—although not obviously finitely generated, but at least with finite K-multiplicity. The importance of the above Lemma for my purposes is that to each pair $v \in V$, $\tilde{v} \in \tilde{V}$, one may associate the function $\langle \pi(g)v, \tilde{v} \rangle$ on G, the *matrix coefficient* of the pair. It is smooth and annihilated by some ideal of differential operators in $Z(\mathfrak{g})$ of finite codimension, hence even real-analytic. (In **[5]** I construct the matrix coefficients directly as solutions of a system of differential equations on G, and deduce the Lemma from that.)

2. Let (π, V) be a finitely generated Harish-Chandra module. For each $n \in N$ define $\mathfrak{n}^n V$ to be the subspace of V spanned by $\{\pi(v_1)\dots\pi(v_n)v \mid v_i \in \mathfrak{n}, v \in V\}$. For $n \geq m$, $\mathfrak{n}^n V \subseteq \mathfrak{n}^m V$ so that there is a canonical projection $V/\mathfrak{n}^n V \to V/\mathfrak{n}^m V$. Define the Jacquet module $V_{[\mathfrak{n}]}$ of V associated to P to be the projective limit of these spaces. Each $V/\mathfrak{n}^n V$ is naturally a $(\mathfrak{g}, K \cap P)$-module, and so is the limit. In fact, since V is finitely generated over $U(\mathfrak{n})$ (see **[6]**) each $V/\mathfrak{n}^n V$ is finite-dimensional; this implies that each $V/\mathfrak{n}^n V$, hence $V_{[\mathfrak{n}]}$, is a P-space. There is a canonical map from V to $V_{[\mathfrak{n}]}$.

The construction of $V_{[\mathfrak{n}]}$ is motivated by an analogous p-adic definition. The reason that things here are more complicated is that in the p-adic case all finite-dimensional P-spaces are trivial on N, so that one need only consider the analogue of $V/\mathfrak{n}V$, while here one only knows that a finite-dimensional P-space is nilpotent on $\mathfrak{n}$.

The most immediate result about $V_{[\mathfrak{n}]}$ is a version of Frobenius reciprocity. Recall that the Harish-Chandra module $\mathrm{Ind}(\sigma|P, G)$ induced from the finite-dimensional representation (σ, U) of P is the right regular representation on the K-finite $f: G \to U$ such that $f(pg) = \sigma\delta^{1/2}(p)f(g)$ for all $p \in P$, $g \in G$. (Here δ is the modulus character of P.) Let $\Omega: \mathrm{Ind}(\sigma) \to U$ be the $(\mathfrak{g}, K \cap P)$-morphism from $\mathrm{Ind}(\sigma)$ to the P-module $\sigma\delta^{1/2}$, $f \to f(1)$. Composition with Ω gives a map

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(V, \mathrm{Ind}(\sigma)) \to \mathrm{Hom}_{(\mathfrak{g}, K \cap P)}(V, \sigma\delta^{1/2}).$$

Since $\mathfrak{n}$ acts nilpotently on U, any $\mathfrak{p}$-map from V to U must factor through $V/\mathfrak{n}^n V$ for $n \gg 0$, hence through $V_{[\mathfrak{n}]}$. It is elementary to see:

Proposition (Frobenius Reciprocity). *Composition with Ω induces an isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(V, \mathrm{Ind}(\sigma)) \cong \mathrm{Hom}_P(V_{[\mathfrak{n}]}, \sigma\delta^{1/2}).$$

In other words, $\mathfrak{p}$-morphisms from V into induced representations are determined by the structure of $V_{[\mathfrak{n}]}$ as a P-space. In particular, maps into principal series are determined by the M-module $V/\mathfrak{n}V$: if V is irreducible, any non-zero M-morphism $V/\mathfrak{n}V \to \sigma\delta^{1/2}$ gives an embedding of V into $\mathrm{Ind}(\sigma)$. (Incidentally, it is important to realize that $V/\mathfrak{n}V$ may not be M-semisimple.)

Although each finite quotient $V/\mathfrak{n}^n V$ is only a P-space, something more can be said about the limit.

Lemma. *There exists $d \geq 0$ such that for every $n \geq d$ and $X \in \mathfrak{g}$*

$$X\mathfrak{n}^n U(\mathfrak{g}) \subseteq \mathfrak{n}^{n-d} U(\mathfrak{g}).$$

As a consequence, the element X induces a sequence of maps $V/\mathfrak{n}^n V \to V/\mathfrak{n}^{n-d} V$, hence an endomorphism of $V_{[\mathfrak{n}]}$. Thus $V_{[\mathfrak{n}]}$ *becomes a module over* $(\mathfrak{g}, P)$.

g-module is $\mathfrak{p}$-finite if every element in it is contained in a finite-dimensional e subspace. If it is in addition finitely generated over $U(\mathfrak{g})$ then it is a quotient e $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U$, U a finite-dimensional $\mathfrak{p}$-module (this is called a *generalized module*). Recall that if U is irreducible then $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} U$ has a unique ible g-quotient $X(U)$.

POSITION. *Let X be a g-module. Then it is finitely generated and $\mathfrak{p}$-finite if 'y if it satisfies these two conditions:* (a) *every $x \in X$ is annihilated by some $\mathfrak{n}^n$; subspace of $x \in X$ annihilated by $\mathfrak{n}$ is finite-dimensional. In this case it has -length and every composition factor is of the form $X(U)$ for certain U.*

tinue to let (π, V) be a finitely generated Harish-Chandra module. Define be the subspace of the linear dual of V consisting of functionals annihilated e $\mathfrak{n}^n$—i.e. trivial on some $\mathfrak{n}^n V$. The Lemma in § 2 implies that it is a g-module. fact simply a sort of dual of $V_{[\mathfrak{n}]}$, the topological dual if $V_{[\mathfrak{n}]}$ is assigned dic topology. And it is not hard to see that $V_{[\mathfrak{n}]}$ is the entire linear dual of n other words, $V \rightsquigarrow V_{[\mathfrak{n}]}$ is a functor from the category of finitely generated -Chandra modules to that of linear duals of finitely generated $\mathfrak{p}$-finite ules.

et R for the moment be the ring $U(\mathfrak{n})$, I the two-sided ideal of R generated The module $V_{[\mathfrak{n}]}$ is nothing but the completion of the finite R-module h respect to powers of I. When $\mathfrak{n}$ is abelian this construction is standard, turns out that even for general nilpotent $\mathfrak{n}$ something can be said about it.

MA. (a) *The completion $R_I = \varprojlim R/I^n$ is a Noetherian ring.*
(ARTIN–REES). *If $A \subseteq B$ are two finite R-modules then there exists $d \geq 0$ such r $n \gg 0$*

$$I^n B \cap A \subseteq I^{n-d} A.$$

ıct this is true for any two-sided ideal of R. (There is a large literature attempt- extend results of commutative algebra to rings like R—see [7], [14] for example.)

POSITION. *The functor $V \rightsquigarrow V_{[\mathfrak{n}]}$ is exact.*

in, the motivation for expecting such a result comes from something similar ıdic groups.
ıt one would really like to do is apply general results about modules over obtain something about the structure of V as an R-module. For example, ; abelian then in [6] Osborne and I were able to show that the associated prime of V are rather restricted, and obtain (implicitly, I am afraid) a useful on of V as a $\mathfrak{p}$-module. But for general $\mathfrak{n}$ the known results on associated ; are not sufficient to say anything. This is reasonable in light of the fact that ogies can occur even for finite R-modules which are at the same time $\mathfrak{p}$-modules allach has described to me).

5. I have given no reason so far to expect that $V_{[\mathfrak{n}]} \neq 0$. An easy non-commutative version of Nakayama's Lemma shows that $V_{[\mathfrak{n}]} = 0$ if and only if $V/\mathfrak{n}V = 0$, but even this apparently simple possibility is hard to rule out. The first proof of this fact involved looking at the asymptotic expansions of matrix coefficients at infinity; this is no accident, and in fact the deepest results about $V_{[\mathfrak{n}]}$ are related to these expansions.

For simplicity, assume G for the moment to be semisimple. Let Δ be the set of simple roots of $\mathfrak{g}$ with respect to A determined by the choice of P (so that $\mathfrak{n}$ is the sum of positive root spaces). Embed A in $\mathbf{C}^\Delta$: $a \to (\alpha(a))$. For each $s \in \mathbf{C}^\Delta$ define functions which are single-valued on A, multivalued on the complement of coordinate hyperplanes in $\mathbf{C}^\Delta$:

$$a^s \log^m a = \prod \alpha(a)^{s_\alpha} \log^{m_\alpha} \alpha(a) \quad (\alpha \in \Delta).$$

LEMMA. *There exist finite sets $S \subseteq \mathbf{C}^\Delta$, $\mathscr{M} \subseteq \mathbf{N}^\Delta$ such that for every $v \in V$, $\tilde{v} \in \tilde{V}$, there exist functions $f_{s,m}$ $(s \in S, m \in \mathscr{M})$ holomorphic in the region $|\alpha| < 1$ with*

$$\langle \pi(a) v, \tilde{v} \rangle = \sum f_{s,m}(a) a^s \log^m a$$

on $\{a \in A \mid \alpha(a) < 1 \text{ for all } \alpha \in \Delta\}$.

The possible sets $S, \mathscr{M}$ depend on V, but are not unique.

This is a restatement of results of Harish–Chandra (see the Appendix to [16], also the forthcoming paper with Milicic).

Let

$$f_{s,m} = \sum f_{s,m,n} a^n \quad (n \in \mathbf{N}^\Delta)$$

be the Taylor's series expansion of f at the origin.

LEMMA. *For $v \in \mathfrak{n}^n V$ or $v \in (\mathfrak{n}^-)^n \tilde{V}$, $f_{s,m,n} = 0$ whenever $\sum n_i < n$.*

Here $\mathfrak{n}^-$ is the opposite of $\mathfrak{n}$.

Neither is this much different from what Harish–Chandra has shown; it depends on the expression of elements of $\mathfrak{n}^n U(\mathfrak{g})$ in so-called radial coordinates in terms of the Cartan decomposition $G = KAK$ (see Chapter 9 of [16], also again the paper with Milicic).

Let $v \in V$ be given, $v \neq 0$. Choose $\tilde{v} \in \tilde{V}$ with $\langle v, \tilde{v} \rangle \neq 0$. Then the restriction of the corresponding matrix coefficient to A is not trivial. Since it is analytic, its expansion around the origin (as in the above Lemma) must not be trivial. The second Lemma implies the existence of some n with $v \notin \mathfrak{n}^n V$.

THEOREM. *The canonical map from V to $V_{[\mathfrak{n}]}$ is an injection.*

The kernel of this map, after all, is the intersection of the $\mathfrak{n}^n V$.

As one consequence, $V/\mathfrak{n}V \neq 0$. If V is irreducible, then, earlier remarks show that V occurs as the subspace of at least one principal series representation. As another, since $\tilde{V}^{[\mathfrak{n}]}$ has finite length one sees immediately that V does too.

$n \gg 0$, these two facts imply that V embeds into Ind $(V/\mathfrak{n}^n V \cdot \delta^{-1/2})$. Wallach :n able to show recently that the closure of the image of V in the correspond- ıce of smooth functions is as a topological vector space independent of n. :fines in this way a canonical extension of V to G which should play an ant role.

statement that V embeds into $V_{[\mathfrak{n}]}$ is purely algebraic, and one might like a purely algebraic proof. For abelian $\mathfrak{n}$ this follows from the result in **[6]** ned earlier, and indeed from a very much easier argument. Recently Wallach ıfford have constructed a satisfactory algebraic proof of this whenever G has nk one, and also when $G = \mathrm{SL}_n$. Still, one should keep in mind that part of ırm of $V_{[\mathfrak{n}]}$ lies in the link to matrix coefficients. This link is even stronger have so far said; Hecht has been able to show that not only do matrix co- ts imply something about $V_{[\mathfrak{n}]}$, but that one has a converse as well. (He result of Milicic **[19]** which is a special case of what I have in mind.) Thus .s criteria for whether V is square-integrable, or tempered, or has matrix ents vanishing at infinity, all purely in terms of the M-module $V/\mathfrak{n}V$.

e considers parabolic subgroups which are not necessarily minimal, one obtains about the asymptotic behaviour of matrix coefficients in directions other than gin in C^A (in Harish-Chandra's terminology, "along the walls").

few other applications and related ideas: (1) The $\mathfrak{n}$-homology of V is the .s that of $V_{[\mathfrak{n}]}$, and dual to the cohomology of $\hat{V}^{[\mathfrak{n}]}$. What my remarks in ount to is that in some sense the M-module $V/\mathfrak{n}V = H_0(\mathfrak{n}, V)$ determines .ding terms of the symptotics matrix coefficients. These observations have ıined to an ingenious sequence of arguments by Schmid to obtain a new proof cohomology vanishing theorem of Borel–Wallach **[2]** and Zuckerman **[17]**, in fact relates the extent of cohomology vanishing to the rapidity of matrix ent decrease of a unitary representation.

)ne can obtain a new proof of some of Zuckerman's results in **[18]** concerning ng Harish-Chandra modules with finite-dimensional ones, and perhaps recover some information about certain extensions which arise in his construc- ıe is able to say something only about semisimplifications of the modules ng, but one may be able to combine the two techniques to say more).

Nhat may seem most amazing, one obtains a proof that *any* $\mathfrak{g}$*-morphism* ı *K-finite induced representations extends continuously to the corresponding representations of* G. (In certain cases, Kashiwara has also proved this.) oofs involved in these results are to some extent a generalization of those in 's thesis, and illuminate Langlands' arguments in **[12]** as well. (In my talk .ed that all intertwining operators have closed images but I was overconfident. lelicate points remain to be shown.)

Iecht and Schmid have proven Osborne's conjecture concerning the restric- characters to noncompact tori. (Refer to **[4]** for a statement of the conjecture as a proof of the p-adic analogue.) This involves $\mathfrak{n}$-homology, and in view

of the relationship between that and matrix coefficients gives what seems to me the most satisfactory proof of the relationship between the asymptotics of matrix coefficients and those of characters (see **[9]**).

(5) One may define the completion of V with respect to other maximal ideals of $U(\mathfrak{n})$, and this is related to results of Kostant (and, in weaker form, of Zuckerman and myself) on Whittaker models for Harish-Chandra modules (see **[11]**).

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