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DIXMIER ALGEBRAS FOR CLASSICAL COMPLEX NILPOTENT ORBITS VIA KRAFT-PROCESI MODELS I

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Dedicated to Professor Alexander Kirillov on his 2^6 birthday

ABSTRACT. We attach a Dixmier algebra $\mathcal{B}$ to the closure $\overline{\mathcal{O}}$ of any nilpotent orbit of G where G is $GL(n,\mathbb{C})$, $O(n,\mathbb{C})$ or $Sp(2n,\mathbb{C})$. This algebra $\mathcal{B}$ is a noncommutative analog of the coordinate ring $\mathcal{R}$ of $\overline{\mathcal{O}}$, in the sense that $\mathcal{B}$ has a G-invariant algebra filtration and $\operatorname{gr}\mathcal{B} = \mathcal{R}$.

We obtain $\mathcal{B}$ by making a noncommutative analog of the Kraft-Procesi construction which modeled $\overline{\mathcal{O}}$ as the algebraic symplectic reduction of a finite-dimensional symplectic vector space L. Indeed $\mathcal{B}$ is a subquotient of the Weyl algebra for L.

$\mathcal{B}$ identifies with the quotient of $\mathcal{U}(\mathfrak{g})$ by a two-sided ideal J, where $\mathfrak{g} = Lie(G)$. Then $\operatorname{gr} J$ is the ideal $\mathfrak{I}(\overline{\mathcal{O}})$ in $S(\mathfrak{g})$ of functions vanishing on $\overline{\mathcal{O}}$. In every case where $\mathcal{O}$ is connected, J is a completely prime primitive ideal.

1. INTRODUCTION

By means of symplectic reduction in the setting of complex algebraic varieties, Kraft and Procesi ([22],[23]) constructed a model of the closure of any nilpotent coadjoint orbit $\mathcal{O}$ of G when G is one of the classical groups $GL(n,\mathbb{C})$, $O(n,\mathbb{C})$ and $Sp(2n,\mathbb{C})$. The symplectic aspect is not actually mentioned, but the construction is clearly symplectic.

In this paper we give a noncommutative analog, or quantization, of the Kraft-Procesi construction. The result is that we attach a Dixmier algebra $\mathcal{B}$ to each orbit closure $\overline{\mathcal{O}}$. Our algebra $\mathcal{B}$ has a G-invariant algebra filtration and we show that $\operatorname{gr}\mathcal{B}$ is isomorphic, as a graded Poisson algebra, to the coordinate ring $\mathcal{R}$ of $\overline{\mathcal{O}}$.

In fact, $\mathcal{B}$ identifies, as a filtered algebra, with the quotient $\mathcal{U}(\mathfrak{g})/J$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g} = Lie(G)$ by some two-sided ideal J. Then $\operatorname{gr} J$ is the ideal $\mathfrak{I}(\overline{\mathcal{O}})$ in $S(\mathfrak{g})$ defining $\overline{\mathcal{O}}$. We find that J is stable under the principal anti-automorphism of $\mathcal{U}(\mathfrak{g})$, and also under the anti-linear automorphism of $\mathcal{U}(\mathfrak{g})$ defined by a Cartan involution of $\mathfrak{g}$.

The Kraft-Procesi construction attaches to $\mathcal{O}$ a complex symplectic vector space L together with a Hamiltonian action of $G \times S$ on L, where S is an auxillary complex reductive Lie group. The actions of G and S lie inside the symplectic group $Sp(L,\mathbb{C})$. Kraft and Procesi show that $\overline{\mathcal{O}}$ is scheme-theoretically the algebraic symplectic reduction of L by S. In this way, they obtain $\mathcal{R}$ as a subquotient of the algebra $\mathcal{P}$ of polynomial functions on L. More precisely, $\mathcal{R}$ is realized as $\mathcal{P}^{inv}/\mathcal{I}^{inv}$ where $\mathcal{I}$ is an ideal in $\mathcal{P}$ and the superscript inv denotes taking S-invariants.

Key words and phrases. nilpotent coadjoint orbit, Dixmier algebra, symplectic reduction.

It is nice from the viewpoint of representation theory to regard $\mathcal{R}$ as a subquotient of the algebra $\mathcal{P}^{even}$ of even polynomials. (We can do this since $\mathcal{P}^{inv}$ lies in $\mathcal{P}^{even}$.) For $\mathcal{P}^{even}$ is the coordinate ring of the closure of the minimal nilpotent orbit $\mathcal{Y}$ of $Sp(L, \mathbb{C})$.

To make a noncommutative analog of the Kraft-Procesi construction, we start from the fact that there is a unique Dixmier algebra attached to $\overline{\mathcal{Y}}$, namely the quotient of $\mathcal{U}(\mathfrak{sp}(L, \mathbb{C}))$ by its Joseph ideal $\mathcal{J}$. We can model $\mathcal{U}(\mathfrak{sp}(L, \mathbb{C}))/\mathcal{J}$ as the even part $\mathcal{W}^{even}$ of the Weyl algebra $\mathcal{W}$ for L and then $\operatorname{gr} \mathcal{W}^{even} = \mathcal{P}^{even}$. There is an obvious quantization of the Hamiltonian S-action on L, namely the natural $(\mathfrak{s} \oplus \mathfrak{s}, S_c)$-module structure on $\mathcal{W}$. Here the subscript c denotes taking a compact real form.

We define $\mathcal{B}$ to be the coinvariants for the $(\mathfrak{s} \oplus \mathfrak{s}, S_c)$-action on $\mathcal{W}^{even}$. A priori, $\mathcal{B}$ is a $(\mathfrak{g} \oplus \mathfrak{g}, G_c)$-module with a G_c-invariant filtration, but $\mathcal{B}$ is not an algebra. However, we easily identify $\mathcal{B}$ as the quotient by a two-sided ideal of $\mathcal{W}^{inv}$ (where the superscript again indicates taking S-invariants, or equivalently, S_c-invariants). In this way, $\mathcal{B}$ becomes a filtered algebra and a subquotient of $\mathcal{W}^{even}$.

Our main result (Theorem 6.3) is to compute the associated graded algebra $\operatorname{gr} \mathcal{B}$. It is easy to see that $\operatorname{gr} \mathcal{B}$ is some quotient of $\mathcal{P}^{inv}/\mathcal{I}^{inv}$, but in fact we prove $\operatorname{gr} \mathcal{B} = \mathcal{P}^{inv}/\mathcal{I}^{inv}$. To do this, we recognize $\mathcal{B}$ as the degree zero part of the relative Lie algebra homology $H(\mathfrak{s} \oplus \mathfrak{s}, S_c; \mathcal{W}^{even})$. We consider the standard complex which computes this homology, introduce a filtration and then apply the spectral sequence for a filtered complex. We compute the E_1 term of the spectral sequence by using the fact proven by Kraft and Procesi that $\mathcal{I}$ is a complete intersection ideal. Then we easily show $E_1 = E_\infty$.

We establish some properties of $\mathcal{B}$ and the corresponding ideal J. If $\mathcal{O}$ is connected then J is a completely prime primitive ideal (Corollary 6.5). In every case, $\mathcal{B}$ admits a unique $\mathfrak{g}^\sharp$-invariant Hermitian inner product $(\cdot|\cdot)$ such that $(1|1) = 1$, where $\mathfrak{g}^\sharp$ is a real form of $\mathfrak{g} \oplus \mathfrak{g}$ with $\mathfrak{g}^\sharp \simeq \mathfrak{g}$ (see Proposition 3.1). This prompts the question as to whether J is "good" in the sense that J is maximal and $\mathcal{U}(\mathfrak{g})/J$ is unitarizable. The latter property means (since $\mathcal{B} \simeq \mathcal{U}(\mathfrak{g})/J$) that $(\cdot|\cdot)$ is positive-definite.

Attaching "good" ideals to $\mathcal{O}$ is an important problem in representation theory and the orbit method. Quite a bit of work has been done on this (see e.g. some of the references and authors cited below) but the problem for nilpotent orbits of a complex semisimple Lie group remains unsolved.

If $G = GL(n, \mathbb{C})$, then our J is good (see Remark 6.6(i) and [8]). But $G = GL(n, \mathbb{C})$ is really a very special case for us as the geometry of $\mathcal{O}$ is incredibly nice, including but not limited to the fact that $\overline{\mathcal{O}}$ is always normal. For $G = O(n, \mathbb{C})$ or $G = Sp(2n, \mathbb{C})$, it is not the case that J is always good. Certainly if $\overline{\mathcal{O}}$ is not normal, we should not expect J to be good.

Our point of view (to be justified in [9]) is that $\mathcal{B}$ is the "canonical" quantization of the Kraft-Procesi construction, and so the failure of J to be good is really a statement about $\overline{\mathcal{O}}$. The next step is then to investigate whether we can make a modification to our quantization process in order to obtain some good ideals in $\mathcal{U}(\mathfrak{g})$ attached to $\mathcal{O}$ (and even its covers).

This paper is the first in a series. In the subsequent papers we make explicit the important role of Howe duality in our project. Indeed, $\mathcal{W}^{even}$ is the Harish-Chandra module of the (even) oscillator representation of $Sp(L, \mathbb{C})$, and the pair (G, S) constitutes a sequence of Howe dual pairs (see Remark 4.1). In taking coinvariants, we are implementing a sequence of Howe duality "operations". Each "operation" is like implementing a Howe duality correspondence, except that we do *not* pass to the the irreducible quotient. In working on this project (which started in earnest in the summer of 2001 – and is part of a program we began in 1994), we have been reading the Howe duality literature. We have been influenced by especially the papers [18], [29], [1] and [25].

Our first construction of the Dixmier algebra $\mathcal{B}$ actually came out of the ideas of Howe duality and quantization by constraints. This is given in [9] and lies more in the realm of harmonic analysis than algebra. Our starting point there is the fact ([21]) that L is hyperkähler and the Kraft-Procesi construction is the algebraic analog of the hyperkähler reduction of L by S_c.

The notion of Dixmier algebra for nilpotent orbits (including their closures and their coverings) was first developed in work of McGovern, Joseph and Vogan. See e.g. [27], [31], and [32]. The motivation for these authors and for most Dixmier algebra theorists is the search for completely prime primitive ideals. This motivation is very important for us too; we also find additional motivations coming from star products and from geometric quantization.

The results in this paper should be compared with the work in [1], [3], [4], [5], [6], [7], [10], [14], [13], [16], [24], [25], [26], [27], [28], [29] and [33]. Some of this comparison work will be done in [8] and [9].

Part of this work was carried out while I was visiting the IML and the CPT of the Université de la Méditerranée in the summer of 2001, and I thank my colleagues there for their hospitality. I especially thank Christian Duval and Valentin Ovsienko for some very valuable discussions.

It is a real pleasure to dedicate this article to Sasha Kirillov whose discoveries have opened up so many new vistas, starting of course with the Orbit Method. I warmly thank him for his friendship and his interest in my own work.

2. Dixmier Algebra for the closure of a complex nilpotent orbit

Let G be a reductive complex algebraic group. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}^*$ be the dual space. Then G acts on $\mathfrak{g}$ and $\mathfrak{g}^*$ by, respectively, the adjoint action and the coadjoint action. The symmetric algebra $S(\mathfrak{g}) = \oplus_{p=0}^{\infty} S^p(\mathfrak{g})$ is the algebra of polynomial functions on $\mathfrak{g}^*$. The G-invariants form the graded subalgebra $S(\mathfrak{g})^G = \oplus_{p=0}^{\infty} S^p(\mathfrak{g})^G$. We can fix some nondegenerate G-invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$.

The *nullcone* in $\mathfrak{g}^*$ is the set of $\lambda \in \mathfrak{g}^*$ which satisfy the following equivalent properties:
(i) the closure of the coadjoint orbit of λ contains zero
(ii) the coadjoint orbit of λ is stable under dilations of the vector space $\mathfrak{g}^*$
(iii) every nonconstant homogeneous G-invariant in $S(\mathfrak{g})$ vanishes on λ
(iv) $\lambda = (x, \cdot)$ where x is a nilpotent in $\mathfrak{g}$
The nullcone is G-stable and breaks into finitely many orbits of G, which are then called the *nilpotent coadjoint orbits*, or simply the *nilpotent orbits*, of G.

Let $\mathcal{O}$ be a nilpotent orbit of G. The closure $\overline{\mathcal{O}}$ is a complex algebraic subvariety of $\mathfrak{g}^*$; but $\overline{\mathcal{O}}$ may be reducible if G is disconnected. The coordinate ring $\mathbb{C}[\overline{\mathcal{O}}]$ of $\overline{\mathcal{O}}$ is the quotient algebra

$$\mathcal{R} = S(\mathfrak{g})/\mathfrak{I}(\overline{\mathcal{O}}) \tag{2.1}$$

where $\mathfrak{I}(\overline{\mathcal{O}})$ is the ideal of functions which vanish on $\overline{\mathcal{O}}$. Then $\mathfrak{I}(\overline{\mathcal{O}})$ is a graded ideal and $\mathcal{R} = \oplus_{p=0}^{\infty}\mathcal{R}^p$ is a graded algebra where $\mathcal{R}^p = S^p(\mathfrak{g})/\mathfrak{I}(\overline{\mathcal{O}})^p$. Each space $\mathcal{R}^p$ is a finite dimensional completely reducible representation of G. Kostant's description of $S(\mathfrak{g})$ as a module over $S(\mathfrak{g})^G$ implies that all G-multiplicities in $\mathcal{R}$ are finite.

$\mathcal{R}$ inherits from $S(\mathfrak{g})$ the structure of a graded Poisson algebra where $\{\mathcal{R}^p, \mathcal{R}^q\} \subseteq \mathcal{R}^{p+q-1}$. This Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{R}$ is G-invariant and corresponds to the holomorphic Kirillov-Kostant-Souriau symplectic form on $\mathcal{O}$.

In this situation, we define Dixmier algebras in the following way. We fix a Cartan involution ς of $\mathfrak{g}$. Then ς corresponds to a compact real form G_c of G with Lie algebra $\mathfrak{g}_c$. Let $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition 2.1. A *Dixmier algebra* for $\overline{\mathcal{O}}$ is a quadruple $(\mathcal{D}, \xi, \tau, \vartheta)$ where

- $\mathcal{D}$ is a filtered algebra with an increasing algebra filtration $\mathcal{D} = \cup_{p\in\mathbb{N}}\mathcal{D}_p$ such that $\operatorname{gr}\mathcal{D}$ is commutative.
- $\xi : \mathfrak{g} \to \mathcal{D}_1$, $x \mapsto \xi^x$, is a homomorphism of Lie algebras and ξ induces an isomorphism of graded Poisson algebras from $S(\mathfrak{g})/\mathfrak{I}(\overline{\mathcal{O}})$ onto $\operatorname{gr}\mathcal{D}$.
- τ is a filtered algebra anti-involution of $\mathcal{D}$ such that $\tau(\xi^x) = -\xi^x$.
- ϑ is an anti-linear filtered algebra involution such that $\vartheta(\xi^x) = \xi^{\varsigma(x)}$.

Here are some explanations about the definition. First, commutativity of $\operatorname{gr}\mathcal{D}$ implies that $\operatorname{gr}\mathcal{D}$ has a natural structure of graded Poisson algebra; here the commutator in $\mathcal{D}$ induces the Poisson bracket on $\operatorname{gr}\mathcal{D}$. Second, ξ extends naturally to a filtered algebra homomorphism

$$\widetilde{\xi} : \mathcal{U}(\mathfrak{g}) \to \mathcal{D} \tag{2.2}$$

Let J be the kernel of $\widetilde{\xi}$. Then $\operatorname{gr} J$ is a Poisson ideal of $S(\mathfrak{g})$, and $\operatorname{gr}\widetilde{\xi}$ induces a 1-to-1 homomorphism $\zeta : S(\mathfrak{g})/\operatorname{gr} J \to \operatorname{gr} D$ of graded Poisson algebras. We require that ζ is surjective and

$$\operatorname{gr} J = \mathfrak{I}(\overline{\mathcal{O}}) \tag{2.3}$$

Notice that ζ is surjective if and only if $\widetilde{\xi}$ is surjective in each filtration degree; then $\widetilde{\xi}$ induces a filtered algebra isomorphism

$$\mathcal{U}(\mathfrak{g})/J \xrightarrow{\sim} \mathcal{D} \tag{2.4}$$

Third, τ satisfies $\tau(cA) = c\,\tau(A)$, $\tau(A+B) = \tau(A) + \tau(B)$, and $\tau(AB) = \tau(B)\tau(A)$ where $A, B \in \mathcal{D}$ and $c \in \mathbb{C}$. Fourth, ϑ satisfies $\vartheta(cA) = \overline{c}\,\vartheta(A)$, $\vartheta(A+B) = \vartheta(A)+\vartheta(B)$, and $\vartheta(AB) = \vartheta(A)\vartheta(B)$. Clearly $\tau\vartheta = \vartheta\tau$.

Notice that $\mathcal{D}$ and ξ (and ς) uniquely determine τ and ϑ, if the latter exist. Indeed, the endomorphisms $x \mapsto -x$ and $x \mapsto \varsigma(x)$ of $\mathfrak{g}$ extend uniquely to $\tau_{\mathfrak{g}}$ and $\vartheta_{\mathfrak{g}}$, where $\tau_{\mathfrak{g}}$ is an algebra anti-involution of $\mathcal{U}(\mathfrak{g})$ and $\vartheta_{\mathfrak{g}}$ is an antilinear algebra involution of $\mathcal{U}(\mathfrak{g})$. Then, via (2.4), $\tau_{\mathfrak{g}}$ and $\vartheta_{\mathfrak{g}}$ induce τ and ϑ.

We have an obvious notion of isomorphism of Dixmier algebras for $\overline{\mathcal{O}}$: $(\mathcal{D},\xi,\tau,\vartheta)$ is isomorphic to $(\mathcal{D}',\xi',\tau',\vartheta')$ if there is a filtered algebra isomorphism $\eta:\mathcal{D}\to\mathcal{D}'$ such that $\xi'=\eta\circ\xi$, $\eta\circ\tau=\tau'\circ\eta$, and $\eta\circ\vartheta=\vartheta'\circ\eta$. We can easily classify Dixmier algebras.

Observation 2.2. *Suppose $(\mathcal{D},\xi,\tau,\vartheta)$ is a Dixmier algebra for $\overline{\mathcal{O}}$. In addition to* (2.3)*, J satisfies*

$$\tau_{\mathfrak{g}}(J)=J \qquad \textit{and} \qquad \vartheta_{\mathfrak{g}}(J)=J \tag{2.5}$$

In this way, we get a bijection between (isomorphism classes of) Dixmier algebras for $\overline{\mathcal{O}}$ and two-sided ideals J of $\mathcal{U}(\mathfrak{g})$ satisfying (2.3) *and* (2.5)*.*

Proof. Clearly (2.3) and (2.5) imply that $(\mathcal{U}(\mathfrak{g})/J,\iota,\tau'_{\mathfrak{g}},\vartheta'_{\mathfrak{g}})$ is a Dixmier algebra for $\overline{\mathcal{O}}$, where ι is the obvious map and $\tau'_{\mathfrak{g}}$ and $\vartheta'_{\mathfrak{g}}$ are induced by $\tau_{\mathfrak{g}}$ and $\vartheta_{\mathfrak{g}}$. Conversely, if $(\mathcal{D},\xi,\tau,\vartheta)$ is given, then $(\mathcal{U}(\mathfrak{g})/J,\iota,\tau'_{\mathfrak{g}},\vartheta'_{\mathfrak{g}})$ is isomorphic to it via (2.4). □

3. Properties of Dixmier algebras

The hopes in constructing a Dixmier algebra are (i) J will be a completely prime primitive ideal of $\mathcal{U}(\mathfrak{g})$, or even better, a completely prime maximal ideal, and (ii) $\mathcal{D}$ will be unitarizable. See [12, 3.1] for the definitions of the terms in (i).

To understand (ii), we observe that Definition 2.1 makes $\mathcal{D}$ into a $(\mathfrak{g}\oplus\mathfrak{g},G_c)$-module. Indeed, the natural $(\mathfrak{g}\oplus\mathfrak{g},G_c)$-module structure on $\mathcal{U}(\mathfrak{g})/J$ transfers over to $\mathcal{D}$ via (2.4). Then $\mathfrak{g}\oplus\mathfrak{g}$ acts on $\mathcal{D}$ though the representation

$$\Pi:\mathfrak{g}\oplus\mathfrak{g}\to\operatorname{End}\mathcal{D}, \qquad (x,y)\mapsto\Pi^{x,y} \tag{3.1}$$

where $\Pi^{x,y}(A)=\xi^x A-A\xi^y$. The action of G_c corresponds to the subalgebra $\{(x,x): x\in\mathfrak{g}_c\}$.

Next consider the subalgebra $\mathfrak{g}^\sharp=\{(x,\varsigma(x)):x\in\mathfrak{g}\}$ of $\mathfrak{g}\oplus\mathfrak{g}$. We say $\mathcal{D}$ is *unitarizable* if $\mathcal{D}$ admits a $\mathfrak{g}^\sharp$-invariant positive definite Hermitian inner product. In this event, by a theorem of Harish-Chandra, the operators $\Pi^{x,\varsigma(x)}$ correspond to a unitary representation of G on the Hilbert space completion of $\mathcal{D}$. This unitary representation is then a quantization of $\mathcal{O}$ in the sense of geometric quantization, if we view $\mathcal{O}$ as a *real* symplectic manifold. (If $\mathbb{C}[\overline{\mathcal{O}}]\neq\mathbb{C}[\mathcal{O}]$, then this might be just a piece of a quantization of $\mathcal{O}$.)

Notice that the following three properties are equivalent: (i) J is maximal, (ii) $\mathcal{D}$ is a simple ring, and (iii) the representation Π is irreducible.

Our formalism gives some partial results pertaining to hopes (i) and (ii). Notice that $\mathcal{D}_0=\mathbb{C}$ by (2.4).

Proposition 3.1. *Suppose $(\mathcal{D},\xi,\tau,\vartheta)$ is a Dixmier algebra for $\overline{\mathcal{O}}$ and $J=\ker\widetilde{\xi}$. Then*

(i) *J has an infinitesimal character.*
(ii) *If $\overline{\mathcal{O}}$ is irreducible then J is a completely prime primitive ideal in $\mathcal{U}(\mathfrak{g})$.*
(iii) *There is a a unique G_c-invariant projection $\mathcal{T}:\mathcal{D}\to\mathbb{C}$. This map $\mathcal{T}$ is a trace, i.e., $\mathcal{T}(AB)=\mathcal{T}(BA)$.*
(iv) *$\mathcal{D}$ admits a unique $\mathfrak{g}^\sharp$-invariant Hermitian inner product $(\cdot|\cdot)$ such that $(1|1)=1$, and it is given by*

$$(A|B)=\mathcal{T}(AB^\vartheta). \tag{3.2}$$

where $B^\vartheta=\vartheta(B)$.

Proof. (i) This means (for any proper two-sided ideal J) that J contains a maximal ideal of the center of $\mathcal{U}(\mathfrak{g})$. This happens if and only $\operatorname{gr} J$ contains $S^+(\mathfrak{g})^G = \oplus_{p>0} S^p(\mathfrak{g})^G$. But $\operatorname{gr} J = \mathfrak{I}(\overline{\mathcal{O}})$ and $\mathfrak{I}(\overline{\mathcal{O}}) \supset S^+(\mathfrak{g})^G$ since $\overline{\mathcal{O}}$ lies in the nullcone. (ii) If $\overline{\mathcal{O}}$ is irreducible then $\mathfrak{I}(\overline{\mathcal{O}})$ is a prime ideal in $S(\mathfrak{g})$ and so J is a completely prime ideal in $\mathcal{U}(\mathfrak{g})$. This together with (i) implies, by a result of Dixmier, that J is primitive. (iii) Since $\mathfrak{I}(\overline{\mathcal{O}}) \supset S^+(\mathfrak{g})^G = S^+(\mathfrak{g})^{G_c}$, we have $\mathcal{R}^{G_c} = \mathbb{C}$ and so $\mathcal{D}^{G_c} = \mathbb{C}$. Thus we get a unique G_c-invariant projection map $\mathcal{T}$. Now G_c-invariance implies $\mathcal{T}([\xi^x, A]) = 0$ where $x \in \mathfrak{g}$. We can write this as $\mathcal{T}(\xi^x A) = \mathcal{T}(A\xi^x)$. Iteration gives $\mathcal{T}(\xi^{x_1} \cdots \xi^{x_k} A) = \mathcal{T}(A\xi^{x_1} \cdots \xi^{x_k})$. This proves $\mathcal{T}(BA) = \mathcal{T}(AB)$ since the ξ^x generate $\mathcal{D}$. (This is the same proof as in [7, Proposition 8.4].) (iv) Suppose $(\cdot|\cdot)$ is an inner product with the desired properties. Then $(A|1) = \mathcal{T}(A)$. Now $\mathfrak{g}^\sharp$-invariance means that the operators $\Pi^{x,\varsigma(x)}$ are skew-hermitian, or equivalently, $(\xi^x A|B) = (A|B\xi^{\varsigma(x)})$. So for $B = \xi^{x_1} \cdots \xi^{x_k}$, we have $(A|B) = (\xi^{\varsigma(x_1)} \cdots \xi^{\varsigma(x_k)} A|1) = (B^\vartheta A|1) = \mathcal{T}(B^\vartheta A)$. The result is now clear. ☐

Corollary 3.2. *$\mathcal{D}$ is unitarizable if and only if the pairing defined by* (3.2) *is positive definite. If $\mathcal{D}$ is unitarizable, then J is maximal.*

Proof. Both statements follow from the uniqueness in Proposition 3.1(iv). ☐

Example 3.3. Suppose $\mathcal{O}$ is the minimal nilpotent orbit in $\mathfrak{g}$ where $\mathfrak{g}$ is simple and $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{C})$. This is a case where $\mathbb{C}[\overline{\mathcal{O}}] = \mathbb{C}[\mathcal{O}]$. Then there is a unique Dixmier algebra for $\overline{\mathcal{O}}$. This follows by Observation 2.2 since there is exactly one choice for J satisfying (2.3) and (2.5). Moreover, (i) J is a completely prime maximal ideal of $\mathcal{U}(\mathfrak{g})$, and (ii) $\mathcal{U}(\mathfrak{g})/J$ is unitarizable if $\mathfrak{g}$ is classical.

Indeed, there is a unique two-sided ideal J satisfying (2.3) and $\tau_{\mathfrak{g}}(J) = J$; see [2, proof of Proposition 3.1]. Since the ideal $\mathfrak{I}(\overline{\mathcal{O}})$ is preserved by the antilinear algebra involution of $S(\mathfrak{g})$ defined by ς, it follows by the uniqueness of J that $\vartheta_{\mathfrak{g}}(J) = J$. Since $\overline{\mathcal{O}}$ is irreducible, Proposition 3.1(ii) implies that J is completely prime. If $\mathfrak{g} \neq \mathfrak{sl}(n, \mathbb{C})$, then J is the Joseph ideal and this is maximal by [19, Theorem 7.4]. If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ ($n \geq 3$), then J is maximal by [30]. Finally, unitarizabilty is known; see [2, Theorem 9.1] for a uniform construction of these unitary representations on spaces of holomorphic functions on $\mathcal{O}$.

4. The Kraft-Procesi Construction

In this section we recall how Kraft and Procesi in [22] and [23] constructed the closures of complex classical nilpotent orbits. We add to their construction the framework of algebraic symplectic reduction.

Let V be a complex vector space with $\mathbf{b}$ a bilinear form on V. Let G be the symmetry group of $\mathbf{b}$. We consider only the following three cases.

(i) $\mathbf{b}$ is identically zero. Then $G = GL(V, \mathbb{C})$.
(ii) $\mathbf{b}$ is nondegenerate and symmetric. Then $G = O(V, \mathbb{C})$.
(iii) $\mathbf{b}$ is nondegenerate and symplectic. Then $G = Sp(V, \mathbb{C})$.

Choose a nilpotent orbit $\mathcal{O}$ of G and an element λ in $\mathcal{O}$. Then λ corresponds (via the trace functional on $\operatorname{End} V$) to $x \in \mathfrak{g}$; so x is in particular a nilpotent endomorphism of V.

We note that $\mathcal{O}$ is connected, and so $\overline{\mathcal{O}}$ is irreducible, except in the following situation. If $G = O(V, \mathbb{C})$ where $\dim V$ is even and also the Jordan block size partition of $\mathcal{O}$ is very even (i.e., all parts are even and occur with even multiplicities), then $\mathcal{O}$ has two connected components and $\overline{\mathcal{O}}$ has two irreducible components. See [23], [11, Chapter 5].

Let V_d be the image of x^d. Then

$$V = V_0 \supset V_1 \supset \cdots \supset V_r \supset V_{r+1} = 0 \tag{4.1}$$

where r is the largest number such that $x^r \neq 0$. We define a complex vector space L by

$$L = L(V_0, V_1) \oplus L(V_1, V_2) \oplus \cdots \oplus L(V_{r-1}, V_r) \tag{4.2}$$

where $L(V_{d-1}, V_d)$ is obtained in the following way. If $G = GL(V, \mathbb{C})$ then

$$L(V_{d-1}, V_d) = \mathrm{Hom}(V_d, V_{d-1}) \oplus \mathrm{Hom}(V_{d-1}, V_d) \tag{4.3}$$

If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, then

$$L(V_{d-1}, V_d) = \mathrm{Hom}(V_d, V_{d-1}) \tag{4.4}$$

Next we construct a complex Lie group S of the form

$$S = S_1 \times S_2 \times \cdots \times S_r \tag{4.5}$$

where S_d is obtained in the following way. To begin with, we put $\mathbf{b}_0 = \mathbf{b}$ and $S_0 = G$. If $G = GL(V, \mathbb{C})$ then for each d we put $\mathbf{b}_d = 0$ and $S_d = GL(V_d, \mathbb{C})$. If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, then V_d admits an intrinsic nondegenerate complex bilinear form $\mathbf{b}_d$ and we define S_d to be the symmetry group of $\mathbf{b}_d$. In more detail, $\mathbf{b}_d$ is the bilinear form on V_d defined by $\mathbf{b}_d(x^d(u), x^d(v)) = \mathbf{b}(u, x^d(v))$. It turns out that $\mathbf{b}_d$ is nondegenerate. If $\mathbf{b}_{d-1}$ is orthogonal then $\mathbf{b}_d$ is symplectic and we put $S_d = Sp(V_d, \mathbb{C})$. If $\mathbf{b}_{d-1}$ is symplectic then $\mathbf{b}_d$ is orthogonal and we put $S_d = O(V_d, \mathbb{C})$.

Next we construct commuting actions of G and S on L. If $G = GL(V, \mathbb{C})$, we make G and S act by

$$\begin{aligned}(g, s_1, \ldots, s_r) &\bullet (A_1, B_1, A_2, B_2, \ldots, A_r, B_r) \\ &= (gA_1 s_1^{-1}, s_1 B_1 g^{-1}, s_1 A_2 s_2^{-1}, s_2 B_2 s_1^{-1}, \ldots, s_{r-1} A_r s_r^{-1}, s_r B_r s_{r-1}^{-1})\end{aligned} \tag{4.6}$$

where $A_d \in \mathrm{Hom}(V_d, V_{d-1})$ and $B_d \in \mathrm{Hom}(V_{d-1}, V_d)$. If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, we make G and S act by

$$(g, s_1, \ldots, s_r) \bullet (C_1, C_2, \ldots, C_r) = (gC_1 s_1^{-1}, s_1 C_2 s_2^{-1}, \ldots, s_{r-1} C_r s_r^{-1}) \tag{4.7}$$

where $C_d \in \mathrm{Hom}(V_d, V_{d-1})$.

L has a (complex) symplectic form Ω given by $\Omega = \Omega_1 + \Omega_2 + \cdots + \Omega_r$ where Ω_d is the symplectic form on $L(V_{d-1}, V_d)$ defined in the following way. If $G = GL(V, \mathbb{C})$, then $\Omega_d(A + B, A' + B') = \mathrm{tr}(AB') - \mathrm{tr}(BA')$. If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, then $\Omega_d(C, C') = \mathrm{tr}(C^* C)$ where $C^* \in \mathrm{Hom}(V_{d-1}, V_d)$ is the adjoint of C defined by $\mathbf{b}_{d-1}(u, C^*(v)) = \mathbf{b}_d(C(u), v)$.

Now G and S act faithfully and symplectically on L. Thereby G and S identify with commuting subgroups of the symplectic group $Sp(L, \mathbb{C})$. The action of $Sp(L, \mathbb{C})$ is Hamiltonian with canonical moment map

$$L \to \mathfrak{sp}(L, \mathbb{C})^* \tag{4.8}$$

Hence our actions of G and S are Hamiltonian with induced moment maps (obtained by projection)

$$\gamma : L \to \mathfrak{g}^* \qquad \text{and} \qquad \sigma : L \to \mathfrak{s}^* \tag{4.9}$$

Then γ is G-equivariant and S-invariant, and σ is S-equivariant and G-invariant.

Here are the explicit formulas for γ and σ. We may write these as $\mathfrak{g}$-valued and $\mathfrak{s}$-valued maps, with the convention that the trace functional on $\operatorname{End} L$ induces isomorphisms $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ and $\mathfrak{s} \xrightarrow{\sim} \mathfrak{s}^*$. If $G = GL(V, \mathbb{C})$, then γ and σ are given by

$$(-A_1B_1) \qquad \text{and} \qquad (B_1A_1 - A_2B_2, \dots, B_{r-1}A_{r-1} - A_rB_r, B_rA_r) \tag{4.10}$$

If $G = O(V, \mathbb{C})$ or $G = Sp(V, \mathbb{C})$, then γ and σ are given by

$$(-C_1C_1^*) \qquad \text{and} \qquad (C_1^*C_1 - C_2C_2^*, \dots, C_{r-1}^*C_{r-1} - C_rC_r^*, C_r^*C_r) \tag{4.11}$$

Remark 4.1. For each $d = 1, \dots, r$, S_{d-1} and S_d act on $L(V_{d-1}, V_d)$ as a Howe dual pair.

Let $\mathcal{P}$ be the algebra of polynomial functions on L. Then $\mathcal{P} = S(L^*)$ is a graded Poisson algebra with respect to the Poisson bracket $\{\cdot,\cdot\}$ defined by Ω. Our grading is $\mathcal{P} = \oplus_{j\in\frac{1}{2}\mathbb{N}} \mathcal{P}^j$ where $\mathcal{P}^j = S^{2j}(L^*)$ and then $\{\mathcal{P}^j, \mathcal{P}^k\} \subseteq \mathcal{P}^{j+k-1}$. (This choice of halving the natural degrees is convenient as the aim is to obtain $\mathcal{R}$ as a subquotient of $\mathcal{P}$.) The momentum functions γ_y ($y \in \mathfrak{g}$) and σ_x ($x \in \mathfrak{s}$) are the component functions of γ and σ; i.e., $\gamma_y(m) = \langle \gamma(m), y\rangle$ and $\sigma_x(m) = \langle \sigma(m), x\rangle$ where $m \in L$. The γ_y and σ_x lie in $\mathcal{P}^1$ and satisfy the bracket relations $\{\gamma_y, \gamma_{y'}\} = \gamma_{[y,y']}$, $\{\sigma_x, \sigma_{x'}\} = \sigma_{[x,x']}$ and $\{\gamma_y, \sigma_x\} = 0$.

Let $\mathcal{I}$ be the ideal in $\mathcal{P}$ generated by the momentum functions σ_x where $x \in \mathfrak{s}$. Then $\mathcal{I} = \oplus_{j\in\frac{1}{2}\mathbb{N}}^{\infty} \mathcal{I}^j$ is a graded ideal stable under both G and S. Hence the quotient algebra $\mathcal{P}/\mathcal{I}$ is a graded algebra on which G and S act by graded algebra automorphisms. The grading is

$$\mathcal{P}/\mathcal{I} = \oplus_{j\in\frac{1}{2}\mathbb{N}} (\mathcal{P}/\mathcal{I})^j$$

where $(\mathcal{P}/\mathcal{I})^j = \mathcal{P}^j/\mathcal{I}^j$. Kraft and Procesi proved that $\mathcal{I}$ is the full ideal of functions vanishing on $\sigma^{-1}(0)$. Thus $\mathcal{P}/\mathcal{I}$ is the coordinate ring $\mathbb{C}[\sigma^{-1}(0)]$ of the zero locus of σ.

The *algebraic symplectic reduction* L^{red} of L by S is the Mumford quotient of $\sigma^{-1}(0)$ by S. Thus L^{red} is the affine complex algebraic variety with coordinate ring

$$\mathbb{C}[L^{red}] = (\mathcal{P}/\mathcal{I})^{inv} = \mathcal{P}^{inv}/\mathcal{I}^{inv} \tag{4.12}$$

where the superscript *inv* denotes taking S-invariants. Moreover $\mathcal{I}^{inv}$ is a Poisson ideal in $\mathcal{P}^{inv}$. So $\mathbb{C}[L^{red}]$ inherits the structure of a graded Poisson algebra.

Notice that S contains the center $\mathbb{Z}_2 = \{1, -1\}$ of $Sp(L, \mathbb{C})$ and the action of $\mathbb{Z}_2$ induces the decomposition $\mathcal{P} = \mathcal{P}^{even} \oplus \mathcal{P}^{odd}$ where the even part is the graded Poisson algebra

$$\mathcal{P}^{even} = \oplus_{d\in\mathbb{N}} \mathcal{P}^d \tag{4.13}$$

So $\mathcal{P}^{inv}$ lies in $\mathcal{P}^{even}$, and thus $\mathcal{P}^{inv}$ and $\mathbb{C}[L^{red}]$ are $\mathbb{N}$-graded.

The symplectic version of the Kraft-Procesi result is

Theorem 4.2. [22, Theorem 3.3], [23, Theorem 5.3] *The algebra homomorphism $\gamma^* : S(\mathfrak{g}) \to \mathcal{P}$ defined by $y \mapsto \gamma_y$ ($y \in \mathfrak{g}$) induces a G-equivariant isomorphism of $\mathbb{N}$-graded Poisson algebras from $\mathcal{R}$ onto $\mathcal{P}^{inv}/\mathcal{I}^{inv}$.*

The cited results of Kraft and Procesi are given in geometric language, and the reader who wants to read all the proofs in [22] and [23] will need some knowledge in algebraic geometry. The statements in [22, Theorem 3.3] and [23, Theorem 5.3] are easy to translate into algebra though, since we are dealing with *affine* varieties. Kraft and Procesi show that γ maps $\sigma^{-1}(0)$ onto $\overline{\mathcal{O}}$, and moreover this surjection $\gamma' : \sigma^{-1}(0) \to \overline{\mathcal{O}}$ is a quotient map for the action of S. In this setting of a reductive group acting on an affine variety, "quotient map" has a very strong meaning coming from Mumford's geometric invariant theory, as explained in [22, §1.4] and [23, §0.11]. Precisely, γ' being a quotient map means that the corresponding map $\mathcal{R} \to \mathcal{P}/\mathcal{I}$ on coordinate rings is injective and has image equal to $(\mathcal{P}/\mathcal{I})^{inv}$.

In symplectic language then, Kraft and Procesi proved that the moment map γ induces a G-isomorphism of affine complex algebraic varieties from L^{red} onto $\overline{\mathcal{O}}$. This isomorphism is also equivariant with respect to the natural $\mathbb{C}^*$-actions on L^{red} and $\overline{\mathcal{O}}$. Finally, since γ is a moment map it follows that γ^* preserves the Poisson brackets. Thus we get Theorem 4.2.

5. Weyl algebra $\mathcal{W}$ for L

The Kraft-Procesi construction realized $\mathcal{R}$ as a subquotient, namely $\mathcal{P}^{inv}/\mathcal{I}^{inv}$, of $\mathcal{P}^{even}$. Our aim is to make a noncommutative analog of their construction.

The image of the moment map (4.8) is the closure $\overline{\mathcal{Y}}$ of the minimal nilpotent orbit $\mathcal{Y}$ of $Sp(L,\mathbb{C})$, and $\mathcal{P}^{even} = \mathbb{C}[\overline{\mathcal{Y}}]$ as graded Poisson algebras. We know by Example 3.3 that $\overline{\mathcal{Y}}$ has a unique Dixmier algebra $(\mathcal{D}, \xi_{\mathcal{D}}, \tau_{\mathcal{D}}, \vartheta_{\mathcal{D}})$, and then $\mathcal{D}$ is the quotient of $\mathcal{U}(\mathfrak{sp}(L,\mathbb{C}))$ by its Joseph ideal. In this section we will give a more concrete model for this Dixmier algebra. Then in §6 we will perform the noncommutative analog of reduction.

Let $\mathcal{W}$ be the Weyl algebra for L^*. This means that $\mathcal{W}$ is the quotient of the tensor algebra of L^* by the two-sided ideal generated by the elements $a\otimes b - b\otimes a - \{a,b\}$ where a and b lie in L^*. Let $a \mapsto \hat{a}$ be the natural map $L^* \to \mathcal{W}$. We can identify $\mathfrak{sp}(L,\mathbb{C})$ with S^2L^* and then we have the Lie algebra embedding

$$\xi : \mathfrak{sp}(L,\mathbb{C}) \longrightarrow \mathcal{W}, \quad \xi^{ab} = \hat{a}\hat{b} + \hat{b}\hat{a} \tag{5.1}$$

There is an increasing algebra filtration $\mathcal{W} = \cup_{j\in\frac{1}{2}\mathbb{N}} \mathcal{W}_j$ where $\mathcal{W}_j$ is the image of the space of tensors of degree at most $2j$. We have $[\mathcal{W}_j, \mathcal{W}_k] \subset \mathcal{W}_{j+k-1}$. Thus the associated graded algebra $\operatorname{gr}\mathcal{W} = \oplus_{j\in\frac{1}{2}\mathbb{N}}\mathcal{W}_j/W_{j-\frac{1}{2}}$ is commutative and the commutator in $\mathcal{W}$ induces a Poisson bracket (of degree -1) on $\operatorname{gr}\mathcal{W}$. In this way $\operatorname{gr}\mathcal{W}$ becomes a graded Poisson algebra. Then $\operatorname{gr}\mathcal{W}$ identifies naturally with $\mathcal{P}$.

The symplectic group $Sp(L,\mathbb{C})$ acts naturally on $\mathcal{W}$ by algebra automorphisms. This action respects ξ, the filtration on $\mathcal{W}$, the Poisson bracket on $\operatorname{gr}\mathcal{W}$, etc. The corresponding action of $\mathfrak{sp}(L,\mathbb{C})$ on $\mathcal{W}$ is given by the operators $[\xi^{ab},\cdot]$.

We next choose a Cartan involution ς of $\mathfrak{sp}(L,\mathbb{C})$. To do this, we go back into the Kraft-Procesi construction. Recall that each space V_d in (4.1) carried a bilinear form $\mathbf{b}_d$ ($d = 0,\dots,r$). We can choose a positive definite hermitian form $\mathbf{h}_d$ on V_d which is compatible with $\mathbf{b}_d$ in the sense that the intersection of S_d with the unitary group of $\mathbf{h}_d$

is a maximal compact subgroup K_d of S_d. (This is an equivalent version of the setup in [21].) These $\mathbf{h}_d$ determine naturally a positive definite hermitian form $\mathbf{h}$ on L. Now we define $\varsigma(T) = -T^\dagger$ for $T \in \mathfrak{sp}(L,\mathbb{C})$, where $T^\dagger$ is the adjoint of T with respect to $\mathbf{h}$.

Corresponding to ς is a compact real form $Sp(L)$ of $Sp(L,\mathbb{C})$ with Lie algebra $\mathfrak{sp}(L)$. For later use (see §6), we notice that $G \cap Sp(L) = K_0$ and $S \cap Sp(L) = K_1 \times \cdots \times K_r$ are compact real forms of G and S, which we will denote by G_c and S_c.

Now $\mathcal{W}$ is a $(\mathfrak{sp}(L,\mathbb{C}) \oplus \mathfrak{sp}(L,\mathbb{C}), Sp(L))$-module, where the representation

$$\mathfrak{sp}(L,\mathbb{C}) \oplus \mathfrak{sp}(L,\mathbb{C}) \to \operatorname{End} \mathcal{W} \tag{5.2}$$

is given by $(x,y)\cdot A = \xi^x A - A\xi^y$ and the action of $Sp(L)$ corresponds to the subalgebra $\{(x,x) : x \in \mathfrak{sp}(L)\}$. The action of the center $\mathbb{Z}_2$ of $Sp(L)$ produces the decomposition

$$\mathcal{W} = \mathcal{W}^{even} \oplus \mathcal{W}^{odd} \tag{5.3}$$

where $\mathcal{W}^{even}$ is space of invariants for $\mathbb{Z}_2$. The induced filtration on $\mathcal{W}^{even}$ satisfies $\mathcal{W}^{even}_{p+\frac{1}{2}} = \mathcal{W}^{even}_p$ if $p \in \mathbb{N}$. So we might as well just consider the algebra filtration

$$\mathcal{W}^{even} = \cup_{d\in\mathbb{N}} \mathcal{W}^{even}_d \tag{5.4}$$

Now we can make $\mathcal{W}^{even}$ into a Dixmier algebra.

Proposition 5.1. *The Dixmier algebra for the closure $\overline{\mathcal{Y}}$ of the minimal nilpotent orbit of $Sp(L,\mathbb{C})$ is the quadruple $(\mathcal{W}^{even}, \xi, \tau, \vartheta)$, for some unique choices of τ and ϑ.*

Proof. The map ξ induces a filtered algebra isomorphism $\pi : \mathcal{U}(\mathfrak{sp}(L,\mathbb{C}))/\mathcal{J} \to \mathcal{W}^{even}$ where $\mathcal{J}$ is the Joseph ideal (see [2, §5]). Clearly $\operatorname{gr} \mathcal{W}^{even} = \mathcal{P}^{even}$ and so π induces a graded isomorphism $S(\mathfrak{sp}(L,\mathbb{C}))/\mathfrak{I}(\overline{\mathcal{Y}}) \xrightarrow{\sim} \mathcal{P}^{even}$. Now everything follows by Example 3.3. ∎

6. Dixmier Algebra for $\overline{\mathcal{O}}$

$\mathcal{W}^{even}$ is, by means of (5.2), both a $(\mathfrak{g}\oplus\mathfrak{g}, G_c)$-module and an $(\mathfrak{s}\oplus\mathfrak{s}, S_c)$-module, and these two actions commute.

Definition 6.1. $\mathcal{B}$ is the $(\mathfrak{g}\oplus\mathfrak{g}, G_c)$-module obtained by taking the coinvariants of $\mathcal{W}^{even}$ in the category of $(\mathfrak{s}\oplus\mathfrak{s}, S_c)$-modules.

This means that $\mathcal{B}$ is the quotient $\mathcal{W}^{even}/\mathcal{M}$ where $\mathcal{M}$ is the subspace spanned by all $\xi^x A - A\xi^y$ and $A - s\cdot A$ where $x, y \in \mathfrak{s}$, $s \in S_c$ and $A \in \mathcal{W}^{even}$ (see [20, Chapter II]). Then $\mathcal{B}$ inherits from $\mathcal{W}^{even}$ an increasing G_c-stable vector space filtration $\mathcal{B} = \cup_{d\in\mathbb{N}} \mathcal{B}_d$.

Let $\mathcal{W}^{inv}$ be the algebra of invariants for S_c. Then $\mathcal{W}^{inv}$ lies in $\mathcal{W}^{even}$ (since S_c contains $\mathbb{Z}_2$) and so $\mathcal{W}^{inv}$ inherits from $\mathcal{W}^{even}$ an algebra filtration $\mathcal{W}^{inv} = \cup_{d\in\mathbb{N}} \mathcal{W}^{inv}_d$.

Lemma 6.2. *The natural map*

$$\phi : \mathcal{W}^{inv} \to \mathcal{B} \tag{6.1}$$

is surjective in each filtration degree and its kernel is a two-sided ideal. In this way, $\mathcal{B}$ becomes a filtered algebra. The corresponding map $\operatorname{gr}\phi : \mathcal{P}^{inv} \to \operatorname{gr}\mathcal{B}$ is a surjective homomorphism of graded Poisson algebras.

Proof. We prove this in §7. ∎

Our main result is

Theorem 6.3. *We have* $\operatorname{gr}\mathcal{B} = \mathcal{P}^{inv}/\mathcal{I}^{inv}$. *So* $\operatorname{gr}\mathcal{B} \simeq \mathcal{R}$ *as graded Poisson algebras.*

Proof. The proof occupies §8. □

Corollary 6.4. *The quadruple* $(\mathcal{B}, \xi_{\mathcal{B}}, \tau_{\mathcal{B}}, \vartheta_{\mathcal{B}})$ *is a Dixmier algebra for* $\overline{\mathcal{O}}$, *where* $\xi_{\mathcal{B}}$, $\tau_{\mathcal{B}}$ *and* $\vartheta_{\mathcal{B}}$ *are the maps induced by* ξ, τ, *and* ϑ.

Proof. We prove this in §9. □

Let J be the kernel of the algebra homomorphism $\widetilde{\xi_{\mathcal{B}}} : \mathcal{U}(\mathfrak{g}) \to \mathcal{B}$ defined by $\xi_{\mathcal{B}}$. Proposition 3.1 gives

Corollary 6.5. *Suppose we exclude the cases where* $\mathcal{O}$ *is disconnected* (*so where* $G = O(2n, \mathbb{C})$ *and the Jordan block size partition of* $\mathcal{O}$ *is very even*). *Then* J *is a completely prime primitive ideal of* $\mathcal{U}(\mathfrak{g})$ *with* $\operatorname{gr} J = \mathfrak{I}(\overline{\mathcal{O}})$ $\tau_{\mathfrak{g}}(J) = J$, *and* $\vartheta_{\mathfrak{g}}(J) = J$.

The methods we have used thus far give no information about the excluded cases.

Remark 6.6. (i) Suppose $G = GL(n, \mathbb{C})$. Then the space L has a $G \times S$-invariant polarization, and using this we can describe $\mathcal{B}$ and ξ in the following way. Let X be the flag manifold of G of flags of the type in (4.1). Let $\mathcal{D}^{\frac{1}{2}}(X)$ be the algebra of twisted differential operators for the (locally defined) square root of the canonical bundle on X as in [7]. We can show ([8]) that $\mathcal{B}$ identifies with $\mathcal{D}^{\frac{1}{2}}(X)$ in such a way that ξ corresponds to the canonical mapping of $\mathfrak{g}$ into $\mathcal{D}^{\frac{1}{2}}(X)$. Then by [7, Corollary 8.5], J is a maximal ideal in $\mathcal{U}(\mathfrak{g})$.

(ii) Suppose $G = O(n, \mathbb{C})$ or $G = Sp(2n, \mathbb{C})$. If $\mathcal{O}$ is the minimal nilpotent orbit, then $\mathcal{B}$ is the quotient of $\mathcal{U}(\mathfrak{g})$ by its Joseph ideal. This follows by the result in Example 3.3.

7. Proof of Lemma 6.2

The action of S_c on $\mathcal{W}$ is completely reducible and locally finite, and S and S_c have the same invariants and the same irreducible subspaces. So we can form the decomposition

$$\mathcal{W}^{even} = \mathcal{W}^{inv} \oplus \mathcal{X} \tag{7.1}$$

where $\mathcal{X}$ is the sum of all non-trivial S_c-isotypic components. Then $\mathcal{X}$ is the span of the elements $A - s \cdot A$ where $A \in \mathcal{W}^{even}$ and $s \in S_c$. Then $\mathcal{M} = \mathcal{M}^{inv} \oplus \mathcal{X}$. Hence the natural map ϕ is surjective and its kernel is $\mathcal{M}^{inv}$. I.e., we have vector space isomorphisms

$$\mathcal{W}^{inv}/\mathcal{M}^{inv} \xrightarrow{\sim} \mathcal{W}^{even}/\mathcal{M} \xrightarrow{\sim} \mathcal{B} \tag{7.2}$$

The decomposition (7.1) is compatible with the filtration on $\mathcal{W}^{even}$, since the filtration is S_c-invariant. Consequently ϕ is surjective in each filtration degree. So $\mathcal{W}^{even}$ and $\mathcal{W}^{inv}$ induce the same filtration on $\mathcal{B}$.

Next we show that $\mathcal{M}^{inv}$ is a two-sided ideal in $\mathcal{W}^{inv}$. To begin with, $\mathcal{M}^{inv}$ lies inside the subspace $\mathcal{M}'$ of $\mathcal{M}$ spanned by all $\xi^x A$ and $A\xi^x$ where $x \in \mathfrak{s}$ and $A \in \mathcal{W}^{even}$. This follows using (7.1). So it suffices to show that $D\mathcal{M}'$ and $\mathcal{M}'D$ lie in $\mathcal{M}'$ if $D \in \mathcal{W}^{inv}$. Obviously $DA\xi^x$ lies in $\mathcal{M}'$. Invariance of D gives $\xi^x D - D\xi^x = 0$ and so $D\xi^x A = \xi^x DA$ lies in $\mathcal{M}'$. Thus $D\mathcal{M}' \subseteq \mathcal{M}'$; similarly $\mathcal{M}'D \subseteq \mathcal{M}'$.

The associated graded algebra $\operatorname{gr}\mathcal{B}$ is the quotient $\operatorname{gr}\mathcal{W}^{inv}/\operatorname{gr}\mathcal{M}^{inv}$. We find $\operatorname{gr}\mathcal{W}^{inv} = \mathcal{P}^{inv}$. Now the final assertion is clear.

Remark 7.1. If we replace $\mathcal{W}^{even}$ by $\mathcal{W}$ in Definition 6.1, then we get the same thing. I.e., if $\widetilde{\mathcal{B}}$ is the module of coinvariants of $\mathcal{W}$, then $\widetilde{\mathcal{B}}$ identifies naturally with $\mathcal{B}$. Indeed $\widetilde{\mathcal{B}} = \mathcal{W}/\widetilde{\mathcal{M}}$ where $\widetilde{\mathcal{M}}$ is the subspace spanned by all $\xi^x A - A\xi^y$ and $A - s\cdot A$ where now $A \in \mathcal{W}$. But then $\widetilde{\mathcal{M}} = \mathcal{W}^{odd} \oplus \mathcal{M}$ and so the natural map $\widetilde{\phi} : \mathcal{W}^{inv} \to \widetilde{\mathcal{B}}$ is surjective with the same kernel $\mathcal{M}^{inv}$. Also the filtration $\widetilde{\mathcal{B}} = \cup_{j\in\frac{1}{2}\mathbb{N}}\widetilde{\mathcal{B}}_j$ induced by $\mathcal{W}$ reduces to the one induced by $\mathcal{W}^{inv}$ in the sense that $\widetilde{\phi}(\mathcal{W}^{inv}_d) = \widetilde{\mathcal{B}}_d = \widetilde{\mathcal{B}}_{d+\frac{1}{2}}$.

8. Proof of Theorem 6.3

We will compute $\operatorname{gr}\mathcal{B}$ by using a homology spectral sequence. We will consider the relative Lie algebra homology $H(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even})$. By definition (see [20, Chapter II, §6-7]), $H_j(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even})$ is the jth derived functor, in the category of $(\mathfrak{s}\oplus\mathfrak{s}, S_c)$-modules, of the coinvariants. So

$$\mathcal{B} = H_0(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even}) \tag{8.1}$$

The idea is that we will introduce a filtration of the complex that computes the homology in such a way that the induced filtration on $H_0(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even})$ is the one we have already defined on $\mathcal{B}$. Then we will use the usual spectral sequence of a filtered complex to compute $\operatorname{gr} H_0(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even})$. The computation will rely on the geometric result of Kraft and Procesi that (in the notation of §4) $\sigma^{-1}(0)$ is a complete intersection.

To begin with, we have $\mathfrak{s}\oplus\mathfrak{s} = \mathfrak{k}\oplus\mathfrak{p}$ where $\mathfrak{k} = \{(x,x) : x\in\mathfrak{s}\}$ and $\mathfrak{p} = \{(x,-x) : x \in \mathfrak{s}\}$. Then $\mathfrak{k}$ is the complexified Lie algebra of S_c. The standard complex ([20, page 163]) for computing $H(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even})$ is

$$0 \longleftarrow \wedge^0\mathfrak{p}\otimes_{S_c}\mathcal{W}^{even} \xleftarrow{\partial} \wedge^1\mathfrak{p}\otimes_{S_c}\mathcal{W}^{even} \xleftarrow{\partial} \cdots \xleftarrow{\partial} \wedge^m\mathfrak{p}\otimes_{S_c}\mathcal{W}^{even} \longleftarrow 0 \tag{8.2}$$

Here $m = \dim\mathfrak{s}$ and $\otimes_{S_c}$ denotes the S_c-coinvariants of the tensor product. We call this complex A where ${}^tA = \wedge^t\mathfrak{p}\otimes_{S_c}\mathcal{W}^{even}$.

The differential ∂ in (8.2) is given by

$$\partial(Y_1\wedge\cdots\wedge Y_t\otimes D) = \sum_{l=1}^{t}(-1)^l Y_1\wedge\cdots\wedge\widehat{Y_l}\wedge\cdots\wedge Y_t\otimes\Pi^{Y_l}(D) \tag{8.3}$$

where the Y_i lie in $\mathfrak{p}$, $D\in\mathcal{W}^{even}$, and Π is the representation (5.2). (Notice the terms involving $[Y_i, Y_j]$ are not present because $[\mathfrak{p},\mathfrak{p}]\subseteq\mathfrak{k}$.) To make the complex more transparent, we identify $\mathfrak{p}$ with $\mathfrak{s}$ so that $(x,-x)$ corresponds to x. Then (8.2) becomes

$$\partial(x_1\wedge\cdots\wedge x_t\otimes D) = \sum_{l=1}^{t}(-1)^l x_1\wedge\cdots\wedge\widehat{x_l}\wedge\cdots\wedge x_t\otimes(\xi^{x_l}D + D\xi^{x_l}) \tag{8.4}$$

Next we define an increasing filtration of A by the spaces

$${}^tA^d = \wedge^t\mathfrak{p}\otimes_{S_c}\mathcal{W}^{even}_{d-t} \tag{8.5}$$

where we set $\mathcal{W}^{even}_j = 0$ if $j<0$. Then $\partial\,{}^tA^d \subseteq {}^{t-1}A^d$. This follows since the ξ^x lie in $\mathcal{W}^{even}_1$ and so $\mathfrak{p}\cdot\mathcal{W}^{even}_d\subseteq\mathcal{W}^{even}_{d+1}$. So we have in hand a filtration of the complex (8.1). We put $A^{d,q} = {}^{d+q}A^d$; then d is the filtration degree and q is the complementary degree. The induced filtration on the homology is $H(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even}) = \cup_{d\in\mathbb{N}} F^d$ where $F^d = \oplus_{q\in\mathbb{Z}} F^{d,q}$ and $F^{d,q}$ is the dth filtration piece of $H_{d+q}(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even})$. The associated graded space $\mathrm{gr}\ H(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even})$ is the direct sum of the spaces

$$\mathrm{gr}^d\ H_{d+q}(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even}) = F^{d,q}/F^{d-1,q+1} \tag{8.6}$$

Notice that ${}^0A = \mathcal{W}^{even}$ and the filtration on ${}^0A$ defined by (8.5) is the same one as in (5.4). So $\mathrm{gr}\ H_0(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even}) = \mathrm{gr}\ \mathcal{B}$. Our goal is to prove

$$\mathrm{gr}^d\ H_0(\mathfrak{s}\oplus\mathfrak{s}, S_c; \mathcal{W}^{even}) = (\mathcal{P}^{inv}/\mathcal{I}^{inv})^d \tag{8.7}$$

Now we consider the spectral sequence $E_0, E_1, \ldots$ associated to our filtered complex. (See e.g., [20, Appendix D] or [15, Chapter I,§4] for the construction of this spectral sequence in the general setting.) The E_0 term is given by $E_0^{d,q} = A^{d,q}/A^{d-1,q+1}$ and so

$$E_0^{d,q} = \wedge^{d+q}\,\mathfrak{p}\ \otimes_{S_c}\ \mathcal{W}^{even}_{-q}/\mathcal{W}^{even}_{-q-1} \tag{8.8}$$

The identification $\mathrm{gr}\,\mathcal{W}^{even} = \mathcal{P}^{even}$ gives

$$E_0^{d,q} = \wedge^{d+q}\,\mathfrak{p}\ \otimes_{S_c}\ \mathcal{P}^{-q} \tag{8.9}$$

(Thus the E_0 term occupies the octant of the d,q plane where $q \le 0$ and $d+q \ge 0$.) So E_0^d is the complex

$$0 \longleftarrow \wedge^0\,\mathfrak{p}\otimes_{S_c}\mathcal{P}^d \xleftarrow{\partial_0} \wedge^1\,\mathfrak{p}\otimes_{S_c}\mathcal{P}^{d-1} \xleftarrow{\partial_0} \cdots \xleftarrow{\partial_0} \wedge^m\,\mathfrak{p}\otimes_{S_c}\mathcal{P}^{d-m} \longleftarrow 0 \tag{8.10}$$

The boundary ∂_0 is induced by ∂. We can easily compute ∂_0 since the natural projection maps $\psi_d : \mathcal{W}_d \to \mathcal{P}^d$ are given by $\psi_d(\hat{a}_1\cdots\hat{a}_{2d}) = a_1\cdots a_{2d}$ where $a_i \in L^*$ (cf. §5). So for $D \in \mathcal{W}_d$ we have $\psi_{d+1}(\xi^x D) = \psi_{d+1}(D\xi^x) = \sigma_x\psi_d(D)$. Thus (8.4) gives

$$\partial_0(x_1\wedge\cdots\wedge x_t\otimes f) = \sum_l (-1)^l x_1\wedge\cdots\wedge\widehat{x_l}\wedge\cdots\wedge x_t\otimes(2\sigma_{x_l}f) \tag{8.11}$$

The total complex E_0 is

$$0 \longleftarrow \wedge^0\,\mathfrak{p}\otimes_{S_c}\ \mathcal{P}^{even} \xleftarrow{\partial_0} \wedge^1\,\mathfrak{p}\otimes_{S_c}\ \mathcal{P}^{even} \xleftarrow{\partial_0} \cdots \xleftarrow{\partial_0} \wedge^m\,\mathfrak{p}\otimes_{S_c}\ \mathcal{P}^{even} \longleftarrow 0 \tag{8.12}$$

The homology $H(E_0)$, together with a differential ∂_1, is the E_1 term of the spectral sequence. More precisely, $E_1^{d,q} = H_{d+q}(E_0^{d,*})$.

To compute E_1, we observe that $H(E_0)$ is the S_c-coinvariants of the homology of the complex

$$0 \longleftarrow \wedge^0\,\mathfrak{p}\otimes\ \mathcal{P}^{even} \xleftarrow{\partial_0} \wedge^1\,\mathfrak{p}\otimes\ \mathcal{P}^{even} \xleftarrow{\partial_0} \cdots \xleftarrow{\partial_0} \wedge^m\,\mathfrak{p}\otimes\ \mathcal{P}^{even} \longleftarrow 0 \tag{8.13}$$

Indeed, E_0 is the S_c-coinvariants of (8.13), and taking coinvariants commutes with taking homology. The latter follows because each space $\wedge^t\mathfrak{p}\otimes\mathcal{P}^{even}$ is a locally finite S_c-representation, and for any such representation $\mathcal{V}$, the natural map $\mathcal{V}^{S_c}\to\mathcal{V}_{S_c}$ from invariants to coinvariants is an isomorphism.

To compute the homology of (8.13), we recognize (8.13) as the Koszul complex K of the sequence $\sigma_{y_1},\ldots,\sigma_{y_m}$ in $\mathcal{P}^{even}$ where $y_1,\ldots,y_m$ is any basis of $\mathfrak{s}$. Recall from §4 that $\mathcal{I}$ is the ideal in $\mathcal{P}$ generated by the σ_{y_i}. Kraft and Procesi proved in [22, Theorem 3.3] and [23, Theorem 5.3] that the subscheme $\sigma^{-1}(0)$ of L is a reduced complete intersection, i.e., $\sigma_{y_1},\ldots,\sigma_{y_m}$ is a regular sequence in $\mathcal{P}$. Let us consider the Koszul complex $\widetilde{K}$ of this sequence in $\mathcal{P}$. By a well known result of commutative algebra (see [17, III, Proposition 7.10A]) the homology of $\widetilde{K}$ is concentrated in degree zero and $H_0(\widetilde{K}) = \mathcal{P}/\mathcal{I}$ as graded algebras. But K is simply obtained from $\widetilde{K}$ by taking $\mathbb{Z}_2$-invariants. Hence the homology of K is concentrated in degree zero and $H_0(K) = \mathcal{P}^{even}/\mathcal{I}^{even}$ as graded algebras. Then the module $H_0(K)_{S_c}$ of coinvariants identifies with $\mathcal{P}^{inv}/\mathcal{I}^{inv}$.

Thus E_1 is the complex

$$0 \longleftarrow \mathcal{P}^{inv}/\mathcal{I}^{inv} \xleftarrow{\partial_1} 0 \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_1} 0 \longleftarrow 0 \tag{8.14}$$

where the differentials $\partial_1^{d,q} : E_1^{d,q} \to E_1^{d-1,q}$ are obviously zero and

$$E_1^{d,-d} = (\mathcal{P}^{inv}/\mathcal{I}^{inv})^d \qquad \text{while} \qquad E_1^{d,q} = 0 \text{ if } q \neq -d \tag{8.15}$$

Now we can compute the rest of the spectral sequence. We know E_{r+1} is the homology of E_r with respect to a differential ∂_r; i.e. $E_{r+1}^{d,q} = \ker \partial_r^{d,q} / \operatorname{im} \partial_r^{d+r,q-r+1}$ where $\partial_r^{d,q}$ maps $E_r^{d,q}$ to $E_r^{d-r,q+r-1}$. For $r \geq 1$, we find that $E_1^{d,q} = E_r^{d,q}$ and the differentials $\partial_r^{d,q}$ are all zero.

The E_∞ term of the spectral sequence satisfies

$$E_\infty^{d,q} = \mathrm{gr}^d \; H_{d+q}(\mathfrak{s} \oplus \mathfrak{s}, S_c; \mathcal{W}^{even}) \tag{8.16}$$

Our final step is to show our spectral sequence converges in that

$$E_1^{d,q} = E_\infty^{d,q} \tag{8.17}$$

This will finish off the proof of Theorem 6.3 because then (8.15) gives the desired result (8.7).

The convergence (8.17) follows formally from the two properties: (i) $A^d = 0$ if $d < 0$ where $A^d = \oplus_{q \in \mathbb{Z}} A^{d,q}$ and (ii) $A^{d,q}$ has finite dimension. Indeed, following the notation in [15, I,§4.2], we have

$$\begin{aligned} E_r^{d,q} &= Z_r^{d,q}/(B_{r-1}^{d,q} + Z_{r-1}^{d-1,q+1}) \\ E_\infty^{d,q} &= Z_\infty^{d,q}/(B_\infty^{d,q} + Z_\infty^{d-1,q+1}) \end{aligned} \tag{8.18}$$

where $Z_r^{d,q} = \{z \in A^{d,q} : \partial z \in A^{d-r}\}$, $Z_\infty^{d,q} = A^{d,q} \cap \ker \partial$, $B_r^{d,q} = A^{d,q} \cap \partial A^{d+r}$ and $B_\infty^{d,q} = A^{d,q} \cap \partial A$. Suppose we fix d and q. Then (i) gives $Z_r^{d,q} = Z_\infty^{d,q}$ and $Z_{r-1}^{d-1,q+1} = Z_\infty^{d-1,q+1}$ if $r > d$, and (ii) gives $B_{r-1}^{d,q} = B_\infty^{d,q}$ if r is large enough. Therefore $E_r^{d,q} = E_\infty^{d,q}$ for r large enough. But we found $E_1^{d,q} = E_r^{d,q}$ $(r \geq 1)$ and so (8.17) follows.

We remark that it also follows that $\mathrm{gr}\, H_j(\mathfrak{s} \oplus \mathfrak{s}, S_c; \mathcal{W}^{even}) = 0$ for $j > 0$. Thus we have proven

Proposition 8.1. *$H_j(\mathfrak{s} \oplus \mathfrak{s}, S_c; \mathcal{W}^{even}) = 0$ if $j > 0$.*

9. Proof of Corollary 6.4

S_c and G_c are commuting subgroups of $Sp(L)$. It follows that ξ maps $\mathfrak{g}$ into $\mathcal{W}^{inv}$ and so ξ induces $\xi_{\mathcal{B}}$ where $\xi_{\mathcal{B}}$ is the composition $\mathfrak{g} \xrightarrow{\xi} \mathcal{W}^{inv} \to \mathcal{B}$. Next τ is $Sp(L)$-invariant and so in particular is S_c-invariant. Then τ preserves $\mathcal{W}^{inv}$. We see that τ preserves $\mathcal{M}$, and so τ preserves $\mathcal{M}^{inv}$. Hence τ induces a filtered algebra anti-involution $\tau_{\mathcal{B}}$ of $\mathcal{B} = \mathcal{W}^{inv}/\mathcal{M}^{inv}$. Finally, ϑ is $Sp(L)$-invariant and so in particular is S_c-invariant. Then

ϑ preserves $\mathcal{W}^{inv}$. We see that ϑ preserves $\mathcal{M}$, and so ϑ induces an antilinear filtered algebra involution $\vartheta_{\mathcal{B}}$ of $\mathcal{B}$.

Thus we have in place our Dixmier algebra data for $\overline{\mathcal{O}}$. It is clear because of Proposition 5.1 that the axioms are satisfied.

References

[1] J. Adams and D. Barbasch, *Reductive dual pair correspondence for complex groups*, J. Funct. Anal. **132** (1995), 1-42.

[2] A. Astashkevich and R. Brylinski, *Non-Local Equivariant Star Product on the Minimal Nilpotent Orbit*, math.QA/0010257 v2, Adv. Math. **171** (2002), 86-102

[3] D. Barbasch, *The unitary dual for complex classical Lie groups*, Inv. Math. **96** (1989), 103–176.

[4] D. Barbasch, *Orbital integrals of nilpotent orbits*, The Mathematical Heritage of Harish-Chandra, R. S. Doran and V. S. Varadarajan, eds., Proc. Symp. Pure Math., vol. 68, AMS, 2000, 97–110.

[5] D. Barbasch and D.A. Vogan, *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. **259** (1982), 152–199.

[6] D. Barbasch and D.A. Vogan, *Unipotent representations of complex semisimple groups*, Ann. of Math. **121** (1985), 41–110.

[7] R. Brylinski, *Equivariant Deformation Quantization for the Cotangent Bundle of a Flag Manifold*, math.QA/0010258, Ann. Inst. Fourier **52**:3 (2002), 881-897.

[8] R. Brylinski, *Dixmier Algebras for Classical Complex Nilpotent Orbits via Kraft-Procesi Models II*, in preparation.

[9] R. Brylinski, *Quantization of Classical Complex Nilpotent Orbits*, in preparation.

[10] M. Cahen and S. Gutt, *An algebraic construction of $*$ product on the regular orbits of semi simple Lie groups*, Gravitation and Geometry, W. Rindler and A. Trautman eds., 73–82, Bibliopolis, 1987.

[11] D.H. Collingwood and W.M. McGovern, Nilpotent Orbits in Semisimple Lie Algebras, Van Nostrand Reinhold, 1993.

[12] J. Dixmier, Enveloping Algebras, North-Holland, 1977.

[13] M. Duflo, *Représentations unitaires irréductibles des groupes simples complexes de rang deux*, Bull. Soc. Math. France **107** (1979), 55–96.

[14] C. Duval, P. Lecomte and V. Ovsienko, *Conformally equivariant quantization: existence and uniqueness*, Ann. Inst. Fourier **49:6** (1999), 1999–2029.

[15] R. Godement, Théorie des Faisceaux, Hermann, Paris, third edition, 1973.

[16] K.I. Gross, *The dual of a parabolic subgroup and a degenerate principal series of $Sp(n,\mathbb{C})$*, Amer Jour Math **93** (1971), 399-428.

[17] R. Hartshorne, Algebraic Geometry, Graduate Texts in Math 52, Springer-Verlag New York, 1977.

[18] R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.

[19] A. Joseph, *The minimal orbit in a simple Lie algebra and associated maximal ideal*, Ann. Scient. Ec. Norm. Sup. **9** (1976), 1–30.

[20] A. Knapp and D.A. Vogan, Cohomological Induction and Unitary Representations, Princeton University Press, 1995.

[21] P.Z. Kobak and A. Swann, *Classical nilpotent orbits as hyperkähler quotients*, Int. J. Math. **7** (1996), 193–210.

[22] H. Kraft and C. Procesi, *Closures of conjugacy classes of matrices are normal*, Inv. Math. **53** (1979), 227–247.

[23] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in clasical groups*, Comment. Math. Helv. **57** (1982), 539–602.

[24] T. Levasseur and J.T. Stafford, *Rings of differential operators on classical rings of invariants*, Mem. Amer. Math. Soc. , number 412, 1989.
[25] J-S Li, *Singular unitary representations of classical groups*, Inv. Math. **97** (1989), 237–255.
[26] W. McGovern, *Quantization of nilpotent orbits and their covers in complex classical groups*, preprint, Yale University (1989).
[27] W. McGovern, *Completely prime maximal ideals and quantization*, Mem. Amer. Math. Soc. , number 519, 1994.
[28] C. Moeglin, *Idéaux complètement premiers de l'algèbre enveloppante de* $gl_n(\mathbb{C})$, J. Alg. **87** (1987), 287–366.
[29] C. Moeglin, *Correspondance de Howe pour les paires réductives duales, quelques calculs dans le cas Archimedien*, J. Funct. Anal. **85** (1989), 1–85.
[30] M. Van den Bergh, *Differential operators on semi-invariants for tori and weighted projective spaces*, Topics in Invariant Theory (Paris 1989/90), 255-272, Lecture Notes in Math. **1478**, Springer, Berlin 1991.
[31] D.A. Vogan, *Noncommutative algebras and unitary representations*, The Mathematical Heritage of Hermann Weyl, 35–60, Proc. Symp. Pure Math., vol. 48, Amer. Math. Soc., Providence, 1988.
[32] D.A. Vogan, *Dixmier algebras, sheets and representation theory*, Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory, 333–396, Progress in Math, vol. 92, Birkhäuser, 1990.
[33] D.A. Vogan, *Associated varieties and unipotent representations*, in Harmonic Analysis on Reductive Lie Groups, 315-388, Progress in Math, vol. 101, Birkhäuser, 1991.

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