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A LIMIT FORMULA FOR ELLIPTIC ORBITAL INTEGRALS

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Abstract

Let $\mathscr{O}$ be a nilpotent orbit for a semisimple Lie group which appears as the leading orbit in the wave-front set of an $A_{\mathfrak{q}}(\lambda)$-module. We establish a limit formula for the computation of the canonical measure on $\mathscr{O}$ through differentiation of the canonical measures on elliptic orbits.

0. Introduction

Fourier inversion of the nilpotent orbital integrals is closely related to the computation of the canonical measure on a nilpotent orbit through differentiation of the canonical measures on the semisimple orbits. This leads naturally to the so-called limit formulas. The importance of Fourier inversion for the harmonic analysis on a semisimple group is already apparent in the work of Harish-Chandra on the Plancherel formula. Since then, the problem of computing the nilpotent measures has appeared in the work of various authors. In [BV1], [BV2], [HK], and [R2] the problem was solved for complex semisimple groups and in [B] for the real semisimple groups of rank one. For a general real semisimple group, the problem appears to be quite hard. However, in the special case of $U(p, q)$, the computation for all the orbits was carried out in [BV3], using character theory and combinatorics. In [R1] a conjectural limit formula was proposed, based on Rossmann's theory of character contours and Weyl group representations (see [R2], [R1], [R3]). The goal of this paper is to establish a limit formula for the class of nilpotent orbits that arise as the leading orbits in the wave-front set of $A_{\mathfrak{q}}(\lambda)$-modules. The main technical tool in our approach is provided by the theory developed by W. Schmid and K. Vilonen [SV1], [SV2], [SV3] in the course of their work on the Barbasch-Vogan conjecture.

To describe the main result of the paper, we have to introduce some notation. Let $G_{\mathbb{R}}$ be a linear, connected, semisimple Lie group, let G be the complexification, and let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$. Denote by $\mathfrak{k}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ the Lie algebras, and denote by $\mathfrak{k} \subset \mathfrak{g}$ the complexified Lie algebras of $K_{\mathbb{R}} \subset G_{\mathbb{R}}$. We choose

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a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ so that the complex conjugate $\overline{\mathfrak{p}}$ is opposite to $\mathfrak{p}$. Then $\mathfrak{l} = \mathfrak{p} \cap \overline{\mathfrak{p}}$ is a Levi factor of $\mathfrak{p}$. Denote by $\mathfrak{c}$ the center of $\mathfrak{l}$, and denote by $(\mathfrak{c} \cap \mathfrak{k}_{\mathbb{R}})^*$ the real linear dual of $\mathfrak{c} \cap \mathfrak{k}_{\mathbb{R}}$. Let $C^+ \subset i(\mathfrak{c} \cap \mathfrak{k}_{\mathbb{R}})^*$ be the positive chamber defined by $\mathfrak{p}/\mathfrak{l}$. If $\mathscr{V} \subset i\mathfrak{g}_{\mathbb{R}}^*$ is a $G_{\mathbb{R}}$-orbit, we denote by $m_{\mathscr{V}}$ the canonical measure on $\mathscr{V}$, which is up to a constant multiple the $(\dim_{\mathbb{R}} \mathscr{V}/2)$-power of the Liouville form. If p is a polynomial on $\mathfrak{c}^*$, write $\partial(p)$ for the corresponding differential operator on $\mathfrak{c}$. Finally, let $\mathscr{O} \subset i\mathfrak{g}_{\mathbb{R}}^*$ be the nilpotent $G_{\mathbb{R}}$-orbit associated via the Sekiguchi correspondence with dense nilpotent K-orbit in $K \cdot (\mathfrak{g}/\mathfrak{p} + \mathfrak{k})^*$. (Here $(\mathfrak{g}/\mathfrak{p} + \mathfrak{k})^*$ stands for $\mathbb{C}$-linear dual.) Now we are ready to state the main result of the paper.

THEOREM 0.1
There exist a polynomial p on $\mathfrak{c}^*$ and a nonzero constant c so that the following limit formula holds:

$$\lim_{\lambda \to 0(C^+)} \partial(p) m_{G_{\mathbb{R}} \cdot \lambda} = c m_{\mathscr{O}}.$$

To prove the limit formula, we study the asymptotic behavior at $\lambda = 0$ of the holomorphic function

$$\lambda \mapsto \int_{G_{\mathbb{R}} \cdot \lambda} \hat{\phi} \sigma_\lambda^m,$$

where $\hat{\phi}$ is the Fourier transform of a test function ϕ on $\mathfrak{g}_{\mathbb{R}}$, σ_λ is the Liouville form on $G_{\mathbb{R}} \cdot \lambda$, and $2m = \dim_{\mathbb{R}} G_{\mathbb{R}} \cdot \lambda$. This is accomplished in several steps. First, we transfer the problem to the cotangent bundle T^*Y of the generalized flag variety Y of $\mathrm{Ad}(G)$-conjugates of $\mathfrak{p}$. We use the twisted moment map $\mu_\lambda : T^*Y \longrightarrow G \cdot \lambda$ to write

$$\int_{G_{\mathbb{R}} \cdot \lambda} \hat{\phi} \sigma_\lambda^m = \int_{\mu_\lambda^{-1}(G_{\mathbb{R}} \cdot \lambda)} \mu_\lambda^*(\hat{\phi} \sigma_\lambda^m).$$

In [R3] W. Rossmann shows how to obtain the Taylor series expansion at $\lambda = 0$ of distributions on the right-hand side of the formula in the case when $\mathfrak{p}$ is a Borel subalgebra. We present in Section 4 an analogue of Rossmann's theory for a generalizied flag variety. Next, we compute in Section 3 the characteristic cycle $CC(\mathscr{G})$ of a standard sheaf $\mathscr{G}$ associated with open orbit $G_{\mathbb{R}} \cdot \mathfrak{p} \subset Y$. In fact, we show that $\mu_\lambda^{-1}(G_{\mathbb{R}} \cdot \lambda)$, with natural orientation, is homologous to $CC(\mathscr{G})$. This result is a slight generalization of [SV2, §7]. Finally, to understand the leading term in the Taylor series expansion of $\int_{CC(\mathscr{G})} \mu_\lambda^*(\hat{\phi} \sigma_\lambda^m)$, we apply the results from [SV3] on the microlocalization of the Matsuki correspondence for sheaves. In Sections 1 and 2, we give a summary of the results from [SV3] needed for our applications in the setting of a generalized flag variety.

We hope the powerful theory developed in [SV1], [SV2], and [SV3] will shed more light on the problem of the Fourier inversion of the nilpotent orbits in the future.

1. Preliminaries

Let $G_{\mathbb{R}}$ be a real semisimple Lie group. In addition, we assume that $G_{\mathbb{R}}$ is connected and linear. We fix a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$ and write K and G for the respective complexifications. We denote further by $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ Lie algebras of $G_{\mathbb{R}}$ and G, respectively. Let θ be the Cartan involution on $\mathfrak{g}_{\mathbb{R}}$ determined by the choice of $K_{\mathbb{R}}$. We denote by the same letter the corresponding involution on $\mathfrak{g}$. We fix a compact real form $U_{\mathbb{R}}$ of G such that $K_{\mathbb{R}} \subset U_{\mathbb{R}}$. Denote by $\mathfrak{u}_{\mathbb{R}}$ the Lie algebra of $U_{\mathbb{R}}$.

Let P be a fixed parabolic subalgebra of G. Denote by $\mathfrak{p}$ the Lie algebra of P. Let $Y = G/P$ be the generalized flag variety. Alternatively, we may view Y as the variety of parabolic subalgebras $\mathrm{Ad}(G)$-conjugated to $\mathfrak{p}$. Let $\mathfrak{h} \subset \mathfrak{p}$ be a Cartan subalgebra. For an $\mathfrak{h}$-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$, we write $\Delta(\mathfrak{m})$ for the set of $\mathfrak{h}$-weights on $\mathfrak{m}$. We use $\mathfrak{h}$ to define Levi decompositions

$$\mathfrak{p} = \mathfrak{l} + \mathfrak{n} \qquad \text{and} \qquad \overline{\mathfrak{p}} = \mathfrak{l} + \overline{\mathfrak{n}}. \tag{1.1}$$

Here $\Delta(\mathfrak{l}) = (-\Delta(\mathfrak{p})) \cap \Delta(\mathfrak{p})$, $\Delta(\mathfrak{n}) = \Delta(\mathfrak{p}) \setminus \Delta(\mathfrak{l})$, $\Delta(\overline{\mathfrak{n}}) = -\Delta(\mathfrak{n})$. Recall that $\overline{\mathfrak{p}}$ is called the opposite parabolic subalgebra of $\mathfrak{p}$. Denote by $2\rho_{\mathfrak{p}}$ the sum of the roots from $\Delta(\mathfrak{n})$. Write B for the Killing form on $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, we may use B to identify $\mathfrak{g}$ with complex linear dual $\mathfrak{g}^*$.

Denote by A the group $G_{\mathbb{R}}$ or K, and write $\mathfrak{a}$ for the Lie algebra of A. As a subgroup of G, the group A acts naturally on Y. We use this action to define the equivariant derived category $D_A(Y)$. Recall that the objects of this category are represented by the complexes of A-equivariant sheaves on Y. By the result of T. Matsuki [M], the group A acts on Y with finitely many orbits that determine a semialgebraic Whitney stratification. The image of an object from $D_A(Y)$ under the forgetful functor is constructible with respect to this stratification.

Next, we recall the construction of the equivalence of equivariant categories,

$$\gamma : D_K(Y) \longrightarrow D_{G_{\mathbb{R}}}(Y),$$

which was conjectured by M. Kashiwara and proved in [MUV]. Consider the maps

$$Y \xleftarrow{a} G_{\mathbb{R}} \times Y \xrightarrow{q} G_{\mathbb{R}}/K_{\mathbb{R}} \times Y \xrightarrow{p} Y$$

given by $a(g, y) = g^{-1}y$, $q(g, y) = (gK_{\mathbb{R}}, y)$, $p(gK_{\mathbb{R}}, y) = y$. These maps are $(G_{\mathbb{R}} \times K_{\mathbb{R}})$-equivariant with respect to the following actions: $(g, k) \cdot y = k \cdot y$, $(g, k) \cdot (g_1, y) = (gg_1k^{-1}, g \cdot y)$, $(g, k) \cdot (g_1K_{\mathbb{R}}, y) = (gg_1K_{\mathbb{R}}, g \cdot y)$, $(g, k) \cdot y = g \cdot y$. By restricting from K to $K_{\mathbb{R}}$, we may view $\mathscr{F} \in D_K(Y)$ as an object from $D_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(Y)$. We have $a^!(\mathscr{F}) \in D_{G_{\mathbb{R}} \times K_{\mathbb{R}}}(G_{\mathbb{R}} \times Y)$, and since $K_{\mathbb{R}}$ acts freely on $G_{\mathbb{R}} \times Y$, there exists $\mathscr{G} \in D_{G_{\mathbb{R}}/K_{\mathbb{R}}}(G_{\mathbb{R}} \times Y)$ such that $a^!(\mathscr{F}) \cong q^!(\mathscr{G})$. Now we define $\gamma(\mathscr{F}) = Rp_!(\mathscr{G})$. The functorial notation for an inverse and direct image is as in [KS].

We proceed to describe the action of γ on standard sheaves. Let S be an A-orbit on Y. Denote by $j : S \subset Y$ the inclusion. Let $\mathscr{L}$ be an A-equivariant local system on S. The standard sheaves associated with $(S, \mathscr{L})$ are defined by

$$\mathscr{I}^*_A(\mathscr{L}) = Rj_*(\mathscr{L}) \qquad \text{and} \qquad \mathscr{I}^!_A(\mathscr{L}) = Rj_!(\mathscr{L}).$$

There exists a natural bijection between K- and $G_{\mathbb{R}}$-orbits on Y, called the Matsuki correspondence. If the K-orbit Z and the $G_{\mathbb{R}}$-orbit S are in Matsuki correspondence, then $Z \cap S$ is a $K_{\mathbb{R}}$-orbit. Moreover, we also have a natural correspondence between local systems $\{\mathscr{L}\}$ on Z and $\{\mathscr{L}'\}$ on S characterized by

$$\mathscr{L}|_{Z\cap S} \cong \mathscr{L}'|_{Z\cap S}.$$

Denote by $\mathbb{C}_Y$ the constant sheaf on Y with stalk isomorphic to $\mathbb{C}$. The following result generalizes [MUV, Th. 6.6]. The proof from [MUV] applies in our situation without any changes.

THEOREM 1.2
The morphism $\gamma : D_K(Y) \longrightarrow D_{G_{\mathbb{R}}}(Y)$ is an equivalence of categories. Suppose that the pairs $(Z, \mathscr{L})$ and $(S, \mathscr{L}')$ are in Matsuki correspondence. Then the action of γ on standard sheaves is given by

$$\gamma\big(\mathscr{I}^*_K(\mathscr{L})\big) = \mathscr{I}^!_{G_{\mathbb{R}}}\big(\mathscr{L}' \otimes j^!(\mathbb{C}_Y)\big).$$

Use the complex structure to orient Y, and view Y as a real analytic manifold. Write T^*Y for the cotangent bundle of Y, and write T^*_AY for the union of conormal bundles of the A-orbits on Y. For a locally compact space Z, we denote by $H_*(Z, \mathbb{Z})$ ($H_*(Z, \mathbb{C})$) the Borel-Moore homology with integral (complex) coefficients. Set $2m = \dim_{\mathbb{R}} Y$. The characteristic cycle construction from [KS] yields a homomorphism from the Grothendieck group of $D_A(Y)$ to the top homology group of T^*_AY:

$$CC : \mathscr{K}\big(D_A(Y)\big) \longrightarrow H_{2m}(T^*_AY, \mathbb{Z}).$$

Define the map $\mu : T^*Y \longrightarrow \mathfrak{g}^*$ as the composition of the natural embedding $T^*Y \hookrightarrow Y \times \mathfrak{g}^*$ and the second projection $Y \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$. We call μ the moment map of Y. Next, we introduce a homomorphism

$$\Phi : H_{2m}(T^*_KY, \mathbb{Z}) \longrightarrow H_{2m}(T^*_{G_{\mathbb{R}}}Y, \mathbb{Z}),$$

which makes the following diagram commutative:

$$\begin{array}{ccc} \mathscr{K}\big(D_K(Y)\big) & \xrightarrow{\gamma} & \mathscr{K}\big(D_{G_{\mathbb{R}}}(Y)\big) \\ {\scriptstyle CC}\big\downarrow & & {\scriptstyle CC}\big\downarrow \\ H_{2m}(T^*_KY, \mathbb{Z}) & \xrightarrow{\Phi} & H_{2m}(T^*_{G_{\mathbb{R}}}Y, \mathbb{Z}) \end{array} \tag{1.3}$$

First, we define a family of bianalytic maps $F_s : T^*Y \longrightarrow T^*Y$, $s \in \mathbb{R}_{>0}$, by setting $F_s(\xi) = \ell\big(\exp(-s^{-1} \operatorname{Re} \mu(\xi))\big)^*(\xi)$, $\xi \in T^*Y$. Here we write $\operatorname{Re} \mu(\xi)$ for the real part of $\mu(\xi)$ relative to the real form $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$, and we write $\ell(g)^*$ for the map induced by the left translation $\ell(g) : Y \longrightarrow Y$, $g \in G$. One can check that for $C \in H_{2m}(T^*_K Y, \mathbb{Z})$, we obtain a family of cycles $(F_s(C), s > 0)$ in the sense of [SV1]. We draw attention now to the notion of the limit of a family of cycles from [SV1]. The proof of the next theorem is completely analogous to [SV3, proof of Th. 3.7].

THEOREM 1.4
For $C \in H_{2m}(T^*_K Y, \mathbb{Z})$, the limit $\Phi(C) = \lim_{s \to 0^+} F_s(C)$ exists and is supported in $T^*_{G_{\mathbb{R}}} Y$. The resulting homomorphism Φ makes diagram (1.3) commutative.

2. Integrals associated with characteristic cycles

In this section we study the convergence properties of integrals of certain differential forms over the cycles from $H_{2m}(T^*_{G_{\mathbb{R}}} Y, \mathbb{Z})$. We also establish results that are needed in order to study the asymptotics of such integrals. This material is completely analogous to [SV2, §3] and [SV3, §5].

Recall that $Y = G/P$, where P is a fixed parabolic subgroup. Write $y_0 \in Y$ for the point determined by P. Then Y is a homogeneous space for $U_{\mathbb{R}}$ and $U_{\mathbb{R}} \cap P$ is the centralizer of a torus in $U_{\mathbb{R}}$. Denote by $\mathfrak{c}_{\mathbb{R}}$ the Lie algebra of this torus. Extend $\mathfrak{c}_{\mathbb{R}}$ to a Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{u}_{\mathbb{R}}$, and write $\mathfrak{c} \subset \mathfrak{h}$ for the complexifications. Use $\mathfrak{h}$ to define Levi decompositions (1.1). The choice of $\mathfrak{l}$ defines a splitting $\mathfrak{p} = \mathfrak{c} \oplus [\mathfrak{p}, \mathfrak{p}]$. Consider the exact sequence

$$0 \longrightarrow (\mathfrak{g}/\mathfrak{p})^* \longrightarrow (\mathfrak{g}/[\mathfrak{p}, \mathfrak{p}])^* \longrightarrow \mathfrak{c}^* \longrightarrow 0.$$

Using a direct sum decomposition $\mathfrak{g} = \mathfrak{c} \oplus ([\mathfrak{p}, \mathfrak{p}] + [\mathfrak{g}, \mathfrak{c}])$, we define a section of $(\mathfrak{g}/[\mathfrak{p}, \mathfrak{p}])^* \longrightarrow \mathfrak{c}^*$. Thus the above sequence splits; that is, $(\mathfrak{g}/[\mathfrak{p}, \mathfrak{p}])^* \cong \mathfrak{c}^* \oplus (\mathfrak{g}/\mathfrak{p})^*$. In particular, we view $\mathfrak{c}^* \oplus \mathfrak{p}^\perp \subset \mathfrak{g}^*$. We say that $\lambda \in \mathfrak{h}^*$is P-regular if $\operatorname{cent}_{\mathfrak{g}}(\lambda) = \operatorname{cent}_{\mathfrak{g}}(\mathfrak{c})$.

Assume in the following definition that λ is P-regular. As in [R2] and [R1], define a twisted moment map $\mu_\lambda : T^*Y \longrightarrow G \cdot \lambda$ by the formula

$$\mu_\lambda\big(u \cdot (y_0, \nu)\big) = u \cdot (\lambda + \nu), \quad u \in U_{\mathbb{R}},\ \nu \in T^*_{y_0} Y \cong \mathfrak{p}^\perp.$$

Since λ is P-regular, $u_1 \cdot y_0 = u_2 \cdot y_0$, $u_1, u_2 \in U_{\mathbb{R}}$, implies $u_1 \cdot \lambda = u_2 \cdot \lambda$, and hence μ_λ is well defined.

PROPOSITION 2.1
If $\lambda \in \mathfrak{h}^*$ is P-regular, then $\mu_\lambda : T^*Y \longrightarrow G \cdot \lambda$ is a real algebraic isomorphism.

Proof
Consider the decomposition $\mathfrak{g} = \mathfrak{l} + \overline{\mathfrak{n}} + \mathfrak{n}$. Denote by h_λ the image of λ under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ defined by the form B. The assumption that λ is P-regular implies that $\mathrm{ad}(h_\lambda)$ is invertible on $\mathfrak{n}$. Thus we have $P \cdot h_\lambda = h_\lambda + \mathfrak{n}$. Applying the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ to this formula and using $\mathfrak{p}^\perp \cong \mathfrak{n}$, we obtain

$$P \cdot \lambda = \lambda + \mathfrak{p}^\perp.$$

We proceed to define the inverse $\epsilon_\lambda : G \cdot \lambda \longrightarrow T^*Y$ of μ_λ. Using the above formula and $G = U_{\mathbb{R}} \cdot P$, we write any $\xi \in G \cdot \lambda_0$ in the form $\xi = u \cdot (\lambda + \nu)$, where $u \in U_{\mathbb{R}}$ and $\nu \in \mathfrak{p}^\perp$. Now we set $\epsilon_\lambda(\xi) = (u \cdot y_0, u \cdot \nu)$. We have to show that ϵ_λ is well defined. Let $\xi = u_1 \cdot (\lambda + \nu_1)$, where $u_1 \in U_{\mathbb{R}}$ and $\nu_1 \in \mathfrak{p}^\perp$. Then we may find $p, p_1 \in P$ such that $\lambda + \nu = p \cdot \lambda$ and $\lambda + \nu_1 = p_1 \cdot \lambda$. The condition $up \cdot \lambda = u_1 p_1 \cdot \lambda$ implies $(u_1 p_1)^{-1} up \in P$. Thus $u_1^{-1} u \in U_{\mathbb{R}} \cap P$, and therefore $u \cdot y_0 = u_1 \cdot y_0$ and $u \cdot \lambda = u_1 \cdot \lambda$. Finally, we conclude that $(u \cdot y_0, u \cdot \nu) = (u_1 \cdot y_0, u_1 \cdot \nu_1)$; hence ϵ_λ is well defined. It follows immediately from the definitions that ϵ_λ is the inverse of μ_λ. □

Suppose that Z is a space with G-action ($U_{\mathbb{R}}$-action), and suppose that $\xi \in \mathfrak{g}$ ($\xi \in \mathfrak{u}_{\mathbb{R}}$). We denote by $l(\xi)$ the vector field on Z defined by the group action. Given a map f of smooth manifolds, we write f_* for its differential. If $u \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, we write $u \cdot \xi$ for the coadjoint action. After these notational preliminaries, we introduce differential forms $\sigma_{\mathscr{O}}$ and τ_λ.

Recall that each orbit $\mathscr{O} \subset \mathfrak{g}^*$ carries a canonical G-equivariant complex symplectic form $\sigma_{\mathscr{O}}$. If $\xi \in \mathscr{O}$ and $u_1 \cdot \xi, u_2 \cdot \xi \in T^*_\xi \mathscr{O}$, then $\sigma_{\mathscr{O}}$ is defined by

$$\sigma_{\mathscr{O}}|_\xi (u_1 \cdot \xi, u_2 \cdot \xi) = \xi([u_1, u_2]).$$

When $\mathscr{O} = G \cdot \lambda$, we simply write σ_λ for $\sigma_{\mathscr{O}}$. Furthermore, we define a $U_{\mathbb{R}}$-invariant two-form τ_λ on Y by the formula

$$\tau_\lambda|_{y_0}\big(l(u), l(v)\big) = \lambda([u, v]), \quad u,\ v \in \mathfrak{u}_{\mathbb{R}}.$$

When λ is P-regular, the differential forms τ_λ, σ_λ, and $\sigma_{\mathscr{O}}$ are related as follows (see [R2, Lem. 7.2], [R3, Lem. 1.3.1]).

PROPOSITION 2.2
Suppose that $\lambda \in \mathfrak{h}^*$ is P-regular, and suppose that $\mathscr{O} \subset \mathfrak{g}^*$ is a nilpotent orbit. Then

$$\mu_\lambda^* \sigma_\lambda = \mu^* \sigma_{\mathscr{O}} + \pi^* \tau_\lambda$$

at the smooth points of $\mu^{-1}(\mathscr{O})$.

Proof

Since Y is $U_{\mathbb{R}}$-homogeneous, it suffices to prove the formula at the smooth point $(y_0, \nu) \in \mu^{-1}(\mathscr{O})$. The tangent space $T_{(y_0,\nu)}T^*Y$ is spanned by $\ker(\pi_*)_{(y_0,\nu)} \cap T_{(y_0,\nu)}(\mu^{-1}\mathscr{O})$ and $l(\mathfrak{u}_{\mathbb{R}})$. We have to show that

$$\sigma_\lambda(\mu_{\lambda*}v_1, \mu_{\lambda*}v_2) = \tau_\lambda(\pi_* v_1, \pi_* v_2) + \sigma_{\mathscr{O}}(\mu_* v_1, \mu_* v_2) \quad \text{if } v_1, v_2 \in T_{(y_0,\nu)}(\mu^{-1}\mathscr{O}).$$

We consider three cases. First, suppose $v_i = l(u_i) \in l(\mathfrak{u}_{\mathbb{R}}) \subset T_{(y_0,\nu)}(\mu^{-1}\mathscr{O})$, $i = 1, 2$. Since the maps μ_λ, π, and μ are $U_{\mathbb{R}}$-equivariant, we have

$$\begin{aligned}\sigma_\lambda\big(\mu_{\lambda*}l(u_1), \mu_{\lambda*}l(u_2)\big) &= \sigma_\lambda\big(l(u_1), l(u_2)\big) = (\lambda+\nu)([u_1, u_2]) \\ &= \tau_\lambda\big(\pi_* l(u_1), \pi_* l(u_2)\big) + \sigma_{\mathscr{O}}\big(\mu_* l(u_1), \mu_* l(u_2)\big) \\ &= \pi^*\tau_\lambda\big(l(u_1), l(u_2)\big) + \mu^*\sigma_{\mathscr{O}}\big(l(u_1), l(u_2)\big),\end{aligned}$$

as desired. Next, we suppose $v_1 \in \ker\pi_* \cap T_{(y_0,\nu)}(\mu^{-1}\mathscr{O})$, $v_2 = l(u_2) \in l(\mathfrak{u}_{\mathbb{R}})$. In this case, we make the following identifications: $\ker\pi_{*(y_0,\nu)} \cong T_{(y_0,\nu)}(T^*_{y_0}Y) \cong T^*_{y_0}Y \cong \mathfrak{p}^\perp$. The restriction of μ_λ to $T^*_{y_0}Y$ is affine; hence $\mu_{\lambda*}v_1 = \mu_* v_1$. It follows that

$$\mu_{\lambda*}v_1 = u_1 \cdot (\lambda + \nu) = u_1' \cdot \nu = \mu_* v_1$$

for some $u_1, u_1' \in \mathfrak{g}$, and thus

$$\sigma_\lambda(\mu_{\lambda*}v_1, \mu_{\lambda*}v_2) = \sigma_\lambda\big(u_1 \cdot (\lambda+\nu), l(u_2)\big) = (\lambda+\nu)([u_1, u_2]) = \nu([u_1', u_2]).$$

On the other hand, $\tau_\lambda(\pi_* v_1, \pi_* v_2) = 0$ and $\sigma_{\mathscr{O}}(\mu_* v_1, \mu_* v_2) = \nu([u_1', u_2])$, so the desired equality holds again. Finally, we suppose $v_1, v_2 \in \ker\pi_* \cap T_{(y_0,\nu)}(\mu^{-1}\mathscr{O})$. Let $\mathfrak{b} \subset \mathfrak{p}$ be a Borel subalgebra of $\mathfrak{g}$. In view of our earlier identifications, we have

$$\mu_{\lambda*}v_1, \mu_{\lambda*}v_2 \in \lambda + \mathfrak{p}^\perp \subset G \cdot \lambda \qquad \text{and} \qquad \mu_* v_1, \mu_* v_2 \in \mathscr{O} \cap \mathfrak{p}^\perp \subset \mathscr{O} \cap \mathfrak{b}^\perp.$$

A direct computation shows that $\lambda + \mathfrak{p}^\perp \subset G \cdot \lambda$ is isotropic. On the other hand, $\mathscr{O} \cap \mathfrak{b}^\perp \subset \mathscr{O}$ is Lagrangian by the result of A. Joseph [J], and therefore $\mathscr{O} \cap \mathfrak{p}^\perp$ is isotropic. In other words,

$$\sigma_\lambda(\mu_{\lambda*}v_1, \mu_{\lambda*}v_2) = 0 = \tau_\lambda(\pi_* v_1, \pi_* v_2) + \sigma_{\mathscr{O}}(\mu_* v_1, \mu_* v_2).$$

This completes the proof. □

Let C be a semialgebraic chain in T^*Y. We say that C is $\mathbb{R}$-bounded if $\operatorname{Re}\mu(\operatorname{supp}(C))$ is a bounded subset of $\mathfrak{g}^*$ (see [SV2, §3]). Recall that the Fourier transform of a test function $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$ is defined by

$$\hat{\phi}(\xi) = \int_{\mathfrak{g}_{\mathbb{R}}} e^{\xi(x)}\phi(x)\,dx, \quad \xi \in \mathfrak{g}^*.$$

The proof of the next proposition comes down to the application of the fact that $\hat{\phi}$ decays rapidly in the imaginary directions.

PROPOSITION 2.3 ([R2, Vol. II, §1.2], [SV2, Lem. 3.16])
If C is a semialgebraic, $\mathbb{R}$-bounded $2m$-chain in T^*Y, then the integral

$$\int_C \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m), \quad \phi \in C_c^\infty(\mathfrak{g}_\mathbb{R}),$$

converges absolutely. The integral depends holomorphically on λ. In particular, if $C \in H_{2m}(T^*_{G_\mathbb{R}}Y, \mathbb{Z})$, the above integral converges.

In order to compare integrals over various cycles, we recall the notion of restricted homology. Let C_1, C_2 be semialgebraic, $\mathbb{R}$-bounded $2m$-cycles in T^*Y. We say that C_1 and C_2 are $\mathbb{R}$-homologous if there exists a semialgebraic, $\mathbb{R}$-bounded $(2m+1)$-chain C in T^*Y such that $C_1 - C_2 = \partial C$.

PROPOSITION 2.4 ([SV2, Lem. 3.19])
Suppose that C_1, C_2 are semialgebraic, $\mathbb{R}$-bounded $2m$-cycles in T^*Y. If C_1 and C_2 are $\mathbb{R}$-homologous, then

$$\int_{C_1} \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m) = \int_{C_2} \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m).$$

Denote by $\mathscr{N}$ the nilpotent cone in $\mathfrak{g}^*$. Observe that a nilpotent orbit has an even complex dimension. Write $\tilde{\mathscr{N}_k}$, respectively, $\mathscr{N}_k$, for the union of nilpotent orbits $\mathscr{O}$ such that $\dim_\mathbb{C} \mathscr{O} \leq 2k$, respectively, $\dim_\mathbb{C} \mathscr{O} = 2k$. Suppose that $\mathscr{O} \subset \mathscr{N}$ is a nilpotent orbit. Then $\mathscr{O} \cap \mathfrak{a}^\perp$ is a union of finitely many A-orbits of real dimension $2k$ (see [KR]).

A simple computation gives $T^*_A Y = \mu^{-1}(\mathfrak{a}^\perp)$. Suppose $\mathscr{N}_k \cap \mu(T^*_A Y) \neq \emptyset$. Then the restriction of the moment map $\mu : \mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp) \longrightarrow \mathscr{N}_k \cap \mathfrak{a}^\perp$ is an A-equivariant fibration whose typical fiber $\mu^{-1}(\xi)$ is a complex projective variety of complex dimension less than or equal to $m-k$. It follows that the real dimension of $\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{a}^\perp)$ is less than or equal to $2m$. Observe that the homomorphism $H_{2m}(\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{a}^\perp), \mathbb{Z}) \longrightarrow H_{2m}(T^*_A Y, \mathbb{Z})$, induced by the closed embedding $\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{a}^\perp) \subset T^*_A Y$, is injective. Hence, we view $H_{2m}(\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{a}^\perp), \mathbb{Z})$ as a subgroup of $H_{2m}(T^*_A Y, \mathbb{Z})$. Since $\mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp)$ is open in $\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{a}^\perp)$, the restriction of homology classes induces the homomorphism

$$H_{2m}\big(\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{a}^\perp), \mathbb{Z}\big) \longrightarrow H_{2m}\big(\mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp), \mathbb{Z}\big), \qquad C \mapsto C|_{\mathscr{N}_k \cap \mathfrak{a}^\perp}.$$

If $\lambda \in \mathfrak{c}^*$, consider the form $(1/(2\pi i))\tau_\lambda \in H^2(Y, \mathbb{C})$. Set

$$e^{\tau_\lambda/2\pi i} = 1 + \frac{1}{2\pi i}\tau_\lambda + \frac{1}{2!(2\pi i)^2}(\tau_\lambda)^2 + \cdots,$$

and write $\pi^* e^{\tau_\lambda/2\pi i}$ for the pullback of $e^{\tau_\lambda/2\pi i}$ under the projection $\pi : T^*Y \longrightarrow Y$. We use the form $e^{\tau_\lambda/2\pi i}$ to descend the cycles in $T^*_A Y$ to the nilpotent cone $\mathscr{N}$. In fact, suppose $C \in H_{2m}(\mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp), \mathbb{Z})$. We take the cap product of C against the component in $\pi^* e^{\tau_\lambda/2\pi i}$ of degree $2m - 2k$. This produces the class $C \cap \pi^* e^{\tau_\lambda/2\pi i} \in H_{2k}(\mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp), \mathbb{C})$. Finally, we define

$$\int_C \pi^* e^{\tau_\lambda/2\pi i} \in H_{2k}(\mathscr{N}_k \cap \mathfrak{a}^\perp, \mathbb{C})$$

as the pushforward of the cycle $C \cap \pi^* e^{\tau_\lambda/2\pi i}$ via the map $\mu : \mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp) \longrightarrow \mathscr{N}_k \cap \mathfrak{a}^\perp$. We remark that the cap product followed by the pushforward agrees with geometric operation of integration over the fiber (see [SV3, §5]). This fact is used in Section 4.

Now, if $H_{2m}(\mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp), \mathbb{Z}) \neq 0$, then $\dim_{\mathbb{R}} \mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp) = 2m$. Hence, $\dim_{\mathbb{R}} \mu^{-1}(\xi) = 2m - 2k$ for $\xi \in \mathscr{N}_k \cap \mathfrak{a}^\perp$, so we obtain the homomorphism $H_{2m}(\mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp), \mathbb{Z}) \longrightarrow H_{2k}(\mathscr{N}_k \cap \mathfrak{a}^\perp, \mathbb{C})$ by assigning $C \mapsto \int_C \pi^* e^{\tau_\lambda/2\pi i}$. Choose $C \neq 0$ from $H_{2m}(T^*_A Y, \mathbb{Z})$, and denote by $k = k(C)$ the least integer such that $\operatorname{supp}(C) \subset \mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{a}^\perp)$. Then $\dim_{\mathbb{R}} \operatorname{supp}(C) = 2m$ implies $\dim_{\mathbb{R}} \mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{a}^\perp) = 2m$, and therefore $H_{2m}(\mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp), \mathbb{Z}) \neq 0$. Finally, we define

$$(\operatorname{gr} \mu_*)_\lambda : H_{2m}(T^*_A Y, \mathbb{Z}) \longrightarrow \bigoplus H_{2k}(\mathscr{N}_k \cap \mathfrak{a}^\perp, \mathbb{C})$$

by the formula $(\operatorname{gr} \mu_*)_\lambda(C) = \int_{C^0} \pi^* e^{\tau_\lambda/2\pi i}$. Here $C^0 = C|_{\mu^{-1}(\mathscr{N}_k \cap \mathfrak{a}^\perp)}$.

The next goal is to define a homomorphism $\phi : \bigoplus H_{2k}(\mathscr{N}_k \cap \mathfrak{k}^\perp, \mathbb{C}) \longrightarrow \bigoplus H_{2k}(\mathscr{N}_k \cap \mathfrak{g}_{\mathbb{R}}^\perp, \mathbb{C})$, which makes the following diagram commutative:

$$\begin{tikzcd} H_{2m}(T^*_K Y, \mathbb{Z}) \arrow[r, "\Phi"] \arrow[d, "(\operatorname{gr} \mu_*)_\lambda"'] & H_{2m}(T^*_{G_{\mathbb{R}}} Y, \mathbb{Z}) \arrow[d, "(\operatorname{gr} \mu_*)_\lambda"'] \\ \bigoplus H_{2k}(\mathscr{N}_k \cap \mathfrak{k}^\perp, \mathbb{C}) \arrow[r, "\phi"] & \bigoplus H_{2k}(\mathscr{N}_k \cap \mathfrak{g}_{\mathbb{R}}^\perp, \mathbb{C}) \end{tikzcd} \tag{2.5}$$

Similarly, as before, consider the family of bianalytic maps $f_s : \mathscr{N} \longrightarrow \mathscr{N}$, $s \in \mathbb{R}_{>0}$, defined by the formula $f_s(\eta) = \operatorname{Ad}(\exp(-s^{-1} \operatorname{Re}(\eta)))\eta$, $\eta \in \mathscr{N}$. For notational simplicity, we denote here the coadjoint action also by Ad.

THEOREM 2.6 ([SV3, Th. 5.10])
Let C be a $2k$ cycle in $\mathscr{N}_k \cap \mathfrak{k}^\perp$. Then the limit of cycles $\phi(c) = \lim_{s \to 0^+} (f_s)_*(c)$ exists as a cycle in $\mathscr{N}_k$ and is supported in $\mathscr{N}_k \cap \mathfrak{g}_{\mathbb{R}}^\perp$. The induced homomorphism $\phi : \bigoplus H_{2k}(\mathscr{N}_k \cap \mathfrak{k}^\perp, \mathbb{C}) \longrightarrow \bigoplus H_{2k}(\mathscr{N}_k \cap \mathfrak{g}_{\mathbb{R}}^\perp, \mathbb{C})$ makes diagram (2.5) commutative.

The homomorphism ϕ has a particularly nice description of the invariant part of homology. First, we explain how to orient A-orbits. Given a G-orbit $\mathscr{O} \subset \mathscr{N}_k$, denote by $\sigma_{\mathscr{O}}$ the complex symplectic form on $\mathscr{O}$. Any K-orbit $\mathscr{O}_K \subset \mathscr{O} \cap \mathfrak{k}^{\perp}$ is a complex manifold of dimension k. We use the underlying complex structure to orient $\mathscr{O}_K$. On the other hand, any $G_{\mathbb{R}}$-orbit $\mathscr{O}_{G_{\mathbb{R}}} \subset \mathscr{O} \cap \mathfrak{g}_{\mathbb{R}}^{\perp}$ is a real manifold of dimension $2k$. Observe that $\sigma_{\mathscr{O}}$ is purely imaginary on $\mathscr{O}_{G_{\mathbb{R}}}$ and thus the restriction of $(1/2\pi i)\sigma_{\mathscr{O}}$ to $\mathscr{O}_{G_{\mathbb{R}}}$ determines a symplectic form. We use it to orient $\mathscr{O}_{G_{\mathbb{R}}}$. Hence, if $\mathscr{V} \subset \mathscr{O} \cap \mathfrak{a}^{\perp}$ is an A-orbit, we have a well-defined cycle $[\mathscr{V}]$ which determines a homology class in $H_{2k}(\mathscr{N}_k \cap \mathfrak{a}^{\perp}, \mathbb{C})$.

There exists a natural bijection between the set of K-orbits in $\mathscr{N} \cap \mathfrak{k}^{\perp}$ and the set of $G_{\mathbb{R}}$-orbits in $\mathscr{N} \cap \mathfrak{g}_{\mathbb{R}}^{\perp}$, called the Sekiguchi correspondence (see [S]). Recall that the orbits associated by the Sekiguchi correspondence lie in the same G-orbit and, in particular, have the same real dimension. The next result is [SV3, Th. 6.3].

THEOREM 2.7 ([SV3, Th. 6.3])
Let $\mathscr{O}_K$ be a K-orbit in $\mathscr{N} \cap \mathfrak{k}^{\perp}$. Then $\phi([\mathscr{O}_K]) = [\mathscr{O}_{G_{\mathbb{R}}}]$, where $\mathscr{O}_{G_{\mathbb{R}}}$ is the image of $\mathscr{O}_K$ under the Sekiguchi correspondence.

3. The case of open orbits

The goal of this section is to compute the characteristic cycle of a standard sheaf associated with an open $G_{\mathbb{R}}$-orbit on Y. The results are analogous to those in [SV2, §7]. We begin by fixing the notation. First, we draw attention to the notation already introduced in Section 1. In addition, assume that $G_{\mathbb{R}}$ has a compact Cartan subgroup $H_{\mathbb{R}}$. Write $\mathfrak{h}_{\mathbb{R}} = \mathrm{Lie}(H_{\mathbb{R}})$ and $\mathfrak{h} = (\mathfrak{h}_{\mathbb{R}})_{\mathbb{C}}$. The results of this section hold in the more general setting when $\mathfrak{h}_{\mathbb{R}}$ is fundamental, but the arguments (most notably the orientation statement in Lem. 3.2) are simpler under the assumption in force. Let $y_0 \in Y$. Denote by $P \subset G$ the parabolic subalgebra that stabilizes y_0. Set $\mathfrak{p} = \mathrm{Lie}(P)$, and assume that $\mathfrak{p} \cap \mathfrak{g}_{\mathbb{R}}$ contains a compact Cartan subalgebra. Without any loss of generality, we may assume $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{p} \cap \mathfrak{g}_{\mathbb{R}}$. Since the roots of $\mathfrak{h}$ are real valued on $i\mathfrak{h}_{\mathbb{R}}$, we have $\mathfrak{l} = \mathfrak{p} \cap \overline{\mathfrak{p}}$. This implies further that $\mathfrak{l}$ is the complexification of the real Lie algebra

$$\mathfrak{l}_{\mathbb{R}} = \mathfrak{l} \cap \mathfrak{g}_{\mathbb{R}} = \mathfrak{p} \cap \mathfrak{g}_{\mathbb{R}}.$$

Observe that $\mathfrak{l}_{\mathbb{R}}$ is the Lie algebra of the subgroup

$$L_{\mathbb{R}} = \left\{ g \in G_{\mathbb{R}} : \mathrm{Ad}(g)\mathfrak{p} = \mathfrak{p} \right\}.$$

Set $S = G_{\mathbb{R}} \cdot y_0$, and identify the tangent space of S, respectively, Y, at y_0 with $\mathfrak{g}_{\mathbb{R}}/\mathfrak{l}_{\mathbb{R}}$, respectively, $\mathfrak{g}/\mathfrak{p}$. A simple computation shows $\dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}/\mathfrak{l}_{\mathbb{R}} = \dim_{\mathbb{R}} \mathfrak{g}/\mathfrak{p}$; hence, the orbit S is open in Y. Conversely, if the orbit S is open, it is not difficult to show that $\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{p}$ contains a fundamental (in particular, compact, under our assumptions) Cartan subalgebra.

Let $\mathfrak{c}_{\mathbb{R}}$ be the center of $\mathfrak{l}_{\mathbb{R}}$. The choice of $\mathfrak{n}$ determines the positive chamber in $\mathfrak{c}^*$:

$$C^+ = \left\{\lambda \in i\mathfrak{c}_{\mathbb{R}}^* : \lambda(h_\alpha) > 0 \text{ if } \alpha \in \Delta(\mathfrak{n})\right\}. \tag{3.1}$$

Here $\mathfrak{c}_{\mathbb{R}}^*$ is the $\mathbb{R}$-linear dual and $h_\alpha \in \mathfrak{h}$ is the element corresponding to $\alpha \in \mathfrak{h}^*$ via the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. Clearly, any $\lambda \in C^+$ is P-regular, so the twisted moment map $\mu_\lambda : T^*Y \longrightarrow G \cdot \lambda$ is a real analytic isomorphism. Let $j : S \hookrightarrow Y$ be the inclusion. The goal is to show that the cycles $CC(Rj_*\mathbb{C}_S)$ and $[\mu_\lambda^{-1}(G_{\mathbb{R}} \cdot \lambda)]$ are $\mathbb{R}$-homologous for any $\lambda \in C^+$. First, we study the case $\lambda = 2\rho_{\mathfrak{p}}$.

Let $V = V_{2\rho_{\mathfrak{p}}}$ be the irreducible G-module with highest weight $2\rho_{\mathfrak{p}}$. Choose a $U_{\mathbb{R}}$-invariant positive definite hermitian form h_u on V. The space V has a natural action of the involution θ. In fact, identify V with the space of regular (algebraic) functions $F : G \longrightarrow \mathbb{C}$ satisfying the condition $F(gp) = e^{-2\rho_{\mathfrak{p}}}(p)F(g)$, $g \in G$, $p \in \overline{P}$. Here $\overline{P}$ is the parabolic subgroup of G opposite to P. Then we define θF by the formula $\theta F(g) = F(\theta g)$. One checks immediately that $\theta F \in V$. Another hermitian form h_r on V is defined by the formula

$$h_r(v_1, v_2) = h_u(v_1, \theta v_2), \quad v_1, v_2 \in V.$$

A short computation shows that h_r is $G_{\mathbb{R}}$-invariant. Write $\mathfrak{p}_y = \mathrm{Ad}(g)\mathfrak{p}$ and $\mathfrak{n}_y = \mathrm{Ad}(g)\mathfrak{n}$, whenever $y = g \cdot y_0$. Observe that $\mathfrak{n}_y$ does not depend on the choice of g. The space of invariant vectors $V^{\mathfrak{n}_y}$ is 1-dimensional, so the formula

$$f(y) = \frac{h_r(v, v)}{h_u(v, v)}, \quad v \in V^{\mathfrak{n}_y},$$

defines a real algebraic function on Y. If necessary, we multiply h_r by a constant so as to make f positive on S. In fact, by the $G_{\mathbb{R}}$-invariance of h_r, it suffices to examine the sign of $h_r(v, v)$, $v \in V^{\mathfrak{n}}$. Since $\mathfrak{n}$ is θ-stable, we have $\theta v \in V^{\mathfrak{n}}$. Hence, it suffices to multiply h_r by -1 if $\theta v = -v$. We denote the modified form also by h_r.

Write $Y^{\mathbb{R}}$ for the underlying real analytic structure on Y, and identify the real cotangent bundle $T^*Y^{\mathbb{R}}$ with the holomorphic cotangent bundle T^*Y via the pairing $(v_y, \xi_y) \mapsto 2\,\mathrm{Re}\langle \xi_y, v_y\rangle$, $v_y \in T_yY$, $\xi_y \in T_y^*Y$. View the differential $d \log f$ as a section of $T^*S^{\mathbb{R}}$, and orient it via the isomorphism $d \log f \cong S$ and the complex structure on S. On the other hand, orient $G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{p}})$ so that the top exterior power of $-i\sigma_{2\rho_{\mathfrak{p}}}$ is positive. Set $s = \dim_{\mathbb{C}} \mathfrak{n} \cap \mathfrak{k}$. The properties of the function f are given by the following lemma.

LEMMA 3.2

The function f is positive on S and vanishes on the boundary of S. The twisted moment map $\mu_{2\rho_{\mathfrak{p}}}$ restricted to $d \log f$ defines a real algebraic isomorphism

$$\mu_{2\rho_{\mathfrak{p}}} : d \log f \longrightarrow G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{p}})$$

which preserves (resp., reverses) orientation if s is even (resp., odd).

Proof
Our argument is completely analogous to [SV2, proof of Lem. 7.10]. Let $y \in \mathrm{Cl}(S) \setminus S$. We show $h_r(v, v) = 0$ for $v \in V^{\mathfrak{n}_y}$. The form h_r is $G_{\mathbb{R}}$-invariant, so conjugating by an element from $G_{\mathbb{R}}$, we may assume that $\mathfrak{p}_y$ contains a Cartan subalgebra $\mathfrak{h}_1$ stable for θ and the conjugation with respect to $\mathfrak{g}_{\mathbb{R}}$. Then $\mathfrak{h}_1$ is also stable for the conjugation with respect to the compact real form $\mathfrak{u}_{\mathbb{R}}$. Let $V(\lambda_1)$ and $V(\lambda_2)$ be $\mathfrak{h}_1$-weight spaces for $\lambda_1 \neq \lambda_2$. We claim $h_u(V(\lambda_1), V(\lambda_2)) = 0$. Let $v_1 \in V(\lambda_1)$, $v_2 \in V(\lambda_2)$, and choose $h \in \mathfrak{h}_1 \cap \mathfrak{u}_{\mathbb{R}}$ such that $\lambda_1(h) \neq \lambda_2(h)$. Since λ_2 is the sum of roots, we have $\lambda_2(h) \in i\mathbb{R}$ and, consequently,

$$\lambda_1(h)h_u(v_1, v_2) = h_u(h \cdot v_1, v_2) = -h_u(v_1, h \cdot v_2) = \lambda_2(h)h_u(v_1, v_2).$$

Hence, $h_u(v_1, v_2) = 0$, as desired. Observe that $V^{\mathfrak{n}_y} = V(2\rho_y)$ and $\theta V^{\mathfrak{n}_y} = V(2\theta\rho_y)$. Here $2\rho_y$ is the sum of roots from $\mathfrak{n}_y$. To prove $h_r(v, v) = 0$ for $v \in V^{\mathfrak{n}_y}$, it suffices to show $\rho_y \neq \theta\rho_y$. Otherwise, $\rho_y = \theta\rho_y$ would imply $\rho_y \in i\mathfrak{k}^*_{\mathbb{R}}$. Then $\mathfrak{p}_y \cap \mathfrak{g}_{\mathbb{R}}$ would contain a compact Cartan subalgebra. However, this is impossible since $G_{\mathbb{R}} \cdot y$ is not open.

Next, we show that $\mu_{2\rho_{\mathfrak{p}}}$ induces a real algebraic isomorphism between $d \log f$ and $G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{p}})$. Let $\nu \in \mathfrak{p}^{\perp}$, $u \in U_{\mathbb{R}}$. Suppose $\mu_{2\rho_{\mathfrak{p}}}(uy_0, \mathrm{Ad}(u)\nu) = \mathrm{Ad}(g)(2\rho_{\mathfrak{p}})$ for some $g \in G_{\mathbb{R}}$. The definition of $\mu_{2\rho_{\mathfrak{p}}}$ implies

$$\mathrm{Ad}(u)\nu = \mathrm{Ad}(g)(2\rho_{\mathfrak{p}}) - \mathrm{Ad}(u)(2\rho_{\mathfrak{p}}).$$

Rewriting this condition and using the fact that $2\rho_{\mathfrak{p}}$ is P-regular, we obtain

$$\mathrm{Ad}(u^{-1}g)(2\rho_{\mathfrak{p}}) = 2\rho_{\mathfrak{p}} + \nu = \mathrm{Ad}(p)(2\rho_{\mathfrak{p}})$$

for some $p \in P$. Then $\mathrm{Ad}(g^{-1}up)\rho_{\mathfrak{p}} = \rho_{\mathfrak{p}}$, and, consequently, $gy_0 = uy_0 = y$. Thus we obtain

$$\mu^{-1}_{2\rho_{\mathfrak{p}}}\big(\mathrm{Ad}(g)(2\rho_{\mathfrak{p}})\big) = \big(gy_0, \mathrm{Ad}(g)(2\rho_{\mathfrak{p}}) - \mathrm{Ad}(u)(2\rho_{\mathfrak{p}})\big), \quad g \in G_{\mathbb{R}},\ gy_0 = uy_0.$$

To establish the statement about the isomorphism, it suffices to prove

$$d \log f|_{gy_0} = \big(gy_0, \mathrm{Ad}(g)(2\rho_{\mathfrak{p}}) - \mathrm{Ad}(u)(2\rho_{\mathfrak{p}})\big), \quad g \in G_{\mathbb{R}},\ gy_0 = uy_0.$$

Regarding $d \log f$ as a section of $T^*Y^{\mathbb{R}}$, we have for $g \in G_{\mathbb{R}}$ and $Z \in \mathfrak{g}$,

$$\frac{d}{dt} \log f(\exp tZ \cdot gy_0)|_{t=0} = 2\,\mathrm{Re}\big\langle \mu(d \log f|_{gy_0}), Z\big\rangle.$$

On the other hand, a simple calculation yields

$$\frac{d}{dt} \log f(\exp tZ \cdot gy_0)|_{t=0} = \frac{h_r(Zv, v) + h_r(v, Zv)}{h_r(v, v)} - \frac{h_u(Zv, v) + h_u(v, Zv)}{h_u(v, v)}$$

for any $v \in V^{\mathfrak{n}_y}$. Since the space $V^{\mathfrak{n}_y}$ is 1-dimensional, we may take $v = gv_0$ in the first term and $v = uv_0$ in the second term on the right-hand side. Here $g \in G_{\mathbb{R}}$, $u \in U_{\mathbb{R}}$, $v_0 \in V^{\mathfrak{n}}$, and $gy_0 = uy_0$. Consider the decomposition $\mathfrak{g} = \mathfrak{c} + [\mathfrak{p}, \mathfrak{p}] + \overline{\mathfrak{n}}$. If $T \in [\mathfrak{p}, \mathfrak{p}] + \overline{\mathfrak{n}}$, then $\mathbb{C} \cdot Tv_0 \cap \mathbb{C} \cdot v_0 = \{0\}$. Since the distinct weight spaces are orthogonal with respect to h_u, we obtain $h_r(Tv_0, v_0) = 0$. On the other hand, if $T \in \mathfrak{c}$, then $Tv_0 = \langle 2\rho_{\mathfrak{p}}, T\rangle v_0$. Using the above decomposition of $\mathfrak{g}$, we regard $2\rho_{\mathfrak{p}} \in \mathfrak{g}^*$ and apply the preceding discussion to $T = \mathrm{Ad}(g^{-1})Z$. We deduce

$$\begin{aligned}\frac{h_r(Zgv_0, gv_0) + h_r(gv_0, Zgv_0)}{h_r(gv_0, gv_0)} &= \frac{h_r(\mathrm{Ad}(g^{-1})Zv_0, v_0) + h_r(v_0, \mathrm{Ad}(g^{-1})Zv_0)}{h_r(v_0, v_0)} \\ &= 2\,\mathrm{Re}\big\langle 2\rho_{\mathfrak{p}}, \mathrm{Ad}(g^{-1})Z\big\rangle.\end{aligned}$$

The analogous calculation for the second term finally yields

$$\mu(d \log f|_{gy_0}) = \mathrm{Ad}(g)(2\rho_{\mathfrak{p}}) - \mathrm{Ad}(u)(2\rho_{\mathfrak{p}}),$$

as desired.

Now we turn to the orientation statement. It suffices to compare the orientations of S and $G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{p}})$ only at points y_0 and $2\rho_{\mathfrak{p}}$. First, we describe the tangent spaces. The holomorphic tangent space $T_{2\rho_{\mathfrak{p}}} G \cdot (2\rho_{\mathfrak{p}})$ is isomorphic to $\mathfrak{g}/\mathfrak{l} \simeq \bigoplus_{\alpha \in \pm\Delta(\mathfrak{n})} \mathfrak{g}^{\alpha}$, and the real tangent space $T_{2\rho_{\mathfrak{p}}} G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{p}})$ is isomorphic to $\mathfrak{g}_{\mathbb{R}}/\mathfrak{l}_{\mathbb{R}} \subset \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}/\mathfrak{l}_{\mathbb{R}} \simeq \mathfrak{g}/\mathfrak{l}$. On the other hand, the real tangent space $T_{y_0} S^{\mathbb{R}}$ is isomorphic to $\mathfrak{g}_{\mathbb{R}}/\mathfrak{l}_{\mathbb{R}}$. The map $I : S \longrightarrow G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{p}})$, $I(gy_0) = \mathrm{Ad}(g)(2\rho_{\mathfrak{p}})$, $g \in G_{\mathbb{R}}$, is a real algebraic isomorphism, and we use it to compare the orientations. Via the above identifications of the tangent spaces, the differential of I at y_0 becomes the identity on $\mathfrak{g}_{\mathbb{R}}/\mathfrak{l}_{\mathbb{R}}$. As before, we regard $\rho_{\mathfrak{p}} \in \mathfrak{g}^*$. Then the symplectic form $\sigma_{2\rho_{\mathfrak{p}}}$ at $2\rho_{\mathfrak{p}}$ is given by the formula

$$\sigma_{2\rho_{\mathfrak{p}}}(T_1, T_2) = \big\langle 2\rho_{\mathfrak{p}}, [T_1, T_2]\big\rangle, \quad T_1,\ T_2 \in \mathfrak{n} + \overline{\mathfrak{n}}.$$

In particular, the root spaces $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{\beta}$ are orthogonal for $\sigma_{2\rho_{\mathfrak{p}}}$ unless $\alpha + \beta = 0$. For $\alpha \in \Delta(\mathfrak{n})$, consider the 3-dimensional subalgebra

$$\mathfrak{g}_{\alpha} = \mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}].$$

Let G_{α}, respectively, $G_{\alpha,\mathbb{R}}$, be the connected subgroup of G with Lie algebra $\mathfrak{g}_{\alpha}$, respectively, $\mathfrak{g}_{\alpha} \cap \mathfrak{g}_{\mathbb{R}}$. Clearly, $G_{\alpha,\mathbb{R}} \subset G_{\alpha}$ is a real form. Denote by B_{α} the Borel subgroup of G_{α} with Lie algebra $\mathfrak{g}^{\alpha} + [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]$. Enumerate the elements of $\Delta(\mathfrak{n})$ as $\alpha_1, \dots, \alpha_m$, and write y_i for the point in the flag variety of $\mathfrak{g}_{\alpha}$ defined by B_{α_i}, $i = 1, \dots, m$. Observe that $\rho_{\mathfrak{p}}$ restricts to a regular weight on $\mathfrak{g}_{\alpha}$, and consider the maps

$$G_{\alpha_1} \times \cdots \times G_{\alpha_m} \longrightarrow Y, \qquad (g_1, \dots, g_m) \mapsto g_1 \cdots g_m y_0$$

and

$$G_{\alpha_1} \times \cdots \times G_{\alpha_m} \longrightarrow G \cdot (2\rho_{\mathfrak{p}}), \qquad (g_1, \ldots, g_m) \mapsto \mathrm{Ad}(g_1 \cdots g_m)(2\rho_{\mathfrak{p}}).$$

They induce further the maps

$$G_{\alpha_1}/B_{\alpha_1} \times \cdots \times G_{\alpha_m}/B_{\alpha_m} \longrightarrow Y$$

and

$$G_{\alpha_1} \cdot (2\rho_{\mathfrak{p}}) \times \cdots \times G_{\alpha_m} \cdot (2\rho_{\mathfrak{p}}) \longrightarrow G \cdot (2\rho_{\mathfrak{p}}).$$

For the real orbits, we obtain the maps

$$G_{\alpha_1,\mathbb{R}} \cdot y_1 \times \cdots \times G_{\alpha_m,\mathbb{R}} \cdot y_m \longrightarrow Y$$

and

$$G_{\alpha_1,\mathbb{R}} \cdot (2\rho_{\mathfrak{p}}) \times \cdots \times G_{\alpha_m,\mathbb{R}} \cdot (2\rho_{\mathfrak{p}}) \longrightarrow G_{\mathbb{R}} \cdot (2\rho_{\mathfrak{p}}).$$

By considering the tangent maps, we deduce immediately that all these maps are local isomorphisms, compatible with complex structures and symplectic forms. In this way the problem is reduced to the special case of $G_{\mathbb{R}} = \mathrm{SU}(1, 1)$ if α is noncompact and $G_{\mathbb{R}} = \mathrm{SU}(2)$ if α is compact. In these cases the orientation statement is easily established by a direct computation. □

THEOREM 3.3

Set $\lambda = 2\rho_{\mathfrak{p}}$, *and consider* $\mu_\lambda^{-1}(G_{\mathbb{R}}\cdot\lambda)$ *as a* $2m$*-cycle on* T^*Y. *Then, for* $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$, *the following formula holds:*

$$\int_{CC(Rj_*\mathbb{C})} \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m) = (-1)^s \int_{\mu_\lambda^{-1}(G_{\mathbb{R}}\cdot\lambda)} \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m).$$

Proof

Set $C_0 = CC(Rj_*\mathbb{C}_S)$, $C_1 = d\log(f|S)$. View C_0 and C_1 as $2m$-cycles on T^*Y. Consider the map $l : (0, 1) \times T^*Y \longrightarrow T^*Y$ defined by $l(t, (y, \xi)) = (y, t\xi)$, $t \in \mathbb{R}$, $y \in Y$, $\xi \in T_y^*Y$. Let $\tilde{C}$ be a $(2m + 1)$-chain on T^*Y with support equal to $l((0, 1) \times \mathrm{supp}(C_1))$ and the orientation determined by the product orientation of $(0, 1) \times \mathrm{supp}(C_1)$. Applying [SV1, Prop. 3.25, Th. 4.2], we obtain

$$C_0 - C_1 = \partial\tilde{C}.$$

Since $\mathrm{Re}\,\mu$ is bounded on $\mathrm{supp}(\tilde{C})$, the statement of the theorem follows from Proposition 2.4 and Lemma 3.2. □

THEOREM 3.4
Suppose that $\lambda \in \mathfrak{h}^*$ lies in the positive chamber C^+. Then the formula from Theorem 3.3 holds.

Proof
Set $\lambda(t) = (1-t)(2\rho_{\mathfrak{p}}) + t\lambda$, $t \in [0,1]$. Clearly, $\lambda(t) \in C^+$, and in particular, $\lambda(t)$ is P-regular. Observe that $S \cong G_{\mathbb{R}} \cdot \lambda$ and that the same orientation statement as in Lemma 3.2 holds. Arguing similarly as in the proof of Theorem 3.3, we find

$$\big[\mu_{\lambda(0)}^{-1}\big(G_{\mathbb{R}} \cdot (\lambda(0))\big)\big] - \big[\mu_{\lambda(1)}^{-1}\big(G_{\mathbb{R}} \cdot (\lambda(1))\big)\big] = \partial \tilde{C},$$

where $\tilde{C}$ is a $(2m+1)$-chain on T^*Y such that $\operatorname{Re}\mu$ is bounded on $\operatorname{supp}(\tilde{C})$. We use brackets to indicate that a given set is considered as a cycle. The claim now follows from Theorem 3.3 and Proposition 2.4. □

4. Elliptic orbital integrals

In this section we study the nilpotent orbital integrals which arise as leading terms in the asymptotic expansion of Fourier transforms of elliptic orbital integrals. The material is analogous to [R2, §§7, 8] and [R3, §1]. We begin with a simple observation. We use the notation introduced in Sections 1 – 3.

LEMMA 4.1
Let $\lambda \in \mathfrak{c}^*$ and $k \in \mathbb{Z}_+$. Then

$$\tau_\lambda^k = \sum_{i=1}^{s} p_i(\lambda)\tau_i,$$

where $p_1, \ldots, p_s$ are homogeneous polynomials on $\mathfrak{c}^*$ of degree k and $\tau_1, \ldots, \tau_s$ are $U_{\mathbb{R}}$-invariant $2k$-forms on Y not depending on λ. The polynomials p_i are real valued on $\mathfrak{c}_{\mathbb{R}}$.

Proof
Recall that $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{u}_{\mathbb{R}} \cap \mathfrak{p}$ is a Cartan subalgebra. For any $\alpha \in \Delta(\mathfrak{g})$ we may find a vector x_α in the α-weight space $\mathfrak{g}^\alpha$ so that

$$x_\alpha - x_{-\alpha},\ i(x_\alpha + x_{-\alpha}) \in \mathfrak{u}_{\mathbb{R}} \qquad \text{and} \qquad [x_\alpha, x_{-\alpha}] = \frac{2}{\alpha(h_\alpha)} h_\alpha.$$

Set $e_\alpha = x_\alpha - x_{-\alpha}$, $f_\alpha = i(x_\alpha + x_{-\alpha})$, and define the spaces $\mathfrak{r}_1 = \sum_{\alpha \in \Delta^+(\mathfrak{l})}(\mathbb{R}e_\alpha + \mathbb{R}f_\alpha)$, $\mathfrak{r}_2 = \sum_{\alpha \in \Delta(\bar{\mathfrak{n}})}(\mathbb{R}e_\alpha + \mathbb{R}f_\alpha)$. We have the direct sum decompositions

$$\mathfrak{u}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} + \mathfrak{r}_1 + \mathfrak{r}_2, \qquad \mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \cap [\mathfrak{l}_{\mathbb{R}}, \mathfrak{l}_{\mathbb{R}}] + \mathfrak{c}_{\mathbb{R}},$$

and we use them to identify $\mathbb{R}$-linear duals $\mathfrak{h}_{\mathbb{R}}^*, \mathfrak{c}_{\mathbb{R}}^*, \mathfrak{r}_1^*, \mathfrak{r}_2^*$ with subspaces of $\mathfrak{u}_{\mathbb{R}}^*$. Denote by $(e_\alpha^*, f_\alpha^*; \alpha \in \Delta(\bar{\mathfrak{n}}))$ the basis in $\mathfrak{r}_2^*$ dual to $(e_\alpha, f_\alpha; \alpha \in \Delta(\bar{\mathfrak{n}}))$. Let $\tilde{e}_\alpha$, respectively, $\tilde{f}_\alpha$, be a unique $U_{\mathbb{R}}$-invariant form on Y whose value at y_0 is e_α^*, respectively, f_α^*. The form τ_λ can then be written as

$$\tau_\lambda = \sum_{\alpha \in \Delta(\bar{\mathfrak{n}})} \lambda(ih_\alpha)\tilde{e}_\alpha \wedge \tilde{f}_\alpha.$$

All the claims of the lemma are immediate consequences of this formula. □

Recall that we have the decomposition $T_{G_{\mathbb{R}}}^* Y = \bigcup_{l \geq 0} \mu^{-1}(\mathscr{N}_l \cap \mathfrak{g}_{\mathbb{R}}^\perp)$, so that $\dim_{\mathbb{R}} \mu^{-1}(\mathscr{N}_l \cap \mathfrak{g}_{\mathbb{R}}^\perp) \leq 2m$. Set

$$I = \{l \in \mathbb{Z} : \dim_{\mathbb{R}} \mu^{-1}(\mathscr{N}_l \cap \mathfrak{g}_{\mathbb{R}}^\perp) = 2m\}.$$

Choose $C \in H_{2m}(T_{G_{\mathbb{R}}}^* Y, \mathbb{Z})$, and let $k = k(C)$ be the minimal $k \in \mathbb{Z}_+$ such that $C \in H_{2m}(\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{g}_{\mathbb{R}}^\perp), \mathbb{Z})$. In Section 1, we remarked that $k \in I$. Set $T_{G_{\mathbb{R}}}^* Y_C = \bigcup_{l \in I, l \leq k} \mu^{-1}(\mathscr{N}_l \cap \mathfrak{g}_{\mathbb{R}}^\perp)$.

LEMMA 4.2

In the group of $2m$-chains on $T_{G_{\mathbb{R}}}^* Y$, we have $C = \sum_{l \in I, l \leq k} C_l$, where C_l is a $G_{\mathbb{R}}$-invariant $2m$-chain on $\mu^{-1}(\mathscr{N}_l \cap \mathfrak{g}_{\mathbb{R}}^\perp)$. If $l = k$, C_k is the restriction of C to $\mu^{-1}(\mathscr{N}_k \cap \mathfrak{g}_{\mathbb{R}}^\perp)$.

Proof

The definition of $T_{G_{\mathbb{R}}}^* Y_C$ implies $\dim_{\mathbb{R}} \mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{g}_{\mathbb{R}}^\perp) \setminus \overline{T_{G_{\mathbb{R}}}^* Y_C} < 2m$, so the exact sequence

$$\begin{aligned} 0 \longrightarrow H_{2m}(\overline{T_{G_{\mathbb{R}}}^* Y_C}, \mathbb{Z}) &\longrightarrow H_{2m}\big(\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{g}_{\mathbb{R}}^\perp), \mathbb{Z}\big) \\ &\longrightarrow H_{2m}\big(\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{g}_{\mathbb{R}}^\perp) \setminus \overline{T_{G_{\mathbb{R}}}^* Y_C}, \mathbb{Z}\big) \longrightarrow \cdots \end{aligned}$$

determines the isomorphism $H_{2m}(\overline{T_{G_{\mathbb{R}}}^* Y_C}, \mathbb{Z}) \cong H_{2m}(\mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{g}_{\mathbb{R}}^\perp), \mathbb{Z})$. Write $\partial T_{G_{\mathbb{R}}}^* Y_C = \overline{T_{G_{\mathbb{R}}}^* Y_C} \setminus T_{G_{\mathbb{R}}}^* Y_C$, and consider the exact sequence

$$0 \longrightarrow H_{2m}(\partial T_{G_{\mathbb{R}}}^* Y_C, \mathbb{Z}) \longrightarrow H_{2m}(\overline{T_{G_{\mathbb{R}}}^* Y_C}, \mathbb{Z}) \longrightarrow H_{2m}(T_{G_{\mathbb{R}}}^* Y_C, \mathbb{Z}) \longrightarrow \cdots .$$

The homomorphism $H_{2m}(\overline{T_{G_{\mathbb{R}}}^* Y_C}, \mathbb{Z}) \longrightarrow H_{2m}(T_{G_{\mathbb{R}}}^* Y_C, \mathbb{Z})$ is injective since $\dim_{\mathbb{R}} \partial T_{G_{\mathbb{R}}}^* Y_C < 2m$. Via this injection, we view C as an element from $H_{2m}(T_{G_{\mathbb{R}}}^* Y_C, \mathbb{Z})$. For $l \in I, l \leq k = k(C)$, denote by U_l the interior of $\mu^{-1}(\mathscr{N}_l \cap \mathfrak{g}_{\mathbb{R}})$ in $T_{G_{\mathbb{R}}}^* Y_C$. Applying repeatedly a Mayer-Vietoris sequence, we construct the injective homomorphism

$$0 \longrightarrow H_{2m}(T_{G_{\mathbb{R}}}^* Y_C, \mathbb{Z}) \longrightarrow \bigoplus_{l \in I, l \leq k} H_{2m}(U_l, \mathbb{Z}).$$

If we identify C with its image in $\bigoplus_{l\in I, l\leq k} H_{2m}(U_l, \mathbb{Z})$, we obtain the required decomposition $C = \sum_{l\in I, l\leq k} C_l$. The cycle C is $G_{\mathbb{R}}$-invariant, and all the constructions involved are $G_{\mathbb{R}}$-invariant; so it follows that C_l is also $G_{\mathbb{R}}$-invariant. Finally, we remark that $U_k = \mu^{-1}(\mathcal{N}_k \cap \mathfrak{g}_{\mathbb{R}}^{\perp})$; hence, C_k is the restriction of C to $\mu^{-1}(\mathcal{N}_k \cap \mathfrak{g}_{\mathbb{R}}^{\perp})$. $\square$

Let $\mathcal{O} \subset \mathcal{N} \cap \mathfrak{a}^{\perp}$ be an A-orbit. The moment map restricts to a fibration $\mu : \mu^{-1}\mathcal{O} \longrightarrow \mathcal{O}$. Let $2d_{\mathcal{O}} = \dim_{\mathbb{R}} \mathcal{O}$, and let $2e_{\mathcal{O}} = \dim_{\mathbb{R}} \mu^{-1}\nu$ ($\nu \in \mathcal{O}$). We orient $\mathcal{O}$ as in Section 2. The orientation on $\mu^{-1}\mathcal{O}$ is defined via the fibration $\mu : \mu^{-1}\mathcal{O} \longrightarrow \mathcal{O}$, using the orientation on $\mathcal{O}$ and the orientation on the fibers induced by the underlying complex structure. Fix $\nu \in \mathcal{O}$, and denote by $Z_A(\nu)$ the centralizer of ν in A. Then for $p \in \mathbb{Z}_+$ there exists a natural bijection between $Z_A(\nu)$-invariant $(2e_{\mathcal{O}} - p)$-chains on $\mu^{-1}\nu$ and A-invariant $(2m - p)$-chains on $\mu^{-1}\mathcal{O}$. On the level of sets, this bijection is given by

$$C(\nu) \mapsto C = A \cdot C(\nu), \qquad C \mapsto C(\nu) = C \cap \mu^{-1}\nu. \tag{4.3}$$

We recall the Lebesgue-Fubini theorem in the form convenient for our applications. Let V and W be C^{∞}-manifolds of dimensions r and s, respectively. Let α and β be top-dimensional differential forms on V and W. Suppose that β is nonzero at any point of W. Let $f : V \longrightarrow W$ be a submersion. Then there exists an $(r-s)$-form γ on V, so that $\alpha = \gamma \wedge f^*\beta$. Here we denote by $f^*\beta$ the pullback of β to V along f. Write α_w and γ_w for the restriction of α and γ to the fiber $f^{-1}w$, and write $\beta(w)$ for the value of β at $w \in W$. Then γ_w is uniquely determined by α_w and $\beta(w)$, so we write $\gamma_w = \alpha_w/\beta(w)$. Suppose that V and W are oriented. Then we orient the fibers $f^{-1}w$ compatibly with f. Further, we assume that β is the orientation form on W and denote by m_β the corresponding measure. If the form α is integrable, then for almost all $w \in W$ with respect to m_β, the form γ_w is integrable on $f^{-1}w$ and (almost everywhere) defined form $w \mapsto (\int_{f^{-1}w} \gamma_w)\beta(w)$ is integrable on W. For the corresponding integrals, the following formula holds:

$$\int_V \alpha = \int_{w\in W} \Big(\int_{f^{-1}w} \gamma_w \Big) \beta(w). \tag{4.4}$$

Next, we describe how (4.4) is applied. Let $\mathcal{O} \subset \mathcal{N} \cap \mathfrak{g}_{\mathbb{R}}^{\perp}$ be a $G_{\mathbb{R}}$-orbit, and let $U \subset \mu^{-1}\mathcal{O}$ be an open $G_{\mathbb{R}}$-invariant set. Let $C \in H_{2m}(U, \mathbb{Z})$. Choose $\nu \in \mathcal{O}$, and denote by $\mu^{-1}\nu_r \subset \mu^{-1}\nu$ the set of regular points. The set $U \cap G_{\mathbb{R}} \cdot \mu^{-1}\nu_r$ is open dense in U and $\dim_{\mathbb{R}}(U \setminus U \cap G_{\mathbb{R}} \cdot \mu^{-1}\nu_r) < 2m$. It follows that the restriction homomorphism $H_{2m}(U, \mathbb{Z}) \longrightarrow H_{2m}(U \cap G_{\mathbb{R}} \cdot \mu^{-1}\nu_r, \mathbb{Z})$ is injective. Any $2m$-cycle on $U \cap G_{\mathbb{R}} \cdot \mu^{-1}\nu_r$ is a $\mathbb{Z}$-linear combination of connected components. If we choose such a component V, then $\mu : V \longrightarrow \mathcal{O}$ is a submersion, and we may apply (4.4) to the forms $\alpha = \mu_\lambda^*(\hat{\phi} c^m \sigma_\lambda^m)|V$ and $\beta = c^{d_{\mathcal{O}}} \sigma_{\mathcal{O}}^{d_{\mathcal{O}}}$. Here $\phi \in C_c^{\infty}(\mathfrak{g}_{\mathbb{R}})$, and we set for simplicity $c = 1/(2\pi i)$. Adding up terms corresponding to the various components of the chain C, we obtain

$$\int_C \mu_\lambda^*(\hat{\phi} c^m \sigma_\lambda^m) = \int_{\nu \in \mathscr{O}} \Big(\int_{C(\nu)} \gamma_\nu \Big) c^{d_{\mathscr{O}}} \sigma_{\mathscr{O}}^{d_{\mathscr{O}}}. \tag{4.5}$$

Since C is $G_{\mathbb{R}}$-invariant, the inner integral converges for any $\nu \in \mathscr{O}$. We can describe (4.5) more explicitly. Taking into account Proposition 2.3, we have on $\mu^{-1}\mathscr{O}$,

$$\mu_\lambda^*(\sigma_\lambda^m) = (\pi^*\tau_\lambda + \mu^*\sigma_{\mathscr{O}})^m = \sum_{i=0}^m \binom{m}{i} \pi^*\tau_\lambda^i \wedge \mu^*\sigma_{\mathscr{O}}^{m-i}.$$

Hence, we obtain for the quotient form

$$\frac{\mu_\lambda^*(\sigma_\lambda^m)}{\mu^*(\sigma_{\mathscr{O}}^{d_{\mathscr{O}}})} = \sum_{i=e_{\mathscr{O}}}^m \binom{m}{i} \frac{\pi^*\tau_\lambda^i \wedge \mu^*\sigma_{\mathscr{O}}^{m-i}}{\mu^*(\sigma_{\mathscr{O}}^{d_{\mathscr{O}}})}. \tag{4.6}$$

For $(y, \nu) \in T^*Y$, set $(y, \nu) = u(y) \cdot (y_0, \nu_0)$, where $u(y) \in U_{\mathbb{R}}$, $\nu_0 \in T_{y_0}^*Y \cong \mathfrak{p}^\perp$. Since $\hat{\phi}$ is holomorphic, we have

$$\mu_\lambda^*\hat{\phi}(y, \nu) = \hat{\phi}\big(u(y)\cdot\lambda + u(y)\cdot\nu_0\big) = \hat{\phi}\big(\nu + u(y)\cdot\lambda\big) = \sum_{j=0}^\infty \frac{1}{j!} D_{u(y)\cdot\lambda}^j \hat{\phi}(\nu). \tag{4.7}$$

Here D_ξ denotes the derivative in the direction of $\xi \in \mathfrak{g}^*$. We set further for $\nu \in \mathscr{O}$,

$$\big(P_{\mathscr{O},k}(C, \lambda)\hat{\phi}\big)(\nu) = \sum_{i=0}^k \frac{d_{\mathscr{O}}! c^{e_{\mathscr{O}}}}{(k-i)! i! (m-i)!} \int_{C(\nu)} (D_{u(y)\cdot\lambda}^{k-i}\hat{\phi})(\nu) \frac{\pi^*\tau_\lambda^i \wedge \mu^*\sigma_{\mathscr{O}}^{m-i}}{\mu^*(\sigma_{\mathscr{O}}^{d_{\mathscr{O}}})}. \tag{4.8}$$

Denote by $\mathbb{C}[\mathfrak{c}^*]^{(k)}$ the set of homogeneous polynomials on $\mathfrak{c}^*$ of degree k. Combining (4.6), (4.7), and (4.8), we obtain the following result.

LEMMA 4.9
Let $U \subset \mu^{-1}(\mathscr{O})$ be a $G_{\mathbb{R}}$-invariant open set, let $C \in H_{2m}(U, \mathbb{Z})$, and let $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$. Then for any $\nu \in \mathscr{O}$, we have $P_{\mathscr{O},k}(C, \cdot)\hat{\phi}(\nu) \in \mathbb{C}[\mathfrak{c}^*]^{(k)}$, and

$$\int_C \mu_\lambda^*\big(\hat{\phi}\frac{1}{m!}c^m\sigma_\lambda^m\big) = \sum_{k=0}^\infty \int_{\mathscr{O}} P_{\mathscr{O},k}(C, \lambda)\hat{\phi}\frac{1}{d_{\mathscr{O}}!}c^{d_{\mathscr{O}}}\sigma_{\mathscr{O}}^{d_{\mathscr{O}}}$$

is a Taylor series expansion at $\lambda = 0$ of the holomorphic function $\lambda \mapsto \int_C \mu_\lambda^*(\hat{\phi}(1/m!)c^m\sigma_\lambda^m)$. The leading term in this expansion is equal to

$$p_{\mathscr{O}}(\lambda)\int_{\mathscr{O}} \hat{\phi}\frac{1}{d_{\mathscr{O}}!}c^{d_{\mathscr{O}}}\sigma_{\mathscr{O}}^{d_{\mathscr{O}}}, \quad \textit{where } p_{\mathscr{O}}(\lambda) = \int_{C(\nu)} \pi^* e^{\tau_\lambda/2\pi i}.$$

Now we turn to the problem of comparing the leading term defined via the map $(\mathrm{gr}\,\mu_*)_\lambda$ and the leading term of the Taylor series expansion for an arbitrary cycle. Let $C \in H_{2m}(T_A^*Y, \mathbb{Z})$, and let $k = k(C)$. Write $C = \sum_{l \le k, l \in I} C_l$ as in Lemma 4.2. In the notation from Section 2, we have $C_k = C^\circ$. Any A-orbit $\mathscr{O} \subset \mathscr{N}_k \cap \mathfrak{a}^\perp$ is open; hence, there exists a unique $C_{k,\mathscr{O}} \in H_{2m}(\mu^{-1}\mathscr{O}, \mathbb{Z})$, so that $C_k = \sum_{\mathscr{O} \subset \mathscr{N}_k \cap \mathfrak{a}^\perp} C_{k,\mathscr{O}}$. Choose $\nu_\mathscr{O} \in \mathscr{O}$. Observe that $p_\mathscr{O}(\lambda) = \int_{C_{k,\mathscr{O}}(\nu_\mathscr{O})} \pi^* e^{\tau_\lambda/2\pi i}$ does not depend on the choice of $\nu_\mathscr{O}$. Now we use the fact that the cup product followed by the pushforward agrees with integration over the fiber. The base component of the cycle $C_{k,\mathscr{O}}$ is $[\mathscr{O}]$, and the fiber component over $\nu_\mathscr{O} \in \mathscr{O}$ is $C_{k,\mathscr{O}}(\nu_\mathscr{O})$, so we deduce

$$(\mathrm{gr}\,\mu_*)_\lambda(C) = \sum_{\mathscr{O} \subset \mathscr{N}_k \cap \mathfrak{a}^\perp} p_\mathscr{O}(\lambda)[\mathscr{O}]. \tag{4.10}$$

On the other hand, if $A = G_\mathbb{R}$, we consider the Taylor series expansion of the holomorphic function $(c^m/m!) \int_C \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m)$ at $\lambda = 0$. By Lemma 4.9 the terms $\int_{C_l} \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m)$ for $l < k$ cannot contribute to the leading term. It follows that the leading term is equal to

$$\sum_{\mathscr{O} \subset \mathscr{N}_k \cap \mathfrak{g}_\mathbb{R}^\perp} p_\mathscr{O}(\lambda) \int_\mathscr{O} \hat{\phi} \frac{c^k}{k!} \sigma_\mathscr{O}^k. \tag{4.11}$$

Next, our goal is to specialize the above considerations to the case of the characteristic cycle computed in Section 3. As in Section 3, we fix a θ-stable parabolic subalgebra $\mathfrak{p}$ and denote by y_0 the corresponding point in Y. Set $Z = K \cdot y_0$, $S = G_\mathbb{R} \cdot y_0$, and denote by $i : Z \longrightarrow Y$, $j : S \longrightarrow Y$ the inclusions. The orbit Z is closed, and the orbit S is open in Y. Let $\mathscr{G} = Ri_*(\mathbb{C}_Z)$ and $\mathscr{F} = Rj_!(\mathbb{C}_S)$ be standard sheaves. Then for the Matsuki correspondence for the sheaves γ, we have $\gamma(\mathscr{G}) = \mathscr{F}$. Recall that the characteristic cycle $CC(\mathscr{L})$ of a sheaf $\mathscr{L}$ is supported on the microsupport $SS(\mathscr{L})$. In our case, $SS(\mathscr{G}) = T_Z^*Y$, and, consequently, $\mu(SS(\mathscr{G}))$ is an irreducible K-invariant complex variety in $\mathscr{N} \cap \mathfrak{k}^\perp$. It follows that $\mu(SS(\mathscr{G})) = \overline{\mathscr{V}}$ for a single K-orbit $\mathscr{V} \subset \mathscr{N} \cap \mathfrak{k}^\perp$. Set $2k = \dim_\mathbb{R} \mathscr{V}$. An application of Theorem 2.6 shows that k is the minimal integer such that $SS(\mathscr{F}) \subset \mu^{-1}(\tilde{\mathscr{N}_k} \cap \mathfrak{g}_\mathbb{R}^\perp)$, and furthermore $\mu(SS(\mathscr{F})) \cap \mathscr{N}_k = \mathscr{O}$ for a unique $G_\mathbb{R}$-orbit $\mathscr{O} \subset \mathscr{N} \cap \mathfrak{g}_\mathbb{R}^\perp$. Finally, we observe that $CC(\mathscr{F}) = -CC(Rj_*(\mathbb{C}_S))$ by [SV1, Th. 4.2]. The preceding discussion and Theorems 2.6, 2.7, and 3.4 imply immediately the following result analogous to [R3, Th. 1.6.1]. For the convenience of the reader, we give a detailed explanation of the notation.

THEOREM 4.12
Let $G_\mathbb{R}$ be a connected, linear, semisimple group. Assume that $G_\mathbb{R}$ has a compact Cartan subgroup. Let $\mathfrak{p}$ be a θ-stable parabolic subalgebra in $\mathfrak{g}$, and let $\overline{\mathfrak{p}}$ be the

complex conjugate. Then $\mathfrak{l} = \mathfrak{p} \cap \overline{\mathfrak{p}}$ *is a Levi factor of* $\mathfrak{p}$. *Denote by* $\mathfrak{c}$ *the center of* $\mathfrak{l}$, *and denote by* C^+ *the positive chamber in* $i\mathfrak{c}_{\mathbb{R}}^*$ *defined by the roots from* $\mathfrak{p}/\mathfrak{l}$. *Let* $\mathscr{V} \subset \mathscr{N} \cap \mathfrak{k}^\perp$ *be the* K*-orbit such that* $K \cdot (\mathfrak{p}^\perp \cap \mathfrak{k}^\perp) = \overline{\mathscr{V}}$. *Set* $2k = \dim_{\mathbb{R}} \mathscr{V}$, *and denote by* $\mathscr{O} \subset \mathscr{N} \cap \mathfrak{g}_{\mathbb{R}}^\perp$ *the* $G_{\mathbb{R}}$*-orbit associated with* $\mathscr{V}$ *via the Sekiguchi correspondence. Let* Z *be the* K*-orbit of the point* y_0 *corresponding to* $\mathfrak{p}$ *in the generalized flag variety* Y. *Denote by* $C \in H_{2m}(T_K^*Y, \mathbb{Z})$ *the fundamental cycle of the conormal bundle* T_K^*Y, *denote by* C^0 *the restriction of* C *to* $\mu^{-1}\mathscr{V}$, *and denote by* $C^0(\nu)$ *the restriction of* C^0 *to* $\mu^{-1}(\nu)$, $\nu \in \mathscr{V}$ *(cf. (4.3)). Let* $s = \dim_{\mathbb{C}} \mathfrak{n} \cap \mathfrak{s}$. *Then* $p(\lambda) = (-1)^{s+1} \int_{C^0(\nu)} \pi^* e^{\tau_\lambda / 2\pi i}$ *is a homogeneous polynomial of degree* $m - k$, *and for* $\lambda \in C^+$, $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$, *the leading term in the Taylor series expansion at* $\lambda = 0$ *of the holomorphic function* $\lambda \mapsto \big(1/(m!(2\pi i)^m)\big) \int_{G_{\mathbb{R}} \cdot \lambda} \hat{\phi} \sigma_\lambda^m$ *is equal to*

$$p(\lambda) \int_{\mathscr{O}} \hat{\phi} \frac{\sigma_{\mathscr{O}}^k}{k!(2\pi i)^k}.$$

In the next proposition we show that the polynomial p is nonzero.

PROPOSITION 4.13
Suppose that $-\lambda \in C^+$ *is integral. Then* $p(\lambda) \neq 0$.

Proof
Let $\mathscr{V}$ be a K-orbit from Theorem 4.12. We show that there exists an irreducible component D of $\mu^{-1}\nu$, $\nu \in \mathscr{V}$, such that $K \cdot D$ is open dense in T_Z^*Y. Let $D_1, \ldots, D_t$ be the irreducible components of $\mu^{-1}\nu$ so that

$$\mu^{-1}\mathscr{V} = K \cdot D_1 \cup \cdots \cup K \cdot D_t$$

and $K \cdot D_i$ are pairwise distinct. Clearly, $K \cdot D_i$, $i = 1, \ldots, t$, are the irreducible components of $\mu^{-1}\mathscr{V}$. Since $T_Z^*Y \cap \mu^{-1}\mathscr{V}$ is a nonempty open set in T_Z^*Y and T_Z^*Y is irreducible, we have

$$T_Z^*Y = \overline{T_Z^*Y \cap \mu^{-1}\mathscr{V}} \subset \overline{K \cdot D_1} \cup \cdots \cup \overline{K \cdot D_t}.$$

Again for the reasons of irreducibility, $T_Z^*Y \subset \overline{K \cdot D_i}$ for some i. Set $D_i = D$. Since $\dim_{\mathbb{C}} T_Z^*Y = \dim_{\mathbb{C}} K \cdot D$, we deduce $T_Z^*Y = \overline{K \cdot D}$. Observe that $K \cdot D$ is closed in $\mu^{-1}\mathscr{V}$; so $\overline{K \cdot D} \cap \mu^{-1}\mathscr{V} = K \cdot D$. Hence, the restriction of $[T_Z^*Y]$ to $\mu^{-1}\mathscr{V}$ is equal to $[K \cdot D]$. The bracket here denotes the fundamental cycle. Let $z_1, \ldots, z_r$ be a full set of representatives for the component group $Z_K(\nu)/Z_K(\nu)^\circ$. Then the fiber of $[K \cdot D]$ over $\nu \in \mathscr{V}$ can be written as $[K \cdot D](\nu) = [z_1 D] + \cdots + [z_r D]$; hence, we obtain $p(\lambda) = r \int_D e^{\tau_\lambda / 2\pi i}$. If $-\lambda \in C^+$ is integral, let V_λ be the finite-dimensional representation of G with P-lowest weight λ. If necessary, we replace G here by an appropriate covering. Denote by i_λ the imbedding of Y into the projective space $\mathbb{P}(V_\lambda)$ as the orbit of a lowest weight vector. A simple computation shows $i_\lambda^*(\omega) = (\tau_\lambda)/(2\pi i)$, where ω is the standard Kähler form on $\mathbb{P}(V_\lambda)$. Finally, we obtain

$$p(\lambda) = \frac{r}{(m-k)!} \int_D \omega^{(m-k)} > 0,$$

as desired. □

Given a polynomial $p \in \mathbb{C}[\mathfrak{c}^*]$, denote by $p(\partial)$ a constant coefficient differential operator on $\mathfrak{c}$ determined uniquely by the property

$$p(\partial)e^\lambda = p(\lambda)e^\lambda, \quad \lambda \in \mathfrak{c}^*.$$

Consider the Taylor series expansion of the holomorphic function $\lambda \mapsto \big(1/(m!(2\pi i)^m)\big) \int_{G_\mathbb{R}\cdot\lambda} \hat{\phi}\sigma_\lambda^m$ at $\lambda = 0$. Differentiating this expansion by $p(\partial)$ and then letting $\lambda \to 0$, we deduce the following formula:

$$\lim_{\lambda\to 0(C^+)} p(\partial)\frac{1}{m!(2\pi i)^m} \int_{G_\mathbb{R}\cdot\lambda} \hat{\phi}\sigma_\lambda^m = \frac{p(\partial)p}{k!(2\pi i)^k} \int_{\mathscr{O}} \hat{\phi}\sigma_{\mathscr{O}}^k. \tag{4.14}$$

Define the measures m_λ and $m_{\mathscr{O}}$ on the orbits $G_\mathbb{R}\cdot\lambda$ and $\mathscr{O}$ by the formulas

$$m_\lambda = \frac{1}{m!(2\pi i)^m}\sigma_\lambda^m \qquad \text{and} \qquad m_{\mathscr{O}} = \frac{1}{k!(2\pi i)^k}\sigma_{\mathscr{O}}^k.$$

Observe that Lemma 4.1 implies $p(\partial)p \neq 0$; hence, taking the Fourier transform in (4.14), we obtain the following theorem.

THEOREM 4.15
With the same assumptions as in Proposition 4.13, the following limit formula for the orbital measures holds:

$$\lim_{\lambda\to 0(C^+)} p(\partial)m_\lambda = cm_{\mathscr{O}}.$$

Here c is a nonzero constant. In fact, we may take $c = p(\partial)p$.

Finally, we relate $p(\lambda)$ to the harmonic polynomials. As in Lemma 4.1 we choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ so that $\mathfrak{h}_\mathbb{R} \subset \mathfrak{l} \cap \mathfrak{u}_\mathbb{R}$. Write $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{c}$, where $\mathfrak{h}_1 \subset [\mathfrak{l}, \mathfrak{l}]$ is a Cartan subalgebra. Taking $\mathbb{C}$-linear duals, we have $\mathfrak{h}^* = \mathfrak{h}_1^* + \mathfrak{c}^*$. Using this splitting, we write for $\lambda \in \mathfrak{h}^*$, $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1 \in \mathfrak{h}_1^*$ and $\lambda_2 \in \mathfrak{c}^*$. Denote by $\mathscr{H}(\mathfrak{h}^*)$ the space of harmonic polynomials on $\mathfrak{h}^*$. Let X be the flag variety of Borel subalgebras of $\mathfrak{g}$. Denote by $q : X \longrightarrow Y$ the natural fibration, and write $l = \dim_\mathbb{C} q^{-1}y_0$. Let $\mu_0 : T^*X \longrightarrow \mathfrak{g}^*$ be the moment map defined analogously to μ. Let $\mathscr{V}$ be the K-orbit defined in Proposition 4.13. Choose $\nu \in \mathscr{V}$, and view $\mu_0^{-1}\nu$ as a subset of X, and $\mu^{-1}\nu$ as a subset of Y. Observe that $\dim_{\mathbb{C}} \mu^{-1}\nu = m - \dim_{\mathbb{C}} \mathscr{O} = e$ implies $\dim_{\mathbb{C}} \mu_0^{-1}\nu = e + l$. Hence, the fibration q determines the pullback homomorphism $q^* : H_{2e}(\mu^{-1}\nu, \mathbb{C}) \longrightarrow H_{2(e+l)}(\mu_0^{-1}\nu, \mathbb{C})$. Denote by $C_G(\nu)$ the component group of the centralizer of ν in G. Let W be the Weyl group of $\mathfrak{g}$. By the Springer theory, the spaces $H_{2(e+l)}(\mu_0^{-1}\nu, \mathbb{C})$ and $H_{2(e+l)}(\mu_0^{-1}\nu, \mathbb{C})^{C_G(\nu)}$ carry a natural structure of W-module. Furthermore, the character of W on $H_{2(e+l)}(\mu_0^{-1}\nu, \mathbb{C})^{C_G(\nu)}$ is the Springer representation χ_ν corresponding to the orbit $G \cdot \nu$ and the trivial character of $C_G(\nu)$. Homology classes and harmonic polynomials are related via the Borel map

$$H_{2(e+l)}(\mu_0^{-1}\nu, \mathbb{C})^{C_G(\nu)} \longrightarrow \mathscr{H}(\mathfrak{h}^*), \qquad \eta \mapsto \int_\eta \tau_\lambda^{e+l}.$$

We remark that the Borel map intertwines W-actions. In the notation from Proposition 4.13, set $C = [T_Z^* Y]^\circ(\nu)$, $\nu \in \mathscr{V}$. Via the homomorphism $H_{2e}(\mu^{-1}\nu, \mathbb{Z}) \longrightarrow H_{2e}(\mu^{-1}\nu, \mathbb{C})$, we view $C \in H_{2e}(\mu^{-1}\nu, \mathbb{C})$. Define $P(\lambda) \in \mathscr{H}(\mathfrak{h}^*)$ by the formula

$$P(\lambda) = \int_{q^*C} \tau_\lambda^{e+l}.$$

Our goal is to relate polynomials $P(\lambda)$ and $p(\lambda)$. We remark that the proof of Lemma 4.1 implies $\tau_\lambda = \tau_{\lambda_1} + \tau_{\lambda_2}$, so we may write

$$\tau_\lambda^{e+l} = \sum_{i=0}^{e+l} p_i(\lambda_1)\tau_i. \tag{4.16}$$

Here $p_i(\lambda_1)$ is a homogeneous polynomial in λ_1 of degree i, and τ_i is a $2(e+l)$-form not depending on λ_1. If $i = e$, then $p_l(\lambda_1)\tau_l = \binom{e+l}{l}\tau_{\lambda_1}^l \wedge \tau_{\lambda_2}^e$. Set $\omega_L(\lambda_1) = \prod_{\alpha\in\Delta^+(\mathfrak{l})} \lambda_1(ih_\alpha)$. Since $\nu \in K \cdot (\mathfrak{p}^\perp \cap \mathfrak{k}^\perp)$, we may assume without any loss of generality $y_0 \in \mu^{-1}\nu$. Hence, from (4.4) we deduce

$$\int_{q^*C} \tau_{\lambda_1}^l \wedge \tau_{\lambda_2}^e = \Big(\int_{q^{-1}y_0} \tau_{\lambda_1}^l\Big) \cdot \Big(\int_C \tau_{\lambda_2}^e\Big) = c_1 \omega_L(\lambda_1) p(\lambda_2),$$

where c_1 is a nonzero constant. After differentiating the integral of (4.16) by $\omega_L(\partial)$ and letting $\lambda_1 \to 0$, we deduce the following proposition.

PROPOSITION 4.17

Let $\mathscr{V}$ be the open K-orbit in $K \cdot (\mathfrak{p}^\perp \cap \mathfrak{k}^\perp)$, and let $\nu \in \mathscr{V}$. Write $\lambda = \lambda_1 + \lambda_2$ according to the splitting $\mathfrak{h}^* = \mathfrak{h}_1^* + \mathfrak{c}^*$. Denote by C the fiber of $[T_Z^* Y]$ at ν. Then $P(\lambda) = \int_{q^*C} \tau_\lambda^{e+l}$ is a harmonic polynomial on $\mathfrak{h}^*$ which transforms under the W-action according to the Springer character χ_ν. The polynomial $p(\lambda_2)$ can be computed from $P(\lambda)$ by the formula

$$p(\lambda_2) = c \lim_{\lambda_1 \to 0} \omega_L(\partial) P(\lambda),$$

where c is a nonzero constant.

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