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Inventiones mathematicae

Differential operators on homogeneous spaces. III

Characteristic varieties of Harish Chandra modules and of primitive ideals

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Summary. Let G be a semi-simple complex algebraic group with Lie algebra $\mathfrak{g}$ and flag variety $X = G/B$. For each primitive ideal J with trivial central character in the enveloping algebra $U(\mathfrak{g})$ we define a characteristic variety in the cotangent bundle of X, which projects under the Springer resolution map $T^*X \to \mathfrak{g}$ onto the closure of a nilpotent orbit. We prove that this characteristic variety is the G-saturation of the characteristic variety of a highest weight module with annihilator J. We conjecture that it is irreducible for $G = SL_n$. Our conjecture would provide a geometrical explanation for the classification of primitive ideals in terms of Weyl group representations, as achieved by A. Joseph. The presentation of these ideas here is simultaneously used to some extent as an opportunity to continue our more general systematic discussion of differential operators on a complete homogeneous space, and to study more generally characteristic varieties of Harish-Chandra modules.

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Introduction

We consider linear differential operators on a complete homogeneous G-space X for a complex semisimple Lie group G with Lie algebra $\mathfrak{g}$, and we apply this

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to the study of primitive ideals in the enveloping algebra $U(\mathfrak{g})$. While our first paper [BoB] mainly dealt with *globally* defined differential operators, trying to get as far as possible by means of ordinary ring and representation theory, the purpose of the present paper is to use the power of the sheaf theoretic point of view in this context (including the use of derived categories at some point in §3), to obtain both stronger results as well as deeper and more natural insights in the applications.

As our point of departure, we review the Beilinson-Bernstein theory [BeBe], which enables us to study modules over the sheaf of differential operators on X, by interpretating them as "localizations" of $\mathfrak{g}$-modules with trivial central characters, and vice versa. By considering twisted differential operators instead, the theory covers the case of arbitrary central characters as well; but to simplify the presentation, let us mainly speak about the untwisted case here. For a finitely generated such $\mathfrak{g}$-module M, or rather its localization on the flag variety X, the notion of a characteristic variety $\mathrm{Ch}(M)$ in the cotangent bundle T^*X is defined as usual in the general theory of systems of linear differential equations ($\mathscr{D}$-modules), and turns out to be a useful proper refinement of the notion of an associated variety $V(M)$ in $\mathfrak{g}^*$ defined as usual in the theory of representations of Lie algebras ($\mathfrak{g}$-modules). In fact, the main point of our §1 is to explain in some detail that the former maps onto the latter under the moment map π from T^*X to $\mathfrak{g}^*$, which is also known as Springer's resolution of singularities of the cone of nilpotent elements in $\mathfrak{g}^*$. A more refined result, concerning also multiplicities, is discussed in appendix A.

For a Harish-Chandra module H, that is to say here a $(\mathfrak{g}, K)$-module with respect to any closed connected subgroup K of G which has only finitely many orbits in X, the finiteness of the action of $\mathfrak{k}=\mathrm{Lie}\,K$ implies that $\mathrm{Ch}(H)$ is contained in $\pi^{-1}\mathfrak{k}^{\perp}$, and then the involutiveness of the characteristic variety implies that $\mathrm{Ch}(H)$ is a union of closures of conormal bundles of K-orbits in X. So in order to describe the characteristic variety of a Harish-Chandra module, one has only to specify the set of K-orbits whose conormal bundle actually occurs. However, to compute this set explicitly for a given module seems to pose a very hard problem in general. In the course of our review of the general Beilinson-Bernstein theory in §2, we collect some general implications on $\mathrm{Ch}(H)$.

In chapters 3 and 4, we apply this theory to our two cases of main interest: First to highest weight modules (case $K=B$ is a Borel subgroup), second to Harish-Chandra bimodules (case K is the diagonal subgroup of $G\times G$). In the first case, the K-orbits are the Schubert (or Bruhat) cells, in the second they are the varieties of pairs of Borel subgroups of G in a given attitude. In both cases, the orbits are classified by the elements of the Weyl group. The two cases are actually closely related by a certain equivalence of categories. On the level of representation theory, this relation is given by tensoring the Harish-Chandra bimodules on one side with the Verma module of dominant highest weight, and was essentially independently realized by Bernstein-Gelfand [BeGe], Enright [E] and Joseph [J1]. The significance of this relation is illustrated by the fact that it implies almost immediately Duflo's fundamental theorem about primitive ideals in $U(\mathfrak{g})$, stating that they are all annihilators of simple highest weight modules. Our main point in §3 is to explain this relation on the level of $\mathscr{D}$-modules geometrically as a most natural operation of restriction of sheaves. – Let us mention that in the case $K=B$, the Beilinson-Bernstein theory was given independently and in more detail by Kashiwara and the second author [BK].

The preceding results may be applied to the case $H=U(\mathfrak{g})/J$, where J denotes a primitive ideal. By definition, J is the annihilator of a simple $\mathfrak{g}$-module L; by Duflo's theorem, we take for L a highest weight module. We define the left (l-)characteristic variety of J to be $\mathrm{Ch}(H^l)$, where H^l means: the bimodule H is considered only as a left module. A similar terminology applies to associated varieties, and to right sides as well. However, the choice of sides does not matter, since it turns out that $V(H^l)=V(H^r)$ and even $\mathrm{Ch}(H^l)=\mathrm{CH}(H^r)$ (although the latter would not hold for a general Harish-Chandra bimodule). Our main point in §4 is to prove that the l-associated variety of J is the G-saturation (union of all G-conjugates) of that of L, i.e.

$$V(H^l)=G\,V(L).$$

This settles an old conjecture, which we could only partially prove before, by the methods exposed in the first two papers in this series. This "G-saturation theorem", in combination with recent results of Joseph [J3] and Hotta [H] about $V(L)$, which were independently proven by Ginsburg, cf. also Kashiwara-Tanisaki [KT], provides a complete and unified proof for the irreducibility of $V(H^l)$. A quite different proof was independently given by Joseph [J4]. This "irreducibility theorem" means that the l-associated variety of J is the closure of a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$.

Note that this result was already proved by our previous, more elementary, methods before [BoB] for integral central characters; however, our previous proof involved case by case checkings, and does not readily extend to nonintegral central characters, as opposed to the proof given here.

The purpose of §5 is to prove several refined versions of the above G-saturation theorem, including the analogous statement for characteristic varieties, i.e.

$$\mathrm{Ch}(H^l)=G\,\mathrm{Ch}(L),$$

and also a version for characteristic cycles, that is to say characteristic varieties with a certain multiplicity attached to each irreducible component. Although each of the versions implies the previous one, we give direct proofs on each level of refinement, presuming that different readers might be interested in different levels. Let us mention at this point that the study of characteristic varieties of Harish-Chandra bimodules involves some understanding of the geometry of Steinberg's variety of triples (consisting of a pair of Borel subgroups and a nilpotent element contained in their Lie algebras) studied in [St1] to relate Weyl group elements to nilpotent orbits. In fact, this variety identifies with $\pi_{X\times X}^{-1}\mathfrak{k}$ in the case of Harish-Chandra bimodules. So the characteristic variety of a Harish-Chandra bimodule H is a union of components of Steinberg's triple variety, and $\mathrm{Ch}(H^l)$ is obtained from $\mathrm{Ch}(H)$ by the forgetful map forgetting one of the Borel subgroups of a triple.

In chapter 6, we give further applications to the study of primitive ideals. Here, that is in §6 only, we really require the integrality of the central character. We conjecture that not only the l-associated variety, but even the l-characteristic variety of a primitive ideal J is always irreducible. Then it will necessarily be the closure of an irreducible component of $\pi^{-1}\mathcal{O}$, the preimage of the corresponding nilpotent orbit $\mathcal{O}$ (as above) under Springer's resolution map. Moreover, the set of all such components will be put into bijection with the set ("clan") of primitive ideals corresponding to this same nilpotent orbit $\mathcal{O}$ (and with the same central character). So our irreducibility conjecture would provide a most natural and completely geometric understanding of the classification of primitive ideals, in terms of characteristic varieties. Unfortunately, we can not quite prove this, but only support it by some partial results. However, we can prove an only slightly weaker substitute: The characteristic variety is indeed sufficient for the purpose of classification, if we take also multiplicities into account, that is to say if we consider characteristic cycles instead. In fact, it turns out that Joseph's Goldie-rank polynomial of J [J2] has a geometric interpretation in terms of the characteristic cycle of J. In conclusion, this approach provides in any case an adequate framework for a direct geometric understanding of Joseph's classification of primitive ideals [J2].

As a final remark, let us point out that our present study emphasises the significance of the varieties which are closures of components of the preimage of a nilpotent orbit under Springer's resolution. Alternatively, these are also the varieties which are closures of G-saturations of conormal bundles of Schubert cells in the flag variety. Alternatively, these are also the projections of components of Steinberg's triple variety under the forgetful map mentioned above. Alternatively, these are also the associated fibre-bundles over G/B of components of $\overline{\mathcal{O}\cap \mathfrak{n}}$, where $\mathcal{O}$ is a nilpotent orbit and $\mathfrak{n}$ a maximal nilpotent subalgebra of $\mathfrak{g}$. Since this type of geometry of nilpotent orbits is fundamental for a deeper understanding of our ideas in chapter 6, we develop this geometrical picture – which mainly goes back to Steinberg's and Spaltenstein's work – in appendix B in some detail.

Let us refer to the varieties characterized above in four alternative equivalent ways as the "*orbital cone bundles*" on the flag variety X – in view of the last characterization. Let us mention here that such cone bundles determine certain characteristic classes in the cohomology ring of X, the Segre classes defined by Fulton-MacPherson in a general context, and that Joseph's Goldie rank polynomial of a primitive ideal J turns out to be nothing else but the Segre class of the l-characteristic cycle of J. We shall explain this in more detail in another paper, joint with R. MacPherson (in preparation), based on our present work. For some announcements and open problems in this context, we refer also to [BBM].

Remark. The main results of this paper were reported in a preliminary version for the first time in the enveloping algebra seminar organized by Joseph-Rentschler-Tauvel at Paris in December 1981. The first (pre-)printed statements of our "G-saturation theorem" (4.10) were circulating since spring 1982, e.g. in [J3], 1.5(*). In the final form, including the more general treatment of

Harish-Chandra bimodules here, the results were first presented at the enveloping algebra summer school of the London Mathematical Society organized by Goldie-McConnel at Durham in July 1983 and at conferences on algebraic groups organized for the Taniguchi foundation by Hotta at Katata and Kyoto (Japan) in September 1983. The final version of the present paper was first preprinted in two parts at Bures-sur-Yvette (chapter 1–5 and appendix A, IHES-preprint M/84/12) and Wuppertal (chapter 6 and appendix B) in spring 1984. There are close relations, and some minor overlaps, of our work and the recent preprints [J3, J4, Gi, KT] by A. Joseph, V. Ginsburg, M. Kashiwara, T. Tanisaki, which we tried to clarify in the text. After circulating our preprints, we received another preprint by Ginsburg, entitled "$\mathfrak{g}$-modules, Springer's representations, and bivariant Chern classes", which has some major overlap with our chapter 4 and 5.8, but does not contain e.g. the stronger results in chapters 5 and 6. We also learnt from C. Moeglin that she has an alternative proof of 5.9.

§ 1. Associated varieties in relation to characteristic varieties

1.1. *Definition of characteristic varieties and cycles*

We maintain our general conventions and notations introduced in part I of this series [BoB], § 1. In particular, we consider a smooth algebraic variety X over an algebraically closed field k of characteristic 0, and we denote by $\mathcal{D}_X$ the sheaf of (algebraic) differential operators on X. Filtering $\mathcal{D}_X$ by the $\mathcal{O}_X$-submodules $\mathcal{D}_X(m)$, the subsheaf of operators of order $\leqq m$, we identify the associated graded sheaf

$$\operatorname{gr}\mathcal{D}_X = \bigoplus_{m\geqq 0} \mathcal{D}_X(m)/\mathcal{D}_X(m-1)$$

with the structure sheaf $\mathcal{O}_{T^*X}$ on the cotangent bundle of X (or more precisely its direct image under the bundle map onto X). By a *coherent $\mathcal{D}_X$-module*, we mean a sheaf of left modules over $\mathcal{D}_X$ which is coherent, i.e. such that the modules are "locally of finite presentation". Filtering a $\mathcal{D}_X$-module $\mathcal{M}$ by $\mathcal{O}_X$-submodules $\mathcal{M}_j$ ($j\in\mathbb{Z}$), we consider the associated graded $\operatorname{gr}\mathcal{D}_X$-module

$$\operatorname{gr}\mathcal{M} = \bigoplus_j \operatorname{gr}_j\mathcal{M},$$

where

$$\operatorname{gr}_j\mathcal{M} := \mathcal{M}_j/\mathcal{M}_{j-1},$$

as an $\mathcal{O}_{T^*X}$-module. If this is a coherent $\mathcal{O}_{T^*X}$-module, then $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. Conversely, if $\mathcal{M}$ is coherent over $\mathcal{D}_X$, then we can filter it by coherent $\mathcal{O}_X$-submodules, such that $\operatorname{gr}\mathcal{M}$ becomes coherent over $\mathcal{O}_{T^*X}$. In this case, the filtration is called *good*, and the support of $\operatorname{gr}\mathcal{M}$ as a module on T^*X is called the *characteristic variety* of $\mathcal{M}$. This is a closed algebraic subvariety of the cotangent bundle T^*X, which is denoted by

$$\operatorname{Ch}(\mathcal{M}) = \operatorname{Supp}(\operatorname{gr}\mathcal{M}).$$

Although the sheaf $\operatorname{gr}\mathcal{M}$ may depend a lot on the choice of a good filtration, the variety $\mathrm{Ch}(\mathcal{M})$ does not, as is well known. Hence the characteristic variety of a coherent $\mathcal{D}_X$-module is a well defined notion.

We may slightly refine this definition to obtain even an algebraic cycle on T^*X as follows. Given any coherent $\mathcal{O}_{T^*X}$-module $\mathcal{F}$ and any component C of its support, the rank of the module $\mathcal{F}$ at a generic point of C is a well defined positive integer $m_C(\mathcal{F})$, called the "*multiplicity*" of C in the support of $\mathcal{F}$. Instead of the variety $\mathrm{Supp}(\mathcal{F})$ we may then consider the algebraic cycle

$$\mathcal{S}upp(\mathcal{F})=\sum_C m_C(\mathcal{F})[C],$$

i.e. the formal linear combination of the components C of $\mathrm{Supp}(\mathcal{F})$ with coefficients $m_C(\mathcal{F})$. – Now we define the *characteristic cycle* of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ by

$$\mathcal{C}h(\mathcal{M}):=\mathcal{S}upp(\operatorname{gr}\mathcal{M}).$$

Here "gr" refers to a good filtration of $\mathcal{M}$. But again it turns out that the characteristic cycle is a well defined notion, independent of the choice of a good filtration, as is well known.

1.2. *Associated varieties of* $\mathfrak{g}$*-modules*

Let G be a connected linear algebraic group over k with Lie algebra $\mathfrak{g}$. Let M be a finitely generated module over the enveloping algebra $U(\mathfrak{g})$. Recall [BoB], 4.1 that the *associated variety* (or *Bernstein variety*) of M is defined by

$$V(M)=V_{\mathfrak{g}}(M)=\mathrm{Supp}(\operatorname{gr}M)$$

as the support in $\mathfrak{g}^*$ of the associated graded module $\operatorname{gr}M$ over the symmetric algebra $S(\mathfrak{g})$, with respect to a good filtration on M.

We may refine this notion of associated variety to that of an "associated cycle" in exactly the same way as we refined in 1.1 the notion of characteristic variety to that of a characteristic cycle. To be slightly more precise, we define the *associated cycle* of M on $\mathfrak{g}^*$ by

$$\mathcal{V}(M)=\mathcal{V}_{\mathfrak{g}}(M)=\mathcal{S}upp(\operatorname{gr}M)$$

where $\operatorname{gr}M$ is the same module as above, but where its support is now considered as an algebraic cycle on $\mathfrak{g}^*$ in the way explained in 1.1. Again it turns out that not only $V(M)$, but even $\mathcal{V}(M)$, is independent of the choice of a good filtration.

1.3. *Associated varieties of* $D(X)$*-modules*

For a finitely generated (left)module M over the ring $D(X)=\Gamma(X,\mathcal{D}_X)$ of differential operators globally defined on X we may define its *associated variety* resp. *cycle* in a completely analogous way by

$$V_{D(X)}(M)=\mathrm{Supp}(\operatorname{gr}M)$$

resp.

$$\mathcal{V}_{D(X)}(M)=\mathcal{S}upp(\operatorname{gr}M).$$

Im more detail, these are the supports in the affine variety $\mathrm{Maxspec}(\operatorname{gr}D(X))$ of the associated graded module $\operatorname{gr}M$ over the associated graded ring $\operatorname{gr}D(X)$ (notation [BoB], 1.1) with respect to some good filtration on M. Again by [Be1], this is well defined, i.e. is independent of the choice of a good filtration on M.

Recall [BoB], 1.1 that we consider principal symbols of differential operators as functions on the cotangent bundle, and hence identify $\operatorname{gr}D(X)$ as a graded subring of the ring $R(T^*X)=\Gamma(T^*X,\mathcal{O}_{T^*X})$ of regular functions globally defined on T^*X. Recall further that for any algebraic variety Y, the maximal spectrum of $R(Y)$ is an affine algebraic variety Y^{aff}, called the *affinization* of Y, and that each morphism of Y into any affine variety factors through the canonical morphism $Y\to Y^{\mathrm{aff}}$. Using this terminology, we have the following commutative diagram of canonical maps α,β,γ:

$$\begin{tikzcd} T^*X \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & (T^*X)^{\mathrm{aff}} \arrow[d, "\beta"] \\ & \mathrm{Maxspec}(\operatorname{gr}D(X)). \end{tikzcd}$$

For instance, let X be a *complete homogeneous space*, which will be the case of our main interest. Then α is the "collapsing morphism" of the cotangent bundle, contracting the zero section to a point, and since β turns out to be an isomorphism in this case by [BoB], lemma 1.4, γ is essentially the same as α. It follows in particular, that γ is *surjective*, and its fibres are connected projective varieties. In conclusion, we may identify in this situation the associated variety of a $D(X)$-module with a subvariety of the collapsed cotangent bundle $(T^*X)^{\mathrm{aff}}$.

1.4. *Factorization of the moment map*

We shall always assume that X is a G-variety, i.e. that our group G (1.2) acts (algebraically) on X. This gives rise to an action of the Lie algebra $\mathfrak{g}$ by vector fields on X, which extends to a representation of $U(\mathfrak{g})$ by global differential operators on X, denoted $\psi=\psi_X\colon U(\mathfrak{g})\to D(X)$ (notation [BoB], 3.1), and hence allows us to consider each $D(X)$-module as a $\mathfrak{g}$-module. Then we have defined two associated varieties of M: $V(M)=V_{\mathfrak{g}}(M)$ in 1.2, and $V_{D(X)}(M)$ in 1.3. The question arises, how the two relate to each other.

Recall that ψ is compatible with the natural filtrations, and that the associated graded ring homomorphism $\phi=\phi_X=\operatorname{gr}\psi$ from $S(\mathfrak{g})$ to $\operatorname{gr}D(X)\subset R(T^*X)$ gives rise to the so called *moment map* $\pi=\pi_X$ from T^*X to

$\mathfrak{g}^*$, cf. [BoB], 3.2 and 2.3. Note that we obtain a commutative diagram of canonical maps as follows:

$$\begin{tikzcd} T^*X \arrow[r, "\alpha"] \arrow[d, "\pi"'] \arrow[dr, "\gamma"] & (T^*X)^{\mathrm{aff}} \arrow[d, "\beta"] \\ \mathfrak{g}^* \hookleftarrow_{i} \operatorname{Im}\pi & \operatorname{Maxspec}(\operatorname{gr} D(X)). \arrow[l, "\delta"'] \end{tikzcd}$$

Now suppose that we know for some reason that the map δ is finite, i.e. that $\operatorname{gr} D(X)$ is of finite type as an $S(\mathfrak{g})$-module. Then any good filtration of M as a $D(X)$-module is also good as a filtration of M as a $\mathfrak{g}$-module: In fact, $\operatorname{gr} M$ of finite type as $\operatorname{gr} D(X)$-module implies by the preceding assumption that $\operatorname{gr} M$ is of finite type as $S(\mathfrak{g})$-module.

We conclude first of all that M is of finite type over $U(\mathfrak{g})$, and furthermore, that we may use the same filtration of M to compute both varieties $V_{\mathfrak{g}}(M)$ and $V_{D(X)}(M)$, as the support $\operatorname{gr} M$ as $S(\mathfrak{g})$-module resp. as $\operatorname{gr} D(X)$-module. Then clearly δ maps the latter support onto the former one. Let us summarize these observations:

Lemma. *Suppose that in the above factorization $\pi = \delta\gamma$ of the moment map, δ is finite. Then:*

a) *Each module M of finite type over $D(X)$ is of finite type over $U(\mathfrak{g})$.*

b) *Each filtration of (such) M which is good relative $D(X)$ is good relative $U(\mathfrak{g})$.*

c) *δ maps $V_{D(X)}(M)$ onto $V_{\mathfrak{g}}(M)$.*

1.5. *Identification of two notions of associated cycles*

Proposition (notation 1.4). *Let X be a complete homogeneous G-space. Then the moment map π is proper, and its factorization $\pi = \delta\gamma$ coincides with the Stein factorization* ([Ha], III.11.5). *In particular, β is an isomorphism, γ has connected fibres, i is closed, and δ is finite.*

Proof. This is merely a reformulation of [BoB], Proposition 2.5, combined with [BoB], Lemma 1.4. Q.e.d.

In important special cases, although not in general, the finite map δ is even an isomorphism, and may be used to identify the two notions of associated cycles $\mathscr{V}_{D(X)}(M)$ resp. $\mathscr{V}_{\mathfrak{g}}(M)$. Let us clarify this point in the following corollary.

Corollary. *Of the following five statements, the last three are equivalent, and are implied by each of the first two.*

(1) *X is the variety of Borel subgroups of G.*

(2) *G is the group $SL(n, k)$.*

(3) *π is birational with normal image.*

(4) *δ is an isomorphism.*

(5) *For all $D(X)$-modules M of finite type, δ maps $V_{D(X)}(M)$ isomorphically onto $V_{\mathfrak{g}}(M)$, and even identifies $\mathscr{V}_{D(X)}(M)$ with $\mathscr{V}_{\mathfrak{g}}(M)$.*

Proof. Let us first note that both (1) and (2) are sufficient for (3): The normality of the image of π is a theorem of Kostant [Ko] in case (1), resp. of Kraft-Procesi [KP] in case (2), while the birationality of π comes from a well known fact about connectedness of centralizers of nilpotent elements in both cases. – Next recall that we have discussed condition (3) already in detail in [BoB], theorem 5.6, characterizing this situation by five equivalent conditions (i)–(v). It is only left to translate the theorem into our present new terminology. For instance, condition (iii) of loc. cit. means precisely that $\delta\beta$ is an isomorphism, or also – since β is an isomorphism – that δ is an isomorphism. Therefore, our present claim (3)$\Leftrightarrow$(4) is a reformulation of (iii)$\Leftrightarrow$(iv) in loc. cit. Similarly, (5) is a disguised form of (i) in loc. cit.: The latter says that the operator filtration and the natural filtration coincide on $\psi(U(\mathfrak{g}))$, which is equal to $D(X)$ by [BoB], Theorem 3.8. But then the definitions of $\mathscr{V}_{\mathfrak{g}}(M)$ resp. $\mathscr{V}_{D(X)}(M)$ refer to (essentially) the same filtered ring, and so must obviously define (essentially) the same object. In conclusion, (4)$\Leftrightarrow$(5) may be viewed as a rephrasement of (i)$\Leftrightarrow$(iii) in [BoB], Theorem 5.6. – But let us point out that it is also directly clear from our discussion in 1.4 above that (4) is sufficient for (5). The converse is trivial: Taking $M = D(X)$ in (5) shows δ must be an isomorphism. Q.e.d.

Convention. Henceforth, we shall freely drop the distinguishing subscripts $\mathfrak{g}$ resp. $D(X)$ in our notation for associated varieties and cycles for $\mathfrak{g}$-resp. $D(X)$-modules: We shall write just $V(M)$ and $\mathscr{V}(M)$, whenever this is unambiguous by the above corollary.

1.6. *The global section functor*

Let us now turn to our main purpose, which is to study the relation of $\mathfrak{g}$-modules to (sheaves of) $\mathscr{D}_X$-modules. Having clarified the relation from $\mathfrak{g}$-modules to $D(X)$-modules in the preceding sections, we may now concentrate on the relation from $\mathscr{D}_X$-modules to $D(X)$-modules.

Starting from a $\mathscr{D}_X$-module $\mathscr{M}$, we consider its global sections $M = \Gamma(X, \mathscr{M})$ as a module over $D(X) = \Gamma(X, \mathscr{D}_X)$. Given a filtration of $\mathscr{M}$ by $\mathscr{O}_X$-submodules $\mathscr{M}_j$, we obtain a filtration of M by subspaces

$$M_j = \Gamma(X, \mathscr{M}_j), \quad j \in \mathbb{Z},$$

which we refer to as the *induced* filtration on M.

Lemma. *Let $\mathscr{M}$ be a $\mathscr{D}_X$-module and $M = \Gamma(X, \mathscr{M})$. With respect to some filtration on $\mathscr{M}$ and the induced filtration on M, we have:*

a) *There is a canonical graded monomorphism*

$$\operatorname{gr} M = \operatorname{gr} \Gamma(X, \mathscr{M}) \hookrightarrow \Gamma(X, \operatorname{gr} \mathscr{M}) \tag{1}$$

of graded $\operatorname{gr} D(X)$-modules.

b) *If the filtrations are good, then we can conclude*

$$\mathscr{V}_{D(X)}(M) \subset \gamma(\overline{\mathrm{Ch}(\mathscr{M})}). \tag{2}$$

c) *Assume X is a complete homogeneous G-space. Then each good filtration on $\mathcal{M}$ induces a good filtration on M. In particular, $\mathcal{M}$ coherent implies M finitely generated, and* (2) *holds.*

Proof. From the short exact sequence of sheaves

$$0 \to \mathcal{M}_{j-1} \to \mathcal{M}_j \to \operatorname{gr}_j \mathcal{M} \to 0$$

we obtain an exact sequence

$$\begin{array}{ccccccc} 0 \to \Gamma(X, \mathcal{M}_{j-1}) & \to & \Gamma(X, \mathcal{M}_j) & \to & \Gamma(X, \operatorname{gr}_j \mathcal{M}) & \to & H^1(X, \mathcal{M}_{j-1}) \\ \| & & \| & & & & \\ M_{j-1} & & M_j & & & & \end{array}$$

and hence a canonical embedding

$$\operatorname{gr}_j M = M_j / M_{j-1} \hookrightarrow \Gamma(X, \operatorname{gr}_j \mathcal{M}) \tag{3}$$

in all degrees j. This gives the graded monomorphism and proves a). From (1), we obtain an inclusion of supports in $Y := \operatorname{Maxspec}(\operatorname{gr} D(X))$:

$$\operatorname{Supp}(\operatorname{gr} M) \subset \operatorname{Supp}(\Gamma(X, \operatorname{gr} \mathcal{M})). \tag{4}$$

Now we consider $\operatorname{gr} \mathcal{M}$ as a sheaf on $T^* X$; then $\Gamma(X, \operatorname{gr} \mathcal{M})$, considered as an $\mathcal{O}_Y$-module, is the direct image under γ: $T^* X \to Y$ of the $\mathcal{O}_{T^* X}$-module $\operatorname{gr} \mathcal{M}$:

$$\Gamma(X, \operatorname{gr} \mathcal{M}) = \Gamma(T^* X, \operatorname{gr} \mathcal{M}) = \Gamma(Y, \gamma_* \operatorname{gr} \mathcal{M}).$$

So the right hand side of (4) is

$$\operatorname{Supp}(\gamma_* \operatorname{gr} \mathcal{M}) = \overline{\gamma(\operatorname{Supp}(\operatorname{gr} \mathcal{M}))} = \overline{\gamma \operatorname{Ch}(\mathcal{M})},$$

provided that the filtration is good. Also, the left hand side of (4) is $V_{D(X)}(M)$, if the filtration on M is good. Hence (4) implies (2), and b) is proved.

Now assume X is a complete homogeneous space. Then γ is proper by 1.5. So if a filtration on $\mathcal{M}$ is good, i.e. if $\operatorname{gr} \mathcal{M}$ is a coherent $\mathcal{O}_{T^* X}$-module, then the direct image $\gamma_*(\operatorname{gr} \mathcal{M})$ is again coherent by Grauert's theorem (or rather Grothendieck's algebraic version of it, see [EGA] III, Théorème 3.2.1 or [Ha] III, Theorem 8.8(b)). In other words, $\Gamma(X, \operatorname{gr} \mathcal{M})$ is a finitely generated module over $\operatorname{gr} D(X)$. Since $\operatorname{gr} M$ is a submodule by (1), it is also finitely generated. Hence the induced filtration on M is also good. This proves the first statement in c), and the rest is then clear. Q.e.d.

Remark. Let us comment on some possibilities of refining the above arguments to get stronger results. Note that it would be particularly nice to have

$$H^1(X, \mathcal{M}_{j-1}) = 0 \tag{5}$$

in the first part of the above proof, because then the inclusion (3) becomes an equality

$$\operatorname{gr}_j M = \Gamma(X, \operatorname{gr}_j \mathcal{M}). \tag{6}$$

If we knew this in all degrees j, then (1) is an equality, and the same proof as above shows that also (4) and hence (2) now hold with equality. In fact, equality in (1) gives even equality of supports *with multiplicities* in (4), so that we can conclude more precisely a description for the associated cycle:

$$\mathscr{V}_{D(X)}(M) = \mathscr{S}\!upp(\gamma_* \operatorname{gr} \mathcal{M}). \tag{7}$$

Such is the case when $M = D(X)$, see [BoB], § 5.

It suffices for this conclusion to know (5) only for j sufficiently large: Then (1) becomes an equality in almost all (all but finitely many) degrees, and the supports in (4) – with multiplicities – cannot differ by any component of positive dimension, and so (7) follows, if we exclude the trivial case were the associated variety of M reduces to a point (i.e. M is finite dimensional). – Let us summarize these additional observations as a complement to the lemma: *complement to the lemma*

d) *If* (5) *holds for almost all degrees j, then*

$$V_{D(X)}(M) = \gamma \operatorname{Ch}(\mathcal{M}). \tag{8}$$

And, unless this variety reduces to a single point, *even the refinement* (7) *holds.*

In conclusion, we may refine our preceding arguments concerning the associated variety in such a way to get also some control over the multiplicities, i.e. over the associated cycle. Unfortunately, we cannot prove the assumption (5) for almost all degrees, in the case of a complete homogeneous space X (probably it is false). However, we can prove slightly modified versions, which suffice to work as a kind of substitute. – But since it is much easier to prove just (8) directly, let us do this now first, and let us return to the refined discussion concerning multiplicities later in 5.9 resp. appendix A.

1.7. *Significance of homogeneity*

For an arbitrary group action of G on X, the tangent map $\mathfrak{g} \to T(X)_x$ at a point $x \in X$ maps the Lie algebra $\mathfrak{g}$ onto the tangent space of the orbit through x, and so $\mathfrak{g}$ surjects onto $T(X)_x$ if and only if the orbit is dense in X (cf. [BoB], 2.1). Consequently, the Lie algebra surjects onto *all* tangent spaces of X, if and only if X is a connected *homogeneous* G-space. Let $\mathscr{T}_X = \Gamma(T(X))$ (notation [BoB], 1.2) denote the *sheaf* of regular vector fields on X. The preceding observation may be rephrased as follows.

Lemma 1. *The G-space X is homogeneous if and only if $\mathscr{T}_X$, as a left $\mathcal{O}_X$-module, is generated by the image of $\mathfrak{g}$.*

We conclude from this simple fact that homogeneous spaces enjoy the following very nice special property:

Lemma 2. *Let X be homogeneous. Then its sheaf of differential operators of degree $\leqq j$, as a left $\mathcal{O}_X$-module, is generated by global sections.*

The claim is, in the notation introduced in [BoB], § 1, that

$$\mathscr{D}_X(j)=\mathscr{O}_X D_j(X) \tag{1}$$

for all $j \geqq 0$, and consequently

$$\mathscr{D}_X=\mathscr{O}_X D(X). \tag{2}$$

Here we interpret a space of global differential operators also as a constant subsheaf of $\mathscr{D}_X$, without a particular notational distinction. In the same notation, Lemma 1 says that

$$\mathscr{T}_X=\mathscr{O}_X \psi(\mathfrak{g}), \tag{3}$$

where ψ denotes the representation by global differential operators on X (notation [BoB], 3.1). Since $\psi(\mathfrak{g}) \subset D_1(X)$, this implies

$$\mathscr{D}_X(1)=\mathscr{O}_X+\mathscr{T}_X=\mathscr{O}_X+\mathscr{O}_X \psi(\mathfrak{g}) \subset \mathscr{O}_X D_1(X).$$

Since the commutator of a vector field and a function is a function, we have

$$\begin{aligned} D_1(X)\mathscr{O}_X &\subset \mathscr{O}_X D_1(X)+[D_1(X),\mathscr{O}_X] \\ &\subset \mathscr{O}_X D_1(X)+\mathscr{O}_X \quad \subset \mathscr{O}_X D_1(X). \end{aligned}$$

By this relation the inclusion $\mathscr{D}_X(j) \subset \mathscr{O}_X D_j(X)$ for any $j \geqq 0$ implies

$$\begin{aligned} \mathscr{D}_X(j+1)=\mathscr{D}_X(1)\,\mathscr{D}_X(j) &\subset \mathscr{O}_X D_1(X)\,\mathscr{O}_X D_j(X) \\ &\subset \mathscr{O}_X D_1(X) D_j(X) \subset \mathscr{O}_X D_{j+1}(X), \end{aligned}$$

so the non trivial inclusion in (1) follows by induction on j, which proves our claim.

Remark. Of course we have (symmetrically) $\mathscr{D}_X(j)=D_j(X)\mathscr{O}_X$ and $\mathscr{D}_X=D(X)\mathscr{O}_X$ as well.

1.8. The localization functor

Starting from a $D(X)$-module M, the sheaf $\mathscr{M}=\mathscr{D}_X \otimes_{D(X)} M$ on X, as a left module over $\mathscr{D}_X$, is called the *localization* (of M on X), following [BeBe]. In this context, we frequently identify a subspace N of M also with a constant subsheaf $1 \otimes N$ of $\mathscr{M}$. Assuming X homogeneous, we notice from 1.7(2) that

$$\mathscr{M}=\mathscr{O}_X M.$$

Moreover, given a filtration of M by subspaces M_j, it follows from 1.7(1) that

$$\mathscr{M}_j=\mathscr{O}_X M_j \quad (j \in \mathbb{Z})$$

defines a filtration of the localization $\mathscr{M}$ by $\mathscr{O}_X$-modules $\mathscr{M}_j$, which we refer to as the *induced* filtration.

Proposition. *For a homogeneous space X, each good filtration of a $D(X)$-module M induces a good filtration on its localization $\mathscr{M}$. In particular, M finitely generated implies $\mathscr{M}$ coherent. If such is the case, then*

$$\gamma(\mathrm{Ch}(\mathscr{M})) \subset V_{D(X)}(M),$$

i.e. the characteristic variety of the localization maps into the associated variety of the module.

Proof. By the definition of a filtration, we have

$$D_1(X) M_j \subset M_{j+1} \tag{1}$$

resp.

$$\mathscr{D}_X(1)\mathscr{M}_j \subset \mathscr{M}_{j+1}, \tag{2}$$

for all degrees j. The filtration on M resp. $\mathscr{M}$ will be *good* if and only if in (1) resp. (2) equality holds for almost all degrees j. But the equality $D_1(X)M_j = M_{j+1}$ in (1) implies

$$\begin{aligned} \mathscr{D}_X(1)\mathscr{M}_j &= \mathscr{O}_X D_1(X)\mathscr{O}_X M_j \\ &= \mathscr{O}_X D_1(X) M_j = \mathscr{O}_X M_{j+1} = \mathscr{M}_{j+1}, \end{aligned}$$

using 1.7, that is equality in (2). Hence a good filtration on M induces, in fact, a good one on $\mathscr{M}$. If M is finitely generated, then it has a good filtration, and so does $\mathscr{M}$ by what has just been proved, so $\mathscr{M}$ is coherent.

To prove the last part of our proposition, let M_j and $\mathscr{M}_j$ be the good filtrations of M and $\mathscr{M}$ as considered before; then it suffices to prove the

Lemma. *If the principal symbol of an operator $P \in D(X)$ annihilates* gr M, *then it annihilates* gr $\mathscr{M}$.

In fact, if P has degree m, then the fact that its principal symbol annihilates gr M means that

$$P M_j \subset M_{j+m-1}, \quad \text{for all } j \in \mathbb{Z}. \tag{3}$$

Since the commutator of P with a local section $f \in \mathscr{O}_X$ is an operator of smaller degree, i.e. $[P,f] \in \mathscr{D}_X(m-1)$, we conclude from (3) that for all $j \in \mathbb{Z}$

$$\begin{aligned} P\mathscr{M}_j = P\mathscr{O}_X M_j &\subset \mathscr{O}_X P M_j + \mathscr{D}_X(m-1) M_j \\ &\subset \mathscr{O}_X M_{j+m-1} = \mathscr{M}_{j+m-1}, \end{aligned}$$

which means exactly that $P(\mathrm{gr}\,\mathscr{M})=0$. This proves the lemma.

To complete the proof of the proposition, note that the lemma states

$$\mathrm{Ann}_{\mathrm{gr}\,D(X)}(\mathrm{gr}\,M) \subset \mathrm{Ann}_{\mathscr{D}_X}(\mathrm{gr}\,\mathscr{M}),$$

and so

$$\mathrm{Ann}_{\mathrm{gr}\,D(X)}(\mathrm{gr}\,M) \subset \mathrm{Ann}_{\mathrm{gr}\,D(X)}(\mathrm{gr}\,\mathscr{M}) = \mathrm{Ann}(\gamma_*(\mathrm{gr}\,\mathscr{M}))$$

as well. This inclusion of annihilator ideals implies a reverse inclusion for the supports of the modules:

$$\begin{aligned}\operatorname{Supp}(\operatorname{gr} M) &\supset \operatorname{Supp}(\gamma_*(\operatorname{gr} \mathcal{M}))\\ &\supset \gamma(\operatorname{Supp}(\operatorname{gr} \mathcal{M})).\end{aligned}$$

Since the filtrations are good, the support of $\operatorname{gr} M$ resp. $\operatorname{gr} \mathcal{M}$ is $V_{D(X)}(M)$ resp. $\operatorname{Ch}(\mathcal{M})$, hence the proposition is proved. Q.e.d.

1.9. The Beilinson-Bernstein equivalence of categories

Theorem. *Let X be a complete homogeneous space. Then:*

a) [BeBe] *The localization resp. global sections functor are quasi-inverse functors establishing an equivalence*

$$\mathcal{M}od(\mathcal{D}_X) \approx \operatorname{Mod}(D(X))$$

of the categories $\operatorname{Mod}(D(X))$ *resp.* $\mathcal{M}od(\mathcal{D}_X)$ *of left $D(X)$ – resp. quasi-coherent left $\mathcal{D}_X$-modules.*

b) [BeBe] *Under this equivalence, a coherent $\mathcal{D}_X$-module $\mathcal{M}$ corresponds to a finitely generated $D(X)$-module M and vice versa.*

c) *For such modules $\mathcal{M}$ and M, we have*

$$V_{D(X)}(M) = \gamma(\operatorname{Ch}(\mathcal{M})).$$

Proof. a) By [BeBe], Théorème principal, the space X is "$\mathcal{D}$-affine"[1], and this implies the equivalence claimed in a) by loc. cit., Proposition 1. Now we can combine 1.6 and 1.8 to conclude first b), and then also

$$\gamma(\operatorname{Ch}(\mathcal{M})) \subset V_{D(X)}(M) \subset \overline{\gamma(\operatorname{Ch}(\mathcal{M}))}.$$

Here equality must hold, since $\gamma(\operatorname{Ch}(\mathcal{M}))$ is closed because γ is proper. This proves c). Q.e.d.

Complementary remarks. Recall that by [BoB], Theorem 3.8 for our complete homogeneous G-space X the operator representation $\psi_X: U(\mathfrak{g}) \to D(X)$ is *surjective*, and so a $D(X)$-module is essentially the same as a $U(\mathfrak{g})$-module annihilated by $I_X = \ker \psi_X$. To be more precise, let $\operatorname{Mod}(\mathfrak{g})$ denote the category of (left) $\mathfrak{g}$-modules, and indicate by a subscript I_X, or just X for simplicity, restriction to the full subcategory of modules annihilated by I_X. Then

$$\operatorname{Mod}(D(X)) \approx \operatorname{Mod}(\mathfrak{g})_X.$$

Composing this with the equivalence a), we obtain the equivalence

$$\mathcal{M}od(\mathcal{D}_X) \approx \operatorname{Mod}(\mathfrak{g})_X. \qquad \text{a}')$$

[1] As remarked by the referee, [BeBe] explicitly states this only for the case $X = G/B$. However, the general case $X = G/P$ can be deduced from this special case by considering the projection map $G/B \to G/P$, which is a fibration by flag varieties P/B, and applying [BeBe] to the pull-back to G/B of a D-module on G/P. The required vanishing of higher cohomology then follows by a degeneration of spectral sequence argument

The composed functors establishing this equivalence are again called *localization* resp. *global section functor*. Similarly, restriction to the full subcategories of finitely generated resp. coherent modules (denoted by superscripts "fg" resp. "coh") gives equivalences

$$\mathcal{M}od(\mathcal{D}_X)^{\mathrm{coh}} \approx \operatorname{Mod}(D(X))^{\mathrm{fg}} \approx \operatorname{Mod}(\mathfrak{g})_X^{\mathrm{fg}}. \qquad \text{b}')$$

Finally, if $\mathcal{M}$ and M are modules corresponding to each other under this latter equivalence, then Lemma 1.4 c) tells us that

$$V(M) = V_{\mathfrak{g}}(M) = \delta(V_{D(X)}(M)).$$

Using this (and $\pi = \delta\gamma$, see 1.4), our theorem c) above translates into the following $\mathfrak{g}$-module version:

$$V(M) = \pi(\mathcal{C}h(\mathcal{M})). \qquad \text{c}')$$

Corollary. *Let M be a finitely generated $\mathfrak{g}$-module with trivial central character. Let $\mathcal{M}$ be its localization on the flag variety $X = G/B$. Then the associated variety of M is the image of the characteristic variety of $\mathcal{M}$ under Springer's resolution map π for the cone of nilpotent elements in $\mathfrak{g}$.*

In fact, for the flag variety X, the kernel I_X of the operator representation ψ_X is the ideal generated by the kernel of the trivial central character [Br], and so $\operatorname{Mod}(\mathfrak{g})_X$ is the category of $\mathfrak{g}$-modules with trivial central characters in this case. On the other hand, the moment map $\pi = \pi_X$ is now Springer's resolution (cf. [BoB], 2.6 and 3.7). So the corollary is just a rephrasement of a special case of c′) above.

§ 2. Characteristic varieties of Harish Chandra modules

2.1. The general setting of $(\mathfrak{g}, K)$*-modules* [BeBe]

From now on, X will always denote the *flag variety* (the variety of all Borel subgroups) of our semisimple group G with Lie algebra $\mathfrak{g}$ (cf. 1.2). Let $K \subset G$ be a closed connected subgroup with Lie algebra $\mathfrak{k}$, and *assume* that K has only finitely many orbits in X. For example, K may be a parabolic subgroup, or also the identity component of the fixed point subgroup of a non-trivial involution of G. We are actually interested later on only in two particular cases of these two examples, which will be discussed in more detail in § 3 resp. § 4. For the sake of clarity, however, we review in the present chapter the general theory of Beilinson-Bernstein [BeBe].

An *algebraic* $(\mathfrak{g}, K)$*-module* is a $\mathfrak{g}$-module M, equipped with an algebraic action of the group K on M, which is compatible with the $\mathfrak{g}$-module structure. In more detail, we may state the following formal definition.

Definition. A k-vector-space M, equipped with two k-linear maps

$$m: U(\mathfrak{g}) \otimes_k M \to M \quad \text{and} \quad \mu: M \to R(K) \otimes_k M$$

satisfying (a)–(d) below is called a $(\mathfrak{g}, K)$-*module*: (Here $R(K)$ is the ring of regular functions on K.)

(a) m defines the structure of a (unitary) left $U(\mathfrak{g})$-module on M.

(b) μ defines the structure of a (co-unitary) $R(K)$-comodule on M (with respect to the co-algebra structure on $R(K)$, cf. [DG]).

(c) The restriction to $\mathfrak{k}$ of the $\mathfrak{g}$-action on M given by m coincides with the differential at 1 of the K-action on M given by μ.

(d) m is a homomorphism of $R(K)$-comodules with respect to the $R(K)$-comodule structure on $U(\mathfrak{g})$ given by the adjoint action of K.

2.2. *K-equivariant $\mathscr{D}_X$-modules* [BeBe]

An (algebraic) $(\mathscr{D}_X, K)$-module is a $\mathscr{D}_X$-module $\mathscr{M}$, equipped with an algebraic action of the group K, which is compatible with the $\mathscr{D}_X$-module structure. In more detail, we may state the following formal definition. Note that $\mathfrak{k}$ acts by vectorfields on X and hence can be considered as a subsheaf in $\mathscr{D}_X$.

Definition. A sheaf of k-vector-spaces $\mathscr{M}$ on X, equipped with two morphisms of sheaves

$$m: \mathscr{D}_X \otimes_k \mathscr{M} \to \mathscr{M} \quad \text{and} \quad \mu: \mathscr{M} \to q_* p^* \mathscr{M}$$

satisfying (a)–(d) below is called a $(\mathscr{D}_X, K)$-*module*. Here p resp. q denote the morphisms $K \times X \to X$ given by projection resp. the K-action on X.

(a) m defines the structure of a left $\mathscr{D}_X$-module on $\mathscr{M}$ (which, by definition, is quasi-coherent as a $\mathscr{O}_X$-module).

(b) μ defines the structure of a sheaf of $R(K)$-comodules on $\mathscr{M}$, (since it defines a structure of a sheaf of comodules over $q_* \mathscr{O}_{K \times X} \supset R(K)$).

(c) The restriction to $\mathfrak{k}$ of the $\mathscr{D}_X$-action on $\mathscr{M}$ given by m coincides with the differential at 1 of the K-action on $\mathscr{M}$ given by μ.

(d) m is a homomorphism of K-modules for the natural K-action on $\mathscr{D}_X$.

Remark. Notice that to give μ amounts to the same as to give an isomorphism $q^* \mathscr{M} \to p^* \mathscr{M}$ of $D_{K \times X}$-modules, and so an algebraic $(\mathscr{D}_X, K)$-module may also be considered as a K-equivariant $\mathscr{D}_X$-module.

2.3. *Localization of $(\mathfrak{g}, K)$-modules* [BeBe]

Recall our notations introduced in 1.9 for various module categories. We extend this notation in an obvious manner to categories of $(\mathfrak{g}, K)$-modules and of $(\mathscr{D}_X, K)$-modules. For example, $\mathrm{Mod}(\mathfrak{g}, K)_X^{fg}$ is the category of $(\mathfrak{g}, K)$-modules which are finitely generated as $\mathfrak{g}$-module and have trivial central character (cf. 1.9). And $\mathscr{M}od(\mathscr{D}_X, K)^{coh}$ is the category of $(\mathscr{D}_X, K)$-modules which are coherent as $\mathscr{D}_X$-modules.

Theorem. *(Beilinson-Bernstein)*

a) *The localization resp. global section functor provide quasi-inverse functors which establish equivalences of categories*

$$\mathscr{M}od(\mathscr{D}_X, K) \approx \mathrm{Mod}(\mathfrak{g}, K)_X \quad \textit{and} \quad \mathscr{M}od(\mathscr{D}_X, K)^{coh} \approx \mathrm{Mod}(\mathfrak{g}, K)_X^{fg}.$$

b) *All coherent $(\mathscr{D}_X, K)$-modules are holonomic, with regular singularities, and have finite length.*

Remark. Concerning b), let us recall from the general theory of differential systems [Be2, Bj, KK] that a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is called *holonomic*, if its characteristic variety $\mathrm{Ch}(\mathscr{M})$ has dimension $\leq \dim X$. It is a general fact, following from the involutivity of $\mathrm{Ch}(\mathscr{M})$ [Ga], that the length of such a module is finite. (In fact, it is bounded by the sume of the coefficients of its characteristic cycle $\mathscr{C}h(\mathscr{M})$; cf. also the discussion in 2.6 below). – For the definition of "regular singularities" let us refer to the general literature on differential system, see e.g. [KK, BK].

2.4. *Conormal bundles of K-orbits*

Recall our basic assumption (2.1) that the flag variety X decomposes into a *finite* number of orbits under the action of K. Let us denote these K-orbits by X_w, $w \in \mathbf{W}$, where $\mathbf{W}$ is a finite index set (a Weyl group in the applications below). Each K-orbit X_w is a locally closed smooth subvariety of X, and as such has a *conormal bundle* $T^*_{X_w}X$. This may be defined as the largest subvariety of the *cotangent bundle* T^*X which maps into X_w under the bundle map $p: T^*X \to X$, and into $\mathfrak{k}^\perp$ under the moment map $\pi = \pi_X: T^*X \to \mathfrak{g}^*$ (1.4):

$$T^*_{X_w}X = p^{-1} X_w \cap \pi^{-1} \mathfrak{k}^\perp. \tag{1}$$

To explain this in more detail, let us recall from [BoB], 2.3 that T^*X may be identified with the closed subvariety of pairs (ξ, x) in $\mathfrak{g}^* \times X$ such that ξ vanishes on the isotropy algebra $\mathfrak{g}_x$. (Then p resp. π become just the forgetful maps forgetting ξ resp. x.) In this notation, a covector $(\xi, x) \in (T^*X)_x$ of X at a point $x \in X_w$ is called conormal to X_w, if it vanishes on the tangent space $(TX_w)_x \cong \mathfrak{k}/\mathfrak{k}_x$, or equivalently, if $\xi \in \mathfrak{k}^\perp$. Hence we find the descriptions

$$p^{-1} X_w = \bigcup_{x \in X_w} \mathfrak{g}_x^\perp \times \{x\}, \tag{2}$$

and

$$p^{-1} X_w \cap \pi^{-1} \mathfrak{k}^\perp = \bigcup_{x \in X_w} (\mathfrak{g}_x^\perp \cap \mathfrak{k}^\perp) \times \{x\} =: T^*_{X_w} X \tag{3}$$

for the variety of all such conormal vectors, i.e. for the conormal bundle of X_w.

Lemma. *The subvariety $\pi^{-1}\mathfrak{k}^\perp$ has the following properties:*

a) *It is the disjoint union of the conormal bundles of all K-orbits,*

$$\pi^{-1} \mathfrak{k}^\perp = \bigcup_{w \in \mathbf{W}} T^*_{X_w} X.$$

b) *It is Lagrangean. In particular, it is equidimensional of dimension* $\dim X$.

c) *Its irreducible components are $\overline{T^*_{X_w}X}$ for $w \in \mathbf{W}$.*

d) *The image of an individual irreducible component under the moment map π is given by*

$$\pi(\overline{T^*_{X_w}X}) = \overline{K(\mathfrak{g}_{x(w)}^\perp \cap \mathfrak{k}^\perp)},$$

i.e. *the closure of the K-saturation of the conormal space* $\mathfrak{g}^{\perp}_{x(w)} \cap \mathfrak{k}^{\perp}$ *of* X_w *in* X *at a point* $x(w) \in X_w$.

Proof. a) is immediate from equation (3) above:

$$\pi^{-1}\mathfrak{k}^{\perp} = \bigcup_{w\in W} (p^{-1}X_w \cap \pi^{-1}\mathfrak{k}^{\perp}) = \bigcup_{w\in W} T^*_{X_w}X.$$

Now each conormal bundle $T^*_{X_w}X$ is clearly Lagrangean (cf. [Bj] for definition), and since they are *finite* in number, their union is too. Moreover, they all have dimension $\dim T^*_{X_w}X = \dim X$. Hence their closure are the irreducible components of $\pi^{-1}\mathfrak{k}^{\perp}$, which is thus equidimensional of dimension $\dim X$. Finally, d) is again immediate from (3):

$$\pi(T^*_{X_w}X) = \bigcup_{x\in X_w} \mathfrak{g}^{\perp}_x \cap \mathfrak{k}^{\perp} = \bigcup_{g\in K} \mathfrak{g}^{\perp}_{gx(w)} \cap \mathfrak{k}^{\perp}$$
$$= \bigcup_{g\in K} g(\mathfrak{g}^{\perp}_{x(w)} \cap \mathfrak{k}^{\perp}) = K(\mathfrak{g}^{\perp}_{x(w)} \cap \mathfrak{k}^{\perp}). \quad \text{Q.e.d.}$$

2.5. Characteristic varieties of $(\mathcal{D}_X, K)$*-modules*

Proposition. *Let* $\mathcal{M}$ *be a coherent* $(\mathcal{D}_X, K)$*-module. Then its characteristic variety is given by a unique subset* $\Sigma(\mathcal{M})$ *of* $\mathcal{W}$ *as*

$$\mathrm{Ch}(\mathcal{M}) = \bigcup_{w\in\Sigma(\mathcal{M})} \overline{T^*_{X_w}X}.$$

Proof. Let $M = \Gamma(X, \mathcal{M})$ be the corresponding $(\mathfrak{g}, \mathfrak{k})$-module. Then the associated variety $V(M)$ of M as a $\mathfrak{g}$-module must be contained in $\mathfrak{k}^{\perp}$: This is an easy consequence of the fact that the action of $\mathfrak{k}$ on M is locally finite. Now Theorem 1.9 c′) implies that

$$\mathrm{Ch}(M) = \mathrm{Ch}(\mathcal{M}) \subset \pi^{-1}\mathfrak{k}^{\perp} = \bigcup_{w\in W} \overline{T^*_{X_w}X}, \tag{$*$}$$

where the last equality (by (2.4) is the decomposition of the variety $\pi^{-1}\mathfrak{k}^{\perp}$ into its irreducible components, which have all the same dimension $\dim X$. But on the other hand, if $\mathcal{M} \neq 0$, then all irreducible components of $\mathrm{Ch}(\mathcal{M})$ must have *at least* dimension $\dim X$, as a consequence of the *involutivity* of characteristic varieties [Ga]. Hence $\mathrm{Ch}(\mathcal{M})$ must be a union of *some* of the irreducible components of $\pi^{-1}\mathfrak{k}^{\perp}$, and so the proposition follows from the inclusion $(*)$. In the case $\mathcal{M} = 0$, the proposition is trivially true with $\Sigma(\mathcal{M}) = \emptyset$. Q.e.d.

Remark. Note that the proposition *implies* the *holonomicity* of coherent $(\mathcal{D}_X, K)$-modules (Theorem 2.3 b)), and is in fact a much more precise statement.

2.6. Characteristic cycles

Let $\mathcal{M}$ be as in 2.5. For each $w \in \mathcal{W}$, we denote $m_w(\mathcal{M}) \in \mathbb{N}$ the *multiplicity* with which $\overline{T^*_{X_w}X}$ occurs as a component in the characteristic cycle of $\mathcal{M}$ (1.1). Note that $\Sigma(\mathcal{M})$ is the set of those w for which $m_w(\mathcal{M}) \neq 0$. By abuse of notation, we

shall write

$$\sum_w m_w(\mathcal{M})[\overline{T^*_{X_w}X}] = \sum_w m_w(\mathcal{M})[w],$$

i.e. we shall identify the algebraic cycle on the left hand side with an element in $\mathbb{Z}[\mathbf{W}]$, the free abelian group with generators $\mathbf{W}$.

Proposition. a) *The characteristic cycle of a coherent* $(\mathcal{D}_X, K)$*-module* $\mathcal{M}$ *has the form*

$$\mathcal{C}h(\mathcal{M}) = \sum_{w\in\mathcal{W}} m_w(\mathcal{M})[\overline{T^*_{X_w}X}] = \sum_w m_w(\mathcal{M})[w].$$

b) $\mathcal{M} \mapsto \mathcal{C}h(\mathcal{M})$ *defines a homomorphism from the Grothendieck group of coherent* $(\mathcal{D}_X, K)$*-modules to* $\mathbb{Z}[\mathbf{W}]$.

c) *This is an isomorphism, provided that all K-orbits* X_w *are simply connected.*

d) *The length of* $\mathcal{M}$ *is bounded by* $\sum_w m_w(\mathcal{M})$.

Proof. a) is merely a reformulation of 2.5. Then b) is a well known general fact in the theory of holonomic systems, and d) follows from b) by the trivial observation that each composition factor $\mathcal{L}$ of $\mathcal{M}$ contributes $\sum_w m_w(\mathcal{L}) \geqq 1$ to the sum of all multiplicities of $\mathcal{M}$. For the proof of c) see 2.7 c) below. Q.e.d.

2.7. Construction of simple modules [BeBe, BK]

As a sheaf on X, a coherent $(\mathcal{D}_X, K)$-module $\mathcal{M}$ has a well defined *support*, which is obviously a K-invariant closed subvariety $\mathrm{Supp}\,\mathcal{M}$ of X. It is easy to see that

$$\mathrm{Supp}\,\mathcal{M} = p(\mathrm{Ch}(\mathcal{M})), \tag{1}$$

where $p\colon T^*X \to X$ denotes the bundle map. It turns out that *for* $\mathcal{M}$ *simple*, $\mathrm{Supp}\,\mathcal{M}$ *is irreducible*, and hence is the closure $\bar{X}_w$ of some K-orbit X_w (see [BeBe]).

Conversely, let us now start from any given K-orbit X_w, and let us construct a simple $(\mathcal{D}_X, K)$-module $\mathcal{L}_w$ with support $\bar{X}_w$ as follows. For any locally closed (irreducible) subvariety Y of X, and any $\mathcal{O}_X$-module $\mathcal{F}$, we have $\mathcal{O}_X$-modules

$$\mathcal{H}^j_{[Y]}(\mathcal{F}) \quad \text{for } j = 0, 1, 2 \ldots,$$

the relative cohomology groups supported in Y, as in [BK], 1.4.[2] If $\mathcal{F}$ has the structure of a $\mathcal{D}_X$-module, then $\mathcal{H}^j_{[Y]}(\mathcal{F})$ has too. If, furthermore, Y is K-invariant, and $\mathcal{F}$ has even the structure of a coherent $(\mathcal{D}_X, K)$-module, then $\mathcal{H}^j_{[Y]}(\mathcal{F})$ has too.

[2] For the following proposition, we have to translate [BK] from the language of complex analytic $\mathcal{D}_X$-modules to the algebraic language, which is possible using a GAGA-principle due to Kashiwara-Kawai implicit in [KK], made explicit in [B2]

Proposition (cf. [BeBe, BK]). a) *The coherent* $(\mathcal{D}_X, K)$*-module*

$$\mathcal{K}_w := \mathcal{H}^{\operatorname{codim}_X X_w}_{[X_w]}(\mathcal{O}_X)$$

has a unique simple submodule $\mathcal{L}_w$.

b) *We have* $\operatorname{Supp}(\mathcal{L}_w) = \bar{X}_w$, *and* $\operatorname{Supp}(\mathcal{K}_w/\mathcal{L}_w) \subset \bar{X}_w \setminus X_w$.

c) *If* X_w *is simply connected, then* $\mathcal{L}_w$ *is the unique simple* $(\mathcal{D}_X, K)$*-module* (up to isomorphism) *with support* $\bar{X}_w$.

d) ([BK], Theorem 8.6): *The De Rham complex of* $\mathcal{L}_w$ *is the Deligne-Goresky-MacPherson intersection homology complex of* $\bar{X}_w$ (cf. loc. cit. and [GoM] for terminology), i.e.:

$$\mathrm{DR}(\mathcal{L}_w) = \mathbf{IC}^{\cdot}(\bar{X}_w).$$

Comments. 1. We are following here the procedure and language used by Brylinsky-Kashiwara [BK], 1.4, 4.2 (where only the special case $B=K$ was discussed explicitly). But let us point out that the construction of simple modules in [BeBe] is essentially the same. In the terminology of [Be2] our module $\mathcal{K}_w$ above is constructed as the direct image of $\mathcal{O}_{X_w}$ under the inclusion map $X_w \hookrightarrow X$.

2. For the purposes of the present paper, we are only interested in the special case where all K-orbits X_w are simply connected. So in this case (cf. c) above), the simple objects in $\mathcal{M}od(\mathcal{D}_X, K)^{\mathrm{coh}}$ are classified by the K-orbits. Let us only mention here that in the general case, treated by Beilinson-Bernstein in [BeBe], the simple modules correspond to pairs (X_w, θ), where θ is an irreducible finite-dimensional representation of the fundamental group of X_w (at least if K is simply-connected).

2.8. *Characteristic cycle of simple modules*

Let $w \in \mathbf{W}$, and let L_w be the simple $(\mathcal{D}_X, K)$-module with support X_w constructed in 2.7. Let us note here some general information about its characteristic variety, which follows from the above geometric construction of $\mathcal{L}_w$ (cf. [BK], 6.4).

Proposition. a) $m_y(\mathcal{L}_w) \neq 0$ *implies* $X_y \subset \bar{X}_w$.

b) $m_w(\mathcal{L}_w) = 1$.

c) *If* $y \neq w$, *then* $m_y(\mathcal{L}_w) \neq 0$ *implies* $\bar{X}_w$ *is singular at* X_y.

Proof. Since $\operatorname{Supp}(\mathcal{L}_w) = \bar{X}_w$ by 2.7 a), we have

$$\operatorname{Ch}(\mathcal{L}_w) \subset \bigcup_{y;\, X_y \subset \bar{X}_w} T^*_{X_y} X$$

by 2.7(1), 2.4(1) and 2.5. This inclusion proves a). Statements b) and c) follow from the fact that the sheaf $\mathcal{L}_w$ coincides with the direct image of $\mathcal{O}_{X_w}$ at all smooth points of $\bar{X}_w$, by proposition 2.7 c). Q.e.d.

Remark. In particular, a) and b) say that, with respect to a suitable ordering of the index set, the $\mathbf{W}$ by $\mathbf{W}$ matrix of integers $m_y(\mathcal{L}_w)$ is upper triangular, with

1's on the diagonal, so is invertible over the integers. Now assume that all K-orbits X_w are simply connected, so that the $\mathcal{L}_w (w \in \mathbf{W})$ form a $\mathbb{Z}$-basis of the Grothendieck-group of coherent $(\mathcal{D}_X, K)$-modules by 2.7 c). Then the preceding remark proves that $\mathcal{M} \mapsto \mathcal{Ch}(\mathcal{M})$ is an isomorphism of this Grothendieck group with $\mathbb{Z}[\mathbf{W}]$, as claimed in 2.6 c). In the above mentioned construction of simple modules, one has then to replace the trivial sheaf $\mathcal{O}_{X_w}$ by a locally trivial sheaf with θ as "monodromy representation".

3. Let us mention at this point, that it follows from the above construction of the simple modules, that they (and hence *all* coherent $(\mathcal{D}_X, K)$-modules) are *holonomic with regular singularities*, cf. [Be2], 4.2, or cf. [Me, KK], as in [BK].

§3. Localisation of highest weight modules and Harish-Chandra Bimodules

3.1. *Highest weight modules* ([Di], *chap.* 7)

We fix a Borel subgroup B of G, and a maximal torus $H \subset B$. We denote by $\mathfrak{h} \subset \mathfrak{b}$ the corresponding Lie algebras, and by W the Weyl group, which acts as a finite reflection group on $\mathfrak{h}^*$, the dual of the Cartan algebra $\mathfrak{h}$. Each linear form $\lambda \in \mathfrak{h}^*$ determines a one-dimensional $\mathfrak{b}$-module k_λ. The "Verma-module of highest weight λ" is, by definition, the $\mathfrak{g}$-module induced from k_λ, and is denoted $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$. It has a unique simple quotient $L(\lambda)$, called the "simple module of highest weight λ". The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on $M(\lambda)$ and $L(\lambda)$ by a character χ, which coincides with the trivial central character χ_0 if and only if $\lambda = w(-\rho) - \rho$ for some $w \in W$, where ρ denotes half the sum of the positive roots (in $\mathfrak{h}^*$ relative $\mathfrak{b}$). We use the short notations

$$M_w = M(w(-\rho)-\rho), \qquad L_w = L(w(-\rho)-\rho)$$

for all $w \in W$, as in [BK], and [BoB]. In the terminology of 2.2, these modules are obviously $(\mathfrak{g}, B)$-modules. Moreover, recalling some basic facts from representation theory, we may state that the modules L_w represent all isomorphism classes of simple $(\mathfrak{g}, B)$-modules with trivial central character. More precisely:

Proposition (cf. [Di], chap. 7). *The modules* L_w (resp. M_w), *for* $w \in W$, *form a basis for the Grothendieck group of* $\operatorname{Mod}(\mathfrak{g}, B)^{\mathrm{fg}}_X$ (Notation 2.3).

3.2. *Localisation of highest weight modules* [BK, BeBe]

Alternatively, the preceding results from representation theory may also be viewed as an application of the theory of $\mathcal{D}$-modules on the flag variety X as follows. The general Beilinson-Bernstein theory, as reviewed in chapter 2, applies to the special case $K=B$, because X decomposes into *finitely many* B-*orbits*. In fact, the B-orbits in X are the so called *Schubert cells*, which are indexed by the elements w in the Weyl group W. Identifying X with G/B, the Schubert cell X_w may be defined as BwB/B. It is well known to be an affine space (a "cell") of dimension $l(w)$, where l denotes the length function on W

[Bou]. In particular, all B-orbits are *simply connected*. Therefore (cf. 2.7) the simple coherent $(\mathscr{D}_X, B)$-modules are classified by the B-orbits: For each Schubert cell X_w, there is a unique simple coherent $(\mathscr{D}_X, B)$-module $\mathscr{L}_w$ with support $\bar{X}_w$ (called a *Schubert variety*), and these modules $\mathscr{L}_w$ $(w\in W)$ form a $\mathbb{Z}$-basis for the Grothendieck group of the category $\mathscr{M}od(\mathscr{D}_X, B)^{coh}$ of coherent $(\mathscr{D}_X, B)$-modules. Translating this statement from the language of $\mathscr{D}$-modules into the language of $\mathfrak{g}$-modules by means of the following result, we obtain Proposition 3.1.

Proposition [BK, BeBe]. *Under the equivalence of categories*

$$\mathrm{Mod}(\mathfrak{g}, B)_X^{fg} \approx \mathscr{M}od(\mathscr{D}_X, B)^{coh}$$

(cf. 2.3) *given by the localization functor, L_w corresponds to $\mathscr{L}_w$ for all $w\in W$.*

In other words, there are natural isomorphisms $\mathscr{L}_w \cong \mathscr{D}_X \otimes_{U(\mathfrak{g})} L_w$ and $L_w \cong \Gamma(X, \mathscr{L}_w)$.

Remark. From now on, throughout the rest of this paper, the notations L_w, $\mathscr{L}_w$ and X_w will always have the particular meaning defined above in 3.1 resp. 3.2. In other words, this notation will be reserved for the special case $K=B$ of the general theory reviewed in §2. Note that this special case has been treated already in much detail in [BK], to which we refer for more background and details. Our present summary should mainly serve the purpose of clarifying notation and terminology, and provide references for the convenience of the reader not familiar with the theory developed in [BK, BeBe].

3.3. *Examples for the geometric interpretation of Verma-modules*

Let us consider the two extreme cases $w=1$ or $w=w_0$, where w_0 denotes the *longest element* of W. The corresponding Schubert cells are the single point $X_1 = \{B\}$, respectively the "big cell" X_{w_0} with closure $\bar{X}_{w_0}=X$. The corresponding Verma-modules are $M_1 = M(-2\rho)$ resp. $M_{w_0}=M(0)$, the simple highest weight modules are $L_1 = M_1$ resp. $L_{w_0}=L(0)$, the trivial (onedimensional) representation of $\mathfrak{g}$. The two "extreme" Verma-modules M_1, M_{w_0} have the following very natural geometric interpretations. For a left $U(\mathfrak{g})$-module M, we define a right $U(\mathfrak{g})$-module $M^\vee$ by defining $mu := u^\vee m$ for $m\in M$, $u\in U(\mathfrak{g})$, where $^\vee$ denotes the principal anti-automorphism of $U(\mathfrak{g})$. Let Ω_X denote the sheaf of algebraic differential forms of order $n=\dim X$ on X. The sheaves $\mathcal{O}_X$ resp. Ω_X are naturally left resp. right $\mathscr{D}_X$-modules, hence the cohomology groups $H^n_x(X, \mathcal{O}_X)$ resp. $H^n_x(X, \Omega_X)$ with support in a point $x\in X$ are naturally left resp. right modules for $\mathscr{D}_{X,x} \supset D(X) \supset \mathfrak{g}$.

Lemma. *For the natural $\mathfrak{g}$-module structures, we have with $x=B$:*

a) $H^n_x(X, \mathcal{O}_X) \cong M_1$ *and* b) $H^n_x(X, \Omega_x) \cong M^\vee_{w_0}$.

Proof. For a detailed discussion of a), we refer to [BoB] (1.5, 1.6, 3.5). Recall in particular the intrinsic interpretation of the $D(X)$-action on $H^n_x(X, \mathcal{O}_X)$ in terms

of the canonical linear maps (see loc. cit., 1.6):

$$\Omega_{X,x} \otimes H^n_x(X, \mathcal{O}_x) \xrightarrow{\mu} H^n_x(X, \Omega_X) \xrightarrow{\text{res}} k.$$

Recall also (loc. cit. 3.5) that the n-th exterior power of the tangent space $T(X)_x \cong \mathfrak{g}/\mathfrak{g}_x = \mathfrak{g}/\mathfrak{b}$ provides a cyclic generator of highest weight -2ρ.

Now observe for the proof of b), that the surjective linear map μ above provides simultaneously an intrinsic description of the right $D(X)$-action on $H^n_x(X, \Omega_X)$. In particular, it becomes obvious that this module is generated by the "canonical class" given by

$$\mu(\Lambda^n(T^*(X)_x) \otimes \Lambda^n(T(X)_x)) \cong \mu(\Lambda^n(\mathfrak{b}^\perp) \otimes \Lambda^n(\mathfrak{g}/\mathfrak{b})),$$

which is obviously a 1-dimensional $\mathfrak{b}$-invariant subspace of weight 0. Now the universal property of the Verma-module $M(0)^\vee = M^\vee_{w_0}$ provides a homomorphism from $M(0)^\vee$ onto $H^n_x(X, \Omega_X)$, and an argument similar to that in loc. cit., 3.5 shows that this is actually an isomorphism. Q.e.d.

Remark. A sheaf theoretic interpretation of the localisation of the right $\mathfrak{g}$-module $M(0)^\vee$ as a right $\mathscr{D}_X$-module on X is now obtained as follows. Let i: $\{x\} \hookrightarrow X$ denote the inclusion of the point $x=B$ into X, and consider – using the terminology of say [Be2], p. 5 – the inverse image of the right $\mathscr{D}_X$-module $\mathscr{D}_X$. This is a right $\mathscr{D}_X$-module denoted

$$i^\Delta(\mathscr{D}_X) = \mathscr{D}_{\{x\} \hookrightarrow X}$$

which identifies with the localized module $M(0)^\vee \otimes_{U(\mathfrak{g})} \mathscr{D}_X$.

3.4. *Harish-Chandra bimodules* [Di, CD, BeGe, J1, Ja]

By a $\mathfrak{g}$-*bimodule* M we mean a left module over $U(\mathfrak{g}\times\mathfrak{g}) = U(\mathfrak{g}) \otimes U(\mathfrak{g})$. We call $\mathfrak{g}^l = \mathfrak{g}\times 0$ (resp. $\mathfrak{g}^r = 0\times\mathfrak{g}$) the *left* (resp. *right*) copy of $\mathfrak{g}$ in $\mathfrak{g}\times\mathfrak{g}$, and denote by M^l resp. M^r the $\mathfrak{g}^l$- resp. $\mathfrak{g}^r$-module obtained from M by restriction of the $\mathfrak{g}\times\mathfrak{g}$-action. Moreover, we call $\mathfrak{g}^d = \{(x,x) \mid x\in\mathfrak{g}\}$ the *diagonal* copy of $\mathfrak{g}$ in $\mathfrak{g}\times\mathfrak{g}$, and we denote $G^l = G\times 1$, $G^r = 1\times G$, $G^d = \{(g,g) \mid g\in G\}$ the corresponding subgroups of $G\times G$. The diagonal copies will also be denoted $\mathfrak{k} = \mathfrak{g}^d$ resp. $K = G^d$. Let M be a $(\mathfrak{g}\times\mathfrak{g}, K)$-module in the sense of 2.2. Then M is in particular locally finite as a $\mathfrak{k}$-module. It is called a *Harish-Chandra bimodule*, if it satisfies the additional finiteness condition

$$\dim \mathrm{Hom}_{\mathfrak{k}}(E, M) < \infty \qquad (*)$$

for all finite-dimensional $\mathfrak{k}$-modules E.

We are primarily concerned here with the category $\mathrm{Mod}(\mathfrak{g}\times\mathfrak{g}, K)^{fg}_{X\times X}$ of finitely generated $(\mathfrak{g}\times\mathfrak{g}, K)$-modules with trivial central character. For such modules, condition (*) is automatically satisfied, so this is *a* category of Harish-Chandra bimodules. It is *the* category of Harish-Chandra bimodules considered in §3–5. There are two reasons, why this category is of particular interest for us: First its close relation to highest weight modules stated in the proposition below, and second its significance for the study of ideals in $U(\mathfrak{g})$, resulting from the example below.

Proposition (cf. Bernstein-Gelfand [BeGe], Joseph [J1]). *The functor $H \mapsto M(0)^\vee \otimes_{U(\mathfrak{g}^r)} H = L$ provides an equivalence of categories*

$$\mathrm{Mod}(\mathfrak{g}\times\mathfrak{g}, K)^{fg}_{X\times X} \xrightarrow{\approx} \mathrm{Mod}(\mathfrak{g}, B)^{fg}_X.$$

For a statement which is very close to this one, and for more details and background, the reader may consult [Ja], 6.27.

Let us mention that the inverse functor sends a highest weight module L to the Harish-Chandra bimodule $\mathscr{L}(M(0), L)$ of $\mathfrak{g}$-finite linear maps from $M(0)$ into L. We shall denote by H_w the simple Harish-Chandra bimodule corresponding to the simple highest weight module L_w under this equivalence (for all $w\in W$). – Again, we are now going to reformulate these well-known facts from representation theory in terms of the theory of $\mathscr{D}$-modules.

Example. We consider $U(\mathfrak{g})$ as a $\mathfrak{g}$-bimodule by defining $(a\otimes b)u = aub^\vee$ for all $a, b, u\in U(\mathfrak{g})$. Its sub-bimodules are the (two-sided) *ideals* I of $U(\mathfrak{g})$. A quotient $U(\mathfrak{g})/I$ is a Harish-Chandra bimodule if (and only if) the center $Z(\mathfrak{g})$ acts through a finite-dimensional quotient. The quotient $U(\mathfrak{g})/I_X \cong D(X)$ is the Harish-Chandra bimodule corresponding to the highest weight module $M(0)$ under the above equivalence. Consequently, the lattice of ideals of $D(X)$ is isomorphic to the lattice of submodules of $M(0)$. (The correspondence being given by $I\mapsto IM(0)$, cf. Joseph [J1].)

3.5. *Localization of bimodules on $Z = X\times X$*

We consider the flag variety $Z = X\times X$ on the product group $G\times G$, acted on by the diagonal copy $K = G^d$ (3.4) of G in $G\times G$. Since K is the subgroup of elements fixed by an involution of $G\times G$ (in fact, of $(g,h)\mapsto(h,g)$), it has only finitely many orbits in Z. In other words, our basic assumption in 2.1 is satisfied, so that the general Beilinson-Bernstein theory reviewed in §2 applies to the present special case of $(\mathscr{D}_Z, K)$-modules.

More precisely, the K-orbits on Z are again classified by the Weyl group elements as follows. Two Borel subgroups B_1, B_2 of G are "in relative position w", notation $B_1 \xrightarrow{w} B_2$, if $B_2 = gB_1g^{-1}$ with $g\in B_1wB_1$ (cf. e.g. [KL], appendix). Then the K-orbits are exactly the varieties of all pairs $(B_1, B_2)\in Z$ in a given position w. Let us denote these "position varieties" by

$$Z_w = \{(B_1, B_2) \mid B_2 \xrightarrow{w} B_1\}$$

for all $w\in W$. They are related to the Schubert cell X_w (3.2) as follows: In Z, we have a *left* (resp. *right*) copy of X and of each X_w, defined by

$$X^l = X\times\{B\} \quad \text{and} \quad X^l_w = X_w\times\{B\} = \{(B_1, B) \mid B \xrightarrow{w} B_1\},$$

resp.

$$X^r = \{B\}\times X \quad \text{and} \quad X^r_w = \{B\}\times X_w = \{(B, B_2) \mid B \xrightarrow{w} B_2\}.$$

Then the following is immediate:

Lemma. *For each $w\in W$, we have*

$$Z_w\cap X^l = X^l_w \quad \textit{and} \quad Z_w\cap X^r = X^r_{w^{-1}}.$$

Moreover, these intersections are transversal. And

$$Z_w = GX^l_w = GX^r_{w^{-1}}.$$

Finally, each K-orbit Z_w is *simply connected*, because it is a fiber bundle over X with fiber an affine space. In fact, the projection $p^l\colon Z\to X$ onto the left factor, i.e. the forgetful map $(B_1, B_2)\mapsto B_1$, restricts to a bundle map of Z_w onto X with fiber X_w. Similarly, Z_w fibers over X with fiber $X_{w^{-1}}$ under the projection $p^r\colon Z\to X$ which forgets B_1.

Now 2.3, 2.7 apply to give the

Corollary. [BeBe]

a) *The category of f.g. Harish-Chandra bimodules with trivial central character is equivalent to the category of coherent $(\mathscr{D}_Z, K)$-modules via the localization functor*

$$H\mapsto \mathscr{D}_Z\otimes_{U(\mathfrak{g}\times\mathfrak{g})} H = \mathscr{H}$$

$$\mathrm{Mod}(\mathfrak{g}\times\mathfrak{g}, K)^{fg}_Z \xrightarrow{\approx} \mathscr{M}od(\mathscr{D}_Z, K)^{coh}.$$

b) *For each $w\in W$, there is a unique simple coherent $(\mathscr{D}_Z, K)$-module $\mathscr{H}_w$ with support $\bar{Z}_w$, and these are all the simple modules in this category.*

Notation. The simple coherent $(\mathscr{D}_Z, K)$-module with support $\bar{Z}_w$ defined by 2.7 will always be denoted $\mathscr{H}_w$. It will turn out in the next section, that $\mathscr{H}_w$ is (up to isomorphism) the localization of the simple Harish-Chandra module H_w as defined in 3.4. In other words, under the above equivalence of categories, $\mathscr{H}_w$ corresponds to H_w.

3.6. *Restriction of $(\mathscr{D}_Z, K)$-modules to $(\mathscr{D}_X, B)$-modules*

The inclusions of X into $X\times X = Z$ as the left resp. right copy X^l resp. X^r are denoted i_l resp. i_r. They give rise to inverse image ("restriction") functors i_l^* resp. i_r^* from coherent $(\mathscr{D}_Z, K)$-modules to coherent $(\mathscr{D}_X, B)$-modules.

Proposition. *Let $\mathscr{H}$ be a $\mathscr{D}_Z$-coherent $(\mathscr{D}_Z, K)$-module. Then:*

a) $i_l^*(\mathscr{H}) = \mathscr{O}_{X^l}\otimes_{\mathscr{O}_Z}\mathscr{H}$ *is a $\mathscr{D}_X$-coherent $(\mathscr{D}_X, B)$-module.*

b) $\mathrm{Tor}_i^{\mathscr{O}_Z}(\mathscr{O}_{X^l}, \mathscr{H}) = 0$ *for all $i>0$.*

c) $\Gamma(X, i_l^*(\mathscr{H}))\cong M(0)^\vee\otimes_{U(\mathfrak{g}^r)}\Gamma(Z, \mathscr{H})$, *canonically.*

d) i_l^* *resp.* i_r^* *is an equivalence of categories:*

$$\mathscr{M}od(\mathscr{D}_Z, K)^{coh} \xrightarrow{\approx} \mathscr{M}od(\mathscr{D}_X, B)^{coh}.$$

e) $i_l^*(\mathscr{H}_w)\cong\mathscr{L}_w$ *for all $w\in W$, i.e. the simple module $\mathscr{H}_w$ with support $\bar{Z}_w$ corresponds to the simple module $\mathscr{L}_w$ with support $\bar{X}_w$ introduced in* 3.6 resp. 3.2.

f) *Analogous results hold for* i_r, *in particular:*

$$i_r^*(\mathscr{H}_w) \cong \mathscr{L}_{w^{-1}} \quad \textit{for all} \quad w \in W.$$

We may summarize the essential content of this proposition as follows.

Corollary. *There is a diagram of equivalences of categories*

$$\begin{tikzcd}
\mathcal{M}od(\mathscr{D}_X, B)^{\mathrm{coh}} & \mathcal{M}od(\mathscr{D}_Z, K)^{\mathrm{coh}} \arrow[l, "i_l^*"'] \arrow[r, "i_r^*"] & \mathcal{M}od(\mathscr{D}_X, B)^{\mathrm{coh}} \\
\mathrm{Mod}(\mathfrak{g}, B)_X^{\mathrm{fg}} \arrow[u, "\approx"] & \mathrm{Mod}(\mathfrak{g}\times\mathfrak{g}, K)_Z^{\mathrm{fg}} \arrow[l] \arrow[r] \arrow[u, "\approx"] & \mathrm{Mod}(\mathfrak{g}, B)_X^{\mathrm{fg}} \arrow[u, "\approx"]
\end{tikzcd}$$

(given by 3.2, 3.5a), 3.4 resp. 3.6) *which commutes* (in the sense that corresponding composed functors are naturally equivalent). *The correspondence of simple objects is* (for all $w \in W$) *given by:*

$$\begin{tikzcd}
\mathscr{L}_w & \mathscr{H}_w \arrow[l, mapsto] \arrow[r, mapsto] & \mathscr{L}_{w^{-1}} \\
L_w \arrow[u, leftrightarrow] & H_w \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, leftrightarrow] & L_{w^{-1}} \arrow[u, leftrightarrow]
\end{tikzcd}$$

We note that e.g. the commutativity of the diagram is just part c) of the proposition. – The *proof* of the proposition (in 3.8–3.9) is prepared by the following lemma, which provides a commutativity result similar to c) on the level of derived categories.

3.7. In the situation of 3.6, let $p^r: Z \to X$ denote the right projection (3.5). We use some notation and terminology from the theory of derived categories (cf. e.g. [Ve], p. 291 ff.). Given an abelian category $\mathcal{M}$, the bounded derived category of bounded complexes of objects in $\mathcal{M}$ is denoted $D^b(\mathcal{M})$. Given a left-exact functor $F: \mathcal{M} \to \mathcal{M}'$ of abelian categories, the left derived functor of F is a functor $D^b(\mathcal{M}) \to D^b(\mathcal{M}')$, denoted $\mathbb{L}F$.

Lemma. *For a coherent* $\mathscr{D}_X$*-module* $\mathcal{M}$, *we have canonically*

$$\Gamma(Z, p_r^{-1}(\mathcal{M}) \otimes^{\mathbb{L}}_{p_r^{-1}(\mathscr{D}_X)} \mathscr{H}) \cong \Gamma(X, \mathcal{M}) \otimes^{\mathbb{L}}_{D(X^r)} \Gamma(Z, \mathscr{H}).$$

Proof. The left and right hand side are exact bi-functors of $\mathcal{M}$ and $\mathscr{H}$. There is a canonical map from the right hand side to the left hand side, which exists before we go to derived categories and functors, namely:

$$\begin{aligned}\Gamma(X, \mathcal{M}) \otimes_{D(X^r)} \Gamma(Z, \mathscr{H}) &\to \Gamma(Z, p_r^{-1}(\mathcal{M})) \otimes_{\Gamma(Z, p_r^{-1}(\mathscr{D}_X))} \Gamma(Z, \mathscr{H}) \\ &\to \Gamma(Z, p_r^{-1}(\mathcal{M}) \otimes_{p_r^{-1}(\mathscr{D}_X)} \mathscr{H}).\end{aligned}$$

Using the fact that any $\mathscr{D}_X$-module $\mathcal{M}$ has a *global free resolution of finite length*, we may now easily reduce the proof of the lemma to the special case $\mathcal{M} = \mathscr{D}_X$. But for this case, we merely have to observe that the above map becomes just the identity map on $\Gamma(Z, \mathscr{H})$. Q.e.d.

3.8. Proof of Proposition 3.6

Statement a) is clear. Statement b) is a consequence of the K-equivariance of $\mathscr{H}$, using the fact that each K-orbit Z_w intersects X^l transversally in X^l_w by Lemma 3.5. Using the language of derived categories (3.7), statement b) may be reformulated as

b') $\mathbb{L}i_l^*(\mathscr{H}) = i_l^*(\mathscr{H})$,

which means to say that $\mathbb{L}i_l^*(\mathscr{H})$ is the complex with $i_l^*(\mathscr{H})$ in degree zero, and with zero in all higher degrees.

In order to prove c), let us apply Lemma 3.7 to the right $\mathscr{D}_X$-module $\mathcal{M}$ corresponding to the Verma-module $M(0)^\vee$ with highest weight 0, i.e. such that (cf. 3.3)

$$\Gamma(X, \mathcal{M}) \cong M(0)^\vee.$$

As explained in Remark 3.3, this module $\mathcal{M}$ is $\mathscr{D}_{\{x\} \hookrightarrow X}$. Observing that

$$p_r^{-1}(\mathcal{M}) \otimes_{p_r^{-1}(\mathscr{D}_X)} \mathscr{D}_Z \cong \mathscr{D}_{X^l \hookrightarrow Z}$$

as a right $\mathscr{D}_Z$- and left $\mathscr{D}_{X^l}$-bimodule, and that our restriction functor i_l^* is just tensoring with this bimodule:

$$\begin{aligned}i_l^*(\mathscr{H}) &\cong \mathscr{D}_{X^l \hookrightarrow Z} \otimes_{\mathscr{D}_Z} \mathscr{H} \cong (p_r^{-1}(\mathcal{M}) \otimes_{p_r^{-1}(\mathscr{D}_X)} \mathscr{D}_Z) \otimes_{\mathscr{D}_Z} \mathscr{H} \\ &\cong p_r^{-1}(\mathcal{M}) \otimes_{p_r^{-1}(\mathscr{D}_X)} \mathscr{H},\end{aligned}$$

we obtain from Lemma 3.7 a canonical isomorphism

c') $\Gamma(X, \mathbb{L}i_l^*(\mathscr{H})) \cong M(0)^\vee \otimes^{\mathbb{L}}_{D(X^r)} \Gamma(Z, \mathscr{H})$.

This implies c), using b') on the left hand side, and a similar fact about vanishing of higher cohomology groups on the right.

This completes the proof of part c) of the proposition, which gives the commutativity of functors statement in the corollary. But this statement says – in different words – just this: The functor i_l^* is the geometric interpretation in terms of $\mathscr{D}$-modules of the functor from Harish-Chandra bimodules to highest weight modules explained in 3.4. Since the latter functor is an equivalence (3.4), we may derive part d) of the proposition by means of c) and the Beilinson-Bernstein equivalences. – Alternatively, one may prove directly d), that i_l^* is an equivalence, and then use c) to obtain a geometric proof of 3.4 (see 3.10 below).

Now let us prove e) and f). First note that the functor i_l^* sends simple objects to simple objects by d) and that the supports are given by

$$\begin{aligned}\mathrm{Supp}\, i_l^*(\mathscr{H}_w) &= (\mathrm{Supp}\, \mathscr{H}_w) \cap X^l \\ &= \bar{Z}_w \cap X^l = \bar{X}^l_w.\end{aligned}$$

Since the simple objects are characterized by their supports (because of the simply connectedness of the Schubert cells, 2.7c), we can conclude that $i_l^*(\mathscr{H}_w)$

$\cong \mathscr{L}_w$. Similarly $i_r^*(\mathscr{H}_w) \cong \mathscr{L}_{w^{-1}}$ because

$$\operatorname{Supp} i_r^*(\mathscr{H}_w) = \bar{Z}_w \cap X^r = \bar{X}^r_{w^{-1}}.$$

This completes the proof of Proposition 3.6. Q.e.d.

3.9. Let us now give *an alternative proof* for the last parts (e–f) of Proposition 3.6. We use the Lefschetz principle to reduce to the case $k = \mathbb{C}$, and then pass to the analytic category using [GAGA]. Then, using that the modules under consideration are holonomic with regular singularities (Theorem 2.3b)), we may apply Mebkhout's results [Me, cf. also Ka], to conclude: It is enough to show that $i_l^*(\mathscr{H})$ has the same DeRham complex as $\mathscr{L}_{w^{-1}}$. Using b) and [Me], we compute the DeRham complex as follows (cf. Proposition 2.7d))

$$\begin{aligned} \mathrm{DR}(i_l^*(\mathscr{H}_w)) &= \mathrm{DR}(\mathbb{L} i_l^*(\mathscr{H}_w)) = i_l^! \,\mathrm{DR}(\mathscr{H}_w)[-n] \\ &= i_l^! \,\mathbf{IC}^{\cdot}(\bar{Z}_w)[-n], \end{aligned}$$

where $[-n]$ denotes a dimension shift by $-n = -\dim X$. Now we use that the K-orbits Z_y with $y \leqq w$ (Bruhat order) form a Whitney stratification of $\bar{Z}_w$, whose strata intersect X^l transversally (3.5). For such a situation [GoM], Theorem 5.4.1 gives that

$$i_l^! \,\mathbf{IC}^{\cdot}(\bar{Z}_w)[-n] = i_l^* \,\mathbf{IC}^{\cdot}(\bar{Z}_w) = \mathbf{IC}^{\cdot}(\bar{X}_w),$$

which is $\mathrm{DR}(\mathscr{L}_w)$ by 2.7 d). This completes our second proof of e). Of course, f) is analogous. Q.e.d.

3.10. A geometric proof of Proposition 3.4

Let us now give an alternative proof for the result of Bernstein-Gelfand and Joseph that Harish-Chandra bimodules are equivalent to highest weight modules (Proposition 3.4). For this purpose, *it suffices to give a direct geometric argument for statement* 3.6d), that the restriction functors i_l^* resp. i_r^* are equivalences (use Proposition 3.6c) in combination with the Beilinson-Bernstein equivalences). Our subsequent proof would apply to other equivariant geometric objects as well. It consists in introducing the map $\delta\colon G \times X \to X (= G/B)$ such that $\delta(g, x) = g^{-1}x$. If $\mathscr{M}$ is any coherent $\mathscr{D}_X$-module, then the $\mathscr{D}_{G \times X}$-module $\delta^*(\mathscr{M})$ is G-equivariant, with respect to the "diagonal" action of G on $G \times X$, i.e. $g(g', x) = (gg', gx)$. If furthermore $\mathscr{M}$ is B-equivariant, then $\delta^{-1}\mathscr{M}$ is B-equivariant, with respect to B acting by right translation on G, and trivially on X. Moreover, these actions of G and B commute, so they define an action of $G \times B$ on $G \times X$, and in this sense $\delta^*(\mathscr{M})$ is $G \times B$-equivariant.

Now the action of B on $G \times X$ is free, the map from $G \times X$ to the quotient $X \times X = Z$ is smooth and even a fibration (with respect to the Zariski topology), so we may *descend* $\delta^*(\mathscr{M})$ to a $\mathscr{D}$-module on Z, which is K-equivariant (recall that K is G, embedded diagonally in $G \times G$). In this fashion, we obtain an exact functor from the category of coherent $(\mathscr{D}_X, B)$-modules to the category of coherent $(\mathscr{D}_Z, K)$-modules, which is easily seen to be a quasi-inverse of i_r^*. This proves Proposition 3.6d) for i_r^*. The proof for i_l^* follows from using the involution $(x, y) \mapsto (y, x)$ of $X \times X = Z$, which preserves $\mathscr{M}od(\mathscr{D}_Z, K)^{\mathrm{coh}}$. Q.e.d.

§ 4. A *G*-saturation theorem for associated varieties

4.1. Conormal bundles of Schubert cells

(We maintain the notations introduced in the preceding chapter). By definition, an element w of the Weyl group W is a coset modulo H of the normalizer $N_G(H)$ of our maximal torus $H \subset B$ (3.1). We write B^w for the conjugate $g_w B g_w^{-1}$ of B by a representative g_w of this coset. We shall use a similar notation for the g_w-conjugates of any Ad H-invariant subgroup of G, and of the corresponding Lie algebras. Of course, such a conjugate depends only on the coset w. Observe that B^w is the unique fixed point of H in X_w. In other words, by our choice of H, we simultaneously fix a particular base point (B^w) in each Schubert cell X_w of the flag variety X. Recall from [BoB] that we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form, and that the orthogonal of $\mathfrak{b} = \operatorname{Lie} B$ is then $\mathfrak{b}^\perp = \mathfrak{n}$, the nilradical of $\mathfrak{b}$. More generally, for each Weyl element w, the co-isotropy space of X at the point B^w is $(\mathfrak{b}^w)^\perp = \mathfrak{n}^w$. This exhibits the following as a special case of 2.4:

Lemma. [BK]

a) $\pi^{-1}(\mathfrak{n}) = \bigcup_{w \in W} T^*_{X_w} X$

b) $\pi(T^*_{X_w} X) = B(\mathfrak{n} \cap \mathfrak{n}^w)$, *for all* $w \in W$.

4.2. Associated and characteristic varieties of highest weight modules

Proposition. *Let L be a finitely generated* $(\mathfrak{g}, B)$*-module with trivial central character. Let* $\mathscr{L}$ *denote its localisation on* X*. Then:*

a) [BK] *There exist non-negative integers* $m_w(\mathscr{L})$, $w \in W$, *such that*

$$\mathscr{C}\!h(\mathscr{L}) = \sum_{w \in W} m_w(\mathscr{L}) [\overline{T^*_{X_w} X}].$$

b) [BK] *There exists a subset* $\Sigma(L)$ *of* W *such that*

$$\mathrm{Ch}(\mathscr{L}) = \bigcup_{w \in \Sigma(L)} \overline{T^*_{X_w} X}.$$

c) *The associated variety of L is then given by*

$$V(L) = \bigcup_{w \in \Sigma(L)} \overline{B(\mathfrak{n} \cap \mathfrak{n}^w)}.$$

Proof. The (equivalent) statements a) and b) are a special case of 2.6. Now apply Theorem 1.9 c) to derive from b)

$$V(L)=\pi(\mathrm{Ch}(\mathscr{L}))=\bigcup_{w\in\Sigma(L)}\pi(\overline{T^*_{X_w}X})=\bigcup_{w\in\Sigma(L)}\overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}.$$

Here the last step uses Lemma 4.1 and the properness of π. Q.e.d.

4.3. *The case of a simple highest weight module*

Corollary 1. $\overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}\subset V(L_w)$ *for each* $w\in W$.

Proof. By Proposition 2.8 b), we have $w\in\Sigma(L_w)$. *Q.e.d.*

Remark 1. This result was independently obtained by Joseph [J3] using completely different methods. Note that our *results extend* immediately *to the case of modules with any integral highest weight*, by using the technique of tensoring with finite-dimensional representations, which does not change associated varieties. In this extended form, the above corollary is precisely Joseph's theorem 8.15 in [J3].

Remark 2. We have reduced here the description of the variety $V(L_w)$ to the computation of certain combinatorial data, viz. of the subset $\Sigma(L_w)$ of the Weyl group. The actual computation of this subset, however, seems to be a hard problem in general. This subset admits a purely geometric interpretation in terms of the singularities of the Schubert variety $\bar{X}_w$, see [BDK] for more back-ground and details.

Some very modest general information about the subsets $\Sigma(L_w)$ results from Proposition 2.8 (cf. also 4.4, 4.5), which refers only to the roughest geometrical information. It seems remarkable to us that already such modest information from the geometric point of view very easily prove results like the corollary above, whereas from the purely algebraic point of view as in [J3] this result appears to be highly non-trivial (cf. [J3]). As another example, let us state the following immediate consequence of 2.8c):

Corollary 2. *If* $\bar{X}_w$ *is smooth, then* $\Sigma(L_w)=\{w\}$, *and so*

$$V(L_w)=\overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}.$$

4.4. *Example.* We take $G=\mathrm{Sp}(4,k)$ and $w=s_1s_2s_1$, where s_1, s_2 are simple reflections generating W, s_1 corresponding to a short root. Then the Schubert variety X_w is singular (precisely) at the points of X_{s_1}: In fact, it is locally on X_{s_1} a product of a quadratic cone of dimension 2 with X_{s_1}. It turns out that in this case

$$\mathscr{Ch}(\mathscr{L}_w)=[\overline{T^*_{X_w}X}]+[\overline{T^*_{X_{s_1}}X}]$$

has precisely two components, both of multiplicity 1; in particular

$$\Sigma(L_w)=\{w,s_1\}.$$

In this case, $\mathfrak{n}\cap\mathfrak{n}^w$ is onedimensional (spanned by a long root vector), and $B(\mathfrak{n}\cap\mathfrak{n}^w)$ has dimension 2, whereas $B(\mathfrak{n}\cap\mathfrak{n}^{s_1})$ has dimension 3. We have a *proper inclusion* (cf. Corollary 4.3.1)

$$V(L_w)=\overline{B(\mathfrak{n}\cap\mathfrak{n}^{s_1})}\supsetneqq\overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}.$$

4.5. Although the preceding example shows that the associated variety of L_w may be bigger than $\overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}$, we nevertheless expect that it should be irreducible. In view of 4.2, this is equivalent to the

Conjecture. For each $w\in W$ *there is a* $y\in W$ *such that*

$$V(L_w)=\overline{B(\mathfrak{n}\cap\mathfrak{n}^y)}.$$

Comments. 1. With respect to the Bruhat order, we have necessarily $y\leqq w$, even for all $y\in\Sigma(L_w)$, by Proposition 2.8a).

2. By a result of Gabber [Lv], the variety $V(L_w)$ is necessarily equidimensional. Therefore, *Proposition* 4.2 c) *holds for a simple module* L_w *even with* $\Sigma(L_w)$ *replaced by* the subset

$$\Gamma(L_w)=\{y\in\Sigma(L_w)\,|\dim B(\mathfrak{n}\cap\mathfrak{n}^y)=\dim V(L_w)\},$$

which may be strictly smaller (see 4.4).

3. The example 4.4 shows that we may have $w\notin\Gamma(L_w)$. As observed by Joseph [J3], 8.15, 9.14, this cannot occur for $G=SL_n$. In other words, for $G=SL_n$, the variety $\overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}$ is at least always a component of $V(L_w)$, and the conjecture is then that there should be no further components, i.e.

$$V(L_w)=\overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}.$$

4. *For* $G=SL_n$, $n\leqq 6$, *the conjecture is true.* In fact, for this range one even knows the much sharper result: $\Sigma(L_w)=\{w\}$ is a single element, i.e. even the characteristic variety $\mathrm{Ch}(\mathscr{L}_w)$ is irreducible (4.2). Conjecturally, even this holds for all n. As proved by Kashiwara-Tanisaki [KT], 5.4 resp. by R. MacPherson (unpublished), this latter conjecture is equivalent to certain conjectures by Kazhdan-Lusztig, stated in [KL2], Section 7 resp. in [KL], 6.3, both concerning Springer's Weyl group representations. Kazhdan and Lusztig have verified the conjecture in [KL], 6.3, for $n\leqq 6$.

5. The conjecture would obviously imply the following fact about the G-saturation (union of all G-conjugates) of $V(L_w)$:

$$GV(L_w)=G\overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}=\overline{G(\mathfrak{n}\cap\mathfrak{n}^w)}=\bar{\mathcal{O}}_x$$

is the closure of a single nilpotent orbit $\mathcal{O}_x$ in $\mathfrak{g}=\mathfrak{g}^*$. This (much weaker) statement is in fact true, as a consequence of recent work of Hotta [H], in combination with Joseph's work in [J3]. Let us state this for later reference as

Theorem (Hotta-Joseph)[3]. *For each $w \in W$, the G-saturation of $V(L_w)$ is the closure of a single nilpotent orbit.*

4.6. Steinberg's variety of triples [St1, KL2]

The famous "*triple variety*" $\mathcal{T}$ consists of all triples (x, B_1, B_2), where x is a nilpotent element of $\mathfrak{g}$, and B_1, B_2 are Borel subgroups of G, whose Lie algebras $\mathfrak{b}_1$, $\mathfrak{b}_2$ contain x. The triples with B_1 in position w relative to B_2 form a subvariety $\mathcal{T}_w$ of $\mathcal{T}$. Then $\mathcal{T}$ is a disjoint union of all $\mathcal{T}_w$ for $w \in W$, and $\mathcal{T}_w$ ($w \in W$) are the $\# W$ irreducible components of $\mathcal{T}$, which is equidimensional of dimension $\dim Z = 2 \dim X$. For a systematic study of the triple variety, we refer to [KL2, St1][4]. The main point to be made here is that $\mathcal{T}$ resp. $\mathcal{T}_w$, up to the notations, are nothing else but the variety $\pi^{-1}\mathfrak{k}^{\perp}$ resp. the conormal bundle $T^*_{Z_w}Z$ of 2.4, for the special case of Harish-Chandra bimodules introduced in 3.4, 3.5.

In fact, to clarify this identification notationally, recall first that the action of a group A on a smooth variety V defines a *moment map* $\pi_V = \pi_{V,A}$ from the cotangent bundle of V into the dual of Lie A (cf. [BoB], § 2). For the case of the complete homogeneous G-space X, the flag variety, we are using the notation $\pi = \pi_X = \pi_{X,G}$ already since § 1. The product map $\pi \times p$: $T^*X \to \mathfrak{g}^* \times X$, where p: $T^*X \to X$ is the bundle map, *identifies* T^*X with the closed subvariety $\tilde{\mathcal{N}} \subset \mathfrak{g} \times X$ of pairs (x, B_1) of a Borel subgroup B_1 and a nilpotent element x in $\mathfrak{b}_1 = \operatorname{Lie} B_1$. Denoting $\mathcal{N} \subset \mathfrak{g}$ the variety of all nilpotent elements, and $\tilde{\mathcal{N}}$ that variety of pairs, the map $\tilde{\mathcal{N}} \to \mathcal{N}$ which forgets B_1 is the well-known interpretation of the moment map π: $T^*X \to \mathfrak{g}^*$ as "*Springer's resolution*" (notation [BoM, cf. S, St2]).

Replacing in this consideration G by $G \times G$, X by $Z = X \times X$, and π by $\pi_Z = \pi_{Z, G \times G}$: $T^*Z \to \mathfrak{g} \times \mathfrak{g}$, we obtain an identification of T^*Z with the closed variety of quadruples (x_1, x_2, B_1, B_2) such that $\mathfrak{b}_i = \operatorname{Lie} B_i$, $x_i \in \mathfrak{b}_i^{\perp}$ $(i = 1, 2)$, and an interpretation of π_Z as the map forgetting both B_1 and B_2. Observe that

$$\mathfrak{k} = \{(x, x) | x \in \mathfrak{g}\}, \quad \mathfrak{k}^{\perp} = \{(x, -x) | x \in \mathfrak{g}\},$$

and therefore, under the above identification with quadruples:

$$\pi_Z^{-1}\mathfrak{k}^{\perp} = T^*Z \cap (\mathfrak{k}^{\perp} \times Z) = \{(x, -x, B_1, B_2) | x \in \mathfrak{b}_1^{\perp} \cap \mathfrak{b}_2^{\perp}\}.$$

Now it is obvious, that the map forgetting x_2 of the quadruple (x_1, x_2, B_1, B_2) will identify $\pi_Z^{-1}\mathfrak{k}$ with Steinberg's triple variety $\mathcal{T}$.

At a particular point $z \in Z_w$, say $z = (B_1, B_2) = (B^w, B)$, the cotangent space is $\mathfrak{b}_1^{\perp} \times \mathfrak{b}_2^{\perp} \times \{z\}$, and so the conormal subspace of the K-orbit Z_w in Z at z is the

[3] We understand from letters of V. Ginsburg, as well as from Kashiwara-Tanisaki [TK], that these authors have independently proved equivalent results

[4] As pointed out by one of the referees, the full variety $\mathcal{T}$ makes its first appearance in the literature in Kazhdan-Lusztig [KL2], see also Springer [S2]. The pieces corresponding to a single conjugary class of the analogue of $\mathcal{T}$ with unipotent elements of G in place of nilpotent elements of $\mathfrak{g}$ however, were first considered resp. systematically studied by Steinberg in [St2, p. 133 resp. St1]

set of $(x, -x, z)$ with $x \in \mathfrak{b}_1^{\perp} \cap \mathfrak{b}_2^{\perp} = \mathfrak{b}^{w\perp} \cap \mathfrak{b}^{\perp} = \mathfrak{n}^w \cap \mathfrak{n}$. Under the above identification with a subvariety of $\mathcal{T}$, this becomes $\mathfrak{n}^w \cap \mathfrak{n} \times \{z\}$, which is also the conormal space of X_w^l in X^l at z. This proves the following

Lemma. *The restriction of a conormal bundle*

$$\mathcal{T}_w = T^*_{Z_w}Z \qquad (w \in W)$$

to the left copy X^l of X in Z coincides with $T^*_{X_w}X$:

$$X^l \times_Z T^*_{Z_w}Z = (X^l \times_Z T^*Z) \cap \mathcal{T}_w = T^*_{X_w}X.$$

The same arguments show for the restriction of *algebraic cycles* to X^l resp. X^r that

$$X^l \times_Z \sum_{w \in W} a_w[\overline{\mathcal{T}_w}] = \sum_{w \in W} a_w[\overline{T^*_{X_w}X}]$$

resp.

$$X^r \times_Z \sum_{w \in W} a_w[\overline{\mathcal{T}_w}] = \sum_{w \in W} a_w[\overline{T^*_{X_{w^{-1}}}X}].$$

4.7. Varieties of Harish Chandra bimodules

Proposition. *Let H be a finitely generated $(\mathfrak{g} \times \mathfrak{g}, K)$-module with trivial central character, and $\mathcal{H}$ the corresponding $(\mathcal{D}_Z, K)$-module (3.5). Then:*

a) There are non-negative integers $m_w(\mathcal{H})$ $(w \in W)$ such that

$$\mathcal{Ch}(\mathcal{H}) = \sum_{w \in W} m_w(\mathcal{H})[\overline{\mathcal{T}_w}].$$

In this equality, we consider $\mathcal{Ch}(\mathcal{H})$ as a cycle on $\mathcal{T}$ rather than on T^*Z.

b) There is a subset $\Sigma(H) \subset W$ such that

$$\operatorname{Ch}(\mathcal{H}) = \bigcup_{w \in \Sigma(H)} \overline{\mathcal{T}_w}.$$

c) The associated variety of H in $\mathfrak{k}^{\perp}$ (identified with $\mathfrak{g}$ by $(x, -x) \mapsto x$) is then given by

$$V(H) = \bigcup_{w \in \Sigma(H)} \overline{G(\mathfrak{n} \cap \mathfrak{n}^w)}.$$

d) Let $\mathcal{L} = i_l^*(\mathcal{H})$ denote the corresponding $(\mathcal{D}_X, B)$-module (cf. 3.6). Then $\mathcal{Ch}(\mathcal{L})$, as an algebraic cycle on T^*X^l, is the pull-back of $\mathcal{Ch}(\mathcal{H})$ by the natural map $T^*X^l = X^l \times_Z \mathcal{T} \to \mathcal{T}$. Let us express this formally by the equality

$$\mathcal{Ch}(\mathcal{L}) = X^l \times_Z \mathcal{Ch}(\mathcal{H})$$

(Similarly for l replaced by r.)

Proof. As in 4.2, statements a), b), c) are a special case of 2.6. For the proof of d), let $\imath_l$: $T^*X \hookrightarrow \mathcal{T}$ denote the inclusion of T^*X as a "left copy" T^*X^l into the triple variety $\mathcal{T}$, given by $\imath_l$ $(B_1, x) = (B_1, B, x)$ (Notation 4.6). Consider the functor $\imath_l^*$ restricting coherent $\mathcal{O}$-modules $\mathcal{F}$ from $\mathcal{T}$ to T^*X^l. Since the intersection $Z_w \cap X^l = X^l_w$ is transversal for all w (3.5), the variety T^*X^l to which $\mathcal{F}$ is restricted meets each $\mathcal{T}_w$ transversally in the generic part of $\mathcal{T}_w$. Hence if $\mathcal{F}$ has support $\mathcal{S}upp\,\mathcal{F} = \sum_{w \in W} m_w[\overline{\mathcal{T}_w}]$, then the multiplicities m_w are preserved by the functor i_l^*:

$$\mathcal{S}upp\, i_l^* \mathcal{F} = \sum_{w \in W} m_w [\overline{T^*_{X_w} X}].$$

Now apply this to the case $\mathcal{F} = \operatorname{gr} \mathcal{H}$, where $\operatorname{gr}\mathcal{H}$ is taken with respect to some good filtration on $\mathcal{H}$. With respect to the induced filtration on $i_l^*(\mathcal{H})$, we have obviously:

$$\operatorname{gr} i_l^*(\mathcal{H}) = i_l^*(\operatorname{gr} \mathcal{H}).$$

Now (cf. Lemma 4.6):

$$\begin{aligned}\mathcal{C}h(\mathcal{L}) &= \mathcal{S}upp(\operatorname{gr}\mathcal{L}) = \mathcal{S}upp(i_l^*(\operatorname{gr}\mathcal{H}))\\ &= \sum_{w\in W} m_w(\mathcal{H})[\overline{T^*_{X_w} X}]\\ &= X^l \times \sum_{w \in W} m_w(\mathcal{H})[\overline{\mathcal{T}_w}] = X^l \times_Z \mathcal{C}h(\mathcal{H}).\end{aligned}$$

Q.e.d.

4.8. *Varieties of simple Harish-Chandra bimodules*

Let us now consider the simple Harish-Chandra bimodules H_w resp. the corresponding $\mathcal{D}$-modules $\mathcal{H}_w$ (Notation 3.5).

Theorem. *For all Weyl group elements $w \in W$, we have:*

a) *The characteristic cycles of $\mathcal{H}_w$ and $\mathcal{L}_w$ have the same multiplicities, i.e.* (Notation 4.2, 4.7)

$$m_y(\mathcal{H}_w) = m_y(\mathcal{L}_w) \qquad \textit{for all } y \in W.$$

a') $$m_y(\mathcal{H}_w) = m_{y^{-1}}(\mathcal{L}_{w^{-1}}) \qquad \textit{for all } y \in W.$$

b) *The characteristic varieties of $\mathcal{H}_w$ resp. $\mathcal{L}_w$ are given* (Notation 4.2, 4.7) *by the same subsets of W:*

$$\Sigma(\mathcal{H}_w) = \Sigma(\mathcal{L}_w).$$

b') $$\Sigma(\mathcal{H}_w) = \Sigma(\mathcal{L}_{w^{-1}})^{-1}.$$

c) *The associated variety of H_w is the G-saturation of that of L_w:*

$$V(H_w) = GV(L_w).$$

d) *There exists an element $v \in W$ such that*

$$V(H_w) = \overline{G(\mathfrak{n} \cap \mathfrak{n}^v)}$$

is the closure of a single nilpotent orbit.

Proof. Claim a) follows from Proposition 4.7 d) (cf. Lemma 4.6), since we have $i_l^*(\mathcal{H}_w) \cong \mathcal{L}_w$ by Proposition 3.6 e). Similarly, $i_r^*(\mathcal{H}_w) = \mathcal{L}_{w^{-1}}$ (3.6f) gives the equation a'). Now a) implies b), because by a):

$$\Sigma(\mathcal{H}_w) = \{y \mid m_y(\mathcal{H}_w) \neq 0\} = \{y \mid m_y(\mathcal{L}_w) \neq 0\} = \Sigma(\mathcal{L}_w).$$

Similarly, a') implies b').

Claim c) follows from the equation $\Sigma(\mathcal{H}_w) = \Sigma(\mathcal{L}_w)$ (part b)), using formula 4.7 c) for $V(H_w)$ and the corresponding formula 4.2 c) for $V(L_w)$. Here we use in addition that

$$\overline{G(B(\mathfrak{n} \cap \mathfrak{n}^w))} = \overline{G(\mathfrak{n} \cap \mathfrak{n}^w)}.$$

In fact, the left hand side is closed, because it is the G-closure of a closed affine variety stable under some parabolic subgroup of G (use [St2], p. 68, Lemma 2). Finally, d) is an immediate consequence of c) and the Hotta-Joseph result formulated in 4.5, Remark 5. Q.e.d.

Remark 1. For $\mathfrak{g} = \mathfrak{sl}_n$, Theorem 4.8d) holds with $v = w$, i.e. $V(H_w) = \overline{G(\mathfrak{n} \cap \mathfrak{n}^w)}$, see 6.8. However, note that 4.8d) does not always hold with $v = w$, as is shown by the example 4.4, where $\mathfrak{g} = \mathfrak{sp}_4$.

Remark 2. One may as well consider the associated variety of the $\mathfrak{g}^l$-module H^l resp. the $\mathfrak{g}^r$-module H^r (notation 3.4), arising from H by forgetting part of the structure. So a priori, we would have to distinguish *three* associated varieties of a Harish-Chandra bimodule: One in $\mathfrak{g}^l$, one in $\mathfrak{g}^r$, and one in $\mathfrak{k}^\perp \subset \mathfrak{g} \times \mathfrak{g}$. However, identifying in this context $\mathfrak{g}^l$ resp. $\mathfrak{g}^r$ resp. $\mathfrak{k}^\perp$ with $\mathfrak{g}$ by making $x \in \mathfrak{g}$ correspond to (x, o) resp. $(o, -x)$ resp. $(x, -x)$, the difference between these three notions of associated varieties disappears completely:

$$V_{\mathfrak{g}^l}(H^l) = V_{\mathfrak{g}^r}(H^r) = V_{\mathfrak{g} \times \mathfrak{g}}(H) =: V(H).$$

This justifies our notation $V(H)$ for all of them. Note, however, that the situation becomes more delicate on the more refined level of *characteristic* varieties (cf. 5.1.).

4.9. *G-saturation theorem for associated varieties of Harish-Chandra modules*

Corollary. *Let H be any finitely generated Harish-Chandra module with trivial central character, and L the corresponding $(\mathfrak{g}, B)$-module* (cf. 3.4). *Then $V(H) = GV(L)$.*

Proof. Since H has a finite Jordan-Hölder series with simple quotients $H_{w_1}, \ldots, H_{w_r}$ for suitable $w_1, \ldots, w_r \in W$, and L has (by 3.4) the corresponding composition factors $L_{w_1}, \ldots, L_{w_r}$, it follows from 4.8 c) that

$$V(H) = \bigcup_{1 \le i \le r} V(H_{w_i}) = \bigcup_{1 \le i \le r} GV(L_{w_i}) = GV(L).$$

Q.e.d.

4.10. **Corollary.** *For each $w \in W$ the primitive ideal $J_w = \operatorname{Ann} L_w$ has associated variety*

$$V(U(\mathfrak{g})/J_w) = GV(L_w).$$

Proof. Considering $H = U(\mathfrak{g})/J_w$ as a Harish-Chandra bimodule (cf. Example 3.4), we have

$$V(H) = V(H^l) = V(H^r),$$

where we identify the left, right, and diagonal copy of $\mathfrak{g}$ (3.4) in the obvious way with $\mathfrak{g}$ $(=\mathfrak{g}^*)$. Since L_w is a homomorphic image of the left $\mathfrak{g}$-module H^l, we have $V(L_w) \subset V(H)$, and so by G-invariance of $V(H)$, the inclusion

$$V(H) \supset GV(L_w)$$

in the corollary is clear. To prove the converse inclusion, we embed H, as a left $\mathfrak{g}$-module H^l, into a finite direct sum of copies of H_w, which is possible by the lemma below. Now Theorem 4.8 c) gives the desired result:

$$V(H) \subset V(H_w) = GV(L_w). \quad \text{Q.e.d.}$$

Lemma. *Let M be a $\mathfrak{g}$-bimodule. Assume the right module M^r finitely generated. Denote the annihilator of the left module M^l by J. Then $U(\mathfrak{g})/J$, as a left $\mathfrak{g}$-module, embeds into a finite direct sum of copies of M.*

This easy (but basic) observation was already used in [V2], p. 199, and in [Bo2], 2.2 for similar considerations about GK-dimensions. One just takes generators $m_1, \dots, m_s$ of M as a right module, and embeds $U(\mathfrak{g})/J$ into the direct sum $M \otimes \dots \otimes M$ (s copies) by mapping u onto $um_1 \otimes \dots \otimes um_s$.

4.11. **Corollary.** *For each primitive ideal with trivial central character, the associated variety is the closure of a nilpotent orbit.*

Proof. This follows from combining Theorem 4.8 c) and d) with Corollary 4.10. Q.e.d.

Remark. 1) This result was first suggested in [Bo], 2.9 and was proved in [BoB], using case by case arguments. It was observed independently by Joseph [J4], and also by Kashiwara-Tanizaki [KT], that our "G-saturation theorem" (Corollary 4.10), if combined with the Joseph-Hotta results (4.5, Remark 5) provides a unified proof of this result, as given above.

2) Our present method of proof for this result extends to arbitrary central characters as well, as does Joseph's [J4].

§5. Refined results on characteristic varieties and cycles

5.1. Left and right characteristic varieties

Let us introduce first some convenient abuses of language. For any $\mathfrak{g}$-module M with trivial central character, we define

$$\operatorname{Ch}(M) := \operatorname{Ch}(\mathscr{M}) \subset T^*X,$$

where $\mathscr{M}$ is the localization of M on the flag variety X. For a Harish-Chandra bimodule H of finite type the characteristic variety $\operatorname{Ch}(H) \subset T^*Z$ in this sense is called its *two-sided* characteristic variety. We know that this is a union of components of Steinberg's triple variety $\mathscr{T} \subset T^*Z$ (recall 4.6, 4.7). We define the *left* resp. *right* characteristic variety of H by forgetting the structure of H on one side, i.e. by (Notation 3.4)

$$\operatorname{Ch}^l(H) := \operatorname{Ch}(H^l) \quad \text{resp.} \quad \operatorname{Ch}^r(H) := \operatorname{Ch}(H^r).$$

These are closed subvarieties of T^*X (of T^*X^l resp. T^*X^r). We use a similar terminology and notation for *characteristic cycles:* $\mathscr{C}\mathscr{h}(H)$, $\mathscr{C}\mathscr{h}^l(H)$, $\mathscr{C}\mathscr{h}^r(H)$ etc. The left, right, two-sided characteristic variety resp. cycle of a *primitive ideal* J with trivial central character is defined to be that of the Harish-Chandra bimodule $H = U(\mathfrak{g})/J$. We shall often formulate statements about *left* characteristic varieties only, the right hand analogue being understood. Our first goal is to relate these different notions of characteristic variety by means of the following proposition to be proved below (cf. 5.4, 5.5). Note that $T^*Z = T^*(X \times X) = T^*X^l \times T^*X^r$. We denote the projection onto the left resp. right hand factor by π^l resp. π^r.

Proposition. *For a finitely generated Harish-Chandra bimodule H with trivial central character, the two-sided characteristic variety projects onto the left one:*

$$\operatorname{Ch}^l(H) = \pi^l(\operatorname{Ch}(H)).$$

This will follow from Proposition 5.5 c), in combination with Proposition 5.4 below.

5.2. Projection of Steinberg's triple variety

Since the characteristic varieties $\operatorname{Ch}(H)$ as above are contained in the subvariety $\pi_Z^{-1}\mathfrak{k}^\perp = \mathscr{T}$ of T^*Z (Notation 4.6), we are interested only in the restriction $\dot{\pi}^l$ of π^l to this subvariety. Recall that a point in $\mathscr{T}$ is interpreted as a *triple* (x, B_1, B_2) of two Borel subgroups and a nilpotent element in the intersection of their Lie algebras, and that a point in T^*X is interpreted similarly as a pair (x, B_1), see 4.6. Then we observe that $\dot{\pi}^l$ resp. $\dot{\pi}^r$ are just the maps $\mathscr{T} \to T^*X$ which forget B_2 resp. B_1.

Note the *diagram of forgetful maps:*

$$\begin{tikzcd} & T^*Z & \\ & \bigcup & \\ T^*X \arrow[d, "\pi"] & \mathscr{T} \arrow[l, "\dot{\pi}^l"'] \arrow[r, "\dot{\pi}^r"] \arrow[d, "\dot{\pi}_Z"] & T^*X \arrow[d, "\pi"] \\ \mathfrak{g} \arrow[r, phantom, "="] & \mathfrak{k}^\perp \arrow[r, phantom, "="] & \mathfrak{g} \end{tikzcd}$$

where $\pi=\pi_X$ resp. π_Z are the moment maps of X resp. Z, and the dots mean restriction to triples. From this description it becomes obvious that π^l is a proper map, since its fibers are projective (being closed subvarieties of a copy of X, which is complete). Let us point this out, as a key lemma for 5.1:

Lemma. *The restriction of π^l resp. π^r to the triple variety $\mathcal{T}$ is a proper map.*

5.3. *Conormal bundles of position varieties*

Now let us consider the subvariety $\mathcal{T}_w \subset \mathcal{T}$ of those triples (x, B_1, B_2) with B_1 and B_2 in a given position w (notation 4.6). We may fiber $\mathcal{T}_w$ over X in two different ways: First over the right copy of X in Z by the map forgetting (x, B_1), second over the left copy of X in Z by the map forgetting (x, B_2). The fiber over $B_2=B$ is $T^*_{X_w}X$ in the first case, and over $B_1=B$ it is $T^*_{X_{w^{-1}}}X$ in the second case, as we already observed in Lemma 4.6. Since these projection maps are K-equivariant morphisms, and the action is transitive on the base X, we may describe these fibrations as associated fiber bundles. Notationally, let us write

$$T^*_{X_w}X \times^B G \qquad \text{resp.} \quad G \times^B T^*_{X_{w^{-1}}}X$$

for the first resp. second of these fibrations of $\mathcal{T}_w$, in order to keep the sides in mind without confusing indices.

What is the effect of our projection π^l on these fibrations? Since π^l forgets B_2, in case of the first fibration it collapses the base to a point, but maps the fibers isomorphically to $T^*_{X_w}X$ resp. its G-conjugates in T^*X, while in case of the second fibration, it maps the base isomorphically, but projects the fibers like the moment map π to $\pi(T^*_{X_{w^{-1}}}X)=B(\mathfrak{n}\cap\mathfrak{n}^{w^{-1}})$ (cf. Lemma 4.1 b) resp. its G-conjugates. More precisely, one can verify in this way that $\pi^l(\mathcal{T}_w)$ is the G-saturation of $T^*_{X_w}X$ in T^*X (first case) resp. the associated fiber bundle $G\times^B B(\mathfrak{n}\cap\mathfrak{n}^{w^{-1}})$ (second case). – Let us summarize our results in the following lemma.

Lemma. *Let $\mathcal{T}_w = T^*_{Z_w}Z$ be the conormal bundle of a position variety* (K-orbit) $Z_w (w\in W)$.

a) *$\mathcal{T}_w$ is fibered by conormal bundles of Schubert cells in two different ways, viz.*

$$\mathcal{T}_w = T^*_{X_w}X \times^B G \qquad \textit{resp.} \quad G\times^B T^*_{X_{w^{-1}}}X.$$

b) *Under π^l resp. π^r, $\mathcal{T}_w$ maps onto the G-saturation of the conormal bundle of X_w resp. $X_{w^{-1}}$, viz.*

$$\pi^l(\mathcal{T}_w) = GT^*_{X_w}X \qquad \textit{resp.} \quad \pi^r(\mathcal{T}_w)=GT^*_{X_{w^{-1}}}X.$$

c) *These images have alternative descriptions as associated fiber bundles, viz.*

$$\pi^l(\mathcal{T}_w) = G\times^B B(\mathfrak{n}\cap\mathfrak{n}^{w^{-1}}) \qquad \textit{resp.}$$
$$\pi^r(\mathcal{T}_w) = B(\mathfrak{n}\cap\mathfrak{n}^{w})\times^B G.$$

Complement. For the Zariski-closure of $\mathcal{T}_w$ we have

a') $$\bar{\mathcal{T}}_w = \overline{T^*_{X_w}X} \times^B G$$

b') $$\pi^l(\bar{\mathcal{T}}_w) = G\overline{T^*_{X_w}X}$$

c') $$\pi^r(\bar{\mathcal{T}}_w) = \overline{B(\mathfrak{n}\cap\mathfrak{n}^w)}\times^B G.$$

This complement follows from the Lemma by using Lemma 5.2, and the fact that the G-saturation of a B-stable closed subvariety is closed ([St2], p. 68, Lemma 2).

5.4. *Projecting $(\mathcal{D}_Z, K)$-modules to $\mathcal{D}_X$-modules*

The projection of $Z = X\times X$ onto the left resp. right factor is denoted p^l resp. p^r. The (ordinary sheaf theoretic) direct image $p^l_*(\mathcal{H})$ of a coherent $(\mathcal{D}_Z, K)$-module $\mathcal{H}$ is a $\mathcal{D}_X$-module. While $\mathcal{H}$ is holonomic (2.3), its "projection" $p^l_*(\mathcal{H})$ is not in general, but *it is coherent*. This can be seen either directly by exhibiting a good filtration (as we do in 5.5), or by interpreting the functor p^l_* in terms of $\mathfrak{g}$-modules, as we do here.

Proposition. *The following diagram of functors commutes* (in the sense that the composed functors are naturally equivalent):

$$\begin{array}{ccc} \mathcal{M}od(\mathcal{D}_Z, K)^{\mathrm{coh}} & \xrightarrow{p^l_*} & \mathcal{M}od(\mathcal{D}_X)^{\mathrm{coh}} \\ \downarrow \approx & & \downarrow \approx \\ \mathrm{Mod}(\mathfrak{g}\times\mathfrak{g}, K)^{\mathrm{fg}}_Z & \longrightarrow & \mathrm{Mod}(\mathfrak{g})^{\mathrm{fg}}_X \end{array}$$

where the vertical arrows are Beilinson-Bernstein equivalences (i.e. the global section functors), *and the horizontal arrows are the projection functor $\mathcal{H}\mapsto p^l_*(\mathcal{H})$, resp. the forgetful functor $H\mapsto H^l$* (Notation 3.4).

Proof. For any $\mathcal{D}_Z$-coherent $(\mathcal{D}_Z, K)$-module $\mathcal{H}$, and any open subset $U\subset X$, we have an isomorphism

$$\Gamma(U, p^l_*(\mathcal{H}) \cong \Gamma((p^l)^{-1}(U), \mathcal{H})$$

by definition of p^l_*. In particular, we have canonically

$$\Gamma(X, p^l_*(\mathcal{H})\cong\Gamma(Z,\mathcal{H}) =: H, \tag{*}$$

at least as vector spaces. But since $p^l\colon Z\to X$ is an equivariant map for the left action of $G^l = G$ on Z, this linear isomorphism (*) must be compatible with the corresponding action of $\mathfrak{g}^l=\mathfrak{g}$. Now (*) gives the commutativity claimed in the proposition. – Now let H be generated as a $\mathfrak{g}\times\mathfrak{g}$-module by the finite-dimensional subspace. E. Then E may be chosen as a $\mathfrak{k}$-submodule, by the K-

equivariance of $\mathscr{H}$. But then $H=U(\mathfrak{g}\times\mathfrak{g})E=U(\mathfrak{g}^l)\,U(\mathfrak{k})E=U(\mathfrak{g}^l)E$, i.e. the $\mathfrak{g}^l$-module H^l is finitely generated by E. Therefore, $p^l_*(\mathscr{H})$ must be coherent as the localization of the finitely generated $\mathfrak{g}$-module H^l (see 1.9). Q.e.d.

5.5. *Characteristic variety of the projected $(\mathscr{D}_Z, K)$-module*

Proposition. *Let $\mathscr{H}$ be a $(\mathscr{D}_Z, K)$-module.*

a) *A good filtration on $\mathscr{H}$ induces a good filtration on $p^l_*(\mathscr{H})$.*
b) *With respect to such filtrations, we have a graded monomorphism*

$$\operatorname{gr} p^l_*(\mathscr{H}) \hookrightarrow \pi^l_*(\operatorname{gr}\mathscr{H}). \tag{$*$}$$

c) *The characteristic variety of the projected module is given by*

$$\operatorname{Ch}(p^l_*(\mathscr{H}))=\pi^l(\operatorname{Ch}(\mathscr{H})).$$

Proof. The proof is parallel to that of Theorem 1.9c). In more detail, we first derive $(*)$ as a graded monomorphism of quasi-coherent graded $\mathscr{O}_{T^*X}$-modules exactly as 1.6(1), (3). Next we use that the filtration on $\mathscr{H}$ is good and so $\operatorname{gr}\mathscr{H}$ is coherent, and is supported only on the triple variety $\mathscr{T}$, cf. 4.7a). Since π^l restricts to a proper map of $\mathscr{T}$ by Lemma 5.2, we may invoke again Grauert's theorem (as in 1.6) to conclude that $\pi^l_*(\operatorname{gr}\mathscr{H})$ is also coherent. Now the monomorphism $(*)$ implies that $\operatorname{gr} p^l_*(\mathscr{H})$ is coherent as well, so that the filtration induced on $p^l_*(\mathscr{H})$ is good. This completes the proof of a) and b), and gives one inclusion in c). – In order to establish the reverse inclusion,

$$\pi^l(\operatorname{Ch}(\mathscr{H}))\subset\operatorname{Ch}(\mathscr{M}) \tag{$**$}$$

for the $\mathscr{D}_X$-module $\mathscr{M}=p^l_*(\mathscr{H})$, we proceed as in Proposition 1.8, combined with Theorem 1.9, as follows. Because of the K-equivariance of $\mathscr{H}$ and the transitivity of G on X^l the Beilinson-Bernstein theory on a fiber of p^l (which is X^r) shows that $\mathscr{H}$ may be reconstructed from $\mathscr{M}$ as $\mathscr{H}=\mathscr{O}_Z\mathscr{M}'$, where $\mathscr{M}'$ is the sheaf obtained from $\mathscr{M}$, which is constant along the fibers of $p^l\colon Z\to X$, that is $(p^l)^{-1}(\mathscr{M})$. Now one starts from a good filtration of $\mathscr{M}$ by $\mathscr{O}_X$-submodules $\mathscr{M}_j$, and observes that the "induced" filtration on $\mathscr{H}$ by $\mathscr{O}_Z$-submodules $\mathscr{H}_j=\mathscr{O}_Z\mathscr{M}'_j$ is also good (parallel to 1.8), so these filtrations may be used for computing characteristic varieties. On the other hand, it is trivial that the principal symbol of an operator in $\mathscr{D}_X$ which annihilates $\operatorname{gr}\mathscr{M}$, also annihilates $\operatorname{gr}\mathscr{H}$ (parallel to Lemma 1.8), and this then proves the desired inclusion $(**)$, as in 1.8. Q.e.d.

Remark. We shall refine part b) of the proposition in the Appendix A.

5.6. *G-saturation theorem for characteristic varieties*

We have now assembled all ingredients to show that the left characteristic variety of a Harish-Chandra bimodule is the *G-saturation* of the characteristic variety of a corresponding highest weight module.

Theorem. *Let H be a finitely generated Harish-Chandra bimodule with trivial central character. Then*

$$\operatorname{Ch}^l(H)=G\operatorname{Ch}(L),$$

where $L=M(0)^\vee\otimes_{U(\mathfrak{g}^r)}H$ is the corresponding $(\mathfrak{g},B)$-module (cf. 3.4).

Proof. We have

$$\begin{aligned}\operatorname{Ch}^l(H)&=\operatorname{Ch}(H^l)=\operatorname{Ch}(p^l_*(\mathscr{H}))=\pi^l(\operatorname{Ch}(\mathscr{H}))\\&=\pi^l\Big(\bigcup_{w\in\Sigma}\bar{\mathscr{T}}_w\Big)=\bigcup_{w\in\Sigma}\pi^l(\bar{\mathscr{T}}_w)=\bigcup_{w\in\Sigma}G\overline{T^*_{X_w}X}\\&=G\bigcup_{w\in\Sigma}\overline{T^*_{X_w}X}=G\operatorname{Ch}(\mathscr{L})=G\operatorname{Ch}(L)\end{aligned}$$

by 5.4, 5.5c), 4.7c), 5.3b'), 4.2b). Here $\mathscr{L}$ denotes the localization of L on X, and the point is that $\Sigma:=\Sigma(\mathscr{H})=\Sigma(\mathscr{L})$ by Theorem 4.8a) (resp. 4.7d)). Q.e.d.

5.7. **Corollary.** *For each $w\in W$, the primitive ideal $J_w=\operatorname{Ann}L_w$ has left characteristic variety*

$$\operatorname{Ch}^l(U(\mathfrak{g})/J_w)=\operatorname{Ch}^l(H_w)=G\operatorname{Ch}(L_w).$$

Proof. The latter equality follows directly from the theorem and the definition of H_w (3.4). The left characteristic variety of $U(\mathfrak{g})/J_w$ is contained in that of H_w, since – as left $\mathfrak{g}$-modules – the former embeds into a finite direct sum of copies of the latter (Lemma 4.10). Finally, the characteristic variety of the left module $U(\mathfrak{g})/J_w$ contains that of its homomorphic image L_w, and so contains also the G-saturation $G\operatorname{Ch}(L_w)$, because of its G-invariance. Q.e.d.

Remark. It follows from the above argument that for any finitely generated Harish-Chandra bimodule H (with trivial central character), the *left characteristic variety depends only on the left annihilator of H.*

5.8. **Corollary.** *Considered as a subset in $G\times^B\mathfrak{n}=T^*X$, the left characteristic variety of the primitive ideal J_w is the associated fiber bundle*

$$\operatorname{Ch}^l(U(\mathfrak{g})/J_w)=G\times^B V(L_{w^{-1}}). \tag{$*$}$$

Proof. Returning to the proof of Theorem 5.6, we conclude that

$$\begin{aligned}\operatorname{Ch}^l(U(\mathfrak{g})/J_w)&=G\operatorname{Ch}(L_w)=\bigcup_{w\in\Sigma}G\overline{T^*_{X_w}X}\\&=\bigcup_{w\in\Sigma}G\times^B\overline{B(\mathfrak{n}\cap\mathfrak{n}^{w^{-1}})}\\&=G\times^B\Big(\bigcup_{w\in\Sigma^{-1}}\overline{B(\mathfrak{n}\cap\mathfrak{n}^{w})}\Big)=G\times^B V(L_{w^{-1}}),\end{aligned}$$

using Lemma 5.3 b') and c') (l-version) plus 4.2c). Here the point is that (by Theorem 4.8)

$$\Sigma^{-1}=\Sigma(L_w)^{-1}=\Sigma(L_{w^{-1}}).\quad \text{Q.e.d.}$$

Remark. Similarly, we can see by using 5.3a') that the two-sided characteristic variety, as a subset in $T^*X\times^B G$ resp. $G\times^B T^*X=T^*Z$, has a fiber-bundle description as

$$\mathrm{Ch}(U(\mathfrak{g})/J_w)=\mathrm{Ch}(L_w)\times^B G \qquad \text{resp. } G\times^B \mathrm{Ch}(L_{w^{-1}}).$$

This description does even hold for characteristic cycles:

$$\mathscr{C}h(U(\mathfrak{g})/J_w)=\mathscr{C}h(L_w)\times^B G=G\times^B \mathscr{C}h(L_{w^{-1}})$$

as is clear from 4.7 d), 4.6. The description of the "left characteristic cycle" $\mathscr{C}h^l(U(\mathfrak{g})/J_w)$ by means of a similar refinement of the above corollary is, however, a more delicate problem, to which we now turn.

5.9. G-Saturation Theorem for Characteristic Cycles

Theorem. *For a finitely generated Harish-Chandra bimodule H with trivial central character, the left resp. right characteristic cycle is given by*

$$\mathscr{C}h^l(H)=G\times^B \mathscr{V}(L^r) \qquad \textit{resp. } \mathscr{C}h^r(H)=\mathscr{V}(L^l)\times^B G,$$

where L^r resp. L^l denote the $(\mathfrak{g},B)$-modules corresponding to H by 3.4, *viz.*

$$L^r=M(0)^\vee\otimes_{U(\mathfrak{g}^l)}H \qquad \textit{resp. } L^l=M(0)^\vee\otimes_{U(\mathfrak{g}^r)}H.$$

The explicit meaning of these formulae (of the first, say) is that

$$\mathscr{V}(L^r)=m_1[V_1]+\ldots+m_n[V_n]$$

implies

$$\mathscr{C}h^l(H)=m_1[G\times^B V_1]+\ldots+m_n[G\times^B V_n].$$

Or in more detail:

a) If $V_1,\ldots,V_n$ are the different irreducible components of $V(L^r)$, then the different irreducible components of $\mathrm{Ch}^l(H)$ are $C_i=G\times^B V_i$ with $i=1,\ldots,n$.

b) If V_i occurs with multiplicity m_i in the associated cycle of L^r, then C_i occurs with the same multiplicity m_i in the left characteristic cycle of H.

The *proof* of the merely set theoretic statement a), and of the equality of $\mathrm{Ch}^l(H)$ and $G\times^B V(L^r)$ as sets, are clear from what has been explained so far, using the same arguments as in the proof of 5.8. What remains to be proven is the equality of multiplicities stated in b). For this purpose, let us denote by

$$f_x:\mathfrak{n}\hookrightarrow T^*X$$

the inclusion of $\mathfrak{n}=\mathfrak{g}_x^\perp=(\mathfrak{g}/\mathfrak{b})^*$ (notation [BoB], 2.2) into the cotangent bundle of X as the fibre at the base point $x=B$. We note that $\mathscr{C}h^l(H)$ is obviously a *G-equivariant* cycle on T^*X, and that the restriction map f_x^* will send such a cycle to a B-equivariant one on $\mathfrak{n}$. Conversely, a B-equivariant cycle on $\mathfrak{n}$ extends to a unique G-equivariant one on the associated fibrebundle $G\times^B\mathfrak{n}=T^*X$. Therefore it suffices to prove the equality with multiplicities (1), which is an immediate consequence of Proposition 5.10 c) below;

$$f_x^*\mathscr{C}h^l(H)=\mathscr{V}(L^r). \tag{1}$$

5.10. Retaining all notations of 5.9, let $\mathscr{H}$ resp. $\mathscr{L}^l$ resp. $\mathscr{L}^r$ denote the localizations of H resp. L^l resp. L^r on Z resp. X^l resp. X^r. Then we have, recalling 3.6,

$$i_r^*(\mathscr{H})=\mathscr{L}^r \quad \text{and} \quad i_l^*(\mathscr{H})=\mathscr{L}^l.$$

Given a good filtration on $\mathscr{H}$ by coherent $\mathcal{O}_Z$-submodules $\mathscr{H}_n$ $(n\in\mathbb{Z})$, we shall consider (as we already did in 4.7) the filtration "induced" on $\mathscr{L}^r$ by the definition

$$\mathscr{L}_n^r:=i_r^*(\mathscr{H}_n) \qquad \text{for all } n\in\mathbb{Z}.$$

If all sheaves occuring in some filtration are equivariant under some group A, let us say – by abuse of language – that the *filtration* is *A-equivariant*. Our object is to prove 5.9 (1), which may now be rewritten as (cf. 5.4)

$$f_x^*\mathscr{C}h(p_*^l\mathscr{H})=\mathscr{V}(\Gamma(X,\mathscr{L}^r)).$$

For this purpose, it will be convenient to work in the categories of G-equivariant coherent sheaves on T^*X, resp. of B-equivariant sheaves on $\mathfrak{n}$, with support in $\mathscr{C}h^l(\mathscr{H})$ resp. $V(L^r)$. Being only interested in multiplicities means that it suffices to consider the corresponding Grothendieck groups.

Proposition. a) *$\mathscr{H}$ admits a K-equivariant good filtration.*

b) *Such a filtration induces a B-equivariant good filtration on $\mathscr{L}^r$.*

c) *With respect to these filtrations, there is an isomorphism*

$$f_x^*(\mathrm{gr}\,p_*^l\mathscr{H})\xrightarrow{\sim}\mathrm{gr}\,\Gamma(X,\mathscr{L}^r).$$

of graded B-equivariant $\mathcal{O}_\mathfrak{n}$-modules.

Here the right hand side is primarily a graded $S(\mathfrak{g})$-module supported on $\mathfrak{n}$, which is identified notationally with the corresponding sheaf on $\mathfrak{n}$, because $\mathfrak{n}$ is an affine variety.

Proof. a) $H=\Gamma(Z,\mathscr{H})$ is generated as a $\mathfrak{g}\times\mathfrak{g}$-module by a finite dimensional subspace $E\subset H$ by 2.3 a), which may be chosen K-invariant since H is K-finite. With such a choice, all vector-spaces

$$H_n:=U_n(\mathfrak{g})E=EU_n(\mathfrak{g}) \qquad (n\in\mathbb{Z})$$

are K-equivariant, and this defines a good filtration of H (as a bimodule as well as a one-sided module). Then

$$\mathscr{H}_n=\mathcal{O}_Z H_n \qquad (n\in\mathbb{Z})$$

defines a filtration of $\mathscr{H}$ by K-equivariant $\mathcal{O}_Z$-subsheaves, which is *good* by 1.8. – Alternatively, one may (at least for $k=\mathbb{C}$) take the *canonical* (good) *filtration* of Kashiwara-Kawai [KK] to obtain a K-equivariant filtration on $\mathscr{H}$.

b) Now we restrict from Z to X^r. Using the same sort of reasoning as in 3.10, we see that the restriction functor i_r^* is *exact* on the category of K-equivariant coherent sheaves on Z. Therefore the $\mathscr{L}_n^r=i_r^*\mathscr{H}_n$ $(n\in\mathbb{Z})$ are subsheaves of $\mathscr{L}^r$, which are clearly B-equivariant and coherent. Moreover, we obtain an isomorphism of B-equivariant graded $\mathcal{O}_{T^*X}$-modules (cf. 4.7)

$$\mathrm{gr}\ \mathcal{L}^r \cong i_r^*(\mathrm{gr}\ \mathcal{H}).$$

c) Observe that X^r is the full fibre of the projection $p^l: Z \to X$ at the base point $x=B$ of X,

$$(p^l)^{-1}(x) = X^r.$$

We prove that the fibre of the coherent sheaf $p^l_* \mathcal{H}_n$ at the base point is equal to

$$\Gamma(X^r, \mathcal{L}^r_n) =: L^r_n \qquad (\text{for } n \in \mathbb{Z}).$$

By E.G.A. IV, 2ème partie, §6.9, the set $\{x \in X \text{ s.t. } \mathcal{H}_{n,y} \text{ is } \mathcal{O}_{X,x}\text{-flat for any } y \in (p^l)^{-1}(x)\}$ is non-empty ("Theorem of generic flatness"); this set is obviously G-equivariant, so is equal to X. Hence each $\mathcal{H}_n$ is $\mathcal{O}_X$-flat. Therefore, we may apply E.G.A. III, 2ème partie, §7.7. In our situation, the functor T_p ($p \in \mathbb{N}$) of loc. cit. is the following functor from $\mathcal{O}_X$-coherent sheaves to $\mathcal{O}_X$-coherent sheaves:

$$M \mapsto R^p p^l_*(p^{l,*} M \otimes_{\mathcal{O}_Z} \mathcal{H}_n).$$

Combining Théorème 7.7.5 and Proposition 7.7.10 of loc. cit., we see that the subset U of the scheme X, defined as the biggest open set of X such that for *any* morphism of schemes $g: U' \to U$ the natural morphism $T_p(\mathcal{O}_U) \otimes_{\mathcal{O}_U} \mathcal{O}_{U'} \xrightarrow{\sim} T_p^{U'}(\mathcal{O}_{U'})$ is an isomorphism, is non-empty, since it contains the generic point of X. Since U is G-invariant, $U = X$. Then for x the base point of X, we have:

$$(R^p p^l \mathcal{H}_n)_x \otimes_{\mathcal{O}_{X,x}} R(\{x\}) = H^p(X^r, \mathcal{H}_n \otimes_{\mathcal{O}_Z} \mathcal{O}_{X^r}) = H^p(X^r, \mathcal{L}^r_n).$$

For $p=0$, this says that $p^l \mathcal{H}_n$ has fibre L^r_n as claimed above. Since p^l converts the K-action on Z into the G-action on X, the coherent $\mathcal{O}_X$-modules $p^l_* \mathcal{H}_n$ here are G-equivariant, and the L^r_n are B-invariant subspaces of L^r. For each degree n we obtain a G-equivariant coherent $\mathcal{O}_X$-module

$$\mathrm{gr}_n p^l_* \mathcal{H} := p^l_* \mathcal{H}_n / p^l_* \mathcal{H}_{n-1}$$

with fibre at $x=B$ a B-module

$$\mathrm{gr}_n \Gamma(X, \mathcal{L}^r) = L^r_n / L^r_{n-1} = \mathrm{gr}_n L^r.$$

Taking direct sums over all degrees, we obtain a graded $\mathrm{gr}\,\mathcal{D}_X$-module $\mathrm{gr}\, p^l_* \mathcal{H}$, which is coherent and G-equivariant, and has fibre at $x=B$ equal to the graded $S(\mathfrak{n})$-module with B-action

$$\mathrm{gr}\, \Gamma(X, \mathcal{L}^r) = \mathrm{gr}\, L^r.$$

(Recall that $\mathrm{gr}\,\mathcal{D}_X = p_* \mathcal{O}_{T^*X}$ for p the bundle map [BoB], 1.2 so that its fibre at x is $S((TX)_x) = S(\mathfrak{n})$.) Now we consider $\mathrm{gr}\, p^l_* \mathcal{H}$ as an $\mathcal{O}_{T^*X}$-module as usual, and $\mathrm{gr}\, L_r$ as a sheaf on $\mathfrak{n}$. Then the above argument shows that the two sheaves on $\mathfrak{n}$ below have the same global sections, and are therefore isomorphic since $\mathfrak{n}$ is affine:

$$f_x^*(\mathrm{gr}\, p^l_* \mathcal{H}) \cong \mathrm{gr}\, \Gamma(X, \mathcal{L}^r). \quad \text{Q.e.d.}$$

§6. Applications to the classification of primitive ideals

In this chapter, J denotes a primitive ideal of $U(\mathfrak{g})$ with trivial central character. We fix an element $w \in W$ such that

$$J = J_w = \mathrm{Ann}\, L_w,$$

which is possible by Duflo's theorem. We consider

$$U_w := U(\mathfrak{g})/J_w$$

as a Harish-Chandra bimodule as in Example 3.4, and denote by

$$\mathcal{U}_w := \mathcal{D}_Z \otimes_{U(\mathfrak{g} \times \mathfrak{g})} U_w$$

its localization on Z. We keep the notations H_w *and* $\mathcal{H}_w$ resp. L_w and $\mathcal{L}_w$ for the simple modules as in §3. For simplicity, we assume $k = \mathbb{C}$ in this chapter.

6.1 *Left-right symmetry*

For a Harish-Chandra bimodule resp. its localization on Z, a superscript l denotes the corresponding left $\mathfrak{g}$-module resp. its localization on X, obtained by forgetting the right action. For example,

$$\mathcal{U}^l_w = \mathcal{D}_X \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})/J_w)^l.$$

Definition (cf. 5.1). The *l-characteristic variety* of the primitive ideal $J = J_w$ is, by definition,

$$\mathrm{Ch}^l(U(\mathfrak{g})/J_w) = \mathrm{Ch}(\mathcal{U}^l_w) = \mathrm{Ch}^l(\mathcal{U}_w).$$

The *r-characteristic variety* is defined analogously. Moreover, the notation and terminology is extended in the obvious way to characteristic *cycles* as well.

Proposition. *The l- and r-characteristic variety of a primitive ideal coincide.*

Proof. In fact, we have seen in §5 that

$$\mathrm{Ch}^l(\mathcal{U}_w) = \mathrm{Ch}^l(\mathcal{H}_w) = G \times^B V(L_{w^{-1}})$$

by 5.7 and 5.9, resp. that

$$\mathrm{Ch}^r(\mathcal{U}_w) = \mathrm{Ch}^r(\mathcal{H}_w) = V(L_w) \times^B G$$

by the right analogues. Now the point is that we may chose for $w \in W$ a self-inverse element by [D], Proposition 9. But for $w = w^{-1}$, the above equations give

$$\mathrm{Ch}^l(\mathcal{U}_w) = \mathrm{Ch}^r(\mathcal{U}_w).$$

(Of course, we identify here both $T^*X^l = G \times^B \mathfrak{n}$, and $T^*X^r = \mathfrak{n} \times^B G$, with T^*X.) Q.e.d.

Remark. The choice of an involution in the Weyl group for the proof above can be made more precise as follows. Let S denote the socle of the Harish-Chandra bimodule $U_w = U(\mathfrak{g})/J$. It is an immediate consequence of the primality of J, that S is simple. So by the classification of simple modules, $S \cong H_y$ for some Weyl group element $y \in W$, which is uniquely determined (by J, say). Then $y = y^{-1}$ is an involution by Duflo (loc. cit.). On the other hand, $S \cong H_y$ is the only Jordan-Hölder factor of U_w of Gelfand-Kirillov dimension $= d(U_w)$, since U_w/S has smaller Gelfand-Kirillov dimension than U_w (again by the primality of J). Therefore U_w and $H_y \cong S$ have the same l- and r-characteristic variety by 5.11d) (or use again 5.7). – We shall refer to the above choice of an involution y as *Duflo's involution of J.*

6.2 *Summary of previous applications to primitive ideals*

Let us list here for convenience the main results concerning primitive ideals which we already proved in Chaps. 4–5 or [BoB]:

$$\mathscr{C}\mathscr{h}^l(\mathscr{U}_w) = \mathscr{C}\mathscr{h}^l(\mathscr{H}_w) = G \times^B \mathscr{V}(L_{w^{-1}}) \tag{1}$$

$$\mathrm{Ch}^l(\mathscr{U}_w) = G\,\mathrm{Ch}(\mathscr{L}_w) = G \times^B V(L_{w^{-1}}) \tag{2}$$

$$\mathscr{V}(\mathrm{gr}\,J) = V(U_w) = G\,V(L_w) \tag{3}$$

$$\mathscr{V}(\mathrm{gr}\,J) = V(U_w) = \bar{\mathcal{O}}_x \quad \text{for a nilpotent orbit } \mathcal{O}_x. \tag{4}$$

We shall keep the notation $\mathcal{O}_x$ for the nilpotent orbit determined by $J = J_w$ for all of §6. Note that we have given direct and independent proofs of (4) in [BoB], 6.8, of (3) in 4.10 here, of (2) in 5.7 here, resp. of (1) in 5.9. But on the other hand, given our strongest result (1) above, one has much easier implications (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4). For (3)$\Rightarrow$(4) for example, one may also proceed as in 4.11, using the Hotta-Joseph results (4.5), which avoids case by case discussions and works for arbitrary central characters as well. – For (2)$\Rightarrow$(3) for example, one has just to apply Theorem 1.9, which gives

$$V(U_w) = \pi(\mathrm{Ch}^l(\mathscr{U}_w)) = \pi(G\,\mathrm{Ch}(\mathscr{L}_w)) = G\,\pi(\mathrm{Ch}(\mathscr{L}_w)) = G\,V(L_w),$$

using the G-equivariance of π. – Let us note at this point, as a corollary of (4) above and Theorem 1.9, that the l-characteristic variety of J satisfies

$$\mathrm{Ch}^l(U(\mathfrak{g})/J) \subset \pi^{-1}\bar{\mathcal{O}}_x. \tag{5}$$

This will be made slightly more precise below (6.4).

6.3 *An application to varieties of highest weight modules*

Let us point out here the following immediate consequence of 5.7 (or 6.2(2)). Recall the definition of the quasi-order relations $\leqq_L$, $\leqq_R$ and $\leqq_{LR}$ on W introduced by Kazhdan and Lusztig [KL] (cf. also [BV2], 2.10, and B.5 in our appendix).

Corollary. *Let $w, y \in W$. Then $w \leqq_R y$ implies $V(L_w) \supset V(L_y)$. In particular, $V(L_w)$ is constant on each right cell.*

Proof. By definition, $w \leqq_R y$ means $w^{-1} \leqq_L y^{-1}$, and this is known to be equivalent to $J_{w^{-1}} \subset J_{y^{-1}}$. This inclusion implies a converse inclusion for the l-characteristic varieties, so by 6.2(2)

$$\mathrm{Ch}^l(U(\mathfrak{g})/J_{w^{-1}}) = G \times^B V(L_w) \supset G \times^B V(L_y) = \mathrm{Ch}^l(U(\mathfrak{g})/J_{y^{-1}}),$$

which gives $V(L_w) \supset V(L_y)$. Q.e.d.

Comments. The corollary is in fact not new: It follows from Vogan's characterization of the order relation $J_{w^{-1}} \subset J_{y^{-1}}$ by the condition that L_y is a subquotient of $L_w \otimes F$ for some finite dimensional representation F (cf. [BV2], 2.9(d)), since the latter implies $V(L_y) \subset V(L_w \otimes F) = V(L_w)$. – However, it seems to us that 6.2(2) gives a more natural explanation for this fact: It says in particular that $V(L_w)$ depends only on the l-characteristic variety of $J_{w^{-1}}$, so is *determined* by $J_{w^{-1}}$. More precisely, 6.2(1) states that *even the associated cycle $\mathscr{V}(L_w)$ is determined by the l-characteristic cycle of $J_{w^{-1}}$.* This explains naturally the fact that $\mathscr{V}(L_w)$ is constant on right cells (depends only on $J_{w^{-1}}$), which was also observed by Joseph in [J3].

6.4. *Equidimensionality of l-characteristic varieties*

Proposition. a) *$\mathrm{Ch}^l(U(\mathfrak{g})/J)$ is equidimensional, of dimension*

$$\dim \mathrm{Ch}^l(U(\mathfrak{g})/J) = \dim X + \tfrac{1}{2}\dim \mathcal{O}_x.$$

b) *It is a union of closures of irreducible components of $\pi^{-1}\mathcal{O}_x$.*

Proof. The equidimensionality follows from 5.11, since the primality of J implies easily that $U(\mathfrak{g})/J$ as a left module is GK-homogeneous. The dimension-formula in a) follows simultaneously. As noted in 6.2, we have certainly

$$\mathrm{Ch}^l(U(\mathfrak{g})/J) \subset \pi^{-1}\bar{\mathcal{O}}_x = \bigcup_u \pi^{-1}\mathcal{O}_u, \tag{$*$}$$

where the union ranges over all nilpotent orbits $\mathcal{O}_u$ in the closure of $\mathcal{O}_x$. But $\pi^{-1}\mathcal{O}_u$ is equidimensional of dimension $\dim X + \frac{1}{2}\dim \mathcal{O}_u$ (see Appendix B.1). Therefore by a) no orbit $\mathcal{O}_u \subsetneqq \bar{\mathcal{O}}_x$ can contribute a component to $\mathrm{Ch}^l(U(\mathfrak{g})/J)$ in ($*$). Hence a) implies b). Q.e.d.

6.5. *Primitive ideals are classified by their l-characteristic cycles*

As a consequence of Proposition 6.4, we may consider the l-characteristic cycle $\mathscr{C}\mathscr{h}^l(U(\mathfrak{g})/J)$ of a primitive ideal J as a *homology class* in the (Borel-Moore) homology groups

$$H_{2d}(\overline{\pi^{-1}\mathcal{O}_x}, \mathbb{Q}) \cong H_{2d}(\pi^{-1}\mathcal{O}_x, \mathbb{Q}) \cong H_{2d_x}(\pi^{-1}x, \mathbb{Q})^{C(x)}, \tag{$*$}$$

where $d = 2\dim X - d_x$. The canonical isomorphisms in (*) are clear resp. explained in detail in Appendix B. (Recall that the right hand side of (*) carries, by definition, Springer's Weyl group representation ρ_x, or more precisely its – isomorphic – contragredient.) This enables us to state our main theorem about the classification of primitive ideals by their l-characteristic cycles in the following more precise form. This result is an easy corollary of Joseph's work [J3] in combination with our Theorem 5.8.

Theorem. *Let* $\mathscr{C}_x := \{J \in \mathscr{X}_{\mathcal{O}} \mid V(U(\mathfrak{g})/J) = \bar{\mathcal{O}}_x\}$ *denote the "clan" of primitive ideals corresponding to a given special nilpotent orbit* $\mathcal{O}_x$ (notation as in [BoB], 6.9). *Then the* $\mathscr{Ch}^l(U(\mathfrak{g})/J)$ *for all* $J \in \mathscr{C}_x$ *form a basis for the vector-space* (*), *and hence for Springer's representation* ρ_x.

Proof. Let $s(J) \in W$ denote Duflo's involution determined by J (6.1). It follows from Joseph's work (cf. [J3], 5.5) that the associated cycles $\mathscr{V}(L_{s(J)})$ for $J \in \mathscr{C}_x$ are linearly independent. In combination with Hotta's results [H], this work (loc. cit.) implies even that those cycles form a basis of $H_{\dim \mathcal{O}_x}(\overline{\mathcal{O}_x \cap \mathfrak{n}}, \mathbb{Q})$. Now we have a canonical isomorphism

$$H_{\dim \mathcal{O}_x}(\overline{\mathcal{O}_x \cap \mathfrak{n}}, \mathbb{Q}) \xrightarrow{\sim} H_{2\dim X + \dim \mathcal{O}_x}(\overline{\pi^{-1}\mathcal{O}_x}, \mathbb{Q}),$$

which sends an irreducible component V of $\overline{\mathcal{O}_x \cap \mathfrak{n}}$ to $G \times^B V$, an irreducible component of $\overline{\pi^{-1}\mathcal{O}_p}$. (See Appendix B for a detailed discussion of the geometrical back-ground involved here.) Our Theorem 5.8, or 6.2(1), states that this isomorphism sends $\mathscr{V}(L_{s(J)})$ to $\mathscr{Ch}^l(U(\mathfrak{g})/J)$. This proves the theorem. Q.e.d.

Remark. Of course, one would like to avoid the reference to Joseph's work in the above proof, and give an independent, direct proof of the theorem by more geometrical methods. This is in fact possible: One may compute Joseph's Goldie rank polynomial of J directly from the l-characteristic cycle of J, using its fibre-bundle description provided by Theorem 5.8. We shall give the details of this alternative proof of the theorem elsewhere (cf. also [BBM]).

6.6. The above formulation of our main theorem includes the following statements.

Corollary 1. *Different primitive ideals have different l-characteristic cycles.* (In fact, they are even linearly independent.)

Corollary 2. *Each irreducible component of* $\overline{\pi^{-1}\mathcal{O}_x}$ *occurs in the l-characteristic variety of some* $J \in \mathscr{C}_x$.

Corollary 3. *A clan* $\mathscr{C}_x$ *has as many members, as* $\pi^{-1}\mathcal{O}_x$ *has irreducible components.*

Comments. Since $\pi^{-1}\mathcal{O}_x$ has $\dim \rho_x$ irreducible components (cf. Appendix B.1 or [BoB], §6), Corollary 3 is merely a restatement of the numerical equality $\# \mathscr{C}_x = \dim \rho_x$, counting primitive ideals in terms of Springer representations, which is of course a well-known consequence of Joseph's work [J2], in combination with results of Barbasch-Vogan [BV1, BV2], resp. of Hotta-Kashiwara [HK]. In fact, the main motivation for our research reported in the present paper was our goal to find a geometric reinterpretation of this well-known numerical equality (and many more similar equalities, as stated in [BoB], Corollary 6.14). We hope that our theorem above will help to make the classification of primitive ideals in terms of Springer's Weyl group representations appear less mysterious and more natural to some readers – as it did to us.

6.7. Some Conjectures (notations as in Theorem 6.5)

Conjecture 1. $\mathrm{Ch}^l(U(\mathfrak{g})/J)$ *is irreducible for all* $J \in \mathscr{C}_x$.

Conjecture 2. $J \mapsto \mathrm{Ch}^l(U(\mathfrak{g})/J)$ *is a bijection of the clan* $\mathscr{C}_x$ *to the set of irreducible components of* $\overline{\pi^{-1}\mathcal{O}_x}$.

Conjecture 3. $V(L_w)$ *is irreducible for each* $w \in W$ *in the KL (Kazhdan-Lusztig) double cell corresponding to* $\mathcal{O}_x$.

Conjecture 4. The basis for Springer's W-representation ρ_x *given by the characteristic cycles* $\mathscr{Ch}^l(U(\mathfrak{g})/J)$ *for* $J \in \mathscr{C}_x$ (cf. Theorem 6.5) *coincides with the canonical basis.*

Conjecture 5. Joseph's Goldie-rank polynomials for all $J \in \mathscr{C}_x$ *coincide, up to scalar factors, with the canonical basis elements for Springer's representation* ρ_x.

Note added in proof. Meanwhile, these conjectures turned out to fail for some clans in types B_3, C_3, as discovered by T. Tanizaki. Hence we can expect the simple picture suggested in 6.7. only for special clans, or for $G = SL_n$, to hold, while in general one has to be content with Theorem 6.5.

Proposition. *These five conjectures are mutually equivalent.*

It is clear from 6.5, that the simple statement about irreducibility of l-characteristic varieties formulated in Conjecture 1, implies the much stronger statement about classification of primitive ideals formulated in Conjecture 2: For example, the injectivity resp. surjectivity of the map in Conjecture 2 comes from Corollary 1 resp. 2 in 6.6 (if Conjecture 1 should hold). – The equivalence of Conjectures 1 and 3 follows from Theorem 5.8, while the equivalence of 1 and 4 is again clear from Theorem 6.5. Finally, Conjecture 5 is equivalent to Conjecture 3 in view of the work of Joseph [J3] and Hotta [H]. Alternatively, Conjecture 5 can be seen to be equivalent to Conjecture 4 in view of Theorem 5.8 and 6.5 (cf. Remark 6.5).

Comments. a) Our main point to be made here is that our *irreducibility-conjecture* (1) would provide a most elegant and simple explanation for various quite different known results: On one hand, it would of course imply (by 1.9) again the irreduciblity of associated varieties of primitive ideals (and even the G-saturation theorem), and on the other hand, the same fact would simultaneously explain the classification of primitive ideals (by the equivalent Conjecture (2)). Although our Theorem 6.5 is already quite satisfactory as a geometric reinterpretation of Joseph's classification, we should like to replace it by Conjecture 2 above, because this would be less sophisticated to formulate.

b) Note that we have discussed a slightly sharper version of Conjecture 3 already in Chap. 4 as Conjecture 4.5. We have noted there (4.5, Comment 4) that this conjecture is in fact true for $G=SL_n$ $(n\leqq 6)$. *Therefore, Conjectures* 1–5 *above are true for* $G=SL_n$ $(n\leqq 6)$.

c) Joseph has given in [J3], 10.1 an example of a simple highest-weight module with non-integral central character in type G_2, whose associated variety is reducible. Hence Conjecture 3 does certainly not hold in the non-integral case, although a modified version may extend to arbitrary central characters, as suggested by Joseph (loc. cit., 10.2).

6.8. *Two cell decompositions of the Weyl group*

Two elements $w, y\in W$ belong to the same (KL) *left* resp. *right* resp. *double cell*, iff the corresponding primitive ideals J_w, J_y resp. $J_{w^{-1}}, J_{y^{-1}}$ resp. their associated varieties coincide (cf. [KL, BV2, BoB], §6). Let us define an analogous cell decomposition of W, suggested by Steinberg's work [St1], as follows: The elements w, y belong to the same *Steinberg left* resp. *right* resp. *double cell*, iff $\overline{B(\mathfrak{n}\cap\mathfrak{n}^{w^{-1}})}$, $\overline{B(\mathfrak{n}\cap\mathfrak{n}^{y^{-1}})}$ resp. $\overline{B(\mathfrak{n}\cap\mathfrak{n}^{w})}$, $\overline{B(\mathfrak{n}\cap\mathfrak{n}^{y})}$ resp. their G-saturations coincide. There are many alternative, equivalent definitions of this cell decomposition of W, as the reader will realize with the general geometrical picture of Steinberg's triple variety in mind, which we have explained in some detail in Appendix B (use the equivalence of B.3, (i) to (v)). We denote the KL resp. Steinberg left (right, double) cell containing w by $\mathscr{C}^L_w$ $(\mathscr{C}^R_w, \mathscr{C}^{LR}_w)$ resp. $\mathscr{C}^l_w$ $(\mathscr{C}^r_w, \mathscr{C}^{lr}_w)$. Note that in this notation, $\mathscr{C}^L_{w^{-1}}=\mathscr{C}^R_w$, $\mathscr{C}^R_{w^{-1}}=\mathscr{C}^L_w$, $\mathscr{C}^{LR}_w=\bigcup_{y\in\mathscr{C}^R_w}\mathscr{C}^L_y=\bigcup_{y\in\mathscr{C}^L_w}\mathscr{C}^R_y$, and the same relations hold with L, R replaced by l, r. The Steinberg resp. KL double cells are in bijection to the set of *all* resp. of *all special* nilpotent orbits; so we use also the notation $\mathscr{C}^{LR}_x=\mathscr{C}^{LR}_w$ resp. $\mathscr{C}^{lr}_x=\mathscr{C}^{lr}_w$, if $\mathcal{O}_x$ is the corresponding nilpotent orbit, which is dense in $\mathscr{V}(\operatorname{gr} J_w)$ resp. in $G(\mathfrak{n}\cap\mathfrak{n}^w)$, to express this bijection notationally. For a slightly more detailed discussion of the analogy between Steinberg and KL cells, we refer to the Appendix (B.5). *For* $G=SL_n$ for example, *the two notions of cells coincide completely* (cf. Spaltenstein [Sp2, Sp4], or also [KL]).

6.9. *The sets* $\Gamma(w)\subset\Sigma(w)$

Given $w\in W$, let us denote by $\Sigma(w):=\Sigma(\mathscr{H}_w)$ the subset of W such that the characteristic variety of $\mathscr{H}_w$ is the union of all $\bar{\mathscr{T}}_y$ for $y\in\Sigma(w)$. Equivalently (recall 4.8), $\Sigma(w)$ is the set of those $y\in W$ such that the characteristic variety of $\mathscr{L}_w$ is the union of all $\overline{T^*_{X_y}X}$ for $y\in\Sigma(w)$. We know from §5 that the l-characteristic variety of J_w is then given by the formula

$$\operatorname{Ch}^l(U(\mathfrak{g})/J_w)=\bigcup_{y\in\Sigma(w)}\pi^l(\bar{\mathscr{T}}_y). \tag{1}$$

Let us call y *big* in $\Sigma(w)$, if its contribution to the union (1) is of maximal dimension. Let

$$\Gamma(w):=\{y\in\Sigma(w)\mid\dim\pi^l(\bar{\mathscr{T}}_y)=\dim\operatorname{Ch}^l(U(\mathfrak{g})/J_w)\} \tag{2}$$

denote the *set of all big elements* in $\Sigma(w)$. Here are a few alternative equivalent definitions of "big":

$$\Gamma(w)=\{y\in\Sigma(w)\mid\dim B(\mathfrak{n}\cap\mathfrak{n}^y)=\dim V(L_w)\} \tag{2'}$$

$$\Gamma(w)=\{y\in\Sigma(w)\mid\pi^l(\bar{\mathscr{T}}_y)\text{ is a component of }\operatorname{Ch}^l(U(\mathfrak{g})/J_w)\} \tag{3}$$

$$\Gamma(w)=\{y\in\Sigma(w)\mid\overline{B(\mathfrak{n}\cap\mathfrak{n}^y)}\text{ is a component of }V(L_w)\} \tag{3'}$$

$$\Gamma(w)=\{y\in\Sigma(w)\mid\pi^l(\bar{\mathscr{T}}_y)\text{ is a component of }\overline{\pi^{-1}\mathcal{O}_x}\} \tag{4}$$

$$\Gamma(w)=\{y\in\Sigma(w)\mid\overline{B(\mathfrak{n}\cap\mathfrak{n}^y)}\text{ is a component of }\mathfrak{n}\cap\bar{\mathcal{O}}_x\}. \tag{4'}$$

In the last two statements, $\mathcal{O}_x$ is the nilpotent orbit corresponding to J_w (Notation 6.1, 6.2). The equivalence of (2) and (2′) follows from the fibre bundle interpretation of l-characteristic varieties: $\operatorname{Ch}^l(U(\mathfrak{g})/J_w)=G\times^B V(L_{w^{-1}})$ and of $\pi^l(\bar{\mathscr{T}}_y)=G\times^B\overline{B(\mathfrak{n}\cap\mathfrak{n}^{w^{-1}})}$ (recall §5), where we also use the formula $\Sigma(w^{-1})=\Sigma(w)^{-1}$ (recall 4.8). The equivalences (3)⇔(3′) and (4)⇔(4′) are similar. The equivalence of (2′) and (3′) follows from the equidimensionality of $V(L_w)$, and the equivalence of (3) and (4) is essentially a rephrasement of Proposition 6.4b). Finally, we note that in the terminology of 6.8, statements (4) and (4′) just mean that

$$\Gamma(w)=\Sigma(w)\cap\mathscr{C}^{lr}_x, \tag{5}$$

is *the intersection of* $\Sigma(w)$ *and the Steinberg double cell for* $\mathcal{O}_x$.

Our interest in the subset $\Gamma(w)$ of "big" elements in $\Sigma(w)$ arises, of course, from the fact that all other elements are redundant in (1), since we have from (3) that

$$\operatorname{Ch}^l(U(\mathfrak{g})/J_w)=\bigcup_{y\in\Gamma(w)}\pi^l(\bar{\mathscr{T}}_y), \tag{6}$$

or equivalent by that

$$V(L_w)=\bigcup_{y\in\Gamma(w)}\overline{B(\mathfrak{n}\cap\mathfrak{n}^y)}. \tag{6'}$$

Consequence. The l-characteristic variety of J_w is irreducible (cf. 6.7.1) *iff $\Gamma(w)$ is contained in a single Steinberg left cell.*

More generally, *the irreducible components correspond to the Steinberg left cells which $\Gamma(w)$ meets.*

Lemma. $\Gamma(w^{-1})=\Gamma(w)^{-1}$.

In fact, from (5) we obtain $\Gamma(w^{-1}) = \Sigma(w^{-1}) \cap \mathscr{C}_x^{lr} = \Sigma(w^{-1}) \cap \mathscr{C}_x^{lr} = \Gamma(w)^{-1}$, using that $\Sigma(w^{-1}) = \Sigma(w)^{-1}$ by 4.8, and that a Steinberg double cell is stable under $y \mapsto y^{-1}$ (cf. B.5).

Proposition. *Assume that the centralizer G_x is connected. Then our irreducibility Conjecture* (6.7.1) *is equivalent to: $\Gamma(w)$ is a single element for all $w \in \mathscr{C}_x^{LR}$.*

Corollary. *Assume $G = PSL_n$. Then Conjecture* 6.7.1 *is equivalent to $\Gamma(w) = \{w\}$ for all $w \in \mathscr{C}_x^{LR}$.*

Proof. The corollary follows from the proposition, since for $G = PSL_n$, the centralizer is automatically connected, and also $w \in \Gamma(w)$ (cf. 4.5.3). To prove the proposition, note that $\# \Gamma(w) = 1$ implies the irreducibility of the l-characteristic variety of J_w by (6), so the condition in the proposition is certainly *sufficient*. To prove *necessity*, let us assume that there are two different "big" elements $y_1, y_2 \in \Gamma(w)$. If y_1 and y_2 belong to *different* Steinberg left cells, then $\mathrm{Ch}^l(U(\mathfrak{g})/J_w)$ is *reducible*. So let us assume that y_1, y_2 are in the same Steinberg left cell.

Claim. Then y_1^{-1}, y_2^{-1} belong to different Steinberg left cells.

This claim will complete the proof of the proposition, since then the above lemma tells us that $\Gamma(w^{-1}) = \Gamma(w)^{-1}$ meets two different Steinberg left cells, and so $J_{w^{-1}}$ (which belongs to $\mathscr{C}_x$) has *reducible* l-characteristic variety. To prove the above claim, we use the assumption that G_x is connected. This tells us (cf. Appendix B.5) that the Steinberg double cell $\mathscr{C}_x^{lr}$ has cardinality $(\dim \rho_x)^2$, and that all $\dim \rho_x$ Steinberg left cells in this double cell have the same cardinality $\dim \rho_x$. Since the *inverses* of a fixed left cell $\mathscr{C}_y^l$ form a right cell $\mathscr{C}_{y^{-1}}^r$, which meets *all* $\dim \rho_x$ left cells, it follows that each left cell can contain *just one* of these $\dim \rho_x$ inverses. This proves our claim, and hence the proposition. Q.e.d.

6.10. *Characterization of primitive ideals by characteristic varieties*

The purpose of the present sections (6.8–6.15) is to support our general philosophy of how the classification of primitive ideals should be geometrically understood by means of their l-characteristic varieties. For this purpose, we are giving some partial results towards our Conjectures 6.7 (in Propositions 6.9, 6.10, 6.14). The next proposition for example sharpens 6.6, Corollary 1 towards 6.7, Conjecture 2.

Proposition. *Let $G = SL_n$. Then different primitive ideals with trivial central character have different l-characteristic varieties.*

Proof. Given any two subsets A and B of W, we say that *A dominates B*, if for each $w \in A$ there is some $y \in B$ such that $w \geq y$ for the Bruhat order. It is clear that "dominance" defines an order relation on the set of KL left cells, or on any other partition of the Weyl group into subsets.

Lemma. *Let $w \in W$, $y \in \Gamma(w)$. Then $\mathscr{C}_w^L$ dominates $\mathscr{C}_y^l$.*

In fact, let $w' \in \mathscr{C}_w^L$. Then the primitive ideals $J_{w'}, J_w$ coincide, so their l-characteristic varieties coincide, so $\Gamma(w')$ and $\Gamma(w)$ meet exactly the same Steinberg l-cells (6.9). Since $\Gamma(w)$ meets $\mathscr{C}_y^l$ in y by assumption, $\Gamma(w')$ meets $\mathscr{C}_y^l$ in some element y'. But then $w' \geq y'$ by 2.8a). This proves the lemma.

To prove the proposition, consider two elements $w, y \in W$ such that J_w and J_y have the same l-characteristic varieties. This means by 6.9 that $\Gamma(w)$ and $\Gamma(y)$ meet the same Steinberg l-cells. Since $G = SL_n$, we have $w \in \Gamma(w)$ (6.9), so $\Gamma(w)$ meets $\mathscr{C}_w^l$ in w, hence $\Gamma(y)$ meets $\mathscr{C}_w^l$ in some $w' \in W$, and the lemma says that $\mathscr{C}_y^L$ dominates $\mathscr{C}_{w'}^l = \mathscr{C}_w^l$. Similarly, the argument shows that $\mathscr{C}_w^L$ dominates $\mathscr{C}_y^l$. But for $G = SL_n$, we have $\mathscr{C}_w^L = \mathscr{C}_w^l$, $\mathscr{C}_y^L = \mathscr{C}_y^l$ by Spaltenstein (last point in 6.8). So we conclude that the left cells of w and y dominate each other. Hence they are equal. This means $J_w = J_y$. Q.e.d.

Remark. The above argument shows, more precisely, that with respect to a suitable total ordering of the irreducible components of $\overline{\pi^{-1}\mathcal{O}_x}$ (notation as in 6.2ff.), the l-characteristic variety of J_w is

$$\mathrm{Ch}^l(U(\mathfrak{g})/J_w) = \pi^l(\bar{\mathscr{T}}_w) \cup S,$$

with S a (conjecturally empty!) union of irreducible components of $\overline{\pi^{-1}\mathcal{O}_x}$ strictly *smaller* than $\pi^l(\bar{\mathscr{T}}_w)$. So even if the l-characteristic variety of J_w should not be irreducible, its irreducible component $\pi^l(\bar{\mathscr{T}}_w)$ would be distinguished from all others as the "biggest". This distinction can also be made by means of the so-called "τ-invariant" as explained at the end of the next section.

6.11. *Computation of τ-invariants from l-characteristic varieties*

The notion of τ-invariant of a primitive ideal was introduced by Duflo [D] and Jantzen and one of us [BoJ], and was generalized by Vogan [V]. Define $\tau(w)$ for $w \in W$ as the set of those simple roots α such that $w\alpha$ is negative. By loc. cit., $\tau(w)$ depends only on J_w: In fact, it is given by the set of those simple reflections s such that $J_s \subset J_w$. The τ-invariant of $J = J_w$ is then defined as $\tau(J) := \tau(w)$. Let us now express $\tau(J)$ as a function of $\mathrm{Ch}^l(U(\mathfrak{g})/J)$. Given any set τ of positive roots, let $\hat{\tau}$ denote the linear span of the corresponding root vectors in $\mathfrak{n}$. With these notations, it is easy to check that we have

$$\overline{B(\mathfrak{n} \cap \mathfrak{n}^w)} \equiv \mathfrak{n} \cap \mathfrak{n}^w \equiv \hat{\tau}(w^{-1} w_0^{-1}) \bmod. [\mathfrak{n}, \mathfrak{n}], \tag{1}$$

where w_0 denotes the longest element in W. (Note that $w_0^{-1} = w_0$, and that $\tau(y w_0)$ is just the complement of $\tau(y)$.) In particular, this shows that the τ-invariant is constant on Steinberg right cells.

Proposition. a) *$\overline{B(\mathfrak{n} \cap \mathfrak{n}^w)} \subset V(L_w)$ with equality modulo $[\mathfrak{n}, \mathfrak{n}]$.*

b) *The τ-invariant of J_w is determined by*

$$\hat{\tau}(w_0 w) \equiv \mathrm{Ch}^l(U(\mathfrak{g})/J_w) \cap \mathfrak{n} \bmod. [\mathfrak{n}, \mathfrak{n}],$$

(where we identify $\mathfrak{n}$ with the fibre of $T^* X$ at the base point B).

Proof. a) The first inclusion means just to recall 4.3, Corollary 1. In view of Eq. (1) above, it remains to be shown that

$$V(L_w) \equiv \hat{\tau}(w^{-1} w_0^{-1}) \bmod . [\mathfrak{n}, \mathfrak{n}]. \tag{2}$$

To verify this, the reader may consult e.g. [Ja], 4.17, or [J6], 2.4: One observes that, in the terminology of loc.cit., L_w is *α-free* for a simple root α, iff α does *not* belong to $\tau(w^{-1})$, and on the other hand iff $V(L_w)$ contains the root space $\mathfrak{g}^\alpha$ corresponding to α.

b) We apply first 5.8, then a) and (1) above, to conclude

$$\begin{aligned}\mathrm{Ch}^l(U(\mathfrak{g})/J_w) \cap \mathfrak{n} &\equiv V(L_{w^{-1}}) \equiv \overline{B(\mathfrak{n} \cap \mathfrak{n}^{w^{-1}})} \equiv \hat{\tau}(w w_0^{-1}) \\ &\equiv \hat{\tau}(w_0 w) \bmod . [\mathfrak{n}, \mathfrak{n}]. \quad \text{Q.e.d.}\end{aligned}$$

Corollary. *For each $y \in \Sigma(w)$, we have $\tau(w_0 y) \subset \tau(w_0 w)$, (or equivalently, $\tau(w) \subset \tau(y)$).*

Remark. This is equivalent to Joseph [J3], Lemma 9.9. For $G = SL_n$, one can refine this to: $\tau(w) \subsetneqq \tau(y)$ unless $w = y$ (cf. loc.cit., Corollary 9.12). Hence $\pi^l(\mathcal{T}_w)$ is characterized as the unique irreducible component of the l-characteristic variety of J_w with the smallest possible "τ-invariant".

6.12. *Induced ideals for example*

Let $P \subset G$ be a parabolic subgroup containing B, with Lie algebra $\mathfrak{p}$. Let $W(P) \subset W$ denote the corresponding parabolic subgroup of W, and w_P the longest element of $W(P)$. Then the Schubert variety $X_w \cong G/P$ is *smooth*, the simple module L_w is *induced* from $\mathfrak{p}$, and the primitive ideal J_w is *induced* from $\mathfrak{p}$ (recalling [BoB], Example 6.6). It follows then by 2.8c) from the smoothness of X_w resp. of Z_w that these characteristic varieties are irreducible:

$$\mathrm{Ch}(L_w) = \overline{T^*_{X_w} X} \quad \text{resp.}$$
$$\mathrm{Ch}(H_w) = \bar{\mathcal{T}}_w.$$

Note that for $w = w_P$ we have $\mathfrak{n} \cap \mathfrak{n}^w = \mathfrak{p}^\perp$, which is P-stable and hence B-stable, so that $\overline{B(\mathfrak{n} \cap \mathfrak{n}^w)} = \mathfrak{p}^\perp$ in the present case. It is now clear (e.g. from 4.2 resp. from 5.8 or 5.5 and 5.3) that:

Proposition. $V(L_{w_P}) = \mathfrak{p}^\perp$ *resp.* $\mathrm{Ch}^l(U(\mathfrak{g})/J_{w_P}) = G \times^B \mathfrak{p}^\perp$. *In particular, the l-characteristic variety of J_{w_P} is irreducible.*

6.13. *A bound for the number of components of a characteristic variety*

We keep the notations of 6.12, and denote by $\mathcal{O}_y$ the Richardson orbit determined by P, i.e. the nilpotent orbit such that $\bar{\mathcal{O}}_y = G\mathfrak{p}^\perp$. As in [BoB], we denote by $\varepsilon^W_{W(P)}$ the representation of W *induced* from the sign-character $\varepsilon_{W(P)}$ of $W(P)$.

Lemma. *For a special nilpotent orbit $\mathcal{O}_x$, the following are equivalent:*

(i) $\mathcal{O}_x \subset \bar{\mathcal{O}}_y$.

(ii) $\mathrm{mtp}(\rho_x, \varepsilon^W_{W(P)}) \neq 0$.

(iii) *The induced ideal J_{w_P} (6.12) is contained in some primitive ideal $J \in \mathscr{C}_x$* (i.e. such that $\mathscr{V}(\mathrm{gr}\, J) = \bar{\mathcal{O}}_x$).

This is merely a restatement of results contained in [BoB], §6. In fact, by loc.cit., Proposition 6.11, the number of choices for J to satisfy (iii) is exactly

$$\#\{J \in \mathscr{C}_x \mid J \supset J_{w_P}\} = \mathrm{mtp}(\rho_x, \varepsilon^W_{W(P)}).$$

Hence (ii)$\Leftrightarrow$(iii). The implication (ii)$\Rightarrow$(i) is explained in loc.cit., 6.13, as a consequence of work of MacPherson and one of us about Springer's representations. The converse (i)$\Rightarrow$(ii) follows from work of G. Kempken for classical groups, resp. of D. Alvis and N. Spaltenstein for exceptional groups, as stated in loc.cit., Propositions 6.9, 6.10 resp. Corollary 6.13, footnote 3.

Proposition. *Let $\mathcal{O}_x$ and J be as in the lemma. Assume that the equivalent conditions of the lemma hold. Then the l-characteristic variety of J has at most $\mathrm{mtp}(\rho_x, \varepsilon^W_{W(P)})$ irreducible components:*

$$\#\,\mathrm{Irr}(\mathrm{Ch}^l(U(\mathfrak{g})/J)) \leqq \mathrm{mtp}(\rho_x, \varepsilon^W_{W(P)}).$$

Proof. Since J contains J_{w_P}, 6.12 implies that

$$\mathrm{Ch}^l(U(\mathfrak{g})/J) \subset \mathrm{Ch}^l(U(\mathfrak{g})/J_{w_P}) = G \times^B \mathfrak{p}^\perp.$$

Therefore, the l-characteristic variety of J can contain only such irreducible components of $\pi^{-1}\mathcal{O}_x$, which occur in $G \times^B \mathfrak{p}^\perp$. In the terminology of [BoB], 6.12, this means that they have to be components "of type P". But the total number of such components is given by [BoB], Proposition 6.12, which concludes our proof as follows:

$$\begin{aligned}\#\,\mathrm{Irr}(\mathrm{Ch}^l(U(\mathfrak{g})/J)) &\leqq \#\,\mathrm{Irr}(\pi^{-1}\mathcal{O}_x \cap G \times^B \mathfrak{p}^\perp) = \#\,\mathrm{Irr}(\mathcal{O}_x \cap \mathfrak{p}^\perp) \\ &= \mathrm{mtp}(\rho_x, \varepsilon^W_{W(P)}). \quad \text{Q.e.d.}\end{aligned}$$

6.14. *A partial result about irreducibility*

Proposition. *Let $\mathcal{O}_x$ be a special nilpotent orbit, and $\mathscr{C}_x$ the corresponding clan of primitive ideals* (notation as before). *Then the clan $\mathscr{C}_x$ has at least one member $J \in \mathscr{C}_x$ such that $\mathrm{Ch}^l(U(\mathfrak{g})/J)$ is irreducible.*

Proof. We chose a parabolic subgroup P such that the Springer representation ρ_x corresponding to $\mathcal{O}_x$ occurs in $\varepsilon^W_{W(P)}$ with multiplicity *one*. This choice is possible by Lemma 6.15 below. Then we chose $J \in \mathscr{C}_x$ with $J \supset J_{w_P}$. This choice is possible by Lemma 6.13. Now Proposition 6.13 implies that $\mathrm{Ch}^l(U(\mathfrak{g})/J)$ must be irreducible. Q.e.d.

6.15. A property of special representations of Weyl groups

Lemma. *For each special representation ρ of W, there exists a parabolic subgroup P of G, such that* $\operatorname{mtp}(\rho, \varepsilon^W_{W(P)})=1$.

Remark. We are grateful to N. Spaltenstein for communicating to us a proof of this lemma for the case of classical groups.

Proof. a) Let $\mathcal{O}_x$ be the special nilpotent orbit of which $\rho=\rho_x$ is the Springer representation. If $\mathcal{O}_x$ is a Richardson orbit, determined by a parabolic group Q, say, then the lemma holds trivially with $P=Q$. Consequently, the lemma is trivial for $G=SL_n$, and for G of rank 2. Furthermore, it suffices to prove the lemma for *simple* G.

b) For classical groups G, we refer to N. Spaltenstein (to appear, cf. also [Sp3]).

c) It remains to verify the lemma for exceptional groups G of type F_4, E_6, E_7 or E_8. The total number of nilpotent orbits in these four cases are 16, 21, 45, 70 respectively, of which resp. 11, 17, 35, 46 are *special*, of which resp. 9, 15, 29, 34 are *Richardson*. By a), we are left with considering only the resp. 2, 2, 6, 12 *special non-Richardson* orbits. We note that resp. 2, 1, 3, 5 (a total of 11) of these are *original* orbits in the sense of [Bo3], while the other resp. 0, 1, 3, 7 (again a total of 11) of them are *induced* (in the sense of Lusztig-Spaltenstein [LS]) from a special orbit in some proper Levi subalgebra of $\mathfrak{g}$. For the 11 special original orbits, we give further details about the verification of the lemma in d) below. The 11 properly induced non-Richardson orbits can be treated by the same method. However, we omit the details for those cases, because they could also be treated by a general induction argument (admitting induction from infinite-dimensional highest weight modules in the considerations of 6.12).

d) The eleven special original orbits in types F_4, E_6, E_7, E_8 are explicitely listed in the table below, which refers to the notation of Spaltenstein's tables in [Sp4], pp. 247–250.

In column 4 of this table, we specify for each case a choice of a parabolic subgroup P satisfying the condition of the lemma.

To verify this claim, one may proceed as follows: First identify Springer's representation ρ_x corresponding to the orbit $\mathcal{O}_x$ in the tables of Alvis and Lusztig [AL], and then check that it occurs only once in $\varepsilon^W_{W(P)}$, using the tables of Alvis [Av]. However, the verification is much easier in most cases. In most cases it is not really necessary to consult [Av], but it suffices to use the last column of our table, in combination with the inclusion relations of nilpotent orbits known from Mizuno, see [Sp4], loc. cit. The last column of our table gives the dimension of the Springer representation ρ_y corresponding to the Richardson orbit $\mathcal{O}_y$ determined by P. (It is computed by combining Elašvili's tables about polarizations reported in [Sp4] again with the tables of Alvis-Lusztig [AL] (resp. Shoji [Sh] for F_4) for Springer's correspondence.

For example, for G of type E_6 and $\mathcal{O}_x$ of type A_1, entries 3, 5, and 6 of our table show that

$$\varepsilon^W_{W(P)}=\rho_y\oplus\rho_x\oplus\varepsilon_W.$$

Table. The 11 special original orbits in type F_4, E_6, E_7, E_8, and a choice of P satisfying the lemma

Type of G	Type of $\mathcal{O}_x$	$\dim\rho_x$	choice of P	$\dim\varepsilon^W_{W(P)}$	$\dim\rho_y$
F_4	$\tilde{A}_1$	4	B_3	24	8
	$A_1+\tilde{A}_1$	9			
E_6	A_1	6	D_5	27	20
E_7	A_1	7	E_6	56	21
	$2A_1$	27			
	$2A_1\times A_2$	120	$D_5\times A_1$	756	168
E_8	A_1	8	E_7	240	112
	$2A_1$	35			
	A_1+A_2	210	D_7	2,160	700
	$2A_1+A_2$	560			
	$A_1+D_4(a_1)$	1,400	A_7	17,280	2,240

because $20+6+1=27$. Similarly, for G of type E_7 and $\mathcal{O}_{x'}$ resp. $\mathcal{O}_{x''}$ of type A_1 resp. $2A_1$, our table gives

$$\varepsilon^W_{W(P)}=\rho_y\oplus\rho_{x'}\oplus\rho_{x''}\oplus\varepsilon_W,$$

because $56=21+27+7+1$ leaves no room for any further constituents (or for multiplicities >1). Q.e.d.

Appendix A

A vanishing result "à la Serre"

A.1. Let us return here once more to our discussion in Chap. 1 (cf. 1.6, remark) about vanishing of cohomology groups $H^1(X, \mathcal{M}_j)$. Here our assumptions are as in §1: The homogeneous space X is complete, the left $\mathcal{D}_x$-module $\mathcal{M}$ is coherent, and the filtration $(\mathcal{M}_j)_{j\in\mathbb{Z}}$ of $\mathcal{M}$ is good. We want to state a vanishing result, for which we need to twist the $\mathcal{M}_j$ by ample line bundles on X. So we first recall the description of the Picard group, i.e. the group of equivalence classes of line bundles on X. We chose a base point in X with isotropy subgroup P, which is some parabolic subgroup of G, and we identify X with G/P. For simplicity, we assume G simply connected. For any character λ of P, we denote by k_λ the corresponding onedimensional P-module, and by $\mathcal{L}(\lambda)$ the line bundle on $X=G/P$ whose total space is the associated bundle $G\times^P k_\lambda$. Recall that this line bundle has a global section if and only if λ is dominant (see e.g. [De] for details). Characters are partially ordered as usual: $\mu\geqq\lambda$ means $\mu\lambda^{-1}$ is dominant.

A.2. The sheaf of differential operators on X with ("twisted") coefficients in the line bundle $\mathcal{L}(\lambda)$ is denoted $\mathcal{D}^\lambda_X$, and may be described as (cf. [B1], §4):

$$\mathcal{D}^\lambda_X=\mathcal{L}(\lambda)\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{L}(\lambda^{-1}).$$

Note that in the case $P=B$, this is a sheaf of twisted differential operators as considered more generally in [BeBe]. It is filtered by the $\mathcal{O}_X$-submodules $\mathcal{D}_X^\lambda(j)$ of differential operators of order $\leqq j$. From the homogeneity of X, we deduce in the same way as in the untwisted case (cf. 1.7):

Lemma. *The sheaf $\mathcal{D}_X^\lambda$ is generated by its global sections. More precisely, for each character λ of P, and for each integer $j\geqq 0$, the sheaf $\mathcal{D}_X^\lambda(j)$, as a left $\mathcal{O}_X$-module, is generated by its global sections.*

A.3. **Lemma.** *With respect to the natural filtration, we have for any character λ of P*

$$H^i(X, \operatorname{gr}\mathcal{D}_X^\lambda)=0$$

in all dimensions $i\geqq 1$.

Proof. This follows from a vanishing result of R. Elkik [El] by similar arguments as in [BoB], 1.4, where we used it already to conclude that $H^i(X,\operatorname{gr}\mathcal{D}_X)=0$. Q.e.d.

A.4. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we put

$$\mathcal{F}(\lambda)=\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}(\lambda).$$

If $\mathcal{F}$ is a left $\mathcal{D}_X$-module, then $\mathcal{F}(\lambda)$ is a left $\mathcal{D}_X^\lambda$-module.

Theorem. (Serre, cf. [Ha], p. 121.) *If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then we can find a character λ of P such that for all $\mu\geqq\lambda$, the sheaf $\mathcal{F}(\mu)$ is generated by its global sections.*

A.5. **Proposition.** *Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module. Let $(\mathcal{M}_j)_{j\in\mathbf{Z}}$ be a good filtration of $\mathcal{M}$. Then we can find an integer j_0 and a character λ of P such that for all $\mu\geqq\lambda$, and for all $i\geqq 1$, we have*

$$H^i(X,\mathcal{M}_j(\mu))=0$$

for all $j\geqq j_0$.

Proof. Since the filtration is good, there exists an integer j_0 such that for all $j\geqq j_0$, we have

$$\mathcal{M}_j=\mathcal{D}_X(1)\mathcal{M}_{j-1}. \tag{1}$$

Without loss of generality, we assume $j_0=0$: In fact, replacing the given good filtration by the good filtration given by $M'_j=M_{j-j_0}$ ("shift" of degrees by $-j_0$), if necessary, we shall proof the proposition except for degrees $j\leqq j_0$, which is enough for our proposition. Now iterated application of (1) gives that

$$\mathcal{M}_j=\mathcal{D}_X(j)\,\mathcal{M}_0\twoheadleftarrow\mathcal{D}_X(j)\otimes_{\mathcal{O}_X}\mathcal{M}_0 \tag{2}$$

is canonically a homomorphic image of the tensor product of $\mathcal{M}_0$ by $\mathcal{D}_X(j)$.

By Serre's theorem (A.4), we find a character λ of P such that for all $\mu\geqq\lambda$, the $\mathcal{O}_X$-module $\mathcal{M}_0(\mu)$ is generated by its global sections. The same is even true for $\mathcal{M}_j(\mu)$, for all $j\geqq 0$. In fact, we first extend (2) to a similar description of the "twisted" sheaf $\mathcal{M}_j(\mu)$ as a quotient of a tensor product:

$$\mathcal{M}_j(\mu)=\mathcal{D}_X^\mu(j)\,\mathcal{M}_0(\mu)\twoheadleftarrow\mathcal{D}_X^\mu(j)\otimes_{\mathcal{O}_X}\mathcal{M}_0(\mu). \tag{3}$$

Rewriting this tensor product as

$$\mathcal{L}(\mu)\otimes_{\mathcal{O}_X}\mathcal{D}_X(j)\otimes_{\mathcal{O}_X}\mathcal{L}(\mu^{-1})\otimes_{\mathcal{O}_X}\mathcal{L}(\mu)\otimes\mathcal{M}_0\cong\mathcal{L}(\mu)\otimes_{\mathcal{O}_X}[\mathcal{D}_X(j)\otimes_{\mathcal{O}_X}\mathcal{M}_0],$$

(3) is an obvious consequence of (2). Now since $\mathcal{M}_0(\mu)$ resp. $\mathcal{D}_X^\mu(j)$ are both generated by global sections, as we know from A.4 resp. A.2, we conclude that so are their tensor product and its quotients. Hence (3) implies that $\mathcal{M}_j(\mu)$ is in fact generated, as an $\mathcal{O}_X$-module, by its global sections.

The preceding argument proves even more. Let

$$M_0^\mu=\Gamma(X,\mathcal{M}_0(\mu))$$

be the vector space of global sections of $\mathcal{M}_0$, which has finite dimension $d(\mu)$ (cf. [Ha], p. 122, Theorem 5.19). Then (3) implies that

$$\mathcal{M}(\mu)=\mathcal{D}_X^\mu M_0^\mu,$$

as a left $\mathcal{D}_X^\mu$-module, is finitely generated by M_0^μ, and is therefore a quotient of

$$\mathcal{D}_X^\mu\otimes_k M_0^\mu\cong(\mathcal{D}_X^\mu)^{d(\mu)},$$

a free $\mathcal{D}_X^\mu$-module of finite rank $d(\mu)$. Thus we obtain an exact sequence of coherent $\mathcal{D}_X^\mu$-modules

$$0\to\mathcal{N}(\mu)\to(\mathcal{D}_X^\mu)^{d(\mu)}\to\mathcal{M}(\mu)\to 0, \tag{4}$$

where the kernel $\mathcal{N}(\mu)$ is the "twist by $\mathcal{L}(\mu)$" of some $\mathcal{D}_X$-module $\mathcal{N}$. Moreover, (3) provides us even with an exact sequence (4) of *filtered* modules, so that we derive an exact sequence of associated graded modules

$$0\to\operatorname{gr}\mathcal{N}(\mu)\to[\operatorname{gr}\mathcal{D}_X^\mu]^{d(\mu)}\to\operatorname{gr}\mathcal{M}(\mu)\to 0. \tag{5}$$

Now we look at the corresponding long exact sequence of cohomology groups. Since all higher cohomology groups of the middle term vanish by Lemma A.3, we obtain an inclusion map

$$0\to H^i(X,\operatorname{gr}\mathcal{M}(\mu))\to H^{i+1}(X,\operatorname{gr}\mathcal{N}(\mu)) \tag{6}$$

for all $i\geqq 1$. Since we know that the cohomology groups vanish in all dimensions $i>\dim X$ for general reasons (cf. [Ha], p. 208), we may argue by descending induction on i as follows: The filtration induced on the "kernel" $\mathcal{N}(\mu)$ is also good, so we may assume by induction hypothesis that there exists a character λ_i of P such that for all $\mu\geqq\lambda_i$, we have

$$H^{i+1}(X,\operatorname{gr}_j\mathcal{N}(\mu))=0$$

in almost all degrees j. Now (6) implies that also $H^i(X,\operatorname{gr}\mathcal{M}(\mu))$ vanishes in almost all degrees. Q.e.d.

A.6. Returning to our description of associated varieties as given in Theorem 1.9c), we are now able to prove a refined version, which takes account of the multiplicities – at least after applying a suitable "twist".

Corollary. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module on a complete homogeneous space X. With respect to a good filtration, we have*

$$\mathcal{V}_{D(X)}(\Gamma(X, \mathcal{M}(\mu))) = \mathcal{S}upp(\gamma_*(\mathrm{gr}\,\mathcal{M}(\mu))), \tag{1}$$

for a suitable (any "sufficiently dominant") *character μ of P.*

Proof. Take λ as in Proposition A.5, so that for all $\mu \geqq \lambda$, we have $H^i(X, \mathcal{M}_j(\mu)) = 0$ for $i \geqq 1$, in almost all degrees j. As explained in 1.6, remark, this (for $i=1$) is exactly what is needed to conclude that the graded monomorphism

$$\mathrm{gr}\,\Gamma(X, \mathcal{M}(\mu)) \hookrightarrow \Gamma(X, \gamma_*(\mathrm{gr}\,\mathcal{M}(\mu)))$$

is an isomorphism in almost all degrees. And this implies (1), except in the case where the associated variety reduces to a single point. But this latter case is easily settled by a separate argument. Q.e.d.

Appendix B

More geometry of nilpotent orbits "à la Steinberg"

Let us return here once more to our discussion of the geometry of Steinberg's triple variety $\mathcal{T}$ (see 4.6, 5.2, 5.3). The purpose of this appendix is to extract and to extrapolate from Steinberg's analysis in [St1] the general geometrical back-ground necessary to understand our study of l-characteristic varieties of primitive ideals in §6. The main point to be clarified here is *Steinberg's surjection* from Weyl group elements w to nilpotent orbits $\mathcal{O}_x$, which provides even a refined surjection from Weyl group elements to irreducible components of $\pi^{-1}\mathcal{O}_x$ (whose closures are the candidates for l-characteristic varieties of primitive ideals by the philosophy developped in our §6).

B.1. Preimage of an orbit under Springer's resolution

We consider here an arbitrary nilpotent orbit $\mathcal{O}_u$ in $\mathfrak{g}$, generated by a nilpotent element $u \in \mathfrak{b}$. By abuse of language, the fibre $\pi^{-1}u$ under Springer's resolution map π, resp. its irreducible components, are referred to as the *Springer fibre* resp. *components* (at u). Such fibres are well known to be equidimensional (Spaltenstein [Sp1]), of dimension

$$d_u := \dim X - \tfrac{1}{2}\dim \mathcal{O}_u = \dim \pi^{-1}u \tag{1}$$

(Steinberg [St1]). The group $C(u) = G_u/G_u^0$ of connected components of the isotropy group G_u of u acts by permutations on the set of Springer components (at u). This action commutes with Springer's W-action on the cohomology group $H^{2d_u}(\pi^{-1}u, \mathbb{Q})$. By definition, Springer's Weyl group representation ρ_u is the restriction of this W-action to the subspace $H^{2d_u}(\pi^{-1}u, \mathbb{Q})^{C(u)}$ of $C(u)$-invariant cohomology classes. (We use notations as in [BoB], where Springer's correspondence from nilpotent orbits to irreducible Weyl group representations was recalled in 6.3.)

Proposition. a) *$\pi^{-1}\mathcal{O}_u$ is equidimensional of dimension*

$$\dim \pi^{-1}\mathcal{O}_u = \dim X + \tfrac{1}{2}\dim \mathcal{O}_u. \tag{2}$$

b) *The number of irreducible components of $\pi^{-1}\mathcal{O}_u$ equals $\dim \rho_u$.*

c) *The map $C \mapsto GC$ which attaches to each Springer component C its G-saturation, induces an isomorphism.*

$$H^{2\dim X + \dim \mathcal{O}_u}(\pi^{-1}\mathcal{O}_u, \mathbb{Q}) \cong H^{2\dim X - \dim \mathcal{O}_u}(\pi^{-1}u, \mathbb{Q})^{C(u)}. \tag{3}$$

Proof. The G-equivariance of π implies that $\pi^{-1}\mathcal{O}_u = G\pi^{-1}u$ is the G-saturation of the Springer fibre $\pi^{-1}u$, and therefore

$$\pi^{-1}\mathcal{O}_u = \bigcup_C GC, \tag{4}$$

where C ranges over all irreducible components of $\pi^{-1}u$.

For each such C, the G-saturation GC is irreducible, because G is connected. Moreover, its dimension is

$$\dim GC = \dim X + \tfrac{1}{2}\dim \mathcal{O}_u. \tag{5}$$

In fact, π (again because of its G-equivariance) maps GC onto $\mathcal{O}_u$ with all fibres containing a G-conjugate of C, so that $\dim GC \geqq \dim C + \dim \mathcal{O}_u$. But since $\dim C = \dim \pi^{-1}u$ by Spaltenstein (loc.cit.), this latter inequality implies (5) by Steinberg's dimension formula (1). From (5) and (4), part a) of the proposition is now clear.

More precisely, we conclude from (4) and (5) that $C \mapsto GC$ defines a surjection

$$\mathrm{Irr}(\pi^{-1}u) \twoheadrightarrow \mathrm{Irr}(\pi^{-1}\mathcal{O}_u), \tag{6}$$

where we write $\mathrm{Irr}(V)$ for the set of irreducible components of a variety V. Now suppose two Springer components C, C' give the same G-saturation $GC = GC'$. Then C' is contained in

$$\pi^{-1}u \cap GC' = \pi^{-1}u \cap GC = G_u C = \bigcup_{z \in C(u)} zC,$$

and so must coincide with one of the (finitely many) $C(u)$-conjugates of C. Conversely, $C' = zC$ for some $z \in C(u)$ implies of course $GC = GC'$. We have shown that (6) induces a bijection

$$\mathrm{Irr}(\pi^{-1}u)/C(u) \xrightarrow{\sim} \mathrm{Irr}(\pi^{-1}\mathcal{O}_u). \tag{7}$$

Here the left hand side denotes the set of $C(u)$-orbits. Now the sets in (7) form canonical bases for the vector-spaces in (3). Therefore part c) of the proposition follows from (7). Finally, c) implies b). Q.e.d.

B.2. Relation of Schubert cells to nilpotent orbits

A closed subvariety of the maximal nilpotent subalgebra $\mathfrak{n}=\mathfrak{b}^{\perp}$ in $\mathfrak{g}$ is called *orbital* (resp. *orbital for* $\mathcal{O}$) if it is an irreducible component of $\overline{\mathcal{O}}\cap\mathfrak{n}$ for some (resp. *the*) nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$.

Lemma. *For each Schubert cell X_w ($w\in W$), the image of $\overline{T^*_{X_w}X}$ under π is orbital.*

To be more precise, it is orbital for $\mathcal{O}_u$, with u a generic element in $\mathfrak{n}\cap\mathfrak{n}^w$. In fact, recall that $\mathfrak{n}\cap\mathfrak{n}^w$ is the fibre of the cotangent bundle $T^*_{X_w}X$, which is therefore mapped onto $B(\mathfrak{n}\cap\mathfrak{n}^w)\subset\mathfrak{n}\cap\overline{\mathcal{O}}_u$ under π. Therefore the lemma will follow from the equality of dimensions

$$\dim B(\mathfrak{n}\cap\mathfrak{n}^w)=\dim\mathfrak{n}\cap\mathcal{O}_u. \tag{1}$$

It follows from the work of Spaltenstein and Steinberg (loc.cit.) that $\mathfrak{n}\cap\mathcal{O}_u$ is equidimensional of dimension

$$\dim\mathfrak{n}\cap\mathcal{O}_u=\tfrac{1}{2}\dim\mathcal{O}_u. \tag{2}$$

On the other hand, the restriction of π to π_w: $T^*_{X_w}X\to B(\mathfrak{n}\cap\mathfrak{n}^w)$ has generic fibre dimension (cf. B.1(1))

$$\dim\pi_w^{-1}u\leqq\dim\pi^{-1}u=d_u=\dim X-\tfrac{1}{2}\dim\mathcal{O}_u. \tag{3}$$

Now (3) and (2) imply (1), completing the proof of the lemma. Q.e.d.

Remarks. a) The argument above also gives that equality must hold in (3), i.e. $\dim\pi_w^{-1}u=\dim\pi^{-1}u$, or in other words: $\pi_w^{-1}u$ contains a full Springer component.

b) All orbital varieties are obtained from Schubert cells in the manner described in the lemma above. This follows from the work of Steinberg [St1], or also from our next proposition B.3 (cf. also [Gi, J3] for this context).

B.3. Alternative descriptions of "orbital cone bundles"

Let us point out here a remarkable coincidence between the following five types of objects, which appear quite different at first glance:

1. Projections of components of Steinberg's triple veriety to T^*X.
2. G-saturations of closures of conormal bundles of Schubert cells.
3. Closures of G-saturations of Springer components.
4. Closures of irreducible components of $\pi^{-1}\mathcal{O}_u$.
5. Associated fibre bundles $G\times^B V$ of orbital varieties V.

To be more precise, let us state the following

Proposition. *Let Y be a closed subvariety of $T^*X=G\times^B\mathfrak{n}$. The following five statements are equivalent. There exists a Weyl group element $w\in W$ resp. a nilpotent orbit $\mathcal{O}_u$ such that:*

(i) *$\mathcal{T}_w$ projects onto a dense subset of Y under π^l.*
(ii) *The G-saturation of $T^*_{X_w}X$ is dense in Y.*
(iii) *The G-saturation of some Springer component at u is dense in Y.*
(iv) *Some irreducible component of $\pi^{-1}\mathcal{O}_u$ is dense in Y.*
(v) *$Y=G\times^B V$ for some irreducible component V of $\overline{\mathcal{O}_u\cap\mathfrak{n}}$.*

Proof. By 5.3, we have $\pi^l(\bar{\mathcal{T}}_w)=G\overline{T^*_{X_w}X}=G\times^B V$, with V the closure of $B(\mathfrak{n}\cap\mathfrak{n}^{w^{-1}})$, which is orbital, say for $\mathcal{O}_u$, by Lemma B.2. This gives (i)$\Rightarrow$(ii)$\Rightarrow$(v). Clearly, $G\times^B V$ is irreducible of dimension $\dim X+\dim V=\dim X+\frac{1}{2}\dim\mathcal{O}_u$. Since π maps it onto $GV=\overline{\mathcal{O}}_u$, it is therefore necessarily the closure of an irreducible component of $\pi^{-1}\mathcal{O}_u$ by Proposition B.1a). This gives (v)$\Rightarrow$(iv). Since (iv)$\Leftrightarrow$(iii) was already explained in detail in B.1, it only remains to prove implication (iii)$\Rightarrow$(i) to establish the proposition.

For this purpose, let us start from an arbitrary nilpotent orbit $\mathcal{O}_u$, and an arbitrary Springer component C at u. We have to find $w\in W$ such that $\pi^l(\mathcal{T}_w)$ is dense in $\overline{GC}$. The existence of such w follows from Steinberg's work [St1]. For the convenience of the interested reader, let us give some more details about this last step in the next remark. Q.e.d.

B.4. Remark. Steinberg decomposes his triple variety $\mathcal{T}$ in two alternative ways: Either with respect to Weyl elements w (the "attitudes"), or with respect to nilpotent orbits:

$$\mathcal{T}=\bigcup_{w\in W}\mathcal{T}_w=\bigcup_{u\in\mathcal{N}/G}\mathcal{T}_u,$$

where $\mathcal{T}_w$ resp. $\mathcal{T}_u$ consists of those triples (x,B_1,B_2) in $\mathcal{T}$ with (B_1,B_2) in the K-orbit Z_w resp. with x in the G-orbit $\mathcal{O}_u$. Now consider any two Springer components C_1, C_2 at u, and write them as $C_i=\{u\}\times D_i$ with $D_i\subset X$ ($i=1,2$). Then $\mathcal{T}_u$ contains $\{u\}\times D_1\times D_2$, hence also its K-saturation. This K-saturation fibres over $\mathcal{O}_u$ with fibre $D_1\times D_2$, and has therefore dimension $=\dim\mathcal{O}_u+\dim C_1+\dim C_2$, which by Spaltenstein's equidimensionality theorem and Steinberg's dimension formula (B.1(1)) equals $\dim\mathcal{O}_u+2d_u=2\dim X=\dim\mathcal{T}$. This argument shows that $\overline{K(\{u\}\times D_1\times D_2)}$ is an irreducible component of $\mathcal{T}$, and so (by 4.6) must be of the form $\bar{\mathcal{T}}_w$ for some $w\in W$. Now this element $w\in W$ satisfies B.3(i), for $Y=\overline{GC}_1$, because

$$\pi^l(\bar{\mathcal{T}}_w)=\overline{\pi^l(K(\{u\}\times D_1\times D_2))}=\overline{G\pi^l(\{u\}\times D_1\times D_2)}=\overline{GC}_1.$$

B.5. Steinberg left cells in the Weyl group

We say that $w,y\in W$ belong to the same *Steinberg left* resp. *right* resp. *double cell*, if and only if the projections of $\mathcal{T}_w$ and $\mathcal{T}_y$ under π^l resp. π^r resp. $\pi_Z=\pi\pi^l=\pi\pi^r$ coincide. The Steinberg left resp. right resp. double cell to which w

belongs is denoted by $\mathscr{C}^l_w$ resp. $\mathscr{C}^r_w$ resp. $\mathscr{C}^{lr}_w$. The corresponding equivalence relations are denoted $\sim_l$, $\sim_r$, resp. $\sim_{lr}$. It is clear from B.3 that $\sim_{lr}$ is the equivalence relation generated by $\sim_l$ and $\sim_r$, and that $w \sim_l y$ if and only if $w^{-1} \sim_r y^{-1}$. Moreover, we can define a quasi-order relation $\leqq_l$ on W, by $w \leqq_l y$ iff $\pi^l(\mathscr{T}_w) \supset \pi^l(\mathscr{T}_y)$, so that $\sim_l$ is given by $w \sim_l y$ iff $w \leqq_l y \leqq_l w$, etc.

Note that all this is completely analogous to the combinatorial notions of Kazhdan-Lusztig left, right or double cells, their quasi-order relations $\leqq_L$, $\leqq_R$, $\leqq_{LR}$, their equivalence relations $\sim_L$, $\sim_R$, $\sim_{LR}$ etc. on the Weyl group W, as introduced in [KL]. We are using completely analogous notions for the two situations, replacing only L resp. R by l resp. r. For example, recall that the Kazhdan-Lusztig left cells are given by

$$\mathscr{C}^L_w = \{y \in W \mid y \sim_L w\} = \{y \in W \mid J_y = J_w\},$$

with notation as in [BoB], 6.11, or also in [BV2].

In general, there are more Steinberg double cells than Kazhdan-Lusztig (KL) double cells. In fact, it follows from 5.3 resp. from [BV1, BV2] (cf. also [HK]) that the Steinberg resp. KL double cells are in bijective correspondence to all nilpotent orbits resp. to *special* nilpotent orbits only. The double cell corresponding to a nilpotent orbit $\mathcal{O}_u$ resp. to a special nilpotent orbit $\mathcal{O}_x$ is given by

$$\mathscr{C}^{lr}_u = \{w \in W \mid \pi\pi^l(\mathscr{T}_w) = \bar{\mathcal{O}}_u\}$$

resp.

$$\mathscr{C}^{LR}_x = \{w \in W \mid \mathscr{V}(\operatorname{gr} J_w) = \bar{\mathcal{O}}_x\}.$$

Let us now list a few properties of Steinberg left cells, for convenience of reference.

Proposition (cf. Spaltenstein [Sp2], Steinberg [St1]).

a) *For $G = SL_n$, the equivalence relations $\sim_l$, $\sim_r$ resp. $\sim_{lr}$ coincide with $\sim_L$, $\sim_R$ resp. $\sim_{LR}$. In particular, $\mathscr{C}^l_w = \mathscr{C}^L_w$ for all $w \in W$.*

b) *Each Steinberg left cell contains a self-inverse element.*

c) *Each Steinberg double cell $\mathscr{C}^{lr}_u$ (corresponding to $\mathcal{O}_u$ as above) decomposes into exactly $\dim \rho_u$ Steinberg left cells.*

d) *The Steinberg double cell $\mathscr{C}^{lr}_u$ is in bijection to the set of $C(u)$-orbits of pairs (C_1, C_2) of Springer components at u:*

$$(\operatorname{Irr} \pi^{-1} u \times \operatorname{Irr} \pi^{-1} u)/C(u) \xrightarrow{\sim} \mathscr{C}^{lr}_u.$$

The bijection is induced by $(C_1, C_2) \mapsto w$, where

$$\overline{K(\{u\} \times C_1 \times C_2)} = \bar{\mathscr{T}}_w$$

as in B.4.

Remark. Note that the analogue of b) for [KL] left cells is also true, by Duflo's result (mentioned already in 6.1). Similarly, the analogue of c) for KL double cells $\mathscr{C}^{LR}_x$ is also true (provided that $\mathcal{O}_x$ is special), as is well known.

Proofs. a) is clear from [Sp2], and b), c) follow from d), which can be found in [St1]. To give a few more details, let us look at the surjective maps

$$(\operatorname{Irr} \pi^{-1} u \times \operatorname{Irr} \pi^{-1} u)/C(u) \rightrightarrows (\operatorname{Irr} \pi^{-1} u)/C(u) \tag{$*$}$$

induced by the forgetful maps $(C_1, C_2) \mapsto C_1$ resp. C_2. Then the fibres of these maps are precisely the Steinberg left resp. right cells. Since the image (the right hand side of $(*)$) is a basis for Springer's representation ρ_u (cf. 6.5), this fact implies c). The element $w \in \mathscr{C}^{lr}_u$ determined by the pair of components (C_1, C_2) is just the "attitude" of C_1 and C_2, considered as subsets of X, that is to say: w is the relative position of Borel subgroups B_1 and B_2 chosen generically in C_1 resp. C_2. From this picture, it is now clear that if (C_1, C_2) represents w, then (C_2, C_1) represents w^{-1}. Consequently, (C_1, C_1) represents an element $w \in W$ such that $w = w^{-1}$, i.e. a self-inverse element. This explains why b) holds. Q.e.d.

TENORIO

Was antwortete er?

DON GONZALO

Er hasse die Heiden nicht.

TENORIO

Junge, Junge!

DON GONZALO

Im Gegenteil, sagte er, wir könnten viel von den Heiden lernen, und wie ich ihn das nächste Mal traf, lag er unter einer Korkeiche und las ein Buch. Ein arabisches.

TENORIO

Geometrie, ich weiß, der Teufel hole die Geometrie.

DON GONZALO

Ich fragte, wozu er das lese.

TENORIO

Was, um Gottes willen, antwortete er?

DON GONZALO

Er lächelte bloß.

(Max Frisch, *Don Juan oder die Liebe zur Geometrie*)

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Inventiones mathematicae

Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups*

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Dedicated to E.B. Dynkin on his 60th birthday

The purpose of the present note is to give a general procedure for calculating the singular cohomology $H^*(K, \mathbb{F}_p)$ with coefficients in an arbitrary finite field $\mathbb{F}_p$ of a connected compact Lie group K. Simultaneously, we calculate the Chow ring $A(G, \mathbb{F}_p) := A(G) \otimes_{\mathbb{Z}} \mathbb{F}_p$ of the corresponding complex algebraic group G, and the degrees of the basic "generalized invariants" of the Weyl group W in arbitrary characteristic p. A mysterious connection between the degrees of generators of $A(G, \mathbb{F}_p)$ and the kernel of the principal nilpotent element in characteristic p is pointed out.

The computation of real cohomology of compact Lie groups started by E. Cartan in 1929 was completed for classical groups by R. Brauer and by L. Pontryagin in 1935. H. Hopf and H. Samelson showed in 1941 that $H^*(K, \mathbb{R})$ is an exterior algebra on generators of degrees $2d_1-1, \ldots, 2d_l-1$, where $l = \operatorname{rank} K$. But only in 1949 C.T. Yen managed to compute the d_i for all exceptional groups, case by case. On the other hand, C. Chevalley and J. Leray showed that the d_i are nothing else but the degrees of the basic invariants over $\mathbb{R}$ of the Weyl group W. A simple procedure for calculating the d_i was given in the late fifties by A.J. Coleman and by B. Kostant. Nowadays this is common knowledge.

The problem of computing the torsion in $H^*(K, \mathbb{Z})$ proved to be more recalcitrant. This was done by several authors: C.E. Miller settled SO_n in 1953 [23], A. Borel settled Spin_n, $\mathrm{Ad\,Spin}_n$, and the quotients of SU_n and Sp_n in 1954 [6], P.F. Baum and W. Browder settled the remaining classical group, the semispin group, in 1965 [4]. G_2 and F_4 were settled in 1954 [6], and the $p \geqq 3$-torsion for E_6 and $p \geqq 5$-torsion for E_7, E_8 in 1961 [7], by A. Borel. S. Araki computed the 2-torsion for E_6, E_7 [1] and the 3-torsion for E_7, E_8 [2], and announced the 2-torsion for E_8 in a joint paper with Y. Shikata in 1961 [3]. The details of the proof for the last result have appeared only recently [18]. There was an error in [1] in the case of $\mathrm{Ad}\, E_7$, which has been fixed in [13].

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