



On the signature character of representations of p -adic general linear groups

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Abstract

In this article we calculate the signature character of certain Hermitian representations of $GL_N(F)$ for a p -adic field F . We further give a conjectural description for the signature character of unramified representations in terms of Kostka numbers.

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0. Introduction

Let F be a finite extension of the field $\mathbb{Q}_p$ of p -adic rational numbers and let $G^\vee$ be a connected reductive group over F . Let $I \subset G^\vee(F)$ be an Iwahori subgroup. We consider irreducible admissible (complex) representations $\mathcal{V}$ of $G^\vee(F)$ such that the space of Iwahori fixed vectors $\mathcal{V}^I$ is non-zero. It is well known that those representations correspond to certain representations of the Hecke algebra $\mathcal{H}$ associated to the extended Weyl group $\tilde{W}$ of $G^\vee$. Moreover, Barbasch and Moy proved in [1] that V is Hermitian (respectively unitary) if and only if $\mathcal{V}^I$ is Hermitian (respectively unitary) with respect to a certain involution of $\mathcal{H}$.

The Hecke algebra $\mathcal{H}$ contains a subalgebra that is isomorphic to the group algebra $\mathbb{C}[W]$ of the Weyl group of $G^\vee$. In particular we can consider for every irreducible $\mathcal{H}$ -module V and for

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every irreducible representation λ of W the space $V_\lambda = \text{Hom}_W(\lambda, V)$. If V is Hermitian, V_λ inherits a Hermitian form. In this article we study the signatures $\sigma_\lambda(V)$ of this induced Hermitian form for $G^\vee = GL_N$. We call the tuple $(\sigma_\lambda(V))_\lambda$ the signature character of V .

To do this we reduce the problem to an analogous one for modules with real central character of a certain graded Hecke algebra $\mathbb{H}$ defined by Lusztig [14]. Again we can consider $\mathbb{C}[W]$ as a subalgebra of $\mathbb{H}$. For these $\mathbb{H}$ -modules there is a classification analogous to the Langlands classification which parametrizes irreducible representations in terms of subsets S of a chosen basis of roots, a tempered representation U of the standard Levi subalgebra M corresponding to S and a dominant real character ν of the center of M . These irreducible $\mathbb{H}$ -modules $L(S, U, \nu)$ are quotients of so-called standard modules $X(S, U, \nu)$. A deformation argument (for the first time used by Casselmann in [6]) allows to determine the W -module structure of $X(S, U, \nu)$.

Our strategy to calculate the signature character is the following. We fix S and U as above. The dominant real characters ν form a cone. Those ν such that $X(S, U, \nu)$ is reducible define affine hyperplanes in this cone. We call these hyperplanes “reducibility walls.” For ν outside the reducibility walls the signature character is locally constant in ν . Moreover, “near zero” Tadić’s classification of unitary representations of GL_N [20] implies that $X(S, U, \nu) = L(S, U, \nu)$ is unitary. As we know that W -module structure of the standard modules $X(S, U, \nu)$, we know the signature character of $L(S, U, \nu)$ for small ν .

On the other hand, a limit argument by Casselmann [6] (see also [3]) also gives an expression of the Hermitian form for ν “near infinity” which allows us to express the signature character purely in terms of characters of the symmetric groups and certain Kostka numbers. For example, our description of the signature at infinity implies that for $S = \emptyset$ we have $\sigma_\lambda = \chi_\lambda(w_0)$ at infinity (where χ_λ denotes the character of W corresponding to λ and w_0 is the longest element in W). Hence to calculate the remaining signature characters we fix a ν_0 lying on a single reducibility wall and a one parameter family $t \mapsto \nu(t)$ for small t with $\nu(0) = \nu_0$ such that $\nu(t)$ does not lie on the reducibility wall for $t \neq 0$. We then would like to express the sum or the difference of the signature characters of $L(S, U, \nu(t))$ for positive and negative t .

We cannot do this for all reducibility walls. Instead we concentrate on the walls that are needed to calculate the signature character in the case of what we call unramified representation, that is, representations with $S = \emptyset$. For these walls we give a precise conjecture for the difference and the sum of signature characters of both sides. Moreover, we prove this conjecture for unramified representations. As a consequence we get an explicit expression of the signature character of unramified representations $L(\emptyset, \mathbf{1}, \nu)$ with $L(\emptyset, \mathbf{1}, \nu) = X(\emptyset, \mathbf{1}, \nu)$ in terms of Kostka numbers. We also give a description of the signature characters on the reducibility walls for unramified representations.

We will now give a short overview of the structure of our work. In Section 1 we describe the equivalence between irreducible representations V of $G^\vee(F)$ with $V^I \neq (0)$ and irreducible representations with real central character of the associated graded Hecke algebra $\mathbb{H}$. Section 2 contains three classifications of irreducible $\mathbb{H}$ -modules. The first is in terms of conjugacy classes of pairs (s, e) of a semisimple element s and a nilpotent element e in the Lie algebra $\mathfrak{gl}_N$. The second one is the Langlands classification already described above. The third classification is the translation of the Bernstein–Zelevinsky classification of representations of $GL_N(F)$ in terms of supercuspidal representations to the setting of graded Hecke algebras. We also explain how to obtain each of the three classification from either of the other two.

In Section 3 we introduce the Zelevinsky involution which will allow us to calculate also the signature character of irreducible representations $L(S, U, \nu)$ which are a proper quotient of

the standard module $X(S, U, \nu)$. Section 4 gives a description of the W -module structure of the standard modules.

Section 5 contains the classification of Hermitian and unitary $\mathbb{H}$ -modules and the formal definition of the signature character of an irreducible $\mathbb{H}$ -module. In Section 6 we express the Hermitian form on standard modules in terms of a certain intertwining operator. Here we follow closely Barbasch and Moy [3]. Section 7 contains the description of the reducibility walls and also the theorem that for unramified representations the isolated unitary representations are precisely those which lie on the intersection of $[N/2]$ reducibility walls. In Section 8 we give an explicit algorithm to compute the signature “near infinity.” We do this by making more precise a description of Barbasch and Moy given in [3].

Section 9 now deals with the topic of crossing the reducibility walls. We define the “height” of such a wall and study reducibility walls of height 1 and 2 in more detail as those are the walls which occur in the unramified case. Then we prove our wall crossing theorems in the unramified case. We further formulate a conjecture for the general case of crossing reducibility walls of height one.

In Section 10 we give a conjecture for crossing certain walls of height bigger than one and use these conjectures to give an explicit description of the signature character for unramified representations in terms of Kostka numbers. Finally in the last section we calculate the signature character for all Hermitian representation of GL_N for $N = 2, 3$ and 4.

Notation. We use the following notations. All algebraic varieties and all representations are assumed to be over the complex numbers $\mathbb{C}$.

If R is any ring, we denote by $M_n(R)$ the ring of $(n \times n)$ -matrices. If $A \in M_n(R)$ and $B \in M_m(R)$ are two matrices we denote by $A \oplus B \in M_{n+m}(R)$ the block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and $\text{diag}(\alpha_1, \dots, \alpha_N) \in M_N(R)$ denotes the diagonal matrix with entries $\alpha_1, \dots, \alpha_N \in R$. Finally let $I_N \in GL_N(R)$ be the identity matrix.

If W is a finite group, we denote by $\hat{W}$ the set of isomorphism classes of irreducible representations of W .

1. Representations of p -adic groups and graded Hecke algebras

1.1. Let $G^\vee$ be a split (connected) reductive group over a p -adic field F and let G be the Langlands dual group over $\mathbb{C}$. For every Borel pair (T, B) (i.e. T is a maximal torus of G and B is a Borel subgroup of G containing T) we denote by $R(T)$ be the sets of roots in $X^*(T)$, by $R^\vee(T) \subset X_*(T)$ the set of coroots, and by $\Pi(T, B) \subset R(T)$ the set of B -simple roots. We call $\mathcal{R}_{(T, B)} = (X^*(T), X_*(T), R(T), R^\vee(T), \Pi(T))$ the associated based root datum. If (T', B') is another Borel pair, we can choose a $g \in G$ such that $(gTg^{-1}, gBg^{-1}) = (T', B')$. This g is unique up to right multiplication with elements of T and left multiplication with elements of T' . Hence conjugation with g induces an isomorphism of based root data $\mathcal{R}_{(T, B)} \xrightarrow{\sim} \mathcal{R}_{(T', B')}$ which is independent of the choice of g . We obtain an inductive system of based root data indexed by the set of Borel pairs. Its inductive limit $\mathcal{R} = (X, Y, R, R^\vee, \Pi)$ is called the *based root datum* of G . By construction it is equipped with an isomorphism $\mathcal{R}_{(T, B)} \xrightarrow{\sim} \mathcal{R}$ for every Borel pair (T, B) .

Let $s_\alpha \in GL(X)$ denote the reflection associated to a root $\alpha \in R$. Let W denote the Weyl group of $\mathcal{R}$, i.e. the group generated by the s_α for $\alpha \in R$. The root base Π defines a partial order on $R^\vee$ by $\alpha_1^\vee \leq \alpha_2^\vee$ if $\alpha_2^\vee - \alpha_1^\vee$ is a linear combination with nonnegative integer coefficients of elements

of $\{\alpha^\vee \mid \alpha \in \Pi\}$. We denote by Π_m the set of all $\beta \in R$ such that $\beta^\vee$ is a minimal element for this partial order.

Elements in the semidirect product $\tilde{W} = W \ltimes X$ are written in the form wa^x for $w \in W$ and $x \in X$. We set

$$\tilde{S} = \{s_\alpha \mid \alpha \in \Pi\} \cup \{s_\alpha a^\alpha \mid \alpha \in \Pi_m\} \subset \tilde{W}.$$

The root base Π defines a system of positive roots $R^+ \subset R$ and a length function

$$l: \tilde{W} \rightarrow \mathbb{N}_0, \\ l(wa^x) = \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^-}} |\langle x, \alpha^\vee \rangle + 1| + \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^+}} |\langle x, \alpha^\vee \rangle|$$

which extends the usual length function on the Coxeter subgroup of $\tilde{W}$ which is generated by $\tilde{S}$.

We further set

$$R^b = \{\alpha \in R \mid \alpha^\vee \in 2Y\}.$$

For example, if $(X, Y, R, R^\vee, \Pi)$ is simple and simply connected (i.e. it is indecomposable and $R^\vee$ generates Y), R^b is nonempty if and only if the Dynkin type is B_n .

1.2. We are interested in the following category of representations of $G^\vee(F)$. Let $I \subset G^\vee(F)$ be an Iwahori subgroup. We call an admissible representation V on a complex vector space I -spherical if it is generated by its I -fixed vectors. This implies that every subquotient of V is generated by its I -fixed vectors.

Denote by $\mathcal{H}(G^\vee//I)$ the Hecke algebra of $G^\vee(F)$ with respect to I . The underlying vector space consists of the $\mathbb{C}$ -valued functions of the (discrete) quotient $I \backslash G^\vee(F)/I$ with finite support, and the algebra structure is given by convolution. For every smooth representation V of $G^\vee(F)$ the space of I -fixed vectors V^I is naturally an $\mathcal{H}(G^\vee//I)$ -module and the functor $V \mapsto V^I$ induces an equivalence between the category of admissible I -spherical representations of $G^\vee(F)$ and the category of left $\mathcal{H}(G^\vee//I)$ -modules which are of finite length (or equivalently finite-dimensional as $\mathbb{C}$ -vector spaces) (see, e.g., [5, 2.2]).

1.3. The Hecke algebra $\mathcal{H}(G^\vee//I)$ can be described directly in terms of generators and relations using the based root datum $\mathcal{R}$. More precisely, it can be considered as a specialization of the affine Hecke algebra $\mathcal{H} = \mathcal{H}_{\mathcal{R}}$ associated to $\mathcal{R}$ which is defined as follows.

Let B be the braid group of $\mathcal{R}$, i.e. the group with generators T_w for $w \in \tilde{W}$ and relations $T_w T_{w'} = T_{ww'}$ whenever $l(w) + l(w') = l(ww')$. Denote by z an indeterminate. Then $\mathcal{H}$ is the $\mathbb{C}[z, z^{-1}]$ -algebra which is the quotient of the group algebra (over $\mathbb{C}[z, z^{-1}]$) of the braid group B by the two-sided ideal generated by the elements

$$(T_s + 1)(T_s - z^2)$$

for $s \in \tilde{S}$. For $w \in \tilde{W}$ we denote the image of T_w in $\mathcal{H}$ again by T_w .

Let q be the number of elements in the residue field (of the ring of integers) of F and denote by $\xi_q: \mathbb{C}[z, z^{-1}] \rightarrow \mathbb{C}$ the $\mathbb{C}$ -algebra homomorphism which sends z to $q^{1/2}$. Then a classical result of Iwahori and Matsumoto [9] shows that $\mathcal{H}(G^\vee//I)$ is isomorphic to $\mathcal{H} \otimes_{\mathbb{C}[z, z^{-1}], \xi_q} \mathbb{C}$.

1.4. We collect some properties of $\mathcal{H}$. Let

$$X_{\text{dom}} = \{x \in X \mid \langle x, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Pi\}$$

be the set of dominant weights. We define for $x \in X$ an element $\bar{T}_x \in B$ as follows. Write $x = x_1 - x_2$ with $x_1, x_2 \in X_{\text{dom}}$ and set

$$\bar{T}_x = T_{a^{x_1}} T_{a^{x_2}}^{-1}.$$

This is easily seen to be well defined. Its image in $\mathcal{H}$ is again denoted by $\bar{T}_x$. Further set

$$\theta_x = z^{-l(\bar{T}_x)} \bar{T}_x \in \mathcal{H},$$

where $l: B \rightarrow \mathbb{Z}$ is the unique extension of the length function l to B .

Let $\mathcal{O}$ be the group algebra of X over the ring $\mathbb{C}[z, z^{-1}]$. Then the map $x \mapsto \theta_x$ defines an embedding of $\mathbb{C}[z, z^{-1}]$ -algebras $\mathcal{O} \rightarrow \mathcal{H}$ [14, 3.4]. In the sequel we consider $\mathcal{O}$ as a subring of $\mathcal{H}$. We further have the following proposition.

Proposition. [14, 3.7 and 3.11] *The Hecke algebra $\mathcal{H}$ is a free left $\mathcal{O}$ -module and a free right $\mathcal{O}$ -module with basis $\{T_w \mid w \in W\}$ in each case. The center $\mathcal{Z}$ of the Hecke algebra $\mathcal{H}$ consists of the W -invariants of $\mathcal{O}$.*

1.5. Every finite-dimensional left $\mathcal{H}$ -module V admits a (unique) primary decomposition with respect to the center $\mathcal{Z}$ of $\mathcal{H}$:

$$V = \bigoplus_{\chi} V_{\chi},$$

where χ runs through the set of characters of the center $\mathcal{Z}$. Here by a character of $\mathcal{Z}$ we mean a $\mathbb{C}$ -algebra homomorphism $\mathcal{Z} \rightarrow \mathbb{C}$. As $\mathcal{Z} = \mathcal{O}^W = \mathbb{C}[z, z^{-1}][X]^W$, the characters of $\mathcal{Z}$ are given by pairs (Wt, z_0) where Wt is a W -orbit of $T = Y \otimes_{\mathbb{Z}} \mathbb{C}^\times$ and where $z_0 \in \mathbb{C}^\times$. This induces a decomposition of the category of finite-dimensional left $\mathcal{H}$ -modules into the direct sum of the categories ${}_{\mathcal{H}}\mathbf{Mod}_{\chi}$ of finite-dimensional left $\mathcal{H}$ -modules V such that $\zeta - \chi(\zeta)$ is nilpotent on V for all $\zeta \in \mathcal{Z}$. If χ corresponds to the pair (Wt, z_0) , we also write ${}_{\mathcal{H}}\mathbf{Mod}_{Wt, z_0}$.

1.6. The categories ${}_{\mathcal{H}}\mathbf{Mod}_{Wt, z_0}$ are equivalent to categories of finite-dimensional left $\mathbb{H}$ -modules where $\mathbb{H}$ is a graded version of $\mathcal{H}$. Instead of explaining how to obtain $\mathbb{H}$ from $\mathcal{H}$ by grading with respect to a certain ideal, we give the abstract definition of the graded Hecke algebra $\mathbb{H} = \mathbb{H}_{\mathcal{R}}$ associated to a reduced root datum $\mathcal{R} = (X, Y, R, R^\vee, \Pi)$.

We set $\mathcal{O} = \mathbb{C}[r] \otimes_{\mathbb{C}} \text{Sym}(X \otimes_{\mathbb{Z}} \mathbb{C})$. It carries an action by W induced by the trivial action on $\mathbb{C}[r]$ and the canonical action on X . As a $\mathbb{C}$ -vector space we have

$$\mathbb{H} = \mathcal{O} \otimes_{\mathbb{C}} \mathbb{C}[W]$$

with a structure of associative $\mathbb{C}$ -algebra with unit $1 \otimes e$, defined by the rules:

- (i) $\mathbb{O} \rightarrow \mathbb{H}$, $\theta \mapsto \theta \otimes e$, is an algebra homomorphism.
- (ii) $\mathbb{C}[W] \rightarrow \mathbb{H}$, $w \mapsto 1 \otimes w$, is an algebra homomorphism.
- (iii) $(\theta \otimes e)(1 \otimes w) = \theta \otimes w$ for $\theta \in \mathbb{O}$ and $w \in W$.
- (iv) For all $\alpha \in \Pi$ and $\theta \in X$ we have

$$(1 \otimes s_\alpha)(\theta \otimes e) - ({}^{s_\alpha}\theta \otimes e)(1 \otimes s_\alpha) = 2r\langle \alpha^\vee, \theta \rangle.$$

- (v) r is in the center of $\mathbb{H}$.

Usually we will omit the $\otimes$ when denoting elements in $\mathbb{H}$ and write t_w instead of $1 \otimes w$. For $\alpha \in R$ we further set $t_\alpha := t_{s_\alpha}$. For example, the relation (iv) becomes

$$t_\alpha \cdot \theta - {}^{s_\alpha}\theta \cdot t_\alpha = 2r\langle \alpha^\vee, \theta \rangle.$$

Using $s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha$ we see that this relation is equivalent to

$$\theta \cdot t_\alpha - t_\alpha \cdot {}^{s_\alpha}\theta = 2r\langle \alpha^\vee, \theta \rangle.$$

1.7. The center of $\mathbb{H}$ consists of the W -invariants of $\mathbb{O}$. As above we get a decomposition of the category of finite-dimensional left $\mathbb{H}$ -modules into categories ${}_{\mathbb{H}}\mathbf{Mod}_{Wu, r_0}$ where (Wu, r_0) runs through the set of pairs consisting of a W -orbit of an elements $u \in Y \otimes_{\mathbb{Z}} \mathbb{C}$ and a complex number r_0 .

In the sequel we will consider a character of the center of $\mathbb{H}$ also as a pair $(\{s\}, r_0)$ where $\{s\}$ is a G -conjugacy class of a semisimple element of $\mathfrak{g}$ and where r_0 is a complex number.

1.8. The relation between $\mathcal{H}$ -modules and $\mathbb{H}$ -modules is the following (see [14, Sections 8–10] and [15, Section 4]). As we are mainly interested in the case $G = GL_N$ we make the following additional assumptions to simplify notations.

- The derived group of G is simply connected (or equivalently the center of $G^\vee$ is connected).
- $R^b = \emptyset$.

Fix a central character of $\mathcal{H}$ corresponding to a pair (Wt, z_0) . To simplify notations further we assume that z_0 is a positive real number, different from 1. Further we choose an auxiliary t in the Weyl orbit Wt . The following constructions will be independent of this choice (up to isomorphism which is given by $w \in W$ if t is replaced by wt). We can decompose $T = \mathbb{C}^\times \otimes_{\mathbb{Z}} Y$ into an elliptic and a hyperbolic part, namely $T = T_{\text{ell}} \times T_{\text{h}}$ where

$$T_{\text{ell}} = \{z \in \mathbb{C}^\times \mid |z| = 1\} \otimes Y, \quad T_{\text{h}} = \mathbb{R}_{>0} \otimes Y.$$

This is a decomposition of real Lie groups. We can therefore write uniquely $t = t_e t_h$ where $t_e \in T_{\text{ell}}$ and $t_h \in T_{\text{h}}$.

Let $\mathcal{R}(t) = (X, Y, R(t), R^\vee(t), \Pi(t))$ be the root datum with

$$R(t) = \{\alpha \in R \mid \alpha(t_e) = 1\},$$

$$R^\vee(t) = \{\alpha^\vee \in R^\vee \mid \alpha \in R(t)\}.$$

Then $R^+(t) = R(t) \cap R^+$ is a system of positive roots and the associated set of simple roots is $\Pi(t)$. We write $W(t)$ for the Weyl group of the root datum $\mathcal{R}(t)$.

Denote by

$$\log_{z_0^2} : T_h = \mathbb{R}_{>0} \otimes_{\mathbb{Z}} Y \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} Y$$

the isomorphism induced by

$$\mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}, \quad x \mapsto \log(x)/2 \log(z_0).$$

Note that the isomorphism class of $\mathcal{R}(t)$ does not depend on the choice of t in its Weyl orbit.

The relation between $\mathcal{H}$ -modules and $\mathbb{H}$ -modules is now given by the next proposition.

1.9. Proposition. *The categories $\mathcal{H}_{\mathcal{R}} \mathbf{Mod}_{W(t), z_0}$ and $\mathbb{H}_{\mathcal{R}(t)} \mathbf{Mod}_{W(t) \log_{z_0^2}(t_h), 1/2}$ are equivalent.*

More precisely, denote by $\hat{\mathcal{H}}$ (respectively $\hat{\mathbb{H}}$) the completion of $\mathcal{H}_{\mathcal{R}}$ (respectively $\mathbb{H}_{\mathcal{R}(t)}$) with respect to the maximal ideal of the center corresponding to (Wt, z_0) (respectively $(W(t) \log_{z_0^2}(t_h), 1/2)$) (hence $\mathcal{H}_{\mathcal{R}} \mathbf{Mod}_{W(t), z_0}$, respectively $\mathbb{H}_{\mathcal{R}(t)} \mathbf{Mod}_{W(t) \log_{z_0^2}(t_h), 1/2}$ is the category of modules of finite length over $\hat{\mathcal{H}}$, respectively $\hat{\mathbb{H}}$). Then $\hat{\mathcal{H}}$ and $\hat{\mathbb{H}}$ are Morita equivalent.

Proof. This follows from the main result of [14, Sections 8–10] and [15, Section 4] with simpler notation due to our assumptions. We also replace the indeterminate r in loc. cit. by $r/2 \log(z_0)$. Moreover, we can replace $\mathcal{R}$ by its derived root datum $\text{der}(\mathcal{R}) = (X_{\text{der}}, Y_{\text{der}}, R_{\text{der}}, R_{\text{der}}^\vee, \Pi_{\text{der}})$, i.e. $Y_{\text{der}} = Y \cap \langle \alpha \mid \alpha \in R \rangle_{\mathbb{R}}$. Then Y_{der} is a direct summand of Y and if we set $T_{\text{der}} = Y_{\text{der}} \otimes \mathbb{C}^\times$, we get a decomposition $T = T' \times T_{\text{der}}$ such that $\alpha(t') = 1$ for all $\alpha \in R$ and hence a decomposition $t = t' t_{\text{der}}$. Now $R(t)$, $R^\vee(t)$ and $W(t)$ depend only on the derived root datum and on t_{der} .

Now our assumption in (1.8) implies that we are in the situation of [15, 4.4]. Then [15, 4.5] shows that the group Γ_t defined in [14, 8.1] is trivial which implies the result as explained in [15, 4.9]. $\square$

1.10. If $\mathcal{R}$ is the root datum of $G = GL_N$ or more generally of a reductive group G whose simple components are of Dynkin type A_n , then for any choice of t the root datum $\mathcal{R}(t)$ is the root datum of a Levi subgroup of G , namely of the centralizer of t_e (after considering T as a maximal torus of G which is well defined up to conjugacy). For other Dynkin types this is not true in general.

2. Classification of $\mathbb{H}$ -modules

2.1. From now on we make the simplifying assumption that G is isomorphic to a Levi subgroup of GL_N (hence isomorphic to a product of groups of the form GL_{N_i}). Let $\mathcal{R} = (X, Y, R, R^\vee, \Pi)$ be the based root datum of G , W its Weyl group, and denote by $\mathbb{H}$ the associated Hecke algebra. We fix $r_0 \in \mathbb{C} \setminus \{0\}$, and denote by $\text{Irr}_{r_0}(\mathbb{H})$ the set of equivalence classes of irreducible $\mathbb{H}$ -modules, where $r \in \mathbb{H}$ acts by r_0 .

2.2. We set $\mathfrak{t} = Y \otimes \mathbb{C}$ and denote by $\mathfrak{t}^* = X \otimes \mathbb{C}$ the dual. Define further

$$\mathfrak{z} = \{\lambda \in \mathfrak{t} \mid \langle \lambda, \alpha \rangle = 0 \text{ for all } \alpha \in R\},$$

$$\mathfrak{z}^* = \{x \in \mathfrak{t}^* \mid \langle \alpha^\vee, x \rangle = 0 \text{ for all } \alpha^\vee \in R^\vee\}.$$

Then $\mathfrak{z}$ is canonically isomorphic to the center of $\text{Lie}(G)$ and the duality of $\mathfrak{t}$ and $\mathfrak{t}^*$ induces a perfect duality of $\mathfrak{z}$ and $\mathfrak{z}^*$. The subspace $\mathfrak{z}^*$ of $X \otimes \mathbb{C}$ has as a complement the space generated by $\alpha \in R$.

Every element $\nu \in \mathfrak{z}$ defines a one-dimensional $\text{Sym}(\mathfrak{z}^*)$ -module $\mathbb{C}_\nu$ where $\xi \in \mathfrak{z}^*$ acts by $\nu(\xi)$.

2.3. Let $S \subset \Pi$ be a subset and denote by $(X, Y, R_S, R_S^\vee, S)$ the corresponding subroot system with Weyl group W_S and associated graded Hecke algebra $\mathbb{H}_S$. For this root system we have the subspaces $\mathfrak{z}_S$ and $\mathfrak{z}_S^*$ as in (2.2). Note that $\mathbb{H}_S$ is isomorphic to the graded Hecke algebra associated to a Levi subgroup M of G and that M again satisfies the condition in (2.1).

We have a canonical embedding $\mathbb{H}_S \subset \mathbb{H}$ which makes $\mathbb{H}$ into a free $\mathbb{H}_S$ -module of rank $\#(W/W_S)$. If U is a $\mathbb{H}_S$ -module we also write $\text{Ind}_{\mathbb{H}_S}^{\mathbb{H}}(U)$ instead of $\mathbb{H} \otimes_{\mathbb{H}_S} U$.

2.4. By [13] there exists a bijection between $\text{Irr}_{r_0}(\mathbb{H})$ and the set of G -conjugacy classes of pairs of the form (s, e) where $s \in \mathfrak{g}$ is semisimple, $e \in \mathfrak{g}$ is nilpotent such that

$$[s, e] = 2r_0 e.$$

Denote by $L_G(s, e, r_0) = L(s, e)$ the irreducible $\mathbb{H}$ -module corresponding to the conjugacy class $\{(s, e)\}$ of (s, e) . By [17, 1.15], $L(s, e)$ is the unique irreducible quotient of a standard module $X_G(s, e, r_0) = X(s, e)$ associated to $\{(s, e)\}$. See [13, Section 8] and [16, Section 10] for the definition of $X(s, e)$ (and see [16, 10.11] for the fact that the module constructed in [13, Section 8] is isomorphic to the module constructed in [16, Section 10]).

2.5. By definition of the standard module $X_G(s, e, r_0)$ [13, Section 8], its central character exists and is given by the pair $(\{s\}, r_0)$ (1.7). In particular this is also true for the irreducible quotient $L(s, e, r_0)$.

2.6. From now on assume that r_0 is a positive real number. Moreover, we will only consider irreducible $\mathbb{H}$ -modules and standard modules associated to pairs (s, e) such that the conjugacy class of s is in $(Y \otimes_{\mathbb{Z}} \mathbb{R})/W$. By this we mean the following. The set of conjugacy classes of semisimple elements of $\mathfrak{g}$ is in canonical bijection to $(Y \otimes_{\mathbb{Z}} \mathbb{C})/W$ and we assume that the conjugacy class of s lies in the subset $(Y \otimes_{\mathbb{Z}} \mathbb{R})/W$.

2.7. Next we will describe the Langlands classification. For this we first define what it means to be tempered in terms of the classification (2.4).

Let $s \in \mathfrak{g} = \mathfrak{gl}_N$ be semisimple and $e \in \mathfrak{gl}_N$ nilpotent such that $[s, e] = 2r_0 e$. Choose a homomorphism of Lie algebras

$$\psi : \mathfrak{sl}_2 \rightarrow \mathfrak{gl}_N$$

such that $e = \psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and such that $[s, f] = -2r_0 f$ where $f = \psi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This implies $[s, h] = [s, [e, f]] = 0$ by the Jacobi identity if $h = \psi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Such a homomorphism exists and is uniquely determined up to conjugation with $z \in G$ such that $\text{Ad}(z)e = e$ and $\text{Ad}(z)s = s$ by a variant of the Jacobson–Morozov theorem (see [10, 2.4(g) and 2.4(h)]).

Define $t := s - r_0 h$. Then t is a semisimple element of $\mathfrak{g}$ whose conjugacy class depends only on the conjugacy class of (s, e) . Note that t commutes with s and with the image of ψ . We call the conjugacy class of (s, e) (or the associated simple module $L(s, e)$) *tempered* if $t = 0$.

If $\{(s, e)\}$ is tempered, we have $L(s, e) = X(s, e)$ by [17, 1.21].

2.8. We state the following version of Langlands classification for $\mathbb{H}$ -modules which is a slight reformulation of [8] (recall that we assume that $r_0 \in \mathbb{R}^{>0}$ and $\{s\} \in (Y \otimes_{\mathbb{Z}} \mathbb{R})/W$).

Theorem. *For every irreducible $\mathbb{H}$ -module V there exists a triple (S, U, ν) where S is a subset of Π , U is an irreducible tempered representation of $\mathbb{H}_S$ and where*

$$\nu \in \mathfrak{z}_S^+ = \{\lambda \in \mathfrak{z}_S \cap Y \otimes \mathbb{R} \mid \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Pi \setminus S\}$$

such that V is the unique irreducible quotient of $\text{Ind}_{\mathbb{H}_S}^{\mathbb{H}}(U \otimes \mathbb{C}_\nu)$. Further S and ν and the isomorphism class of U are uniquely determined by the isomorphism class of V . We set $L(S, U, \nu) = V$ and $X(S, U, \nu) = \text{Ind}_{\mathbb{H}_S}^{\mathbb{H}}(U \otimes \mathbb{C}_\nu)$.

If $L(S', U', \nu')$ is any other irreducible subquotient of $\text{Ind}_{\mathbb{H}_S}^{\mathbb{H}}(U \otimes \mathbb{C}_\nu)$ coming from a triple (S', U', ν') , we have $\nu' < \nu$ (with respect to the extended Bruhat order on $\tilde{W}$).

The triple (S, U, ν) is called *Langlands data* associated to V .

2.9. We keep the notations of (2.7). In particular we have the G -conjugacy class $\{(s, e)\}$ and the associated irreducible $\mathbb{H}$ -module $L(s, e)$. We are now going to explain how to obtain the Langlands data (S, U, ν) corresponding to $L(s, e)$. We follow [17, 3.9ff] choosing for τ in loc. cit. the homomorphism $\mathbb{C} \rightarrow \mathbb{R}$ which associates to each complex number its real part. We use a somewhat more explicit but less canonical description. For this we fix a Borel subgroup B of G and a maximal torus T of G contained in B which gives an identification of the abstract based root datum $\mathcal{R}$ with the based root datum of (G, B, T) . In particular we have an identification $\text{Lie}(T) = \mathfrak{t} = Y \otimes \mathbb{C}$. The conjugacy class of s is then an element in $(Y \otimes \mathbb{R})/W$ (because we assumed that s is real). After G -conjugation we can assume that s is a diagonal matrix and that $(s, e) = (s_1, e_1) \oplus \cdots \oplus (s_l, e_l)$ where $e_i \in M_{m_i}(\mathbb{C})$ is of the form

$$e_i = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

Let σ_i be the first entry in the diagonal matrix s_i . We can assume that

$$\sigma_1 + \frac{1}{2}m_1 \geq \cdots \geq \sigma_l + \frac{1}{2}m_l.$$

Now we can choose ψ as in (2.7) such that $h \in Y \otimes \mathbb{R} \subset \mathfrak{t}$, in particular $t = s - r_0 h \in Y \otimes \mathbb{R}$. We have $\langle t, \alpha \rangle \geq 0$ for all $\alpha \in \Pi$.

We set

$$S = \{\alpha \in \Pi \mid \langle t, \alpha \rangle = 0\}.$$

Then $\mathbb{H}_S$ is isomorphic to the graded Hecke algebra associated to the Levi subalgebra $\mathfrak{m} = \text{Cent}_{\mathfrak{g}}(t)$. Note that s and the image of ψ are contained in $\mathfrak{m}$. Let M be the corresponding Levi subgroup of G . As the derived group of G is simply connected, we have $M = Z_G(t)$. Further we have $[r_0 h, e] = 2r_0 e$. Hence the conjugacy class of $(r_0 h, e)$ defines an irreducible representation

$$U := L_M(r_0 h, e)$$

of $\mathbb{H}_S$. By definition this is a tempered representation of $\mathbb{H}_S$. Finally let ν be the dominant representative of the W -orbit of t . By definition we have $\nu \in \mathfrak{z}_S^+$. We then have $U \otimes \mathbb{C}_\nu = L_M(s, e) = X_M(s, e)$ and by [17, 3.38] we have $X_G(s, e) = \text{Ind}_{\mathbb{H}_M}^{\mathbb{H}_G}(U \otimes \mathbb{C}_\nu)$.

2.10. For the rest of Section 2 let us assume that $G = GL_N$. Hence the p -adic group $G^\vee$ is isomorphic to GL_N and we have the classification of representations of the p -adic group $GL_N(F)$ by Bernstein–Zelevinsky in terms of supercuspidal representations. By a theorem of Casselman (see, e.g., [4, 3.8]) an irreducible admissible representation of $GL_N(F)$ admits a nontrivial fixed vector under an Iwahori subgroup if and only if its supercuspidal support consists of unramified quasi-characters. Using (1.9) we obtain a classification of irreducible $\mathbb{H}$ -modules where $r \in \mathbb{H}$ acts by $1/2$ (or equivalently a classification of irreducible modules of the algebra $\mathbb{H}^{1/2} = (\mathbb{H} \otimes_{\mathbb{C}[r], r \mapsto 1/2} \mathbb{C})$) which we call the Bernstein–Zelevinsky classification. We further can assume that the central character is real. It is described in the next sections.

2.11. For $G = GL_1$, $\mathbb{H}_G^{1/2}$ is nothing but a polynomial algebra in one indeterminate and we consider any complex number as a one-dimensional representation of $\mathbb{H}_{GL_1}^{1/2}$.

Let $\vec{N} = (N_1, \dots, N_m)$ denote a finite tuple of positive integers and set $GL_{\vec{N}} = GL_{N_1} \times \dots \times GL_{N_m}$. If V_i is an $\mathbb{H}_{GL_{N_i}}^{1/2}$ -module ($i = 1, \dots, m$), we set $V_1 \boxtimes \dots \boxtimes V_m$ for the module of

$$\mathbb{H}_{GL_{\vec{N}}}^{1/2} = \mathbb{H}_{GL_{N_1}}^{1/2} \otimes \dots \otimes \mathbb{H}_{GL_{N_m}}^{1/2}$$

whose underlying vector space is $V_1 \otimes \dots \otimes V_m$ and which is endowed with the componentwise action. We further set

$$V_1 \boxdot \dots \boxdot V_m = \mathbb{H}_{GL_N}^{1/2} \otimes_{\mathbb{H}_{GL_{\vec{N}}}^{1/2}} (V_1 \boxtimes \dots \boxtimes V_m),$$

where $N = N_1 + \dots + N_m$.

If in particular $(\sigma_1, \dots, \sigma_m)$ is a tuple of complex numbers, $\sigma_1 \boxdot \dots \boxdot \sigma_m$ is a $\mathbb{H}_{GL_m}^{1/2}$ -module.

2.12. Given $m \in \mathbb{N}$ and σ a real number, define the *segment*

$$\Delta(\sigma, m) = [\sigma, \sigma + 1, \dots, \sigma + m - 1].$$

The real number $\sigma + (m - 1)/2$ is called the *center* of $\Delta(\sigma, m)$, and the integer $m \geq 1$ is called the *length* of $\Delta(\sigma, m)$.

Consider $\Delta_1 = \Delta(\sigma_1, m_1)$ and $\Delta_2 = \Delta(\sigma_2, m_2)$ two segments. We will say that Δ_1 and Δ_2 are *linked* if $\Delta_1 \not\subseteq \Delta_2$ and $\Delta_2 \not\subseteq \Delta_1$ and $\Delta_1 \cup \Delta_2$ is of the form $\Delta(\tau, m')$, for some $\tau \in \{\sigma_1, \sigma_2\}$ and some $m' \in \mathbb{N}$.

Further, we say that Δ_1 *precedes* Δ_2 , if Δ_1 and Δ_2 are linked and $\tau = \sigma_1$.

2.13. With these definitions we can state the following facts [10,11,22].

- (1) Take $\Delta = \Delta(\sigma, m)$ as above. Then $\sigma \boxdot (\sigma + 1) \boxdot \cdots \boxdot (\sigma + m - 1)$ is reducible for $m > 1$ and has a unique irreducible quotient $L(\Delta)$.
- (2) Let $(\Delta_1, \dots, \Delta_l)$ be a tuple of segments as above, and assume that Δ_i does not precede Δ_j for $i < j$, then $L(\Delta_1) \boxdot \cdots \boxdot L(\Delta_l)$ admits a unique irreducible quotient $L(\Delta_1, \dots, \Delta_l)$.
- (3) Every irreducible admissible representation of $\mathbb{H}_{GL_N}^{1/2}$ is isomorphic to some $L(\Delta_1, \dots, \Delta_l)$ where $(\Delta_1, \dots, \Delta_l)$ is a tuple as in (2). If $(\Delta'_1, \dots, \Delta'_k)$ is any other tuple as in (2) such that $L(\Delta_1, \dots, \Delta_l) \cong L(\Delta'_1, \dots, \Delta'_k)$ then $l = k$ and $\Delta'_i = \Delta_{\pi(i)}$ for some permutation $\pi \in S_l$.
- (4) $L(\Delta_1) \boxdot \cdots \boxdot L(\Delta_l)$ is irreducible if and only if no two segments Δ_i and Δ_j are linked.

We set

$$X(\Delta_1, \dots, \Delta_l) := L(\Delta_1) \boxdot \cdots \boxdot L(\Delta_l)$$

such that $L(\Delta_1, \dots, \Delta_l)$ is the unique irreducible quotient of $X(\Delta_1, \dots, \Delta_l)$.

We call this classification of the irreducible admissible representation of $\mathbb{H}_{GL_N}^{1/2}$ the Bernstein–Zelevinsky classification.

2.14. Let us connect the Bernstein–Zelevinsky classification with the classification by conjugacy classes of pairs (s, e) such that $[s, e] = e$ (note that to simplify we are still in the case $r_0 = 1/2$ which we can assume anyway because of (1.9)).

Let $V = L(\Delta(\sigma_1, m_1), \dots, \Delta(\sigma_l, m_l))$ be an irreducible representation of $\mathbb{H}_{GL_N}^{1/2}$ with $N = m_1 + \cdots + m_l$. Denote by $\gamma_i = \sigma_i + \frac{1}{2}(m_i - 1)$ the center of $\Delta(\sigma_i, m_i)$. We assume that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_l$ which in particular implies that $\Delta(\sigma_i, m_i)$ does not precede $\Delta(\sigma_j, m_j)$ for $i < j$. We set

$$s = \bigoplus_{i=1}^l \text{diag}(\sigma_i, \sigma_i + 1, \dots, \sigma_i + m_i - 1), \quad e = \bigoplus_{i=1}^l n_{m_i},$$

where n_d is the nilpotent $(d \times d)$ -matrix

$$n_d = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

Then we have $[s, e] = e$ and the irreducible $\mathbb{H}_{GL_N}^{1/2}$ -module associated to the conjugacy class of (s, e) is isomorphic to V .

Now we can use (2.9) to compute the corresponding Langlands triple (S, U, ν) . For this we have to construct the element t . We assume that the based root datum is given by the Borel pair $T \subset B$ of GL_N where T is the diagonal torus and B the Borel subgroup of upper triangular matrices. The simple roots in Π are then given by the linear forms $\alpha_i : \text{diag}(x_1, \dots, x_N) \mapsto x_i - x_{i+1}$ for $i = 1, \dots, N - 1$.

As homomorphism $\psi : \mathfrak{sl}_2 \rightarrow \mathfrak{gl}_N$ we choose the unique ψ such that $\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e$ and such that $\psi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \text{diag}(h_1, \dots, h_l)$, where $h_i = \text{diag}(-m_i + 1, -m_i + 3, \dots, m_i - 1)$. Hence we have

$$t = s - \frac{1}{2}h = \bigoplus_{i=1}^l \text{diag}(\underbrace{\gamma_i, \dots, \gamma_i}_{m_i \text{ times}}).$$

2.15. We give an example. Let V be the irreducible $\mathbb{H}_{GL_6}^{1/2}$ -module given by the sequence of segments

$$([2, 3], [0, 1, 2], [1]).$$

Then we have $V = L(s, e)$ where $s = \text{diag}(2, 3, 0, 1, 2, 1)$ and

$$e = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & & 0 & & & \\ & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & & 0 \end{pmatrix}.$$

Further $t = \text{diag}(5/2, 5/2, 1, 1, 1, 1)$ and hence $S = \Pi \setminus \{\alpha_2\}$. The tempered representation U can be considered as an $(\mathbb{H}_{GL_2}^{1/2} \otimes \mathbb{H}_{GL_4}^{1/2})$ -module and it is the tensor product of the $\mathbb{H}_{GL_2}^{1/2}$ -module U_1 and the $\mathbb{H}_{GL_4}^{1/2}$ -module U_2 where U_1 is given (with respect to the Bernstein–Zelevinsky classification) by $[-1/2, 1/2]$ and U_2 is given by $([-1, 0, 1], [0])$. We have $\mathfrak{z}_S^+ = \{(x_1, x_1, x_2, x_2, x_2, x_2) \in \mathbb{R}^6 \mid x_1 > x_2\}$ and $\nu = (5/2, 5/2, 1, 1, 1, 1)$.

2.16. We remark that we can also check the irreducibility of $X(s, e)$ directly. The Levi subgroup $Z_G(s)$ acts on the vector space $\{n \in \mathfrak{gl}_N \mid [s, n] = n\}$ by conjugation and $X(s, e)$ is irreducible if and only if e lies in the unique open orbit of that action.

2.17. Let $\mathfrak{M}$ be a multiset of segments and let $X(\mathfrak{M})$ be the corresponding standard module. The irreducible subquotients of $X(\mathfrak{M})$ can be described as follows [22].

An elementary operation on a multiset $\mathfrak{M}$ is by definition to take two segments Δ_1 and Δ_2 which are linked from $\mathfrak{M}$ and to replace them by $\Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2$. For two multisets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ we say that $\mathfrak{M}_1 \preceq \mathfrak{M}_2$ if $\mathfrak{M}_1$ can be obtained from $\mathfrak{M}_2$ by elementary operations.

With this definition we have that $L(\mathfrak{M}')$ occurs as irreducible subquotient of $X(\mathfrak{M})$ if and only if $\mathfrak{M}' \preceq \mathfrak{M}$.

Further, $X(\mathfrak{M})$ has a unique irreducible quotient and a unique irreducible submodule and the isomorphism classes of both of them occur with multiplicity one in $X(\mathfrak{M})$.

Zelevinsky [22] shows that if all real numbers which occur in all segments of $\mathfrak{M}$ are pairwise distinct, $L(\mathfrak{M}')$ occurs in $X(\mathfrak{M})$ with multiplicity one for all $\mathfrak{M}' \preccurlyeq \mathfrak{M}$.

2.18. Let $V = L(s, e) = L(\Delta_1, \dots, \Delta_l)$ be an irreducible $\mathbb{H}_{GL_N}^{1/2}$ -module with associated Langlands data (S, U, ν) . Then we call V *unramified* if the following equivalent conditions are satisfied.

- (1) We have $e = 0$ and s is a regular semisimple element.
- (2) The length of all segments Δ_i is equal to 1 and their centers are pairwise different.
- (3) $S = \emptyset$.

2.19. Let $V = L(\Delta_1, \dots, \Delta_l)$ be an irreducible $\mathbb{H}_{GL_N}^{1/2}$ -module with associated Langlands data (S, U, ν) . Then the following assertions are equivalent.

- (1) V is tempered.
- (2) All centers of the segments Δ_i are equal to zero.
- (3) $S = \Pi$ and $\nu = 0$.

3. The Zelevinsky involution

3.1. Let G be a reductive group over $\mathbb{C}$ with based root datum $(X, Y, R, R^\vee, \Pi)$ and let $\mathbb{H}_G$ be the associated graded Hecke algebra. We define an involution on $\mathbb{H}_G$ which we show to be induced by the Zelevinsky involution on the affine Hecke algebra $\mathcal{H}_G$ for $G = GL_N$. By abuse of notation we will call it also Zelevinsky involution and denote it by ζ . It is defined as

$$\begin{aligned}\zeta(r) &= r, \\ \zeta(t_w) &= (-1)^{l(w)} t_{w_0 w w_0} \quad \text{for } w \in W, \\ \zeta(\theta) &= {}^{w_0}\theta \quad \text{for } \theta \in X,\end{aligned}$$

where w_0 is the element of maximal length in the Weyl group W of G .

It is easy to check that ζ preserves the relations in (1.6) defining the graded Hecke algebra.

For every $\mathbb{H}_G$ -module V (where the $\mathbb{H}_G$ -module structure is given by a $\mathbb{C}$ -algebra homomorphism $\rho: \mathbb{H}_G \rightarrow \text{End}(V)$) we write $\zeta(V)$ for the $\mathbb{H}_G$ -module given by $\rho \circ \zeta$. This defines an involutive endofunctor of the category of $\mathbb{H}_G$ -modules $\mathbb{H}_G\text{Mod}$.

3.2. Note that from the definition we get the following observation

Remark. Let V be an $\mathbb{H}_G$ -module which admits a central character χ . Then $\zeta(V)$ also admits a central character χ' , and we have $\chi = \chi'$.

3.3. We want to describe the effect of ζ on irreducible $\mathbb{H}_{GL_N}$ -modules given by the Bernstein–Zelevinsky classification. For this we show that ζ induces Zelevinsky’s involution. To prove this

we make the following definition. Let $\mathcal{R}_N$ be the Grothendieck group of the category of finite-dimensional $\mathbb{H}_{GL_N}^{1/2}$ -modules with real central character and set $\mathcal{R} = \bigoplus_{N \geq 0} \mathcal{R}_N$ where GL_0 is by definition the trivial group (and hence $\mathcal{R}_0 = \mathbb{Z}$). The map

$$\mathcal{R}_{N_1} \times \mathcal{R}_{N_2} \rightarrow \mathcal{R}_{N_1+N_2}, \quad ([V_1], [V_2]) \mapsto [V_1 \boxplus V_2]$$

makes $\mathcal{R}$ into a graded ring. For every segment Δ as in (2.12) we have the corresponding irreducible representation $[L(\Delta)] \in \mathcal{R}$ and the same arguments given for the analogous statement for representations of $GL_N(F)$ in [22, Section 7] show that this makes $\mathcal{R}$ into the polynomial algebra over $\mathbb{Z}$ in indeterminates Δ where Δ runs through all segments.

3.4. The Zelevinsky involution defines an involutive automorphism of the graded ring $\mathcal{R}$. We define another involution which is by definition the unique involutive automorphism ζ' of $\mathcal{R}$ such that

$$\zeta'(L([\sigma, \sigma + 1, \dots, \sigma + m - 1])) = L([\sigma + m - 1, [\sigma + m - 2], \dots, [\sigma]])$$

which is the analog of the automorphism constructed by Zelevinsky in [22, 9.12].

Proposition. *The involutions ζ and ζ' of $\mathcal{R}$ coincide.*

Proof. This follows from results of Mœglin and Waldspurger [19, I] for the affine Hecke algebra using (1.9). We remark that the elements X_i (respectively S_j) of [19, I] are those which are called θ_{e_i} (respectively T_{s_j}) in (1.4) where $e_i \in \mathbb{Z}^N = X$ is the i th standard base vector and s_j is the reflection corresponding to the base root $(x_1, \dots, x_N) \mapsto x_j - x_{j+1}$. Further note that for the image t_{s_j} of S_j in the graded Hecke algebra we have $t_{s_j}^{-1} = t_{s_j}$. Finally note that q in [19, I] is equal to z^2 in (1.3) and hence that under the transition to the graded Hecke algebra as described in (1.9) the factor q becomes 1. $\square$

3.5. Now let $V = L(\Delta(\sigma_1, m_1), \dots, \Delta(\sigma_l, m_l))$ be an irreducible $\mathbb{H}_{GL_N}^{1/2}$ -module. We want to explain the effect of ζ on V . For this we follow [19, II]. Set $\Delta_i = \Delta(\sigma_i, m_i)$ and let $\mathfrak{M}$ be the multiset (i.e. the set with multiplicities) of the segments Δ_i . By [19, II], $\zeta(V)$ is the irreducible representation associated to the multiset $\mathfrak{M}^\#$ of segments where $\mathfrak{M}^\#$ is defined as follows.

For $t \in \mathbb{R}/\mathbb{Z}$ write $\mathfrak{M}_t = \{\Delta_i \in \mathfrak{M} \mid \sigma_i \equiv t \pmod{\mathbb{Z}}\}$. Then $\mathfrak{M}$ is the disjoint union of the $\mathfrak{M}_t$, where t runs through $\mathbb{R}/\mathbb{Z}$ and we set

$$\mathfrak{M}^\# = \bigcup_{t \in \mathbb{R}/\mathbb{Z}} \mathfrak{M}_t^\#.$$

Hence we will from now on assume that $\sigma_1 \equiv \dots \equiv \sigma_l \pmod{\mathbb{Z}}$. We introduce a total order on the set of segments by saying $\Delta(\sigma_1, m_1) \geq \Delta(\sigma_2, m_2)$ if

$$\begin{aligned} \sigma_1 &> \sigma_2, \quad \text{or} \\ \sigma_1 &= \sigma_2 \quad \text{and} \quad \sigma_1 + m_1 \geq \sigma_2 + m_2. \end{aligned}$$

Further, if $\Delta = \Delta(\sigma, m)$, we set

$$\Delta^- = \Delta(\sigma, m - 1).$$

Let δ be the biggest real number appearing in one of the segments of $\mathfrak{M}$ and let Δ_{i_0} be a segment containing δ which is maximal with this property. Necessarily we have $\delta = \sigma_{i_0} + m_{i_0} - 1$. Now define inductively integers $i_1, \dots, i_r$ as follows.

1. Δ_{i_s} is a segment of $\mathfrak{M}$ preceding $\Delta_{i_{s-1}}$ such that $\sigma_{i_s} + m_{i_s} - 1 = \delta - s$ and such that Δ_{i_s} is maximal with this property.
2. i_r is the last integer which can be defined this way.

We set

$$\mathfrak{M}^- = (\Delta'_1, \dots, \Delta'_l),$$

where

$$\Delta'_i = \begin{cases} \Delta_i^-, & \text{if } i \in \{i_0, \dots, i_r\}, \\ \Delta_i, & \text{otherwise.} \end{cases}$$

Note that Δ'_i can be empty. Now we define

$$\mathfrak{M}^\# = \{\Delta(\delta - r, r + 1)\} \cup (\mathfrak{M}^-)^\#$$

and proceed inductively.

3.6. We give an example of the effect of ζ . For

$$\mathfrak{M} = ([3, 4], [2, 3, 4], [1, 2], [1/2], [0], [-1/2], [-1, 0, 1])$$

we get

$$\mathfrak{M}^\# = ([4], [4], [3], [1, 2, 3], [0, 1, 2], [0], [-1/2, 1/2], [-1]).$$

3.7. There has also been given another combinatorial description of ζ in [12].

4. W -structure of standard modules

4.1. We assume in this section that $G = GL_N$ hence we have $W = S_N$. As $\mathbb{H}_G$ contains $\mathbb{C}[W]$ as a subalgebra, every $\mathbb{H}_G$ -module V has also the structure of a representation of W . We are interested in the $\mathbb{C}[W]$ -module structure of the standard modules $X(s, e)$ and their irreducible quotients $L(s, e)$.

4.2. Let us briefly recall some parts of the representation theory of $W = S_N$. The set of isomorphism classes of irreducible representations of S_N is denoted by $\hat{S}_N$. We have two distinguished (irreducible) representations of S_N , namely the trivial representation **1** and the sign representation sgn . These are the only 1-dimensional representations of S_N .

Denote by $\mathcal{P}(N)$ the set of partitions of N . Given a partition $\mathbf{d} = [d_1 \geq \dots \geq d_N \geq 0] \in \mathcal{P}(N)$, define the transpose of $\mathbf{d}$ as $\mathbf{d}' = [d'_1 \geq \dots \geq d'_N \geq 0]$ with $d'_i = \#\{j \mid d_j \geq i\}$. For $\mathbf{d} \in \mathcal{P}(N)$ we set $S_{\mathbf{d}} = S_{d_1} \times \dots \times S_{d_N}$ which we embed into S_N in the usual way.

The set $\hat{S}_N$ is in bijection to $\mathcal{P}(N)$. The representation $\pi_{\mathbf{d}}$ corresponding to $\mathbf{d} \in \mathcal{P}(N)$ is the unique irreducible representation of S_N such that:

- (a) The restriction of $\pi_{\mathbf{d}}$ to the subgroup $S_{\mathbf{d}}$ of S_N contains a copy of the trivial representation.
- (b) The restriction to $S_{\mathbf{d}'}$ contains a copy of the sign representation.

The tensor product with the sign representation defines an involution on $\hat{S}_N$. More precisely, we have $\pi_{\mathbf{d}'} \cong \pi_{\mathbf{d}} \otimes \text{sgn}$. The dimension of $\pi_{\mathbf{d}}$ is the number of standard Young tableaux of shape $\mathbf{d}$.

On the other hand, there exists a bijective correspondence between the set of GL_N -orbits of nilpotent elements in $\mathfrak{gl}_N$ and the set $\mathcal{P}(N)$ given by the block sizes of the Jordan normal form of the nilpotent element. Combining these two facts we obtain a bijection between nilpotent orbits in $\mathfrak{gl}_N$ and $\hat{S}_N$.

Via this bijection, the principal nilpotent orbit corresponds to the trivial representation, and the zero orbit corresponds to the sign representation.

On the set of nilpotent G -orbits of $\mathfrak{g}$ there is a partial order where we say that $\mathcal{O} \leq \mathcal{O}'$ if and only if $\mathcal{O}$ is contained in the closure of $\mathcal{O}'$. This corresponds to a partial order on $\mathcal{P}(N)$ which is given by $\mathbf{d} \leq \mathbf{d}'$ if and only if $d_1 + \dots + d_k \leq d'_1 + \dots + d'_k$ for all $k = 1, \dots, N$. Hence we get also a partial order on $\hat{S}_N$ such that $\mathbf{1}$ is the greatest element and sgn is the smallest element.

4.3. Let $e \in \mathfrak{gl}_N$ be a nilpotent element, and $\mathcal{B}_e$ be the variety of Borel subalgebras of $\mathfrak{gl}_N$ that contain e . The Springer correspondence tells us that $H^*(\mathcal{B}_e) = H^*(\mathcal{B}_e, \mathbb{C})$ carries an action of the Weyl group W such that $H^{\dim(\mathcal{B}_e)}(\mathcal{B}_e)$ is isomorphic to the irreducible W -representation corresponding to the G -orbit of e .

We want to describe the W -action of the standard module $X_{GL_N}(s, e, r_0)$. If we let r_0 vary, these standard modules are by definition the fibres of a vector bundle with W -action over the affine line. As representations of a finite group cannot be deformed, the W -structure of $X_{GL_N}(s, e, r_0)$ is independent of r_0 . Hence we can assume $r_0 = 0$ and we have an isomorphism of W -modules [16, 10.13]

$$X(s, e) \cong H^*(\mathcal{B}_e) \otimes \text{sgn}.$$

On the other hand, if $\mathbf{d}$ is the partition corresponding to the GL_N -orbit of e , there is an isomorphism of W -modules (e.g., [7])

$$H^*(\mathcal{B}_e) \cong \text{Ind}_{S_{\mathbf{d}}}^{S_N}(\mathbf{1}).$$

Finally the multiplicity of $\pi_{\mathbf{d}'}$ in $\text{Ind}_{S_{\mathbf{d}}}^{S_N}(\mathbf{1})$ is given by the Kostka number $K_{\mathbf{d}', \mathbf{d}}$ (see, e.g., [18, I, 6] for a definition).

Altogether we obtain the following proposition.

Proposition. Fix an $r_0 \in \mathbb{C}$. Let s in $\mathfrak{gl}_N$ be a semisimple element and $e \in \mathfrak{gl}_N$ be a nilpotent element such that $[s, e] = 2r_0e$. Let $\mathbf{e}$ be the partition corresponding to the GL_N -orbit of e . Then the W -structure of the standard module $X(s, e)$ is given by

$$[X(s, e) : \pi_{\mathbf{d}}] = K_{\mathbf{d}', \mathbf{e}}, \quad \mathbf{d} \in \mathcal{P}(N).$$

In particular, $X(s, 0)$ is isomorphic to $\mathbb{C}[W]$ as a W -module.

4.4. We want to compare the underlying W -module structures of an $\mathbb{H}_{GL_N}$ -module V and its image under the Zelevinsky involution as defined in (3.1). The involution ζ on $\mathbb{H}_{GL_N}$ restricts to an involution on the subalgebra $\mathbb{C}[W]$ which we denote again by ζ and which induces an involutive endofunctor ζ of the category $\mathbf{Rep}(W)$ of representations of W . Its effect is described by the following result:

Proposition. *Let V be a representation of W . Then we have*

$$\zeta(V) \cong V \otimes \text{sgn}.$$

Proof. This follows directly from the definitions as the endofunctor on $\mathbf{Rep}(W)$ given by ζ is isomorphic to the one induced by the involution $w \mapsto (-1)^{l(w)}w$ on $\mathbb{C}[W]$. $\square$

4.5. Let $X(s, e)$ be a standard module. By (4.3) the $\lambda \in \hat{W}$ corresponding to the dual partition of the Jordan type of e is the unique maximal $\lambda \in \hat{W}$ occurring in $X(s, e)$ and we have $[X(s, e) : \lambda] = 1$.

Moreover, it follows from [1] that the sum X' of all $\mathbb{H}_G^{1/2}$ -submodules of $X(s, e)$ which do not contain λ is a maximal $\mathbb{H}_G^{1/2}$ -submodule of $X(s, e)$ and that we have $X/X' = L(s, e)$.

5. Hermitian and unitary $\mathbb{H}$ -modules

5.1. We return briefly to the general notations of Section 1. The $\mathbb{C}$ -vector space $X \otimes_{\mathbb{Z}} \mathbb{C}$ has a conjugation coming from the complex conjugation $\mathbb{C}$ and this induces a complex anti-linear algebra involution on $\text{Sym}(X \otimes_{\mathbb{Z}} \mathbb{C})$ which we denote by $\theta \mapsto \bar{\theta}$. For $w \in W$ we denote by t_w the corresponding element in $\mathbb{C}[W] \subset \mathbb{H}_G$. Let $w_0 \in W$ be the longest element.

Define the $*$ -operation on $\mathbb{H}_G$ as follows:

$$\begin{aligned} t_w^* &= t_{w^{-1}}, \quad \text{for } w \in W, \\ \theta^* &= (-1)^{\deg \theta} t_{w_0} (w_0 \bar{\theta}) t_{w_0}, \quad \text{for } \theta \in \text{Sym}(X \otimes_{\mathbb{Z}} \mathbb{C}), \\ r^* &= r. \end{aligned}$$

It is easy to check that this defines a complex anti-linear involution on the algebra $\mathbb{H}_G$. We call a finite-dimensional $\mathbb{H}$ -module V *Hermitian* if there is a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$ on V such that

$$\langle H \cdot x_1, x_2 \rangle = \langle x_1, H^* \cdot x_2 \rangle$$

for $H \in \mathbb{H}$, and $x_1, x_2 \in V$. By [2, Section 5] this notion of being Hermitian corresponds to the obvious one if V comes from an admissible representation of $G^\vee(F)$ by the procedure described in (1.2) and (1.9).

5.2. Now let us again assume that $G = GL_N$ and let V be an irreducible $\mathbb{H}_{GL_N}^{1/2}$ -module with real central character. We want to express the property that V is Hermitian in terms of the Langlands and the Bernstein–Zelevinsky classification.

First let (S, U, ν) be the Langlands data associated to V . Then it follows from [3, 1.5] that V is Hermitian if and only if there exists a $w \in W$ satisfying:

- (1) $w(\nu) = -\nu$;
- (2) $w(S) = S$;
- (3) $w(U) \cong U$.

Because of (1) and $\nu \in \mathfrak{z}_S^+$ we have necessarily $w \in w_0 W_S$ where W_S is the subgroup of W generated by s_α for $\alpha \in S$. If we write $\Pi = \{\alpha_1, \dots, \alpha_{N_1}\}$ as in (2.14), the identity (2) then implies that $\alpha_i \in S$ if and only if $\alpha_{N-i} \in S$ for all $i = 1, \dots, N-1$.

Now assume that $V = L(\Delta_1, \dots, \Delta_l)$. By [20] its Hermitian dual is given by $L(\Delta_1^h, \dots, \Delta_l^h)$ where

$$[x, x+1, \dots, x+m-1]^h = [-(x+m-1), \dots, -x].$$

In particular we see:

Proposition. *Let $V = L(\Delta_1, \dots, \Delta_l)$ with $\Delta_i = \Delta(\sigma_i, m_i)$ be given as above. Then V is Hermitian if and only if we can group together the segments to pairs Δ_{i_1} and Δ_{i_2} (i_1 not necessarily different from i_2) such that*

- (1) *The center of Δ_{i_1} is the negative of the center of Δ_{i_2} .*
- (2) *We have $m_{i_1} = m_{i_2}$.*

5.3. We now recall Tadić's description of unitary representations [20] transferred to the setting of graded Hecke algebras. We phrase this in terms of the Kazhdan–Lusztig classification. For any integer $d \geq 1$ we define the $(d \times d)$ -matrix

$$n_d = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

Let $L(s, e)$ be an irreducible $\mathbb{H}_{GL_N}^{1/2}$ -module. Then $L(s, e)$ is unitary if and only if (s, e) is conjugate to a direct sum $\bigoplus (s_i, e_i)$ where (s_i, e_i) is of one of the following forms:

- (I) $s_i = s(l_i, d_i)$ and $e_i = n_{d_i}^{\oplus l_i}$ with

$$s(l, d) = \bigoplus_{j=1}^l \text{diag} \left(\frac{-l-d}{2} + j, \frac{-l-d}{2} + j + 1, \dots, \frac{-l+d}{2} + j - 1 \right).$$

- (II) $s_i = (s(l_i, d_i) + \alpha I_{l_i d_i}) \oplus (s(l_i, d_i) - \alpha I_{l_i d_i})$ and $e_i = n_{d_i}^{\oplus 2l_i}$ for some real number α_i with $0 < \alpha_i < \frac{1}{2}$.

5.4. It follows that an unramified irreducible $\mathbb{H}_{GL_N}^{1/2}$ -module $L(s, 0)$ is unitary if and only if s is conjugated to a direct sum of diagonal matrices s_i which are of one of the following forms:

- (I) $s_i = \text{diag}(\frac{1-l_i}{2}, \frac{1-l_i}{2} + 1, \dots, \frac{l_i-1}{2})$;
 (II) $s_i = \text{diag}(\frac{1-l_i}{2} - \alpha_i, \frac{1-l_i}{2} + \alpha_i, \dots, \frac{l_i-1}{2} - \alpha_i, \frac{l_i-1}{2} + \alpha_i)$ for some real number $0 < \alpha_i < \frac{1}{2}$.

5.5. Let V be a finite-dimensional Hermitian $\mathbb{H}_G$ -module and let $\langle, \rangle : V \times V \rightarrow \mathbb{C}$ be a non-degenerate Hermitian form on V such that $\langle hv, v' \rangle = \langle v, h^* v' \rangle$ for all $h \in \mathbb{H}_G$ and all $v, v' \in V$. Decompose $V = \bigoplus_{i \in I} V_i$ into an orthogonal sum of irreducible W -representations. The restriction $\langle, \rangle_i$ of $\langle, \rangle$ to V_i is either positive or negative definite. For each $\lambda \in \hat{W}$ we set

$$\begin{aligned}\sigma_\lambda^+(V, \langle, \rangle) &= \#\{i \in I \mid V_i \cong \lambda, \langle, \rangle_i \text{ is positive definite}\}, \\ \sigma_\lambda^-(V, \langle, \rangle) &= \#\{i \in I \mid V_i \cong \lambda, \langle, \rangle_i \text{ is negative definite}\}, \\ \sigma_\lambda(V, \langle, \rangle) &= \sigma_\lambda^+(V, \langle, \rangle) - \sigma_\lambda^-(V, \langle, \rangle).\end{aligned}$$

These numbers are independent of the choice of the orthogonal decomposition of V into irreducible W -representations.

Assume that V is irreducible as an $\mathbb{H}_G$ -module. In this case $\langle, \rangle$ is uniquely determined up to a non-zero real number. Hence the class of $(\sigma_\lambda(V, \langle, \rangle))_{\lambda \in \hat{W}} \in \mathbb{Z}^{\hat{W}}$ in $\mathbb{Z}^{\hat{W}}/\{\pm 1\}$ is independent of the choice of $\langle, \rangle$ (here $\{\pm 1\}$ acts on $\mathbb{Z}^{\hat{W}}$ by $\varepsilon \cdot (\sigma_\lambda) = (\varepsilon \sigma_\lambda)$). We call this class $\bar{\Sigma}(V)$.

5.6. If $\bar{\Sigma} \in \mathbb{Z}^{\hat{W}}/\{\pm 1\}$ is the signature character of some irreducible Hermitian representation we define a lift $\Sigma \in \mathbb{Z}^{\hat{W}}$ as follows. For every irreducible $\mathbb{H}_G^{1/2}$ -module V there exists a unique maximal $\lambda \in \hat{W}$ such that $[V : \lambda] > 0$, and, moreover, we have $[V : \lambda] = 1$ (4.5). We let Σ be the unique lift of $\bar{\Sigma}$ such that $\Sigma_\lambda = 1$. Using this normalization, we get a map

$$\Sigma : \{\text{irreducible Hermitian } \mathbb{H}_G^{1/2}\text{-modules}\} \rightarrow \mathbb{Z}^{\hat{W}}.$$

5.7. The bijection $\hat{W} \rightarrow \hat{W}$ which sends U to $U \otimes \text{sgn}$ defines a $\mathbb{Z}$ -linear automorphism of $\mathbb{Z}^{\hat{W}}$ (by taking the corresponding permutation matrix) and this induces a bijection of order 2 on $\mathbb{Z}^{\hat{W}}/\{\pm 1\}$ which we denote again by $[(\sigma_\lambda)] \mapsto [(\sigma_\lambda)] \otimes \text{sgn}$.

Proposition. For every irreducible $\mathbb{H}_G^{1/2}$ -module V we have

$$\bar{\Sigma}(\zeta(V)) = \bar{\Sigma}(V) \otimes \text{sgn}.$$

Proof. It follows directly from the definitions that the involution $*$ and the Zelevinsky involution ζ , see (3.1), commute with each other. Hence (4.4) implies the proposition. $\square$

6. Intertwining operators and the Hermitian form

6.1. In the sequel we will use the following notations. We set $G = GL_N$ and fix a standard parabolic subgroup. It is given by a subset S of the set of the simple roots which corresponds to an ordered partition $(N_1, \dots, N_r)$ of N . We denote by M the associated standard Levi subgroup. Instead of $\mathbb{H}_S$ (2.3) we often write $\mathbb{H}_M$.

Let U be a fixed irreducible tempered representation of $\mathbb{H}_M^{1/2}$. Thus it will be of the form

$$U = U_1 \boxtimes \dots \boxtimes U_r$$

for irreducible tempered representations U_i of $\mathbb{H}_{GL_{N_i}}^{1/2}$. We denote by $\langle \cdot, \cdot \rangle_{U_i}$ the unitary form (unique up to a positive scalar) on U_i and by $\langle \cdot, \cdot \rangle_U$ its tensor product on U .

We will consider (real) Hermitian representations with Langlands data (S, U, ν) (2.8). Hence we have for all i , $N_i = N_{r+1-i}$, and U_i is isomorphic to U_{r+1-i} . In the sequel we choose an identification of unitary $\mathbb{H}_{GL_{N_i}}$ -modules $U_i \cong U_{r+1-i}$. Finally ν will be given by an element in $\mathfrak{z}_S^+$ which we can consider as an r -tuple of real numbers $\nu \in C_{(N_1, \dots, N_r)}$ where

$$C_{(N_1, \dots, N_r)} := \{(\nu_1, \dots, \nu_r) \in \mathbb{R}^r \mid \nu_1 > \dots > \nu_r, \nu_i + \nu_{r+1-i} = 0 \text{ for } i = 1, \dots, r\}.$$

We denote by $X(S, U, \nu)$ the associated standard module and by $L(S, U, \nu)$ its unique irreducible quotient, and we call ν the *Hermitian parameter*.

Let W_M be the Weyl group of M , and we set $W = W_G$. We identify W_M with $S_{N_1} \times \dots \times S_{N_r}$, embedded in $W = S_N$ as usual. Let w_0 be the element of maximal length in W and let $w_{0,M}$ be the element of minimal length in the double coset $W_M w_0 W_M$. Note that as w_0 normalizes W_M , we have $W_M w_0 W_M = w_0 W_M = W_M w_0$ and $w_{0,M}^2 = 1$. Our identification of U_i with U_{r+1-i} then gives an identification $U \cong w_{0,M}(U)$.

We fix the following isomorphism of $\mathbb{H}_M$ -modules preserving unitary forms:

$$\begin{aligned} \tau : U_r \boxtimes \dots \boxtimes U_1 &\xrightarrow{\sim} U_1 \boxtimes \dots \boxtimes U_r = U, \\ u_r \otimes \dots \otimes u_1 &\mapsto u_1 \otimes \dots \otimes u_r. \end{aligned}$$

Note that we can consider τ as an isomorphism $w_{0,M}(U) \xrightarrow{\sim} U$.

6.2. We now introduce certain elements following [3, 1.6 and 1.7]. Fix $w \in W$ and let $w = s_1 \dots s_l$ be a reduced decomposition. Then define $\rho_w = \rho_1 \rho_2 \dots \rho_l$ where $\rho_i = t_{\alpha_i} \alpha_i - 2r$ if s_i corresponds to the simple root α_i . Using a result of Lusztig [14, 5.2], it is shown in [3, 1.6] that ρ_w does not depend on the choice of the reduced decomposition of w and that we have for all $\theta \in \mathbb{O}$

$$\theta \rho_w = \rho_w ({}^{w^{-1}} \theta). \quad (6.2.1)$$

6.3. There is a unique $\mathbb{O}$ -linear map $\varepsilon_M : \mathbb{H}_G \rightarrow \mathbb{H}_M$ such that $\varepsilon(t_w) = t_w$ for $w \in W_M$ and $\varepsilon(t_w) = 0$ for $w \in W \setminus W_M$.

We now define an Hermitian form $\beta_\nu = \beta_{S, U, \nu}$ on $X(S, U, \nu)$ as follows. For $w, w' \in W$ and $u, u' \in U \otimes \mathbb{C}_\nu$ we set

$$\beta_{S,U,v}(t_w \otimes u, t_{w'} \otimes u') = \langle \varepsilon(t_{w'^{-1}} t_w \rho_{w_0,M}) \tau(u), u' \rangle_U.$$

This is well defined by (6.2.1).

6.4. The Hermitian form $(\cdot, \cdot)$ on $X(S, U, v)$ defined by

$$(t_w \otimes u, t_{w'} \otimes u') = \langle \varepsilon(t_{w'^{-1}} t_w) u, u' \rangle_U$$

is unitary as we have

$$(t_w \otimes u, t_w \otimes u) = \langle u, u \rangle_U.$$

Hence the signature character of $\beta_v = \beta_{S,U,v}$ can be computed as follows. Consider the $\mathbb{H}_G$ -linear map

$$\begin{aligned} A_{w_0,M} : X(S, U, v) = \mathbb{H}_G \otimes_{\mathbb{H}_M} (U \otimes \mathbb{C}_v) &\rightarrow \mathbb{H}_G \otimes_{\mathbb{H}_M} (U \otimes \mathbb{C}_{-v}) = X(S, U, -v), \\ h \otimes u &\mapsto h \rho_{w_0,M} \otimes \tau(u). \end{aligned}$$

By definition of $\mathbb{C}_v$, source and target of $A_{w_0,M}$ are canonically isomorphic as $\mathbb{C}[W]$ -modules, hence we can also consider $A_{w_0,M}$ as an endomorphism of a $\mathbb{C}[W]$ -module X , and for each $\lambda \in \hat{W}$ we get an induced endomorphism $A_{w_0,M,\lambda}$ on $\text{Hom}_W(\lambda, X)$. We denote the number of positive eigenvalues minus the number of negative eigenvalues of $A_{w_0,M,\lambda}$ by $\sigma_\lambda(A_{w_0,M})$ and set $\Sigma(A_{w_0,M}) = (\sigma_\lambda(A_{w_0,M}))_{\lambda \in \hat{W}}$. Then we have

$$\Sigma(A_{w_0,M}) = \Sigma(\beta_v).$$

Proposition. Assume that $A_{w_0,M}$ is non-zero. Then its image is isomorphic to $L(S, U, v)$ and β_v is up to a scalar the Hermitian form given by the involution $*$ (5.1).

Proof. Let $X' \subset X(S, U, v)$ be the maximal submodule. Then we have $X/X' = L(S, U, v)$ (2.17). Let $A' : X' \rightarrow X(S, U, v)^h$ be the restriction of $A_{w_0,M}$ to X' . We have to show that $A' = 0$. As $X(S, U, v)$ has a unique irreducible quotient (namely $L(S, U, v)$), $X(S, U, v)^h$ has a unique irreducible submodule (namely $L(S, U, v)^h$). But $L(S, U, v)^h$ is isomorphic to $L(S, U, v)$ as $L(S, U, v)$ is Hermitian. As $X(S, U, v)^h$ is of finite length, the image of A' would have to contain this unique irreducible submodule if A' were non-zero. This would imply that X' has a subquotient which is isomorphic to $L(S, U, v)$ but this is a contradiction to the fact that $L(S, U, v)$ occurs with multiplicity one in $X(S, U, v)$. $\square$

7. Reducibility walls

7.1. We keep the notations from (6.1).

For fixed S and U we call $v \in C_{(N_1, \dots, N_r)}$ *irreducible* if the $\mathbb{H}_{GL_N}^{1/2}$ -module $X(S, U, v)$ is irreducible and denote by $C_{(N_1, \dots, N_r)}^0$ the set of irreducible v .

Every irreducible Hermitian irreducible $\mathbb{H}_{GL_N}^{1/2}$ -module V defines a signature character $\Sigma(V) \in \mathbb{Z}^{\hat{W}}$ (5.6). We obtain a map

$$C_{(N_1, \dots, N_r)} \rightarrow \mathbb{Z}^{\hat{W}}, \quad v \mapsto \Sigma(L(S, U, v)).$$

By (6.3) and (6.4) this map is locally constant on $C_{(N_1, \dots, N_r)}^0$.

7.2. Write $L(S, U, v) = L(\Delta_1, \dots, \Delta_s)$ (2.13). Note that the number and the length of the segments do not depend on v . The standard module $X(S, U, v)$ is irreducible if and only if no two of the segments can be linked. Hence the reducibility locus, i.e. $C_{(N_1, \dots, N_r)} \setminus C_{(N_1, \dots, N_r)}^0$, is a union of hyperplanes of the form

$$H_\alpha = \{v \in C_{(N_1, \dots, N_r)} \mid \langle \alpha, v + \chi \rangle = 1\},$$

where $\alpha \in R^+$ runs through a certain set of positive roots (cf. (9.4)) and where χ is the central character of the tempered representation U .

7.3. We make this more concrete in the case of an unramified representations, i.e. for the case $S = \emptyset$ and hence U the trivial representation. In this case all segments have length 1 and $v = (v_1, \dots, v_N)$ is irreducible if and only if $\langle v, \alpha \rangle \neq 1$ for all $\alpha \in R^+$, i.e. $v_i - v_j \neq 1$ for all $i < j$.

For every root $\alpha \in R^+$ we define the corresponding reducibility wall

$$H_\alpha = \{v \in C_N \mid \langle \alpha, v \rangle = 1\}.$$

Set $M = \lfloor N/2 \rfloor$. Via $(v_1, \dots, v_N) \mapsto (v_1, \dots, v_M)$ we can identify C_N with D_M where

$$D_M = \{(x_1, \dots, x_M) \in \mathbb{R}^M \mid x_1 > \dots > x_M > 0\}.$$

Via this identification the reducibility walls can be described as follows. Assume first that N is even. Here we have that all nonempty reducibility walls are the following.

$$H_{ij}^\pm = \{(x_1, \dots, x_M) \in D_M \mid x_i \pm x_j = 1\}, \quad \text{for } 1 \leq i < j \leq M$$

and

$$H_{i, \frac{1}{2}} = \left\{ (x_1, \dots, x_M) \in D_M \mid x_i = \frac{1}{2} \right\}, \quad \text{for } 1 \leq i \leq M.$$

For N odd, we have in addition to the walls above those of the form

$$H_{j,1} = \{(x_1, \dots, x_M) \in D_M \mid x_j = \pm 1\}, \quad \text{for } 1 \leq j \leq M.$$

We will give a description of the reducibility walls in terms of the roots, namely, we will determine a subset of the set of roots such that each reducibility plane is determined by one element in this subset. The first easy remark is that

$$H_\alpha^+ = H_{-\alpha}^-, \quad H_\alpha^- = H_{-\alpha}^+$$

for any α in the root lattice. Denote by

$$K = \{\alpha_{i,j}^{\pm} = e_i \pm e_j \mid 1 \leq i < j \leq M\} \cup \{\alpha_i = e_i - e_{N-i+1} \mid i = 1, \dots, M\} \\ \cup \{\alpha_{i,M} = e_i - e_{M+1} \mid i = 1, \dots, M\}.$$

We have a one to one correspondence between the set of all vanishing walls and elements in K , namely $H_{i,j}^{\pm}$ maps to $\alpha_{i,j}^{\pm}$, and also between each $H_{i,\frac{1}{2}}$ to $\alpha_i = e_i - e_{N-i+1}$, if N is even. Now if N is odd, we also have that $H_{j,1}^{\pm}$ corresponds to $\alpha_{i,M} = e_i - e_{M+1}$.

We have the following trivial intersection rules for the walls:

- (1) $H_{ij}^{-} \cap H_{ik}^{-} = H_{ik}^{-} \cap H_{jk}^{-} = H_{ij}^{+} \cap H_{ik}^{+} = H_{ik}^{+} \cap H_{jk}^{+} = \emptyset$.
- (2) $H_{ij}^{+} \cap H_{jk}^{+} = \emptyset$.
- (3) $H_{ij}^{+} \cap H_{ij}^{-} = \emptyset$.
- (4) $H_{i,\frac{1}{2}} \cap H_{j,\frac{1}{2}} = H_{i,1} \cap H_{j,1} = \emptyset$.
- (5) $H_{ij}^{+} \cap H_{jk}^{-} = \emptyset$.
- (6) $H_{ij}^{-} \cap H_{i,\frac{1}{2}} = H_{ij}^{-} \cap H_{i,1} = \emptyset$.
- (7) $H_{ij}^{+} \cap H_{i,\frac{1}{2}} = H_{ij}^{+} \cap H_{j,\frac{1}{2}} = H_{ij}^{+} \cap H_{i,1} = H_{ij}^{+} \cap H_{j,1} = \emptyset$.

Now let $R' \subset R^{+}$ be a set of positive roots. We set

$$H_{R'} = \bigcap_{\alpha \in R'} H_{\alpha}.$$

Proposition. Let $R' \subset R^{+}$ be such that $H_{R'}$ consists of a single point $v = (v_1, \dots, v_N)$. Then $L(v)$ is unitary.

Proof. Let $J \subset R'$ be a minimal subset such that $H_J = H_{R'}$. In particular we have $\#J = M$. We say that two walls H and H' in J are equivalent if there exist walls $H = H_0, H_1, \dots, H_{r-1}, H_r = H'$ in J such that the set of indices for the wall H_k has nonempty intersection with the set of indices of H_{k+1} . Let $J = \bigcup J_{\alpha}$ be the decomposition into equivalence classes with respect to this relation. J_{α} has to be a set of walls of one of the following forms:

- (i) $J_{\alpha} = \{H_{i_1,i_2}^{-}, H_{i_2,i_3}^{-}, \dots, H_{i_{m-1},i_m}^{-}\}$.
- (ii) $J_{\alpha} = \{H_{i_1,i_2}^{-}, H_{i_2,i_3}^{-}, \dots, H_{i_{m-1},i_m}^{-}, H_{i_m,\frac{1}{2}}\}$.
- (iii) $J_{\alpha} = \{H_{i_1,i_2}^{-}, H_{i_2,i_3}^{-}, \dots, H_{i_{m-1},i_m}^{-}, H_{i_m,1}\}$.
- (iv) $J_{\alpha} = \{H_{i_1,i_2}^{-}, H_{i_2,i_3}^{-}, \dots, H_{i_{m-1},i_m}^{-}, H_{i_m,i_{m+1}}^{+}\}$.

As we have $\#J = \sum \#J_{\alpha} = M = \#\{\text{indices of walls occurring in } J\}$, in J_{α} have to be $\#J_{\alpha}$ indices, and this means that only the cases (ii) and (iii) are possible. As the v_i are pairwise different, each of the cases (ii) and (iii) can occur only once. Now it follows from Tadić's description of unitary representations (5.4) that $L(\emptyset, \mathbf{1}, v)$ has to be unitary. $\square$

8. Signature at infinity

8.1. Recall from (7.1) that the signature character is constant on the connected components of $C_{(N_1, \dots, N_r)}^0$. It is easy to see that there is a unique connected component C_{∞} of $C_{(N_1, \dots, N_r)}^0$ such

that for all roots $\alpha \in \Pi \setminus S$ the set of real numbers $\{\nu(\alpha) \mid \nu \in C\}$ is not bounded. In this section we want to describe the signature of $X(S, U, \nu) = L(S, U, \nu)$ for $\nu \in C_\infty$. For this we follow Barbasch and Moy [3].

8.2. As a W -module we have $X(S, U, \nu) \cong \mathbb{C}[W] \otimes_{\mathbb{C}[W_M]} U$. Let $\varepsilon: \mathbb{C}[W] \rightarrow \mathbb{C}[W_M]$ be the $\mathbb{C}$ -linear map with $\varepsilon(t_w) = t_w$ for $w \in W_M$ and $\varepsilon(t_w) = 0$ for $w \in W \setminus W_M$ and define a $\mathbb{C}[W]$ -invariant Hermitian form on $X(S, U, \nu)$ by

$$\beta_\infty(t_w \otimes u, t_{w'} \otimes u') = \langle \varepsilon(t_{w'^{-1}t_w t_{w_0, M}}) \tau(u), u' \rangle_U.$$

As in (5.5) this defines a signature character $\Sigma_\infty \in \mathbb{Z}^{\hat{W}}$ which depends only on S and U .

A limit argument using (7.1) (cf. [3, 2.3]) shows that for $\nu \in C_\infty$ the signature character of β_ν and the signature character of β_∞ coincide. Hence we get that for $\nu \in C_\infty$ the class of Σ_∞ in $\mathbb{Z}^{\hat{W}}/\{\pm 1\}$ is equal to the class of the signature of β_ν in $\mathbb{Z}^{\hat{W}}/\{\pm 1\}$.

8.3. To calculate the signature character of β_∞ we make the following general remark. Let $(V, \langle, \rangle)$ be any finite-dimensional complex unitary space and $f \in \text{End}(V)$ be a self-adjoint endomorphism (in particular f is semisimple). We define a Hermitian form β_f on V by

$$\beta_f(v, v') = \langle f v, v' \rangle.$$

This form is non-degenerate if and only if f is invertible and in this case its signature is equal to

$$\#\{\text{positive eigenvalues of } f\} - \#\{\text{negative eigenvalues of } f\},$$

where we count eigenvalues with multiplicity. Now assume further that $f^2 = \text{id}_V$. Then f has only eigenvalues $+1$ or -1 , and the signature of β_f is nothing but $\text{Tr}(f)$.

8.4. Let us apply this to the unitary form $\langle t_w \otimes u, t_{w'} \otimes u' \rangle = \langle \varepsilon(t_{w'^{-1}t_w}) u, u' \rangle_U$ on $X(S, U, \nu)$ and to $f = r_{w_0, M}$ given by

$$r_{w_0, M}: t_w \otimes u \mapsto t_w t_{w_0, M} \otimes \tau(u).$$

Then it follows that $\Sigma_\infty = (\sigma_{\infty, \lambda}) \in \mathbb{Z}^{\hat{W}}$ with

$$\sigma_{\infty, \lambda} = \dim(\lambda)^{-1} \text{Tr}(r_{w_0, M}|_{X_\lambda}), \quad (8.4.1)$$

where $\dim(\lambda)$ denotes the complex dimension of the irreducible W -representation corresponding to $\lambda \in \hat{W}$ and X_λ denotes the λ -isotypical component of the left W -module $X = X(S, U, \nu)$.

Now let χ_λ be the character on W corresponding to the irreducible representation λ . The projection p_λ from X onto its isotypical component X_λ is given by

$$t \otimes u \mapsto \frac{\dim(\lambda)}{\#W} \sum_{w \in W} \chi_\lambda(w) (wt \otimes u).$$

Hence if we define for $w \in W$ a $\mathbb{C}$ -linear endomorphism f^w of X by

$$t \otimes u \mapsto wt w_{0,M} \otimes \tau(u),$$

we have

$$\sigma_{\infty, \lambda} = \dim(\lambda)^{-1} \operatorname{Tr}(p_{\lambda} \circ r_{w_{0,M}}) = (\#W)^{-1} \sum_{w \in W} \chi_{\lambda}(w) \operatorname{Tr}(f^w).$$

For $z \in W$ let $\ell_z : X \rightarrow X$ be the left multiplication with z . Then we have

$$\ell_z \circ f^w \circ \ell_z^{-1} = f^{zwz^{-1}}.$$

In particular we see that $\operatorname{Tr}(f^w)$ depends only on the conjugacy class of w . Identifying conjugacy classes in W with irreducible representations of W , we get

$$\sigma_{\infty, \lambda} = (\#W)^{-1} \sum_{\mu \in \hat{W}} N(\mu) \chi_{\lambda}(\mu) \operatorname{Tr}(f^{w(\mu)}), \quad (8.4.2)$$

where $N(\mu)$ is the number of elements of W in the conjugacy class μ and where $w(\mu)$ is some element in μ .

8.5. It remains to calculate the trace of f^w . Let $[W/W_M]$ be a system of representatives in W of the quotient W/W_M . As a $\mathbb{C}$ -vector space, X is isomorphic to $\bigoplus_{w \in [W/W_M]} \mathbb{C} t_w \otimes U$. Hence

$$\operatorname{Tr}(f^w) = \sum_{\substack{z \in [W/W_M], \\ z^{-1} w z w_{0,M} \in W_M}} \operatorname{Tr}(z^{-1} w z w_{0,M} \tau|U). \quad (8.5.1)$$

It remains to determine for $x = (x_1, \dots, x_r) \in W_M = S_{N_1} \times \dots \times S_{N_r}$ the trace of the endomorphism $u \mapsto x\tau(u)$ of U . For this we set $U'_i = U_i \otimes U_{r+1-i}$ for $i = 1, \dots, [r/2]$ and also $U'_{(r+1)/2} = U_{(r+1)/2}$ if r is odd. Then we have

$$U = U'_1 \otimes \dots \otimes U'_{[(r+1)/2]}$$

and the endomorphism $x\tau|U$ is the tensor product of the endomorphisms $x'_i \tau_i|U'_i$ where $x'_i = (x_i, x_{r+1-i})$ (and $x'_{[(r+1)/2]} = x'_{[(r+1)/2]}$ if r is odd) and τ_i is the endomorphism of U'_i which switches the two components (and is the identity on $U'_{(r+1)/2}$ if r is odd). Therefore we have

$$\operatorname{Tr}(x\tau|U) = \prod_{i=1}^{[(r+1)/2]} \operatorname{Tr}(x'_i \tau_i|U'_i). \quad (8.5.2)$$

8.6. Hence we are reduced to the following situation. For an integer $M \geq 1$ let V be a finite-dimensional S_M -module and set $U = V \otimes V$. For $(x_1, x_2) \in S_M \times S_M$ we want to determine the trace of $\ell_{(x_1, x_2)} \circ \tau$ with

$$\begin{aligned}\tau : v_1 \otimes v_2 &\mapsto v_2 \otimes v_1, \\ \ell_{(x_1, x_2)} : v_1 \otimes v_2 &\mapsto x_1 v_1 \otimes x_2 v_2.\end{aligned}$$

We claim that

$$\mathrm{Tr}(\ell_{(x_1, x_2)} \circ \tau) = \sum_{\lambda \in \hat{S}_M} [V : \lambda] \chi_\lambda(x_1 x_2).$$

First note that if we have a decomposition of S_M -modules $V = V' \oplus V''$, a concrete matrix multiplication shows that

$$\mathrm{Tr}((\ell_{(x_1, x_2)} \circ \tau)|_{(V \otimes V)}) = \mathrm{Tr}((\ell_{(x_1, x_2)} \circ \tau)|_{(V' \otimes V')}) + \mathrm{Tr}((\ell_{(x_1, x_2)} \circ \tau)|_{(V'' \otimes V'')}).$$

Therefore we can assume that the S_M -module V is irreducible, say of isomorphism class $\lambda \in \hat{S}_M$. Of course, then we know that the trace of left multiplication with $x_1 x_2$ on V is simply $\chi_\lambda(x_1 x_2)$. Hence the claim follows if we prove the following lemma.

Lemma. *Let k be a field, let V be a finite-dimensional k -vector space and let f_1 and f_2 be two endomorphisms of V and let $\tau : V \otimes V \rightarrow V \otimes V$ the map that switches components. Then we have*

$$\mathrm{Tr}((f_1 \otimes f_2) \circ \tau) = \mathrm{Tr}(f_1 \circ f_2).$$

Proof. We can assume that k is algebraically closed. Using the fact that the set of semisimple endomorphisms is dense within the space of all endomorphisms with respect to the Zariski topology, we can also assume that f_1 is semisimple. Choose a basis (e_i) of V such that f_1 is given by a diagonal matrix A_1 with respect to this basis. Denote by A_2 the matrix of f_2 . Then the map $(f_1 \otimes f_2) \circ \tau$ sends $e_i \otimes e_j$ to $(A_1)_{jj} e_j \otimes (\sum_l (A_2)_{li} e_l)$. Hence we see that the trace of $(f_1 \otimes f_2) \circ \tau$ is nothing but $\sum_i (A_1)_{ii} (A_2)_{ii}$ which is the same as the trace of $f_1 \circ f_2$. $\square$

8.7. In (8.4) we described an algorithm which reduced the computation of the signature character of $X(S, U, \nu)$ for $\nu \in C_\infty$ to the computation of the isomorphism class of U as a $\mathbb{C}[W_M]$ -module (which we know by (4.3)) and the computation of the characters of the symmetric group (which is well known).

In the case of unramified representations (i.e. where S is empty and U is the trivial representation) we have:

Corollary. *For unramified representations the signature character of β_∞ is equal to $(\sigma_{\infty, \lambda})_{\lambda \in \hat{W}} \in \mathbb{Z}^{\hat{W}}$ with*

$$\sigma_{\infty, \lambda} = \chi_\lambda(w_0).$$

Proof. This is a straightforward application of the algorithm above. Note that we not only have equality in $\mathbb{Z}^{\hat{W}}/\{\pm 1\}$ but even in $\mathbb{Z}^{\hat{W}}$ as $\chi_1(w_0) = 1$. $\square$

9. Wall crossing

9.1. We continue to use the notations of (6.1). Also denote by χ the central character of U . Let (s, e) be a pair as in (2.4) corresponding to $L(S, U, \nu)$. We can conjugate (s, e) such that $s = \chi + \nu \in \mathfrak{t}$. Note that the conjugacy class of e depends only on S and U . Let $R_M \subset R$ be the root system of the Levi subgroup M . We further assume that we are not in the trivial case that the cone $C_{(N_1, \dots, N_r)}$ of Hermitian parameters is empty.

The tempered representations U_i are of the form $L(\Delta_1^i, \dots, \Delta_{r_i}^i)$ where $\Delta_j^i = \Delta(-(m_j^i - 1)/2, m_j^i)$ is a segment with center 0. We order these segments such that $j \mapsto m_j^i$ is non-increasing.

Let $\Delta_1, \dots, \Delta_m$ be a tuple of segments such that $L(\Delta_1, \dots, \Delta_m) = L(S, U, \nu)$. We call the number of pairs (Δ_i, Δ_j) such that Δ_i precedes Δ_j the *height* of ν . Hence ν is irreducible if and only if the height of ν is zero. For every reducibility wall H the height of $\nu \in H$ is constant equal to some positive integer $\text{height}(H)$ outside those points that lie on an intersection of H with some other reducibility wall. We call $\text{height}(H)$ the *height* of the reducibility wall.

Our goal in this section is to cross these walls. More precisely let ν_0 be a Hermitian parameter, such that there exists a unique reducibility wall H with $\nu_0 \in H$. Then we can find a small positive real number ε and a nonconstant map

$$(-\varepsilon, \varepsilon) \rightarrow C_{(N_1, \dots, N_r)}, \quad t \mapsto \nu(t)$$

which is the restriction of an affine map $\mathbb{R} \rightarrow \mathbb{R}^N$ such that

- (i) $\nu(0) = \nu_0$;
- (ii) $\nu(t)$ is irreducible for all $t \neq 0$.

Then for $t > 0$ (respectively $t < 0$), the signature character of $L(S, U, \nu(t))$ is constant and we call it Σ^+ (respectively Σ^-). The goal of this section is to find an expression for $\Sigma^+ + \Sigma^-$ and for $\Sigma^- - \Sigma^+$.

9.2. To do this we use a filtration which is called in [21, Section 3] the Jantzen filtration. Let $\mathbb{H}$ be a $\mathbb{C}$ -algebra with a complex anti-linear involution $*$ and let E be a left H -module which is a finite-dimensional $\mathbb{C}$ -vector space. Let β_t be a real analytic family of Hermitian forms on E defined for small real t such that $\beta_t(hx, y) = \beta_t(x, h^*y)$ for all $h \in \mathbb{H}$, $x, y \in E$. Assume that β_t is non-degenerate for $t \neq 0$. Then there is a unique sequence of subspaces

$$E = E^0 \supset E^1 \supset \dots \supset E^n = (0)$$

such that the meromorphic family of Hermitian forms $\beta_t^i = \frac{1}{t^i} \beta_t(x, y)|_{E^i}$ can be extended to an analytic family of Hermitian forms on E^i and such that the radical of β_0^i is equal to E^{i+1} . In particular β_0^i induces a non-degenerate pairing β^i on $\text{gr}^i(E) = E^i/E^{i+1}$.

Explicitly E^i can be defined as

$$E^i = \{x \in E \mid \beta_t(x, y) \text{ vanishes at least to order } i \text{ at } t = 0, \text{ for any } y \in E\}.$$

Note that our definition of the E^i varies a little bit from the definition given in [21, Section 3], but it is easily seen that both definitions are equivalent. Moreover, it follows from the

definition that the E^i are also $\mathbb{H}$ -submodules of E and that β_t^i are Hermitian with respect to the involution $*$.

Let σ^i be the signature of β^i , and denote by σ^+ (respectively σ^-) the signature of β_t for small positive (respectively negative) t . Then we have [21, Section 3]

$$\begin{aligned}\sigma^+ - \sigma^- &= 2 \sum_{i \geq 1 \text{ odd}} \sigma^i, \\ \sigma^+ + \sigma^- &= 2 \sum_{i \geq 0 \text{ even}} \sigma^i.\end{aligned}$$

9.3. Now let $\langle, \rangle$ be a fixed unitary form on E and let $f_t: E \rightarrow E$ be the analytic family of self-adjoint endomorphisms such that $\beta_t(x, y) = \langle f_t(x), y \rangle$. Then $\det(f_t)$ is non-zero for $t \neq 0$ and the analytic map $t \mapsto \det(f_t)$ has a zero of order

$$\sum_{i > 0} i (\dim(E^i) - \dim(E^{i-1})) = \sum_{i > 0} i \dim(\operatorname{gr}^i(E))$$

in $t = 0$.

9.4. Let $\bar{O}_e$ be the closure of the orbit of e in $\mathcal{N}(s) = \{n \in \mathfrak{g} \mid [s, n] = n\}$ under the action of $Z_G(s)$. For all ν we define R_ν as the set of roots $\alpha \in R^+$ such that $\mathfrak{g}^\alpha \subset \mathcal{N}(s)$ and such that there exists a non-zero element $e_\alpha \in \mathfrak{g}^\alpha$ such that $e + e_\alpha \notin \bar{O}_e$. These are those roots which “link” segments of $X(S, U, \nu)$. We have $R_\nu = \emptyset$ if and only if ν is irreducible. If ν lies on a unique reducibility wall H , R_ν is independent of ν , and we call it R_H .

Then for all $\alpha \in R_H$, $\operatorname{sgn}(\alpha(\nu(t) + \chi) - 1)$ is independent of $\alpha \in R_H$ and we have $\operatorname{sgn}(\alpha(\nu(t) + \chi) - 1) = -\operatorname{sgn}(\alpha(\nu(-t) + \chi) - 1)$ for all $t \in (-\varepsilon, \varepsilon)$. After a possible substitution of $t \mapsto -t$ we can and will from now on assume that $\operatorname{sgn}(\alpha(\nu(t) + \chi) - 1) = \operatorname{sgn}(t)$.

9.5. Let H be a reducibility wall of height one and fix $t \mapsto \nu(t)$ as above. We write $L(S, U, \nu(0))$ as $L(\mathfrak{M})$ where $\mathfrak{M}$ is a multiset of segments $\Delta_1, \dots, \Delta_m$. As H is of height one, there exists a unique pair (Δ_i, Δ_j) of segments, such that Δ_i precedes Δ_j . We denote $\mathfrak{M}'$ the multiset consisting of the segments Δ_l for $l \neq i, j$ and the segments $\Delta_i \cap \Delta_j$ and $\Delta_i \cup \Delta_j$. By (2.13) and (2.17) we have an exact sequence

$$0 \rightarrow L(\mathfrak{M}') \rightarrow X(\mathfrak{M}) \rightarrow L(\mathfrak{M}) \rightarrow 0 \quad (9.5.1)$$

and $L(\mathfrak{M}')$ is again Hermitian, and we have $L(\mathfrak{M}') = X(\mathfrak{M}')$, otherwise $\nu(0)$ would not lie on a unique reducibility wall.

We now apply (9.2) to $E = X(\nu(t))$ which as a $\mathbb{C}[W]$ -module does not depend on t and its Hermitian form β_t . It is of the form

$$X(\nu(t)) = E^0 \supsetneq L(\mathfrak{M}') = E^1 = \dots = E^\omega \supsetneq (0) = E^{\omega+1}$$

for some $\omega \geq 1$, and we have

$$\operatorname{gr}^0(E) = L(\nu(0)).$$

As $L(\mathfrak{M}')$ is an irreducible module, its signature character has to be equal to the signature character of β_ω up to a sign ε .

If we set $\Sigma^+ = \lim_{t \rightarrow 0^+} \Sigma(L(S, U, v(t)))$ and $\Sigma^- = \lim_{t \rightarrow 0^-} \Sigma(L(S, U, v(t)))$ we will therefore have

$$\begin{aligned}\Sigma^+ + \Sigma^- &= \begin{cases} 2\Sigma(L(S, U, v(0))), & \text{if } \omega \text{ is odd,} \\ 2\Sigma(L(S, U, v(0))) + 2\varepsilon\Sigma(L(\mathfrak{M}')), & \text{if } \omega \text{ is even,} \end{cases} \\ \Sigma^+ - \Sigma^- &= \begin{cases} 2\varepsilon\Sigma(L(\mathfrak{M}')), & \text{if } \omega \text{ is odd,} \\ 0, & \text{if } \omega \text{ is even.} \end{cases}\end{aligned}$$

Hence if ω is even, the signature characters on both sides of the wall are equal. If ω is odd, we can calculate Σ^+ (respectively Σ^-) if we know $\Sigma(L(\mathfrak{M}'))$ and Σ^- (respectively Σ^+) as this allows us also to calculate ε . Indeed, by (4.3) we know that the sign representation occurs with multiplicity one in $X(S, U, v(t))$ for all t and $L(\mathfrak{M}')$. Hence the equality

$$(\Sigma^+)_{\text{sgn}} - (\Sigma^-)_{\text{sgn}} = 2\varepsilon\Sigma(L(\mathfrak{M}'))_{\text{sgn}}$$

implies that we have

$$(\Sigma^+)_{\text{sgn}} = -(\Sigma^-)_{\text{sgn}} \quad \text{and} \quad \varepsilon = \Sigma(L(\mathfrak{M}'))_{\text{sgn}}(\Sigma^+)_{\text{sgn}}.$$

9.6. In fact we conjecture the following.

Conjecture for reducibility walls of height one. *We always have that ω is odd. Hence*

$$\begin{aligned}\Sigma^+ - \Sigma^- &= \varepsilon 2\Sigma(L(\mathfrak{M}')), \\ \Sigma^+ + \Sigma^- &= 2\Sigma(L(\mathfrak{M}))\end{aligned}$$

with $\varepsilon = \Sigma(L(\mathfrak{M}'))_{\text{sgn}}(\Sigma^+)_{\text{sgn}} = -\Sigma(L(\mathfrak{M}'))_{\text{sgn}}(\Sigma^-)_{\text{sgn}}$.

9.7. Now let H be a reducibility wall of height two and again fix $t \mapsto v(t)$ as above. We write $L(S, U, v(0))$ as $L(\mathfrak{M})$ where $\mathfrak{M}$ is a multiset of segments $\Delta_1, \dots, \Delta_m$. Now there exist two pairs $(\Delta_{i_1}, \Delta_{j_1})$ and $(\Delta_{i_2}, \Delta_{j_2})$ of segments, such that Δ_{i_k} precedes Δ_{j_k} for $k = 1, 2$. We can assume that Δ_{j_2} does not precede Δ_{i_1} . Note that it is possible that Δ_{j_1} precedes Δ_{i_2} . We denote $\mathfrak{M}_1$ (respectively $\mathfrak{M}_2$, respectively $\mathfrak{M}'$) the multiset of segments which we get from $\mathfrak{M}$ by linking $(\Delta_{i_1}, \Delta_{j_1})$ (respectively $(\Delta_{i_2}, \Delta_{j_2})$, respectively both $(\Delta_{i_1}, \Delta_{j_1})$ and $(\Delta_{i_2}, \Delta_{j_2})$).

By (2.17) we know that $L(\mathfrak{M})$ is the unique irreducible quotient of $X(\mathfrak{M})$, $L(\mathfrak{M}')$ is the unique irreducible submodule of $X(\mathfrak{M})$ and that the other irreducible subquotients of $X(\mathfrak{M})$ are isomorphic to $L(\mathfrak{M}_1)$ and $L(\mathfrak{M}_2)$. We assume that $L(\mathfrak{M}_1)$ and $L(\mathfrak{M}_2)$ both occur with multiplicity one in $X(\mathfrak{M})$. The standard modules satisfy the inclusions $X(\mathfrak{M}') \subset X(\mathfrak{M}_i) \subset X(\mathfrak{M})$ for $i = 1, 2$. $L(\mathfrak{M}') = X(\mathfrak{M}')$ is the unique irreducible submodule of $X(\mathfrak{M})$, we have exact sequences

$$0 \rightarrow L(\mathfrak{M}') \rightarrow X(\mathfrak{M}_i) \rightarrow L(\mathfrak{M}_i) \rightarrow 0,$$

and $L(\mathfrak{M}) = X(\mathfrak{M})/(X(\mathfrak{M}_1) + X(\mathfrak{M}_2))$.

Further $L(\mathfrak{M}')$ is Hermitian and as $v(0)$ lies only on a unique reducibility wall, we have $L(\mathfrak{M}') = X(\mathfrak{M}')$. $L(\mathfrak{M}_1)$ is non-Hermitian, its Hermitian dual is $L(\mathfrak{M}_2)$.

Again we apply (9.2) to $E = X(\mathfrak{M})$ with its family of Hermitian forms β_t . As the Hermitian forms induced on the graded pieces of the Jantzen filtration are non-degenerate, it follows that the Jantzen filtration will be of the form

$$\begin{aligned} X(\mathfrak{M}) = E^0 &\supsetneq X(\mathfrak{M}_1) + X(\mathfrak{M}_2) = E^1 = \dots = E^{\omega_1} \\ &\supsetneq X(\mathfrak{M}') = E^{\omega_1+1} = \dots = E^{\omega_2} \\ &\supsetneq (0) = E^{\omega_2+1} \end{aligned}$$

for integers $\omega_2 \geq \omega_1 > 0$. Here $\omega_2 = \omega_1$ means that only $L(\mathfrak{M})$ and $X(\mathfrak{M}_1) + X(\mathfrak{M}_2)$ occur as graded pieces in the filtration.

If $\omega_2 > \omega_1$, the signature character of the form β^{ω_2} induced on the irreducible module $\text{gr}^{\omega_2}(E) = L(\mathfrak{M}')$ is equal to the signature character of $L(\mathfrak{M}')$ up to a sign ε .

Again we can use (9.2) to compute difference and sum of $\lim_{t \rightarrow 0^+} \Sigma(L(S, U, v(t)))$ and $\lim_{t \rightarrow 0^-} \Sigma(L(S, U, v(t)))$ in terms of $\Sigma(L(\mathfrak{M}))$, $\Sigma(L(\mathfrak{M}'))$ and $\Sigma(L(\mathfrak{M}_1) \oplus L(\mathfrak{M}_2))$. Note that the last signature is zero as $L(\mathfrak{M}_1)$ and $L(\mathfrak{M}_2)$ are dual to each other.

9.8. We now consider wall crossing in the unramified case. Hence from now on we assume that S is empty. Hermitian representations are then given by elements $v \in C_{(1, \dots, 1)} = \{(v_i) \in \mathbb{R}^N \mid v_i + v_{N+1-i} = 0\}$. For simplicity we write $X(v)$ for the corresponding standard module and $L(v)$ for its unique irreducible quotient.

We now consider the $\mathbb{H}_G$ -linear homomorphism

$$A_{w_0} : h \otimes 1 \mapsto h \rho_{w_0} \otimes 1$$

defined in (6.4). As the leading term of ρ_{w_0} is $t_{w_0} \prod_{\alpha \in R^+} \alpha$, A_{w_0} is non-zero as we have $\prod_{\alpha \in R^+} \langle \alpha, v \rangle \neq 0$. Hence we can calculate the signature character of $L(v)$ using the form β_v defined in (6.3). If we write $w_0 = s_k s_{k-1} \dots s_1$ as a product of simple reflections, we get a decomposition $A_{w_0} = T_k \circ T_{k-1} \circ \dots \circ T_1$ where T_i is an $\mathbb{H}_G$ -linear map

$$T_i : \mathbb{H}_G \otimes_{\mathbb{H}_T} \mathbb{C}_{s_{i-1} \dots s_1(v)} \rightarrow \mathbb{H}_G \otimes_{\mathbb{H}_T} \mathbb{C}_{s_i \dots s_1(v)}.$$

Source and target of each T_i are canonically isomorphic as $\mathbb{C}[W]$ -modules and hence we can consider T_i as an endomorphism of $\mathbb{C}[W]$ -modules.

For each simple root $\alpha \in \Pi$ we define the Levi subgroup G^α of G by $\text{Lie}(G^\alpha) = \mathfrak{t} \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$. If the simple reflection s_i corresponds to the simple root α_i , T_i can be written as $\text{id}_{\mathbb{H}_G} \otimes T_i^{\alpha_i}$ with

$$\begin{aligned} T_i^{\alpha_i} : \mathbb{H}_{G^{\alpha_i}} \otimes_{\mathbb{H}_T} \mathbb{C}_{s_{i-1} \dots s_1(v)} &\rightarrow \mathbb{H}_{G^{\alpha_i}} \otimes_{\mathbb{H}_T} \mathbb{C}_{s_i \dots s_1(v)}, \\ h \otimes 1 &\mapsto h(t_{\alpha_i} \alpha_i - 1) \otimes 1. \end{aligned}$$

Similarly as above we can consider $T_i^{\alpha_i}$ as an endomorphism of $\mathbb{C}[S_2]$ -modules. Further by (4.3) we know source and target are isomorphic to $\mathbb{C}[S_2] = \text{sign} \oplus \mathbf{1}$. An easy calculation shows that $T_i^{\alpha_i}$ acts on sign by the scalar $1 - \langle \alpha_i, s_{i-1} \dots s_1(v) \rangle$ and on $\mathbf{1}$ by the scalar $-1 - \langle \alpha_i, s_{i-1} \dots s_1(v) \rangle$.

9.9. Now fix a reducibility wall H and a map $t \mapsto v(t)$ as above using the normalization in (9.4). We necessarily have $R_H = \{\alpha_0, -w_0(\alpha_0)\}$ for some $\alpha_0 \in R^+$, in particular the height of H is at most 2.

We apply the discussion in (9.8) to $v = v(t)$ and get $A_{w_0}(t)$. For $t = 0$ we see that $T_i^{\alpha_i}$ (and hence T_i) is invertible for all i except for

$$i \in J_H := \{j \mid s_1 s_2 \dots s_{j-1} \alpha_j \in R_H\}.$$

Further, for $i \in J_H$, $T_i^{\alpha_i}$ acts on the $\mathbf{1}$ -component by the scalar

$$-\langle \alpha_i, s_{i-1} \dots s_1(v(t)) \rangle - 1$$

which is equal to -2 at $t = 0$ and it acts on the sign-component by the scalar

$$\langle \alpha_i, s_{i-1} \dots s_1(v(t)) \rangle - 1$$

which is a linear function of t , in particular its vanishing order is 1.

9.10. We keep the notation of (9.8) and assume that we are in the situation of (9.5), in particular H is a reducibility wall of height one and J_H consists of a single element j . By the arguments above we have that

$$\text{ord}_{t=0} \det(A_{w_0}(t)) = \dim L(\mathfrak{M}') = \dim E^1.$$

Hence it follows from (9.3) that $\omega = 1$. We will also show that in this case we always have $\varepsilon = -1$. Let $\lambda \in \hat{W}$ be the maximal element occurring in $L(\mathfrak{M}')$. As the multiplicity of λ in $L(\mathfrak{M}')$ is 1, the signature of the Hermitian form induced by β_1 on the one-dimensional space $V = \text{Hom}_{\hat{W}}(\lambda, L(\mathfrak{M}'))$ is equal to ± 1 . We have to show that it is equal to -1 .

By the definition we have to show that the derivative of the function

$$\rho : t \mapsto 1 - \langle \alpha_j, s_{j-1} \dots s_1(v(t)) \rangle$$

is negative in $t = 0$. As this is a linear function it suffices to show that $\text{sgn}(\rho(t)) = -\text{sgn}(t)$. By definition of $v(t)$ (9.4), we have $\text{sgn}(\langle \alpha, v(t) \rangle - 1) = \text{sgn}(t)$ for the unique $\alpha \in R_H$. By (9.9) we have $\alpha = s_1 s_2 \dots s_{j-1} \alpha_j$. Hence we see that

$$\begin{aligned} \text{sgn}(\rho(t)) &= -\text{sgn}(\langle \alpha_j, s_{j-1} \dots s_1(v(t)) \rangle - 1) \\ &= -\text{sgn}(\langle \alpha, v(t) \rangle) \\ &= -\text{sgn}(t). \end{aligned}$$

Hence we get:

9.11. Theorem. *For unramified representations and for $v(0)$ lying in a reducibility wall of height one, we have*

$$\begin{aligned}\lim_{t \rightarrow 0^-} \Sigma(\beta_{v(t)}) + \lim_{t \rightarrow 0^+} \Sigma(\beta_{v(t)}) &= 2\Sigma(L(v(0))), \\ \lim_{t \rightarrow 0^-} \Sigma(\beta_{v(t)}) - \lim_{t \rightarrow 0^+} \Sigma(\beta_{v(t)}) &= 2\Sigma(L(\mathfrak{M}')).\end{aligned}$$

9.12. Note that we also have

$$\Sigma(L(v(0))) = \Sigma(L(\mathfrak{M}')) \otimes \text{sgn}$$

as $L(v(0))$ is the Zelevinsky dual of $L(\mathfrak{M}')$ (4.4).

9.13. We keep the notation of (9.9) and assume that we are in the situation of (9.7), in particular H is a reducibility wall of height two and J_H consists of two elements j_1 and j_2 .

If we set $m := \#W$, we have $\dim(X(\mathfrak{M})) = m$ and it follows from (9.9) that

$$\text{ord}_{t=0} \det(A_{w_0}(t)) = m.$$

If we define ω_1 and ω_2 as in (9.7), we have by (9.3)

$$m = \omega_1 \dim(X((M)_1) + X((M)_2)) + (\omega_2 - \omega_1) \dim(X(\mathfrak{M}')).$$

There exist two pairs (i_1, j_1) and (i_2, j_2) of indices such that $v_{j_k}(0) - v_{i_k}(0) = 1$ for $k = 1, 2$. We can assume that $i_1 < i_2$. We distinguish the cases $j_1 \neq i_2$ and $j_1 = i_2$.

By (4.3) we have in the first case (respectively in the second case)

$$\begin{aligned}\dim(X(\mathfrak{M}_i)) &= m/2, & \dim(X(\mathfrak{M}')) &= m/4 \\ (\text{respectively } \dim(X(\mathfrak{M}_i)) &= m/2, & \dim(X(\mathfrak{M}')) &= m/6)\end{aligned}$$

and it follows that

$$\begin{aligned}m &= \omega_1 \frac{3}{4}m + (\omega_2 - \omega_1) \frac{1}{4}m \\ \left(\text{respectively } m &= \omega_1 \frac{5}{6}m + (\omega_2 - \omega_1) \frac{1}{6}m \right).\end{aligned}$$

As $\omega_2 \geq \omega_1$ are positive integers, this implies in both cases

$$\omega_1 = 1, \quad \omega_2 = 2.$$

Therefore (9.2) implies

$$\Sigma^- + \Sigma^+ = 2\Sigma(L(v(0))) + \varepsilon 2\Sigma(L(\mathfrak{M}')),$$

where

$$\Sigma^+ = \lim_{t \rightarrow 0^+} \Sigma(\beta_{v(t)}), \quad \Sigma^- = \lim_{t \rightarrow 0^-} \Sigma(\beta_{v(t)}).$$

Moreover, it also follows that $\Sigma^- - \Sigma^+$ is nothing but two times the signature character of $L(\mathfrak{M}_1) \oplus L(\mathfrak{M}_2)$ which is zero as those two modules are Hermitian duals of each other.

Similarly as in (9.5) this implies

$$\varepsilon = (\Sigma^+)_{\text{sign}} \Sigma(L(\mathfrak{M}'))_{\text{sgn}} = (\Sigma^-)_{\text{sign}} \Sigma(L(\mathfrak{M}'))_{\text{sgn}}.$$

Hence we get:

9.14. Theorem. *For unramified representations and for $\nu(0)$ lying in a reducibility wall of height two, we have*

$$\begin{aligned} \Sigma^+ - \Sigma^- &= 0, \\ \Sigma^+ + \Sigma^- &= 2\Sigma(L(\nu(0))) + \varepsilon 2\Sigma(L(\mathfrak{M}')) \\ &= 2(\Sigma(L(\mathfrak{M}')) \otimes \text{sign} + \varepsilon \Sigma(L(\mathfrak{M}'))), \end{aligned}$$

where

$$\varepsilon = (\Sigma^+)_{\text{sign}} \Sigma(L(\mathfrak{M}'))_{\text{sgn}} = (\Sigma^-)_{\text{sign}} \Sigma(L(\mathfrak{M}'))_{\text{sgn}}.$$

9.15. Again we have

$$\Sigma(L(\nu(0))) = \Sigma(L(\mathfrak{M}')) \otimes \text{sgn}$$

as $L(\nu(0))$ is the Zelevinsky dual of $L(\mathfrak{M}')$.

10. Signature character for unramified representations

10.1. We will now give a conjectural inductive procedure to calculate the signature character for unramified representations.

For this we consider Bernstein–Zelevinsky parameter giving rise to standard irreducible representations of $\mathbb{H}_{GL_N}^{1/2}$ of the following form. If N is even, we set for $0 \leq m \leq N/2$ and real numbers $\nu_1 > \dots > \nu_m > 0$:

$$\mathfrak{M}_{\nu_1, \dots, \nu_m}^N := \left(\nu_1, \dots, \nu_m, \underbrace{\left[-\frac{1}{2}, \frac{1}{2} \right], \dots, \left[-\frac{1}{2}, \frac{1}{2} \right]}_{N/2-m \text{ times}}, -\nu_m, \dots, -\nu_1 \right).$$

If N is odd, we set for $0 \leq m \leq (N-1)/2$

$$\mathfrak{M}_{\nu_1, \dots, \nu_m}^N := \left(\nu_1, \dots, \nu_m, \underbrace{\left[-\frac{1}{2}, \frac{1}{2} \right], \dots, \left[-\frac{1}{2}, \frac{1}{2} \right]}_{(N-1)/2-m \text{ times}}, 0, -\nu_m, \dots, -\nu_1 \right).$$

We conjecture the following.

Conjecture. Assume that $X(\mathfrak{M}_{v_1, \dots, v_m}^N)$ is irreducible. Then $\Sigma(X(\mathfrak{M}_{v_1, \dots, v_m}^N))$ depends only on the cardinality of $\{v_i \mid v_i > \frac{1}{2}\}$.

It follows from (9.14) that the conjecture is true in the unramified case.

10.2. From now on we will assume that the conjectures (9.6) and (10.1) hold. In particular the following is well defined. For integers $N \geq 1$, $0 \leq m \leq N/2$ and $0 \leq r \leq (N - 2m)/2$ we set

$$\Sigma^N(m, r) := \Sigma(X(\mathfrak{M}_{v_1, \dots, v_m}^N)),$$

where $v_1 > \dots > v_r > \frac{1}{2} > v_{r+1} > \dots > v_m$.

Let $\mathbf{e}$ be the partition $(2^m, 1^{N-2m})$ of N . Further let S be the set of simple roots corresponding to the ordered partition $\lambda := (1^m, N - 2m, 1^m)$. Finally let U be the tempered representation of the standard Levi subgroup $GL_1^m \times GL_{N-2m} \times GL_1^m$ corresponding to λ which is given due to the Bernstein–Zelevinsky classification by

$$\underbrace{(0) \boxtimes \dots \boxtimes (0)}_{m \text{ times}} \boxtimes \underbrace{\left(\left[-\frac{1}{2}, \frac{1}{2} \right], \dots, \left[-\frac{1}{2}, \frac{1}{2} \right] \right)}_{N/2-m \text{ times}} \boxtimes \underbrace{(0) \boxtimes \dots \boxtimes (0)}_{m \text{ times}}$$

if N is even and by

$$\underbrace{(0) \boxtimes \dots \boxtimes (0)}_{m \text{ times}} \boxtimes \underbrace{\left(\left[-\frac{1}{2}, \frac{1}{2} \right], \dots, \left[-\frac{1}{2}, \frac{1}{2} \right], 0 \right)}_{(N-1)/2-m \text{ times}} \boxtimes \underbrace{(0) \boxtimes \dots \boxtimes (0)}_{m \text{ times}}$$

if N is odd.

Proposition. For $r = 0$ and $r = m$ we can calculate $\Sigma^N(m, r)$ as follows:

(1) For $\mathbf{d} \in \hat{S}_N$ we have

$$\Sigma^N(m, 0)_{\mathbf{d}} = K_{\mathbf{d}', \mathbf{e}}.$$

(2) Let $\Sigma_{\infty}(S, U)$ the signature at infinity (8.2) corresponding to (S, U) . Then we have

$$\Sigma^N(m, m) = \Sigma_{\infty}(S, U).$$

Proof. By (5.3) we know that $X(\mathfrak{M}_{v_1, \dots, v_m}^N)$ is unitary if all $v_i < \frac{1}{2}$. Hence (1) follows from (4.3).

By (10.1) $\Sigma^N(m, m)$ is nothing but the signature at infinity with respect to (S, U) . Therefore we know that the classes of $\Sigma^N(m, m)$ and $\Sigma_{\infty}(S, U)$ in $\mathbb{Z}^{\hat{W}}/\{\pm 1\}$ are equal. A calculation using the algorithm in (8.4) then gives equality even in $\mathbb{Z}^{\hat{W}}$. $\square$

10.3. By (9.6) we have the equality

$$\Sigma^N(m, r-1) - \Sigma^N(m, r) = 2\Sigma^N(m+1, r-1).$$

By induction this implies for $0 \leq k \leq r$

$$\Sigma^N(m, r) = \sum_{i=0}^k (-2)^i \binom{k}{i} \Sigma^N(m+i, r-k).$$

In particular:

Proposition. For N , m and r as above:

$$\Sigma^N(m, r) = \sum_{i=0}^r (-2)^i \binom{r}{i} \Sigma^N(m+i, 0)$$

which allows us to calculate $\Sigma^N(m, r)$ as we know the right-hand side by (10.2).

10.4. As we know not only the signature character of unitary modules but also of representations at infinity, our conjectures implies in particular the following equality. Let $w_0 \in S_N$ be the longest element and let λ be a partition of N . Set $r = \lfloor N/2 \rfloor$. Then we have by (10.3) and (8.7):

$$\chi_\lambda(w_0) = \sum_{i=0}^r (-2)^i \binom{r}{i} \Sigma^N(i, 0)_\lambda = \sum_{i=0}^r (-2)^i \binom{r}{i} K_{\lambda', (2^i, 1^{N-2i})}.$$

We do not know how to prove this equality directly using purely combinatorial methods.

11. Examples

11.1. We conclude with the calculation of signature characters of GL_N for $N = 2, 3, 4$. Each time we will classify irreducible $\mathbb{H}_{GL_N}$ -representations by the Langlands data (S, U, ν) .

We will describe S by the corresponding ordered partition $(\sigma_1, \dots, \sigma_r)$ of N . As we consider only Hermitian modules, we always have that $\sigma_i = \sigma_{r+1-i}$.

The tempered representation U of $GL_{\sigma_1} \times \dots \times GL_{\sigma_r}$ is a tensor product of irreducible tempered representations U_i of GL_{σ_i} and each U_i will be described by its Bernstein–Zelevinsky datum. The condition of being Hermitian implies that $U_i \cong U_{r+1-i}$.

Finally ν will be considered as an r -tuple of real numbers $(\nu_1, \dots, \nu_r)$ with $\nu_1 > \dots > \nu_r$. We have $\nu_i + \nu_{r+1-i} = 0$ because of the property of being Hermitian.

If (S, U, ν) is a Langlands datum, we denote by $\mathfrak{M}(S, U, \nu)$ the corresponding Bernstein–Zelevinsky datum.

11.2. We now consider the case GL_2 .

Let $S = (2)$, $U = \left(\left[-\frac{1}{2}, \frac{1}{2} \right] \right)$. In this case we necessarily have $\nu = (0)$ and $\mathfrak{M}(S, U, \nu) = \left(\left[-\frac{1}{2}, \frac{1}{2} \right] \right)$ and $X(S, U, \nu)$ is irreducible (2.13) and is unitary (5.3). Hence by (4.3)

$$\Sigma(L(S, U, \nu)) = (1^2) = \text{sgn}.$$

Let $S = (2)$, $U = (0, 0)$. Again ν is (0) , $X(S, U, \nu)$ is irreducible and unitary, hence

$$\Sigma(L(S, U, \nu)) = (1^2) + (2) = \text{sgn} + 1.$$

Let $S = (1, 1)$. U is necessarily $(0) \boxtimes (0)$, and ν is of the form $(\nu_1, -\nu_1)$ with $\nu_1 > 0$. We have $\mathfrak{M}(S, U, \nu) = (\nu_1, -\nu_1)$ and $X(S, U, \nu)$ is irreducible if and only if $\nu_1 \neq \frac{1}{2}$. For $\nu_1 < \frac{1}{2}$, $L(S, U, \nu)$ is unitary, for $\nu_1 > \frac{1}{2}$, we are near infinity and can apply (8.7). Alternatively we can use (9.11). Finally for $\nu_1 = \frac{1}{2}$ we have $L(\frac{1}{2}, -\frac{1}{2}) = \zeta(L([- \frac{1}{2}, \frac{1}{2}]))$ by (3.4). Hence we get

$$\Sigma(L(S, U, (\nu_1, -\nu_1))) = \begin{cases} (1^2) + (2), & \nu_1 < \frac{1}{2}, \\ (2), & \nu_1 = \frac{1}{2}, \\ -(1^2) + (2), & \nu_1 > \frac{1}{2}. \end{cases}$$

11.3. We now consider the case $N = 3$.

Let $S = (3)$. Again $\nu = (0)$ and we have

$$\Sigma(L(S, U, \nu)) = \begin{cases} (1^3), & U = ([-1, 0, 1]), \\ (1^3) + (2, 1), & U = ([-\frac{1}{2}, \frac{1}{2}], 0), \\ (1^3) + (2, 1) + (3), & U = (0, 0, 0). \end{cases}$$

Let $S = (1, 1, 1)$. We have $U = (0) \boxtimes (0) \boxtimes (0)$ and $\nu = (\nu_1, 0, -\nu_1)$. The standard module $X(S, U, \nu)$ is irreducible for $\nu_1 \neq \frac{1}{2}, 1$ and it is unitary for $\nu_1 < \frac{1}{2}$. At $\nu_1 = \frac{1}{2}$ we have a reducibility wall of height one and at $\nu_1 = 1$ a reducibility wall of height two. Further we have $L(\frac{1}{2}, 0, -\frac{1}{2}) = \zeta(L([- \frac{1}{2}, \frac{1}{2}], 0))$ and $L(1, 0, -1) = \zeta(L([-1, 0, 1]))$. Hence we get

$$\Sigma(S, U, (\nu_1, 0, -\nu_1)) = \begin{cases} (1^3) + 2(2, 1) + (3), & \nu_1 < \frac{1}{2}, \\ (2, 1) + (3), & \nu_1 = \frac{1}{2}, \\ -(1^3) + 0(2, 1) + (3), & \frac{1}{2} < \nu_1 < 1, \\ (3), & \nu_1 = 1, \\ (1^3) + 0(2, 1) + (3), & \nu_1 > 1. \end{cases}$$

11.4. Finally consider $N = 4$.

Let $S = (4)$. We have $\nu = (0)$ and

$$\Sigma(S, U, \nu) = \begin{cases} (1^4), & U = ([-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}]), \\ (1^4) + (2, 1^2), & U = ([-1, 0, 1], 0), \\ (1^4) + (2, 1^2) + (2^2), & U = ([-\frac{1}{2}, \frac{1}{2}], [-\frac{1}{2}, \frac{1}{2}]), \\ (1^4) + 2(2, 1^2) + (2^2) + (3, 1), & U = ([-\frac{1}{2}, \frac{1}{2}], 0, 0), \\ (1^4) + 3(2, 1^2) + 2(2^2) + 3(3, 1) + (4), & U = (0, 0, 0, 0). \end{cases}$$

Let $S = (2, 2)$, $U = ([-\frac{1}{2}, \frac{1}{2}]) \boxtimes ([-\frac{1}{2}, \frac{1}{2}])$. We have $\nu = (\nu_1, -\nu_1)$ and

$$\mathfrak{M}(S, U, \nu) = \left(\left[-\frac{1}{2} + \nu_1, \frac{1}{2} + \nu_1 \right], \left[-\frac{1}{2} - \nu_1, \frac{1}{2} - \nu_1 \right] \right).$$

Hence $X(S, U, \nu)$ is reducible for $\nu_1 = \frac{1}{2}$ and $\nu_1 = 1$, and both are reducibility walls of height one. For $\nu_1 < \frac{1}{2}$, $L(S, U, \nu)$ is unitary and for $\nu_1 > 1$ we are near infinity. Hence we can calculate both signature characters ((4.3) and (8.4)ff):

$$\Sigma(L(S, U, \nu)) = \begin{cases} (1^4) + (2, 1^2) + (2^2), & \nu_1 < \frac{1}{2}, \\ (1^4) - (2, 1^2) + (2^2), & \nu_1 > 1. \end{cases}$$

For $\frac{1}{2} < \nu_1 < 1$ we can use the conjecture (9.6) to cross one of the reducibility walls. Hence the conjecture implies in this case the following both equalities:

$$\begin{aligned} \Sigma(L(S, U, \nu)) &\stackrel{?}{=} (1^4) - (2, 1^2) + (2^2) - 2\Sigma\left(L\left(\left[-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right]\right)\right), \\ \Sigma(L(S, U, \nu)) &\stackrel{?}{=} (1^4) + (2, 1^2) + (2^2) - 2\Sigma(L([-1, 0, 1], 0)). \end{aligned}$$

The right-hand sides coincide and we get conjecturally for $\nu = (\nu_1, -\nu_1)$ and $\frac{1}{2} < \nu_1 < 1$

$$\Sigma(L(S, U, \nu)) \stackrel{?}{=} -(1^4) - (2, 1^2) + (2^2).$$

Finally we can now again use (9.6) to compute

$$\Sigma(L(S, U, (\nu_1, -\nu_1))) \stackrel{?}{=} \begin{cases} (2^2), & \nu_1 = \frac{1}{2}, \\ -(2, 1^2) + (2^2), & \nu_1 = 1. \end{cases}$$

Let $S = (2, 2)$, $U = (0, 0) \boxtimes (0, 0)$. We have $\mathfrak{M}(S, U, \nu) = (\nu_1, \nu_1, -\nu_1, -\nu_1)$ for $\nu_1 > 0$. The only reducibility wall is at $\nu_1 = \frac{1}{2}$ and it is of height 4. For $\nu_1 < \frac{1}{2}$ we are in the unitary case, for $\nu_1 > \frac{1}{2}$ we are near infinity, and for $\nu_1 = \frac{1}{2}$ we have $L(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) = \zeta(L([- \frac{1}{2}, \frac{1}{2}], [- \frac{1}{2}, \frac{1}{2}]))$. Hence

$$\Sigma(L(S, U, (\nu_1, -\nu_1))) = \begin{cases} (1^4) + 3(2, 1^2) + 2(2^2) + 3(3, 1) + (4), & \nu_1 < \frac{1}{2}, \\ (2^2) + (3, 1) + (4), & \nu_1 = \frac{1}{2}, \\ (1^4) - (2, 1^2) + 2(2^2) - (3, 1) + (4), & \nu_1 > \frac{1}{2}. \end{cases}$$

Let $S = (1, 2, 1)$, $U = (0) \boxtimes [-\frac{1}{2}, \frac{1}{2}] \boxtimes (0)$. We have $\mathfrak{M}(S, U, \nu) = (\nu_1, [-\frac{1}{2}, \frac{1}{2}], -\nu_1)$ for $\nu_1 > 0$. The reducibility walls are at $\nu_1 = \frac{1}{2}$ (height one) and $\nu_1 = \frac{3}{2}$ (height two). As above we get

$$\Sigma(L(S, U, \nu)) \begin{cases} = (1^4) + 2(2, 1^2) + (2^2) + (3, 1), & \nu_1 < \frac{1}{2}, \\ \stackrel{?}{=} (2, 1^2) + 0(2^2) + (3, 1), & \nu_1 = \frac{1}{2}, \\ \stackrel{?}{=} -(1^4) + 0(2, 1^2) - (2^2) + (3, 1), & \frac{1}{2} < \nu_1 < \frac{3}{2}, \\ \stackrel{?}{=} (2^2) - (3, 1), & \nu_1 = \frac{3}{2}, \\ = -(1^4) + 0(2, 1^2) - (2^2) + (3, 1), & \nu_1 > \frac{3}{2}. \end{cases}$$

Let $S = (1, 2, 1)$, $U = (0) \boxtimes (0, 0) \boxtimes (0)$. Here is $\mathfrak{M}(S, U, \nu) = (\nu_1, 0, 0, -\nu_1)$ for $\nu_1 > 0$. The reducibility walls are at $\nu_1 = \frac{1}{2}$ (height one) and $\nu_1 = 1$ (height 4). We get

$$\Sigma(L(S, U, \nu)) \begin{cases} = (1^4) + 3(2, 1^2) + 2(2^2) + 3(3, 1) + (4), & \nu_1 < \frac{1}{2}, \\ \stackrel{?}{=} (2, 1^2) + (2, 2) + 2(3, 1) + (4), & \nu_1 = \frac{1}{2}, \\ \stackrel{?}{=} -(1^4) - (2, 1^2) + 0(2, 2) + (3, 1) + (4), & \frac{1}{2} < \nu_1 < 1, \\ = (3, 1) + (4), & \nu_1 = 1, \\ = -(1^4) - (2, 1^2) + 0(2, 2) + (3, 1) + (4), & \nu_1 > 1. \end{cases}$$

Let $S = (1, 1, 1, 1)$. Here we have $\mathfrak{M}(S, U, \nu) = (\nu_1, \nu_2, -\nu_2, -\nu_1)$ with $\nu_1 > \nu_2$. There are four reducibility walls, namely those given by the conditions $\nu_1 = \frac{1}{2}$ (height one), $\nu_1 + \nu_2 = 1$ (height two), $\nu_2 = \frac{1}{2}$ (height one), and $\nu_1 - \nu_2 = 1$ (height two). We know that the signature character does not change if we cross walls of height two (9.14) and we can use (9.11) to calculate $\Sigma(L(S, U, \nu))$ for those ν such that $X(S, U, \nu)$ is irreducible:

$$\Sigma(L(S, U, \nu)) = \begin{cases} (1^4) + 3(2, 1^2) + 2(2^2) + 3(3, 1) + (4), & \nu_2 < \nu_1 < \frac{1}{2}, \\ -(1^4) - (2, 1^2) + 0(2^2) + (3, 1) + (4), & \nu_2 < \frac{1}{2} < \nu_1, \\ (1^4) - (2, 1^2) + 2(2^2) - (3, 1) + (4), & \frac{1}{2} < \nu_2 < \nu_1. \end{cases}$$

For representations lying on a single reducibility wall we can use (9.11) if this wall is of height one:

$$\Sigma(L(S, U, \nu)) = \begin{cases} (2, 1^2) + (2^2) + 2(3, 1) + (4), & \nu_2 < \nu_1 = \frac{1}{2}, \\ (2^2) + 0(3, 1) + (4), & \frac{1}{2} = \nu_2 < \nu_1 < \frac{3}{2}, \\ (2^2) + 0(3, 1) + (4), & \frac{1}{2} = \nu_2 < \frac{3}{2} < \nu_1. \end{cases}$$

For representations lying on a single reducibility we use the Zelevinsky involution if this wall is of height two:

$$\Sigma(L(S, U, \nu)) \begin{cases} = (2^2) + (3, 1) + (4), & \nu_2 = 1 - \nu_1 < \frac{1}{2}, \\ \stackrel{?}{=} -(2^2) + (3, 1) + (4), & \nu_2 = \nu_1 - 1 < \frac{1}{2}, \\ = (2^2) - (3, 1) + (4), & \nu_2 = \nu_1 - 1 > \frac{1}{2}. \end{cases}$$

Finally there is a unique representation which lies on two reducibility walls, namely

$$L\left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$$

and this representation is unitary. It is the Zelevinsky dual of $L\left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$ and hence we have in this case

$$\Sigma(L(S, U, \nu)) = (4).$$

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References

- [1] D. Barbasch, A. Moy, A unitarity criterion for p -adic groups, *Invent. Math.* 98 (1989) 19–37.
- [2] D. Barbasch, A. Moy, Reduction to real infinitesimal character in affine Hecke algebras, *J. Amer. Math. Soc.* 6 (3) (1993) 611–635.
- [3] D. Barbasch, A. Moy, Unitary spherical spectrum for p -adic classical groups, *Acta Appl. Math.* 44 (1996) 3–37.
- [4] P. Cartier, Representations of p -adic groups: A survey, in: *Proc. Sympos. Pure Math.*, vol. 33(1), Amer. Math. Soc., 1979, pp. 111–155.
- [5] W. Casselmann, Introduction to the theory of admissible representations of p -adic reductive groups, www.math.ubc.ca/people/faculty/cass/research.html.
- [6] W. Casselmann, A new non-unitarity argument for p -adic representations, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 28 (1981) 907–928.
- [7] C. de Concini, C. Procesi, Symmetric functions, conjugacy classes and the flag variety, *Invent. Math.* 64 (1981) 203–219.
- [8] S. Evens, The Langlands classification for graded Hecke algebras, *Proc. Amer. Math. Soc.* 124 (1996) 1285–1290.
- [9] N. Iwahori, H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups, *Inst. Hautes Études Sci. Publ. Math.* 25 (1965) 5–48.
- [10] D. Kazhdan, G. Lusztig, Proof of Deligne–Langlands conjecture for Hecke algebras, *Invent. Math.* 87 (1987) 153–215.
- [11] S. Kudla, The local Langlands correspondence: The non-Archimedean case, in: *Proc. Sympos. Pure Math.*, vol. 55(2), 1994, pp. 365–391.
- [12] H. Knight, A. Zelevinsky, Representations of quivers of type A and the multisegment duality, *Adv. Math.* 117 (1996) 273–293.
- [13] G. Lusztig, Cuspidal local systems and graded Hecke algebras I, *Publ. Math. Inst. Hautes Études Sci.* 67 (1988) 145–202.
- [14] G. Lusztig, Affine Hecke algebras and their graded version, *J. Amer. Math. Soc.* 2 (1989) 599–635.
- [15] G. Lusztig, Classification of unipotent representations of simple p -adic groups, *Int. Math. Res. Not.* 11 (1995) 517–589.
- [16] G. Lusztig, Cuspidal local systems and graded Hecke algebras II, in: B. Allison, G. Cliff (Eds.), *Representations of Groups*, in: *Canad. Math. Soc. Conf. Proc.*, vol. 16, Amer. Math. Soc., 1995, pp. 217–275.
- [17] G. Lusztig, Cuspidal local systems and graded Hecke algebras III, preprint MIT, RT/0108173, 2001.
- [18] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Oxford Univ. Press, 1995.
- [19] C. Moeglin, J.-L. Waldspurger, Sur l’involution de Zelevinski, *J. Reine Angew. Math.* 372 (1986) 136–177.
- [20] M. Tadić, Classification of unitary representations in irreducible representation of general linear group (non-Archimedean case), *Ann. Sci. École Norm. Sup. (4)* 19 (1986) 335–382.
- [21] D.A. Vogan Jr., Unitarizability of certain series of representations, *Ann. of Math.* 120 (1984) 141–187.
- [22] A.V. Zelevinsky, Induced representations of reductive p -adic groups II, on irreducible representations of $GL(n)$, *Ann. Sci. École Norm. Sup. (4)* 13 (1980) 165–210.

