Introduction

In [4], Enright developed an infinitesimal approach to study the fundamental series. This was based on his earlier work with Varadarajan ([6]) and Wallach ([5]). The present paper is an attempt to understand Enright's construction in terms of geometry of the flag variety.

Let G_0 be a connected semisimple real Lie group with finite center, K_0 a maximal compact subgroup of G_0 and ϑ the corresponding Cartan involution. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebras of G_0 and K_0 respectively, $\mathfrak{g}$ and $\mathfrak{k}$ their complexifications, and K the complexification of K_0. Denote by X the flag variety of Borel subalgebras in $\mathfrak{g}$.

The action of K on X defines finitely many affinely imbedded orbits. In particular, we consider closed K-orbits. They correspond to ϑ-stable Borel subalgebras. Thus we fix a ϑ-stable Borel subalgebra $\mathfrak{b}$. Then $\mathfrak{b}$ contains a ϑ-stable fundamental Cartan subalgebra $\mathfrak{h}$. Denote by x_0 a point in X representing $\mathfrak{b}$ and put $Y = K \cdot x_0$. Then Y can be identified with the flag variety of $\mathfrak{k}$, and therefore Y decomposes into finitely many Bruhat cells $C_{\mathfrak{k}}(w)$, where w runs over the Weyl group $W_{\mathfrak{k}}$ of $(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{h})$. Suppose that $\mathrm{Int}(\mathfrak{g})$-homogeneous line bundle $\mathcal{L}$ on X is defined by a linear form λ on $\mathfrak{h}$. Using the Cousin complex methods developed by Grothendieck [11], we can deduce the existence of the resolution

$$0 \to H^c_Y(X, \mathcal{L}) \to \cdots \to \bigoplus_{\ell(w)=p} H^{c+p}_{C_{\mathfrak{k}}(w)}(X, \mathcal{L}) \to \cdots \to H^n_{\{x_0\}}(X, \mathcal{L}) \to 0, \quad (0.1)$$

here $n = \dim X$, $c = \mathrm{codim}(Y, X)$ and w_0 is the longest element in $W_{\mathfrak{k}}$. By the above remark, the terms in the resolution are $(\mathcal{U}(\mathfrak{g}), B_{\mathfrak{k}})$-modules ($B_{\mathfrak{k}}$ is the stabilizer of x_0 in $\mathrm{Int}(\mathfrak{k})$).

On the other hand, starting from the pair $(\mathfrak{b}, \lambda)$ and using Enright's completion functor (see 3.1.) we obtain a family of $\mathcal{U}(\mathfrak{g})$-modules $\{C_w(M(\lambda))\,;\, w \in W_{\mathfrak{k}}\}$. Here

$$C_1(M(\lambda)) = M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

(ρ is a half sum of positive roots); and if $w \geq v$ in the Bruhat order in $W_{\mathfrak{k}}$, we have natural imbedding $C_v(M(\lambda)) \hookrightarrow C_w(M(\lambda))$. Therefore, we may put

$$E(M(\lambda)) = C_{w_0}(M(\lambda)) \Big/ \left(\sum_{w<w_0} C_w(M(\lambda)) \right).$$

Then we have the homological resolution of $E(M(\lambda))$ generalizing the Bernstein-Gelfand-Gelfand resolution of a finite dimensional module,

$$0 \to M(\lambda) \to \cdots \to \bigoplus_{\ell(w)=p} C_w(M(\lambda)) \to \cdots \to C_{w_0}(M(\lambda)) \to E(M(\lambda)) \to 0. \tag{0.2}$$

In [15], Zuckerman posed a problem (which he attributed to Phillip Trauber) of constructing a duality relating (0.1) and (0.2), and this served as the motivation for the present work.

As an illustration, we discuss the example $G_0 = \mathrm{SU}(2,1)$. In this case X is a three-dimensional projective variety and there are exactly three closed K-orbits, each of them being isomorphic to a projective line. We have $W_{\mathfrak{k}} = \{1, s_\alpha\}$, where α is a compact root and s_α the corresponding reflection. Thus $Y = \{x_0\} \cup \mathbb{A}^1$ is the Bruhat decomposition of Y ($\mathbb{A}^1$ denotes the affine line) and (0.2) specializes to

$$0 \to H^2_Y(X, \mathcal{L}) \to H^2_{\mathbb{A}^1}(X, \mathcal{L}) \to M(\lambda) \to 0\ . \tag{0.3}$$

For two orbits α is a simple root of the pair $(\mathfrak{g}, \mathfrak{h})$ (observe that in all three cases $\mathfrak{h}$ is a compact Cartan subalgebra), and (0.3) can be described explicitly (compare 2.7.). This is closely related to the construction of holomorphic discrete series. In the remaining case the situation is more complicated. Although we can determine $\mathfrak{h}$-module structure of $H^2_{\mathbb{A}^1}(X, \mathcal{L})$, this doesn't seem to be as useful as for the highest weight modules, since $\mathfrak{h}$-weight spaces are infinite-dimensional. On the other hand, the homological resolution in our example takes the form

$$0 \to M(\lambda) \to C_{s_\alpha}(M(\lambda)) \to E(M(\lambda)) \to 0\ . \tag{0.4}$$

Again, in the cases when α is simple, the previous remark applies. In the third case, for sufficiently negative λ, we construct $C_{s_\alpha}(M(\lambda))$ as follows. Notice first that for $\nu = \lambda - \rho + \frac{\alpha}{2}$ we have the inclusion of $\mathfrak{k}$-Verma modules $M_{\mathfrak{k}}(\nu) \hookrightarrow M_{\mathfrak{k}}(s_\alpha \nu)$. This induces further an inclusion

$$M(\lambda) \hookrightarrow V = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} M_{\mathfrak{k}}(s_\alpha \nu)/I,$$

where I is the kernel of the natural surjective map $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} M_{\mathfrak{k}}(\nu) \longrightarrow M(\lambda)$. One can show that

$$J = \{v \in V \mid y \cdot v = 0 \text{ for some } y \in \mathcal{U}(\mathfrak{k}_{-\alpha}) - \{0\}\}$$

(here $\mathfrak{k}_{-\alpha}$ is a root space in $\mathfrak{k}$) is $\mathcal{U}(\mathfrak{g})$-module. Finally, we put $C_{s_\alpha}(M(\lambda) = V/J$. It turns out that it is quite difficult to compare directly modules $H^2_{\mathbb{A}^1}(X, \mathcal{L})$ and $C_{s_\alpha}(M(\lambda))$.

To overcome these difficulties we have followed a suggestion of D. Miličić to consider costandard modules associated with Bruhat cells $C_{\mathfrak{k}}(w)$ (compare 1. for a definition). In this way we obtain a geometric realization of modules $C_w(M(\lambda))$ (5.4.). We use this to give a geometric proof for the existence of (0.2) and to construct a natural contravariant duality functor on certain full subcategory of a category of $\mathcal{U}(\mathfrak{g})$-modules that maps (0.1) into (0.2) (see 2.6. and 5.5.).

This paper is a part of authors thesis completed at the University of Utah, Salt Lake City. I want to express gratidude to my thesis adviser D. Miličić for the help and encouragement. I am also indebted to H. Hecht for showing me his unpublished manuscript [8] and many instructive conversations.

1. Preliminaries

For a smooth complex algebraic variety X we denote by $\mathcal{O}_X$, by $\mathcal{D}_X$ the sheaf of local differential operators on X and by ω_X the canonical sheaf on X (i. e. top exterior power of the cotangent sheaf on X). If $\mathcal{D}$ is a twisted sheaf of differential operators on X we denote by $\mathcal{D}^\circ$ the sheaf of rings opposite to $\mathcal{D}$, by $\mathcal{M}od(\mathcal{D})$ a category of left $\mathcal{D}$-modules that are quasicoherent as $\mathcal{O}_X$-modules and by $\mathcal{M}od_{hol}(\mathcal{D})$ the full subcategory in $\mathcal{M}od(\mathcal{D})$ consisting of holonomic $\mathcal{D}$-modules. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{D}^{\mathcal{L}}$ denotes a twist of $\mathcal{D}$ by $\mathcal{L}$. Let $f : Y \longrightarrow X$ be a morphism of smooth algebraic varieties. Put

$$\mathcal{D}_{Y\to X} = f^*(\mathcal{D}) = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}$$

and denote by $\mathcal{D}^f$ a sheaf of differential endomorphisms of $\mathcal{D}_{Y\to X}$ that commute with right $f^{-1}\mathcal{D}$ action. Then we view the inverse image and ($0^{th}-$)direct image as functors

$$f^+ : \mathcal{M}od(\mathcal{D}) \longrightarrow \mathcal{M}od(\mathcal{D}^f) \quad \textit{resp.} \quad R^0_+ f : \mathcal{M}od(\mathcal{D}) \longrightarrow \mathcal{M}od(\mathcal{D}^f).$$

Further we define the duality functor $\mathbb{D}_X : \mathcal{M}od_{hol}(\mathcal{D}) \longrightarrow \mathcal{M}od_{hol}((\mathcal{D}^\circ)^{\omega_X^{-1}})$ by the formula

$$\mathbb{D}_X(\mathcal{M}) = \mathcal{E}xt^{\dim X}(\mathcal{M}, \mathcal{D}) \otimes_{\mathcal{O}_X} \omega^{-1}.$$

Finally we define $R^0 f_! : \mathcal{M}od_{hol}(\mathcal{D}^f) \longrightarrow \mathcal{M}od_{hol}(\mathcal{D})$ by

$$R^0 f_! = \mathbb{D}_X \circ R^0 f_+ \circ \mathbb{D}_Y.$$

Next we want to summarize some well known results on the category of highest weight modules.

Let G be a complex reductive and connected algebraic group. Denote by $\mathfrak{g}$ a Lie algebra of G and by X a flag variety of Borel subalgebras in $\mathfrak{g}$. Fix a point $x_0 \in X$. Let $\mathfrak{b}$ be the corresponding Borel subalgebra in $\mathfrak{g}$ and B its normalizer in G. Let N be the unipotent radical of B and $\mathfrak{n}$ its Lie algebra. Choose further a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ and denote by Σ the root system of $(\mathfrak{g}, \mathfrak{h})$ and by Σ^+ a positive subsystem in Σ determined by $\mathfrak{b}$. We shall denote by Π the set of simple roots in Σ^+, by W the Weyl group of Σ and by ρ the half-sum of positive roots. We also fix a W-invariant positive bilinear form $(.,.)$ on $\mathfrak{h}^*$ (and dually on $\mathfrak{h}$). Then s_α denotes a reflection with respect to $\alpha \in \Sigma$. Further, $\ell(w)$ will denote the length of $w \in W$, and $P(\Sigma)$ the group of integral weights in $\mathfrak{h}^*$.

Let $\mathcal{Z}(\mathfrak{g})$ be a center of $\mathcal{U}(\mathfrak{g})$. Given $\lambda \in \mathfrak{h}^*$, χ_λ will denote the character of $\mathcal{Z}(\mathfrak{g})$ determined by $\lambda + \rho$. It is well known that χ_λ depends only on the Weyl group orbit $\theta = W \cdot \lambda$. Therefore we put $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})/\ker\chi_\lambda \cdot \mathcal{U}(\mathfrak{g})$.Denote by $\mathfrak{M}(\mathcal{U}_\theta)$ the category of (left) $\mathcal{U}_\theta$-modules.

Any linear form $\lambda \in \mathfrak{h}^*$ defines a G-homogeneous twisted sheaf of differential operators $\mathcal{D}_{X,\lambda+\rho}$ on X [9]. We shall denote it by $\mathcal{D}_\lambda$. Recall that $\Gamma(X, \mathcal{D}_\lambda) \cong \mathcal{U}_\theta$ [1] and therefore we may introduce the localization functor

$$\Delta_\lambda : \mathcal{M}od(\mathcal{U}_\theta) \to \mathcal{M}od(\mathcal{D}_\lambda), \quad \Delta_\lambda(M) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} M.$$

A finitely generated $\mathcal{U}(\mathfrak{g})$-module M is said to be a highest weight module (with respect to $\mathfrak{b}$) if $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{b}) \cdot m < \infty$ for any $m \in M$. Denote by $\mathcal{M}od_{fg}(\mathcal{U}_\theta, N)$ a category of finitely generated $(\mathcal{U}_\theta, N)$-modules. Then the highest weight modules with infinitesimal character χ_λ are precisely the modules from $\mathcal{M}od_{fg}(\mathcal{U}_\theta, N)$ [12]. Given highest weight module M we denote by $M\check{\ }$ its contravariant dual. It is known that $\check{\ }$ preserves $\mathcal{M}od_{fg}(\mathcal{U}_\theta, N)$.

The important example of highest weight module is furnished by the Verma modules. Recall that Verma module with highest weight $\lambda - \rho$ is defined by

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}\ .$$

Put $I(\lambda) = M(\lambda)\check{\ }$ and denote by $L(\lambda)$ the unique irreducible quotient of $M(\lambda)$. Observe that $M(\lambda)$, $I(\lambda) \in \mathcal{M}od_{fg}(\mathcal{U}_\theta, N)$. Let $\mathcal{M}od_{coh}(\mathcal{D}_\lambda, N)$ be the category of coherent $\mathcal{D}_\lambda$-modules with compatible N-action [9]. Then we have $\Delta_\lambda(M(\lambda))$, $\Delta_\lambda(I(\lambda)) \in \mathfrak{M}_{coh}(\mathcal{D}_\lambda, N)$. In fact, these modules can be identified with standard modules associated with Bruhat cells on X. To be more precise, recall that the action of N on X induces Bruhat decomposition $X = \bigcup_{w \in W} C(w)$ of the flag variety X, here $C(w) = Nwx_0 \cong \mathbb{C}^{l(w)}$. Let $i_w : C(w) \to X$ be the natural inclusion. Then the only N-homogeneous irreducible $\mathcal{D}_\lambda^{i_w}$-connection on $C(w)$ is $\mathcal{O}_{C(w)}$. Therefore the only standard resp. costandard $\mathcal{D}_\lambda$-module associated with $C(w)$ is

$$\mathcal{I}(w, \lambda) = R^0 i_{w,+}(\mathcal{O}_{C(w)}) \text{ resp. } \mathcal{M}(w, \lambda) = R^0 i_{w,!}(\mathcal{O}_{C(w)}).$$

Let $\mathcal{L}(w, \lambda)$ be the unique irreducible submodule (quotient) of $\mathcal{I}(w, \lambda)$ ($\mathcal{M}(w, \lambda)$).

Recall that $\lambda \in \mathfrak{h}^*$ is said to be antidominant if $\check{\alpha} \notin \mathbb{N}$ for any $\alpha \in \Sigma$.

Proposition 1.1. *Suppose $\lambda \in \mathfrak{h}^*$ is antidominant. Then*

(i) $\Gamma(X, \mathcal{I}(w, \lambda)) = I(w\lambda)$

(ii) $\Gamma(X, \mathcal{M}(w, \lambda)) = M(w\lambda)$

(iii) If in addition λ is regular, then $\Gamma(X, \mathcal{L}(w, \lambda)) = L(w\lambda)$.

The proof can be found in [12].

Notice that each $\mu \in P(\Sigma)$ defines an invertible sheaf $\mathcal{O}_X(\mu)$ on X equipped with $\mathcal{D}_{\mu-\rho}$-action. In fact, denote by Z the center of G and put $G_1 = G/Z$. Let $\widetilde{G}_1$ be a universal cover of G_1. Notice that $\widetilde{G}_1$ is an algebraic group. Then we construct $\mathcal{O}_X(\mu)$ as $\widetilde{G}_1$-homogeneous invertible sheaf on X. The differential of $\widetilde{G}_1$-action determines $\mathcal{U}(\mathfrak{g}_1)$-module structure on $\mathcal{O}_X(\mu)$ ($\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$). Then we can extend this to $\mathcal{U}(\mathfrak{g})$-module structure so that $\mathcal{O}_X(\mu)$ becomes $\mathcal{D}_{\mu-\rho}$-module.

Next we review some facts related to local cohomology groups. Given a closed subset $Z \subseteq X$ and a sheaf $\mathcal{F}$ of abelian groups on X denote by $\mathit{\Gamma}_Z(\mathcal{F})$ a subsheaf of sections of $\mathcal{F}$ that are supported on Z. If $Z_2 \subseteq Z_1$ are closed, put

$$\mathit{\Gamma}_{Z_1/Z_2}(\mathcal{F}) = \mathit{\Gamma}_{Z_1}(\mathcal{F})/\mathit{\Gamma}_{Z_2}(\mathcal{F}).$$

The following lemma summarizes Cousin complex techniques [11] that we shall use.

Lemma 1.2.. *Let $X \supseteq Z_0 \supseteq Z_1 \cdots \supseteq Z_{n+1} = \emptyset$ be a finite filtration of X by closed subsets. Then there is a spectral sequence with the first term*

$$E_1^{pq} = \mathcal{H}^{p+q}_{Z_p/Z_{p+1}}(\mathcal{F})$$

converging to $\mathcal{H}^{\cdot}_{Z_0}(\mathcal{F})$. The first differential is given by $d_1^{pq} : E_1^{pq} \longrightarrow E_1^{p+1,q}$. Further, for a fixed $c \in \mathbb{Z}_+$, the natural map $\mathcal{H}^c_{Z_0}(\mathcal{F}) \longrightarrow \mathcal{H}^c_{Z_0/Z_1}(\mathcal{F})$ and differentials d_1^{pc} make a sequence

$$0 \longrightarrow \mathcal{H}^c_{Z_0}(\mathcal{F}) \longrightarrow \mathcal{H}^c_{Z_0/Z_1}(\mathcal{F}) \longrightarrow \ldots \longrightarrow \mathcal{H}^{c+i}_{Z_i/Z_{i+1}}(\mathcal{F}) \longrightarrow \ldots \tag{1.2.1}$$

into a complex. If

$$\mathcal{H}^{c+i}_{Z_0}(\mathcal{F}) = 0 \textit{ for } i \neq 0 \textit{ and } \mathcal{H}^{c+i}_{Z_p/Z_{p+1}}(\mathcal{F}) = 0 \textit{ for } i \neq p$$

then (1.2.1) becomes a resolution of $\mathcal{H}^c_{Z_0}(\mathcal{F})$.

Proof. Let $\mathcal{F} \longrightarrow \mathcal{C}^{\cdot}(\mathcal{F})$ be a canonical resolution (see [7] for a definition). Then the filtered complex

$$\mathit{\Gamma}_{Z_0}(\mathcal{C}^{\cdot}(\mathcal{F})) \supseteq \mathit{\Gamma}_{Z_1}(\mathcal{C}^{\cdot}(\mathcal{F})) \supseteq \mathit{\Gamma}_{Z_1}(\mathcal{C}^{\cdot}(\mathcal{F})) \supseteq \ldots$$

defines the desired spectral sequence. The fact that (1.2.1) is a complex is now evident. To prove the second statement observe that by assumption $E_1^{pq} = 0$ if

$q \neq c$ and this implies $E_2^{pc} = E_\infty^{pc}$ and $\mathcal{H}^n_{Z_0}(\mathcal{F}) = E_2^{n-c,c}$. Since $\mathcal{H}^n_{Z_0}(\mathcal{F}) = 0$ for $n \neq c$ we obtain

$$E_2^{pc} = 0 \text{ for } p \neq 0 \text{ and } E_2^{0c} = \ker(E_1^{0c} \longrightarrow E_1^{1c}) = \mathcal{H}^c_{Z_0}(\mathcal{F})$$

proving the exactness of (1.2.1). □

Denote by $\bar{Y}$ a closure of $Y \subseteq X$ and write $\partial Y = \overline{Y} - Y$. Fix a simple root $\alpha \in \Pi$ and consider the flag variety X_α of parabolic subalgebras of type α and a natural projection $p_\alpha : X \longrightarrow X_\alpha$. Recall that p_α is locally trivial fibration with fibres isomorphic to a projective line $\mathbb{P}^1$. Pick $v, w \in W$ such that $w = s_\alpha v$ and $\ell(w) = \ell(v) + 1$. Then $C(v) \subseteq \overline{C(w)}$ for the corresponding Bruhat cells. In that case we usually write $w \geq v$. Put $Z = p_\alpha^{-1} p_\alpha(C(v))$. Then $Z = C(v) \cup C(w)$ and $C(v)$ is closed (of codimension 1) and $C(w)$ open subvariety of Z. Moreover, local triviality of p_α implies that Z is a smooth affinely imbedded subvariety of X. Assume now that $\mathcal{L} = \mathcal{O}_X(\lambda + \rho)$ for some $\lambda \in P(\Sigma)$. Then we have the exact sequence of $\mathcal{D}_\lambda$-modules

$$0 \longrightarrow \mathit{\Gamma}_{\overline{C(w)}/\partial C(w)}(\mathcal{L}) \longrightarrow \mathit{\Gamma}_{\overline{C(w)}/\partial Z}(\mathcal{L}) \longrightarrow \mathit{\Gamma}_{\partial C(w)/\partial Z}(\mathcal{L}) \longrightarrow 0.$$

Notice that $\partial C(w) - \partial Z = C(v)$ and $\overline{C(w)} - \partial Z = C(w) \cup C(v)$. Thus we obtain from the corresponding long exact sequence (using $\mathcal{I}(w, \lambda) \cong j_{w,*} \mathcal{H}^{n-l(w)}_{c(w)}(j_w^* \mathcal{L})$, where $j_w : X - C(w) \longrightarrow X$, $n = \dim X$ [2]) a natural surjective map $\mathcal{I}(w, \lambda) \longrightarrow \mathcal{I}(v, \lambda)$. Dually, there is a natural injective map $\mathcal{M}(v, \lambda) \longrightarrow \mathcal{M}(w, \lambda)$. We summarize this discussion in the following lemma:

Lemma 1.3. *Let $v, w \in W$ be such that $w = s_\alpha v$ ($\alpha \in \Pi$) and $\ell(w) = \ell(v) + 1$. Then for $\lambda \in P(\Sigma)$ we have a natural surjective (injective) map*

$$\mathcal{I}(w, \lambda) \longrightarrow \mathcal{I}(v, \lambda) \quad (\mathcal{M}(v, \lambda) \longrightarrow \mathcal{M}(w, \lambda))$$

For $p \in \mathbb{Z}_+$ denote by Z_p the union of Bruhat cells whose codimension in X is $\geq p$. Then we have

$$Z_p - Z_{p+1} = \bigcup_{\ell(w) = \ell(w_0) - p} C(w)$$

where w_0 denotes the longest element in W.

Proposition 1.4. *Let $\lambda \in P(\Sigma)$ and $\mathcal{L} = \mathcal{O}_X(\lambda+\rho)$. Then the following sequences are exact:*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{I}(w_0,\lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=p} \mathcal{I}(w,\lambda) \longrightarrow \cdots \longrightarrow \mathcal{I}(1,\lambda) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{M}(1,\lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=p} \mathcal{M}(w,\lambda) \longrightarrow \cdots \longrightarrow \mathcal{M}(w_0,\lambda) \longrightarrow \mathcal{L} \longrightarrow 0.$$

In both cases the differentials are induced by the natural maps described in 1.3.

Proof. Notice that $\mathcal{H}^j_{Z_p/Z_{p+1}}(\mathcal{L}) = 0$ if $j \neq p$ and

$$\mathcal{H}^p_{Z_p/Z_{p+1}}(\mathcal{L}) = \bigoplus_{\ell(w)=\ell(w_0)-p} \mathcal{I}(w,\lambda).$$

Thus we may apply 1.2. to obtain the first sequence. Moreover, for $w, v \in W$ such that $w \geq v$ and $\ell(w) = \ell(w_0) - p$, the map $\mathcal{I}(w,\lambda) \longrightarrow \mathcal{I}(v,\lambda)$ induced by

$$\mathcal{H}^p_{Z_p/Z_{p+1}}(\mathcal{L}) \longrightarrow \mathcal{H}^{p+1}_{Z_{p+1}/Z_{p+2}}(\mathcal{L})$$

coincides with the natural surjection $\mathcal{I}(w,\lambda) \longrightarrow \mathcal{I}(v,\lambda)$ constructed in 1.3.

The second sequence is obtained from the first one by observing that $\mathbb{D}_{C(w)}(\mathcal{O}_{C(w)}) \cong \mathcal{O}_{C(w)}$ as N-homogeneous $\mathcal{D}^{i_w}_{-\lambda-2\rho}$-connections and by applying exact functor

$$\mathbb{D}_X : \mathcal{M}od_{hol}(\mathcal{D}_{-\lambda-2\rho}) \longrightarrow \mathcal{M}od_{hol}(\mathcal{D}_\lambda) \quad \square$$

2. Two resolutions

Let G_0 be a real semisimple connected Lie group with finite center and $\mathfrak{g}$ a complexified Lie algebra of G_0. Denote by G a simply connected complex group with Lie algebra $\mathfrak{g}$. The choice of a maximal compact subgroup in G_0 determines an involution $\vartheta : \mathfrak{g} \longrightarrow \mathfrak{g}$. We denote by the same symbol the induced involution on G. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the corresponding decomposition into ± 1-eigenspaces of ϑ. Denote by K the set of fixed points of ϑ in G. Thus K is a connected reductive algebraic group. In the following we use the notation established in 1.

We want to recall how to describe closed K-orbits in X. First of all, observe that we can always find a ϑ-stable Cartan subalgebra in $\mathfrak{b}$ and any two Cartan subalgebras with this property are conjugate by an element from $K \cap N$ ([10]). Thus we may assume that the Cartan subalgebra $\mathfrak{h}$ from 1. is ϑ-stable. We write

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$$

where $\mathfrak{h}_1 = \mathfrak{k} \cap \mathfrak{h}$ and $\mathfrak{h}_2 = \mathfrak{p} \cap \mathfrak{h}$. The involution ϑ induces an involution of the root system Σ to be denoted also by ϑ. As usual we denote by Σ_{CI},$\Sigma_{\mathbb{R}}$ and $\Sigma_{\mathbb{C}}$ the sets of compact imaginary, real and complex roots respectively in Σ. The following result is well known [10].

Lemma 2.1. *Let $x_0 \in X$ be a point corresponding to $\mathfrak{b}$. Then the following statements are equivalent:*
(i) The orbit $K \cdot x_0$ is closed;
(ii) $\mathfrak{k} \cap \mathfrak{b}$ is a Borel subalgebra in $\mathfrak{k}$;
(iii) Σ^+ is ϑ-stable.
If these conditions are satisfied, $\mathfrak{h}$ is a fundamental Cartan subalgebra in $\mathfrak{g}$ and therefore $\Sigma_{\mathbb{R}} = \emptyset$.

In the sequel we assume that the conditions from 2.1. are fulfilled. Denote by $\Sigma_{\mathfrak{k}}$ the root system of $(\mathfrak{k}, \mathfrak{h}_1)$ and by $\Sigma_{\mathfrak{k}}^+$ a positive subsystem determined by $\mathfrak{b} \cap \mathfrak{k}$. Let $W_{\mathfrak{k}}$ be a Weyl group of $\Sigma_{\mathfrak{k}}$ and $\rho_{\mathfrak{k}}$ a half sum of positive roots. Also $\mathbb{L}_{\mathfrak{k}}$ is defined analogously as $\mathbb{L}$. For $\lambda \in \mathfrak{h}^*$, we denote $\lambda_1 = \lambda | \mathfrak{h}_1$. We may assume that the invariant form $(\cdot, \cdot)$ from 1. is ϑ-invariant.

Lemma 2.2. *(i) If $\lambda \in P(\Sigma)$, then $\lambda_1 \in P(\Sigma_{\mathfrak{k}})$.*
(ii) If $\lambda \in P(\Sigma)$ is Σ^+-antidominant, then λ_1 is $\Sigma_{\mathfrak{k}}^+$-antidominant.

Proof. We may identify $\mathfrak{h}_1^*$ with the subspace $\{\lambda \in \mathfrak{h}^* : \vartheta\lambda = \lambda\}$. For $\alpha_1 \in \Sigma_{\mathfrak{k}}$ we have to find $\check{\alpha_1} \in (\mathfrak{h}_1^*)^* = \mathfrak{h}_1$ such that $\check{\alpha_1}(\alpha) = 2$. Notice that $\check{\alpha_1}$ is unique modulo center of $\mathfrak{k}$. We distinguish several cases.

(a) If $\alpha \in \Sigma_{CI}$, then $\alpha\check{} \in \mathfrak{h}_1^*$ and $\alpha\check{}(\alpha) = \alpha\check{}(\alpha_1)$. Thus we may put $\check{\alpha_1} = \alpha\check{}$.

For the remaining cases observe that $(\alpha, \vartheta\alpha) \leq 0$ if $\alpha \in \Sigma_{\mathbb{C}}$ (since $\alpha - \vartheta\alpha$ is not in Σ) and $(\alpha, \alpha) = (\vartheta\alpha, \vartheta\alpha)$. We conclude that $(\alpha\check{}, \vartheta\alpha) = 0, -1$.

(b) If $(\alpha\check{}, \vartheta\alpha) = 0$ we may put $\check{\alpha_1} = \alpha\check{} + (\vartheta\alpha)\check{}$.

(c) If $(\alpha\check{}, \vartheta\alpha) = -1$ we may put $\alpha_1\check{} = 2(\alpha\check{} + (\vartheta\alpha)\check{})$. Choose $\lambda \in P(\Sigma)$. Examining all three cases above we deduce that $\check{\alpha_1}(\lambda_1) \in \mathbb{Z}$ for any $\alpha_1 \in \Sigma_{\mathfrak{k}}$, as desired. $\square$

Put $Y = K \cdot x_0$ and let $Y \hookrightarrow X$ be the imbedding. We view Y as a flag variety of $\mathfrak{k}$ and denote by $\widetilde{\mathcal{D}}_\nu$ a K-homogeneous twisted sheaf of differential operators on Y determined by a linear form $\nu + \rho_\mathfrak{k}$ on $\mathfrak{h}_1$. Then we have

$$\mathcal{D}^i_\lambda = \widetilde{\mathcal{D}}_\nu \quad \text{where} \quad \nu = \lambda_1 + \rho_1 - \rho_\mathfrak{k}.$$

We assume in the following that $\lambda \in \mathfrak{h}^*$ is such that $\lambda_1 \in P(\Sigma_\mathfrak{k})$. Since $\tau = \mathcal{O}_Y(\nu + \rho_\mathfrak{k})$ is an irreducible $\widetilde{\mathcal{D}}_\nu$-connection by Kashiwara's theorem $\mathcal{I}(Y, \tau) = R^\circ i_+(\tau)$ is irreducible $\mathcal{D}_\lambda$-module.

The action of $K \cap N$ on Y induces the Bruhat decomposition on Y. Denote by $C_\mathfrak{k}(w)$ the Bruhat cell associated with $w \in W_\mathfrak{k}$ and by $i_w : C_\mathfrak{k}(w) \to Y$ the corresponding imbedding. Let w_0 be the longest element in $W_\mathfrak{k}$. For $w \in W_\mathfrak{k}$ we put

$$\mathcal{I}_\mathfrak{k}(w, \nu) = R^0 i_{w,+}(\mathcal{O}_{C_\mathfrak{k}(w)}), \quad \mathcal{M}_\mathfrak{k}(w, \nu) = R^0 i_{w,!}(\mathcal{O}_{C_\mathfrak{k}(w)}),$$

$$\mathcal{T}(w, \lambda) = R^0 (i \circ i_w)_+(\mathcal{O}_{C_\mathfrak{k}(w)}), \quad \mathcal{Z}(w, \lambda) = R^0 (i \circ i_w)_!(\mathcal{O}_{C_\mathfrak{k}(w)}),$$

$$L(Y, \tau) = \Gamma(X, \mathcal{I}(Y, \tau)), \quad T(w, \lambda) = \Gamma(X, \mathcal{T}(w, \lambda)), \quad Z(w, \lambda) = \Gamma(X, \mathcal{Z}(w, \lambda)).$$

Here we view $\mathcal{O}_{C_\mathfrak{k}(w)}$ as $\mathcal{D}^{i \circ i_w}_\lambda$-module. Then we have

$$\mathcal{T}(w, \lambda) = R^0 i_+(\mathcal{I}_\mathfrak{k}(w, \nu)), \quad \mathcal{Z}(w, \lambda) = R^0 i_!(\mathcal{M}_\mathfrak{k}(w, \nu)).$$

Notice that in the second equality we have used that i is a proper map. We conclude from 1.3. that for $w, v \in W_\mathfrak{k}$ such that $w \geq v$ there is a natural surjective (resp. injective) map

$$\mathcal{T}(w, \lambda) \to \mathcal{T}(v, \lambda) \quad (\text{ resp. } \mathcal{Z}(v, \lambda) \to \mathcal{Z}(w, \lambda)).$$

The following lemma is easily proved:

Lemma 2.3. *For $\mu \in P(\Sigma)$ we have*
(i) $\mathcal{T}(w, \lambda)(\mu) = \mathcal{T}(w, \lambda + \mu)$
(ii) $\mathcal{Z}(w, \lambda)(\mu) = \mathcal{Z}(w, \lambda + \mu)$.

Proposition 2.4. *Suppose $\lambda \in \mathfrak{h}^*$ is antidominant and such that $\lambda_1 \in P(\Sigma_\mathfrak{k})$. Let $\tau = \mathcal{O}_Y(\nu + \rho_\mathfrak{k})$ $(\nu = \lambda_1 + \rho_1 - \rho_\mathfrak{k})$. Then*

$$0 \to L(Y, \tau) \to T(w_0, \lambda) \to \cdots \to \bigoplus_{\ell(w) = p} T(w, \lambda) \to \cdots \to T(1, \lambda) \to 0$$

and

$$0 \to Z(1,\lambda) \to \cdots \to \bigoplus_{\ell(w)=p} Z(w,\lambda) \to \cdots \to Z(w_0,\lambda) \to L(Y,\tau) \to 0$$

are exact sequences of $(\mathcal{U}_\theta, K \cap N)$-modules of finite length.

Proof. First we apply 1.5. in the $\mathfrak{k}$-setting. Then we act on the obtained sequences by the exact functor $R^0 i_+ : \mathfrak{M}(\widetilde{\mathcal{D}}_\nu) \to \mathfrak{M}(\mathcal{D}_\lambda)$. Finally, the result follows by taking global sections, since the higher cohomologies of $\mathcal{D}_\lambda$- modules for λ antidominant vanish. $\square$

Now we shall construct a duality operation on a certain full subcategory of $\mathfrak{M}(\mathcal{U}_\theta)$ that relates exact sequences from 2.4. Let λ be antidominant and ν and τ be as before. Then a twist by τ^{-1} induces an equivalence of categories

$$\mathfrak{M}_{hol}(\widetilde{\mathcal{D}}_\nu, K \cap N) \to \mathfrak{M}_{hol}(\mathcal{D}_Y, K \cap N).$$

Using this, we transfer the duality functor

$$\mathbb{D}_Y : \mathfrak{M}_{hol}(\mathcal{D}_Y, K \cap N) \to \mathfrak{M}_{hol}(\mathcal{D}_Y, K \cap N)$$

from $\mathfrak{M}_{hol}(\mathcal{D}_Y, K \cap N)$ to $\mathfrak{M}_{hol}(\widetilde{\mathcal{D}}_\nu, K \cap N)$. Denote by

$$\check{} : \mathfrak{M}_{hol}(\widetilde{\mathcal{D}}_\nu, K \cap N) \to \mathfrak{M}_{hol}(\widetilde{\mathcal{D}}_\nu, K \cap N)$$

the duality functor obtained in this way. Let $\mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N)$ be the full subcategory of $\mathfrak{M}_{hol}(\mathcal{D}_\lambda, K \cap N)$ consisting of modules supported in Y. By Kashiwara's theorem we extend $\check{}$ to $\mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N)$. Thus we obtain a contravariant exact functor

$$\check{} : \mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N) \to \mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N)$$

such that $(\mathcal{M}\check{})\check{} \cong \mathcal{M}$ for $\mathcal{M} \in \mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N)$.

For $\mathcal{M} \in \mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N)$ denote by $\mathcal{M}_0$ the largest $(\mathcal{D}_\lambda, K \cap N)$-submodule with trivial global sections. Put

$$\mathfrak{M}_Y(\mathcal{U}_\theta, K \cap N) = \{M \in \mathfrak{M}_{fg}(\mathcal{U}_\theta, K \cap N) : \Delta_\lambda(M)/\Delta_\lambda(M)_0 \in \mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N)\}.$$

Notice that if $\alpha : M \to N$ is a map of $(\mathcal{U}_\theta, K \cap N)$-modules then $\Delta_\lambda(\alpha)(\Delta_\lambda(M)_0) \subseteq \Delta_\lambda(N)_0$. It follows that $\mathfrak{M}_Y(\mathcal{U}_\theta, K \cap N)$ is a full subcategory of $\mathfrak{M}_{fg}(\mathcal{U}_\theta, K \cap N)$. For $M \in \mathfrak{M}_Y(\mathcal{U}_\theta, K \cap N)$ set

$$M\tilde{} = \Gamma(X, (\Delta_\lambda(M)/\Delta_\lambda(M)_0)\check{}).$$

We have to show that $M\tilde{} \in \mathfrak{M}_Y(\mathcal{U}_\theta, K \cap N)$.

Lemma 2.5. *Suppose $\mathcal{M} \in \mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N)$ is such that $\Gamma(X, \mathcal{M}) = 0$. Then $\Gamma(X, \mathcal{M}\check{}) = 0$ as well.*

Proof. If $\mathcal{N}$ is a composition factor of $\mathcal{M}$ then $\Gamma(X, \mathcal{N}) = 0$. Further $\mathcal{M}$ and $\mathcal{M}\check{}$ have isomorphic composition series and therefore the induction on the length of $\mathcal{M}$ yields $\Gamma(X, \mathcal{M}\check{}) = 0$. $\square$

Now we can show $M\tilde{} \in \mathfrak{M}_Y(\mathcal{U}_\theta, K \cap N)$. Put $\mathcal{M} = (\Delta_\lambda(M)/\Delta_\lambda(M)_0)\check{}$. Then there is a natural map $\varphi : \Delta_\lambda(M\tilde{}) \to \mathcal{M}$. Let $\mathcal{K} = \ker \varphi$, $\mathcal{I} = \operatorname{im} \varphi$, $\mathcal{C} = \operatorname{coker} \varphi$. Then $\mathcal{K} \subset \Delta_\lambda(M\tilde{})_0$ and $\Delta_\lambda(M\tilde{})/\mathcal{K} \in \mathfrak{M}_Y(\mathcal{D}_\lambda, K \cap N)$ imply $M\tilde{} \in \mathfrak{M}_Y(\mathcal{U}_\theta, K \cap N)$. Further we show that $(M\tilde{})\tilde{} \cong M$. In fact, from the sequence

$$0 \to \mathcal{I} \to \mathcal{M} \to \mathcal{C} \to 0$$

we obtain

$$0 \to \mathcal{C}\check{} \to \Delta_\lambda(M)/\Delta_\lambda(M)_0 \to \mathcal{I}\check{} \to 0.$$

Using 2.5. we deduce $(\Delta_\lambda(M\tilde{})/\mathcal{K})\check{} = \Delta_\lambda(M)/\Delta_\lambda(M)_0$. On the other hand, dualizing

$$0 \to \Delta_\lambda(M\tilde{})_0/\mathcal{K} \to \Delta_\lambda(M\tilde{})/\mathcal{K} \to \Delta_\lambda(M\tilde{})/\Delta_\lambda(M\tilde{})_0 \to 0$$

and applying 2.5. again we conclude

$$\begin{aligned}(M\tilde{})\tilde{} = \Gamma(X, (\Delta_\lambda(M\tilde{})/\Delta_\lambda(M\tilde{})_0)\check{}) &= \Gamma(X, (\Delta_\lambda(M\tilde{})/\mathcal{K})\check{}) \\ &= \Gamma(X, \Delta_\lambda(M)/\Delta_\lambda(M)_0) = M.\end{aligned}$$

Using similar arguments we show further that

$$T(w, \lambda)\tilde{} = Z(w, \lambda), \quad L(Y, \tau)\tilde{} = L(Y, \tau).$$

Moreover, for $w, v \in W_{\mathfrak{k}}$ such that $w \geq v$ a natural surjection $T(w, \lambda) \to T(v, \lambda)$ is transferred under $\tilde{}$ into a natural injection $Z(v, \lambda) \to Z(w, \lambda)$. We summarize our discussion in the following proposition.

Proposition 2.6. *Suppose $\lambda \in \mathfrak{h}$ is antidominant. Then there exists a full subcategory $\mathfrak{M}_Y(\mathcal{U}_\theta, K \cap N)$ of $\mathfrak{M}_{fg}(\mathcal{U}_\theta, K \cap N)$ containing modules $T(w, \lambda)$, $Z(w, \lambda)$, $L(Y, \tau)$ and a contravariant involutive functor $\tilde{}$ on $\mathfrak{M}_Y(\mathcal{U}_\theta, K \cap N)$ transforming the exact sequences from 2.4. into each other.*

Example 2.7. Assume that $\operatorname{rank} G_0 = \operatorname{rank} K_0$ and that symmetric space G_0/K_0 has a Hermitian structure. Let $\mathfrak{h}_0$ be the maximal abelian subspace in $\mathfrak{k}_0$ and $\mathfrak{h}$ its comlexification. Then $\mathfrak{h}$ is a common Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{g}$. Put

$\Sigma_{NCI} = \{\alpha \in \Sigma : \mathfrak{g}_\alpha \cap \mathfrak{k} = \emptyset\}$. It is known ([13]) that the the choice of a positive system $\Sigma^+ \subset \Sigma$ with the property

$$\alpha,\ \beta \in \Sigma^+ \cap \Sigma_{NCI} \quad \Rightarrow \quad \alpha + \beta \notin \Sigma^+$$

determines the invariant complex structure on G_0/K_0. In particular, in this case $\Pi_\mathfrak{k} \subset \Pi$ and $C_\mathfrak{k}(w) = C(w)$ for $w \in W_\mathfrak{k}$. Notice that the second statement follows from the first by induction on $\ell(w)$. In the present case the discrete series (holomorphic discrete series) can be realiezed on the global sections of certain holomorphic vector bundles on G_0/K_0. Their infinitesimal description is particulary simple and we recall it bellow following [14]. Choose λ antidominant regular and such that $\lambda + \rho \in \mathbb{L}_\mathfrak{k}$. Then $M(v\lambda) \hookrightarrow M(w\lambda)$ if $w \geq v$, $v,\ w \in W_\mathfrak{k}$ and hence we may put

$$D(\lambda) = M(w_0\lambda)) \Big/ \left(\sum_{w < w_0} M(w\lambda) \right).$$

The module $D(\lambda)$ is closely related to the holomorphic discrete series. More details about the identification can be found in [14]. Observe that by our choices $\mathfrak{q} = \mathfrak{k} + \mathfrak{b} \cap \mathfrak{p}$ is a parabolic subalgebra in $\mathfrak{g}$. Thus if we view Bernstein-Gelfand-Gelfand resolution

$$0 \longrightarrow M_\mathfrak{k}(\nu) \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=p} M_\mathfrak{k}(w\nu) \longrightarrow \cdots \longrightarrow M_\mathfrak{k}(w_0\nu) \longrightarrow L_\mathfrak{k}(w_0\nu) \longrightarrow 0 \ \ (\nu = \lambda - \rho + \rho_\mathfrak{k})$$

as the resolution of $\mathfrak{q}$-modules by letting $\mathfrak{b} \cap \mathfrak{p}$ act trivially, we obtain after tensoring with $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} -$ a resolution of $D(\lambda)$ by $\mathfrak{g}$-modules

$$0 \longrightarrow M(\lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=p} M(w\lambda) \longrightarrow \cdots \longrightarrow M(w_0\lambda) \longrightarrow D(\lambda) \longrightarrow 0.$$

On the other hand since $C_\mathfrak{k}(w) = C(w)$ we have $Z(w, \lambda) = M(w\lambda)$ and $T(w, \lambda) = I(w\lambda)$. Moreover, the duality $^\sim$ acts on (2.14) as contragredient duality (2.1) and maps it into

$$0 \longrightarrow D(\lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=p} I(w\lambda) \longrightarrow \cdots \longrightarrow I(w_0\lambda) \longrightarrow I(\lambda) \longrightarrow 0.$$

3. A resolution of $\mathcal{D}_{Y\to X}$

The aim of this section is to produce a K-equivariant resolution of $\mathcal{D}_{Y\to X}$ in the case when Y is a K-orbit on a flag variety X. I learned about this result from H. Hecht. The approach taken up here is different then the original one in [8].

First of all we refer to [9] for a definition and construction of a homogeneous twisted sheaf of differential operators.

Put $Z = G/(K\cap B)$, $Y = K/(K\cap B)$ and let $j : Y \to Z$ be the inclusion. Choose $(K\cap B)$-invariant form $\mu \in (\mathfrak{k}\cap\mathfrak{b})^*$. Then μ determines a homogeneous twisted sheaf of differential operators $\mathcal{D}_{Z,\mu}$ on Z. Suppose $\mathcal{V}$ is a G-equivariant locally free sheaf on Z. Then we form $\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V}$, where we use right $\mathcal{O}_Z$-module structure on $\mathcal{D}_{Z,\mu}$ to form tensor product. Notice that $\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V}$ is a left $\mathcal{D}_{Z,\mu}$-module for the left multiplication on the first factor. Further, the differentiation of G-action on $\mathcal{V}$ defines left $\mathcal{U}(\mathfrak{g})$-module structure on $\mathcal{V}$. Now, if $d\in\mathcal{D}_{Z,\mu},\ v\in\mathcal{V}$ and $\xi\in\mathfrak{g}$, we put

$$(d\otimes v)\cdot\xi = d\xi\otimes v - d\otimes\xi\cdot v.$$

It is easy to check that this determines the structure of a right $\mathcal{U}(\mathfrak{g})$-module on $\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V}$. The previous discussion shows that $\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V}$ is $(\mathcal{D}_{Z,\mu},\mathcal{U}(\mathfrak{g}))$-bimodule and this two module structures commute.

Let $\mathcal{D}_{Y,\mu}$ be a K-homogeneous twisted sheaf of differential operators on Y determined by μ. In other words, $\mathcal{D}_{Y,\mu} = \mathcal{D}^{j}_{Z,\mu}$. We want to explain how $(\mathcal{D}_{Z,\mu},\mathcal{U}(\mathfrak{g}))$-bimodule structure on $\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V}$ induces $(\mathcal{D}_{Y,\mu},\mathcal{U}(\mathfrak{g}))$-bimodule structure on $j^*(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$.

Notice that $j^{-1}(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$ is a left $j^{-1}\mathcal{O}_Z$-module and a right $\mathcal{U}(\mathfrak{g})$-module. For $f\in\mathcal{O}_Y$, $d\in j^{-1}(\mathcal{D}_{Z,\mu})$, $v\in j^{-1}(\mathcal{V})$, and $\xi\in\mathfrak{g}$, we have

$$f(g\circ j)\otimes(d\otimes v)\cdot\xi = f\otimes g\circ j((d\otimes v)\cdot\xi) = f\otimes((g\circ j)d\otimes v)\cdot\xi.$$

We conclude that

$$(f\otimes(d\otimes v))\cdot\xi = f\otimes(d\otimes v)\cdot\xi$$

gives well defined $\mathcal{U}(\mathfrak{g})$-action on $j^*(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$.

To define $\mathcal{D}_{Y,\mu}$-action on $j^*(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$ notice first that we have a map

$$j^{-1}(l) : j^{-1}(\mathfrak{g}^\circ_Z)\times j^{-1}(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V}) \to j^{-1}(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$$

($\mathfrak{g}^\circ_Z = \mathcal{O}_Z\otimes_{\mathbb{C}}\mathfrak{g}$) induced by the left multiplication on $\mathcal{D}_{Z,\mu}$. This induces the map

$$\alpha : j^*(\mathfrak{g}^\circ_Z)\times j^{-1}(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V}) \to j^*(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$$

defined by the formula

$$\alpha(f \otimes z, u) = f \otimes j^{-1}(l)(z, u), \quad f \in \mathcal{O}_Y,\ z \in j^{-1}(\mathfrak{g}_Z^\circ),\ u \in j^{-1}(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}).$$

Using inclusion $\mathfrak{k}_Y^\circ \longrightarrow j^*(\mathfrak{g}_Z^\circ)$ ($\mathfrak{k}_Y^\circ = \mathcal{O}_Y \otimes_{\mathbb{C}} \mathfrak{k}$) we define the map

$$\beta : \mathfrak{k}_Y^\circ \times \mathcal{O}_Y \times j^{-1}(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}) \longrightarrow j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}),$$

$$\beta(s, f, u) = \tau_1(s)f \otimes u + f\alpha(s, u) \quad s \in \mathfrak{k}_Y^\circ,\ f \in \mathcal{O}_Y,\ u \in j^{-1}(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}),$$

here $\tau_1 : \mathfrak{k}_Y^\circ \longrightarrow \mathcal{T}_Y$ is a homomorphism into the Lie algebra of vector fields on Y. It follows easily that β is $j^{-1}\mathcal{O}_Z$-linear in the last two terms and therefore it induces the action

$$\gamma : \mathfrak{k}_Y^\circ \times j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}) \longrightarrow j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}).$$

One checks easily that γ has the following properties:
(i) $\gamma([s, t], \cdot) = [\gamma(s, \cdot), \gamma(t, \cdot)]$;
(ii) $\gamma(fs, \cdot) = f\gamma(s, \cdot)$;
(iii) $[\gamma(s, \cdot), f] = \tau_1(s)f$;
here $s, t \in \mathfrak{k}_Y^\circ$, $f \in \mathcal{O}_Y$. Using (i)-(iii), the standard inductive argument shows that γ extends to the action

$$\gamma : \mathcal{U}_Y^\circ(\mathfrak{k}) \times j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}) \longrightarrow j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}).$$

Let $\tau : \mathfrak{g}_Z^\circ \longrightarrow \mathcal{T}_Z$ be a natural morphism into the Lie algebra of vector fields on Z. Denote by $\mathfrak{b}_Z^\circ$ (resp. $\mathfrak{b}_Y^\circ$) the kernel of τ (resp. τ_1) and by $\sigma_\mu : \mathfrak{b}_Z^\circ \longrightarrow \mathcal{O}_Z$ (resp. $\sigma'_\mu : \mathfrak{b}_Y^\circ \longrightarrow \mathcal{O}_Y$) the G-equivariant (K-equivariant) morphism determined by μ. Notice that $j^*(\mathfrak{b}_Z^\circ) = \mathfrak{b}_Y^\circ$, and that $s \in \mathfrak{b}_Y^\circ$ can be written as $s = \sum f_i \otimes s_i$, $f_i \in \mathcal{O}_Y$, $s_i \in j^{-1}(\mathfrak{b}_Z^\circ)$. Thus we have

$$\begin{aligned}\gamma(s, f \otimes u) &= \tau_1(s)f \otimes u + f\alpha(s, u) = \sum ff_i\alpha(s_i, u)\\ &= \sum ff_i(\sigma_\mu \circ j)(s_i) \otimes u = \left(\sum f_i\sigma'_\mu(s_i)\right)(f \otimes u) = \sigma'_\mu(s)(f \otimes u).\end{aligned}$$

This implies finally that there is a map

$$\delta : \mathcal{D}_{Y,\mu} \times j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}) \longrightarrow j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}).$$

It remains to check that δ commutes with the right $\mathcal{U}(\mathfrak{g})$-action.

Let $\xi \in \mathfrak{g}$, $s \in \mathfrak{k}_Y^\circ$, $d \in j^{-1}(\mathcal{D}_{Z,\mu})$, $v \in j^{-1}\mathcal{V}$. We have

$$\delta(s,(f\otimes(d\otimes v))\cdot\xi) = \tau_1(s)f\otimes(d\xi\otimes v) - \tau_1(s)f\otimes(d\otimes\xi v) + f\alpha(s,d\xi\otimes v) - f\alpha(s,d\otimes\xi v)$$
$$= (\tau_1(s)f\otimes(d\otimes v))\cdot\xi + (f\alpha(s,(d\otimes v))\cdot\xi = (\delta(s,(f\otimes(d\otimes v)))\cdot\xi,$$

as desired. For simplicity we write in the following $\delta(s,D) = s\cdot D$, if $s\in\mathcal{D}_{Y,\mu}$ and $D \in j^*(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$.

Observe that locally free sheaf $j^*(\mathcal{V})$ is K-equivariant and thus the differentiation of K-action yields left $\mathcal{U}(\mathfrak{k})$-action on $j^*(\mathcal{V})$. As before we define right $\mathcal{U}(\mathfrak{k})$-module structure on $\mathcal{D}_{Y,\mu}\otimes_{\mathcal{O}_Y} j^*(\mathcal{V})$ by putting

$$(d\otimes u)\cdot\xi = d\xi\otimes u - d\otimes\xi\cdot u,\ d\in\mathcal{D}_{Y,\mu},\ u\in j^*(\mathcal{V}),\ \xi\in\mathfrak{k}.$$

Recall also that $\mathcal{D}_{Y,\mu}\otimes_{\mathcal{O}_Y} j^*(\mathcal{V})$ is left $\mathcal{D}_{Y,\mu}$-module for the left multiplication on the first factor and this action carries over to $(\mathcal{D}_{Y,\mu}\otimes_{\mathcal{O}_Y} j^*(\mathcal{V}))\otimes_{\mathcal{U}(\mathfrak{k})}\mathcal{U}(\mathfrak{g})$. Our goal is to show:

Lemma 3.1. *As $(\mathcal{D}_{Y,\mu},\mathcal{U}(\mathfrak{g}))$-bimodule $j^*(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$ is isomorphic to*

$$(\mathcal{D}_{Y,\mu}\otimes_{\mathcal{O}_Y} j^*(\mathcal{V}))\otimes_{\mathcal{U}(\mathfrak{k})}\mathcal{U}(\mathfrak{g}).$$

To prove this we need a little preparation.

Lemma 3.2. *Let $x_0 \in Y$ and let $T_{x_0}(\mathcal{F})$ denotes a geometric fibre of $\mathcal{O}_Z(\mathcal{O}_Y)$-module $\mathcal{F}$. Then*
(i) $T_{x_0}(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V}) = (\mathbb{C}_{-\mu}\otimes_{\mathbb{C}} V)\otimes_{\mathcal{U}(\mathfrak{b}\cap\mathfrak{k})}\mathcal{U}(\mathfrak{g})$ as a right $\mathcal{U}(\mathfrak{g})$-module.
(ii) $T_{x_0}(\mathcal{D}_{Y,\mu}\otimes_{\mathcal{O}_Y} j^*\mathcal{V}) = (\mathbb{C}_{-\mu}\otimes_{\mathbb{C}} V)\otimes_{\mathcal{U}(\mathfrak{b}\cap\mathfrak{k})}\mathcal{U}(\mathfrak{k})$ as a right $\mathcal{U}(\mathfrak{k})$-module.

Proof. This is Lemma 3.6. from [9].

Now we return to the proof of 3.1. It is clear from the previous discussion that we can define the map

$$\kappa : (\mathcal{D}_{Y,\mu}\otimes_{\mathcal{O}_Y} j^*\mathcal{V})\times\mathcal{U}(\mathfrak{g}) \longrightarrow j^*(\mathcal{D}_{Z,\mu}\otimes_{\mathcal{O}_Z}\mathcal{V})$$

by

$$\kappa(d\otimes(f\otimes u),\xi) = (d\cdot(f\otimes(1\otimes u)))\cdot\xi,$$

$d\in\mathcal{D}_{Y,\mu}$, $f\in\mathcal{O}_Y$, $u\in j^{-1}\mathcal{V}$, $\xi\in\mathcal{U}(\mathfrak{g})$ and $1\in j^{-1}(\mathcal{D}_{Z,\mu})$ is the identity. If $\eta\in\mathfrak{k}$, we have

$$\kappa(((d\otimes(f\otimes u))\cdot\eta,\xi) = \kappa(d\eta\otimes(f\otimes u) - d\otimes\eta\cdot(f\otimes u),\xi)$$
$$= (d\eta\cdot(f\otimes(1\otimes u))) - (d\tau_1(\eta)f\otimes(1\otimes u)))\cdot\xi - (d(f\otimes 1\otimes\eta\cdot u)))\cdot\xi$$
$$= (d(\tau_1(\eta)f\otimes(1\otimes u)))\cdot\xi + ((d(f\otimes(\eta\otimes u)))\cdot\xi - (d(\tau_1(\eta)f\otimes(1\otimes u)))\cdot\xi - ((d(f\otimes(1\otimes\eta\cdot u)))\cdot\xi$$
$$= (d(f\otimes(1\otimes u)))\cdot\eta\xi = \kappa(d\otimes(f\otimes u),\eta\xi).$$

We conclude that κ induces the map

$$\psi : (\mathcal{D}_{Y,\mu} \otimes_{\mathcal{O}_Y} j^*\mathcal{V}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \longrightarrow j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V})$$

by

$$\psi((d \otimes (f \otimes u)) \otimes \xi) = (d(f \otimes (1 \otimes u)) \cdot \xi.$$

First we check that ψ is surjective. It suffices to show that $f \otimes ((1 \otimes \xi) \otimes u)$ is in the image of ψ, here $f \in \mathcal{O}_Y$, $u \in j^{-1}\mathcal{V}$ and $1 \otimes \xi$, $\xi \in \mathfrak{g}$, is viewed as an element of $j^{-1}(\mathcal{D}_{Z,\mu})$. In fact this follows from

$$\begin{aligned} f \otimes ((1 \otimes \xi) \otimes u) &= (f \otimes (1 \otimes 1) \otimes u) \cdot \xi + f \otimes (1 \otimes 1) \otimes \xi \cdot u \\ &= \psi(1 \otimes (f \otimes u) \otimes \xi) + \psi(1 \otimes (f \otimes \xi \cdot u) \otimes 1). \end{aligned}$$

For $r \in \mathbb{Z}_+$ let $\mathcal{U}(\mathfrak{p})_r$ denotes the r^{th} subspace in the standard filtration of $\mathcal{U}(\mathfrak{p})$. Notice that ψ induces the map

$$\psi_r : (\mathcal{D}_{Y,\mu} \otimes_{\mathcal{O}_Y} j^*\mathcal{V}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p})_r \longrightarrow j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}).$$

It will suffice to show that ψ_r is injective for any $r \in \mathbb{Z}_+$. Let F' and F'' denote the filtrations on $\mathcal{D}_{Y,\mu}$ and $\mathcal{D}_{Z,\mu}$ respectively determined by the degree of differential operators. Put

$$\begin{aligned} \mathrm{F}_{n+r}((\mathcal{D}_{Y,\mu} \otimes_{\mathcal{O}_Y} j^*\mathcal{V}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p})_r) &= (\mathrm{F}'_n\, \mathcal{D}_{Y,\mu} \otimes_{\mathcal{O}_Y} j^*\mathcal{V}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p})_r \\ \mathrm{F}_n\, j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}) &= j^*(\mathrm{F}''_n\, \mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V}) \quad n, r \in \mathbb{Z}_+. \end{aligned}$$

Then ψ_r is compatible with filtrations, and it suffices to prove that

$$\mathrm{Gr}\,\psi_r : \mathrm{Gr}((\mathcal{D}_{Y,\mu} \otimes_{\mathcal{O}_Y} j^*\mathcal{V}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p})_r) \longrightarrow \mathrm{Gr}\, j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V})$$

is injective. Clearly, this follows if we show that $\mathrm{Gr}\,\psi_r$ is injective on geometric fibres. Finally, we are reduced to show that ψ_r is injective on geometric fibres. By 3.2. we have isomorphisms

$$\phi_1 : ((\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} V) \otimes_{\mathcal{U}(\mathfrak{b}\cap\mathfrak{t})} \mathcal{U}(\mathfrak{k})) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p})_r \longrightarrow T_{x_0}(\mathcal{D}_{Y,\mu} \otimes_{\mathcal{O}_Y} j^*\mathcal{V}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p})_r,$$

$$\phi_2 : (\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} V) \otimes_{\mathcal{U}(\mathfrak{b}\cap\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \longrightarrow T_{x_0}(j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V})).$$

From the construction of ϕ_1 and ϕ_2 (compare the proof of loc. cit.) we deduce further that the following diagram commutes

$$\begin{array}{ccc} T_{x_0}(\mathcal{D}_{Y,\mu} \otimes_{\mathcal{O}_Y} j^*\mathcal{V}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p})_r & \xrightarrow{\psi_r} & T_{x_0}(j^*(\mathcal{D}_{Z,\mu} \otimes_{\mathcal{O}_Z} \mathcal{V})) \\ \uparrow{\scriptstyle \phi_1} & & \uparrow{\scriptstyle \phi_2} \\ ((\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} V) \otimes_{\mathcal{U}(\mathfrak{b}\cap\mathfrak{t})} \mathcal{U}(\mathfrak{k})) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{p})_r & \xrightarrow{\phi} & (\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} V) \otimes_{\mathcal{U}(\mathfrak{b}\cap\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \end{array} ;$$

here the bottom map is given by

$$\phi((v \otimes \eta)\xi) = v \otimes \eta\xi, \ v \in V, \ \eta \in \mathcal{U}(\mathfrak{k}), \ \xi \in \mathcal{U}(\mathfrak{p})_r.$$

Since ϕ is injective, we conclude that ψ_r is also injective. This completes the proof of 3.1.

Let X be a flag variety of $\mathfrak{g}$ and $\mathcal{D}_\lambda = \mathcal{D}_{X,\lambda+\rho}$ a G-homogeneous twisted sheaf of differential operators on X. Let $Y = K \cdot x_0$ be a K-orbit and $i : Y \longrightarrow X$ the inclusion. With the preliminaries we have developed we proceed to construct the resolution of $i^*(\mathcal{D}_\lambda)$ by $(\mathcal{D}^i_\lambda, \mathcal{U}(\mathfrak{g}))$-bimodules. Denote by $p : Z \longrightarrow X$ the natural projection. Then we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & Z \\ \Big\| & & {\scriptstyle p}\big\downarrow \\ Y & \xrightarrow{i} & X \end{array} .$$

Let $\mathcal{T}_{Z|X}$ be a locally free $\mathcal{O}_Z$-module consisting of vector fields on Z tangent to the fibres of p. Notice that $\mathcal{T}_{Z|X}$ is G-equivariant sheaf with the geometric fibre isomorphic to $\mathfrak{b}/(\mathfrak{b}\cap\mathfrak{k})$. Also $\mathcal{T}_{Z|X} \subset \mathcal{D}^p_\lambda$. Recall that p defines a relative de Rham complex

$$\cdots \longrightarrow \mathcal{D}^p_\lambda \otimes_{\mathcal{O}_Z} \wedge^k \mathcal{T}_{Z|X} \xrightarrow{d} \mathcal{D}^p_\lambda \otimes_{\mathcal{O}_Z} \wedge^{k-1} \mathcal{T}_{Z|X} \longrightarrow \ \cdots \ \longrightarrow \mathcal{D}^p_\lambda \longrightarrow p^*(\mathcal{D}_\lambda) \longrightarrow 0,$$

with the differential d given by the formula

$$\begin{aligned} d(D \otimes v_1 \wedge \cdots \wedge v_k) = &\sum (-1)^i D v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots v_k \\ &+ \sum_{i<j} (-1)^{i+j} D \otimes [v_i, v_j] \wedge \cdots \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k, \end{aligned}$$

$D \in \mathcal{D}^p_\lambda$, $v_i \in \mathcal{T}_{Z|X}$. It is well known that de Rham complex is a resolution of $p^*(\mathcal{D}_\lambda)$ by locally free $\mathcal{D}^p_\lambda$-modules. As explained above the terms in de Rham complex are naturally right $\mathcal{U}(\mathfrak{g})$-modules. Moreover, it can be checked that the differential $d_k : \mathcal{D}^p_\lambda \otimes_{\mathcal{O}_Z} \wedge^k \mathcal{T}_{Z|X} \to \mathcal{D}^p_\lambda \otimes_{\mathcal{O}_Z} \wedge^{k-1}\mathcal{T}_{Z|X}$ is a morphism of right $\mathcal{U}(\mathfrak{g})$-modules. It follows that

$$j^*(d) : j^*(\mathcal{D}^p_\lambda \otimes_{\mathcal{O}_Z} \wedge^k \mathcal{T}_{Z|X}) \to j^*(\mathcal{D}^p_\lambda \otimes_{\mathcal{O}_Z} \wedge^{k-1}\mathcal{T}_{Z|X})$$

is a morphism of $(\mathcal{D}^i_\lambda, \mathcal{U}(\mathfrak{g}))$-bimodules. Since all terms in the de Rham complex are locally free as $\mathcal{O}_Z$-modules the functor j^* preserves exactness. Applying 3.1., we obtain a variant of H. Hecht's result:

Proposition 3.3. *The sequence*

$$\cdots \to (\mathcal{D}^i_\lambda \otimes_{\mathcal{O}_Y} j^*(\wedge^k \mathcal{T}_{Z|X})) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g}) \to (\mathcal{D}^i_\lambda \otimes_{\mathcal{O}_Y} j^*(\wedge^{k-1} \mathcal{T}_{Z|X})) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g})$$
$$\cdots \to \mathcal{D}^i_\lambda \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g}) \to i^*(\mathcal{D}_\lambda) \to 0$$

is a resolution of $i^*(\mathcal{D}_\lambda)$ by $(\mathcal{D}^i_\lambda, \mathcal{U}(\mathfrak{g}))$-modules.

Proposition 3.4. *Let $\mathcal{V}$ be a right $\mathcal{D}^i_\lambda$-module. Then the sequence*

$$\cdots \to i_*(\mathcal{V} \otimes_{\mathcal{O}_Y} j^*(\wedge^k \mathcal{T}_{Z|X})) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g}) \to i_*(\mathcal{V} \otimes_{\mathcal{O}_Y} j^*(\wedge^{k-1} \mathcal{T}_{Z|X})) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g}) \to$$
$$\cdots \to i_*(\mathcal{V}) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g}) \to R^0 i^R_+(V) \to 0$$

represents a resolution of $R^0 i^R_+(\mathcal{V})$ by right $\mathcal{U}(\mathfrak{g})$-modules. (The superscript R indicates here that the direct image functor is defined in the category of right $\mathcal{D}$-modules.)

Proof. Since each module in 3.3. is a flat $\mathcal{D}^i_\lambda$-module, we deduce after tensoring with $\mathcal{V} \otimes_{\mathcal{D}^i_\lambda} -$ that

$$\cdots \to (\mathcal{V} \otimes_{\mathcal{O}_Y} j^*(\wedge^k \mathcal{T}_{Z|X})) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g}) \to (\mathcal{V} \otimes_{\mathcal{O}_Y} j^*(\wedge^{k-1} \mathcal{T}_{Z|X})) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g}) \to$$
$$\cdots \to \mathcal{V} \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{g}) \to \mathcal{V} \otimes_{\mathcal{D}^i_\lambda} \mathcal{D}_{Y\to X} \to 0,$$

is an exact sequence of right $\mathcal{U}(\mathfrak{g})$-modules. Finally, we obtain a desired resolution after applying the functor i_* on the last sequence. $\square$

From now on assume that Y is a closed K-orbit and use the notation from 2. In particular, x_0 corresponds to a ϑ-stable Borel subalgebra $\mathfrak{b}$.

DEFINITION 3.5. We say that $\lambda \in \mathfrak{h}^*$ is in a *good position* if $(\lambda_1 - \rho_1 + \rho_{\mathfrak{k}}) + \mu$ is $\Sigma^+_{\mathfrak{k}}$-antidominant whenever μ is a weight of $\wedge^{\cdot}(\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{k}))$.

Lemma 3.6. *Suppose λ is in a good position. Let $\mathcal{V}$ be a left $\mathcal{D}^i_\lambda$-module. Then*

$$H^i(Y, \mathcal{V} \otimes_{\mathcal{O}_Y} \omega_{Y|X} \otimes_{\mathcal{O}_Y} j^*(\wedge^k \mathcal{T}_{Z|X})) = 0$$

for any $i > 0$ and $k \geq 0$.

Proof. First we choose $\mathfrak{b} \cap \mathfrak{k}$-invariant filtration

$$0 = \mathrm{F}_0(\wedge^k(\mathfrak{b}/(\mathfrak{b}\cap\mathfrak{k}))) \subset \mathrm{F}_1(\wedge^k(\mathfrak{b}/(\mathfrak{b}\cap\mathfrak{k}))) \subset \cdots \subset \mathrm{F}_m(\wedge^k(\mathfrak{b}/(\mathfrak{b}\cap\mathfrak{k}))) = \wedge^k(\mathfrak{b}/\mathfrak{b}\cap\mathfrak{k})$$

such that

$$\mathrm{F}_i(\wedge^k(\mathfrak{b}/(\mathfrak{b}\cap\mathfrak{k})))/\,\mathrm{F}_{i-1}(\wedge^k(\mathfrak{b}/(\mathfrak{b}\cap\mathfrak{k}))) = \mathbb{C}_{\nu_i}$$

for some $\mathfrak{h}_1$ -weight ν_i of $\wedge^k(\mathfrak{b}/(\mathfrak{b}\cap\mathfrak{k}))$. This will determine a filtration on $\mathcal{W} = \mathcal{V} \otimes_{\mathcal{O}_Y} \omega_{Y|X} \otimes_{\mathcal{O}_Y} j^*(\wedge^k \mathcal{T}_{Z|X}))$,

$$0 = \mathrm{F}_0\,\mathcal{W} \subset \mathrm{F}_1\,\mathcal{W} \subset\ \cdots\ \subset \mathrm{F}_m\,\mathcal{W} = \mathcal{W}$$

such that

$$\mathrm{F}_i\,\mathcal{W}/\,\mathrm{F}_{i-1}\,\mathcal{W} = \mathcal{V}(2(\rho_\mathfrak{k} - \rho_1) + \nu_i).$$

The statement follows now easily using Beilinson-Bernstein vanishing theorem. □

Proposition 3.7. *Suppose $\lambda \in \mathfrak{h}$ is antidominant and in a good position. Let $\mathcal{V}$ be a left $\mathcal{D}^i_\lambda$-module. Then as a left $\mathcal{U}(\mathfrak{g})$-module $\Gamma(X, R^0 i_+(\mathcal{V}))$ is generated by $\Gamma(Y, \mathcal{V} \otimes_{\mathcal{O}_Y} \omega_{Y|X})$.*

Proof. We may view $R^0 i_+(\mathcal{V})$ as a right $\mathcal{D}_{-\lambda}$-module and $\mathcal{V} \otimes_{\mathcal{O}_Y} \omega_{Y|X}$ as a right $\mathcal{D}^i_{-\lambda}$-module. Moreover, if $R^0 i^R_+ : \mathfrak{M}^R(\mathcal{D}^i_{-\lambda}) \longrightarrow \mathfrak{M}^R(\mathcal{D}_{-\lambda})$ is a direct image functor for the right modules, then $R^0 i^R_+(\mathcal{V} \otimes_{\mathcal{O}_Y} \omega_{Y|X}) = R^0 i_+(\mathcal{V})$. Therefore we have to show that as a right $\mathcal{U}(\mathfrak{g})$-module $\Gamma(X, R^0 i^R_+(\mathcal{V} \otimes_{\mathcal{O}_Y} \omega_{Y|X}))$ is generated by $\Gamma(Y, \mathcal{V} \otimes_{\mathcal{O}_Y} \omega_{Y|X})$. But this follows directly from 3.4. and 3.6. □

4. Enright's construction

In this section we recall briefly the properties of Enright's completion functor.

Let $\{X, Y, H\}$ be a standard basis of $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$. If M is an $\mathfrak{a}$-module we denote by M^X the space of X-invariants and by $M[c]$ the c-eigenspace of H. Put $M[c]^X = M[c] \cap M^X$. Assume in addition that M is a weight module for $\mathfrak{a}$. Then M is *complete* if

$$Y^{n+1} : M[n]^X \longrightarrow M[-n-2]^X$$

is isomorphism for any $n \in \mathbb{Z}_+$. An $\mathfrak{a}$-module M' is a *completion* of M if there is $\mathfrak{a}$-module injection $i : M \hookrightarrow M'$ such that

(i) $M'/i(M)$ is $\mathfrak{a}$-finite,

(ii) M' is complete.

To establish the existence of completion one restricts to the category $\mathcal{C}(\mathfrak{a})$ of $\mathfrak{a}$-modules M with the following properties:

(a) M is a weight module for H;

(b) M has no Y-torsion;

(c) X acts locally nilpotently on M.

Suppose further that $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{a} \subset \mathfrak{g}$. Then we consider the category $\mathcal{C}(\mathfrak{g}, \mathfrak{a})$ of $\mathfrak{g}$-modules M such that $M \in \mathcal{C}(\mathfrak{a})$ for the underlying $\mathfrak{a}$-module structure.

Proposition 4.1. *Let $M \in \mathcal{C}(\mathfrak{g}, \mathfrak{a})$.*

(i) There is a $\mathfrak{g}$-module $C(M)$ such that as $\mathfrak{a}$-module $C(M)$ is a completion of M and the inclusion $i : M \hookrightarrow C(M)$ is a morphism of $\mathfrak{g}$-modules. The last condition determines $\mathfrak{g}$-module structure on $C(M)$ uniquely.

(ii) For any morphism $\varphi : M_1 \longrightarrow M_2$ in $\mathcal{C}(\mathfrak{g}, \mathfrak{a})$ there is a unique $\mathfrak{g}$-module map $C(\varphi) : C(M_1) \longrightarrow C(M_2)$ extending φ.

(iii) If F is a finite dimensional $\mathfrak{g}$-module then $M \otimes_{\mathbb{C}} F \in \mathcal{C}(\mathfrak{g}, \mathfrak{a})$ and $C(M \otimes_{\mathbb{C}} F) = C(M) \otimes_{\mathbb{C}} F$.

(iv) $C(M_{[\lambda]}) = C(M)_{[\lambda]}$.

Proof. (i)-(iii) are proved in [4]. To prove (iv), choose $v \in C(M_{[\lambda]})$. Since $C(M_{[\lambda]})/M_{[\lambda]}$ is $\mathfrak{a}$-finite there is $n \in \mathbb{Z}_+$ such that $Y^n \cdot v \in M_{[\lambda]}$. It follows that $(z - \chi_\lambda(z))^m (Y^n \cdot v) = 0$ for some $m \in \mathbb{Z}_+$. On the other hand, Y acts without torsion on $C(M)$ and hence $(z - \chi_\lambda(z))^m \cdot v = 0$. In other words, $v \in C(M)_{[\lambda]}$. To conclude the proof it suffices to show that $C(M)_{[\lambda]}/C(M_{[\lambda]})$ is $\mathfrak{a}$-finite (compare loc.cit., 3.9). Let $v \in C(M)_{[\lambda]}$. Then $Y^n \cdot v \in M$ for some $n \in \mathbb{Z}_+$. In this case necessarily $Y^n \cdot v \in M_{[\lambda]}$, as desired. □

Suppose that a pair $(\mathfrak{g}, \mathfrak{k})$ is choosen as in 2. Put in addition

$$\mathfrak{n}_{\mathfrak{k}} = \sum_{\alpha \in \Sigma_{\mathfrak{k}}^+} \mathfrak{k}_\alpha, \quad \bar{\mathfrak{n}}_{\mathfrak{k}} = \sum_{\alpha \in \Sigma_{\mathfrak{k}}^+} \mathfrak{k}_{-\alpha}.$$

Definition 4.2. We consider the category $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ of $\mathfrak{g}$-modules M with the following properties:

(i) M is a weight module for $\mathfrak{h}_1$ with integral weights;

(ii) M is $\mathcal{U}(\overline{\mathfrak{n}}_{\mathfrak{k}})$-torsion free;

(iii) M is $\mathcal{U}(\mathfrak{n}_{\mathfrak{k}})$-finite.

For each $\alpha \in \Pi_{\mathfrak{k}}$ we choose $x_{\pm\alpha} \in \mathfrak{k}_{\pm\alpha}$ and $h_\alpha \in \mathfrak{h}_1$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$, $\alpha(h_\alpha) = 2$. Denote by $\mathfrak{a}^\alpha$ a 3-dimensional subalgebra in $\mathfrak{k}$ spanned by x_α, $x_{-\alpha}$ and h_α. Then we can consider a completion functor C_α defined with respect to $\mathfrak{a}^\alpha$. It can be shown that completion functors C_α for $\alpha \in \Pi_{\mathfrak{k}}$ preserve $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$. Let $w \in W_{\mathfrak{k}}$ and let $w = s_{\alpha_1} \cdots s_{\alpha_m}$, $\alpha_i \in \Pi_{\mathfrak{k}}$, be a reduced expression. Then for $M \in \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ we put

$$C_w(M) = C_{\alpha_1}(\cdots(C_{\alpha_m}(M))\cdots).$$

It is known that $C_w(M)$ depends only on w and not on the particular reduced expression [3].

The following proposition generalizes 4.1.

Proposition 4.3. *Let $M \in \mathcal{C}(\mathfrak{g}, \mathfrak{k})$.*

(i) For any $w \in W_{\mathfrak{k}}$, there is a unique $\mathfrak{g}$-module structure on $C_w(M)$ such that $M \hookrightarrow C_w(M)$ is a $\mathfrak{g}$-module map.

(ii) Suppose $(M_w; w \in W_{\mathfrak{k}})$ is a family of $\mathfrak{g}$-modules such that $M_1 = M$, and for $w = s_\alpha v$, $w \geq v$, $\alpha \in \Pi_{\mathfrak{k}}$, $C_\alpha(M_v) = M_w$. Then we have a unique family of isomorphisms $\varphi_w : C_w(M) \to M_w$, $w \in W_{\mathfrak{k}}$, making the following diagram commutative

$$\begin{tikzcd} C_v(M) \arrow[r] \arrow[d, "\varphi_v"'] & C_w(M) \arrow[d, "\varphi_w"'] \\ M_v \arrow[r] & M_w \end{tikzcd}.$$

(iii) If F is a finite dimensional $\mathfrak{g}$-module then $M \otimes_{\mathbb{C}} F \in \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ and $C_w(M \otimes_{\mathbb{C}} F) = C_w(M) \otimes_{\mathbb{C}} F$.

(iv) $C_w(M_{[\lambda]}) = C_w(M)_{[\lambda]}$.

We know that the sequence

$$0 \to E_m \to \cdots \to E_i \to \cdots \to E_0 \to M(\lambda) \to 0,$$

where

$$E_i = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}\cap\mathfrak{k})} (\wedge^i(\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{k})) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_1 - \rho_1})$$

and $m = \dim(\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{k}))$, is a resolution of $M(\lambda)$ by $\mathcal{U}(\mathfrak{g})$-modules [4]. It can be shown (loc.cit., 5.), that under certain restrictions on the parameter λ this resolution can be lifted to a resolution of $C_w(M(\lambda))$.

Proposition 4.4. *Suppose $\lambda \in \mathfrak{h}^*$ is antidominant and in a good position, $\lambda_1 \in P(\Sigma_{\mathfrak{k}})$ and $\nu = \lambda_1 - \rho_1 + \rho_{\mathfrak{k}}$. Then, for any $w \in W_{\mathfrak{k}}$, there is an exact sequence of $\mathcal{U}(\mathfrak{g})$-modules*

$$0 \to C_w(E_m) \to \cdots \to C_w(E_i) \to \cdots \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} M_{\mathfrak{k}}(w(\nu+\mu)) \to C_w(M(\lambda)) \to 0. \blacksquare$$

Examining the last map in the above sequence we conclude that the family of modules $E_w = C_w(M(\lambda))$, $w \in W_{\mathfrak{k}}$, has the following four properties:

(i) $Z_1 = M(\lambda)$;

(ii) For any $w \in W_{\mathfrak{k}}$ there is an injective map $\varepsilon_w : M_{\mathfrak{k}}(w\nu) \hookrightarrow E_w$ such that

$$E_w = \mathcal{U}(\mathfrak{g}) \cdot \varepsilon_w(M_{\mathfrak{k}}(w\nu));$$

(iii) For $w, v \in W_{\mathfrak{k}}$, $w \geq v$, there is an injective map $Z_v \hookrightarrow Z_w$ such that the following diagram commutes

$$\begin{array}{ccc} M_{\mathfrak{k}}(v\nu) & \longrightarrow & M_{\mathfrak{k}}(w\nu) \\ {\scriptstyle \varepsilon_w}\big\downarrow & & {\scriptstyle \varepsilon_v}\big\downarrow \\ E_v & \longrightarrow & E_w \end{array} \quad ;$$

(iv) For any $w \in W_{\mathfrak{k}}$, the module E_w is $\mathcal{U}(\bar{\mathfrak{n}}_{\mathfrak{k}})$-torsion free.

On the other hand, these four properties characterize family $(E_w; w \in W_{\mathfrak{k}})$. In fact, we have [6]:

Proposition 4.5. *Let $\lambda \in \mathfrak{h}^*$ be such that $\nu = \lambda_1 - \rho_1 + \rho_{\mathfrak{k}}$ is regular and $\Sigma_{\mathfrak{k}}^+$-antidominant, and $\lambda_1 \in P(\Sigma_{\mathfrak{k}})$. Then there is a unique family of modules $(E_w; w \in W_{\mathfrak{k}})$ with the properties (i)-(iv). More precisely, if $(E'_w; w \in W_{\mathfrak{k}})$ is another family of $\mathfrak{g}$-modules with the same properties, there is a family of isomorphisms $\varphi : E'_w \to E_w$ such that for any $w \geq v$ the following diagram commutes*

$$\begin{array}{ccc} E'_v & \longrightarrow & E'_w \\ {\scriptstyle \varphi_v}\big\downarrow & & {\scriptstyle \varphi_w}\big\downarrow \\ E_v & \longrightarrow & E_w \end{array} \quad .$$

5. Main theorem

We show here that the modules $Z(w,\lambda)$ coincide with the modules $C_w(M(\lambda))$ defined via Enright's completion functor. This will be done in two steps. First we identify the aforementioned modules when parameter λ is sufficiently negative. The general case of the antidominant parameter is then reduced to the first one using translation functors. We turn now to the details.

Lemma 5.1. *Suppose $\lambda \in \mathfrak{h}^*$ is antidominant, in a good position and such that $\lambda_1 \in P(\Sigma_{\mathfrak{k}})$. Then a family of modules $(Z(w,\lambda); w \in W_{\mathfrak{k}})$ has the properties (i)-(iv) from 4.4.*

Proof. (i) We have $Z(1,\lambda) = M(\lambda)$ by 1.1.

For the rest of the proof put $\nu = \lambda_1 - \rho_1 + \rho_{\mathfrak{k}}$ and $\nu' = \lambda_1 + \rho_1 - \rho_{\mathfrak{k}}$. Recall that

$$\mathcal{Z}(w,\lambda) = R^0 i_+(\mathcal{M}_{\mathfrak{k}}(w,\nu'))$$

can be filtered by normal degree as explained in [9]. Since we are considering left $\mathcal{D}$- modules our filtration differs from the one in loc. cit. in a twist by

$$\omega_{Y|X} = \omega_Y \otimes_{\mathcal{O}_Y} i^*(\omega_X^{-1}).$$

Moreover, the filtered submodules $\mathrm{F}_p\, R^0 i_+(\mathcal{M}_{\mathfrak{k}}(w,\nu'))$, $p \in \mathbb{Z}_+$, carry the additional structure of $\mathcal{U}(\mathfrak{k})$-modules. In fact, this holds since the differentiation with the operators from $\mathcal{U}(\mathfrak{k})$ preserves the ideal of regular functions vanishing on Y.

(ii) Notice that $\omega_{Y|X} = \mathcal{O}_Y(2(\rho_{\mathfrak{k}} - \rho_1))$ and therefore

$$\mathrm{F}_0\, R^0 i_+(\mathcal{M}_{\mathfrak{k}}(w,\nu')) = i_*(\mathcal{M}_{\mathfrak{k}}(w,\nu') \otimes_{\mathcal{O}_Y} \omega_{Y|X}) = i_*(\mathcal{M}_{\mathfrak{k}}(w,\nu)).$$

Since ν is $\Sigma_{\mathfrak{k}}^+$-antidominant, this implies

$$M_{\mathfrak{k}}(w\nu) = \Gamma(X, \mathrm{F}_0\, R^0 i_+(\mathcal{M}_{\mathfrak{k}}(w,\nu'))) \subset Z(w,\lambda).$$

Moreover, applying 3.7. we conclude that $M_{\mathfrak{k}}(w\nu)$ generates $Z(w,\lambda)$ as $\mathcal{U}(\mathfrak{g})$-module. ∎

(iii) Let $w \geq v$. Then we have injective map $\mathcal{Z}(v,\lambda) \hookrightarrow \mathcal{Z}(w,\lambda)$. Thus the diagram

$$\begin{tikzcd}
\mathrm{F}_0\, \mathcal{Z}(v,\lambda) \arrow[r] \arrow[d] & \mathrm{F}_0\, \mathcal{Z}(w,\lambda) \arrow[d] \\
\mathcal{Z}(v,\lambda) \arrow[r] & \mathcal{Z}(v,\lambda)
\end{tikzcd}$$

is commutative. After taking global sections we obtain finally

$$\begin{array}{ccc} M_{\mathfrak{k}}(v\nu) & \longrightarrow & M_{\mathfrak{k}}(w\nu) \\ \big\downarrow & & \big\downarrow \\ Z(v,\lambda) & \longrightarrow & Z(w,\lambda) \end{array} .$$

To prove the fourth property we need a little preparation.

Lemma 5.2. *Let A be a ring and Z a topological space. Let $\mathcal{F}$ be a sheaf of A-modules and*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_p \subset \ \cdots, \quad \bigcup_{p \in \mathbb{Z}_+} \mathcal{F}_p = \mathcal{F},$$

an exhaustive filtration of $\mathcal{F}$ by subsheaves of A-modules. Then there is a spectral sequence with the first term

$$E_1^{pq} = H^{p+q}(X, \mathcal{F}_p/\mathcal{F}_{p-1})$$

abutting to $H^{\cdot}(X, \mathcal{F})$. The r^{th}-differentials

$$d_r : E_r^{pq} \longrightarrow E_r^{p-r, q+r+1}$$

are morphisms of A-modules.

We will also need the following result from [12]:

Lemma 5.3. *For $\mu \in P(\Sigma)$ put*

$$n(\mu) = \min\{\ell(w) : w \in W, \ w\mu \textit{ is antidominant}\}.$$

Let $\mathcal{M}$ be $\mathcal{D}_\mu$-module. If $n(\mu) > 0$, there is a finite dimensional $\mathfrak{g}$-module F, $\mu' \in P(\Sigma)$ such that $n(\mu + \mu') < n(\mu)$, and an injective map

$$\mathcal{M} \to \mathcal{M}(\mu') \otimes_{\mathcal{O}_X} \mathcal{F} \qquad (\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F).$$

We continue now with the proof of 5.1. Suppose $\mathcal{M}$ is $\mathcal{D}_\lambda^i$-module. Then the filtration by normal degree of $R^0 i_+(\mathcal{M})$ yields

$$\operatorname{Gr} R^0 i_+(\mathcal{M}) = i_*(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathrm{S}(\mathcal{N}_{Y|X}) \otimes_{\mathcal{O}_Y} \omega_{Y|X})$$

as $\mathcal{U}(\mathfrak{k})$-modules, here $\mathcal{N}_{Y|X}$ is the normal sheaf of Y in X. Each K-homogeneous sheaf $\mathrm{S}^p(\mathcal{N}_{Y|X})$, $p \in \mathbb{Z}_+$, can be filtered further by K-homogeneous subsheaves. In fact, observe that $\mathcal{N}_{Y|X}$ is associated to the adjoint representation of $K \cap B$ on $\mathfrak{p}/(\mathfrak{p} \cap \mathfrak{b})$. Thus we can choose $(K \cap B)$-invariant filtration on $V = \mathrm{S}^p(\mathfrak{p}/(\mathfrak{p} \cap \mathfrak{b}))$:

$$0 = \mathrm{F}_0\, V \subset \mathrm{F}_1\, V \subset \ \cdots\ \subset \mathrm{F}_n\, V = V,$$

such that $\mathrm{F}_i\, V/\,\mathrm{F}_{i-1}\, V = \mathbb{C}_{\mu_i}$, $i = 1, \cdots, n$, for some $\mathfrak{h}_1$-weight μ_i of $\mathrm{S}^p(\mathfrak{p}/(\mathfrak{p} \cap \mathfrak{b}))$. This will induce the filtration of $\mathcal{V} = \mathrm{S}^p(\mathcal{N}_{Y|X})$,

$$0 = \mathrm{F}_0\, \mathcal{V} \subset \mathrm{F}_1\, \mathcal{V} \subset \cdots \subset \mathrm{F}_n\, \mathcal{V} = \mathcal{V}$$

such that $\mathrm{F}_i\, \mathcal{V}/\,\mathrm{F}_{i-1}\, \mathcal{V} = \mathcal{O}_Y(\mu_i)$, $i = 1, \cdots, n$. Using these filtrations we can refine the filtration $(\mathrm{F}_p\, R^0 i_+(\mathcal{M});\ p \in \mathbb{Z}_+)$, to obtain a new increasing and exhaustive filtration $(\mathrm{F}'_p\, R^0 i_+(\mathcal{M});\ p \in \mathbb{Z}_+)$ of $R^0 i_+(\mathcal{M})$ by $\mathcal{U}(\mathfrak{k})$-modules. Notice that

$$\mathrm{Gr}'\, R^0 i_+(\mathcal{M}) = \bigoplus_{p \geq 0} \mathcal{M}(\mu_p + 2(\rho_{\mathfrak{k}} - \rho_1))$$

where $\mu_0 = 0$ and $\mu_p \in \mathfrak{h}_1^*$, $p \geq 1$, exhaust all $\mathfrak{h}_1$-weights of $\mathrm{S}(\mathfrak{p}/(\mathfrak{p} \cap \mathfrak{b}))$ counted with multiplicity. We are now prepared to apply 5.2. It follows that there is a spectral sequence with the first term

$$E_1^{pq} = H^{p+q}(Y, \mathcal{M}(\mu_p + 2(\rho_{\mathfrak{k}} - \rho_1)))$$

abutting to $H^{\cdot}(X, R^0 i_+(\mathcal{M}))$. Since λ is antidominant, $H^n(X, R^0 i_+(\mathcal{M})) = 0$ for $n \neq 0$. Further, the E_∞-term of the spectral sequence corresponds to a filtration of $\Gamma(X, R^0 i_+(\mathcal{M}))$ by $\mathcal{U}(\mathfrak{k})$-modules such that

$$\mathrm{Gr}_p\, \Gamma(X, R^0 i_+(\mathcal{M})) = E_\infty^{p,-p} = \Gamma(Y, \mathcal{M}(\mu_p + 2(\rho_{\mathfrak{k}} - \rho_1))).$$

In the special case when $\mathcal{M} = \mathcal{M}_{\mathfrak{k}}(w, \nu')$, our goal is to show that $\mathcal{U}(\bar{\mathfrak{n}}_{\mathfrak{k}})$ acts without torsion on $E_\infty^{p,-p}$. We have

$$\begin{aligned} E_{r+1}^{p,-p} &= \ker(E_r^{p,-p} \to E_r^{p-r,p+r+1})/\operatorname{im}(E_r^{p+r,-p-r-1} \to E_r^{p,-p}) \\ &= \ker(E_r^{p,-p} \to E_r^{p-r,p+r+1}), \end{aligned}$$

since $E_r^{p+r,-p-r-1} = 0$. Moreover, $E_1^{ab} = 0$ for $a < 0$ implies that $E_r^{p-r,p+r+1}$ for $r \geq p + 1$. We conclude that

$$E_1^{p,-p} \supseteq E_2^{p,-p} \supseteq \cdots \supseteq E_{r+1}^{p,-p} = E_\infty^{p,-p}.$$

Thus, it will suffice to show that $\mathcal{U}(\bar{\mathfrak{n}}_{\mathfrak{k}})$ has no torsion on $E_1^{p,-p} = \Gamma(Y, \mathcal{M}_{\mathfrak{k}}(w, \nu + \mu_p))$. Arguing by induction on $n(\nu + \mu_p)$, it follows from 5.3., that there is a finite dimensional $\mathfrak{k}$-module F, $\mu \in P(\Sigma_{\mathfrak{k}})$, such that $\nu + \mu_p + \mu$ is $\Sigma_{\mathfrak{k}}^+$-antidominant, and an injective map

$$\mathcal{M}_{\mathfrak{k}}(w, \nu + \mu_p) \to \mathcal{M}_{\mathfrak{k}}(w, \nu + \mu_p + \mu) \otimes_{\mathcal{O}_Y} \mathcal{F}.$$

After taking global sections we obtain injective map

$$\Gamma(Y, \mathcal{M}_{\mathfrak{k}}(w, \nu + \mu_p)) \to \Gamma(Y, \mathcal{M}_{\mathfrak{k}}(w, \nu + \mu_p + \mu)) \otimes_{\mathbb{C}} F.$$

On the other hand, we have

$$\Gamma(Y, \mathcal{M}_{\mathfrak{k}}(w, \nu + \mu_p + \mu)) = M_{\mathfrak{k}}(w(\nu + \mu_p + \mu)).$$

Hence, we are reduced to show that $\mathcal{U}(\bar{\mathfrak{n}}_{\mathfrak{k}})$ has no torsion on $M_{\mathfrak{k}}(w(\nu+\mu_p+\mu)) \otimes_{\mathbb{C}} F$. But this is satisfied, since $M_{\mathfrak{k}}(w(\nu + \mu_p + \mu)) \otimes_{\mathbb{C}} F$ has composition series with subquotients of the form $M_{\mathfrak{k}}(w(\nu + \mu_p + \mu) + \alpha)$, where α runs over the set of weights of F. This concludes the proof of 5.1.

Now we can state the main result.

Theorem 5.4. *Suppose $\lambda \in \mathfrak{h}^*$ is antidominant and such that $\lambda_1 = \lambda|\mathfrak{h}_1 \in P(\Sigma_{\mathfrak{k}})$. Then there is a family of $\mathfrak{g}$-module isomorphisms*

$$\varphi_w : C_w(M(\lambda)) \to Z(w, \lambda), \quad w \in W_{\mathfrak{k}};$$

such that, for any $w \geq v$, the diagram

$$\begin{array}{ccc} C_v(M(\lambda)) & \longrightarrow & C_w(M(\lambda)) \\ {\scriptstyle\varphi_v}\big\downarrow & & {\scriptstyle\varphi_w}\big\downarrow \\ Z(v, \lambda) & \longrightarrow & Z(w, \lambda) \end{array}$$

commutes.

Proof. Pick $\mu \in P(\Sigma)$ dominant and such that $\lambda - \mu$ is in a good position. This is possible since $\mu_1 = \mu|\mathfrak{h}_1$ is $\Sigma_{\mathfrak{k}}^+$-dominant and integral. By 5.1. we can construct a family of isomorphisms

$$\varphi'_w : C_w(M(\lambda - \mu)) \to Z(w, \lambda - \mu), \quad w \in W_{\mathfrak{k}};$$

satisfying the required compatibility conditions. Let F be a finite dimensional irreducible $\mathfrak{g}$-module with the highest weight μ. Then

$$(M(\lambda-\mu)\otimes_{\mathbb{C}} F)_{[\lambda]} = M(\lambda),$$

and therefore by 2.3., we have

$$C_w(M(\lambda)) = C_w((M(\lambda-\mu)\otimes_{\mathbb{C}} F)_{[\lambda]}) = (C_w(M(\lambda-\mu))\otimes_{\mathbb{C}} F)_{[\lambda]}.$$

On the other hand, for antidominant λ, any $\mathcal{D}_\lambda$-module $\mathcal{M}$ and $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$ we have (see [12])

$$\mathcal{M} = (\mathcal{M}(-\mu)\otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}.$$

This implies

$$\Gamma(X,\mathcal{M}) = (\Gamma(X,\mathcal{M}(-\mu))\otimes_{\mathbb{C}} F)_{[\lambda]}.$$

In particular, for $\mathcal{M} = \mathcal{Z}(w,\lambda)$ this yields

$$Z(w,\lambda) = (Z(w,\lambda-\mu)\otimes_{\mathbb{C}} F)_{[\lambda]}.$$

Thus we can extend isomorphisms φ'_w to the isomorphisms

$$\varphi_w : C_w(M(\lambda)) \longrightarrow Z(w,\lambda),\ w \in W_{\mathfrak{k}}.$$

It follows from the construction that compatibility conditions are again satisfied. $\square$

Using this theorem we obtain easily the main results of [4]. First of all, we put

$$E(M(\lambda)) = C_{w_0}(M(\lambda)) \Big/ \left(\sum_{w<w_0} C_w(M(\lambda)) \right).$$

Theorem 5.5. *Suppose λ is antidominant and $\lambda_1 \in P(\Sigma_{\mathfrak{k}})$.*
(i) $E(M(\lambda))$ is either irreducible or zero.
(ii) There is a homological resolution of $E(M(\lambda))$ of the form

$$0 \longrightarrow M(\lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=p} C_w(M(\lambda)) \longrightarrow \cdots \longrightarrow C_{w_0}(M(\lambda)) \longrightarrow E(M(\lambda)) \longrightarrow 0\ ,$$

where the differentials are induced by the natural injections $C_v(M(\lambda)) \to C_w(M(\lambda))$ for $w \geq v$, $w,v \in W_{\mathfrak{k}}$. This resolution is dual to the geometric resolution

$$0 \longrightarrow L(Y,\tau) \longrightarrow T(w_0,\lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=p} T(w,\lambda) \longrightarrow \cdots \longrightarrow T(1,\lambda) \longrightarrow 0$$

via the duality operation described in 2.6.

Proof. The first statement is consequence of the fact that, for antidominant λ, global sections of an irreducible $\mathcal{D}_\lambda$-module are either zero or irreducible $\mathfrak{g}$-module [12]. The second statement is clear. $\square$

Finally we want to examine more closely the case $E(M(\lambda)) = 0$. We denote by $I_{s_\alpha} : \mathfrak{M}(\mathcal{D}_\lambda) \to \mathfrak{M}(\mathcal{D}_{s_\alpha\lambda})$ the *intertwining functor* attached to $\alpha \in \Pi$ and by $L^i I_{s_\alpha}, i \in \mathbb{Z}$, the left derived functors of I_{s_α} [12]. Recall that $L^i I_{s_\alpha} = 0$ for $i \neq 0, -1$.

Next result is proved in [10].

Lemma 5.6. *Suppose $\alpha \in \Pi \cap \Sigma_{CI}$ and $i_Q : Q \hookrightarrow X$ is a K-orbit. Let τ be irreducible K-homogeneous $\mathcal{D}_\lambda^{i_Q}$-connection on X. Then:*

(i) $I_{s_\alpha}\mathcal{I}(Q,\tau) = 0$;

(ii) $L^{-1} I_{s_\alpha}\mathcal{I}(Q,\tau) = \mathcal{I}(Q,\tau)(s_\alpha\lambda - \lambda)$.

We will also need a following lemma [11]:

Lemma 5.7. *Let $\alpha \in \Pi$ and suppose $\alpha\check{}(\lambda) = -p \in \mathbb{Z}$. Let $\mathcal{M} \in \mathfrak{M}(\mathcal{D}_\lambda)$ be such that $I_{s_\alpha}\mathcal{M} = 0$ and $L^{-1} I_{s_\alpha}(\mathcal{M}) = \mathcal{M}(p\alpha)$. Then $\Gamma(X, \mathcal{M}(p\alpha)) = 0$.*

Proposition 5.8. *Suppose $i : Y \hookrightarrow X$ is a closed K-orbit and τ an irreducible K-homogeneous $\mathcal{D}_\lambda^i$-connection on X. When λ is antidominant we have:*

(i) If $\alpha\check{}(\lambda) = 0$ for some $\alpha \in \Pi \cap \Sigma_{CI}$ then $L(Y,\tau) = 0$.

(ii) If $\mathrm{Re}(\lambda, \alpha + \vartheta\alpha) \leq 0$ for any $\alpha \in \Sigma_{\mathbb{C}}^+$, and $(\lambda,\alpha) \neq 0$ for any $\alpha \in \Pi \cap \Sigma_{CI}$, then $L(Y,\tau) \neq 0$.

Proof. (i) This follows directly from 5.6. and 5.7.

(ii) Notice first that

$$\Gamma(Y, \mathcal{O}_Y(\lambda_1 - \rho_1 + 2\rho_{\mathfrak{k}})) \subset L(Y,\tau).$$

By Borel-Weil theorem it suffices to check that $\lambda_1 - \rho_1 + \rho_{\mathfrak{k}}$ is $\Sigma_{\mathfrak{k}}^+$-antidominant and regular. Pick $\alpha \in \Pi_{\mathfrak{k}}$. Then $\alpha = \beta|\mathfrak{h}_1$ for some $\beta \in \Sigma_{CI}^+ \cup \Sigma_{\mathbb{C}}^+$. At this point we draw the attention to 2.2. If $\beta \in \Sigma_{\mathbb{C}}^+$ we have $\alpha\check{} = c(\beta\check{} + (\vartheta\beta)\check{})$, $c = 1, 2$. Hence

$$\alpha\check{}(\lambda_1) = c(\beta\check{} + (\vartheta\beta)\check{})(\lambda_1) = c\,\mathrm{Re}((\beta\check{} + (\vartheta\beta)\check{})(\lambda_1)) \leq 0.$$

Observe that we have used $\alpha\check{}(\lambda_1) \in \mathbb{Z}$. Further,

$$\alpha\check{}(\rho_1) = c(\beta\check{} + (\vartheta\beta)\check{})(\rho) \geq 2.$$

Therefore, in this case $\alpha\check{}(\lambda_1 - \rho_1 + \rho_{\mathfrak{k}}) < 0$. If $\beta \in \Sigma_{CI}^+$ we have two cases. In the first, $\beta \in \Pi$ and

$$\alpha\check{}(\lambda_1 - \rho_1 + \rho_{\mathfrak{k}}) = \beta\check{}(\lambda) < 0$$

by the assumption. In the second $\beta \notin \Pi$. Thus $\beta\check{}(\rho) \geq 2$, and this implies again that $\alpha\check{}(\lambda_1 - \rho_1 + \rho_{\mathfrak{k}}) < 0$. □

Corollary 5.9. *Suppose λ is as in 5.8.*

(i) If $\alpha\check{}(\lambda) = 0$ for some $\alpha \in \Pi \cap \Sigma_{CI}$, then $E(M(\lambda)) = 0$.

(ii) If $\mathrm{Re}(\lambda, \alpha + \vartheta\alpha) \leq 0$ for any $\alpha \in \Sigma_{\mathbb{C}}^+$, and $(\lambda, \alpha) \neq 0$ for any $\alpha \in \Pi \cap \Sigma_{CI}$, then $E(M(\lambda)) \neq 0$.

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