Projective representations of symmetric groups via Sergeev duality

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1 Introduction

In this article, we determine the irreducible projective representations of the symmetric group S_d and the alternating group A_d over an algebraically closed field of characteristic $p \neq 2$. These matters are well understood in the case $p = 0$, thanks to the fundamental work of Schur [19] in 1911, as well as the much more recent work of Nazarov [16, 17] and others. So the focus here is primarily on the case of positive characteristic, where surprisingly little is known as yet. In particular, we obtain a natural combinatorial labelling of the irreducibles in terms of a certain set $\mathfrak{RP}_p(d)$ of *restricted p-strict partitions* of d. Such partitions arose recently in work of Kashiwara *et. al.* [9] and Leclerc and Thibon [12] on the q-deformed Fock space of the Kac-Moody algebra of type $A^{(2)}_{p-1}$. In particular, Leclerc and Thibon proposed that $\mathfrak{RP}_p(d)$ should label the irreducible projective representations in some natural way, and we show here how this can be done. Note that for $p = 3, 5$, the labelling problem was solved in [1, 2], and if $p = 2$ the irreducible projective representations are just ordinary, non-projective representations so do not need to be considered further here.

To be more precise, recall that λ is a partition of d if $\lambda = (\lambda_1, \lambda_2, \dots)$ is a non-increasing sequence of non-negative integers summing to d. Call λ a *p-strict partition* if

$$0 < \lambda_i - \lambda_{i+1} + \delta_i \qquad \text{for } i = 1, 2, \dots, \quad \text{where } \delta_i = \begin{cases} 1 & \text{if } p \mid \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

(To include $p = 0$, our convention is that 0 only divides 0.) Call λ a *restricted p-strict partition* if either $p = 0$, or $p > 0$ and

$$0 < \lambda_i - \lambda_{i+1} + \delta_i \leq p \qquad \text{for } i = 1, 2, \dots, \quad \text{where } \delta_i = \begin{cases} 1 & \text{if } p \mid \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{P}_p(d)$ denote the set of all p-strict partitions of d, and $\mathfrak{RP}_p(d) \subseteq \mathfrak{P}_p(d)$ denote the restricted p-strict partitions of d. Also, define $h_{p'}(\lambda)$ to be the number of parts of λ not divisible by p. Then, our construction leads to a labelling of the irreducible projective representations of S_d over an algebraically closed field of characteristic $p \neq 2$ by pairs (λ, ε) where $\lambda \in \mathfrak{RP}_p(d)$ and $\varepsilon = 0$ if $d - h_{p'}(\lambda)$ is even or ± 1 if $d - h_{p'}(\lambda)$ is odd. For A_d, the labelling is by pairs (λ, ε) where $\lambda \in \mathfrak{RP}_p(d)$ and $\varepsilon = \pm 1$ if $d - h_{p'}(\lambda)$ even or 0 if $d - h_{p'}(\lambda)$ is odd.

The construction is based closely on ideas of Sergeev and Nazarov in the characteristic 0 theory. In particular, the key step in our approach is to determine the irreducible "polynomial" representations of the strange Lie supergroup $Q(n)$ in characteristic p. These turn out to be labelled naturally according to high-weight theory by *all* p-strict partitions with at most n non-zero parts. ¿From this, we use Sergeev's superalgebra analogue [20] of Schur-Weyl duality to determine the irreducible spin representations of a certain double cover of the hyperoctahedral group, then pass from there to the symmetric group using a Clifford theory argument due to Nazarov [17].

2 Associative superalgebras

In this section, we record a number of standard, generally well-known results about the representation theory of finite dimensional associative superalgebras. As useful general references, but sometimes with different conventions to us, we cite [6, 15] and [14, ch.3].

Let $\mathbb{k}$ be an algebraically closed field of characteristic $p \neq 2$. By a *superspace* we mean a $\mathbb{Z}_2$-graded $\mathbb{k}$-vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Given a homogeneous vector $0 \neq v \in V$, we denote its degree by $\partial(v) \in \mathbb{Z}_2$. A *superspace map* $f : V \to W$ between two superspaces means a linear map with $f(V_i) \subseteq W_i$ for each $i \in \mathbb{Z}_2$; note as a general rule, we are writing scalars on the right and maps on the left, unless we explicitly say otherwise. A *subsuperspace* $U \subset V$ means a subspace U of V such that $U = (U \cap V_{\bar{0}}) \oplus (U \cap V_{\bar{1}})$. Define the superspace map $\delta_V : V \to V$ on homogeneous vectors by $\delta_V(v) = (-1)^{\partial(v)} v$. Then obviously, a subspace $U \subset V$ is a subsuperspace if and only if U is stable under δ_V.

Let V and W be superspaces. We view the direct sum $V \oplus W$ as a superspace with $(V \oplus W)_i = V_i \oplus W_i$, and the tensor product $V \otimes W$ as a superspace with $(V \otimes W)_{\bar{0}} = V_{\bar{0}} \otimes W_{\bar{0}} \oplus V_{\bar{1}} \otimes W_{\bar{1}}$ and $(V \otimes W)_{\bar{1}} = V_{\bar{0}} \otimes W_{\bar{1}} \oplus V_{\bar{1}} \otimes W_{\bar{0}}$. Also, we make $\mathrm{Hom}_{\mathbb{k}}(V, W)$ into a superspace with $\mathrm{Hom}_{\mathbb{k}}(V, W)_i$ consisting of the *homogeneous maps of degree* i, that is, the maps $\theta : V \to W$ with $\theta(V_j) \subseteq W_{i+j}$ for $j \in \mathbb{Z}_2$. Given in addition superspaces V', W' and homogeneous maps $f \in \mathrm{Hom}_{\mathbb{k}}(V, W)$ and $f' \in \mathrm{Hom}_{\mathbb{k}}(V', W')$, we write $f \otimes f'$ for the map $V \otimes V' \to W \otimes W'$ with $(f \otimes f')(v \otimes v') = (-1)^{\partial(f')\partial(v)} f(v) \otimes f'(v')$ for all homogeneous $v \in V, v' \in V'$; this gives us a natural superspace map $\mathrm{Hom}_{\mathbb{k}}(V, W) \otimes \mathrm{Hom}_{\mathbb{k}}(V', W') \to \mathrm{Hom}_{\mathbb{k}}(V \otimes V', W \otimes W')$. The dual superspace V^* means the superspace $\mathrm{Hom}_{\mathbb{k}}(V, \mathbb{k})$, where we view $\mathbb{k}$ as a superspace concentrated in degree $\bar{0}$. So as a special case of the preceeding definition, we obtain a natural injective superspace map $(V^*) \otimes (W^*) \hookrightarrow (V \otimes W)^*$, which is an isomorphism if V and W are both finite dimensional.

An *associative superalgebra* is a superspace A with the additional structure of an associative, unital $\mathbb{k}$-algebra such that $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$. A superalgebra homomorphism (resp. antihomomorphism) $\theta : A \to B$ is a superspace map that is an algebra homomorphism (resp. antihomomorphism) in the usual sense, and its kernel is a *superideal*, i.e. an ordinary two-sided ideal that is also a subsuperspace. Most importantly, given two superalgebras A and B, we view the tensor product $A \otimes B$ as a superalgebra with the induced grading and multiplication defined by $(a \otimes b)(a' \otimes b') = (-1)^{\partial(b)\partial(a')}(aa') \otimes (bb')$ for homogeneous elements $a, a' \in A, b, b' \in B$. We note that $A \otimes B \cong B \otimes A$, the isomorphism being given by the *supertwist map* $T_{A,B} : A \otimes B \to B \otimes A, a \otimes b \mapsto (-1)^{\partial(a)\partial(b)} b \otimes a$ for homogeneous $a \in A, b \in B$.

2.1. **Example.** Let $V = V_{\bar 0} \oplus V_{\bar 1}$ be a superspace of dimension $m+n$. The *tensor superalgebra* is the tensor algebra $T(V)$ regarded as a superalgebra with the induced grading. As a quotient of $T(V)$, we have the *symmetric superalgebra*, namely,

$$S(V) = T(V)/\langle v \otimes w - (-1)^{\partial(v)\partial(w)} w \otimes v \mid \text{for all homogeneous vectors } v, w \in V\rangle.$$

If we have in mind fixed bases $v_1, \dots, v_m$ for $V_{\bar 0}$ and $v_{m+1}, \dots, v_n$ for $V_{\bar 1}$, we denote the superalgebras $T(V)$ and $S(V)$ instead by $T(m,n)$ and $S(m,n)$, respectively the free superalgebra and the free commutative superalgebra on $m+n$ generators. Set $S(m) := S(m,0)$, just the usual polynomial algebra on m generators concentrated in degree $\bar 0$, and $\bigwedge(n) := S(0,n)$, just the usual exterior algebra but with generators assigned the degree $\bar 1$. The superalgebra $\bigwedge(n)$ is called the *Grassmann superalgebra*. We have that

$$\begin{aligned} S(m) &\cong S(1) \otimes \cdots \otimes S(1) \quad (m \text{ times}),\\ \textstyle\bigwedge(n) &\cong \textstyle\bigwedge(1) \otimes \cdots \otimes \bigwedge(1) \quad (n \text{ times}),\\ S(m,n) &\cong S(m) \otimes \textstyle\bigwedge(n). \end{aligned}$$

2.2. **Example.** Another basic example that we will meet is the *Clifford superalgebra*, namely, the associative superalgebra $C(n)$ on generators $c_1, \dots, c_n$ all of degree $\bar 1$, subject to the relations $c_i^2 = 1$ for $i = 1, \dots, n$ and $c_i c_j = -c_j c_i$ for all $i \neq j$. If, slightly more generally, one has in mind *non-zero* scalars $\lambda_1, \dots, \lambda_n \in \Bbbk^\times$, the superalgebra with generators $b_1, \dots, b_n$ subject to the relations $b_i^2 = \lambda_i, b_i b_j = -b_j b_i$ is isomorphic to $C(n)$, the obvious isomorphism sending $b_i \mapsto \sqrt{\lambda_i} c_i$. The crucial point is that $C(n_1 + n_2) \cong C(n_1) \otimes C(n_2)$. Indeed, the generators $c_1 \otimes 1, \dots, c_{n_1} \otimes 1, 1 \otimes c_1, \dots, 1 \otimes c_{n_2}$ of $C(n_1) \otimes C(n_2)$ satisfy the same relations as the generators $c_1, \dots, c_{n_1}, c_{n_1+1}, \dots, c_{n_1+n_2}$ of the left hand algebra. It follows at once that $C(n) \cong C(1) \otimes \cdots \otimes C(1) \quad (n \text{ times})$.

Let A be an associative superalgebra. A left A-*supermodule* is a superspace M which is a left A-module in the usual sense, such that $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$ (there is of course an analogous notion of right supermodule, which we omit). Saying that M is an A-supermodule is equivalent to saying that the associated representation $\rho : A \to \mathrm{End}_\Bbbk(M)$ is a homomorphism of associative superalgebras. A homomorphism between two A-supermodules means the same as an ordinary A-module homomorphism; it is important now however to write homomorphisms between *left* A-supermodules on the *right* (and vice versa). So, if M and N are left A-supermodules, an A-supermodule homomorphism $f : M \to N$ means a linear map such that $a(mf) = (am)f$ for all $a \in A, m \in M$. Writing $f = f_{\bar 0} + f_{\bar 1}$ for unique homogeneous maps f_i of degree i, both of $f_{\bar 0}$ and $f_{\bar 1}$ are A-supermodule homomorphisms. So, the space $\mathrm{Hom}_A(M,N)$ of all A-supermodule homomorphisms from M to N decomposes as $\mathrm{Hom}_A(M,N)_{\bar 0} \oplus \mathrm{Hom}_A(M,N)_{\bar 1}$, where $\mathrm{Hom}_A(M,N)_i$ is the set of all homogeneous A-supermodule homomorphisms of degree i from M to N. Define the category $\mathbf{mod}(A)$ to be the category of all left A-supermodules, morphisms being the A-supermodule homomorphisms as just defined. It is a *superadditive category* in the sense of [14, §3.7], i.e. an additive category such that each $\mathrm{Hom}_A(M,N)$ is $\mathbb{Z}_2$-graded in a way that is compatible with composition of morphisms. When talking about functors between superadditive categories, we always mean functors which preserve the grading of morphisms.

A subsupermodule of an A-supermodule means an A-submodule in the usual sense that is a subsuperspace. An A-supermodule M is called *irreducible* if it has no proper A-subsupermodules, and *absolutely irreducible* if it is irreducible when viewed just as an ordinary A-module. Let M be a finite dimensional A-supermodule that is irreducible but not absolutely irreducible. Then, we can find an irreducible A-submodule N of M that is not a subsupermodule, i.e. is not δ_M-stable. It is elementary to check that $\delta_M(N)$ is also an irreducible A-submodule of M. Hence, $N \oplus \delta_M(N)$ is an A-submodule of M, even a subsupermodule since it is now δ_M-stable. Since M was an irreducible supermodule, we deduce that in fact $M = N \oplus \delta_M(N)$. Let $u_1, \dots, u_n$ be a basis for N. Then, $\delta_M(u_1), \dots, \delta_M(u_n)$ is a basis for $\delta_M(N)$ so $u_1 + \delta_M(u_1), \dots, u_n + \delta_M(u_n)$ is a basis for $M_{\bar{0}}$ and $u_1 - \delta_M(u_1), \dots, u_n - \delta_M(u_n)$ is a basis for $M_{\bar{1}}$. The following lemma now follows easily:

2.3. **Lemma.** *If M is a finite dimensional irreducible but not absolutely irreducible A-supermodule, then there exist bases $v_1, \dots, v_n$ for $M_{\bar{0}}$ and $v_{-1}, \dots, v_{-n}$ for $M_{\bar{1}}$ such that*

$$M \cong \mathrm{span}\{v_1 + v_{-1}, \dots, v_n + v_{-n}\} \oplus \mathrm{span}\{v_1 - v_{-1}, \dots, v_n - v_{-n}\}$$

as a direct sum of two non-isomorphic irreducible A-submodules. Moreover, the endomorphism $J_M : M \to M, v_i \mapsto v_{-i}$ commutes with the action of A on M.

If M is an A-supermodule, $\mathrm{End}_A(M)$ denotes the superalgebra of all A-supermodule endomorphisms of M. We stress again that we write the action of elements of $\mathrm{End}_A(M)$ on M on the opposite side to the action of A. We have the following analogue of Schur's lemma, which is easily proved (given Lemma 2.3) in the same way as the classical version:

2.4. **Lemma (Schur's lemma)**. *Let M be a finite dimensional irreducible A-supermodule. Then,*

$$\mathrm{End}_A(M) = \begin{cases} \mathrm{span}\{\mathrm{id}_M\} & \textit{if } M \textit{ is absolutely irreducible,} \\ \mathrm{span}\{\mathrm{id}_M, J_M\} & \textit{otherwise,} \end{cases}$$

where J_M is as in Lemma 2.3.

We say that an A-supermodule M is completely reducible if it can be decomposed into a direct sum of irreducible A-supermodules. Call A a *simple superalgebra* if A has no non-trivial superideals, and a *semisimple superalgebra* if A is completely reducible viewed as a left A-supermodule. Equivalently, A is semisimple if *every* left A-supermodule is completely reducible. We have:

2.5. **Lemma (Wedderburn's theorem)**. *Let A be a finite dimensional associative superalgebra. The following are equivalent:*

(i) *A is simple;*

(ii) *A is semisimple with only one irreducible supermodule up to isomorphism;*

(iii) *there is a finite dimensional superspace V such that* either $A \cong \mathrm{End}_{\Bbbk}(V)$ or $A \cong \{\theta \in \mathrm{End}_{\Bbbk}(V) \mid \theta \circ J = J \circ \theta\}$ *for some involution $J \in \mathrm{End}_{\Bbbk}(V)_{\bar{1}}$.*

Moreover, if A is semisimple then it is isomorphic to a direct sum of simple superalgebras.

Notice in view of Lemma 2.3 that if A is a semisimple superalgebra, then it is a semisimple algebra. The converse is also true, and is proved e.g. in [15, (1.4c)]; it can also be deduced from Wedderburn's theorem by considering the effect of the map δ_A on the simple ideals of A. Somewhat more generally, we have:

2.6. **Lemma.** *Let A be a finite dimensional associative superalgebra. Then, the Jacobson radical of A (viewed just as an ordinary algebra) can be characterized as the smallest superideal K of A such that A/K is a semisimple superalgebra.*

Proof. Let J be the Jacobson radical of A viewed as an ordinary algebra, and K be the unique smallest superideal of A such that A/K is a semisimple superalgebra. We know that A/K is semisimple as an ordinary algebra by Lemma 2.3, so $J \subseteq K$. Conversely, we observe that J is a superideal since J is invariant under the algebra automorphism δ_A of A. So, A/J is a superalgebra that is semisimple as an algebra. Hence, by [15, (1.4c)], it is a semisimple superalgebra, and $K \subseteq J$. $\square$

We point out another immediate consequence of Wedderburn's theorem and Lemma 2.6:

2.7. **Corollary.** *Let A be a finite dimensional associative superalgebra, and $\{V_1, \dots, V_n\}$ be a complete set of irreducible A-supermodules such that $V_1, \dots, V_m$ are absolutely irreducible and $V_{m+1}, \dots, V_n$ are not. For $i = m+1, \dots, n$, decompose V_i as $V_i^+ \oplus V_i^-$ as a direct sum of two non-isomorphic irreducible A-modules. Then, $\{V_1, \dots, V_m, V_{m+1}^{\pm}, \dots, V_n^{\pm}\}$ is a complete set of irreducible A-modules.*

Given left supermodules M and N over arbitrary associative superalgebras A and B respectively, the (outer) tensor product $M \otimes N$ is an $A \otimes B$-supermodule with action defined by $(a \otimes b)(m \otimes n) = (-1)^{\partial(b)\partial(m)} am \otimes bn$ for all homogeneous $a \in A, b \in B, m \in M, n \in N$. (Analogously, if M and N are right supermodules, the action of $A \otimes B$ on $M \otimes N$ is defined instead by $(m \otimes n)(a \otimes b) = (-1)^{\partial(a)\partial(n)} ma \otimes nb$ for all homogeneous $a \in A, b \in B, m \in M, n \in N$.) If $f : M \to M'$ (resp. $g : N \to N'$) is a homomorphism between two left A- (resp. B-) supermodules, then $f \otimes g : M \otimes N \to M' \otimes N'$ is an $A \otimes B$-supermodule homomorphism; this works precisely because of our convention to write the homomorphisms f, g and $f \otimes g$ on the right, i.e. $f \otimes g$ means the map with $(m \otimes n)(f \otimes g) = (-1)^{\partial(n)\partial(f)} mf \otimes ng$. The following lemma gives the other basic facts about outer tensor products that we need:

2.8. **Lemma.** *Suppose that A and B are finite dimensional associative superalgebras, and that M, N are irreducible supermodules over A, B respectively.*

(i) *If both M and N are absolutely irreducible, then $M \otimes N$ is an absolutely irreducible $A \otimes B$-supermodule.*

(ii) *If exactly one of the modules M or N is absolutely irreducible, then $M \otimes N$ is an irreducible but not absolutely irreducible $A \otimes B$-supermodule.*

(iii) *If neither M or N are absolutely irreducible, then $M \otimes N$ decomposes as a direct sum of two isomorphic, absolutely irreducible $A \otimes B$-supermodules.*

Moreover, all irreducible $A \otimes B$-supermodules arise as constituents of $M \otimes N$ for some choice of M, N.

Combining Lemma 2.8 with Wedderburn's theorem, it follows in particular that if A and B are finite dimensional semisimple associative superalgebras then $A \otimes B$ is also a semisimple superalgebra.

2.9. **Example.** The Grassmann algebra $\bigwedge(1)$ has just one irreducible supermodule up to isomorphism, namely, $\Bbbk$ itself with elements of $\bigwedge(1)_{\bar{1}}$ acting as zero. This is absolutely irreducible, so it follows by induction on n using Lemma 2.8 that $\bigwedge(n) = \bigwedge(n-1) \otimes \bigwedge(1)$ has just one irreducible supermodule, namely, $\Bbbk$ itself with elements of $\bigwedge(n)_{\bar{1}}$ acting as zero. Note however that $\bigwedge(n)$ is not a semisimple superalgebra, indeed even $\bigwedge(1)$ is not semisimple, being isomorphic as an algebra to the algebra $\Bbbk[x]/(x^2)$ of truncated polynomials.

2.10. **Example.** Consider the Clifford algebra $C(n)$ again. First, observe that $C(1)$ is just the simple superalgebra of 2×2 matrices of the form $\left\{ \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix} \right) \,\middle|\, a, b \in \Bbbk \right\}$, the generator c_1 of $C(1)$ corresponding to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. So $C(1)$ has precisely one irreducible supermodule $U(1)$ which is irreducible but not absolutely irreducible, of dimension 2, as in the second case of Lemma 2.5(iii). Hence, applying Lemma 2.8, $C(2) = C(1) \otimes C(1)$ has one irreducible supermodule $U(2)$, namely the unique irreducible appearing with multiplicity two in the $C(2)$-supermodule $U(1) \otimes U(1)$, and $U(2)$ is absolutely irreducible of dimension 2. Explicitly, $U(2)$ can be described as the module on basis u_+, u_- with action defined by $c_1 u_+ = u_-, c_1 u_- = u_+, c_2 u_+ = \sqrt{-1} u_-, c_2 u_- = -\sqrt{-1} u_+$. Finally, for $n > 2$, $C(n) = C(n-2) \otimes C(2)$, so by Lemma 2.5(i) and (ii), it has just one irreducible supermodule $U(n)$, defined inductively by $U(n) = U(n-2) \otimes U(2)$. This is absolutely irreducible if and only if $U(n-2)$ is absolutely irreducible, which is if and only if n is even. Observe that we have just shown that $C(n)$ is a semisimple superalgebra with a unique irreducible supermodule. So by Lemma 2.5, $C(n)$ is in fact a simple superalgebra, indeed, up to isomorphism, it must be the unique simple superalgebra of dimension 2^n. Its unique irreducible supermodule $U(n)$ has dimension $2^{\lfloor (n+1)/2 \rfloor}$.

Following [20, §1.4], a $\mathbb{Z}_2$*-graded group* is a pair (G, ∂) where G is a finite group and $\partial : G \to \mathbb{Z}_2$ is a group homorphism. If (G, ∂) is a $\mathbb{Z}_2$-graded group, we can regard the group algebra $\Bbbk G$ as a superalgebra, the degree of $g \in G$ being $\partial(g)$. We are interested next in counting the number of irreducible $\Bbbk G$-supermodules in terms of conjugacy classes. Define $n_{p'}(G, \bar{0})$ to be the number of G-conjugacy classes of p'-elements of degree $\bar{0}$ and $n_{p'}(G, \bar{1})$ to be the number of G-conjugacy classes of p'-elements of degree $\bar{1}$.

2.11. **Lemma.** *Let (G, ∂) be a $\mathbb{Z}_2$-graded group. Then, there are $n_{p'}(G, \bar{0})$ pairwise non-isomorphic irreducible $\Bbbk G$-supermodules. Of these, $n_{p'}(G, \bar{0}) - n_{p'}(G, \bar{1})$ of them are absolutely irreducible, and the remaining $n_{p'}(G, \bar{1})$ are irreducible but not absolutely irreducible.*

Proof. We follow the proof of the analogous classical result for ordinary group algebras, see [10, §13]. For an arbitrary superalgebra A, write $Z(A) = \{a \in A \,|\, ab = ba \text{ for all } b \in A\}$ for its centre and $S(A) = \operatorname{span}\{ab - ba \,|\, a, b \in A\}$. These are both subsuperspaces of A. Now let J denote the Jacobson radical of the group algebra $\Bbbk G$. By Lemma 2.6, J is a superideal and $A := \Bbbk G/J$ is the largest semisimple superalgebra quotient of $\Bbbk G$. So $\Bbbk G$ and A have the same number of irreducible supermodules. Combining Lemma 2.4 and Lemma 2.5, we deduce that the number of irreducible $\Bbbk G$-supermodules is equal to $\dim Z(A)_{\bar{0}}$ and the number of irreducible but not absolutely irreducible $\Bbbk G$-supermodules is equal to $\dim Z(A)_{\bar{1}}$. By [10, 13.3], $A = Z(A) \oplus S(A)$, so $\dim[Z(A)]_i = \dim[A/S(A)]_i$ for $i = \bar{0}, \bar{1}$. Finally, to count this dimension in either case, use formula (14) in the proof of [10, 13.8]; this tells us at once that $\dim[A/S(A)]_i = n_{p'}(G, i)$. $\square$

Now suppose that (G, ∂) is a $\mathbb{Z}_2$-graded group and that $\pi : \widehat{G} \twoheadrightarrow G$ is a double cover, so that $\ker \pi = \{1, \zeta\}$ for some $1 \neq \zeta \in Z(\widehat{G})$. Lift ∂ to $\widehat{G}$ to make $\widehat{G}$ into a $\mathbb{Z}_2$-graded group with degree function satisfying $\partial(\zeta) = \bar{0}$. The elements $\zeta_+ = (1-\zeta)/\sqrt{2}$ and $\zeta_- = (1+\zeta)/\sqrt{2}$ are orthogonal central idempotents of the group superalgebra $\Bbbk\widehat{G}$ summing to the identity, so

$$\Bbbk\widehat{G} = \zeta_+(\Bbbk\widehat{G}) \oplus \zeta_-(\Bbbk\widehat{G}) \tag{2.12}$$

as a direct sum of two-sided superideals. Obviously, $\zeta_+(\Bbbk\widehat{G}) \cong (\Bbbk\widehat{G})/\langle \zeta - 1 \rangle \cong \Bbbk G$; the algebra $\zeta_-(\Bbbk\widehat{G}) \cong (\Bbbk\widehat{G})/\langle \zeta + 1 \rangle$ is a *twisted group algebra*. Since the number of irreducible $\Bbbk\widehat{G}$-supermodules is equal to the number of irreducible $\zeta_+(\Bbbk\widehat{G})$-supermodules plus the number of irreducible $\zeta_-(\Bbbk\widehat{G})$-supermodules, we have:

2.13. **Lemma.** *The number of irreducible $\zeta_-(\Bbbk\widehat{G})$-supermodules is $n_{p'}(\widehat{G}, \bar{0}) - n_{p'}(G, \bar{0})$.*

To conclude this preliminary section on associative superalgebras, we give a brief review of "Schur functors" arising from idempotents in this setting. Suppose that A is an arbitrary finite dimensional superalgebra. and that $e \in A$ is a *homogeneous* idempotent, necessarily of degree $\bar{0}$. Then, the ring eAe is a superalgebra in its own right, its identity element being the idempotent e. We have the (exact) *Schur functor*

$$R_e : \mathbf{mod}(A) \to \mathbf{mod}(eAe)$$

given on objects by left multiplication by the idempotent e and by restriction on morphisms. Given an A-supermodule M, let $O_e(M)$ (resp. $O^e(M)$) denote the largest (resp. smallest) subsupermodule N of M such that N (resp. M/N) is annihilated by e. Finally, let $\mathbf{mod}_e(A)$ denote the full subcategory of $\mathbf{mod}(A)$ consisting of all A-supermodules M with $O_e(M) = 0$ and $O^e(M) = 0$. The following basic result is proved as in the classical case, see [8, §2]:

2.14. **Lemma.** *The restriction of the functor R_e to $\mathbf{mod}_e(A)$ gives an equivalence of categories between $\mathbf{mod}_e(A)$ and $\mathbf{mod}(eAe)$.*

Suppose that $\{L(\lambda) \mid \lambda \in \Lambda\}$ be a complete set of pairwise non-isomorphic irreducible A-supermodules, and set $\Lambda_1 = \{\lambda \in \Lambda \mid R_e L(\lambda) \neq 0\}$. Then, as an immediate consequence of Lemma 2.14, we have:

2.15. **Corollary.** *The eAe-supermodules $\{R_e L(\lambda) \mid \lambda \in \Lambda_1\}$ give a complete set of pairwise non-isomorphic irreducible eAe-supermodules. Moreover, for $\lambda \in \Lambda_1$, $R_e L(\lambda)$ is absolutely irreducible if and only if $L(\lambda)$ is absolutely irreducible.*

3 Double covers

Our primary interest is in projective representations of the symmetric group S_d. However, most of the remainder of the article will be taken up with studying the representation theory of a certain finite dimension superalgebra called the *Sergeev algebra*, originally introduced in [20]. In this section, we define this superalgebra, and establish a functorial connection between it and the projective representations of the symmetric group.

Start with the symmetric group S_d acting naturally on the left on the set $\{1, \dots, d\}$. For $i = 1, \dots, d-1$, let $s_i \in S_d$ denote the basic transposition $(i\ i+1)$, and recall that the $s_1, \dots, s_{d-1}$ generate S_d subject to the well-known Coxeter relations. Define the group $\widehat{S}_d$ to be the group with generators $\zeta, \hat{s}_1, \dots, \hat{s}_{d-1}$ subject to the relations

$$\zeta^2 = \hat{s}_i^2 = 1, \qquad \zeta \hat{s}_i = \hat{s}_i \zeta,$$
$$\hat{s}_i \hat{s}_{i+1} \hat{s}_i = \hat{s}_{i+1} \hat{s}_i \hat{s}_{i+1}, \qquad \hat{s}_i \hat{s}_j = \zeta \hat{s}_j \hat{s}_i$$

for all $1 \leq i \leq d-1$ and all $1 \leq j \leq d-1$ with $|i-j| > 1$. The map sending $\zeta \mapsto 1$, $\hat{s}_i \mapsto s_i$ gives a surjective homomorphism $\widehat{S}_d \to S_d$, and $\widehat{S}_d$ is a double cover of S_d (see [21, p.100]).

Make S_d into a $\mathbb{Z}_2$-graded group with degree function $\partial : S_d \to \mathbb{Z}_2$ being the usual signature of a permutation. So, $\ker \partial = A_d$, the alternating group. Lifting, $\widehat{S}_d$ is also a $\mathbb{Z}_2$-graded group and we again denote the degree function by $\partial : \widehat{S}_d \to \mathbb{Z}_2$; its kernel is denoted $\widehat{A}_d$, a double cover of the alternating group. As in (2.12), the superalgebra $\Bbbk \widehat{S}_d$ is isomorphic to $\Bbbk S_d \oplus S(d)$, where $S(d)$ is the superalgebra $\zeta_-(\Bbbk \widehat{S}_d) = \Bbbk \widehat{S}_d / \langle 1 + \zeta \rangle$. We are primarily interested here in studying the representation theory of this superalgebra $S(d)$. Recall the definition of the set $\mathfrak{RP}_p(d)$ of restricted p-strict partitions of d from the introduction.

3.1. **Lemma.** *The number of irreducible $S(d)$-supermodules is equal to $|\mathfrak{RP}_p(d)|$.*

Proof. Using Lemma 2.13 and the known labelling of the conjugacy classes of S_d and $\widehat{S}_d$, see e.g. [21, Theorem 2.1] or [19, p.172], one easily shows that the number of irreducible $S(d)$-supermodules is equal to the number of partitions λ of d with all non-zero parts of λ being odd and not divisible by p. Now we appeal to the following partition identity obtained by Leclerc and Thibon [12, (40)]:

$$\sum_{d \geq 0} |\mathfrak{RP}_p(d)| t^d = \prod_{i\ odd, p \nmid i} \frac{1}{1-t^i}, \tag{3.2}$$

which shows that the number of partitions λ of d with all non-zero parts of λ being odd and not divisible by p is equal to $|\mathfrak{RP}_p(d)|$. □

We turn our attention next to the hyperoctahedral group and its double cover. Denoting elements of the Abelian group $\mathbb{Z}_2^d$ as d-tuples $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$ with each $\varepsilon_i \in \mathbb{Z}_2$, the symmetric group acts on the right on $\mathbb{Z}_2^d$ by $\varepsilon \cdot w = (\varepsilon_{w1}, \varepsilon_{w2}, \dots, \varepsilon_{wd})$ for $w \in S_d, \varepsilon \in \mathbb{Z}_2^d$. The *hyperoctahedral group* W_d is then the semidirect product $S_d \ltimes \mathbb{Z}_2^d$. So, W_d is the set of all pairs (w, ε) with $w \in S_d, \varepsilon \in \mathbb{Z}_2^d$, and the product of two such elements is defined by $(x, \varepsilon)(y, \delta) = (xy, \varepsilon \cdot y + \delta)$. Henceforth, we will identify $w \in S_d$ (resp. $\varepsilon \in \mathbb{Z}_2^d$) with the element $(w, 1) \in W_d$ (resp. $(1, \varepsilon) \in W_d$). Extend the action of S_d on $\mathbb{Z}_2^d$ to an action of all of W_d on $\mathbb{Z}_2^d$, so that $\varepsilon \cdot (w, \delta) = \varepsilon \cdot w + \delta$ for $\varepsilon \in \mathbb{Z}_2^d, (w, \delta) \in W_d$.

The *Clifford group* C_d is the group with generators $\{\zeta, z_1, \dots, z_d\}$ subject to the relations

$$\zeta^2 = 1, \qquad \zeta z_i = z_i \zeta,$$
$$z_i^2 = 1, \qquad z_i z_j = \zeta z_j z_i$$

for all $1 \leq i \neq j \leq d$. The group C_d consists of the distinct elements $\{z^\varepsilon, \zeta z^\varepsilon \mid \varepsilon \in \mathbb{Z}_2^d\}$, where for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{Z}_2^d$, z^ε denotes $z_1^{\varepsilon_1} z_2^{\varepsilon_2} \dots z_d^{\varepsilon_d}$. We view the group algebra $\Bbbk C_d$ as a superalgebra, grading the group elements so that $\partial(\zeta) = \bar{0}$ and $\partial(z_i) = \bar{1}$ for each $i = 1, \dots, d$. Observe then that the Clifford superalgebra $C(d)$ on generators $c_1, \dots, c_d$ from Example 2.2 is a superalgebra quotient of $\Bbbk C_d$, the quotient map sending $z_i \mapsto c_i, \zeta \mapsto -1$. We write $c^\varepsilon = c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_d^{\varepsilon_d}$, the image of the element z^ε under the quotient map, so that the $\{c^\varepsilon \mid \varepsilon \in \mathbb{Z}_2^d\}$ give a basis for $C(d)$. The product of two such basis elements is given explicitly by the rule

$$c^\varepsilon c^\delta = \alpha(\varepsilon; \delta) c^{\varepsilon + \delta} \qquad \text{where} \qquad \alpha(\varepsilon; \delta) = \prod_{1 \leq s < t \leq d} (-1)^{\delta_s \varepsilon_t}$$

for $\varepsilon, \delta \in \mathbb{Z}_2^d$. It is worth remarking for later calculations that $\alpha(\varepsilon + \varepsilon'; \delta) = \alpha(\varepsilon; \delta)\alpha(\varepsilon'; \delta)$ and $\alpha(\varepsilon; \delta + \delta') = \alpha(\varepsilon; \delta)\alpha(\varepsilon; \delta')$.

Now, there is a unique right action of S_d on C_d by automorphisms so that $\zeta \cdot w = \zeta, z_i \cdot w = z_{w^{-1}i}$ for all $i = 1, \dots, d$ and $w \in S_d$. Define $\widehat{W}_d$ to be the resulting semidirect product $S_d \ltimes C_d$, that is, the set $\{(w, z) \mid w \in S_d, z \in C_d\}$ with multiplication given by the rule $(w, z)(w', z') = (ww', (z \cdot w')z')$. The element ζ lies in the centre of $\widehat{W}_d$, and it is easy to see that there is a well-defined surjective group homomorphism $\widehat{W}_d \to W_d$ with $\zeta \mapsto 1$ and $(w, z^\varepsilon) \mapsto (w, \varepsilon)$ for all $w \in S_d, \varepsilon \in \mathbb{Z}_2^d$. Thus, $\widehat{W}_d$ is a double cover of the hyperoctahedral group W_d. Make $\widehat{W}_d$ into a $\mathbb{Z}_2$-graded group with degree $\partial : \widehat{W}_d \to \mathbb{Z}_2$ defined by $\partial(w) = \bar{0}$ for all $w \in S_d$, $\partial(\zeta) = \bar{0}$ and $\partial(z_i) = \bar{1}$ for $i = 1, \dots, d$. As in (2.12), we obtain the *Sergeev superalgebra*

$$W(d) := \zeta_-(\Bbbk \widehat{W}_d) = \Bbbk \widehat{W}_d / \langle \zeta + 1 \rangle.$$

In particular, as $W(d)$ is a quotient of the group algebra $\Bbbk \widehat{W}_d$, we deduce by Maschke's theorem that:

3.3. **Lemma.** *If $p = 0$ or $p > d$, then $W(d)$ is a semisimple (super)algebra.*

The Sergeev superalgebra $W(d)$ can be constructed more directly as a twisted tensor product. Start from the right action of S_d on $C(d)$ by superalgebra automorphisms such that $c_i \cdot w = c_{w^{-1}i}$ for each $i = 1, \dots, d$ and $w \in S_d$. So for $w \in S_d$ and $\varepsilon \in \mathbb{Z}_2^d$,

$$c^\varepsilon \cdot w = \alpha(\varepsilon; w) c^{\varepsilon w} \qquad \text{where} \qquad \alpha(\varepsilon; w) = \prod_{\substack{1 \leq s < t \leq d \\ w^{-1}s > w^{-1}t}} (-1)^{\varepsilon_s \varepsilon_t}.$$

Then, $W(d)$ is the superspace $\Bbbk S_d \otimes C(d)$, where $\Bbbk S_d$ is concentrated in degree $\bar{0}$, with the product of two basis elements given by the formula

$$(x \otimes c^\varepsilon)(y \otimes c^\delta) = \alpha(x, \varepsilon; y, \delta) xy \otimes c^{\varepsilon \cdot y + \delta} \qquad \text{where} \qquad \alpha(x, \varepsilon; y, \delta) = \alpha(\varepsilon; y)\alpha(\varepsilon \cdot y; \delta).$$

Note the resulting function $\alpha : W_d \times W_d \to \{\pm 1\}$, $((x, \varepsilon), (y, \delta)) \mapsto \alpha(x, \varepsilon; y, \delta)$ is a 2-cocycle, i.e. satisfies $\alpha(g, 1) = \alpha(1, g) = 1$ and $\alpha(g, hk)\alpha(h, k) = \alpha(g, h)\alpha(gh, k)$ for all $g, h, k \in W_d$. We record a technical property about this cocycle for later use.

3.4. **Lemma.** *For all $\varepsilon, \delta \in \mathbb{Z}_2^d$ and $g = (w, \gamma) \in W_d$,*

$$\alpha(\varepsilon + \delta; w) = \alpha(\varepsilon; g)\alpha(\delta; g)\alpha(\varepsilon + \delta; \delta)\alpha(\varepsilon \cdot g + \delta \cdot g; \delta \cdot g).$$

Proof. Expand the equation $(c^{\varepsilon+\delta} c^{\delta}) \cdot w = (c^{\varepsilon+\delta} \cdot w)(c^{\delta} \cdot w)$ in two different ways to show that $\alpha(\varepsilon + \delta; w) = \alpha(\varepsilon; w)\alpha(\delta; w)\alpha(\varepsilon + \delta; \delta)\alpha(\varepsilon \cdot w + \delta \cdot w; \delta \cdot w)$. Now expand the definition of $\alpha(\varepsilon; g)\alpha(\delta; g)\alpha(\varepsilon \cdot g + \delta \cdot g; \delta \cdot g)$ to see that it equals

$$\begin{aligned}
&\alpha(\varepsilon; w)\alpha(\varepsilon \cdot w; \gamma)\alpha(\delta; w)\alpha(\delta \cdot w; \gamma)\alpha(\varepsilon \cdot w + \delta \cdot w; \delta \cdot w + \gamma) \\
&= \alpha(\varepsilon; w)\alpha(\delta; w)\alpha(\varepsilon \cdot w + \delta \cdot w; \delta \cdot w)\alpha(\varepsilon \cdot w; \gamma)\alpha(\delta \cdot w; \gamma)\alpha(\varepsilon \cdot w + \delta \cdot w; \gamma) \\
&= \alpha(\varepsilon; w)\alpha(\delta; w)\alpha(\varepsilon \cdot w + \delta \cdot w; \delta \cdot w),
\end{aligned}$$

completing the proof. □

We can count the number of irreducible $W(d)$-supermodules using Lemma 2.13:

3.5. **Lemma.** *The number of irreducible $W(d)$-supermodules is equal to $|\mathfrak{RP}_p(d)|$.*

Proof. By Lemma 2.13 and the information on conjugacy classes in [20, Lemma 5], the number of irreducible $W(d)$-supermodules is equal to the number of partitions λ of d with all non-zero parts of λ being odd and not divisible by p. Now use the Leclerc-Thibon partition identity (3.2) as in the proof of Lemma 3.1. □

Combining Lemma 3.1 and Lemma 3.5, we have seen that:

3.6. **Corollary.** *The superalgebras $S(d)$ and $W(d)$ have the same number of irreducible supermodules.*

Underlying Corollary 3.6 is a more precise functorial connection between $S(d)$- and $W(d)$-supermodules, which we now construct. Define t_i to be the image of the generator $\hat{s}_i$ in the superalgebra $S(d)$. Then, $S(d)$ can be defined directly as the superalgebra on degree $\bar{1}$ generators $t_1, \dots, t_{d-1}$ subject to the relations

$$t_i^2 = 1, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = -t_j t_i$$

for all $1 \le i \le d-1$ and all $1 \le j \le d-1$ with $|i - j| > 1$. For each $w \in S_d$, fix a choice of a preimage $\hat{w} \in \widehat{S}_d$ and let t_w denote the image of $\hat{w}$ in $S(d)$. Then, the elements $\{t_w \,|\, w \in S_d\}$ give a basis for $S(d)$. Multiplication is given by $t_x t_y = \beta(x, y) t_{xy}$ where $\beta : S_d \times S_d \to \{\pm 1\}$ is a 2-cocycle, uniquely determined given the choice of the preimages $\hat{w}$. We will need the following variation on [17, Proposition 1.2]:

3.7. **Lemma.** *There is a unique superspace map $\eta : \Bbbk S_d \to W(d)$ such that*

(1) $\eta(s_i) = \sqrt{\left(-\tfrac{1}{2}\right)}(c_i - c_{i+1})$ *for $i = 1, \dots, d-1$;*

(2) $\eta(xy) = (-1)^{\partial(x)\partial(y)}\beta(x, y)\eta(x)\eta(y)$ *for all $x, y \in S_d$.*

Moreover, $wcw^{-1} = (-1)^{\partial(w)\partial(c)}\eta(w)c\eta(w)^{-1}$ for all $w \in S_d, c \in C(d)$.

Proof. The map $\beta' : S_d \times S_d \to \{\pm 1\}, (x,y) \mapsto (-1)^{\partial(x)\partial(y)}\beta(x,y)$ is a 2-cocycle on S_d, as follows from the fact that β is a 2-cocycle. There is a corresponding twisted group algebra $S(d)'$, namely, the superalgebra with basis $\{t'_w \mid w \in S_d\}$ with multiplication satisfying $t'_x t'_y = \beta'(x,y)t'_{xy}$. This twisted group algebra is generated by the elements $t'_1, \dots, t'_{d-1}$ subject to the relations

$$(t'_i)^2 = -1, \qquad t'_i t'_{i+1} t'_i = t'_{i+1} t'_i t'_{i+1}, \qquad t'_i t'_j = -t'_j t'_i$$

for all $1 \le i \le d-1$ and all $1 \le j \le d-1$ with $|i-j| > 1$. So to prove the existence of the map η, it just suffices to check that the elements $\eta(s_i) \in C(d)$ satisfy these same relations, which is routine. For the second part, we can write an arbitrary $\eta(w)$ as a product $\varepsilon\eta(s_{i_1})\dots\eta(s_{i_r})$ for $1 \le i_1, \dots, i_r < d$ and some a sign ε. Then, $\eta(w)^{-1} = \varepsilon\eta(s_{i_r})^{-1}\dots\eta(s_{i_1})^{-1}$. So it suffices to check that $s_i c_j s_i^{-1} = -\eta(s_i)c_j\eta(s_i)^{-1}$ for generators $s_i \in S_d$ and $c_j \in C(d)$, which is a short calculation. □

Let $U(d)$ be the $C(d)$-supermodule defined in Example 2.10. Now define an exact functor

$$Y : \mathbf{mod}(S(d)) \to \mathbf{mod}(W(d))$$

as follows. On an object $N \in \mathbf{mod}(S(d))$, define $Y(N)$ to be the superspace $U(d) \otimes_{\Bbbk} N$, regarded as a left $W(d)$-supermodule so that $c \in C(d)$ acts as $c \otimes \mathrm{id}_N$, and $w \in S_d$ acts as $\eta(w) \otimes t_w$. To check that this does make $U(d) \otimes N$ into a well-defined $W(d)$-supermodule, we use Lemma 3.7:

$$\begin{aligned}
(\eta(x) \otimes t_x)(\eta(y) \otimes t_y) &= (-1)^{\partial(x)\partial(y)}\eta(x)\eta(y) \otimes t_x t_y \\
&= \beta(x,y)\eta(xy) \otimes \beta(x,y)t_{xy} = \eta(xy) \otimes t(xy); \\
(\eta(w) \otimes t_w)(c \otimes \mathrm{id}_N) &= (-1)^{\partial(w)\partial(c)}(\eta(w)c \otimes t_w) \\
&= (-1)^{\partial(w)\partial(c)}(\eta(w)c\eta(w)^{-1}\eta(w) \otimes t_w) \\
&= (wcw^{-1}\eta(w)) \otimes t_w = (wcw^{-1} \otimes 1)(\eta(w) \otimes t_w)
\end{aligned}$$

for $w, x, y \in S_d, c \in C(d)$. On a homomorphism $f : N \to N'$ of left $S(d)$-supermodules, we define $Y(f) : Y(N) \to Y(N')$ to be the linear map $\mathrm{id}_U \otimes f$ (recall we are writing homomorphisms on the right, so $(u \otimes n)Y(f) = u \otimes nf$ for all $u \in U(d), n \in N$).

We will show that if d is even, then Y is an equivalence of categories; if d is odd, Y is something very close to an equivalence. The proof follows the standard argument of Clifford theory, see [5, §51]. Given a $C(d)$-(super)module M and $w \in S_d$, write ${}^wM$ for the new $C(d)$-(super)module equal to M as a vector space, but with action defined by $c \cdot m = (wcw^{-1})m$ for all $c \in C(d), m \in M$.

3.8. **Lemma.** *Let $w \in S_d$. The map $\theta_w : U(d) \to {}^wU(d), u \mapsto (-1)^{\partial(w)\partial(u)}\eta(w)u$ is a $C(d)$-supermodule isomorphism, homogeneous of degree $\partial(w)$.*

Proof. We need to check that $c \cdot (u\theta_w) = (cu)\theta_w$ for all homogeneous $c \in C(d), u \in U(d)$. We have that

$$\begin{aligned}
(cu)\theta_w &= (-1)^{(\partial(u)+\partial(c))\partial(w)}\eta(w)cu = (-1)^{\partial(u)\partial(w)+\partial(c)\partial(w)}\eta(w)c\eta(w)^{-1}\eta(w)u \\
&= (-1)^{\partial(u)\partial(w)}wcw^{-1}\eta(w)u = wcw^{-1}(u\theta_w) = c \cdot (u\theta_w),
\end{aligned}$$

applying Lemma 3.7. $\square$

3.9. **Lemma.** *Suppose that $w \in S_d$. Let N and N' be superspaces and regard $U(d) \otimes N$ and ${}^wU(d) \otimes N'$ as left $C(d)$-supermodules, $c \in C(d)$ acting as $c \otimes \mathrm{id}_N$. Let*

$$f : U(d) \otimes N \to {}^wU(d) \otimes N'$$

be a $C(d)$-supermodule homomorphism. Then,

(i) *if d is even, there exists a unique linear map $\phi_f : N \to N'$ such that $f = \theta_w \otimes \phi_f$;*

(ii) *if d is odd, there exist unique linear maps $\phi_f, \phi'_f : N \to N'$ such that $f = \theta_w \otimes \phi_f + J_U \circ \theta_w \otimes \phi'_f$, where J_U is the unique element of $\mathrm{End}_{C(d)}(U(d))_{\bar{1}}$ with $J_U^2 = \mathrm{id}$.*

Proof. Write $f = \sum_{j \in J} \pi_j \otimes \phi_j + \sum_{k \in K} \pi'_k \otimes \phi'_k$ for homogeneous maps $\pi_j : U(d) \to U(d)$ of degree $\partial(w)$, $\pi'_k : U(d) \to U(d)$ of degree $\partial(w) + \bar{1}$, and maps $\phi_j, \phi'_k : N \to N'$ such that the ϕ_j (resp. the ϕ'_k) are linearly independent. For all homogeneous $c \in C(d), n \in N, u \in U(d)$, we have that

$$(cu \otimes n)f = \sum_{j \in J} (-1)^{\partial(\pi_j)\partial(n)} (cu)\pi_j \otimes n\phi_j + \sum_{k \in K} (-1)^{\partial(\pi'_k)\partial(n)} (cu)\pi'_k \otimes n\phi'_k,$$

$$c \cdot [(u \otimes n)f] = \sum_{j \in J} (-1)^{\partial(\pi_j)\partial(n)} c \cdot (u\pi_j) \otimes n\phi_j + \sum_{k \in K} (-1)^{\partial(\pi'_k)\partial(n)} c \cdot (u\pi'_k) \otimes n\phi'_k.$$

Since this is true for all $n \in N$, we deduce that

$$c \cdot (u\pi_j) = (cu)\pi_j \qquad \text{and} \qquad c \cdot (u\pi'_k) = (cu)\pi'_k$$

for each $j \in J, k \in K$. We deduce at once by Schur's lemma and Lemma 3.8 that π_j is a scalar multiple of θ_w, so we may rescale to assume that each $\pi_j = \theta_w$. Similarly, if d is even, each π'_k must be zero, while if d is odd, each π'_k must equal $J_U \circ \theta_w$ after rescaling. $\square$

3.10. **Lemma.** *The functor Y is faithful. Moreover, given $N, N' \in \mathbf{mod}(S(d))$,*

$$\dim \mathrm{Hom}_{W(d)}(Y(N), Y(N')) = \begin{cases} \dim \mathrm{Hom}_{S(d)}(N, N') & \text{if } d \text{ is even,} \\ 2 \dim \mathrm{Hom}_{S(d)}(N, N') & \text{if } d \text{ is odd.} \end{cases}$$

Proof. Obviously Y is faithful, by the definition of Y on morphisms. Now we prove the statement about homomorphisms in the case d is odd, the case d even being similar. Take $N, N' \in \mathbf{mod}(S(d))$. Let $\phi, \phi' : N \to N'$ be linear maps. A short calculation reveals that:

3.11. *The map $\mathrm{id} \otimes \phi + J_U \otimes \phi' : Y(N) \to Y(N')$ is a $W(d)$-supermodule homomorphism if and only if both ϕ and ϕ' are $S(d)$-supermodule homomorphisms.*

Hence, in particular, if $\phi : N \to N'$ is an $S(d)$-supermodule homomorphism, both $\mathrm{id} \otimes \phi$ and $J_U \otimes \phi$ are $W(d)$-supermodule homomorphisms. So to complete the proof, we need to show that *every* $W(d)$-supermodule homomorphism $f : Y(N) \to Y(N')$ can be written as $\mathrm{id} \otimes \phi + J_U \otimes \phi' : Y(N) \to Y(N')$ for $S(d)$-homomorphisms $\phi, \phi' : N \to N'$. According to Lemma 3.9(ii), $f = \mathrm{id} \otimes \phi_f + J_U \otimes \phi'_f$ for unique linear maps ϕ_f, ϕ'_f. By (3.11) these are $S(d)$-supermodule homomorphisms. $\square$

3.12. **Theorem.** *Suppose that d is even. Then, the functor $Y : \mathbf{mod}(S(d)) \to \mathbf{mod}(W(d))$ is an equivalence of categories.*

Proof. In view of Lemma 3.10, Y is full and faithful, so it just remains to show that Y is dense (see e.g. [4, 1.3.1]). Take an arbitrary $W(d)$-supermodule M. Since $C(d)$ has a unique irreducible supermodule $U(d)$ up to isomorphism, we can find a superspace N such that

$$M \downarrow_{C(d)} \cong U(d) \otimes N$$

as a $C(d)$-supermodule, where $c \in C(d)$ acts on $U(d) \otimes N$ as $c \otimes \mathrm{id}_N$. Using this isomorphism, transfer the action of $W(d)$ on M to $U(d) \otimes N$, so that $M \cong U(d) \otimes N$ as a $W(d)$-supermodule by construction.

Now take $w \in S_d$. Let $f_w : U(d) \otimes N \to {}^wU(d) \otimes N$ be the $C(d)$-supermodule homomorphism determined by left multiplication by w. Note f_w is of degree $\bar{0}$ as w has degree $\bar{0}$ as an element of $W(d)$. By Lemma 3.9, there exists a unique map $\phi_{f_w} : N \to N'$, necessarily of degree $\partial(w)$, such that $f_w = \theta_w \otimes \phi_{f_w}$. Now we make N into an $S(d)$-supermodule by defining the action of $t_w \in S(d)$ on homogeneous $n \in N$ by

$$t_w n = (-1)^{\partial(w)\partial(n)} n\phi_{f_w}.$$

To check that this is well-defined, we have for $x, y \in S_d$ that

$$\begin{aligned}(u \otimes n)(\theta_y \otimes \phi_{f_y})(\theta_x \otimes \phi_{f_x}) &= (-1)^{\partial(u)\partial(y)}(\eta(y)u \otimes t_y n)(\theta_x \otimes \phi_{f_x}) \\ &= (-1)^{\partial(u)\partial(y)+\partial(u)\partial(x)+\partial(x)\partial(y)}(\eta(x)\eta(y)u) \otimes (t_x t_y n) \\ &= (-1)^{\partial(u)\partial(y)+\partial(u)\partial(x)}\beta(x,y)(\eta(xy)u) \otimes (t_x t_y n).\end{aligned}$$

On the other hand, $f_y f_x = f_{xy}$ (writing maps on the right!), so this is equal to

$$(u \otimes n)(\theta_{xy} \otimes \phi_{f_{xy}}) = (-1)^{\partial(u)\partial(x)+\partial(u)\partial(y)}\eta(xy)u \otimes t_{xy} n.$$

So we deduce that $t_{xy} = \beta(x,y)t_x t_y$ as required. Finally, we check that $Y(N) \cong M$ as $W(d)$-supermodules. For $w \in S_d$, we have that

$$\begin{aligned}w(u \otimes n) &= (u \otimes n)f_w = (u \otimes n)(\theta_w \otimes \phi_{f_w}) \\ &= (-1)^{\partial(u)\partial(w)}(\eta(w)u) \otimes (t_w n) = (\eta(w) \otimes t_w)(u \otimes n).\end{aligned}$$

This shows that the two actions of w on $Y(N) = U(d) \otimes N$ agree, and evidently, the two actions of $c \in C(d)$ do, so $Y(N) \cong U(d) \otimes N \cong M$. □

3.13. **Corollary.** *For even d, the functor Y gives a 1–1 correspondence between the irreducible (resp. absolutely irreducible) supermodules of $S(d)$ and $W(d)$.*

To understand the case d odd, we argue a little further. Certainly, we have the following:

3.14. **Lemma.** *Suppose that d is odd and let D be an irreducible $S(d)$-supermodule. Then,*

(i) *if D is absolutely irreducible, then $Y(D)$ is irreducible but not absolutely irreducible;*

(ii) *if D is not absolutely irreducible, then $Y(D)$ decomposes as a direct sum of two isomorphic, absolutely irreducible $W(d)$-supermodules.*

Proof. Let $A \subseteq \mathrm{End}_{\Bbbk}(U(d))$ denote the image of the Clifford algebra $C(d)$ in its representation on $U(d)$, and $B \subseteq \mathrm{End}_{\Bbbk}(D)$ denote the image of $S(d)$ in its representation on D. Evidently, B is spanned by invertible elements, namely the images of the group elements $t_w \in S(d)$. Since $\eta(w) \in C(d)$ is invertible, we can find an element $c_w \in C(d)$ such that, $(c_w \otimes 1)(\eta(w) \otimes t_w) = 1 \otimes t_w$ as elements of the superalgebra $A \otimes B$. It follows that the image of $W(d)$ in its representation on $Y(D)$ is equal to $A \otimes B \subseteq \mathrm{End}_{\Bbbk}(U(d)) \otimes \mathrm{End}_{\Bbbk}(D)$. Now the lemma follows from Lemma 2.8. □

In view of Lemma 3.10, we deduce from Corollary 3.6 and Lemma 3.14 that:

3.15. **Corollary.** *For odd d, the functor Y gives a 1–1 correspondence between the irreducible supermodules of $S(d)$ and $W(d)$, absolutely irreducibles corresponding to non-absolutely irreducibles and non-absolutely irreducibles corresponding to absolutely irreducibles.*

4 The strange Schur superalgebra

We introduce some further notation. Suppose that $0 \neq i, j \in \mathbb{Z}$. Define $\partial_i = \bar{0}$ if $i > 0$ or $\bar{1}$ if $i < 0$; define $\partial_{i,j} = \partial_i + \partial_j \in \mathbb{Z}_2$. More generally, given d-tuples $\underline{i} = (i_1, \dots, i_d)$ and $\underline{j} = (j_1, \dots, j_d)$ of non-zero integers, let

$$\partial_{\underline{i}} = \partial_{i_1} + \cdots + \partial_{i_d} \in \mathbb{Z}_2, \qquad \partial_{\underline{i},\underline{j}} = \partial_{\underline{i}} + \partial_{\underline{j}} \in \mathbb{Z}_2,$$
$$\varepsilon_{\underline{i}} = (\partial_{i_1}, \partial_{i_2}, \dots, \partial_{i_d}) \in \mathbb{Z}_2^d, \qquad \varepsilon_{\underline{i},\underline{j}} = \varepsilon_{\underline{i}} + \varepsilon_{\underline{j}} \in \mathbb{Z}_2^d.$$

Let $\mathbb{Z}_2^d$ act on the left on $\{\pm 1, \dots, \pm d\}$ so that for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{Z}_2^d$ and $s = 1, \dots, d$, $\varepsilon(\pm s) = (-1)^{\varepsilon_s}(\pm s)$. Extend the natural action of S_d on $\{1, \dots, d\}$ to an action on $\{\pm 1, \dots, \pm d\}$ so that $w(-s) = -(ws)$ for $s = 1, \dots, d$. These two actions combine to give a well-defined left action of the hyperoctahedral group W_d on the left on the set $\{\pm 1, \dots, \pm d\}$.

Now let $I(n,d)$ denote the set of all functions $\underline{i} : \{\pm 1, \dots, \pm d\} \to \{\pm 1, \dots, \pm n\}$ such that $\underline{i}(-s) = -\underline{i}(s)$ for $s = 1, \dots, d$. We often denote the value $\underline{i}(s)$ of the function $\underline{i} \in I(n,d)$ at $s \in \{\pm 1, \dots, \pm_d\}$ by i_s. Then, the element $\underline{i} \in I(n,d)$ can be thought of simply as the d-tuple $(i_1, \dots, i_d)$: the original function $\underline{i}$ can be recovered uniquely from knowledge of this d-tuple since $\underline{i}(-s) = -\underline{i}(s)$. The group W_d acts on the right on $I(n,d)$ by composition of functions, so $(\underline{i} \cdot g)(s) = \underline{i}(gs)$ for $\underline{i} \in I(n,d), g \in W_d$ and $s \in \{\pm 1, \dots, \pm d\}$. Write $\underline{i} \sim \underline{j}$ if $\underline{i}, \underline{j} \in I(n,d)$ lie in the same W_d-orbit. Also let W_d act diagonally on the right on the set $I(n,d) \times I(n,d)$ of *double indexes*, and write $(\underline{i}, \underline{j}) \sim (\underline{k}, \underline{l})$ if the double indexes $(\underline{i}, \underline{j})$ and $(\underline{k}, \underline{l})$ lie in the same orbit.

Let V denote the superspace with basis $v_{\pm 1}, \dots, v_{\pm n}$, where $\partial(v_i) = \partial_i$. Then, the tensor product $V^{\otimes d}$ is also a superspace with the induced grading. A basis is given by the monomials $v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_d}$ for all $\underline{i} \in I(n,d)$, and $\partial(v_{\underline{i}}) = \partial_{\underline{i}}$. We make $V^{\otimes d}$ into a *right* $W(d)$-supermodule by setting

$$v_{\underline{i}}(w \otimes c^{\delta}) = \alpha(\varepsilon_{\underline{i}}; w, \delta) v_{\underline{i} \cdot (w,\delta)}$$

for all $\underline{i} \in I(n,d), (w, \delta) \in W_d$. The fact that this is well-defined follows from the fact that α is a 2-cocycle. To be more explicit, the action of the generator s_i of $S_d \subset W(d)$ is as the linear map $\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes T_{V,V} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$ where the supertwist map $T_{V,V}$ is in the ith position, and the generator c_j of $C(d) \subset W(d)$ acts on the right as the linear map $\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes J_V \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$ where the map $J_V : v_i \mapsto v_{-i}$ is in the jth tensor.

Now define the (strange) *Schur superalgebra*

$$\dot{Q}(n,d) := \mathrm{End}_{W(d)}(V^{\otimes d}).$$

So, $\dot{Q}(n,d)$ acts on $V^{\otimes d}$ on the *left*. We observe right away by Lemma 3.3 that:

4.1. **Lemma.** *If $p = 0$ or $p > d$, then $\dot{Q}(n,d)$ is a semisimple (super)algebra.*

The initial goal is to describe an explicit basis for $\dot{Q}(n,d)$.

4.2. **Lemma.** *For $(\underline{i},\underline{j}) \in I(n,d) \times I(n,d)$, the following properties are equivalent:*
(i) $\partial_{i_s,j_s}\partial_{i_t,j_t} = \bar{0}$ *whenever $|i_s| = |i_t|$ and $|j_s| = |j_t|$ for some $1 \leq s < t \leq d$;*
(ii) $\alpha(\varepsilon_{\underline{i},\underline{j}}; w) = 1$ *for all $(w,\delta) \in \mathrm{Stab}_{W_d}(\underline{i},\underline{j})$.*

Proof. Using the fact that $\mathrm{Stab}_{S_d}(\underline{i},\underline{j})$ is generated by transpositions and that α is a 2-cocycle, property (ii) is easily seen to be equivalent to the weaker condition that $\alpha(\varepsilon_{\underline{i},\underline{j}}; w) = 1$ for all $(w,\delta) \in \mathrm{Stab}_{W_d}(\underline{i},\underline{j})$ with w a transposition. This weaker statement is precisely the condition (i), by the definition of α. $\square$

Call the double index $(\underline{i},\underline{j}) \in I(n,d) \times I(n,d)$ *strict* if it satisfies the properties in the lemma, and let $I^2(n,d)$ denote the set of all strict double indexes. Observe using Lemma 4.2(i) that $I^2(n,d)$ is W_d-stable. Given $(\underline{i},\underline{j}) \sim (\underline{k},\underline{l}) \in I^2(n,d)$, choose $(w,\delta) \in W_d$ such that $(\underline{i},\underline{j}) \cdot (w,\delta) = (\underline{k},\underline{l})$ and define the sign $\sigma(\underline{i},\underline{j};\underline{k},\underline{l})$ to be $\alpha(\varepsilon_{\underline{i},\underline{j}}; w)$. In view of Lemma 4.2(ii), this definition of $\sigma(\underline{i},\underline{j};\underline{k},\underline{l})$ is independent of the choice of (w,δ).

Given $i,j \in \{\pm 1,\dots,\pm d\}$, let $\dot{e}_{i,j} \in \mathrm{End}_{\Bbbk}(V)$ denote the linear map with $\dot{e}_{i,j} v_k = \delta_{j,k} v_i$ for all k. Given $\underline{i},\underline{j} \in I(n,d)$, let

$$\dot{e}_{\underline{i},\underline{j}} = \dot{e}_{i_1,j_1} \otimes \dot{e}_{i_2,j_2} \otimes \cdots \otimes \dot{e}_{i_d,j_d} \in \mathrm{End}_{\Bbbk}(V)^{\otimes d}.$$

Recall that the superalgebras $\mathrm{End}_{\Bbbk}(V)^{\otimes d}$ and $\mathrm{End}_{\Bbbk}(V^{\otimes d})$ are naturally isomorphic. Under the isomorphism, our element $\dot{e}_{\underline{i},\underline{j}}$ corresponds to the linear map with

$$\dot{e}_{\underline{i},\underline{j}} v_{\underline{k}} = \delta_{\underline{j},\underline{k}} \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{k}}) v_{\underline{i}}. \tag{4.3}$$

We will henceforth identify $\mathrm{End}_{\Bbbk}(V)^{\otimes d}$ and $\mathrm{End}_{\Bbbk}(V^{\otimes d})$ in this way. Given *strict* $(\underline{i},\underline{j}) \in I^2(n,d)$, define the linear map $\dot{\xi}_{\underline{i},\underline{j}} \in \mathrm{End}_{\Bbbk}(V^{\otimes d})$ by

$$\dot{\xi}_{\underline{i},\underline{j}} = \sum_{(\underline{k},\underline{l}) \sim (\underline{i},\underline{j})} \sigma(\underline{i},\underline{j};\underline{k},\underline{l}) \dot{e}_{\underline{k},\underline{l}}. \tag{4.4}$$

Obviously, if $(\underline{i},\underline{j}) \sim (\underline{k},\underline{l}) \in I^2(n,d)$, then $\dot{\xi}_{\underline{i},\underline{j}} = \sigma(\underline{i},\underline{j};\underline{k},\underline{l}) \dot{\xi}_{\underline{k},\underline{l}}$. Now choose some set $\Omega(n,d)$ of orbit representatives for the action of W_d on $I^2(n,d)$. Then:

4.5. **Theorem.** *The elements* $\{\dot{\xi}_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in \Omega(n,d)\}$ *give a basis for* $\dot{Q}(n,d)$. *Moreover, given* $(\underline{i},\underline{j}),(\underline{k},\underline{l}) \in I^2(n,d)$,

$$\dot{\xi}_{\underline{i},\underline{j}}\dot{\xi}_{\underline{k},\underline{l}} = \sum_{(\underline{s},\underline{t})\in\Omega(n,d)} a_{\underline{i},\underline{j},\underline{k},\underline{l},\underline{s},\underline{t}}\dot{\xi}_{\underline{s},\underline{t}}$$

where

$$a_{\underline{i},\underline{j},\underline{k},\underline{l},\underline{s},\underline{t}} = \sum_{\substack{\underline{h}\in I(n,d) \text{ with} \\ (\underline{s},\underline{h})\sim(\underline{i},\underline{j}),(\underline{h},\underline{t})\sim(\underline{k},\underline{l})}} \sigma(\underline{i},\underline{j};\underline{s},\underline{h})\sigma(\underline{k},\underline{l};\underline{h},\underline{t})\alpha(\varepsilon_{\underline{s},\underline{h}};\varepsilon_{\underline{h},\underline{t}}).$$

Proof. Obviously, the given elements are linearly independent. To show that they span $\mathrm{End}_{W(d)}(V^{\otimes d})$, let

$$\theta = \sum_{\underline{i},\underline{j}\in I(n,d)} a_{\underline{i},\underline{j}}\dot{e}_{\underline{i},\underline{j}}$$

be an arbitrary element of $\mathrm{End}_{\mathbb{k}}(V^{\otimes d})$. Take $w \in S_d, \delta \in \mathbb{Z}_2^d$ and set $g = (w,\delta) \in W_d$. For $\underline{j} \in I(n,d)$, we have that $(\theta v_{\underline{j}})(w \otimes c^\delta) = \theta(v_{\underline{j}}(w \otimes c^\delta))$ if and only if

$$\sum_{\underline{i}\in I(n,d)} a_{\underline{i},\underline{j}}\alpha(\varepsilon_{\underline{i},\underline{j}};\varepsilon_{\underline{j}})\alpha(\varepsilon_{\underline{i}};g)v_{\underline{i}\cdot g} = \sum_{\underline{i}\in I(n,d)} a_{\underline{i}\cdot g,\underline{j}\cdot g}\alpha(\varepsilon_{\underline{i}\cdot g,\underline{j}\cdot g};\varepsilon_{\underline{j}\cdot g})\alpha(\varepsilon_{\underline{j}};g)v_{\underline{i}\cdot g}$$

Simplifying using Lemma 3.4, we see that $\theta \in \mathrm{End}_{W(d)}(V^{\otimes d})$ if and only if

$$a_{\underline{i}\cdot g,\underline{j}\cdot g} = \alpha(\varepsilon_{\underline{i},\underline{j}};w)a_{\underline{i},\underline{j}}$$

for all $\underline{i},\underline{j} \in I(n,d)$ and $g = (w,\delta) \in W_d$. So by Lemma 4.2(ii), we must have that $a_{\underline{i},\underline{j}} = 0$ unless $(\underline{i},\underline{j})$ is strict, and for strict $(\underline{h},\underline{k}) \sim (\underline{i},\underline{j})$, we have that $a_{\underline{h},\underline{k}} = \sigma(\underline{i},\underline{j};\underline{h},\underline{k})a_{\underline{i},\underline{j}}$. This shows that $\theta \in \dot{Q}(n,d)$ if and only if $\theta = \sum_{(\underline{i},\underline{j})\in\Omega(n,d)} a_{\underline{i},\underline{j}}\dot{\xi}_{\underline{i},\underline{j}}$, completing the proof of the first part of the theorem.

Now we show how to deduce the product rule. To calculate $a_{\underline{i},\underline{j},\underline{k},\underline{l},\underline{s},\underline{t}}$ in the product expansion, we need by (4.4) to determine the coefficient of $\dot{e}_{\underline{s},\underline{t}}$ in

$$\dot{\xi}_{\underline{i},\underline{j}}\dot{\xi}_{\underline{k},\underline{l}} = \sum_{(\underline{i}',\underline{j}')\sim(\underline{i},\underline{j})} \sum_{(\underline{k}',\underline{l}')\sim(\underline{k},\underline{l})} \sigma(\underline{i},\underline{j};\underline{i}',\underline{j}')\sigma(\underline{k},\underline{l};\underline{k}',\underline{l}')\dot{e}_{\underline{i}',\underline{j}'}\dot{e}_{\underline{k}',\underline{l}'}.$$

We have that $\dot{e}_{\underline{i}',\underline{j}'}\dot{e}_{\underline{k}',\underline{l}'} = \delta_{\underline{j}',\underline{k}'}\alpha(\varepsilon_{\underline{i}',\underline{j}'};\varepsilon_{\underline{k}',\underline{l}'})\dot{e}_{\underline{i}',\underline{l}'}$. Using this the $\dot{e}_{\underline{s},\underline{t}}$-coefficient of $\dot{\xi}_{\underline{i},\underline{j}}\dot{\xi}_{\underline{k},\underline{l}}$ is therefore precisely as in the theorem (with $\underline{h} = \underline{j}' = \underline{k}'$). $\square$

5 The coordinate ring

Now we proceed to give an entirely different construction of the Schur superalgebra in the spirit of Green's monograph [7]. We begin by reviewing some basic facts about cosuperalgebras and bisuperalgebras.

A *cosuperalgebra* is a superspace A with the additional structure of a $\mathbb{k}$-coalgebra, such that the comultiplication $\Delta : A \to A \otimes A$ and the counit $\epsilon : A \to \mathbb{k}$ are superspace maps. Given two cosuperalgebras A and B, $A \otimes B$ is a cosuperalgebra with comultiplication $\mathrm{id}_A \otimes T_{A,B} \otimes \mathrm{id}_B \circ (\Delta_A \otimes \Delta_B)$. A cosuperalgebra homomorphism (resp. antihomomorphism) $\theta : A \to B$ means a superspace map that is a coalgebra homomorphism (resp. antihomomorphism) in the usual sense. Cosuperideals and subcosuperalgebras are also the obvious graded version of the usual notions.

Given a cosuperalgebra A, a right A-cosupermodule is a superspace M together with a superspace map $\eta : M \to M \otimes A$, called the *structure map* of M, which makes M into a right A-comodule in the usual sense. A homomorphism between two A-cosupermodules means an A-comodule homomorphism in the usual sense; note we write homomorphisms between right A-cosupermodules on the *left* (and vice versa). We let $\mathbf{comod}(A)$ denote the (superadditive) category of all right A-cosupermodules.

A *bisuperalgebra* is a superspace A that is both an associative superalgebra and a cosuperalgebra, such that the comultiplication $\Delta : A \to A \otimes A$ (recall how $A \otimes A$ is viewed as a superalgebra!) and counit $\epsilon : A \to \mathbb{k}$ are superalgebra homomorphisms. If A is a bisuperalgebra, we have a natural notion of (inner) tensor product of two left A-supermodules M, N, namely, the supermodule $M \otimes N$ with multiplication defined by $a(m \otimes n) = \Delta(a)(m \otimes n)$ (recall how we view $M \otimes N$ as an $A \otimes A$-supermodule!). The fact that the comultiplication is coassociative implies that, given A-supermodules M, N, P, the canonical isomorphism $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ is an isomorphism of supermodules. Similarly, we view the tensor product $M \otimes N$ of two right A-cosupermodules as a right A-cosupermodule, with structure map defined by the composition $M \otimes N \xrightarrow{\eta_M \otimes \eta_N} M \otimes A \otimes N \otimes A \xrightarrow{\mathrm{id} \otimes T_{A,N} \otimes \mathrm{id}} M \otimes N \otimes A \otimes A \xrightarrow{\mu} M \otimes N \otimes A$, where $\eta_M : M \to M \otimes A$ and $\eta_N : N \to N \otimes A$ are the structure maps of M, N respectively and $\mu : A \otimes A \to A$ denotes the multiplication in A.

Let A be a finite dimensional cosuperalgebra. Then, the dual A^* is naturally a superalgebra, with the product $f_1 f_2$ of $f_1, f_2 \in A^*$ being defined as the unique element f of A^* such that $f(a) = (f_1 \otimes f_2)(\Delta(a))$ for all $a \in A$ (take care interpreting the right hand side!). Given a finite dimensional right A-cosupermodule M with structure map $\eta : M \to M \otimes A$, we can view M as a left A^*-supermodule, with action $fm = (\mathrm{id}_{\mathrm{M}} \otimes f)(\eta(m))$ for $f \in A^*, m \in M$ (care!). Now suppose that $\theta : M \to N$ is a morphism of right A-cosupermodules. Let $\tilde{\theta} : M \to N$ be the map $m \mapsto (-1)^{\partial(\theta)\partial(m)}\theta(m)$. Then, viewing M and N as left A^*-supermodules as just explained, the map $\tilde{\theta}$ (now written on the right) is a morphism of left A^*-supermodules. We have now defined a functor which gives an isomorphism between $\mathbf{comod}(A)$ and $\mathbf{mod}(A^*)$.

Finally in this review of definitions, we mention a standard general result about direct sums of cosuperalgebras. Suppose A is a (possibly infinite dimensional) cosuperalgebra and that $A = \bigoplus_{i \in I} A_i$ as a direct sum of subcosuperalgebras. Then, as in [7, p.20] we have:

5.1. **Lemma.** *With the preceeding notation, let M be a right A-cosupermodule with structure map $\eta : M \to M \otimes A$. Then, $M = \bigoplus_{i \in I} M_i$ where M_i is the unique maximal subcosupermodule of V with $\eta(M_i) \subseteq M_i \otimes A_i$.*

As a corollary, one can show that the category of right A-cosupermodules is equivalent to the direct product of the categories of right A_i-cosupermodules for all $i \in I$.

Now we begin the alternative construction of the Schur superalgebra. Start with the free associative superalgebra $F(n)$ on non-commuting generators $\{f_{i,j} \mid i, j = \pm 1, \dots, \pm n\}$, where $\partial(f_{i,j}) = \partial_{i,j}$. Then, $F(n)$ is naturally $\mathbb{Z}$-graded by degree as

$$F(n) = \bigoplus_{d \geq 0} F(n,d).$$

Given a double index $(\underline{i},\underline{j}) \in I(n,d) \times I(n,d)$, define $f_{\underline{i},\underline{j}} = f_{i_1,j_1} f_{i_2,j_2} \dots f_{i_d,j_d}$. The elements $\{f_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in I(n,d) \times I(n,d)\}$ form a basis for $F(n,d)$. One checks that the unique superalgebra maps $\epsilon : F(n) \to \mathbb{k}$ and $\Delta : F(n) \to F(n) \otimes F(n)$ defined on generators by

$$\begin{aligned}
\epsilon(f_{i,j}) &= \delta_{i,j}, \\
\Delta(f_{i,j}) &= \sum_{k \in \{\pm 1, \dots, \pm n\}} (-1)^{\partial_{i,k}\partial_{k,j}} f_{i,k} \otimes f_{k,j}
\end{aligned}$$

make $F(n)$ into a bisuperalgebra. We point out that for $(\underline{i},\underline{j}) \in I(n,d) \times I(n,d)$,

$$\Delta(f_{\underline{i},\underline{j}}) = \sum_{\underline{k} \in I(n,d)} (-1)^{\partial_{\underline{i},\underline{k}}\partial_{\underline{k},\underline{j}}} \alpha(\varepsilon_{\underline{k},\underline{j}}; \varepsilon_{\underline{i},\underline{k}}) f_{\underline{i},\underline{k}} \otimes f_{\underline{k},\underline{j}}.$$

Hence, each $F(n,d)$ is a finite dimensional subcosuperalgebra of $F(n)$. Make V into a right $F(n)$-cosupermodule with structure map $V \to V \otimes F(n)$ defined by

$$v_j \mapsto \sum_{i \in \{\pm 1, \dots, \pm n\}} (-1)^{\partial_i \partial_{i,j}} v_i \otimes f_{i,j}.$$

Then, for each $d \geq 1$, $V^{\otimes d}$ is also automatically a right $F(n)$-cosupermodule with structure map $V^{\otimes d} \to V^{\otimes d} \otimes F(n)$ given explicitly by the formula

$$v_{\underline{j}} \mapsto \sum_{\underline{i} \in I(n,d)} (-1)^{\partial_{\underline{i}}\partial_{\underline{i},\underline{j}}} \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{i}}) v_{\underline{i}} \otimes f_{\underline{i},\underline{j}}.$$

In particular, $V^{\otimes d}$ can be viewed as a right $F(n,d)$-cosupermodule.

Let $E(n,d) = F(n,d)^*$ be the dual superalgebra. Let $e_{\underline{i},\underline{j}}$ denote the element of $E(n,d)$ with

$$e_{\underline{i},\underline{j}}(f_{\underline{i},\underline{j}}) = \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{i},\underline{j}}), \qquad e_{\underline{i},\underline{j}}(f_{\underline{k},\underline{l}}) = 0 \text{ for } (\underline{k},\underline{l}) \neq (\underline{i},\underline{j}).$$

Then, the $\{e_{\underline{i},\underline{j}} \mid \underline{i},\underline{j} \in I(n,d)\}$ give a basis for $E(n,d)$. The right $F(n,d)$-cosupermodule $V^{\otimes d}$ is a left $E(n,d)$-supermodule in a natural way. Let $\rho_d : E(n,d) \to \mathrm{End}_{\mathbb{k}}(V^{\otimes d})$ be the resulting representation.

5.2. **Lemma.** *The representation ρ_d is an isomorphism between $E(n,d)$ and $\mathrm{End}_{\mathbb{k}}(V^{\otimes d})$. Moreover, $\rho_d(e_{\underline{i},\underline{j}}) = \dot{e}_{\underline{i},\underline{j}}$ for all $\underline{i},\underline{j} \in I(n,d)$.*

Proof. It suffices to check that $e_{\underline{i},\underline{j}} v_{\underline{k}} = \dot{e}_{\underline{i},\underline{j}} v_{\underline{k}}$ for all $\underline{i},\underline{j},\underline{k} \in I(n,d)$. By the definition of the action of $E(n,d)$, we have that

$$\begin{aligned}
e_{\underline{i},\underline{j}} v_{\underline{k}} &= (\mathrm{id} \otimes e_{\underline{i},\underline{j}}) \left(\sum_{\underline{h} \in I(n,d)} (-1)^{\partial_{\underline{h}}\partial_{\underline{h},\underline{k}}} \alpha(\varepsilon_{\underline{h},\underline{k}}; \varepsilon_{\underline{h}}) v_{\underline{h}} \otimes f_{\underline{h},\underline{k}} \right) \\
&= \delta_{\underline{j},\underline{k}} \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{i}}) \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{i},\underline{j}}) v_{\underline{i}} = \delta_{\underline{j},\underline{k}} \alpha(\varepsilon_{\underline{i},\underline{j}}; \varepsilon_{\underline{j}}) v_{\underline{i}} = \dot{e}_{\underline{i},\underline{j}} v_{\underline{k}}.
\end{aligned}$$

This completes the proof. $\square$

Now consider the superideal $\mathfrak{I}(n)$ of $F(n)$ generated by the elements

$$\{f_{i,j} - f_{-i,-j}, f_{i,j}f_{k,l} - (-1)^{\partial_{i,j}\partial_{k,l}} f_{k,l}f_{i,j} \mid i,j,k,l = \pm 1, \dots, \pm n\}.$$

A short calculation reveals that this is actually a bisuperideal, so the quotient

$$B(n) := F(n)/\mathfrak{I}(n)$$

is a bisuperalgebra quotient of $F(n)$. Let $b_{i,j} = f_{i,j} + \mathfrak{I}(n)$. Then, $B(n)$ is just the free commutative superalgebra on the degree $\bar{0}$ generators $b_{i,j} = b_{-i,-j}$ and degree $\bar{1}$ generators $b_{i,-j} = b_{-i,j}$, for all $1 \le i,j \le n$. The superideal $\mathfrak{I}(n)$ is homogeneous, so graded as $\mathfrak{I}(n) = \bigoplus_{d \ge 0} \mathfrak{I}(n,d)$. So $B(n)$ is also $\mathbb{Z}$-graded by degree as $B(n) = \bigoplus_{d\ge 0} B(n,d)$, with $B(n,d) \cong F(n,d)/\mathfrak{I}(n,d)$. Moreover, $B(n,d)$ is spanned by all monomials $b_{\underline{i},\underline{j}} = b_{i_1,j_1} \dots b_{i_d,j_d}$ for $\underline{i},\underline{j} \in I(n,d)$. The monomial $b_{\underline{i},\underline{j}}$ is non-zero if and only if $(\underline{i},\underline{j})$ is strict, and for strict $(\underline{i},\underline{j}) \sim (\underline{k},\underline{l})$, we have that

$$b_{\underline{i},\underline{j}} = \sigma(\underline{i},\underline{j};\underline{k},\underline{l}) b_{\underline{k},\underline{l}}.$$

It follows that $B(n,d)$ has basis $\{b_{\underline{i},\underline{j}} \,|\, (\underline{i},\underline{j}) \in \Omega(n,d)\}$, where $\Omega(n,d)$ is the choice of W_d-orbit representatives in $I^2(n,d)$ made earlier.

Now, let $Q(n,d)$ denote the dual superalgebra $B(n,d)^*$. Since $B(n,d) = F(n,d)/\mathfrak{I}(n,d)$, $Q(n,d)$ is naturally identified with the annihilator $\mathfrak{I}(n,d)^\circ \subseteq E(n,d)$. For $(\underline{i},\underline{j}) \in I^2(n,d)$, let $\xi_{\underline{i},\underline{j}} \in Q(n,d) \subseteq E(n,d)$ denote the unique function with

$$\xi_{\underline{i},\underline{j}}(b_{\underline{i},\underline{j}}) = \alpha(\varepsilon_{\underline{i},\underline{j}};\varepsilon_{\underline{i},\underline{j}}), \qquad \text{and} \qquad \xi_{\underline{i},\underline{j}}(b_{\underline{k},\underline{l}}) = 0 \text{ for } (\underline{k},\underline{l}) \not\sim (\underline{i},\underline{j}).$$

So, the $\{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in \Omega(n,d)\}$ give a basis for $Q(n,d)$.

We can regard the $F(n,d)$-cosupermodule $V^{\otimes d}$ instead as a $B(n,d)$-cosupermodule by restriction. Dualizing, we obtain a natural representation $Q(n,d) \to \mathrm{End}_{\Bbbk}(V^{\otimes d})$, which is nothing more than the restriction of the representation $\rho_d : E(n,d) \xrightarrow{\sim} \mathrm{End}_{\Bbbk}(V^{\otimes d})$ defined earlier to the subsuperalgebra $Q(n,d) \subseteq E(n,d)$. Then:

5.3. **Theorem.** *The representation ρ_d gives an isomorphism between $Q(n,d)$ and the Schur superalgebra $\dot{Q}(n,d)$. Moreover, $\rho_d(\xi_{\underline{i},\underline{j}}) = \dot{\xi}_{\underline{i},\underline{j}}$ for all $(\underline{i},\underline{j}) \in I^2(n,d)$.*

Proof. Pick $(\underline{i},\underline{j}) \in I^2(n,d)$. Since $Q(n,d) \subseteq E(n,d)$, we can write

$$\xi_{\underline{i},\underline{j}} = \sum_{\underline{k},\underline{l} \in I(n,d)} a_{\underline{k},\underline{l}} e_{\underline{k},\underline{l}}$$

for coefficients $a_{\underline{k},\underline{l}} \in \Bbbk$. To calculate the coefficient $a_{\underline{k},\underline{l}}$, evaluate both sides at the element $f_{\underline{k},\underline{l}} \in F(n,d)$ to see that $a_{\underline{k},\underline{l}} \alpha(\varepsilon_{\underline{k},\underline{l}};\varepsilon_{\underline{k},\underline{l}}) = \xi_{\underline{i},\underline{j}}(f_{\underline{k},\underline{l}}) = \xi_{\underline{i},\underline{j}}(b_{\underline{k},\underline{l}})$. So by the definition of $\xi_{\underline{i},\underline{j}}$, $a_{\underline{k},\underline{l}}$ is zero unless $(\underline{k},\underline{l}) \sim (\underline{i},\underline{j})$, in which case, $a_{\underline{k},\underline{l}} = \alpha(\varepsilon_{\underline{k},\underline{l}};\varepsilon_{\underline{k},\underline{l}}) \sigma(\underline{i},\underline{j};\underline{k},\underline{l}) \xi_{\underline{i},\underline{j}}(b_{\underline{i},\underline{j}}) = \sigma(\underline{i},\underline{j};\underline{k},\underline{l})$. This shows that

$$\xi_{\underline{i},\underline{j}} = \sum_{(\underline{k},\underline{l}) \sim (\underline{i},\underline{j})} \sigma(\underline{i},\underline{j};\underline{k},\underline{l}) e_{\underline{k},\underline{l}}.$$

Now the theorem follows at once from Lemma 5.2, Theorem 4.5 and (4.4). $\square$

We will henceforth *identify* $Q(n,d)$, which we defined as the dual of the cosuperalgebra $B(n,d)$, with $\dot{Q}(n,d)$, which we defined as the commutant of $W(d)$ on tensor space $V^{\otimes d}$. So the dual basis element $\xi_{\underline{i},\underline{j}} \in Q(n,d)$ is identified with the linear transformation $\dot{\xi}_{\underline{i},\underline{j}} \in \dot{Q}(n,d)$.

6 Weights and idempotents

Let $\Lambda(n,d)$ denote the set of all tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative integers with $\lambda_1 + \dots + \lambda_n = d$. We partially order $\Lambda(n,d)$ by the usual *dominance order*, so $\lambda \geq \mu$ if and only if $\sum_{s=1}^t \lambda_s \geq \sum_{s=1}^t \mu_s$ for each $t = 1, \dots, n$. For $\underline{i} \in I(n,d)$, define its *weight* $\mathrm{wt}(\underline{i})$ to be the composition $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n,d)$ where $\lambda_s = |\{t \mid 1 \leq t \leq d, |i_t| = s\}|$. Conversely, given $\lambda \in \Lambda(n,d)$, let $\underline{i}_\lambda$ denote the element $(1, \dots, 1, 2, \dots, 2, 3, \dots) \in I(n,d)$ where there are λ_1 ones, λ_2 twos, etc..., so that $\mathrm{wt}(\underline{i}_\lambda) = \lambda$. Define

$$\xi_\lambda := \xi_{\underline{i}_\lambda, \underline{i}_\lambda} \in Q(n,d).$$

We call the elements $\{\xi_\lambda \mid \lambda \in \Lambda(n,d)\}$ *weight idempotents*, motivated by the following lemma:

6.1. **Lemma.** *For* $(\underline{i}, \underline{j}) \in I^2(n,d)$,

$$\xi_\lambda \xi_{\underline{i},\underline{j}} = \begin{cases} \xi_{\underline{i},\underline{j}} & \text{if } \mathrm{wt}(\underline{i}) = \lambda, \\ 0 & \text{otherwise.} \end{cases} \qquad \xi_{\underline{i},\underline{j}} \xi_\lambda = \begin{cases} \xi_{\underline{i},\underline{j}} & \text{if } \mathrm{wt}(\underline{j}) = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\{\xi_\lambda \mid \lambda \in \Lambda(n,d)\}$ *is a set of mutually orthogonal idempotents whose sum is the identity element of* $Q(n,d)$.

Proof. It is elementary to check that the matrix units $\{e_{\underline{h},\underline{h}} \mid \underline{h} \in I(n,d)\}$ in $E(n,d)$ are a set of mutually orthogonal idempotents whose sum is the identity, with $e_{\underline{h},\underline{h}} e_{\underline{i},\underline{j}} = \delta_{\underline{h},\underline{i}} e_{\underline{i},\underline{j}}$ and $e_{\underline{i},\underline{j}} e_{\underline{h},\underline{h}} = \delta_{\underline{h},\underline{j}} e_{\underline{i},\underline{j}}$ for all $\underline{h}, \underline{i}, \underline{j} \in I(n,d)$. Now according to (4.4), $\xi_\lambda = \sum_{\underline{h}} e_{\underline{h},\underline{h}}$ summing over all $\underline{h} \in I(n,d)$ with $\mathrm{wt}(\underline{h}) = \lambda$, as an element of $E(n,d)$. The lemma follows easily from these remarks. $\square$

Let ω denote the weight (1^d), which is an element of $\Lambda(n,d)$ providing $n \geq d$. Assuming this, the weight idempotent ξ_ω is a well-defined element of $Q(n,d)$, and $\xi_\omega Q(n,d) \xi_\omega$ is naturally a superalgebra in its own right, its identity element being the idempotent ξ_ω. We have the following double centralizer property:

6.2. **Theorem.** *Assume that* $n \geq d$.

(i) *The map* $\phi : Q(n,d)\xi_\omega \to V^{\otimes d}$, $\xi_{\underline{i},\underline{i}_\omega} \mapsto v_{\underline{i}}$ *for* $\underline{i} \in I(n,d)$ *is an isomorphism of* $Q(n,d)$*-supermodules. In particular,* $V^{\otimes d}$ *is a projective* $Q(n,d)$*-supermodule.*

(ii) *The map* $\psi : W(d) \to \xi_\omega Q(n,d) \xi_\omega$, $x \otimes c^\delta \mapsto \xi_{\underline{i}_\omega \cdot (x,\delta), \underline{i}_\omega}$ *for all* $(x, \delta) \in W_d$, *is a superalgebra isomorphism.*

(iii) $\mathrm{End}_{Q(n,d)}(V^{\otimes d}) \cong W(d)$.

Proof. For (i), we first claim that $\xi_{\underline{i},\underline{i}_\omega} v_{\underline{i}_\omega} = v_{\underline{i}}$. Well, $\xi_{\underline{i},\underline{i}_\omega} = \sum_{(\underline{k},\underline{l})\sim(\underline{i},\underline{i}_\omega)} e_{\underline{k},\underline{l}}$, and $e_{\underline{k},\underline{l}} v_{\underline{i}_\omega} = \delta_{\underline{l},\underline{i}_\omega} v_{\underline{k}}$. Now observe that $(\underline{k},\underline{i}_\omega) \sim (\underline{i},\underline{i}_\omega)$ if and only if $\underline{k} = \underline{i}$, since $\mathrm{Stab}_{W_d}(\underline{i}_\omega) = 1$. It now follows easily that $\xi_{\underline{i},\underline{i}_\omega} v_{\underline{i}_\omega} = v_{\underline{i}}$ as claimed. So in particular, $\xi_\omega v_{\underline{i}_\omega} = v_{\underline{i}_\omega}$, so there is a well-defined $Q(n,d)$-module homomorphism $Q(n,d)\xi_\omega \to V^{\otimes d}$ such that $\xi_\omega \mapsto v_{\underline{i}_\omega}$. By the claim, this is precisely the map ϕ. Finally, observe that $Q(n,d)\xi_\omega$ has as basis the elements $\{\xi_{\underline{i},\underline{i}_\omega} \mid \underline{i} \in I(n,d)\}$, so that ϕ is a superspace isomorphism.

For (ii) and (iii), ξ_ω is an idempotent, so the superalgebras $\mathrm{End}_{Q(n,d)}(Q(n,d)\xi_\omega)$ and $\xi_\omega Q(n,d)\xi_\omega$ are naturally isomorphic. Also, there is a natural map $W(d) \to \mathrm{End}_{Q(n,d)}(V^{\otimes d})$ given by the representation of $W(d)$ on $V^{\otimes d}$. Combining this with (i), we obtain a natural superalgebra homomorphism $\psi : W(d) \to \xi_\omega Q(n,d)\xi_\omega$. By definition, it maps the element $x \otimes c^\delta \in W(d)$ to the unique element ξ of $\xi_\omega Q(n,d)\xi_\omega$ with $\xi\phi = v_{\underline{i}_\omega}(x \otimes c^\delta)$. But $v_{\underline{i}_\omega}(x\otimes c^\delta) = v_{\underline{i}_\omega\cdot(x,\delta)}$, so $\psi(x \otimes c^\delta) = \xi_{\underline{i}_\omega\cdot(x,\delta),\underline{i}_\omega}$ as in the lemma. It remains to observe that the elements $\{\xi_{\underline{i}_\omega\cdot(x,\delta),\underline{i}_\omega} \mid (x,\delta) \in W_d\}$ give a basis for $\xi_\omega Q(n,d)\xi_\omega$, so that ψ is an isomorphism. $\square$

Using Theorem 6.2(ii), Corollary 2.15 and Lemma 3.5, we deduce:

6.3. **Lemma.** *For $n \geq d$, the number of irreducible $Q(n,d)$-supermodules not annihilated by ξ_ω is equal to $|\mathfrak{RP}_p(d)|$.*

There is one other situation where Schur functors arising from weight idempotents will be useful. Suppose now that $m \geq n$. We embed $\Lambda(n,d)$ into $\Lambda(m,d)$ as the set of all weights of the form $(\lambda_1,\dots,\lambda_n,0,\dots,0)$, and $I(n,d)$ into $I(m,d)$ as the set of all $\underline{i} \in I(m,d)$ with $i_s \in \{\pm1,\dots,\pm n\}$ for each $s = 1,\dots,d$. To avoid confusion with the corresponding elements of $Q(n,d)$, we denote the elements $\xi_\lambda, \xi_{\underline{i},\underline{j}} \in Q(m,d)$ for $\lambda \in \Lambda(m,d), (\underline{i},\underline{j}) \in I^2(m,d)$ instead by $\widehat{\xi}_\lambda, \widehat{\xi}_{\underline{i},\underline{j}}$ respectively. Let $e \in Q(m,d)$ denote the idempotent

$$e = \sum_{\lambda\in\Lambda(n,d)\subseteq\Lambda(m,d)} \widehat{\xi}_\lambda. \tag{6.4}$$

If $\underline{i},\underline{j} \in I(n,d) \subseteq I(m,d)$, the element $\widehat{\xi}_{\underline{i},\underline{j}} \in Q(m,d)$ lies in $eQ(m,d)e$.

6.5. **Lemma.** *The map $\iota : Q(n,d) \to eQ(m,d)e$, $\xi_{\underline{i},\underline{j}} \mapsto \widehat{\xi}_{\underline{i},\underline{j}}$ for all $(\underline{i},\underline{j}) \in I^2(n,d)$, is a superalgebra isomorphism.*

Proof. Consider the $\mathbb{Z}$-graded superideal $\mathfrak{J}(m) = \bigoplus_{d\geq0} \mathfrak{J}(m,d)$ of $B(m)$ generated by the elements

$$\{b_{i,j} \mid i \text{ or } j \text{ equals } \pm(n+1), \pm(n+2), \dots, \pm m\}.$$

One checks easily that $\Delta(\mathfrak{J}(m)) \subseteq \mathfrak{J}(m) \otimes B(m) + B(m) \otimes \mathfrak{J}(m)$, so that the comultiplication Δ on $B(m)$ induces a well-defined comultiplication on $B(m)/\mathfrak{J}(m)$ (though $\mathfrak{J}(m)$ is *not* a cosuperideal). Evidently, $B(m)/\mathfrak{J}(m) \cong B(n)$ as superalgebras, the induced comultiplication on $B(m)/\mathfrak{J}(m)$ corresponding to the usual comultiplication on $B(n)$ under the isomorphism. Dualizing, we obtain a *multiplicative* superspace isomorphism between $Q(n,d)$ and $\mathfrak{J}(m)^\circ \subseteq eQ(m,d)e$, being precisely the map ι. Finally, observe that $eQ(m,d)e = \mathfrak{J}(m)^\circ$ to complete the proof. $\square$

Next, we introduce a natural subalgebra of $Q(n,d)$ which plays the role of Cartan subalgebra. Let $\mathfrak{J}_0(n) = \bigoplus_{d\geq 0}\mathfrak{J}_0(n,d)$ denote the $\mathbb{Z}$-graded superideal of $B(n)$ generated by the elements

$$\{b_{i,i}\mid i,j=\pm 1,\dots,\pm n, |i|\neq |j|\}.$$

It is elementary to check that $\mathfrak{J}_0(n)$ is a bisuperideal of $B(n)$, so we can form the bisuperalgebra quotient $B_0(n) := B(n)/\mathfrak{J}_0(n)$. For $i=1,\dots,n$, let x_i denote the image of $b_{i,i}=b_{-i,-i}$ in $B_0(n)$, and x_i' denote the image of $b_{i,-i}=b_{-i,i}$. Then $B_0(n)$ is precisely the free commutative superalgebra on the generators $x_1,\dots,x_n,x_1',\dots,x_n'$. Comultiplication $\Delta: B_0(n)\to B_0(n)\otimes B_0(n)$ is given explicitly on these generators by

$$\Delta(x_i)=x_i\otimes x_i - x_i'\otimes x_i',\qquad \Delta(x_i')=x_i\otimes x_i'+x_i'\otimes x_i.$$

As usual, $B_0(n)$ is $\mathbb{Z}$-graded by degree as $\bigoplus_{d\geq 0}B_0(n,d)$, with $B_0(n,d)\cong B(n,d)/\mathfrak{J}_0(n,d)$ being a subsupercoalgebra of $B_0(n)$ for each $d\geq 0$. The dual superalgebra $Q_0(n,d)=B_0(n,d)^*$ can be identified with the annihilator $\mathfrak{J}_0(n,d)^\circ\subseteq Q(n,d)$, giving us a natural subsuperalgebra of $Q(n,d)$.

Consider the special case $Q_0(1,d)$ for $d\geq 1$ in more detail (obviously, $Q_0(1,0)=\mathbb{k}$). Writing $x=x_1, x'=x_1'$, the elements $\{x^d, x^{d-1}x'\}$ give a basis for $B_0(1,d)$, with comultiplication $\Delta: B_0(n,d)\to B_0(n,d)\otimes B_0(n,d)$ is given explicitly by

$$\Delta(x^d)=x^d\otimes x^d - dx^{d-1}x'\otimes x^{d-1}x',\qquad \Delta(x^{d-1}x')=x^{d-1}x'\otimes x^d + x^d\otimes x^{d-1}x'.$$

As a basis for $Q_0(1,d)$, take the dual basis $\{y_d, y_d'\}$ to the basis $\{x^d, x^{d-1}x'\}$ of $B_0(1,d)$. The algebra multiplication, dual to the comultiplication in $B_0(1,d)$, is then given by $y_dy_d = y_d, y_dy_d' = y_d' = y_d'y_d, y_d'y_d' = dy_d$. Hence, for $d\geq 1$,

$$Q_0(1,d)\cong\begin{cases} C(1) & \text{if } p\nmid d,\\ \bigwedge(1) & \text{if } p|d,\end{cases}$$

recalling Example 2.2.

Now in general, the subsuperalgebra $Q_0(n,d)\subseteq Q(n,d)$ contains each weight idempotent ξ_λ for $\lambda\in\Lambda(n,d)$ in its center. So,

$$Q_0(n,d)\cong\bigoplus_{\lambda\in\Lambda(n,d)}\xi_\lambda Q_0(n,d). \tag{6.6}$$

Moreover, one can see that

$$\xi_\lambda Q_0(n,d)\cong Q_0(1,\lambda_1)\otimes\cdots\otimes Q_0(1,\lambda_n)\cong C(h_{p'}(\lambda))\otimes\bigwedge(h_p(\lambda)) \tag{6.7}$$

where $h_p(\lambda)$ denotes the number of non-zero parts of λ that are divisible by p, and $h_{p'}(\lambda)$ denotes the number of parts of λ that are coprime to p. We deduce immediately using Lemma 2.8, Example 2.9 and Example 2.10 that $\xi_\lambda Q_0(n,d)$ has a unique irreducible supermodule which we denote by $U(\lambda)$, of dimension $2^{\lfloor (h_{p'}(\lambda)+1)/2\rfloor}$. Moreover, the supermodule $U(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even. Finally, regarding $U(\lambda)$ as an $Q_0(n,d)$-supermodule by inflation, we have shown:

6.8. **Lemma.** *The modules $\{U(\lambda) \mid \lambda \in \Lambda(n,d)\}$ give a complete set of pairwise non-isomorphic irreducible $Q_0(n,d)$-supermodules. The dimension of $U(\lambda)$ is $2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor}$, and $U(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even.*

Recalling Lemma 5.1, we have thus determined the irreducible $B_0(n)$-cosupermodules, namely, the $B_0(n)$-cosupermodules $\{U(\lambda) \mid \lambda \in \Lambda(n)\}$, where $\Lambda(n) := \bigcup_{d \geq 0} \Lambda(n,d)$. Now let M be an arbitrary $B(n)$-cosupermodule with structure map $\eta : M \to M \otimes B(n)$. By Lemma 5.1, M decomposes as $M = \bigoplus_{d \geq 0} M_d$ where M_d is the largest subcosupermodule with $\eta(M_d) \subseteq M_d \otimes B(n,d)$. Each M_d is naturally a $B(n,d)$-cosupermodule, hence a $Q(n,d)$-supermodule. Then, for $\lambda \in \Lambda(n,d)$, we define the λ-*weight space* of M to be the space $M_\lambda := \xi_\lambda M_d$. Recalling (6.6), M_λ is a $Q_0(n,d)$-subsupermodule of M_d. Equivalently, M_λ is a $B_0(n)$-subcosupermodule of M, viewing M as a $B_0(n)$-cosupermodule by restriction, and

$$M = \bigoplus_{\lambda \in \Lambda(n)} M_\lambda.$$

Let $X(n)$ denote the free polynomial algebra $\mathbb{Z}[x_1, \dots, x_n]$ and for $\lambda \in \Lambda(n)$, set $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$. Define the *formal character*

$$\operatorname{ch} M = \sum_{\lambda \in \Lambda(n)} \dim M_\lambda x^\lambda \in X(n).$$

Note that for $B(n)$-cosupermodules M, N, we have that $\operatorname{ch}(M \oplus N) = \operatorname{ch} M + \operatorname{ch} N$ and $\operatorname{ch}(M \otimes N) = \operatorname{ch} M . \operatorname{ch} N$. In other words, the map $\operatorname{ch} : \operatorname{Grot}(B(n)) \to X(n)$ is a ring homomorphism from the Grothendieck ring of the category of finite dimensional right $B(n)$-cosupermodules to $X(n)$.

7 The "big cell"

Let $\mathfrak{J}_\flat(n) = \bigoplus_{d \geq 0} \mathfrak{J}_\flat(n,d)$ and $\mathfrak{J}_\sharp(n) = \bigoplus_{d \geq 0} \mathfrak{J}_\sharp(n,d)$ denote the $\mathbb{Z}$-graded superideals of $B(n)$ generated by the elements

$$\{b_{i,j} \mid i,j = \pm 1, \dots, \pm n, |i| < |j|\}, \quad \{b_{i,j} \mid i,j = \pm 1, \dots, \pm n, |i| > |j|\}$$

respectively. One easily checks that these are cosuperideals. Hence, we can form the bisuperalgebras quotients

$$B_\flat(n) := B(n)/\mathfrak{J}_\flat(n), \qquad B_\sharp(n) := B(n)/\mathfrak{J}_\sharp(n).$$

Both $B_\flat(n)$ and $B_\sharp(n)$ are $\mathbb{Z}$-graded with degree d component, denoted $B_\flat(n,d)$ and $B_\sharp(n,d)$ respectively, being cosuperalgebra quotients of $B(n,d)$. The corresponding dual superalgebras to these, namely $Q_\flat(n,d) = \mathfrak{J}_\flat(n,d)^\circ$ and $Q_\sharp(n,d) = \mathfrak{J}_\sharp(n,d)^\circ$, are therefore subsuperalgebras of $Q(n,d)$, called the negative Borel and positive Borel subsuperalgebras respectively. They are spanned by the elements

$$\{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in I^2(n,d), |\underline{i}| \geq |\underline{j}|\} \quad \text{and} \quad \{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in I^2(n,d), |\underline{i}| \leq |\underline{j}|\}$$

respectively, where $|\underline{i}| \geq |\underline{j}|$ means that $|i_k| \geq |j_k|$ for each $k = 1, \dots, d$. Let $\pi_\flat : B(n) \to B_\flat(n)$ and $\pi_\sharp : B(n) \to B_\sharp(n)$ denote the natural quotient maps and set $b^\flat_{\underline{i},\underline{j}} = \pi_\flat(b_{\underline{i},\underline{j}})$, $b^\sharp_{\underline{i},\underline{j}} = \pi_\sharp(b_{\underline{i},\underline{j}})$ for $\underline{i},\underline{j} \in I(n,d)$. In particular, $b^\flat_{\underline{i},\underline{j}} = 0$ unless $|\underline{i}| \geq |\underline{j}|$. Similarly, $b^\sharp_{\underline{i},\underline{j}} = 0$ unless $|\underline{i}| \leq |\underline{j}|$. Let

$$\pi : B(n) \to B_\flat(n) \otimes B_\sharp(n)$$

be the map $(\pi_\flat \otimes \pi_\sharp) \circ \Delta$. The next goal is to prove an analogue of the existence of the big cell crucial for high-weight theory:

7.1. **Theorem.** *π is injective.*

Proof. We proceed in a number of steps. Observe right away that it is enough to prove that π is injective on each $B(n,d)$ separately. So, fix $d \geq 1$ and consider the restriction $\pi : B(n,d) \to B_\flat(n,d) \otimes B_\sharp(n,d)$. Let

$$Y = \{(\underline{i},\underline{k},\underline{l},\underline{j}) \in I(n,d) \times I(n,d) \times I(n,d) \times I(n,d) \mid |\underline{i}| \geq |\underline{k}|, |\underline{l}| \leq |\underline{j}|\}.$$

Write $(\underline{i},\underline{k},\underline{l},\underline{j}) \approx (\underline{i}',\underline{k}',\underline{l}',\underline{j}')$ if both $(\underline{i},\underline{k}) \sim (\underline{i}',\underline{k}')$ and $(\underline{l},\underline{j}) \sim (\underline{l}',\underline{j}')$. Also call $(\underline{i},\underline{k},\underline{l},\underline{j})$ *strict* if both $(\underline{i},\underline{k})$ and $(\underline{l},\underline{j})$ are strict in the sense of Lemma 4.2. Then:

7.2. *If Z is a choice of representatives for the $\approx$-equivalence classes of strict $(\underline{i},\underline{k},\underline{l},\underline{j}) \in Y$, then $\{b^\flat_{\underline{i},\underline{k}} \otimes b^\sharp_{\underline{l},\underline{j}} \mid (\underline{i},\underline{k},\underline{l},\underline{j}) \in Z\}$ is a basis for $B_\flat(n,d) \otimes B_\sharp(n,d)$.*

Now define $\underline{m}(\underline{i},\underline{j})$, for any $\underline{i},\underline{j} \in I(n,d)$, to be the unique element $\underline{m} \in I(n,d)$ with

$$m_s = \begin{cases} i_s & \text{if } |i_s| < |j_s| \\ j_s & \text{if } |i_s| \geq |j_s| \end{cases}$$

for all $s = 1, \dots, d$. Observe that $\underline{m}(\underline{i} \cdot g, \underline{j} \cdot g) = \underline{m}(\underline{i},\underline{j}) \cdot g$ for all $g \in W_d$. We claim:

7.3. *Suppose $\underline{i},\underline{j} \in I(n,d)$ and $g \in W_d$ are such that $\underline{m}(\underline{i},\underline{j}) = \underline{m}(\underline{i},\underline{j} \cdot g) = \underline{m}(\underline{i} \cdot g, \underline{j} \cdot g)$. Then, $(\underline{i},\underline{j}) \sim (\underline{i},\underline{j} \cdot g)$.*

We prove (7.3) by induction on d. Let $\underline{m} = \underline{m}(\underline{i},\underline{j})$. If $d = 1$, then the assumption that $\underline{m} \cdot g = \underline{m}$ forces $g = 1$, and the lemma follows trivially. Now suppose that $d > 1$ and that we have proved (7.3) for all smaller d. Write $\{\pm 1, \dots, \pm d\} = I \sqcup J$ where

$$I = \{\pm s \mid 1 \leq s \leq d, |i_s| \geq |j_s|\},$$
$$J = \{\pm s \mid 1 \leq s \leq d, |i_s| < |j_s|\}.$$

Suppose first that g stabilizes I. Then, we can write $g = xy$ where x fixes J pointwise and y fixes I pointwise. The assumption that $\underline{m} = \underline{m} \cdot g$ implies that both $\underline{m} = \underline{m} \cdot x$ and $\underline{m} = \underline{m} \cdot y$. For $s \in J$, $m_s = i_s$ and $m_{ys} = i_{ys}$, so since $m_s = m_{ys}$, we see that $i_s = i_{ys}$. Hence $\underline{i} \cdot y = \underline{i}$, and a similar argument gives that $\underline{j} \cdot x = \underline{j}$. So, $(\underline{i},\underline{j} \cdot g) = (\underline{i} \cdot y, \underline{j} \cdot y) \sim (\underline{i},\underline{j})$ as required.

Now suppose that g does not stabilize I. Then, we can pick $s \in I$ such that $gs \in J$. Let $t = gs \in J$ and define x to be the unique element of W_d with $xs = t, xt = s$ and fixing all other elements of $\{\pm 1, \dots, \pm d\} \setminus \{\pm s, \pm t\}$. Set $g' = xg, \underline{j}' = \underline{j} \cdot x$, so $\underline{j}' \cdot g' = \underline{j}g$. Using that $\underline{m} \cdot g = \underline{m}$, we have that $j_s = m_s = m_t = i_t$. So, $|j_t| > |i_t| = |m_t| = |m_s|$. Using $\underline{m} = \underline{m}(\underline{i}, \underline{j} \cdot g)$, we must therefore have that $m_s = i_s = i_t = m_t$. This shows that $\underline{i} \cdot x = \underline{i}$ and $\underline{m} \cdot x = \underline{m}$. Now,

$$\begin{aligned} \underline{m}(\underline{i}, \underline{j}) &= \underline{m}(\underline{i} \cdot x, \underline{j} \cdot x) = \underline{m}(\underline{i}, \underline{j}'), \\ \underline{m}(\underline{i}, \underline{j} \cdot g) &= \underline{m}(\underline{i}, \underline{j}' \cdot g'), \\ \underline{m}(\underline{i} \cdot g, \underline{j} \cdot g) &= \underline{m}(\underline{i} \cdot g', \underline{j}' \cdot g'). \end{aligned}$$

So by our assumption, $\underline{m}(\underline{i}, \underline{j}') = \underline{m}(\underline{i}, \underline{j}' \cdot g') = \underline{m}(\underline{i} \cdot g', \underline{j}' \cdot g')$. Now, $g's = s$, so we deduce by induction that $(\underline{i}, \underline{j}') \sim (\underline{i}, \underline{j}' \cdot g')$. Hence, $(\underline{i}, \underline{j}) \sim (\underline{i} \cdot x, \underline{j} \cdot x) = (\underline{i}, \underline{j}') \sim (\underline{i}, \underline{j}' \cdot g') = (\underline{i}, \underline{j} \cdot g)$ as required to complete the proof of (7.3).

Now we apply (7.3) to show:

7.4. *Let $\underline{i}, \underline{j}, \underline{i}', \underline{j}' \in I(n,d)$ and $\underline{m} = \underline{m}(\underline{i}, \underline{j})$, $\underline{m}' = \underline{m}(\underline{i}', \underline{j}')$. If $(\underline{i}, \underline{m}, \underline{m}, \underline{j}) \approx (\underline{i}', \underline{m}', \underline{m}', \underline{j}')$ then $(\underline{i}, \underline{j}) \sim (\underline{i}', \underline{j}')$.*

Indeed, take $g, h \in W_d$ such that $(\underline{i}, \underline{m}) = (\underline{i}' \cdot g, \underline{m}' \cdot g)$ and $(\underline{m}, \underline{j}) = (\underline{m}' \cdot gh, \underline{j}' \cdot gh)$. Set $\underline{k} = \underline{j}' \cdot g$. Now,

$$\begin{aligned} \underline{m} &= \underline{m}(\underline{i}, \underline{j}) = \underline{m}(\underline{i}, \underline{j}' \cdot gh) = \underline{m}(\underline{i}, \underline{k} \cdot h), \\ \underline{m}' \cdot g &= \underline{m}(\underline{i}' \cdot g, \underline{j}' \cdot g) = \underline{m}(\underline{i}, \underline{k}), \\ \underline{m}' \cdot gh &= \underline{m}(\underline{i}' \cdot gh, \underline{j}' \cdot gh) = \underline{m}(\underline{i} \cdot h, \underline{k} \cdot h). \end{aligned}$$

So, observing that $\underline{m} = \underline{m}' \cdot g = \underline{m}' \cdot gh$, we have that $\underline{m}(\underline{i}, \underline{k}) = \underline{m}(\underline{i}, \underline{k} \cdot h) = \underline{m}(\underline{i} \cdot h, \underline{k} \cdot h)$. Hence by (7.3), $(\underline{i}, \underline{k}) \sim (\underline{i}, \underline{k} \cdot h)$. So $(\underline{i}', \underline{j}') \sim (\underline{i}' \cdot g, \underline{j}' \cdot g) = (\underline{i}, \underline{k}) \sim (\underline{i}, \underline{k} \cdot h) = (\underline{i}, \underline{j})$.

Next we claim:

7.5. *Let $\underline{i}, \underline{j} \in I(n,d)$ and $\underline{m} = \underline{m}(\underline{i}, \underline{j})$. If $(\underline{i}, \underline{j})$ is strict, then $(\underline{i}, \underline{m}, \underline{m}, \underline{j})$ is strict.*

To prove this, take $(\underline{i}, \underline{j})$ strict and suppose that $(\underline{i}, \underline{m})$ is not strict. Then, there exist $1 \leq s < t \leq d$ with $|i_s| = |i_t|, |m_s| = |m_t|$ and $\partial_{i_s, m_s} \partial_{i_t, m_t} = \bar{1}$. So, $i_s \neq m_s, i_t \neq m_t$, hence by the definition of $\underline{m}$, $m_s = j_s, m_t = j_t$. But this contradicts the fact that $(\underline{i}, \underline{j})$ is strict. Hence, $(\underline{i}, \underline{m})$ is strict, and a similar argument shows that $(\underline{m}, \underline{j})$ is strict.

Recall that $\Omega(n,d)$ is some set of representatives of the $\sim$-equivalence classes of strict $(\underline{i}, \underline{j}) \in I(n,d) \times I(n,d)$. In view of (7.4) and (7.5), all $\{(\underline{i}, \underline{m}, \underline{m}, \underline{j}) \mid (\underline{i}, \underline{j}) \in \Omega(n,d), \underline{m} = \underline{m}(\underline{i}, \underline{j})\}$ are strict and lie in different $\approx$-equivalence classes. So they are linearly independent by (7.2), and we have now proved:

7.6. *The elements $\{b^{\natural}_{\underline{i}, \underline{m}} \otimes b^{\sharp}_{\underline{m}, \underline{j}} \mid (\underline{i}, \underline{j}) \in \Omega(n,d), \underline{m} = \underline{m}(\underline{i}, \underline{j})\}$ are linearly independent.*

Now we can prove the theorem. Call $(\underline{i}, \underline{k}, \underline{l}, \underline{j}) \in Y$ *special* if there exists $g \in W_d$ such that

$$\begin{aligned} i_{gs} = k_{gs} = l_s &\text{ whenever } |l_s| < |j_s|, \\ l_s = j_s = k_{gs} &\text{ whenever } |l_s| = |j_s| \end{aligned}$$

for all $s = 1, \dots, d$. We point out that if $\underline{m} = \underline{m}(\underline{i}, \underline{j})$, then $(\underline{i}, \underline{m}, \underline{m}, \underline{j})$ is special. Now, if $(\underline{i}, \underline{k}, \underline{l}, \underline{j}) \approx (\underline{i}', \underline{k}', \underline{l}', \underline{j}')$ and $(\underline{i}, \underline{k}, \underline{l}, \underline{j})$ is special, then $(\underline{i}', \underline{k}', \underline{l}', \underline{j}')$ is too. So the property of being special is a property of $\approx$-equivalence classes. Choose a total order $\succ$ on the set of all special $\approx$-equivalence classes such that the following hold for all special $(\underline{i}, \underline{k}, \underline{l}, \underline{j}), (\underline{i}', \underline{k}', \underline{l}', \underline{j}') \in Y$:

(1) if $\mathrm{wt}(\underline{k}') > \mathrm{wt}(\underline{k})$ (in the dominance order) then $(\underline{i}', \underline{k}', \underline{l}', \underline{j}') \succ (\underline{i}, \underline{k}, \underline{l}, \underline{j})$;

(2) if $\mathrm{wt}(\underline{k}) = \mathrm{wt}(\underline{k}')$ and $|\{s \mid 1 \le s \le d, i_s = k_s\}| > |\{s \mid 1 \le s \le d, i'_s = k'_s\}|$ then $(\underline{i}', \underline{k}', \underline{l}', \underline{j}') \succ (\underline{i}, \underline{k}, \underline{l}, \underline{j})$.

We need one more claim:

7.7. *Let $\underline{i}, \underline{j} \in I(n,d)$ and $\underline{m} = \underline{m}(\underline{i}, \underline{j})$. Then,*

$$\pi(b_{\underline{i},\underline{j}}) = \pm b^\flat_{\underline{i},\underline{m}} \otimes b^\sharp_{\underline{m},\underline{j}} + A + B$$

where A is a linear combination of terms of the form $b^\flat_{\underline{i},\underline{k}} \otimes b^\sharp_{\underline{k},\underline{j}}$ with $(\underline{i}, \underline{k}, \underline{k}, \underline{j})$ special and $(\underline{i}, \underline{k}, \underline{k}, \underline{j}) \succ (\underline{i}, \underline{m}, \underline{m}, \underline{j})$, and B is a linear combination of terms of the form $b^\flat_{\underline{i},\underline{k}} \otimes b^\sharp_{\underline{k},\underline{j}}$ with $(\underline{i}, \underline{k}, \underline{k}, \underline{j})$ not special.

To prove (7.7), we have from the definition of π that

$$\pi(b_{i,j}) = \pm b^\flat_{i,m} \otimes b^\sharp_{m,j} \pm b^\flat_{i,-m} \otimes b^\sharp_{-m,j} + (\text{a linear combination of } b^\flat_{i,k} \otimes b^\sharp_{k,j} \text{ with } |k| < |m|)$$

where $m = \min(|i|, |j|)$. So, writing $\underline{m} = \underline{m}(\underline{i}, \underline{j})$,

$$\pi(b_{\underline{i},\underline{j}}) = \sum_{\delta \in \mathbb{Z}_2^d} \pm b^\flat_{\underline{i},\underline{m}\cdot\delta} \otimes b^\sharp_{\underline{m}\cdot\delta,\underline{j}} + (\text{a linear combination of } b^\flat_{\underline{i},\underline{k}} \otimes b^\sharp_{\underline{k},\underline{j}} \text{ with } \mathrm{wt}(\underline{k}) > \mathrm{wt}(\underline{m}).)$$

Therefore, we just need to show that for all $(\bar{0}, \bar{0}, \dots, \bar{0}) \ne \delta \in \mathbb{Z}_2^d$ such that $(\underline{i}, \underline{m}\cdot\delta, \underline{m}\cdot\delta, \underline{j})$ is special, we have that $|\{s \mid 1 \le s \le d, i_s = m_s\}| > |\{s \mid 1 \le s \le d, i_s = m_{\delta s}\}|$. Take $\delta \in \mathbb{Z}_2^d$ such that $(\underline{i}, \underline{m}\cdot\delta, \underline{m}\cdot\delta, \underline{j})$ is special. Then certainly we have that $m_{\delta s} = j_s$ whenever $|m_s| = |j_s|$, when $m_s = j_s$ by definition of $\underline{m}$. So for s with $|m_s| = |j_s|$, we have that $m_{\delta s} = m_s$, whence $\delta_s = \bar{0}$. Instead, take t with $|m_t| < |j_t|$. Then, $m_t = i_t$ so $m_t = i_{\delta t}$ if and only if $\delta_t = \bar{0}$. These observations establish that

$$|\{s \mid 1 \le s \le d, i_s = m_s\}| \ge |\{s \mid 1 \le s \le d, i_s = m_{\delta s}\}|$$

with equality if and only if $\delta = (\bar{0}, \bar{0}, \dots, \bar{0})$. This completes the proof of (7.7).

Now the theorem follows easily from (7.6), (7.7) and a unitriangular argument involving the order $\succ$. $\square$

7.8. **Corollary.** *The natural multiplication map $\mu : Q_\flat(n,d) \otimes Q_\sharp(n,d) \to Q(n,d)$ is surjective.*

8 High-weight theory

Now we can classify the irreducible $Q(n,d)$-supermodules using high-weight theory. Recall that $Q_\sharp(n,d)$ denotes the positive Borel subsuperalgebra of $Q(n,d)$. We begin by determining the irreducible $Q_\sharp(n,d)$-supermodules.

The ideal $\mathfrak{J}_\sharp(n)$ from §7 is contained in the ideal $\mathfrak{J}_0(n)$ from §6. It follows that $Q_0(n,d) \subseteq Q_\sharp(n,d)$. On the other hand, let $Q_+(n,d)$ denote the subsuperspace of $Q_\sharp(n,d)$ spanned by the elements

$$\{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in I^2(n,d), |\underline{i}| \leq |\underline{j}|, |i_s| < |j_s| \text{ for some } s\}.$$

It follows from Lemma 6.1 that $Q_+(n,d)$ is a superideal of $Q_\sharp(n,d)$. Moreover, $Q_\sharp(n,d) = Q_0(n,d) \oplus Q_+(n,d)$ as a superspace, and $Q_\sharp(n,d)/Q_+(n,d) \cong Q_0(n,d)$. Analogously, $Q_-(n,d)$ denotes the superideal spanned by the elements $\{\xi_{\underline{i},\underline{j}} | (\underline{i},\underline{j}) \in I^2(n,d), |\underline{i}| \geq |\underline{j}|, |i_s| > |j_s| \text{ for some } s\}$, and $Q_\flat(n,d) = Q_0(n,d) \oplus Q_-(n.d)$.

If M is any $Q_0(n,d)$-supermodule, we can view it as a $Q_\sharp(n,d)$-supermodule by inflation along the quotient map $Q_\sharp(n,d) \twoheadrightarrow Q_0(n,d)$. In particular, we obtain irreducible $Q_\sharp(n,d)$-modules denoted $\{U(\lambda) \mid \lambda \in \Lambda(n,d)\}$, namely, the inflations of the irreducible $Q_0(n,d)$-supermodules constructed in Lemma 6.8.

Now suppose that M is a $Q_\sharp(n,d)$-supermodule and $\lambda \in \Lambda(n,d)$. By Lemma 6.1, for $\xi \in Q_+(n,d)$, $\xi M_\lambda \subseteq \bigoplus_{\mu > \lambda} M_\mu$. It follows at once that for any weight λ maximal in the dominance order such that $M_\lambda \neq 0$ (such a weight certainly exists as there are finitely many weights!), the weight space M_λ is annihilated by $Q_+(n,d)$. So M_λ is a $Q_\sharp(n,d)$-subsupermodule of M and the action of $Q_\sharp(n,d)$ on M_λ factors through the quotient $Q_0(n,d)$. In particular, if M is an irreducible $Q_\sharp(n,d)$-supermodule, $M \cong U(\lambda)$.

Given an arbitrary weight λ, we call a $Q(n,d)$-supermodule M a *high-weight module* of *high-weight* λ if the following conditions hold:

(1) M_λ is a $Q_\sharp(n,d)$-subsupermodule of M isomorphic to $U(\lambda)$;

(2) M is generated as an $Q(n,d)$-supermodule by M_λ.

For $\lambda \in \Lambda(n,d)$, define the *standard module*

$$\Delta(\lambda) := Q(n,d) \otimes_{Q_\sharp(n,d)} U(\lambda). \tag{8.1}$$

Call the weight λ an *admissible weight* if $\Delta(\lambda) \neq 0$.

8.2. **Lemma.** *For admissible λ, $\Delta(\lambda)$ is a high-weight module of high-weight λ. Moreover, $\Delta(\lambda)_\mu = 0$ unless $\mu \leq \lambda$.*

Proof. Recalling Corollary 7.8, we certainly have that

$$\Delta(\lambda) = Q_\flat(n,d) \otimes U(\lambda) = Q_-(n,d) \otimes U(\lambda) \oplus Q_0(n,d) \otimes U(\lambda).$$

All weights of $Q_-(n,d) \otimes U(\lambda)$ are strictly lower than λ in the dominance order. So the λ-weight space of $\Delta(\lambda)$ is equal to $1 \otimes U(\lambda)$, a homomorphic image of $U(\lambda)$. The assumption that λ is admissible is equivalent to $1 \otimes U(\lambda)$ being non-zero, in which case it is isomorphic to $U(\lambda)$ as $U(\lambda)$ is irreducible. $\square$

The admissible $\Delta(\lambda)$ have the usual universal property:

8.3. **Lemma.** *Suppose that M is a high-weight module of high-weight λ. Then, λ is admissible and M is a homomorphic image of $\Delta(\lambda)$. In particular, $M_\mu = 0$ unless $\mu \leq \lambda$.*

Proof. There is a natural isomorphism

$$\mathrm{Hom}_{Q_\sharp(n,d)}(U(\lambda), M\downarrow) \xrightarrow{\sim} \mathrm{Hom}_{Q(n,d)}(\Delta(\lambda), M).$$

Choose an isomorphism $\theta : U(\lambda) \to M_\lambda \subseteq M$ of $Q_\sharp(n,d)$-supermodules and let $\theta\uparrow: \Delta(\lambda) \to M$ be the corresponding $Q(n,d)$-supermodule homomorphism. This is non-zero, hence λ is admissible, and is surjective as M is generated by M_λ. This shows that M is a quotient of $\Delta(\lambda)$, and the final statement about weights follows from Lemma 8.2. $\square$

For admissible λ, define $L(\lambda)$ to be the head of $\Delta(\lambda)$, i.e. $L(\lambda)$ is the largest completely reducible quotient supermodule of $\Delta(\lambda)$. We remark that if $p = 0$ or $p > d$, then $Q(n,d)$ is semisimple by Lemma 4.1, so that $L(\lambda) = \Delta(\lambda)$ in these cases.

8.4. **Lemma.** *The set $\{L(\lambda) \mid$ for all admissible $\lambda \in \Lambda(n,d)\}$ is a complete set of pairwise non-isomorphic irreducible $Q(n,d)$-supermodules. Moreover, the module $L(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even.*

Proof. Let λ be admissible. We first claim that $\Delta(\lambda)$ has a unique maximal subsupermodule, so that $L(\lambda)$ is irreducible. For let M, N be two maximal subsupermodules of $\Delta(\lambda)$. Since $\Delta(\lambda)_\lambda$ is irreducible over $Q_0(n,d)$ and generates $\Delta(\lambda)$ over $Q(n,d)$, we must have that $M_\lambda = N_\lambda = 0$, so $(M+N)_\lambda = 0$. This shows that $M+N$ is a proper subsupermodule of $\Delta(\lambda)$. Hence, $M = M+N = N$ by maximality, as required.

Evidently, for admissible $\lambda \neq \mu$, $L(\lambda)$ and $L(\mu)$ are not isomorphic, as they have different high-weights. Now suppose that L is an arbitrary irreducible $Q(n,d)$-supermodule. Choose λ maximal in the dominance order such that $L_\lambda \neq 0$. Then, by irreducibility, L must be a high-weight module of high-weight λ, so a quotient of $\Delta(\lambda)$ by Lemma 8.3. Hence, $L \cong L(\lambda)$.

It remains to prove the statement about absolute irreducibility. First observe by adjointness that $\mathrm{Hom}_{Q(n,d)}(\Delta(\lambda), L(\lambda)) \cong \mathrm{Hom}_{Q_\sharp(n,d)}(U(\lambda), L(\lambda)\downarrow) \cong \mathrm{End}_{Q_0(n,d)}(U(\lambda))$. Now there is a natural embedding $\mathrm{Hom}_{Q(n,d)}(L(\lambda), L(\lambda)) \hookrightarrow \mathrm{Hom}_{Q(n,d)}(\Delta(\lambda), L(\lambda))$. To see that it is an isomorphism, observe that any $Q(n,d)$-homomorphism $\Delta(\lambda) \to L(\lambda)$ annihilates the unique maximal submodule of $\Delta(\lambda)$, hence induces a well-defined homomorphism $L(\lambda) \to L(\lambda)$. We have shown that $\mathrm{End}_{Q(n,d)}(L(\lambda)) \cong \mathrm{End}_{Q_0(n,d)}(U(\lambda))$. Now the final part of the lemma follows from Lemma 6.8. $\square$

9 Classification of admissible weights

We now proceed to give a combinatorial description of the admissible weights, to complete the classification of the irreducible $Q(n,d)$-supermodules. We make some definitions. Let $\Lambda^+(n,d)$ denote the set of all $\lambda \in \Lambda(n,d)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, i.e. λ is a partition of d with at most n non-zero parts. Let $\Lambda_p^+(n,d)$ denote the set of all $\lambda \in \Lambda(n,d)$ such that

$$0 < \lambda_i - \lambda_{i+1} + \delta_i \qquad \text{for } i = 1, \dots, n, \quad \text{where } \delta_i = \begin{cases} 1 & \text{if } p \mid \lambda_i, \\ 0 & \text{otherwise,} \end{cases}$$

so that λ is a p-strict partition as in the introduction. Call $\lambda \in \Lambda_p^+(n,d)$ *restricted* if either $p = 0$ or $p > 0$ and $\lambda_i - \lambda_{i+1} + \delta_i \leq p$ for $i = 1, \dots, n$. Let $\Lambda_p^+(n,d)_{\mathrm{res}}$ denote the set of all restricted $\lambda \in \Lambda_p^+(n,d)$. We will show that λ is admissible if and only if $\lambda \in \Lambda_p^+(n,d)$.

We construct another natural subsuperalgebra of $Q(n,d)$. Let $\mathfrak{K}(n) = \bigoplus_{d \geq 0} \mathfrak{K}(n,d)$ denote the $\mathbb{Z}$-graded superideal of $B(n)$ generated by the elements

$$\{b_{i,j} \mid i = 1, \dots, n, j = -1, \dots, -n\}.$$

It is a bisuperideal, so we can form the bisuperalgebra quotient

$$A(n) = B(n)/\mathfrak{K}(n),$$

this being $\mathbb{Z}$-graded as $A(n) = \bigoplus_{d \geq 0} A(n,d)$ where $A(n,d) \cong B(n,d)/\mathfrak{K}(n,d)$. For $i, j = 1, \dots, n$, set $c_{i,j} = b_{i,j} + \mathfrak{K}(n)$. Observing that each $c_{i,j}$ has degree $\bar{0}$, $A(n) = A(n)_{\bar{0}}$ is precisely the free polynomial algebra on the generators $\{c_{i,j} \mid 1 \leq i, j \leq n\}$. So the dual superalgebra $S(n,d) = A(n,d)^*$ is just the usual classical Schur algebra as in [7]. We can identify $S(n,d)$ with the subsuperalgebra $\mathfrak{K}(n,d)^\circ \subseteq Q(n,d)_{\bar{0}} \subseteq Q(n,d)$.

Now we treat the case $n = 2$, copying an argument due to Penkov [18, §7] in our setting.

9.1. **Lemma.** *Suppose that $n = 2$ and that $\lambda \in \Lambda(2,d)$ is an admissible weight. Then, either $\lambda_1 > \lambda_2$, or $\lambda_1 = \lambda_2 = c$ for some $c \geq 0$ with $p \mid c$.*

Proof. The restriction of $L(\lambda)$ to the ordinary Schur algebra $S(2,d) \subseteq Q(2,d)$ gives us an $S(2,d)$-module with maximal weight λ. We deduce from the classical theory that $\lambda_1 \geq \lambda_2$. To complete the proof, suppose for a contradiction that $\lambda_1 = \lambda_2 = c$ but that $p \nmid c$. So $d = 2c$. Now, there are no $\mu \in \Lambda^+(2,2c)$ with $\mu < \lambda$. Since we also know that $\dim L(\lambda)_\lambda = \dim U(\lambda) = 2$, we deduce by the classical representation theory of $S(2,2c)$ that $L(\lambda) \downarrow S(2,2c)$ splits as a direct sum of two irreducible $S(2,2c)$-modules both of high-weight λ. But such $S(2,2c)$-modules are one dimensional (being just a tensor power of the determinant module). This shows that $L(\lambda) = L(\lambda)_\lambda$, of dimension exactly two. Hence, $L(\lambda)_\nu = 0$ for all $\nu \neq \lambda$.

Define the following elements of $I(2,2c)$:

$$\begin{aligned}
\underline{i} &= (1, \dots, 1, -2; 2, \dots, 2, 2), & \underline{j} &= (1, \dots, 1, 2; 2, \dots, 2, 2), \\
\underline{k} &= (1, \dots, 1, 1; 2, \dots, 2, -1), & \underline{l} &= (1, \dots, 1, 1; 2, \dots, 2, 1), \\
\underline{s} &= (1, \dots, 1, -1; 2, \dots, 2, 2), & \underline{t} &= (1, \dots, 1, 1; 2, \dots, 2, -2), \\
\underline{u} &= (1, \dots, 1, -2; 2, \dots, 2, 1), & \underline{i}_\lambda &= (1, \dots, 1, 1; 2, \dots, 2, 2)
\end{aligned}$$

where the symbol ; is between the cth and $(c+1)$th entries. Now an explicit calculation using the product rule Theorem 4.5 shows that

$$\xi_{\underline{i}_\lambda, \underline{j}} \xi_{\underline{i}, \underline{i}_\lambda} = \xi_{\underline{s}, \underline{i}_\lambda} + \xi_{\underline{u}, \underline{i}_\lambda} \qquad \text{and} \qquad \xi_{\underline{i}_\lambda, \underline{k}} \xi_{\underline{l}, \underline{i}_\lambda} = \xi_{\underline{t}, \underline{i}_\lambda} + \xi_{\underline{u}, \underline{i}_\lambda}.$$

Hence,

$$\xi_{\underline{i}_\lambda, \underline{j}} \xi_{\underline{i}, \underline{i}_\lambda} - \xi_{\underline{i}_\lambda, \underline{k}} \xi_{\underline{l}, \underline{i}_\lambda} = \xi_{\underline{s}, \underline{i}_\lambda} - \xi_{\underline{t}, \underline{i}_\lambda}.$$

Using the previous paragraph and a weight argument, both terms on the left hand side of this equation act as zero on $L(\lambda)_\lambda$. Hence, the term $\xi_{\underline{s},\underline{i}_\lambda} - \xi_{\underline{t},\underline{i}_\lambda} \in \xi_\lambda Q_0(n,d)$ on the right hand side acts as zero on $L(\lambda)_\lambda \cong U(\lambda)$. But $\xi_\lambda Q_0(n,d) \cong C(2)$ according to (6.7), so as $U(2)$ is a faithful $C(2)$-supermodule, the non-zero element $\xi_{\underline{s},\underline{i}_\lambda} - \xi_{\underline{t},\underline{i}_\lambda}$ of $\xi_\lambda Q_0(n,d)$ cannot act as zero on $U(\lambda)$, a contradiction. $\square$

Now observe that for $\lambda \in \Lambda(n,d)$, λ lies in $\Lambda_p^+(n,d)$ if and only if for each $i = 1, \dots, n-1$ $(\lambda_i, \lambda_{i+1})$ lies in $\Lambda_p^+(2, \lambda_i + \lambda_{i+1})$. So by an argument involving restriction to various quotients of $B(n)$ isomorphic to $B(2)$, we have the following corollary of Lemma 9.1:

9.2. **Corollary.** *If $\lambda \in \Lambda(n,d)$ is admissible, then $\lambda \in \Lambda_p^+(n,d)$.*

It remains to prove that every $\lambda \in \Lambda_p^+(n,d)$ is admissible, i.e. that there does exist some high-weight module of high-weight λ for each $\lambda \in \Lambda_p^+(n,d)$. To do this, we first give a construction of some high-weight modules in the case $p > 0$ using a Frobenius twist argument. Recall from earlier in the section that $A(n)$ denotes the free polynomial algebra on generators $\{c_{i,j} \mid 1 \leq i,j \leq n\}$, viewed as a bialgebra as in the classical polynomial representation theory of $GL(n)$ [7]. In particular, we can view $A(n)$ is a bisuperalgebra concentrated in degree $\bar{0}$.

9.3. **Lemma.** *If $p > 0$, the unique algebra map $\sigma : A(n) \to B(n)$, such that $c_{i,j} \mapsto b_{i,j}^p$ for all $1 \leq i,j \leq n$, is a bisuperalgebra embedding.*

Proof. This is a routine check of relations. $\square$

In view of the lemma, there is a natural restriction functor

$$\mathrm{Fr} : \mathbf{mod}(A(n)) \to \mathbf{mod}(B(n)).$$

On objects, Fr is defined by sending an $A(n)$-cosupermodule M with structure map $\eta : M \to M \otimes A(n)$ to the $B(n)$-cosupermodule equal to M as a superspace with structure map $(\mathrm{id} \otimes \sigma) \circ \eta$; we call $\mathrm{Fr}\, M$ the *Frobenius twist* of M. On morphisms, Fr sends a morphism to the same linear map but regarded instead as a $B(n)$-cosupermodule map. We note that if M is a polynomial $A(n)$-cosupermodule of degree d, then $\mathrm{Fr}\, M$ is a $B(n, pd)$-cosupermodule. Also, let $\mathrm{Fr} : X(n) \to X(n)$ be the linear map determined by $\mathrm{Fr}(x^\lambda) = x^{p\lambda}$ for each $\lambda \in \Lambda(n)$, where $p\lambda$ denotes $(p\lambda_1, \dots, p\lambda_n)$. Then, the formula

$$\mathrm{ch}(\mathrm{Fr}\, M) = \mathrm{Fr}(\mathrm{ch}\, M)$$

describes the effect of the functor Fr at the level of characters.

9.4. **Lemma.** *Suppose that $\lambda \in \Lambda(n,d_1)$ is an admissible weight, and that $\mu \in \Lambda^+(n,d_2)$ is arbitrary. Then, $\lambda + p\mu \in \Lambda(n, d_1 + pd_2)$ is an admissible weight. Moreover, all non-zero weights of $L(\lambda + p\mu)$ are of the form $\lambda' + p\mu'$ for $\lambda' \leq \lambda$ and $\mu' \leq \mu$.*

Proof. If $p = 0$, there is nothing to prove. Otherwise, by the classical theory, there exists an irreducible $A(n)$-comodule $L'(\mu)$ of high-weight μ. Regard $L'(\mu)$ instead as an $A(n)$-cosupermodule concentrated in degree $\bar{0}$ (say) and consider the $B(n)$-cosupermodule

$$M = L(\lambda) \otimes \mathrm{Fr}\, L'(\mu).$$

It is a $B(n, d_1 + pd_2)$-cosupermodule, hence a $Q(n, d_1 + pd_2)$-supermodule. Its non-zero weights are of the form $\lambda' + p\mu'$ for $\lambda \leq \lambda$ and $\mu' \leq \mu$, and the weight $\lambda + p\mu$ definitely appears as a weight of M. Hence, there exists a high-weight module of high-weight $\lambda + p\mu$, so $\lambda + p\mu$ is admissible. The statement about weights follows because $L(\lambda + p\mu)$ must then be a subquotient of M. $\square$

Now we are in a position to complete the classification of admissible weights by a counting argument. Recall the definition of the idempotent ξ_ω from §6.

9.5. **Theorem.** (i) *$\lambda \in \Lambda(n,d)$ is admissible if and only if $\lambda \in \Lambda_p^+(n,d)$.*

(ii) *Assuming that $n \geq d$ and $\lambda \in \Lambda_p^+(n,d)$, we have that $\xi_\omega L(\lambda) \neq 0$ if and only if $\lambda \in \Lambda_p^+(n,d)_{\mathrm{res}}$.*

Proof. Recalling Corollary 9.2, we just need to show for (i) that if $\lambda \in \Lambda_p^+(n,d)$, then λ is admissible. We consider first the case $n \geq d$, and proceed by induction on $d = 0, 1, \dots, n$. The result is trivially true in case $d = 0$. For $n \geq d > 0$, take $\lambda \in \Lambda_p^+(n,d)$. Suppose first that $\lambda \notin \Lambda_p^+(n,d)_{\mathrm{res}}$. Then, we can write $\lambda = \lambda_1 + p\lambda_2$ where $\lambda_1 \in \Lambda_p^+(n,d_1)$ and $\lambda_2 \in \Lambda^+(n,d_2)$ for some d_1, d_2 with $d = d_1 + pd_2$ and $d_2 \neq 0$. By induction, λ_1 is admissible, so we deduce from Lemma 9.4 that λ is admissible, and moreover that $\xi_\omega L(\lambda) = 0$. But by Lemma 6.3, there are exactly $|\Lambda_p^+(n,d)_{\mathrm{res}}|$ non-isomorphic irreducible $Q(n,d)$-supermodules not annihilated by ξ_ω. In view of Corollary 9.2, this means that all $\lambda \in \Lambda_p^+(n,d)_{\mathrm{res}}$ must both be admissible and satisfy $\xi_\omega L(\lambda) \neq 0$, else we end up with too few such modules.

Now suppose that $n < d$ and choose $m \geq d$. Let $e \in Q(m,d)$ be the idempotent defined in (6.4), and also recall the embedding $\Lambda(n,d) \hookrightarrow \Lambda(m,d)$ there. Take $\lambda \in \Lambda_p^+(n,d)$. Then, viewing λ as an element of $\Lambda_p^+(m,d)$, we have already shown in the previous paragraph that λ is admissible for $Q(m,d)$, so that there exists an irreducible $Q(m,d)$-supermodule $L(\lambda)$ of high-weight λ. In view of Lemma 6.5, we have that $eL(\lambda)_\lambda \neq 0$ as $\lambda \in \Lambda(n,d)$, so $eL(\lambda)$ is an irreducible $Q(n,d)$-supermodule of high-weight λ, as required. $\square$

10 Consequences

In Theorem 9.5(i) and Lemma 8.4, we have classified the irreducible $Q(n,d)$-supermodules; they are precisely the supermodules $\{L(\lambda) \mid \lambda \in \Lambda_p^+(n,d)\}$. Applying Lemma 5.1, we have equivalently determined the irreducible $B(n)$-cosupermodules. Let $\Lambda_p^+(n) = \bigcup_{d \geq 0} \Lambda_p^+(n,d)$ denote the set of all p-strict partitions with at most n non-zero parts. Then, we have shown:

10.1. **Theorem.** *The $B(n)$-cosupermodules $\{L(\lambda) \mid \lambda \in \Lambda_p^+(n)\}$ give a complete set of pairwise non-isomorphic irreducible $B(n)$-cosupermodules. Moreover, $L(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even.*

It is immediate from high-weight theory that the character map $\mathrm{ch} : \mathrm{Grot}(B(n)) \to X(n)$ described at the end of §6 is an embedding of the Grothendieck ring of the category of $B(n)$-cosupermodules into $X(n)$. We have two natural bases for the image: $\{\mathrm{ch}\, L(\lambda) \mid \lambda \in \Lambda_p^+(n)\}$ and $\{\mathrm{ch}\, \Delta(\lambda) \mid \lambda \in \Lambda_p^+(n)\}$. Moreover, for $\lambda \in \Lambda_p^+(n)$,

$$\mathrm{ch}\, \Delta(\lambda) = \mathrm{ch}\, L(\lambda) + \sum_{\mu < \lambda} f_{\lambda,\mu}\, \mathrm{ch}\, L(\mu)$$

for unique non-negative integers $f_{\lambda,\mu}$. This gives us a well-defined *p-decomposition matrix* $F = (f_{\lambda,\mu})_{\lambda,\mu\in\Lambda_p^+(n)}$. It is a unitriangular matrix if rows and columns are ordered in some way refining dominance.

If μ is restricted, one can hope that the p-decomposition number $f_{\lambda,\mu}$ equals the specialization $d_{\lambda,\mu}(1)$ of the polynomials defined by Leclerc and Thibon in [12, Theorem 4.1] (with $h = p$) for sufficiently large p. It would be interesting to extend the construction of [12] to arbitrary (i.e. not necessarily restricted) weights μ, as was done in [11] for the Fock space of $A_{p-1}^{(1)}$. Another basic problem here is the explicit computation of the character $\operatorname{ch}\Delta(\lambda)$ for *all* $\lambda \in \Lambda_p^+(n)$. For $p = 0$, this problem was solved by Sergeev [20, Theorem 4], who showed that $\operatorname{ch}\Delta(\lambda) = 2^{-\lfloor h(\lambda)/2\rfloor}\sum_{\nu\in\Lambda(n,d)} K'_{\lambda,\nu}x^\nu$, where $K'_{\lambda,\nu}$ is as in [13, III(8.16)$'$].

We point out at least for arbitrary p that the character of $\Delta(\lambda)$ is stable as $n \to \infty$, so that $\operatorname{ch}\Delta(\lambda)$ can be regarded as a symmetric function. To be more precise, suppose that $m \geq n$ and let e denote the idempotent from (6.4), embedding $\Lambda(n,d)$ into $\Lambda(m,d)$ as there. For $\lambda \in \Lambda_p^+(n,d)$, denote the standard $Q(n,d)$-supermodule (resp. the standard $Q(m,d)$-supermodule) of high-weight λ by $\Delta_n(\lambda)$ (resp. $\Delta_m(\lambda)$) to avoid ambiguity, and similarly let $L_n(\lambda)$ (resp. $L_m(\lambda)$) denote the irreducible supermodule of high-weight λ. Then, as $Q(n,d)$-supermodules, $L_n(\lambda) \cong eL_m(\lambda)$ and $\Delta_n(\lambda) \cong e\Delta_m(\lambda)$; the first of these formulae follows immediately from Corollary 2.15 and weight considerations, while the second can be proved directly from the definition of $\Delta(\lambda)$ as an induced module. Stability of weight multiplicities, i.e. that $\dim L_m(\lambda)_\mu = \dim L_n(\lambda)_\mu$ and $\dim\Delta_m(\lambda)_\mu = \dim\Delta_n(\lambda)_\mu$ for all $\mu \in \Lambda(n,d)$, follows immediately.

Next we turn our attention to constructing the irreducible representations of the Sergeev superalgebra $W(d)$. Let $n \geq d$, and identify $\Lambda_p^+(n,d)$ with the set $\mathfrak{P}_p(d)$ of all p-strict partitions of d. Then, $\Lambda_p^+(n,d)_{\mathrm{res}}$ is identified with $\mathfrak{RP}_p(d) \subseteq \mathfrak{P}_p(d)$. Also let $\xi_\omega \in Q(n,d)$ be the idempotent from §6. For $\lambda \in \mathfrak{RP}_p(d)$, define the $W(d)$-supermodule

$$V(\lambda) := \xi_\omega L(\lambda).$$

We should note that this definition is independent of the particular choice of $n \geq d$ up to natural isomorphism (this is proved in a standard way, see e.g. [3, §3.5]). The following result is immediate from Theorem 9.5(ii) and Corollary 2.15:

10.2. **Theorem.** *The modules $\{V(\lambda) \mid \lambda \in \mathfrak{RP}_p(d)\}$ give a complete set of pairwise non-isomorphic irreducible $W(d)$-supermodules. Moreover, $V(\lambda)$ is absolutely irreducible if and only if $h_{p'}(\lambda)$ is even.*

In order to obtain a labelling for *all* irreducible $W(d)$-modules, not just supermodules, we know by Lemma 2.3 that if $V(\lambda)$ is not absolutely irreducible, it decomposes as $V(\lambda,+) \oplus V(\lambda,-)$ for two non-isomorphic irreducible $W(d)$-modules $V(\lambda,+), V(\lambda,-)$. By Corollary 2.7, the modules

$$\{V(\lambda) \mid \lambda \in \mathfrak{RP}_p(d), h_{p'}(\lambda) \text{ even}\} \cup \{V(\lambda,+), V(\lambda,-) \mid \lambda \in \mathfrak{RP}_p(d), h_{p'}(\lambda) \text{ odd}\}$$

then give a complete set of pairwise non-isomorphic irreducible $W(d)$-modules.

To pass to the projective representations of the symmetric group, we use Corollary 3.13 and Corollary 3.15. Suppose first that d is even. Then, for each $\lambda \in \mathfrak{RP}_p(d)$, there is a unique irreducible $S(d)$-supermodule $D(\lambda)$ such that $V(\lambda) \cong YD(\lambda)$. Moreover, $D(\lambda)$ is absolutely irreducible if and only if $V(\lambda)$ is absolutely irreducible, which is if and only if $h_{p'}(\lambda)$ is even. In the case that d is odd, take $\lambda \in \mathfrak{RP}_p(d)$. If $h_{p'}(\lambda)$ is odd, then there is a unique absolutely irreducible $S(d)$-supermodule $D(\lambda)$ such that $V(\lambda) \cong YD(\lambda)$. If $h_{p'}(\lambda)$ is even, then there is a unique non-absolutely irreducible $S(d)$-supermodule $D(\lambda)$ such that $YD(\lambda) \cong V(\lambda) \oplus V(\lambda)$. Then:

10.3. **Theorem.** *The modules $\{D(\lambda) \mid \lambda \in \mathfrak{RP}_p(d)\}$ give a complete set of pairwise non-isomorphic irreducible $S(d)$-supermodules. Moreover, $D(\lambda)$ is absolutely irreducible if and only if $d - h_{p'}(\lambda)$ is even.*

If $\lambda \in \mathfrak{RP}_p(d)$ and $d - h_{p'}(\lambda)$ is odd, we can decompose $D(\lambda) \cong D(\lambda, +) \oplus D(\lambda, -)$ as a direct sum of two non-isomorphic irreducible $S(d)$-modules, and by Corollary 2.7 the modules

$$\{D(\lambda) \mid \lambda \in \mathfrak{RP}_p(d), d - h_{p'}(\lambda) \text{ even}\} \cup \{D(\lambda, +), D(\lambda, -) \mid \lambda \in \mathfrak{RP}_p(d), d - h_{p'}(\lambda) \text{ odd}\}$$

then give a complete set of pairwise non-isomorphic irreducible $S(d)$-modules. We have thus determined the irreducible projective representations of S_d.

The next theorem explains how to obtain the irreducible projective representations of A_d from these. Let $A(d) = S(d)_{\bar{0}}$ denote the twisted group algebra of the alternating group. The following theorem follows easily by Clifford theory for groups with normal subgroups of index two.

10.4. **Theorem.** *Let $\lambda \in \mathfrak{RP}_p(d)$. If $d - h_{p'}(\lambda)$ is even, $D(\lambda) \downarrow_{A(d)} \cong E(\lambda, +) \oplus E(\lambda, -)$ for two non-isomorphic irreducible $A(d)$-modules $E(\lambda, +), E(\lambda, -)$. If $d - h_{p'}(\lambda)$ is odd, $D(\lambda) \downarrow_{A(d)} \cong E(\lambda) \oplus E(\lambda)$ for a single irreducible $A(d)$-module $E(\lambda)$. The modules*

$$\{E(\lambda) \mid \lambda \in \mathfrak{RP}_p(d), d - h_{p'}(\lambda) \text{ odd}\} \cup \{E(\lambda, +), E(\lambda, -) \mid \lambda \in \mathfrak{RP}_p(d), d - h_{p'}(\lambda) \text{ even}\}$$

then give a complete set of pairwise non-isomorphic irreducible $A(d)$-modules.

We end with some comments about decomposition numbers. So suppose now that $(\Bbbk, R, K)$ is a p-modular system with K sufficiently large (specifically, containing square roots of $\pm 1, \dots, \pm d$). So, R is a complete DVR, K is its field of fractions of characteristic 0 and our fixed algebraically closed field $\Bbbk$ of characterisitc p is its residue field.

The bisuperalgebra $B(n)$ can be defined in exactly the same as in §5 but over the ground ring R, giving us an R-free R-bisuperalgebra $B(n)_R$ such that $B(n) \cong B(n)_R \otimes_R \Bbbk$. Set $Q(n, d)_R = \mathrm{Hom}_R(B(n, d)_R, R)$ to obtain an R-form of the Schur superalgebra $Q(n, d)$. So, $Q(n, d)_R$ is R-free as an R-module and $Q(n, d) \cong \Bbbk \otimes_R Q(n, d)_R$; we will from now on identify the two. Also, set $Q(n, d)_K = Q(n, d)_R \otimes_R K$, the analogous Schur superalgebra over the ground field K. Similarly, we can define an R-form $Q_0(n, d)_R$ of $Q_0(n, d)$, and set $Q_0(n, d)_K = Q_0(n, d)_R \otimes_R K$. We will always view $Q(n, d)_R$ and $Q_0(n, d)_R$ as R-subsuperalgebras of $Q(n, d)_K$.

For $\lambda \in \Lambda_0^+(n, d)$, let $\Delta(\lambda)_K$ denote the standard $Q(n, d)_K$-supermodule of high-weight λ, constructed as in (8.1). Denote the high-weight space of $\Delta(\lambda)_K$ by $U(\lambda)_K$; this is precisely the $Q_0(n,d)_K$-supermodule defined as in §6. Now, the construction of $U(\lambda)_K$ can be carried out over R instead, because R contains square roots of each $\pm\lambda_i$, giving us a finitely generated R-free $Q_0(n,d)_R$-subsupermodule $U(\lambda)_R$ of $U(\lambda)_K$ such that $U(\lambda)_K \cong U(\lambda)_R \otimes_R K$. Let $\Delta(\lambda)_R$ denote the $Q(n,d)_R$-subsupermodule of $\Delta(\lambda)_K$ generated by $U(\lambda)_R$. Then, $\Delta(\lambda)_R$ is a finitely generated R-free R-module such that $\Delta(\lambda)_K \cong \Delta(\lambda)_R \otimes_R K$. Now set $\bar{\Delta}(\lambda) := \Bbbk \otimes_R \Delta(\lambda)_R$. This gives us a $Q(n,d)$-supermodule such that

$$\operatorname{ch} \bar{\Delta}(\lambda) = \operatorname{ch} \Delta(\lambda)_{\mathbb{C}}.$$

So there are unique integer matrices $D = (d_{\lambda,\mu})$ and $E = (e_{\lambda,\mu})$ for $\lambda \in \Lambda_0^+(n,d), \mu \in \Lambda_p^+(n,d)$ such that

$$\operatorname{ch} \bar{\Delta}(\lambda) = \sum_{\mu \in \Lambda_p^+(n,d)} e_{\lambda,\mu} \operatorname{ch} \Delta(\mu), \qquad \operatorname{ch} \bar{\Delta}(\lambda) = \sum_{\mu \in \Lambda_p^+(n,d)} d_{\lambda,\mu} \operatorname{ch} L(\mu)$$

for all $\lambda \in \Lambda_0^+(n,d)$. The matrix D is the *decomposition matrix* for the reduction of irreducible $Q(n,d)_K$-modules from characteristic 0 to characteristic p. Evidently, $D = EF$ gives a factorization of the decomposition matrix into a product of the matrix E and the (square) p-decomposition matrix F. The following at least follows from high-weight theory and Lemma 6.8:

10.5. **Lemma.** *For $\lambda \in \Lambda_0^+(n,d)$, $d_{\lambda,\lambda} = e_{\lambda,\lambda} = 2^{\lfloor (h(\lambda)+1)/2 \rfloor - \lfloor (h_{p'}(\lambda)+1)/2 \rfloor}$ where $h(\lambda)$ is the number of non-zero parts of λ. Given in addition $\mu \in \Lambda_p^+(n,d)$ with $\mu \not\leq \lambda$, $d_{\lambda,\mu} = e_{\lambda,\mu} = 0$.*

Now we relate these decomposition matrices of $Q(n,d)$ to those of $W(d)$ and $S(d)$. Using the subscript K to indicate that we are working over the ground field K instead of our usual $\Bbbk$, we have irreducible $W(d)_K$- (resp. $S(d)_K$-) supermodules labelled by strict partitions $\lambda \in \mathfrak{P}_0(d)$, which we denote by $V(\lambda)_K$ and $D(\lambda)_K$ respectively. By a minor variation on Brauer's theory, we can reduce these modulo p to obtain $W(d)$- (resp. $S(d)$-) supermodules $\bar{V}(\lambda)$ and $\bar{D}(\lambda)$. These are not uniquely determined up to isomorphism, but at least the multiplicities of composition factors are unique. So we obtain the *decomposition matrices* $D^S = (d^S_{\lambda,\mu})$ and $D^W = (d^W_{\lambda,\mu})$ of $S(d)$ and $W(d)$ respectively, for $\lambda \in \mathfrak{P}_0(d), \mu \in \mathfrak{RP}_p(d)$, determined by the equations

$$[\bar{V}(\lambda)] = \sum_{\mu \in \mathfrak{RP}_p(d)} d^W_{\lambda,\mu}[V(\mu)], \qquad [\bar{D}(\lambda)] = \sum_{\mu \in \mathfrak{RP}_p(d)} d^S_{\lambda,\mu}[D(\mu)]$$

written in the Grothendieck groups of $\mathbf{mod}(W(d))$ and $\mathbf{mod}(S(d))$ respectively. The final theorem relates these decomposition numbers to those of the Schur superalgebra:

10.6. **Theorem.** *Let $\lambda \in \mathfrak{P}_0(d)$ and $\mu \in \mathfrak{RP}_p(d)$. Then, $d^W_{\lambda,\mu} = d_{\lambda,\mu}$. Similarly, if d is even, $d^S_{\lambda,\mu} = d_{\lambda,\mu}$, while if d is odd,*

$$d^S_{\lambda,\mu} = \begin{cases} d_{\lambda,\mu} & \text{if } h(\lambda) - h_{p'}(\mu) \text{ is even,} \\ 2d_{\lambda,\mu} & \text{if } h(\lambda) \text{ is even and } h_{p'}(\mu) \text{ is odd,} \\ \frac{1}{2}d_{\lambda,\mu} & \text{if } h(\lambda) \text{ is odd and } h_{p'}(\mu) \text{ is even,} \end{cases}$$

where $h(\lambda)$ denotes the number of non-zero parts of λ.

Proof. The Schur functor coming from the idempotent ξ_ω can be defined over the ground ring R, using an R-integral version of Theorem 6.2. Using that Schur functors commute with base change, one sees that $[\xi_\omega \bar{\Delta}(\lambda)] = [\bar{V}(\lambda)]$ (equality written in the Grothendieck group). In particular, it follows from this by exactness of Schur functors that $d^W_{\lambda,\mu} = d_{\lambda,\mu}$. Similarly, the functor Y from §3 can be defined over the ground ring R, and Y commutes with base change evidently. One sees in the case that d is even that $[Y\bar{D}(\lambda)] = [\bar{V}(\lambda)]$ and $YD(\mu) = V(\mu)$ by Theorem 3.12 over K or $\Bbbk$ respectively, so that $d^S_{\lambda,\mu} = d^W_{\lambda,\mu}$. Finally, suppose that d is odd. Applying Lemma 3.14 over K or $\Bbbk$, we have that

$$[Y\bar{D}(\lambda)] = \begin{cases} [\bar{V}(\lambda)] & \text{if } h(\lambda) \text{ is odd,} \\ 2[\bar{V}(\lambda)] & \text{if } h(\lambda) \text{ is even.} \end{cases}$$

We also know that

$$[YD(\mu)] = \begin{cases} [V(\mu)] & \text{if } h_{p'}(\mu) \text{ is odd,} \\ 2[V(\mu)] & \text{if } h_{p'}(\lambda) \text{ is even.} \end{cases}$$

The theorem follows from these equations together with exactness of Y. $\square$

Thus our results reduce the problem of determining the decomposition matrices of the twisted group algebras $S(d)$ and $W(d)$ to the problem of determining the decomposition matrices of the Schur superalgebras $Q(n, d)$.

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