First draft

~~Localization of $\mathfrak{g}$-modules II~~

A proof of ~~the~~ Jantzen conjectures

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This paper concerns with some applications of Weil conjectures to representation theory. The main result claims that the localization functor transforms the Jantzen filtration on Verma modules or standard Harish-Chandra modules into the weight filtration on the corresponding perverse sheaves. This fact implies in a moment the remarkable properties of Jantzen filtration: namely, the hereditary property (conjectured by Jantzen), the socle and cosocle properties (the socle one was also proved in [1] by purely algebraic means), and the generalized Kazhdan-Lusztig algorithm for computation of multiplicities in successive quotients (conjectured in [15], [17]).

Here is a brief outline of the paper. The first two paragraphs review the representations - $\mathcal{D}$-modules dictionary; we tried to fix some topics available in folklore only. The §1 contains the discussion round the equivariant $\mathcal{D}$-modules with no representations mentioned; in particular it contains some points about the equivariant derived category and Langlands classification. The §2 deals with localization construction: it covers, in particular, the geometric description of representations with degenerate weight, and the proof of Vogan's semisimplicity conjecture for wall crossing functor. The §3 contains the proof of some geometric properties of K-orbits that are needed for the definition of Jautzen filtration; one finds there also the geometric description of contravariant duality for standard modules. In §4 we define the Jantzen filtration and describe its close relation with monodromy filtration on vanishing cycles. Now in §5 the mixed perverse sheaves appear. We present a proof of a Gabber's theorem about the weight filtration on vanishing cycles, which, being joined with results of previous §, immediately implies the basic fact that geometric Jantzen filtration coincides with weight one. The second point of the § is a remark that pointwise purity of irreducible perverse sheaves (= Deligne - Goreski - Macpherson complexes) implies the socle property for weight filtration in any case when it has chance to occur. Transmitting this stuff to representations world one gets the Jantzen conjecture and all that.

The main results where found in the first half of 1981; the most part of work was done jointly, and the final step was taken separately in Moscow and Bures respectively. The version presented below, which follows the lectures given at a seminar in Moscow in spring 1982, was written down by the first named author; he bears all the responsibility for the gaps and defects of exposition.

§1. Equivariant and monodromic $\mathcal{D}$-modules

1.1. Preliminary notations. In what follows (except §5) "variety" = "scheme" = "quasicompact separated scheme of finite type over $\mathbb{C}$"; in fact one may replace $\mathbb{C}$ by any field of char 0 in any place which has nothing to do with Riemann-Hilbert correspondence. If $\pi : X \to Y$ is a morphism of varieties then $\pi_\cdot$, $\pi^\cdot$ denote the direct and inverse image functors on the category of all sheaves on these varieties. If A is a $\mathbb{C}$-algebra, then $\mathcal{M}(A)$ will denote category of left A-modules, and ${}^r\mathcal{M}(A)$ will be the category of right ones. A subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is Serre subcategory if $\mathcal{B}$ is strictly full subcategory closed under extensions and subquotients; we have the quotient category $\mathcal{A}/\mathcal{B}$.

1.1.1. Let $i : R \to A$ be a morphism of $\mathbb{C}$-algebras such that R is commutative $\mathbb{C}$-algebra of finite type. A good filtration on (A, i) is a ring filtration $A_o \subset A_1 \subset \ldots$, $\bigcup A_i = A$, $A_i \cdot A_j \subset A_{i+j}$, such that $i(R) \subset A_o$ and $Gr.A$ is commutative $\mathbb{C}$-algebra of finite type; we will say that (A, i) is good R-ring if it admits a good filtration. A global version of these is a variety X together with a sheaf A of $\mathbb{C}$-algebras on X equipped with a $\mathbb{C}$-algebras map $i : \mathcal{O}_X \to A$. A good filtration on (A, i) is a ring filtration $A_o \subset A_1 \subset \ldots$ on A such that $i(\mathcal{O}_X) \subset A_o$ and $Gr.A$ is a commutative quasicoherent $\mathcal{O}_X$-algebra of finite type; (A, i) (or simply A) is good if it admits a good filtration. Any good A is quasicoherent as $\mathcal{O}_X$-module. If X is affine, then the global sections functor defines one-to-one correspondence between good A's on X and good $\Gamma(X, \mathcal{O}_X)$-rings. If B is another good algebra on a variety Y, then we have a good algebra $A \boxtimes B$ on $X \times Y$ with $A \boxtimes B(U \times V) = A(U) \otimes_{\mathbb{C}} B(V)$ for affine opens $U \subset X$, $V \subset Y$. The basic example of a good algebra is sheaf $\mathcal{D}_X$ of differential operators on X (assumed to be smooth) with standard embedding $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$.

Let A be a good algebra. Then "A-module" will mean "sheaf of A-modules quasicoherent as $\mathcal{O}_X$-module". Usually we will use left A-modules, and call them simply A-modules; these form an abelian category $\mathcal{M}(A)$. The category of right A-modules will be denoted ${}^r\mathcal{M}(A)$. An A-module is coherent if it is locally finitely generated; these form a full subcategory $\mathcal{M}(A)_{coh} \subset \mathcal{M}(A)$ which is a Serre subcategory.

Lemma-definition. Let A be a good algebra; put $R = \Gamma(X, A)$. We will say that X is A-affine if the following equivalent conditions hold

(i) For each $M \in \mathcal{M}(A)$ one has $H^i(X, M) = 0$ for $i > 0$ and M is generated by global sections

(ii) The functor $\Gamma(X, \cdot) : \mathcal{M}(A) \to \mathcal{M}(R)$ is equivalence of categories. $\square$

If X is affine, then it is A-affine with respect to any A. If X is any quasiprojective variety, then one may choose an affine morphism $f : Y \to X$ with Y affine, hence X is $f_*(\mathcal{O}_Y)$-affine. In this paper we will deal with non-commutative versions of this phenomenon.

1.1.2. Let $S \subset \Gamma(X, A)$ be a subalgebra. We will say that an A-module M is S-finite if for any local section v of M one has $\dim_{\mathbb{C}} Sv < \infty$; denote by $\mathcal{M}(A)^f \subset \mathcal{M}(A)$ the full subcategory of S-finite modules. Assume that S lies in the center of A. If $\chi : S \to \mathbb{C}$ is a $\mathbb{C}$-point of S, let m_χ be the corresponding maximal ideal of S; for $n = 1, 2, \ldots$ put $A_{\chi,n} := A/m_\chi^n A$. Clearly $\mathcal{M}(A)_{\chi,n} := \mathcal{M}(A_{\chi,n})$ is a full subcategory of $\mathcal{M}(A)^f$; denote by $\mathcal{M}(A)_{\chi,\infty} \subset \mathcal{M}(A)$ the full subcategory of A-modules M such that for any local section v and $N \gg 0$ one has $m_\chi^N v = 0$. Any S-finite module has χ-isotypic decomposition: $M = \oplus M_\chi$, $M_\chi \in \mathcal{M}(A)_{\chi,\infty}$, so $\mathcal{M}(A)^f = \prod_\chi \mathcal{M}(A)_{\chi,\infty}$. We will need also the corresponding derived categories $D^b\mathcal{M}(A)$ etc. ~~Clearly~~ It is well known that $D^b\mathcal{M}(A)_{\chi,\infty} \subset D^b\mathcal{M}(A)^f \subset D^b\mathcal{M}(A)$ and $D^b\mathcal{M}(A)_{coh} \subset D^b\mathcal{M}(A)$ are fully faithful embeddings.

1.1.3. Let X be a smooth variety. An $\mathcal{O}_X$-Lie algebra is a sheaf of $\mathcal{O}_X$-modules $\mathcal{P}$ together with morphisms $[\,,\,] : \mathcal{P} \otimes_{\mathbb{C}} \mathcal{P} \to \mathcal{P}$, $\varepsilon : \mathcal{P} \to \mathcal{T}_X$ (here $\mathcal{T}_X$ denotes the sheaf of vector fields) such that

(i) $\mathcal{P}$ is coherent $\mathcal{O}_X$-module, ε is $\mathcal{O}_X$-linear map.

(ii) $[\,,\,]$ is Lie algebra bracket, and ε is Lie algebras map.

(iii) One has $[\xi, f\tau] = \varepsilon(\xi)(f)\tau + f[\xi,\tau]$ for $\xi, \tau \in \mathcal{P}$, $f \in \mathcal{O}_X$.

Let $\mathcal{P}$ be an $\mathcal{O}_X$-Lie algebra. Its twisted universal enveloping algebra $\mathcal{U}(\mathcal{P})$ is the sheaf of associative $\mathbb{C}$-algebras equipped with $\mathbb{C}$-algebras map $i : \mathcal{O}_X \to \mathcal{U}(\mathcal{P})$ and $\mathbb{C}$-Lie algebras map $j : \mathcal{P} \to \mathcal{U}(\mathcal{P})$ such that $\mathcal{U}(\mathcal{P})$ is generated by $i(\mathcal{O}_X)$ and $j(\mathcal{P})$ with the only relations $i(f)j(\xi) = j(f\xi)$, $[j(\xi), i(f)] = i(\varepsilon(\xi)(f))$. Clearly $(\mathcal{U}(\mathcal{P}), i)$ has canonical good filtration $\mathcal{U}(\mathcal{P})_0 = i(\mathcal{O}_X)$, $\mathcal{U}(\mathcal{P})_1 = j(\mathcal{P}) + i(\mathcal{O}_X)$, $\mathcal{U}(\mathcal{P})_n = (\mathcal{U}(\mathcal{P})_1)^n$. If $\mathcal{P}$ is locally free and ε is surjective then $Gr.\mathcal{U}(\mathcal{P}) = Sym^{\bullet}(\mathcal{P})$. If $\mathcal{P} = \mathcal{T}_X$, then $\mathcal{U}(\mathcal{P}) = \mathcal{D}_X$.

1.1.4. Let H be an (affine) algebraic group, X be a smooth variety, and $\pi : \tilde{X} \to X$ be a principal left H-bundle over X. Let $\mathfrak{h} :=$ Lie H be Lie algebra of H, and $\mathfrak{h}^{\sim}$ be $\mathfrak{h} \otimes \mathcal{O}_X$ twisted by means of $\tilde{X}$ with respect to adjoint action of H; one identifies

$\mathfrak{h}^{\sim}$ with the Lie algebra of vertical H-invariant vector fields on X. The current algebra $\widetilde{\mathcal{T}}$ of $\widetilde{X}$ is $\widetilde{\mathcal{T}}_X := \pi.(\mathcal{T}_{\widetilde{X}})^H \subset \pi.\mathcal{T}_{\widetilde{X}}$, the Lie algebra of all H-invariant vector fields on $\widetilde{X}$. One has a canonical exact sequence $0 \to \mathfrak{h}^{\sim} \to \widetilde{\mathcal{T}}_X \xrightarrow{d\pi} \mathcal{T}_X \to 0$ of Lie algebras on X. Clearly $\varepsilon = d\pi$ together with obvious $\mathcal{O}_X$-module structure defines $\mathcal{O}_X$-Lie algebra structure on $\widetilde{\mathcal{T}}_X$. Let $\widetilde{\mathcal{D}}_X := (\pi.\mathcal{D}_{\widetilde{X}})^H \subset \pi.\mathcal{D}_X$ be the algebra of H-invariant differential operators on $\widetilde{X}$. One has

(i) The obvious embeddings $i : \mathcal{O}_X \to \widetilde{\mathcal{D}}_X$, $j : \widetilde{\mathcal{T}}_X \to \widetilde{\mathcal{D}}_X$ identify $\widetilde{\mathcal{D}}_X$ with twisted universal enveloping algebra of $\widetilde{\mathcal{T}}_X$.

(ii) $\pi.\mathcal{D}_{\widetilde{X}} = \pi.\mathcal{O}_{\widetilde{X}} \underset{\mathcal{O}_X}{\otimes} \widetilde{\mathcal{D}}_X$ with ring structure that comes from (i) via the $\widetilde{\mathcal{T}}_X$-action on $\pi.\mathcal{O}_{\widetilde{X}}$.

(iii) $d\pi$ identifies the quotient $\widetilde{\mathcal{D}}_X/\mathfrak{h}^{\sim}\widetilde{\mathcal{D}}_X$ with $\mathcal{D}_X$.

(iv) Any (local) section of $\widetilde{X}$, i.e. an H-isomorphism $\widetilde{X} \simeq X \times H$, induces isomorphism $\widetilde{\mathcal{D}}_X \simeq \mathcal{D}_X \underset{\mathbb{C}}{\otimes} \mathcal{U}(\mathfrak{h})$.

If H is commutative, say H is a torus, then $\mathfrak{h}^{\sim} = \mathfrak{h} \otimes \mathcal{O}_X$, hence we have the embedding $\mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}) \hookrightarrow \widetilde{\mathcal{D}}_X$ which identifies $S(\mathfrak{h})$ with the center of $\widetilde{\mathcal{D}}_X$. According to 1.1.2 one gets the categories $\mathcal{M}(\widetilde{\mathcal{D}}_X)_{\chi,1} \subset \mathcal{M}(\widetilde{\mathcal{D}}_X)_{\chi,2} \subset \dots \subset \mathcal{M}(\widetilde{\mathcal{D}}_X)_{\chi,\infty} \subset \mathcal{M}(\mathcal{D}_X)^f \subset \mathcal{M}(\widetilde{\mathcal{D}}_X)$; here $\chi \in \mathfrak{h}^*$.

1.2. <u>Equivariant $\mathcal{D}$-modules</u>. We use [9] as reference book for $\mathcal{D}$-modules. The category $\mathcal{M}(\mathcal{D}_Y)$ of $\mathcal{D}_Y$-modules will be denoted simply $\mathcal{M}(Y)$; same for the derived category: $D^b(Y) := D^b\mathcal{M}(Y)$ etc.

Let G be an algebraic group and Y be a smooth variety with G-action $m : G \times Y \to Y$. Denote by $m_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{T}_Y$ the corresponding infinitezimal action of $\mathfrak{g} :=$ Lie G. Let $p_Y : G \times Y \to Y$, $p_G : G \times Y \to G$ be projections.

1.2.1. Definition. (i) A $(\mathcal{D}_Y, G)$-module M is $\mathcal{D}_Y$-module together with isomorphism $m^{\circ}M \simeq p_Y^{\circ}M$ of $D_{G\times Y}$-modules such that the usual compatibilities hold.

(ii) A weak $(\mathcal{D}_Y, G)$-module M is $\mathcal{D}_Y$-module together with isomorphism (with usual compatibilities) $m^{\circ}M \simeq p_Y^{\circ} M$ considered as $\mathcal{O}_G \boxtimes \mathcal{D}_Y$-modules (note that $\mathcal{O}_G \boxtimes \mathcal{D}_Y$ is subring of $\mathcal{D}_G \boxtimes \mathcal{D}_Y = \mathcal{D}_{G\times Y}$). Equivalently, this is $\mathcal{D}_Y$-module M together with G-action on M considered as $\mathcal{O}_Y$-module such that for $g \in G$ and local sections $\partial \in \mathcal{D}_Y$, $v \in M$ one has $g(\partial v) = g(\partial)g(v)$. □

Clearly these form the abelian categories denoted by $\mathcal{M}(Y, G)$, $\mathcal{M}(Y, G)_{weak}$; we have the faithful embedding $\mathcal{M}(Y, G) \hookrightarrow \mathcal{M}(Y, G)_{weak}$ and forgetting functor $o_G : M(Y, G) \to M(Y)$.

Let M be a weak $(\mathcal{D}_Y, G)$-module. Then $\mathfrak{g}$ acts on $\mathcal{O}_Y$-module M in two ways: first, we have the infinitezimal action that comes from G-action on M, and the second one comes from $\mathcal{D}_Y$-module structure via $\mathfrak{g} \xrightarrow{m_{\mathfrak{g}}} \mathcal{T}_Y \subset \mathcal{D}_Y$. Let $\mathfrak{g} \ni \xi \mapsto \xi^i$, $i = 1, 2$, denote these actions. Consider the map $w : \mathfrak{g} \to \mathrm{End}\, M$, $w(\xi) = \xi^1 - \xi^2$

1.2.2. Lemma. (i) $w(\xi)$ commutes with $\mathcal{D}_Y$-action, $w(Ad_g(\xi)) = gw(\xi)g^{-1}$, and $w : \mathfrak{g} \to \mathrm{End}_{\mathcal{D}_Y} M$ is Lie algebras map.

(ii) A weak $(\mathcal{D}_Y, G)$-module ~~lies in $M(Y, G)$~~ is $(\mathcal{D}_Y, G)$-module iff $w(\mathfrak{g}) = 0$. □

Now let N be a $\mathcal{D}_Y$-module. Then $p_Y N$ is $(\mathcal{D}_{G\times Y}, G)$-module (here G acts on G Y along the first multiple). Put $\mathrm{Ind}_{weak}(N) := m.p_Y^{\circ}N$. This is a weak $(\mathcal{D}_Y, G)$-module: namely, the $(\mathcal{O}_Y, G)$-module structure comes from $(\mathcal{O}_{G\times Y}, G)$-module structure on $p_Y N$, and a vector field $\tau \in \mathcal{T}_Y \subset \mathcal{D}_Y$ acts on $\mathrm{Ind}_{weak}N$ by means of $\tilde{\tau} \in \mathcal{T}_{G\times Y} \subset \mathcal{D}_{G\times Y}$ with $dp_G(\tilde{\tau}) = 0$, $dm(\tilde{\tau}) = \tau$. This way we get the exact functor $\mathrm{Ind}_{weak} : \mathcal{M}(Y) \to \mathcal{M}(Y, G)_{weak}$

(exactness follows since m is affine). One has

(i) Consider the G-action $*$ on $G \times Y$ defined by formula $g * (\ell, y) = (g\ell, \ell^{-1}g^{-1}\ell\, y)$, $g, \ell \in G$, $y \in Y$ (so $*$ is a free action along the fibers of m). Let $\mathfrak{g} \to \mathcal{T}_{G \times Y}$, $\xi \mapsto \hat{\xi}$, be the corresponding infinitezimal action. Then $w(\xi) \in \mathrm{End}(\mathrm{Ind}_{weak} N)$ is the action of $\hat{\xi}$ on $P_Y^{\circ} N$.

(ii) Consider the embedding $i_{(1)} : Y \to G \times Y$, $i_{(1)}(y) = (1, y)$. The isomorphism $i_{(1)}^{\circ} P_Y^{\circ} N \simeq N$ defines a canonical surjection $\alpha : o_G\ \mathrm{Ind}_{weak}(N) \to N$ of $\mathcal{D}_Y$-modules.

(iii) A weak $(\mathcal{D}_Y, G)$-module structure on N defines the injection $\beta : N \hookrightarrow \mathrm{Ind}_{weak} N$ which comes from G-action (the $\mathcal{O}_G \boxtimes \mathcal{D}_Y$-isomorphism) $m^{\circ} N \simeq P_Y^{\circ} N$ joined with canonical map $N \hookrightarrow m_{\cdot} m^{\circ} N$.

1.2.3. Lemma. (i) The functor $\mathrm{Ind}_{weak} : \mathcal{M}(Y) \to \mathcal{M}(Y, G)_{weak}$ is right adjoint to the forgetting functor o_G by means of adjunction maps (α, β) of (ii), (iii) above.

(ii) The faithful embedding $\mathcal{M}(Y, G) \to \mathcal{M}(Y, G)_{weak}$ has left and right adjoints defined by formulas $M \mapsto M_{w(\mathfrak{g})}$, $M \mapsto M^{w(\mathfrak{g})}$ respectively. □

Examples. (i) Assume that Y is a point. Then $\mathcal{M}(Y, G)$ = representations of finite group G/G° (where G° := the connected component of G), $\mathcal{M}(Y, G)_{weak}$ = algebraic representations of G (here an algebraic representation of G on a vector space V is representation such that V is a union of finite dimensional subspaces with algebraic G-action). The functor Ind_{weak} transforms $\mathbb{C}$ to the regular representation $\mathcal{O}(G)$ of G.

(ii) Let Y be any G-variety. Then $\mathcal{D}_Y$ with usual G-action is weak $(\mathcal{D}_Y, G)$-module with $w(\xi)$, $\xi \in \mathfrak{g}$, equals to the right multiplication by $-m_{\mathfrak{g}}(\xi)$.

(iii) Assume we are in a situation 1.1.3. Since π is affine the functor $\pi_\cdot : \mathcal{M}(X) \to \mathcal{M}(\pi_\cdot \mathcal{D}_X)$ is equivalence of categories. Consider the adjoint functors $\mathcal{M}(\tilde{X}, H)_{weak} \underset{\pi^\#}{\overset{\pi_\#}{\rightleftarrows}} \mathcal{M}(\tilde{\mathcal{D}}_X)$ defined by formulas $\pi_\#(M) := \pi_\cdot(M)^H$, $\pi^\#(N) :=$ $\pi_\cdot^{-1}(\pi_\cdot \mathcal{D}_{\tilde{X}} \underset{\tilde{\mathcal{D}}_X}{\otimes} N) = \mathcal{D}_{\tilde{X}} \underset{\pi^\cdot \tilde{\mathcal{D}}_X}{\otimes} \pi^\cdot N = \mathcal{O}_{\tilde{X}} \underset{\pi^\cdot \mathcal{O}_X}{\otimes} \pi^\cdot N$.

1.2.4. Lemma. (i) $\pi^\#$, $\pi_\#$ are mutually inverse equivalences of categories.

(ii) $\pi_\#$ identifies $\mathcal{M}(\tilde{X}, H)$ with $\mathcal{M}(X)$ (embedded in $\mathcal{M}(\tilde{\mathcal{D}}_X)$ via 1.1.3 (iii))

(iii) $w(\xi)$ acts on $\pi^\#(N) = \mathcal{O}_{\tilde{X}} \underset{\pi^\cdot \mathcal{O}_X}{\otimes} \pi^\cdot N$ as multiplication by $\xi \in \mathfrak{h} \subset \mathcal{O}_{\tilde{X}} \otimes \mathfrak{h} = \mathcal{O}_{\tilde{X}} \underset{\pi^\cdot \mathcal{O}_X}{\otimes} \mathfrak{h}^{\sim}$. □

1.2.5. The main body of this paper requires only a bit of a naive equivariant functoriality to be sketched in this n°. For the general framework see ~~n°1.3.~~ [ ]

Let $a : G \to G'$ be a morphism of algebraic groups, Y, Y' be a smooth G- and G'-variety respectively, and $f : Y \to Y'$ be an a-equivariant map.

(i) If N is $(\mathcal{D}_{Y'}, G')$-module, then $\mathcal{D}_Y$-module $f^\circ(N)$ carries an obvious G-action. Hence one has the right exact functor $f^\circ : M(Y', G') \to M(Y, G)$. More generally, $f^!(N) \in D^b(Y)$ carries the "naive" G-action (we have the isomorphism $p_Y^\circ f^! N \simeq m_Y^\circ f^! N$ in $D^b(G \times Y)$), and this way we get the functors $H^a f^! : \mathcal{M}(Y', G') \to \mathcal{M}(Y, G)$.

(ii) Assume that $a : G \to G'$ is isomorphism, and M is $(\mathcal{D}_Y, G)$-module. Then the smooth base change gives the "naive" G'-action on $f_+(M) \in D^b(Y')$, hence one gets the functors $H^a f_+$:

$: \mathcal{M}(Y, G) \to \mathcal{M}(Y', G')$.

(iii) If a is isomorphism and f is closed embedding, then $H^0 f_+$ is faithful embedding that identifies $\mathcal{M}(Y, G)$ with the category of $(\mathcal{D}_{Y'}, G')$-modules supported on Y; the inverse functor is $H^0 f^!$. This follows immediately from usual Kashiwara theorem [9] VI, 7.13.

(iv) The duality D_Y transforms a coherent $(\mathcal{D}_Y, G)$-module to a complex with "naive" G-action, since D_Y commutes with smooth base change. Hence we have the functors $H^a D_Y : \mathcal{M}(Y, G)_{coh} \to \mathcal{M}(Y, G)_{coh}$, and the holonomic duality $D_Y : \mathcal{M}(Y, G)_{hol} \to \mathcal{M}(Y, G)_{hol}$.

(v) If $M_1, M_2 \in \mathcal{M}(Y, G)$, then $M_1 \otimes_{\mathcal{O}_Y} M_2 \in \mathcal{M}(Y, G)$.

(vi) If $G = G'$ and f is locally closed embedding, then one has the right exact functor $H^0 f_! := D_{Y'} H^0 f_* D_Y : \mathcal{M}(Y, G)_{hol} \to \mathcal{M}(Y', G')_{hol}$ together with a natural morphism $H^0 f_! \to H^0 f_*$. Put $f_{!*} := \operatorname{Im}(H^0 f_! \to H^0 f_*)$; this functor transforms irreducible $(\mathcal{D}_Y, G)$-modules to irreducible $(\mathcal{D}_{Y'}, G')$-ones.

The same functoriality (i)-(vi) holds for weak $(\mathcal{D}_Y, G)$-modules.

1.2.6. Lemma. Assume we are in a situation 1.2.4, and either of the following conditions holds:

- (descent) $a : G \to G'$ is surjective, $G'' := \operatorname{Ker} a$ acts on Y in a free way, and f is projection $Y \to G\backslash Y = Y'$

- (induction) $a : G \to G'$ is injective, and f is embedding such that Y' is a-induced G'-variety: $Y' = G' \times_G Y$

Then $f^0 : \mathcal{M}(Y', G') \to \mathcal{M}(Y, G)$ is equivalence of categories.

Proof. The descent property follows easily from, say, the usual descent for $\mathcal{O}$ -modules; the induction case follows from the descent applied to the maps $\rho_Y : G' \times Y \to Y$, $m : G' \times Y \to Y'$ (here $G' \times Y$ is considered as $G' \times G$-variety according to formula $(g', g)(g'', y) := (g'g''g^{-1}, gy)$). □

Omit up to n° 1.4

The results of the next n will not be seriously used in the main body of the paper, so the reader may omit it.

1.3. Equivariant derived category. As usually one acquires a functorial freedom by means of appropriate derived categories. It is easy to see that in case of equivariant $\mathcal{D}$-modules the naive derived category $D^b \mathcal{M}(Y, G)$ is too rigid to work with, and should be replaced by a different, more flexible, t-category $D^b(Y, G)$ with core $\mathcal{M}(Y, G)$. We know two constructions of this equivariant derived category. The first one, due to Lunts and the second named author (see appendix to [28]) works in every geometric situation: for $\mathcal{D}$-modules or perverse sheaves. It inherits automatically all the functoriality of its non-equivariant ancestor. The second construction, due to Ginsburg and first named author (see [18]), is a version of BRST method in physics. It works in algebraic situations: say, for D-modules and Harish-Chandra modules. In this n° will be presented this second construction (for Harish-Chandra case, implicit in [13], see 2.4 below); we will show that it is equivalent to the first one on the common ground of $\mathcal{D}$-modules.

Assume we are in situation 1.2. In what follows "weak $(\mathcal{D}_Y, G)$-complex" = "complex of weak $(\mathcal{D}_Y, G)$-modules".

1.3.1. Definition. A $(\mathcal{D}_Y, G)$-complex is a weak $(\mathcal{D}_Y, G)$-complex $M^\cdot$ together with a map $\mathfrak{g} \to \mathrm{Hom}_{\mathcal{M}(Y)}(M^\cdot, M^{\cdot -1})$, $\mathfrak{g} \ni \xi \mapsto i_\xi$, such that $i_{\xi_1} i_{\xi_2} + i_{\xi_2} i_{\xi_1} = 0$, $g\, i_\xi g^{-1} = i_{Ad_g(\xi)}$, $di_\xi + i_\xi d = w(\xi)$ for $\xi_i \in \mathfrak{g}$, $g \in G$. A $(\mathcal{D}_Y, G)$-complex $M^\cdot$ is coherent if it is bounded and each M^i is coherent $\mathcal{D}_Y$-module. □

The (D_Y, G)-complexes form an abelian category $C^\cdot(Y, G)$; here $\cdot$ is "boundary condition": $\bullet = \pm$, $\emptyset$ or b. Let $C(Y,G)_{coh} \subset C^b(Y, G)$ be the full subcategory of coherent complexes. Any bounded below complex $M^\cdot \subset C^+(Y, G)$ is direct limit of its coherent subcomplexes (this follows by the similar fact for weak $(\mathcal{D}_Y, G)$-modules, which is clear since any quasicoherent $(\mathcal{O}_Y, G)$-module is union of coherent G-invariant subsheaves).

Let $\bar{\mathfrak{g}}^\cdot$ be the differential graded Lie algebra with $\bar{\mathfrak{g}}^0 := \mathfrak{g}$ with usual bracket, $\bar{\mathfrak{g}}^{-1} = \mathfrak{g}$, $\bar{\mathfrak{g}}^i = 0$ for $i \neq 0, -1$ and $d : \bar{\mathfrak{g}}^{-1} \to \bar{\mathfrak{g}}^0$ is identity map; the group G acts on $\bar{\mathfrak{g}}^\cdot$ via adjoint representation. The universal enveloping algebra $\mathfrak{U}(\bar{\mathfrak{g}}^\cdot)$ is just the standard complex $C.(\mathfrak{g}, \mathfrak{U}(\mathfrak{g}))$. Note that any weak $(\mathcal{D}_Y, G)$-module is naturally a $\mathcal{D}_Y \otimes \mathfrak{U}(\mathfrak{g})$-module where $\mathfrak{g} \subset \mathfrak{U}(\mathfrak{g})$ acts via w. A $(\mathcal{D}_Y, G)$-complex is just a weak $(\mathcal{D}_Y, G)$-complex with the $\mathcal{D}_Y \otimes \mathfrak{U}(\mathfrak{g})$-action extended to $\mathcal{D}_Y \otimes \mathfrak{U}(\bar{\mathfrak{g}}^\cdot)$-action in a G-equivariant way: $\xi \in \mathfrak{g} = \bar{\mathfrak{g}}^{-1}$ acts as i_ξ.

One may define a homotopy between morphisms of $(\mathcal{D}_Y, G)$-complexes and homotopy category $K^\cdot(Y, G)$ in a usual manner. More precisely, for $(\mathcal{D}_Y, G)$-complexes $M_1^\cdot$, $M_2^\cdot$ one has the complex $Hom^\cdot(M_1, M_2)$, $Hom^i(M_1, M_2) = \{(f_\ell) \in \prod Hom_{M(Y,G)_{weak}}(M_1^\ell, M_2^{\ell+i}) : f_\ell i_\xi - (-1)^\ell i_\xi f_\ell = 0\}$, $d^i(f_\ell) = (df_\ell - (-1)^i f_{\ell-1} d)$. Then $Hom(M_1^\cdot, M_2^\cdot)$ coincides with degree zero cycles in $Hom^\cdot$, and $Hom_{K^\cdot(Y,G)}(M_1^\cdot, M_2^\cdot) = H^0 Hom^\cdot(M_1^\cdot, M_2^\cdot)$. The category $K^\cdot(Y, G)$ is a triangulated category in a usual manner (note that for a $(\mathcal{D}_Y, G)$-

complex $M^\cdot$ the operators i_ξ act on shifted complex $M^\cdot[1]$ as minus i_ξ on $M^\cdot$). According to 1.2.2 the cohomologies $H^i(M^\cdot)$ of $(\mathcal{D}_Y, G)$-complex $M^\cdot$ are $(\mathcal{D}_Y, G)$-modules; hence we have the cohomology functor $H : K^\cdot(Y, G) \to \mathcal{M}(Y, G)$. Localising by H-quasiisomorphisms one gets the desired equivariant derived category $D^\cdot(Y, G)$. It has natural t-structure with core $\mathcal{M}(Y, G)$ (embedded in $D^\cdot(Y, G)$ as the subcategory of complexes acyclic off degree 0) and cohomology functor H. Clearly the similar procedure applied to $C(Y, G)_{coh}$ gives us the full subcategory $D^b(Y, G)_{coh}$ of $D^b(Y, G)$ equivalent to the subcategory of complexes with coherent cohomology.

1.3.2. Example. Assume that G acts on Y in a free way, so we have a quotient $\pi : Y \to G\backslash Y$, the equivalence of categories $\pi^\circ : \mathcal{M}(G\backslash Y) \to \mathcal{M}(Y, G)$ and the t-exact functor $\pi^\circ : D^b(G\backslash Y) \to D^b(Y, G)$. This functor is equivalence of t-categories. (Proof. Since π° is equivalence on cores, it suffice to verify that $\pi^\circ : \mathrm{Ext}^i_{D^b(G\backslash Y)}(M, N) \to \mathrm{Ext}^i_{D^b(Y,G)}(\pi^\circ M, \pi^\circ N)$ is isomorphism for any $\mathcal{D}_{G\backslash Y}$-modules M, N. Using a Cech resolvent for N one verifies that it suffice to consider the case when Y is affine and π has section. Then 1.2.4 identifies $C(Y, G)$ with the category of differential graded $\mathcal{D}_{G\backslash Y} \times \mathcal{U}(\tilde{\mathfrak{g}}^\cdot)$-modules, and one verifies the isomorphism on Ext's computing the right Ext by means of free $\mathcal{D}_{G\backslash Y} \times \mathcal{U}(\tilde{\mathfrak{g}}^\cdot)$-resolvent of $\pi^\circ M$). Note that the inverse equivalence $\pi^{\circ\,-1}$ is right derived functor to $L^\cdot \mapsto$ Kernel of $\tilde{\mathfrak{g}}$-action on $\pi_\# L^\cdot$ (see 1.2.4). For $L^\cdot \in C^b(Y, G)$ let $DR(L^\cdot)_{G\backslash Y} \subset \pi.\, DR(L^\cdot)$ be the subcomplex that consists of G-invariant forms killed by i_ξ (here $DR(L^\cdot) = \Omega^\cdot_Y \otimes_{\mathcal{O}_Y} L^\cdot$ is usual de Rham complex equipped with G-action and operators i_ξ, $i_\xi(\omega \otimes \ell) =$

$= (m_{\mathfrak{g}}(\xi) \lrcorner\, \omega) \otimes \ell + (-1)^{\deg \omega} \omega \otimes i_{\xi} \ell$). Then $DR(L^{\cdot})_{G\backslash Y}$ represents, in the derived category, the de Rham complex $DR(\pi^{o\,-1} L^{\cdot})$.

The following functors are quite convenient.

For a weak $(\mathcal{D}_Y, G)$-complex $L^{\cdot}$ put $C_{\mathfrak{g}}(L^{\cdot}) := \mathcal{U}(\bar{\mathfrak{g}}^{\cdot}) \underset{\mathcal{U}(\mathfrak{g})}{\otimes} L^{\cdot}$, $C^{\mathfrak{g}}(L^{\cdot}) := \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\bar{\mathfrak{g}}^{\cdot}), L^{\cdot})$. These are just the standard complexes of $\mathfrak{g}$ with coefficients in $\mathfrak{g}$-module L ($\mathfrak{g}$ acts via w). Clearly $C_{\mathfrak{g}}(L^{\cdot})$, $C^{\mathfrak{g}}(L^{\cdot})$ are $(\mathcal{D}_Y, G)$-complexes, and $C_{\mathfrak{g}}, C^{\mathfrak{g}}$: $: C^{\cdot}(Y, G)_{weak} \to C^{\cdot}(Y, G)$ are exact functors. Denote by $O_{\Lambda} : C(Y, G) \to$ $\to C^{\cdot}(Y, G)_{weak}$ the forgetting (of i_{ξ}) functor.

1.3.3. Lemma. The functors $C_{\mathfrak{g}}, C^{\mathfrak{g}}$ are respectively left and right adjoint to O_{Λ}. Same holds on the level of homotopy and derived categories.

Proof. Here are the formulas for adjunction morphisms. The
identifies $L^{\cdot}$ with $\Lambda^0(\mathfrak{g}) \otimes L^{\cdot} \subset C_{\mathfrak{g}}(L^{\cdot})$; the one $\gamma^{\mathfrak{g}} : O_{\ast} C^{\mathfrak{g}} L^{\cdot} \to L^{\cdot}$ projects $C^{\mathfrak{g}}(L^{\cdot}) = \Lambda^{\cdot}(\mathfrak{g}^*) \otimes L^{\cdot}$
one $\gamma_{\mathfrak{g}} : L^{\cdot} \to O_{\Lambda} C_{\mathfrak{g}} L^{\cdot}$ to $L^{\cdot} = \Lambda^0(\mathfrak{g}) \otimes L^{\cdot}$. The morphisms $\delta_{\mathfrak{g}} : C_{\mathfrak{g}} O_{\Lambda}(M^{\cdot}) \to M^{\cdot}$, $\delta^{\mathfrak{g}} : M^{\cdot} \to C^{\mathfrak{g}} O_{\Lambda} M^{\cdot}$ are $\delta_{\mathfrak{g}}(\xi_1 \wedge \ldots \wedge \xi_{\ell} \otimes m) =$ $= i_{\xi_1} \ldots i_{\xi_{\ell}}(m) = \delta^{\mathfrak{g}}(m)(\xi_1 \wedge \ldots \wedge \xi_{\ell})$. $\square$

1.3.4. Remarks. (i) Denote by ν the determinant of adjoint representation; this is rank 1 G-module. One has canonical isomorphism $C_{\mathfrak{g}}(L) = C^{\mathfrak{g}}(L \otimes \nu)[\dim \mathfrak{g}]$.

(ii) The exact functor $\mathrm{Ind}_{weak} : \mathcal{M}(Y) \to \mathcal{M}(Y, G)_{weak}$ defines the same noted functor between homotopy and derived categories right adjoint to the forgetting functor O_G.

(iii) Put $\mathrm{Ind} := C^{\mathfrak{g}} \mathrm{Ind}_{weak} : C^{\cdot}(Y) \to C^{\cdot}(Y, G)$. This functor is right adjoint to the forgetting functor $O := O_G\ O_{\Lambda}$: $: C^{\cdot}(Y, G) \to C^{\cdot}(Y)$. Same holds on homotopy and derived categories level.

(iv) If $I^{\cdot}$ is injective complex of $\mathcal{D}_Y$-modyles, then $\mathrm{Ind}(I^{\cdot})$ is injective $(\mathcal{D}_Y, G)$-complex. This implies that any $M^{\cdot} \in C^{+}(Y, G)$

may be quasiisomorphically embedded in an injective complex which is convenient for computing derived functors.

Now we may compare 1.3.1 with Bernstein-Lunts construction [28]. Recall the basic definition. Choose a sequence $T_0 \overset{i_0}{\hookrightarrow} T_1 \overset{i_1}{\hookrightarrow} T_2 \overset{i_2}{\hookrightarrow} \dots$ of smooth affine connected free G-varieties such that $\pi_1(T_j) = 0$ and for any $j > 0$ one has $H^j_{DR}(T_i) = 0$ for $i \gg 0$ (so $T = \cup T_j$ is free contractible G-space and $G\backslash T$ is classifying space of G). The object of Bernstein-Lunts equivariant derived category $D^b(Y, G)'$ is a sequence $(M; M_0, M_1, \dots)$, $M \in D^b(Y)$, $M_a \in D^b(G\backslash Y \times T_a)$ (note that G acts on $Y \times T_a$ in a free way) together with a system of isomorphisms $\pi_a^\circ M_a \simeq \pi_Y^\circ M$ in $D^b(Y \times T_a)$ compatible with respect to i_a's (here $G\backslash Y \times T_a \xleftarrow{\pi_a} Y \times T_a \xrightarrow{\pi_Y} Y$ are projections). This $D^b(Y, G)'$ is a t-category with core $\mathcal{M}(Y, G)$. Now if $M^\cdot$ is a $(\mathcal{D}_Y, G)$-complex, then the $(\mathcal{D}_{Y \times T_a}, G)$-complex $\pi_Y^\circ M^\cdot$ descends, according to 1.3.2, to $\mathcal{D}_{G\backslash Y \times T_a}$-complex $M_a^\cdot$, hence $(M^\cdot, M_a^\cdot)$ is an object of $D^b(Y, G)'$. This way we get a t-exact functor $F : D^b(Y, G) \to D^b(Y, G)'$ that induces equivalence on cores.

1.3.5. Lemma. This functor is equivalence of t-categories.

Proof. It suffice to show that $\mathrm{Hom}_{D^b(Y,G)}(M^\cdot, N^\cdot) \to \mathrm{Hom}_{D^b(Y,G)'}(F(M^\cdot), F(N^\cdot))$ is isomorphism for any bounded $(\mathcal{D}_Y, G)$-complexes $M^\cdot$, $N^\cdot$. We may assume that $N^\cdot = \mathrm{Ind}\, L^\cdot$ for a $\mathcal{D}_Y$-complex $L^\cdot$ (since any complex has a resolvent with induced terms, see 1.3.4 (iii), (iv)). Then $\pi_Y^\circ N^\cdot = \mathrm{Ind}\, \pi_Y^\circ L^\cdot$ (since Ind commutes with inverse image), hence $\mathrm{Hom}_{D^b(G\backslash Y \times T_a)}(M_a^\cdot, N_a^\cdot) =$

$= \mathrm{Hom}_{D^b(Y \times T_a, G)}(\pi_Y^\circ M^\cdot, \mathrm{Ind}\, \pi_Y^\circ L^\cdot) = \mathrm{Hom}_{D^b(Y \times T_a)}(\pi_Y^\circ M^\cdot, \pi_Y^\circ L^\cdot) =$

$= \oplus H^j_{DR}(Y) \otimes \mathrm{Hom}_{D^b(Y)}(M^\cdot, L^\cdot[-j]) = \oplus H^j_{DR}(Y) \otimes \mathrm{Hom}_{D^b(Y,G)}(M^\cdot, N^\cdot[-j])$,
When a is large enough, $\mathrm{Hom}_{D^b(G\backslash Y\times T_a)}(M^\cdot_a, N^\cdot_a) =$
$= \mathrm{Hom}_{D^b(Y,G)}(M^\cdot, N^\cdot)$, hence the desired fact. $\square$

Since $D^b(Y, G)'$ inherits the best functorial properties from ordinary $\mathcal{D}$-modules (see [28]) the same holds for $D^b(Y, G)$. In fact, one may define all the functors in a intrinsic way. For example, the functor Ind above is just the direct image functor for the equivariant morphism $(Y, 1) \xrightarrow{id} (Y, G)$ (here 1 is one element group). The lemma 1.2.6 also holds with cores replaced by t-categories $D^b(Y, G)$ themselves.

If Y is a point, then 1.3.5 identifies $D^b(Y, G)$ with $D^b(B_G)$:= the full subcategory of derived category of sheaves on the classifying space B_G of G that consists of complexes with locally constant cohomology. The core of B_G is category of G/G°-modules. If V_1, V_2 are G/G°-modules then $\mathrm{Ext}^i_{B_G}(V_1, V_2) :=$
$:= \mathrm{Hom}_{D^b(B_G)}(V_1, V_2[i])$ may be computed as follows. The finite group G/G° acts on B_{G° with quotient space equal to B_G, hence $H^\cdot(B_{G^\circ})$ is G/G°-module, and one has $\mathrm{Ext}^i_{B_G}(V_1, V_2) =$
$= [V_1^* \otimes V_2 \otimes H^i(B_{G^\circ})]^G$. If Y is any G-space and $M_1, M_2 \in \mathcal{M}(Y, G)$, then the general functoriality implies that one has a natural complex $R\,\mathrm{Hom}_{D^b(Y)}(M_1, M_2) \in D^b(B_G)$ such that $R\,\mathrm{Hom}_{D^b(Y,G)}(M_1, M_2) =$
$= R\,\mathrm{Hom}_{B_G}(\mathbb{C}, R\,\mathrm{Hom}_{D^b(Y)}(M_1, M_2))$. Hence we get a canonical spectral sequence that converges to $\mathrm{Ext}_{D^b(Y,G)}(M_1, M_2)$ with second term $E_2^{p,q} = H^p(B_G, \mathrm{Ext}^q_{D^b(Y)}(OM_1, OM_2))$.

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1.4. <u>Monodromic $\mathcal{D}$-modules</u> (see [32], [33]). Assume we are in a situation 1.1.3 and H is a torus. Such $\pi : \tilde{X} \to X$ will be called H-monodromic space. Put $\tilde{M}(X) := M(\tilde{D}_X)$. According to 1.1.3, 1.2.3 we have the categories $\tilde{M}(X)_{\chi,n} \subset \tilde{M}(X)^f \subset \tilde{M}(X)$ (here $\chi \in \mathfrak{h}^*$), and the equivalence of categories $\pi^{\#}: \tilde{M}(X) \to M(\tilde{X}, H)_{weak}$. Put $\tilde{\pi}^{\cdot} := O\,\pi^{\#}: \tilde{M}(X) \to M(\tilde{X})$.

If M_1, M_2 are $\tilde{\mathcal{D}}_X$-modules, then $M_1 \underset{\mathcal{O}_X}{\otimes} M_2$ is $\tilde{\mathcal{D}}_X$-module according to Leibnitz rule (one has $\tau(m_1 \otimes m_2) = \tau(m_1) \otimes m_2 + m_1 \otimes \tau(m_2)$, $m_i \in M_i$, $\tau \in \tilde{\mathcal{T}}_X$, see 1.1.3 (i)). One has $\pi^{\#}(M_1 \underset{\mathcal{O}_X}{\otimes} M_2) = \pi^{\#} M_1 \underset{\mathcal{O}_{\tilde{X}}}{\otimes} \pi^{\#} M_2$, same for $\tilde{\pi}^{\cdot}$.

For a $\mathcal{D}_{\tilde{X}}$-module M let $\tilde{\pi}.M$ be $\pi.M$ considered as $\tilde{\mathcal{D}}_X$-module; $\tilde{\pi}. : M(\tilde{X}) \to \tilde{M}(X)$ is exact faithful functor right adjoint to $\tilde{\pi}^{\cdot}$. Say that M is monodromic if $\tilde{\pi}.M$ is S( )-finite. Denote by $M(\tilde{X})^m \subset M(\tilde{X})$ the full subcategory of such modules; this is Serre subcategory.

For a monodromic M and $\chi \in \mathfrak{h}^*$ let $\pi_\chi(M) \in \tilde{M}(X)_{\chi\infty}$ be χ-component of $\tilde{\pi}.(M)$; one has $\tilde{\pi}.(M) = \underset{\chi}{\oplus} \pi_\chi(M)$. For $\bar{\chi} \in \mathfrak{h}^*/\mathfrak{h}^*_{\mathbb{Z}}$ (here $\mathfrak{h}^*_{\mathbb{Z}}$:= characters of H embedded in $\mathfrak{h}^*$ in a usual way) put $\pi_{\bar{\chi}}(M) := \underset{\chi = \bar{\chi} \bmod \mathfrak{h}^*_{\mathbb{Z}}}{\oplus} \pi_\chi(M)$; this is $\pi.\mathcal{D}_{\tilde{X}}$-submodule of $\pi.M$, hence $\pi_{\bar{\chi}}(M) = \pi.(M_{\bar{\chi}})$, and we have direct sum decomposition $M = \underset{\bar{\chi} \in \mathfrak{h}^*/\mathfrak{h}^*_{\mathbb{Z}}}{\oplus} M_{\bar{\chi}}$. It is easy to see that $M_{\bar{\chi}} = \tilde{\pi}^{\cdot}\pi_\chi(M)$. Say that M is $(\bar{\chi}, n)$-monodromic if $M = M_{\bar{\chi}}$ and $\pi_\chi(M) \in \tilde{M}(X)_{\chi,n}$; denote by $M(\tilde{X})_{\bar{\chi},n} \subset M(\tilde{X})^m$ the full subcategory of such ones.

For $\varphi \in \mathfrak{h}^*_{\mathbb{Z}}$ put $\mathcal{L}_\varphi := \pi_\varphi(\mathcal{O}_{\tilde{X}})$ (this is a line bundle that corresponds to φ) and define the translation functor $T_\varphi : \tilde{M}(X) \to \tilde{M}(X)$ by formula $T_\varphi(N) := \mathcal{L}_\varphi \underset{\mathcal{O}_X}{\otimes} N$. Clearly $T_{\varphi_1} T_{\varphi_2} = T_{\varphi_1+\varphi_2}$

(in particular T_φ is autoequivalence) and $T_\varphi(\tilde{M}(X)_\chi) = \tilde{M}(X)_{\chi+\varphi}$. The above may be summarised as follows.

1.4.1. Lemma. (i) One has $M(\tilde{X})^m = \prod\limits_{\bar\chi} M(\tilde{X})_{\bar\chi}$

(ii) For $\chi = \bar\chi \bmod \mathfrak{f}^*_{\mathbb{Z}}$ the functors $M(\tilde{X})_{\bar\chi} \underset{\pi^\cdot}{\overset{\pi_\chi}{\rightleftarrows}} \tilde{M}(X)_\chi$ are mutually inverse equivalences of categories.

(iii) One has $\tilde{\pi}^\cdot T_\varphi = \tilde{\pi}^\cdot$ for $\varphi \in \mathfrak{f}^*_{\mathbb{Z}}$. $\square$

Remarks. (i) In a less formal language 1.4.1 says that $M(\tilde{X})^m$ is quotient of $\tilde{M}(X)^f$ module the action T of $\mathfrak{f}^*_{\mathbb{Z}}$.

(ii) $\mathfrak{f}^*/\mathfrak{f}^*_{\mathbb{Z}}$ is character group of the fundamental group of torus H. The Riemann-Hilbert correspondence identifies tame $\bar\chi$-monodromic $\mathcal{D}_{\tilde{X}}$-modules with the $\bar\chi$ -monodromic perverse sheaves on X, i.e. the ones which are lisse along the fibers having $\bar\chi$ as eigenvalues of fiberwise monodromy (see [31], [32]).

(iii) Let $Q \subset H$ be a finite subgroup, $H' := H/Q$. Then $\tilde{X}' := Q\backslash\tilde{X}$ is H'-monodromic space and the $\tilde{D}$'s for $\tilde{X}$ and $\tilde{X}'$ are canonically isomorphic. Hence $\tilde{M}(X)$ depends only on $\tilde{X}$ up to isogeny.

In the monodromic situation we have two candidates for the appropriate derived category of monodromic modules, namely $D^b(\tilde{X})^m$:= full subcategory of $D^b(\tilde{X})$ of complexes with monodromic cohomology, and the derived category $D^b M(\tilde{X})^m$ of $M(\tilde{X})^m$. Similarly one has $D^b(\tilde{X})_{\bar\chi\infty}$ and $D^b M(\tilde{X})_{\bar\chi\infty}$. Fortunately they coincide:

1.4.2. Lemma. The obvious exact functors $D^b M(\tilde{X})^m \to D^b(\tilde{X})^m$, $D^b M(\tilde{X})_{\bar\chi\infty} \to D^b(\tilde{X})_{\bar\chi\infty}$ are equivalences of categories.

Proof. These are t-exact functors between t-categories with the same cores, so it suffice to verify that these induce isomorphism on Hom's. The Cech resolvent shows that the problem is local along X, hence we may assume X to be affine and $\tilde{X} = X \times H$. It suffice to verify that Hom's are the same for generating family. The one is formed by $\tilde{\pi}^{\cdot}(\mathcal{D}_X \otimes V)$, where V is an $S(\mathfrak{f})$-module. The Kunneth formula reduces the problem to the case X = point, dim $H = 1$ where it is obvious. □

1.4.3. It is easy to see that $D^b(\tilde{X})^m \subset D^b(\tilde{X})$ is stable with respect to all the standard functors (duality, $\underset{\mathcal{O}_{\tilde{X}}}{\otimes}$, ...). More precisely, the direct and inverse image functoriality holds for morphisms of monodromic spaces, which are, by definition, maps equivariant with respect to isomorphisms of corresponding toruses. This may be proved by local arguments, similar to 1.4.2. The similar functoriality holds also for $\tilde{\mathcal{D}}$-modules (in a way compatible with $\tilde{\pi}^{\cdot}$).

1.4.4. In the rest of the § we will deal with equivariant monodromic modules. So let H be a torus, G be an algebraic group, and $\alpha : G \to \mathrm{Aut}\, H$, $g \mapsto \alpha_g$, be a fixed homomorphism (α factors through the finite quotient: $G/G^\circ \to \mathrm{Aut}\, H$). Let $\pi : \tilde{X} \to X$ be an H-monodromic variety and $\tilde{m} : G \times \tilde{X} \to \tilde{X}$ be a G-action such that $gh\tilde{x} = \alpha_g(h) g\tilde{x}$ for $g \in G$, $h \in H$, $\tilde{x} \in \tilde{X}$ (in particular $\tilde{m}$ descends to $m : G \times X \to X$). We will call such an object H-monodromic G-variety; this is the same as $G \underset{\alpha}{\ltimes} H$-variety X such that H acts in a free way. The infinitezimal action $m_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{T}_{\tilde{X}}$ reduces to a Lie algebras map $\tilde{m}_{\mathfrak{g}} : \mathfrak{g} \to \tilde{\mathcal{T}}_X$ $(\subset \pi_{\cdot}\mathcal{T}_{\tilde{X}})$. Also G acts on $\tilde{\mathcal{T}}_X$ and $\tilde{\mathcal{D}}_X$ in an obvious way; this action coincides with α on $\mathfrak{f} \subset \tilde{\mathcal{T}}_X$ and is compatible with $\tilde{m}_{\mathfrak{g}}$ in a sense that

$\tilde{m}_{\mathfrak{g}}(\mathrm{Ad}_g(\gamma)) = g\tilde{m}_{\mathfrak{g}}(\gamma)g^{-1}$ for $g \in G$, $\gamma \in \mathfrak{g}$.

Define $(\tilde{D}_X, G)$-module to be a $\tilde{D}_X$-module M together with G-action on M as on $\mathcal{O}_X$-module such that one has $g(\partial v) = g(\partial)g(v)$ for $g \in G$, $\partial \in \tilde{D}_X$, $v \in M$, and the two actions of $\mathfrak{g}$ on M, the first that comes from G-action and the second that comes from $\tilde{D}_X$-action via $\tilde{m}_{\mathfrak{g}}$, coincide (compare with 1.2.1 (ii), 1.2.2 (ii)). The $(\tilde{D}_X, G)$-modules form an abelian category $\tilde{M}(X, G)$. It contains a full subcategory $\tilde{M}(X, G)^f$ of $S(\mathfrak{h})$-finite modules. For $M \in \tilde{M}(X, G)^f$ consider the $\tilde{D}_X$-decomposition $M = \bigoplus_{\chi \in \mathfrak{h}^*} M_\chi$. For any $\alpha(G)$-orbit $\dot{\chi} \in \alpha(G)\backslash\mathfrak{h}^*$ the submodule $M_{\dot{\chi}} := \bigoplus_{\chi \in \dot{\chi}} M_\chi$ of M is G-invariant, so we have the decomposition $M = \bigoplus M_{\dot{\chi}}$ in $\tilde{M}(X, G)$. Let $\tilde{M}(X, G)_{\dot{\chi}\infty}$ be the full subcategory of M's with $M = M_{\dot{\chi}}$; then $\tilde{M}(X, G)^f = \prod_{\dot{\chi}} \tilde{M}(X, G)_{\dot{\chi}\infty}$. We have the easy lemma that reduces the study of arbitrary $S(\mathfrak{h})$-finite $(\tilde{D}_X, G)$-modules to the ones supported on orbit $\dot{\chi}$ that reduces to a single element:

1.4.5. Lemma. For $\chi \in \mathfrak{h}^*$ let $G_\chi \subset G$ be stabilisator of χ (with respect to α), $\dot{\chi} = \alpha(G)\chi$. Then the functor $\tilde{M}(X, G)_{\dot{\chi}\infty} \longrightarrow \tilde{M}(X, G_\chi)_{\chi\infty}$, $M \longmapsto M_\chi$, is equivalence of categories. $\square$

1.4.6. We also have a category $M(\tilde{X}, G)^m$ of monodromic $(D_{\tilde{X}}, G)$-modules together with adjoint functors $M(\tilde{X}, G)^m \underset{\pi^\cdot}{\overset{\pi_\cdot}{\rightleftarrows}} \tilde{M}(X, G)^f$. The category $M(\tilde{X}, G)^m$ decomposes by the subcategories indexed by elements of $\alpha(G)\backslash\mathfrak{h}^*/\mathfrak{h}^*_{\mathbb{Z}}$. If $\chi \in \mathfrak{h}^*$ is such weight that $\alpha(G)\chi = \chi$, then $M(\tilde{X}, G)_{\chi\infty} \underset{\pi^\cdot}{\overset{\pi_\chi}{\rightleftarrows}} \tilde{M}(X, G)_{\chi\infty}$ are mutually inverse equivalences of categories. These easy facts may be shown the same way as 1.4.1.

1.4.7. Both $(\tilde{D}_X, G)$- and $(D_{\tilde{X}}, G)$-modules have the same elementary functoriality 1.2.4 (i)-(vi) as ordinary equivariant modules (see 1.4.3); the lemma 1.2.5, as well as its proof, remains valid in this situation. The proper way to understand the functoriality is to introduce the right equivariant derived category. One can do this in the same manner as in n 1.3. Both 1.4.1 and 1.4.2 remain valid in this context.

1.5. Langlands classification. Let $\tilde{X}$ be a monodromic G-variety. Our aim is explicit classification of irreducible $(\tilde{\mathcal{D}}_X, G)$-modules in case when there are only finitely many orbits on X.

1.5.1. Let $x \in X$ be a point, $G_x \subset G$ be stabilizer of x. Consider the action of G_x on H-torsor $\pi^{-1}(x)$. Since G°- and H-actions commute, the group $(G_x)^{\circ}$ acts on $\pi^{-1}(x)$ by means of H-translations via the morphism $\varphi_x : (G_x)^{\circ} \to H$. The kernel of φ_x is a normal subgroup of G_x; put $G_{(x)} := G_x/\mathrm{Ker}\,\varphi_x$. The connected component $G^{\circ}_{(x)}$ of $G_{(x)}$ is the subtorus of H via φ, hence one has the embedding $i : \mathrm{Lie}\, G_{(x)} \hookrightarrow \mathfrak{h}$. If we put $\mathrm{Ad} := \alpha|_{G_x} : G_{(x)} \to \mathrm{Aut}\, H$, then $(\mathfrak{h}, G_{(x)})$ together with i, Ad form a Harish-Chandra pair (see 2.4 for definitions).

1.5.2 Assume that X is a single orbit. Take any $x \in X$; then $\tilde{X} = G \times_{G_x} \pi^{-1}(x)$, and by 1.2.5 (ii), 1.4.7 one has canonical equivalence $\tilde{M}(X, G) = \tilde{M}(x, G_x)$. But $(\tilde{D}_x, G_x)$-modules are the same as $(\mathfrak{h}, G_{(x)})$-modules (see 2.4), hence the canonical equivalence $F_x : \tilde{M}(X, G) \xrightarrow{\sim} M(\mathfrak{h}, G_{(x)})$. Note that any coherent monodromic $\mathcal{D}_{\tilde{X}}$-module is tame and lisse (= rs holonomic $\mathcal{O}_{\tilde{X}}$-coherent in terminology of [9]), since this is obvious for monodromic $\mathcal{D}_{\tilde{X}}$-modules and 1.2.5 preserves the property of being tame and lisse. Similarly, if $\chi \in \mathfrak{h}^*_{\mathbb{Q}} := \mathbb{Q}\mathfrak{h}^*_{\mathbb{Z}} \subset \mathfrak{h}^*$ is a rational weight, then any $\bar{\chi}$-monodromic $\mathcal{D}_{\tilde{X}}$-module has finite monodromy group hence has geometric origin [5] (6.2.4). For $\chi \in \mathfrak{f}^*$, $Ad(G)(\chi) = \chi$, the category $\mathcal{M}(\mathfrak{f}, G_{(x)})_{\chi,1}$ is semisimple and the set Irr of isomorphism classes of irreducible objects is finite (e.g. $\mathcal{M}(\mathfrak{f}, G_{(x)})_{0,1}$ is the category of $G_{(x)}/G^{c}_{(x)}$-modules). If $Ad(G) \subset \mathrm{Aut}\,\mathfrak{f}$ fixes the n-th infinitezimal neighbourhood of χ, then $\mathcal{M}(\mathfrak{f}, G_{(x)})_{\chi,n}$ has no Ext^1's between nonisomorphic irreducibles and $\mathcal{M}(\mathfrak{f}, G_{(x)})_{\chi,n}$ is equivalent to the category of $\prod_{\mathrm{Irr}} S(\mathfrak{f}/\mathrm{Lie}\,G_{(x)})/m^n$-modules (here $m = \mathfrak{f}S$ is maximal ideal).

1.5.3. Assume now that X has finitely many G-orbits; such monodromic G-variety will be called finite. Let $I_{X,G}$ be the set of orbits on X. This is a partially ordered set: we say that $\alpha_1 \leq \alpha_2$ iff $Q_{\alpha_1} \subset \overline{Q}_{\alpha_2}$ (here for $\alpha \in I_{X,G}$ Q_α is the corresponding orbit). Let I be any partially ordered set; for $\alpha \in I$ put $\bar{\alpha} := \{\beta : \beta \leq \alpha\} \subset I$. Put $A_I := \{a \subset I : \alpha \in a \Rightarrow \bar{\alpha} \in a\}$: if a,b A_I, then $a \cap b$, $a \cup b \in A_I$. Clearly A_I is the lattice of all closed G-invariant subsets of X: for $a \in A_{I_{X,G}}$ we put $X_a := \bigcup_{\alpha \in a} Q_\alpha$ hence $X_{\bar{\alpha}} = \overline{Q}_\alpha$. For $a \in A_{I_{X,G}}$ let $\tilde{\mathcal{M}}(X, G)_a$ be the full subcategory of $\check{\mathcal{M}}(X, G)$ that consists of modules supported on X_a; similarly we have $\mathcal{M}(\check{X}, G)_a$ etc. Let us fix this kind of frame.

1.5.4. <u>Definition</u>. Let C be an abelian category. An I - stratification on C is a set C_a, $a \in A_I$, of Serre subcategories of C such that (i)-(iii) below holds:

(i) $C_{a_1} \subset C_{a_2}$ if $a_1 \subset a_2$; $C_{a_1 \cap a_2} = C_{a_1} \cap C_{a_2}$; $C_{a_1 \cup a_2}$ is the least Serre subcategory that contains C_{a_1}, C_{a_2}.

(ii) For $a_1, a_2 \in A_I$ the embeddings

$$\begin{tikzcd}[row sep=small] & C_{a_1} \arrow[dr] & \\ C_{a_1 \cap a_2} \arrow[ur] \arrow[dr] & & C_{a_1 \cup a_2} \\ & C_{a_2} \arrow[ur] & \end{tikzcd}$$

induce equival ce of categories $C_{a_1}/C_{a_1 \cap a_2} \times C_{a_2}/C_{a_1 \cap a_2} \to C_{a_1 \cup a_2}/C_{a_1 \cap a_2}$.

(iii) For $a_1 \subset a_2$ the projection $C_{a_2} \to C_{a_2}/C_{a_1}$ has right and left adjoints denoted $j_{a_2 - a_1 *}$, $j_{a_2 - a_1 !}$ respectively. $\square$

Our $\widetilde{M}(X, G)$, $M(\check{X}, G)$ are I-stratified categories (with $\mathcal{M}_\lambda$ as above); same for the subcategories of monodromic, coherent modules.

In any I-stratified category C one has the standard devissage pattern. Namely, for $\alpha \in I$ put $C_\alpha := C_{\bar{\alpha}}/C_{\alpha'}$ where $\alpha' := \bar{\alpha} \setminus \{\alpha\}$; we will call this subquotient the α stratum. Then one has the functors $j_{\alpha *}, j_{\alpha !} : C_\alpha \to C_{\bar{\alpha}}$ right and left adjoint to the projection $j_\alpha^* : C_{\bar{\alpha}} \to C_\alpha$. Since $j_\alpha^* j_{\alpha *} = j_\alpha^* j_{\alpha !} = Id_{C_\alpha}$, one gets the natural morphism $j_{\alpha !} \to j_{\alpha *}$ and we put $j_{\alpha ! *} := Im(j_{\alpha !} \to j_{\alpha *})$; this functor transforms irreducibles to irreducibles. One has $\widetilde{M}(X, G)_\alpha = \widetilde{M}(X_\alpha, G)$, $M(\widetilde{X}, G)_\alpha = M(\widetilde{X}_\alpha, G)$.

We will say that C is finite if any object of C has finite length. The devissage shows that this is equivalent to the property that objects of each C_α have finite length.

We may sum up the above discussion, joining it with 1.4.2:

1.5.5. <u>Lemma</u>. (i) $\widetilde{M}(X, G)$ is $I_{X,G}$-stratified category. If we choose a point $x_\alpha \in X_\alpha$, then one has canonical equivalence $F_{x_\alpha} : \widetilde{M}(X, G)_\alpha \rightleftarrows M(\mathcal{f}, G_{(x_\alpha)})$.

(ii) The categories $\widetilde{M}(X, G)^f_{coh}$, $M(\check{X}, G)^m_{coh}$ are finite.

(iii) Each monodromic coherent $(\mathcal{D}_{\widetilde{X}}, G)$-module is tame.

(iv) For any irreducible $(\mathfrak{g}, G_{(x_\alpha)})$-module V the $(\tilde{\mathcal{D}}_X, G)$-module $j_{\alpha!*}(V) := j_{\alpha!*} F^{-1}_{x_\alpha}(V)$ is irreducible. This way the isomorphism classes of irreducibles $(\mathcal{D}_X, G)$-modules become in 1-1 correspondence with pairs (α, V), where $\alpha \in I_{X,G}$ is an orbit, an V is isomorphism class of irreducible $(\mathfrak{g}, G_{(x_\alpha)})$-modules.

(v) If $\chi \in \mathfrak{h}^*_{\mathbb{Q}}$, then any irreducible $\bar{\chi}$-monodromic $(\mathcal{D}_{\check{X}}, G)$-module has geometric origin [5] (6.2.4). □

1.5.6. Remarks. (i) The modules $j_{\alpha!}(V), j_{\alpha*}(V)$, where $V \in \mathcal{M}(\mathfrak{g}, G_{(x_\alpha)})$, are called !- and $*$- standard modules respectively.

(ii) If the embedding $Q_\alpha \hookrightarrow X$ is affine, then the functors $j_{\alpha!}, j_{\alpha*}$ are exact.

(iii) Let $\dim : I \to \mathbb{Z}$ be a function such that $\alpha \leq \beta$, $\alpha \neq \beta \Rightarrow \dim \alpha < \dim \beta$ $(\alpha, \beta \in I)$; e.g. $\dim \alpha = \dim Q_\alpha$ is such a function on $I_{X,G}$. For $n \in Z$ put $\bar{n} := \{\alpha : \dim \alpha \leq n\} \in A_I$. Then $C_{\overline{n-1}} \subset C_{\bar{n}}$ and $C_{\bar{n}}/C_{\overline{n-1}} = \bigoplus_{\dim \alpha = n} C_\alpha$.

(iv) If C is I-stratified category, then the dual category C° is I-stratified by C°_a; the duality interchanges functors $j_{\alpha!}$ and $j_{\alpha*}$.

(v) Let C_i, i = 1, 2, be I_i-stratified categories. Let $\varphi : I_1 \to I_2$ be a morphism of partially ordered sets and $F : C_1 \to C_2$ is a functor. We will say that F is φ-stratified functor if $F(C_{1\bar{\alpha}}) \subset C_{2\overline{\varphi(\alpha)}}$; such F induces the functors $F_\alpha : C_{1\alpha} \to C_{2\varphi(\alpha)}$, $\alpha \in I$; call F_α the α-stratum of F. Say that F is stratied equivalence if φ is isomorphism, F is equivalence of categories and $F(C_{1\alpha}) = C_{2\varphi(\alpha)}$ for any $\alpha \in I$. Clearly, any stratified equivalence commutes with $j_{\alpha!}$ and $j_{\alpha*}$.

§2. Localization construction

This § is a somewhat swollen review of localization theory [3], [4], [19], [27], [29]; it is also an attempt to fix some topics skipped in these references.

2.1. $\mathfrak{g}$-modules. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Denote by G the algebraic group of automorphisms of $\mathfrak{g}$, so G^c is adjoint group, $\mathfrak{g} = \mathrm{Lie}\, G$; the action of $g \in G$ on $\mathfrak{g}$ will be denoted Ad_g. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra, $\mathcal{Z} \subset \mathcal{U}(\mathfrak{g})$ be the center, $\mathfrak{h}$ be the Cartan algebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{h}^*$ be the root system, Δ^+ be the positive roots, $\Sigma = \Sigma(\Delta^+)$ be the set of simple roots, W be the Weyl group, $\rho := \frac{1}{2} \sum_{\nu \in \Delta^+} \nu$; for $\alpha \in \Delta$ let h_α be the corresponding coroot and $\sigma_\alpha \in W$ the corresponding reflection. So for any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and $n = n_{\mathfrak{b}} := [\mathfrak{b}, \mathfrak{b}]$ we have canonical identification $\mathfrak{h} = \mathfrak{b}/n$ invariant under G^c-conjugation, and Δ^+ are weights of $\mathfrak{h}$-action on $\mathfrak{g}/\mathfrak{b} - n^*$ *). We will consider W as the group of affine transformations of $\mathfrak{h}^*$ that leave $-\rho$ fixed, hence $W \subset \mathrm{Aut}\, S(\mathfrak{h})$. One has Harish-Chandra isomorphism $\gamma : \mathcal{Z} \xrightarrow{\sim} S(\mathfrak{h})^W$; let $\gamma^{\cdot} :$ $: \mathfrak{h}^* = \mathrm{Spec}\, S(\mathfrak{h}) \to \mathrm{Spec}\, \mathcal{Z}$ be the corresponding W-sheeted map of spectra.

Denote by $U := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{Z}} S(\mathfrak{h})$ the extended universal enveloping algebra; then $S(\mathfrak{h})$ is the center of U, the group W acts

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*) Often they use (see e.g. [19], [27]) the opposite ordering of Δ; we choose the one for which a positive root corresponds to a positive line bundle on flag space.

on U (via $S(\mathfrak{h})$), and $\mathcal{U}(\mathfrak{g}) = U^W$. The algebras $\mathcal{U}(\mathfrak{g})$, $S(\mathfrak{h})$, U carry canonical involutions (antiautomorphisms of order 2), denoted $x \mapsto {}^t x$, compatible with standard embeddings: these are determined by properties ${}^t g = -g$, ${}^t h = -2\rho(h) - h$ for $g \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$ U, $h \in \mathfrak{h} \subset S(\mathfrak{h}) \subset U$; clearly t commutes with W action. Denote by $S(\mathfrak{h})^{reg}$ the localisation of $S(\mathfrak{h})$ off the non-regular hyperplanes for W action (so $\mathbb{C}$-points of $S(\mathfrak{h})^{reg}$ are regular weights); if A is any $S(\mathfrak{h})$-algebra put $A^{reg} := S(\mathfrak{h})^{reg} \underset{S(\mathfrak{h})}{\otimes} A$. In particular we have U^{reg} and t extends to U^{reg}. The group G acts on all above objects in a compatible way; the action on $\mathfrak{h}$, Δ and W factors through the finite quotient G/G^o, and the action of G/G^o on $\mathfrak{h}$ is faithful.

Let $\mathcal{M}(\mathfrak{g})$, $\mathcal{M}(U)$ be categories of left $\mathcal{U}(\mathfrak{g})$- and U-modules and $\mathcal{M}(U)^{f.g.} \subset \mathcal{M}(U)$ be the subcategory of finitely generated modules; we have also the categories ${}^r\mathcal{M}$ of right ones, but one may identify ${}^r\mathcal{M}$ with $\mathcal{M}$ in a canonical way using t. For $\chi \in \mathfrak{h}^*$, $n = 1, 2, \ldots,$ one has the categories $\mathcal{M}(U)_{\chi,n}$, $\mathcal{M}(\mathfrak{g})_{\gamma(\chi),n}$ (see 1.1.2). The embedding $\mathcal{U}(\mathfrak{g}) \hookrightarrow U$ induces isomorphism $\mathcal{U}(\mathfrak{g})_{\gamma(\chi),1} = U_{\chi,1}$ for any $\chi \in \mathfrak{h}^*$; if χ is regular then $\mathcal{U}(\mathfrak{g})_{\gamma(\chi),n} = U_{\chi,n}$ for any n. Hence the corresponding functor $\mathcal{M}(U)_{\chi,1} \to \mathcal{M}(\mathfrak{g})_{\gamma(\chi),1}$ is equivalence of categories; in case of regular χ the categories $\mathcal{M}(U)_{\chi,n}$ and $\mathcal{M}(\mathfrak{g})_{\gamma(\chi),n}$ are equivalent for any n.

2.2. <u>Flag variety</u>. Let $X = X_{\mathfrak{g}}$ be the flag variety of $\mathfrak{g}$; the points of X are Borel subalgebras of $\mathfrak{g}$. For $x \in X$ let b_x be the corresponding Borel subalgebra, $B_x \subset G^o$ be the corresponding Borel subgroup, $N_x \subset B_x$ be the maximal nilpotent subgroup, so Lie $B_x = b_x$, Lie $N_x = n_x := [b_x, b_x]$ and $\mathfrak{h} = b_x/n_x$. Put $H := B_x/N_x$: this torus, the Cartan group of G, does not depends on the choice of x by the same reason as $\mathfrak{h}$ was, Lie $H = \mathfrak{h}$. The group G acts on X and on H, compatible with above actions on Lie algebras; the action of G^o on X is transitive with the stabilizer of $x \in X$ equal to B_x, so $X = G^c/B_x$.

Let $\check{X} = \check{X}_{\mathfrak{g}}$ be the enhanced flag variety (or "base affine space") of G: its point $\check{x}$ is pair $(b_x, \{a^{\alpha}_{\check{x}}\})$, where $b_x \subset \mathfrak{g}$ is a Borel subalgebra, and $a^{\alpha}_{\check{x}}$, $\alpha \in \sum(\Delta^+)$, is a generator of α-root subspace in $\mathfrak{g}/b_x$. The groups G and H act on $\check{X}$ from the left according to formulas $g\check{x} := (Ad_g(b_x), \{Ad_g(a^{\alpha})\})$, $hx = (b_x, \{\exp\alpha(h) \cdot a^{\alpha}\})$. One has $ghx = Ad_g(h)gx$ in particular G^c commutes with H. The H-action is free, $H\backslash\check{X} = X$, and G^c-action is transitive; for $\check{x} \in \check{X}$ the stabiliser $G^o_{\check{x}}$ equals to N_x; so, as $G^o \times H$ -space, X is G^o/N_x (with H-action $h\cdot(g \bmod N_x) = gh^{-1} \bmod N_x$).

We will consider $\check{X}$ as H-monodromic G-variety (with the compatibility morphism $Ad : G \to \mathrm{Aut}\, H$). Hence we get the algebra $\tilde{\mathcal{D}}_X$ on X with G-action. In particular we have the Lie algebras map $\mathfrak{g} \times \mathfrak{h} \to \tilde{\mathcal{D}}_X$, hence by universality, the morphism of algebras $\mathfrak{U}(\mathfrak{g} \times \mathfrak{h}) = \mathfrak{U}(\mathfrak{g}) \otimes S(\mathfrak{h}) \xrightarrow{\tilde{\delta}} \tilde{\mathcal{D}}_X$. It is easy to see that $\tilde{\delta}(z \otimes 1) = \tilde{\delta}(1 \otimes \gamma(z))$ for $z \in \mathcal{Z}$, hence $\tilde{\delta}$ factors through the same-noted map $\tilde{\delta} : U \to \tilde{\mathcal{D}}_X$. Also for $\chi \in \mathfrak{h}^*$ we have $\tilde{\delta}_{\chi,n} := \tilde{\delta} \bmod m^n_{\chi} : U_{\chi,n} \to \Gamma(X, \tilde{\mathcal{D}}_{X\chi,n})$. Same way the $\mathfrak{g}$-action $m_{\mathfrak{g}}$ on X defines $\delta : \mathfrak{U}(\mathfrak{g}) \to \mathcal{D}_X$ which factors through $\delta : \mathfrak{U}(\mathfrak{g})_{\gamma(0),1} \to \mathcal{D}_X$; this δ coincides with $\tilde{\delta}_{0,1}$ under the identifications $\mathcal{D}_X = \tilde{\mathcal{D}}_{X01}, \mathcal{U}(\mathfrak{g})_{\chi(0),1} = U_{0,1}$, see 1.1.3 (iii),

2.1. The algebra $\tilde{\mathcal{D}}_X$ carries a canonical involution ^{t} such that ^{t} is identity on $\mathcal{O}_X \subset \tilde{\mathcal{D}}_X$ and $\tilde{\delta}$ commutes with ^{t}'s.

2.2.1. Remarks. (i) This ^{t} identifies $\tilde{\mathcal{D}}_{X0,1} = \mathcal{D}_X$ with $\tilde{\mathcal{D}}_{X-2\rho,1} = \mathcal{D}_{\omega_X}$ with reversed multiplication. This is just the standard isomorphism [9] VI.3.

(ii) Consider the Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathcal{O}_X$ on X with bracket $[g_1 \otimes f_1, g_2 \otimes f_2] := [g_1, g_2] \otimes f_1 f_2 - g_1 \otimes m_{\mathfrak{g}}(g_2)(f_1) f_2 + g_2 \otimes m_{\mathfrak{g}}(g_1)(f_2) f_1$, where $g_i \in \mathfrak{g}$, $f_i \in \mathcal{O}_X$. Put $\tilde{\mathfrak{n}} := \{\varphi \in \tilde{\mathfrak{g}} : \varphi(x) \in n_x \ \forall x \in X\}$. The morphism $\tilde{m}_{\mathfrak{g}} : \mathfrak{g} \to \tilde{\mathcal{T}}_X$ extends by $\mathcal{O}$-linearity to surjective Lie algebra map $\tilde{\mathfrak{g}} \to \tilde{\mathcal{T}}_X$ with kernel $\tilde{\mathfrak{n}}$. Hence $\tilde{\mathcal{D}}_X$ is twisted universal enveloping algebra of $\tilde{\mathfrak{g}}/\tilde{\mathfrak{n}}$, see [3]. □

2.2.2. Lemma. For any $S(\mathfrak{h})$-module V the arrow $\tilde{\delta} \otimes \mathrm{Id}_V : U \otimes_{S(\mathfrak{h})} V \to \Gamma(X, \tilde{\mathcal{D}}_X \otimes_{S(\mathfrak{h})} V)$ is isomorphism, and $H^i(X, \tilde{\mathcal{D}}_X \otimes_{S(\mathfrak{h})} V) = 0$ for $i > 0$. In particular $U = \Gamma(X, \tilde{\mathcal{D}}_X)$, $U_{\chi,n} = \Gamma(X, \tilde{\mathcal{D}}_{X\chi,n})$. □

For the proof of the first statement, based on Kostant's theorem, see [29]. The vanishing follows similarly from Elkik's theorem [14].

2.3. Localisation. Now $\tilde{\delta}$ give rise to the adjoint $S(\mathfrak{h})$-linear functors $\mathcal{M}(U) \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \tilde{\mathcal{M}}(X)$, $\Delta(V) = V \otimes_{\hat{U}} \tilde{\mathcal{D}}_X$, $\Gamma(M) := \Gamma(X, M)$.
By restriction to the appropriate subcategories we get

$$\mathcal{M}(U)^f \underset{\Gamma^f}{\overset{\Delta^f}{\rightleftarrows}} \tilde{\mathcal{M}}(X)^f, \quad \mathcal{M}(U)_{\chi,n} \underset{\Gamma_{\chi,n}}{\overset{\Delta_{\chi,n}}{\rightleftarrows}} \tilde{\mathcal{M}}(X)_{\chi,n}, \quad \mathcal{M}(U)^{reg} \underset{\Gamma^{reg}}{\overset{\Delta^{reg}}{\rightleftarrows}} \tilde{\mathcal{M}}(X)^{reg}.$$

We may do the same for δ, and get the functors $\mathcal{M}(\mathfrak{g}) \rightleftarrows \mathcal{M}(X)$

which coincide with $(\Delta_{0,1}, \Gamma_{0,1})$. Same way arise the same=noted functors between categories of right modules; one may identify them with above ones using ^{t}. The functor Δ is right exact and Γ is left exact, so we have the corresponding derived functors $L\Delta$, $R\Gamma$. The functor $R\Gamma$ has finite cohomological dimension (equal to dim X) and one may easily see that the same holds for $L\Delta$, so we may use bounded derived categories. Note that $D^b\mathcal{M}(U)^f$, $D^b\mathcal{M}(U)_{\chi,\infty}$ are full subcategories of $D^b(\dot{U})$. Now 2.2.2 implies in a moment (use free resolvents):

Corollary 2.3.1. One has $R\Gamma \circ L\Delta = Id_{D^b\mathcal{M}(\dot{U})}$, $R\Gamma_{\chi,n} \circ L\Delta_{\chi,n} = Id_{D^b\mathcal{M}(\dot{U})_{\chi,n}}$. □

Recall that $\chi \in \mathfrak{h}^*$ is dominant if $(\chi+\rho)(h_\gamma) \notin \{-1,-2,\ldots\}$ for any positive coroot $h_\gamma \in \mathfrak{h}$. We have the basic

2.3.2. Theorem [3], [4]. Let $n = 1, 2, \ldots, \infty$. Then

(i) If χ is regular dominant, then $\mathcal{M}(U)_{\chi,n} \underset{\Gamma_{\chi,n}}{\overset{\Delta_{\chi,n}}{\rightleftarrows}} \tilde{\mathcal{M}}(X)_{\chi,n}$ are mutually inverse equivalences of categories.

(ii) If χ is arbitrary regular, then $D^b\mathcal{M}(U)_{\chi,n} \underset{R\Gamma_{\chi,n}}{\overset{L\Delta_{\chi,n}}{\rightleftarrows}} D^b\tilde{\mathcal{M}}(X)_{\chi,n}$ are mutually inverse equivalences of categories. □

Remarks. (i) These equivalences interchange coherent and finitely generated modules.

(ii) In [3] (i) was proved for $n = 1$. The general case goes the same line: use 2.2.2 joined with $\tilde{\mathcal{D}}_{X\,\chi,n}$-affinity of X, which follows from $\check{\mathcal{D}}_{X\,\chi,1}$-affinity by obvious devissage.

Now the easy commutative algebra implies

2.3.3. Corollary. $D^b\mathcal{M}(U^{reg}) \underset{R\Gamma}{\overset{L\Delta}{\rightleftarrows}} D^b\tilde{\mathcal{M}}(X)^{reg}$ are mutually inverse equivalences of derived categories. □

To reformulate 2.3.2 in more geometric terms, using the ordinary $\mathcal{D}$-modules only, consider the adjoint functors $\mathcal{M}(U) \underset{\tilde{\Gamma}}{\overset{\tilde{\Delta}}{\rightleftarrows}} \mathcal{M}(\check{X})$ defined by formulas $\tilde{\Delta}(V) := \mathcal{D}_{\check{X}} \underset{\hat{U}}{\otimes} V = \tilde{\pi}^{\cdot}\Delta(V)$, $\tilde{\Gamma}(M) = \Gamma(\check{X}, M) = \Gamma\tilde{\pi}.M$ (see 1.3.). These functors interchange $S(\mathfrak{f})$-finite and monodromic modules, hence for $\chi \in \mathfrak{f}^*$, $\bar{\chi} = \chi \bmod \mathfrak{f}^*_{\mathbb{Z}}$, we get the adjoint functors $\mathcal{M}(U)_{\chi,n} \underset{\tilde{\Gamma}_{\chi,n}}{\overset{\check{\Delta}_{\chi,n}}{\rightleftarrows}} \mathcal{M}(\tilde{X})_{\bar{\chi},n}$, $\tilde{\Delta}_{\chi,n} = \tilde{\Delta}$, $\tilde{\Gamma}_{\chi,n} = \Gamma\tilde{\pi}_\chi = \tilde{\Gamma}_\chi$. We also have the corresponding derived functors $D^b\mathcal{M}(U)_{\chi,n} \underset{R\tilde{\Gamma}_{\chi,n}}{\overset{L\check{\Delta}_{\chi,n}}{\rightleftarrows}} D^b\mathcal{M}(\tilde{X})_{\bar{\chi},n}$. Since $\tilde{\pi}^{\cdot}, \tilde{\pi}_\chi$ are exact, one has $L\check{\Delta}_{\chi,n} = \tilde{\pi}^{\cdot} L\Delta$, $R\tilde{\Gamma}_{\chi,n} = R\Gamma\tilde{\pi}_\chi$. Hence 2.3.2 joined with 1.3.1 (ii) implies

2.3.4. <u>Corollary</u>. If χ is regular dominant then $\mathcal{M}(U)_{\chi,n} \underset{\tilde{\Gamma}_{\chi,n}}{\overset{\tilde{\Delta}_{\chi,n}}{\rightleftarrows}} \mathcal{M}(\check{X})_{\bar{\chi},n}$ are mutually inverse equivalences of categories. Some holds for arbitrary regular χ on derived category level. $\square$

To formulate the localisation theorem in non-regular case one should start with some preliminaries.

2.3.5. For any subset $S \subset \Sigma(\Delta^+)$ of simple roots let $W_S \subset W$ be the subgroup generated by S-reflections. For $x \in X$ let P_{Sx}, $B_x \subset P_{Sx} \subset G^\circ$, be the corresponding parabolic subgroup. Put $P'_{Sx} := [P_{Sx}, P_{Sx}]$, so $P_{Sx}/P'_{Sx} = H_S$ is the quotient of H by the subtorus H^S generated by h_ξ, $\xi \in S$. We have canonical fibrations $X = G^\circ/B_x \xrightarrow{\tau_S} X_S = G^\circ/P_{Sx}$, $\tilde{X} = G^\circ/N_x \longrightarrow H^S\backslash\tilde{X} \xrightarrow{\tilde{\tau}_S} \tilde{X}_S = G^\circ/P'_S$; the space X_S is H_S-monodromic with $\tilde{\pi}_S : \check{X}_S \to H_S\backslash\check{X}_S = X_S$ and carries $G^{(S)}$-action, where $G^\circ \subset G^{(S)} \subset G$ is the subgroup that leaves S invariant. More generally, for a subsets $S_1 \subset S_2$ of simple roots we have the projections

$\tilde{r}_{S_2 S_1} : \tilde{X}_{S_1} \to \tilde{X}_{S_2}$ such that $\tilde{r}_{S_3 S_2} \tilde{r}_{S_2 S_1} = \tilde{r}_{S_3 S_1}$ and $\tilde{r}_{S \emptyset} = \tilde{r}_S$ above; same for $r_{S_2 S_1}$. Consider the categories $\tilde{M}(X_S) := M(\tilde{D}_{X_S})$, $M(\tilde{X}_S)^m, \ldots$, that correspond to these monodromic spaces. The functors $\tilde{r}_S^o : \tilde{M}(X_S) \to \tilde{M}(X)$, $M(\tilde{X}_S)^m \to M(\tilde{X})^m$, ... are fully faithful embeddings and $\tilde{M}(X_S)_\chi \neq 0$ iff $\chi - \rho$ is W_S-fixed. This way $\tilde{M}(X_S)$, $M(\tilde{X}_S)^m, \ldots$ appear as Serre subcategories of the corresponding categories on X. Clearly $\tilde{M}(X_{S_2}) \subset \tilde{M}(X_{S_1}), \ldots$ if $S_1 \subset S_2$. One may see that $\tilde{M}(X_S)$ coincides with the subcategory of $\tilde{\mathcal{D}}$-modules with singular support in (conormal bundle to fibers of $r_S) \times \mathfrak{h}_S^* \subset \Omega_X \times \mathfrak{h}^*$. This implies, according to integrability of characteristic variety theorem, that $\tilde{M}(X_{S_1}) \cap \tilde{M}(X_{S_2}) = \tilde{M}(X_{S_1 \cup S_2})$ or $\tilde{M}(X_S) = \bigcap_{\alpha \in S} \tilde{M}(X_{\{\alpha\}})$. Same holds for $M(\tilde{X}_S)$.

Now define $\tilde{M}(X)^S$ to be the least Serre subcategory of $\tilde{M}(X)$ that contains all $\tilde{M}(X_{\{\xi\}})$, $\xi \in S$; define the Serre subcategories $M(\tilde{X})^{mS} \subset M(\tilde{X})^m$, $M(\tilde{X})^S_{\tilde{\chi},n} \subset M(\tilde{X})_{\tilde{\chi},n}$ etc. in a similar way. So their objects are $\mathcal{D}$-modules that admit a finite filtration with subquotients in either of $\tilde{M}(X_{\{\xi\}})$, $\xi \in S$.

Say that a weight $\chi \in \mathfrak{h}^*$ is good if the stabilizer $W_\chi \subset W$ of χ is W_{Σ_χ} for some set Σ_χ of simple roots. Any χ is W-conjugate to a good dominant weight.

2.3.6. Theorem. (i) If χ is dominant then $\Gamma_{\chi,n} : \tilde{M}(X)_{\chi,n} \to M(U)_{\chi,n}$ is exact functor and $\Gamma_\chi \Delta_\chi = \mathrm{id}_{M(U)_{\chi,n}}$. Hence $\Gamma_{\chi,n}$ identifies $M(U)_{\chi,n}$ with the quotient category $\tilde{M}(X)_{\chi,n}/\tilde{M}(X)^o_{\chi,n}$, where $\tilde{M}(X)^o_{\chi,n}$ is Serre subcategory of modules killed by $\Gamma_{\chi,n}$. Same holds for $\tilde{\Gamma}_{\chi,n} : M(\tilde{X})_{\tilde{\chi},n} \to M(U)_{\chi,n}$.

(ii) If χ is good dominant, $n < \infty$, then $M(\tilde{X})^o_{\tilde{\chi},n} = M(\tilde{X})^S_{\tilde{\chi},n}$,

$\tilde{M}(X)_{\chi,n} = T_{-\rho}\,\tilde{M}(X)^{S}_{\chi+\rho,n}$ where $S = \Sigma_\chi$ (for $T_{-\rho}$ see 1.3.1). □

Part (i) was proven in [3] (there only $\Gamma_{\chi,1}$ was considered, but the statement for $\Gamma_{\chi,n}$ follows by devissage, and $\tilde{\Gamma}$ version follows by 1.3.1 (ii)). For part (ii) see 2.7.3 below.

2.4. Harish-Chandra modules. Recall that a Harish-Chandra pair consists of a Lie algebra $\mathfrak{g}$ and an algebraic group K together with embedding $i : k := \mathrm{Lie}\, K \to \mathfrak{g}$ and a K-action on $\mathfrak{g}$ denoted by $\mathrm{Ad} : K \to \mathrm{Aut}\,\mathfrak{g}$. These i and Ad should be compatible that is the two k-actions on $\mathfrak{g}$, the adjoint action via i and Lie(Ad), coincide. Clearly Ad extends to the same noted K-action on $\mathfrak{U}(\mathfrak{g})$ and, in case $\mathfrak{g}$ is semisimple, to K-action on U.

A Harish-Chandra module, or $(\mathfrak{g}, K)$-module, is a vector space V with $\mathfrak{U}(\mathfrak{g})$-action and algebraic K-action which are compatible in a sense that

(i) For $k \in K$, $g \in \mathfrak{U}(\mathfrak{g})$, $v \in V$ one has $k(gv) = \mathrm{Ad}_K(g)\, k(v)$.

(ii) The k-actions on V that come from K- and $\mathfrak{g}$-actions coincide.

We may replace $\mathfrak{U}(\mathfrak{g})$ by $\dot{U}$ to get the definition of (U, K)-module. Clearly $(\mathfrak{g}, K)$- and (U, K)-modules form abelian categories; denote them $\mathcal{M}(\mathfrak{g}, K)$, $\mathcal{M}(\dot{U}, K)$ respectively. Note that the action of K on centers $\mathcal{Z}$, $S(\mathfrak{h})$ may be non-trivial (certainly K acts via the finite quotient), and so we have $\mathcal{M}(U, K)^f = \prod_{\dot{\chi} \in \mathrm{Ad}K\backslash \mathfrak{h}^*} \mathcal{M}(\dot{U}, K)_{\dot{\chi},\infty}$ in obvious notations. For $\chi \in \mathfrak{h}^*$ let K_χ, $K^0 \subset K_\chi \subset K$, be the stabilizer of χ. If $V \in \mathcal{M}(U, K)_{\dot{\chi}\infty}$, where $\dot{\chi} = \mathrm{Ad}\,K \cdot \chi$, then we have U-modules decomposition

$V = \bigoplus_{\chi' \in \dot{\chi}} V_{\chi'}$. Clearly V_χ is K_χ-invariant subspace, hence (U, K_χ)-module, and we get (cf. 1.3.5).

2.4.1. Lemma. For a K-orbit $\dot{\chi}$ in $\mathfrak{h}^*$ and $\chi \in \dot{\chi}$ the functor $\mathcal{M}(U, K)_{\dot{\chi}\infty} \to \mathcal{M}(U, K_\chi)_{\chi\infty}$, $V \longmapsto V_\chi$, is equivalence of categories. □

The similar fact holds for $(\mathcal{U}, K)$-modules. Hence the study of Harish-Chandra modules with locally finite action of the center reduces trivially to the study of $(U, K)_{\chi\infty}$-modules with the K-fixed $\chi \in \mathfrak{h}^*$.

It is easy to see that Γ and Δ extend in a moment to the equivariant situation: one has the adjoint functors $\mathcal{M}(U, K) \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \tilde{\mathcal{M}}(X, K)$ and, for a K-fixed weight $\chi \in \mathfrak{h}^*$, the corresponding $\Delta_{\chi,n}$, $\Gamma_{\chi,n}$.

2.4.2. Corollary. If χ is regular dominant weight, then $\mathcal{M}(U, K)_{\chi,n} \underset{\Gamma_{\chi,n}}{\overset{\Delta_{\chi,n}}{\rightleftarrows}} \tilde{\mathcal{M}}(X, K)_{\chi,n}$ are equivalences of categories. If χ is good dominant weight, then $\tilde{\Gamma}_{\chi,n}$ identifies $\mathcal{M}(U, K)_{\chi,n}$ with the quotient category $\mathcal{M}(\tilde{X}, K)_{\tilde{\chi},n}/\mathcal{M}(\tilde{X}, K)^{\Sigma_\chi}_{\tilde{\chi},n}$. □

We will say that $(\mathfrak{g}, K)$ is finite Harish-Chandra pair if K acts on X with finitely many orbits. Using 2.4.2 we may translate the pattern from 1.5 to $\mathfrak{g}$-modules to get (we use notations from 1.5 with "G" replaced by "K"):

2.4.3. Corollary. Assume that $(\mathfrak{g}, K)$ is a finite pair. Then

(i) Any finitely generated $\mathcal{Z}$-finite $(\mathfrak{g}, K)$-module has finite length and there are finitely many irreducibles for each central character.

(ii) Let χ be a good dominant weight. Then $\mathcal{M}(U, K)_{\chi,n}$

carries canonical $I_{X,K}$-stratification.

(iii) If χ is regular dominant then the strata $\mathcal{M}(U, K)_{\chi,n\alpha}$ are canonically equivalent to the categories $\mathcal{M}(\mathfrak{f}, K_{(x_\alpha)})_{\chi,n}$ (recall that x_α is a point on K-orbit Q_α, see 1.5).

(iv) If χ is any good dominant weight, then $\mathcal{M}(U, K)_{\chi,n\alpha}$ is equivalent to the quotient of $\mathcal{M}(\mathfrak{f}, K_{(x_\alpha)})_{\chi,n}$ by the subcategory $\mathcal{M}(\mathfrak{f}, K_{(x_\alpha)})^S_{\chi,n}$ generated by those irreducibles V for which $T_\rho j_{\alpha!*}(V)$ comes from a certain quotient $\tilde{X}_{\{\xi\}}$, $\xi \in S = \Sigma_\chi$. $\square$

2.4.4. Remark. In particular in situation (iii) above the irreducible (U, K)-modules are $j_{\alpha!*}(V)$, where V is irreducible $(\mathfrak{f}, K_{(x_\alpha)})$-module, and we have the corresponding !-and $*$- standard modules $j_{\alpha!}(V)$, $j_{\alpha*}(V)$ with canonical morphism $j_{\alpha!}(V) \to j_{\alpha*}(V)$ with image $j_{\alpha!*}(V)$. Note that $j_{\alpha!}(V)$ is just the projective covering of $j_{\alpha!*}(V)$ in $\mathcal{M}(U, K)_{\chi,\alpha,\bar{\alpha}} =$
$= \Gamma(\tilde{\mathcal{M}}(X_\alpha, K)_{\chi,l})$, and $j_{\alpha*}(V)$ is the injective envelope of $j_{\alpha!*}(V)$ in this subcategory. Now assume that χ is not integral weight. Then one may find another dominant weight χ' W-conjugate to χ i.e. such that $\gamma(\chi) = \gamma(\chi')$. Via the equivalences $\mathcal{M}(U, K)_\chi = \mathcal{M}(\mathcal{U}, K)_{\gamma(\chi)} = \mathcal{M}(U, K)_{\chi'}$ each irreducible $(\mathcal{U}, K)_{\gamma(\chi)}$-module gets two pairs of corresponding standard modules: from χ- and χ'-sides. These pairs coincide if $K = N$ (since they are just the Verma and dual to Verma modules, see 3.3); it is natural to suppose that this holds in any case but I do not know the proof. $\square$

One may translate the geometric definition of $\mathcal{M}(\mathfrak{f}, K_{(x_\alpha)})^{\{\xi\}}_\chi$, $\xi \in \Sigma_\chi$, as follows. Assume that $Ad(K)$ leaves ξ fixed, if not, replace K by the stabilizer of ξ. Let $P_\xi \supset b = b_{x_\alpha}$ be the

parabolic subgroup that corresponds to ε , $P'_\varepsilon := [P_\varepsilon, P_\varepsilon]$.
Put $K_{x_\alpha\varepsilon} := \mathrm{Ad}^{-1} P'_\varepsilon$. Assume that

$(*)_\varepsilon$ $\dim K_{x_\alpha\varepsilon} > \dim K_{x_\alpha} \cap K_{x_\alpha\varepsilon}$ (in this case the difference between dimensions is 1)

Put $K_{(x_\alpha\varepsilon)}$:= image of $(K_{x_\alpha\varepsilon})^\circ \cap K_{x_\alpha}$ in $K_{(x_\alpha)}$. It is easy to see that $K_{(x_\alpha\varepsilon)}$ is a central subgroup in $K_{(x_\alpha)}$. Since $\mathrm{Ad}(K_{x_\alpha\varepsilon})^\circ \subset G^\circ$ one has $\mathrm{Ad}((K_{x_\alpha\varepsilon})^\circ \cap K_{x_\alpha}) \subset B_{x_\alpha}$, and we have the character $\exp\varepsilon$ on $(K_{x_\alpha\varepsilon})^\circ \cap K_{x_\alpha}$, $\exp\varepsilon\,(k) :=$ $:= \exp\varepsilon\,(\mathrm{Ad}\,k \bmod N_x)$. One may see that $\exp\varepsilon$ factors through its quotient $K_{(x_\alpha\varepsilon)}$.

Let V be an irreducible $(\mathfrak{g}, K_{(x_\alpha)})_\chi$ -module. Consider the following condition

$(**)_\varepsilon$ The subgroup $K_{(x_\alpha\varepsilon)}$ acts on V via the character which is a square root of $\exp^{-1}\varepsilon$.

2.4.5. <u>Lemma</u>. The module V belongs to $\mathcal{M}(\mathfrak{g}, K_{(x_\alpha)})_\chi^{\{\varepsilon\}}$ iff $(*)_\varepsilon$ and $(**)_\varepsilon$ hold. $\square$

This way we arrive to a Langlands classification of irreducible $(\mathfrak{g}, K)$-modules. Namely, fix a good dominant χ . Then for any K-orbit $\alpha \in I_{X,G}$ and any irreducible $(\mathfrak{g}, K_{(x_\alpha)})_\chi$ -module V we have standard modules $j_{\alpha!}(V)$, $j_{\alpha*}(V)$ and the irreducible module $j_{\alpha!*}(V) = \mathrm{Im}(\, j_{\alpha!}(V) \to j_{\alpha*}(V))$. If χ is regular, we get this way all the irreducible $(\mathfrak{g}, K)$ - modules, each once. If χ is arbitrary good dominant, then for certain V's, namely, for such that $(*)_\varepsilon$, $(**)_\varepsilon$ hold for some $\varepsilon \in \Sigma_\chi$, the module $j_{\alpha!*}(V)$ equals to zero. To get the classification one has just to delete them from the classification list.

2.4.6. Here is another application of localisation function. First note the following fact. Assume we have a variety X with a stratification $\{Q_\alpha\}$. Then the constant sheaf $\mathbb{C}_X$ has an obvious filtration $\Phi_0 \subset \Phi_1 \subset \ldots$ with successive quotients $\Phi_i/\Phi_{i-1} = \bigoplus_{\operatorname{codim} Q_\alpha = i} j_{\alpha !}(\mathbb{C}_{Q_\alpha})$ (here $j_\alpha : Q_\alpha \hookrightarrow X$ is the embedding). Now assume that X and all Q_α's are smooth and j_α are affine embeddings. Then $\mathbb{C}_X[\dim X]$, $j_{\alpha !}(\mathbb{C}_{Q_\alpha})[\dim Q_\alpha]$ are perverse sheaves, and our $(\mathbb{C}_X[\dim X], \Phi_\cdot)$, considered as an object of filtered derived category, is, from the perverse point of view, just a complex of perverse sheaves ... —

$$\to \Phi_i/\Phi_{i-1}[\dim X - i] \to \Phi_{i-1}/\Phi_{i-2}[\dim X - i + 1] \to \ldots$$

with stupid filtration. Hence it is just a resolvent of $\mathbb{C}_X[\dim X]$ in the category of perverse sheaves. We may translate this to $\mathcal{D}$-modules by means of Riemann-Hilbert correspondence to get the resolvent of $\mathcal{O}_X$ with i's term $\bigoplus_{\operatorname{codim} Q_\alpha = i} j_{\alpha !}(\mathcal{O}_{Q_\alpha})$; in fact, it is dual to Cousin resolvent.

Now, returning to representations, assume that our pair $(\mathfrak{g}, K)$ is finite and for each K-orbit Q the embedding j_α is affine. The above construction gives us a canonical resolvent of $\mathcal{O}_{\tilde{X}}$ by standard modules $j_{\alpha !}(\mathcal{O}_{\tilde{Q}_\alpha})$. The functor $\tilde{\Gamma}_\chi$, where χ is dominant integral, transforms it to the resolvent of finite dimensional $\mathfrak{g}$-module by means of standard ones. If $K = N$ this is Bernstein-Gelfand-Gelfand resolvent; if K is symmetric this is Gabber-Joseph resolvent [16] (see § 3 or [19] for affinity of j_α).

2.4.7. The rest of the n° contains the construction of Harish-Chandra derived categories. We follow closely the pattern of n 1.3. The constructions below are implicit in [13]. The subject will not be of much use for the sequel.

Let $(\mathfrak{g}, K)$ is any Harish-Chandra pair ($\mathfrak{g}$ is arbitrary Lie algebra). The definition of weak Harish-Chandra module coincides with that of usual Harish-Chandra module with qxiom (ii) deleted (see the beginning of 2.4 above). Denote by $\mathcal{M}(\mathfrak{g}, K)_{weak}$ the corresponding abelian category. For a weak Harish-Chandra module V and $\xi \in k$ let $\xi^1 \in \mathrm{End}_{\mathbb{C}}V$ be the action of ξ that comes from K-action on V, and $\xi^2 \in \mathrm{End}_{\mathbb{C}}V$ be the action of $i(\xi) \in \mathfrak{g}$; put $w(\xi) := \xi^1 - \xi^2$. Similarly to 1.2.2 one has

(i) $w : k \to \mathrm{End}_{\mathfrak{g}}(V)$ is Lie algebras map that commutes with adjoint K-action

(ii) V is usual Harish-Chandra module iff $w = 0$.

Now define a Harish-Chandra complex to be a complex $M^{\cdot}$ of weak Harish-Chandra modules equipped with a family of operators $i_\xi : M^i \to M^{i-1}$, $\xi \in k$, such that i_ξ commute with $\mathfrak{g}$-action, $\xi \mapsto i_\xi$ commutes with adjoint K-action, $i_{\xi_1} i_{\xi_2} + i_{\xi_2} i_{\xi_1} = 0$, and $di_\xi + i_\xi d = w(\xi)$. The cohomology spaces $H^i(M^{\cdot})$ are usual Harish-Chandra modules by (ii) above. Denote by $C^{\cdot}(\mathfrak{g}, K)$ the category of such complexes. One immediately gets the corresponding homotopy category $K^{\cdot}(\mathfrak{g}, K)$ and, localising by H-quasiisomorphisms, the Harish-Chandra derived category $D^{\cdot}(\mathfrak{g}, K)$. The latter is t-category with core $\mathcal{M}(\mathfrak{g}, K)$ and cohomology functor H. We will also use the corresponding categories $\mathcal{M}(\mathfrak{g}, K)^{f.g.}, \ldots$ of finite generated modules.

2.4.8. Let $O_\wedge : C^{\cdot}(\mathfrak{g}, K) \to C^{\cdot}(\mathfrak{g}, K)_{weak}$ $(:= C^{\cdot}(\mathcal{M}(\mathfrak{g}, K)_{weak}))$ be the forgetting of i_ξ-action functor. It admits left and right adjoints C_k and C^k respectively which are just the standard complexes for k-action: $C_k(M^\cdot) :=$ $:= \mathcal{U}(\bar{k}^\cdot) \underset{\mathcal{U}(k)}{\times} M^\cdot$, $C^k(M^\cdot) := \mathrm{Hom}_{\mathcal{U}(k)}(\mathcal{U}(\bar{k}^\cdot), M^\cdot)$ (see 1.3.3). The corresponding functors between derived categories $D^b(\mathfrak{g}, K) \rightleftarrows D^b(\mathfrak{g}, K)_{weak}$ will be denoted by the same latters; the adjunction property remains valid.

The forgetting of K-action functor $O_K : \mathcal{M}(\mathfrak{g}, K)_{weak} \longrightarrow \mathcal{M}(\mathfrak{g})$ admits right adjoint functor Ind_{weak}. One has $\mathrm{Ind}_{weak}(V) := \mathcal{O}(K) \underset{\mathbb{C}}{\otimes} V$ - the space of V-valued functions on K, and the $\mathfrak{g}$ - and K-action are defined by formulas

$$[\xi(f \otimes v)](k) := f(k)\, \mathrm{Ad}_k(\xi)(v), \quad [\ell(f \otimes v)](k) := $$
$$:= f(k\ell)v \quad (\text{here } f \in \mathcal{O}(K);\ v \in V;\ \xi \in \mathfrak{g}\ ;\ k, \ell \in K)$$

The adjunction maps: $O_K \mathrm{Ind}_{weak} V \to V$ is $f \otimes v \mapsto f(1)v$, $M \to \mathrm{Ind}_{weak}\, O_K\, M$ is $m \mapsto$ (the function $k \mapsto km$).

The forgetting of $\mathfrak{g}$-action functor $O_{\mathfrak{g}} : \mathcal{M}(\mathfrak{g}, K)_{weak} \longrightarrow \mathcal{M}(K)$ admits left adjoint functor P_{weak}. For a K-module (= vector space with algebraic K-action) S one has $P_{weak}(S) = U(\mathfrak{g}) \underset{\mathbb{C}}{\otimes} S$ with $\mathfrak{g}$- and K-action defined by formulas $\xi(u \otimes s) := \xi u \otimes s$, $k(u \otimes s) = \mathrm{ad}_k(u) \otimes ks$ (here $\xi \in \mathfrak{g}$, $k \in K$, $u \in U(\mathfrak{g})$, $s \in S$); the adjunction maps $P_{weak} O_{\mathfrak{g}} M \to M$, $S \longrightarrow O_{\mathfrak{g}} P_{weak} S$ are $u \otimes m \mapsto um$, $s \mapsto 1 \otimes s$ respectively.

The functors Ind_{weak}, O_K, P_{weak}, $O_{\mathfrak{g}}$ are exact and define the same noted functors between derived categories.

The functor $\mathrm{Ind} := C^k\, \mathrm{Ind}_{weak} : C^\cdot(\mathfrak{g}) \to C^\cdot(\mathfrak{g}, K)$ is exact and right adjoint to the forgetting functor $O_{K\Lambda} := O_K O_\Lambda$. It transforms injective complexes to injective ones. Since the adjunction map $M^\cdot \to \mathrm{Ind}\, O_{K\Lambda} M^\cdot$ is embedding, one may use

Ind for construction of injective resolvents of bounded below Harish-Chandra complexes. Similarly the functor $P :=$ $:= C_{\mathbf{k}} P_{weak}: C^{\cdot}(K) \to C^{\cdot}(\mathfrak{g}, K)$ is exact and left adjoint to the forgetting functor $O_{\mathfrak{g}\wedge} := O_{\mathfrak{g}} O_{\wedge}$. The adjunction map $PO_{\mathfrak{g}\wedge} M^{\cdot} \to M^{\cdot}$ is surjective so one may use resolvents of this type to compute left derived functors.

2.4.9. Let $M^{\cdot} \in C^{b}(\mathfrak{g}, K)$ be a finite complex of finitely generated modules, and $N \in C^{+}(\mathfrak{g}, K)$ be any bounded below complex. Consider the complex of $\mathfrak{g}$-maps $Hom^{\cdot}_{\mathfrak{g}}(M^{\cdot}, N^{\cdot})$. It has obvious adjoint K-action and the action of operators i_{ξ} (for $\varphi \in$ Hom one has $i_{\xi}(\varphi) := i_{\xi}\varphi - \varphi i_{\xi}$), which means that $Hom^{\cdot}$ is a functor with values in $C^{+}(B_K) := C^{+}(\text{point}, K)$ (see 1.3).

Consider the derived functor $R\,Hom_{\mathfrak{g}} : D^{b}(\mathfrak{g}, K)^{fg} \times D^{+}(\mathfrak{g}, K) \to D^{+}(B_K)$. It may be computed by localisation of $Hom^{\cdot}$ by either of the variables, i.e. one may use either injective resolvents (constructed by means of Ind) for N or projective resolvents (produced by P) of M. Namely, for $S \in C^{b}(K)^{fg}$ the functor $Hom^{\cdot}_{\mathfrak{g}}(P(S), \cdot) = C^{k}(\cdot \otimes_{\mathbb{C}} S^{*})$ is exact, and for an injective complex $I^{\cdot} \in C^{+}(\mathfrak{g})$ the functor $Hom^{\cdot}_{\mathfrak{g}}(*, Ind\ I) = Ind_{B_K}(Hom_{\mathfrak{g}}(O_{K\wedge}{*}, I))$ is also exact, hence $R\,Hom_{\mathfrak{g}}(P(S), \cdot) = Hom_{\mathfrak{g}}(P(S), \cdot)$, $R\,Hom^{\cdot}_{\mathfrak{g}}(*, Ind\ I) =$ $= Hom^{\cdot}_{\mathfrak{g}}(*, Ind\ I)$.

One has $Hom_{C^{+}(\mathfrak{g}, K)}(M^{\cdot}, N^{\cdot}) = \Gamma(B_K, Hom^{\cdot}_{\mathfrak{g}}(M^{\cdot}, N^{\cdot})) :=$ $Hom_{C^{+}(B_K)}(\mathbb{C}, Hom^{\cdot}_{\mathfrak{g}}(M^{\cdot}, N^{\cdot}))$, which gives the morphism $R\,Hom(M^{\cdot}, N^{\cdot}) \to R\Gamma(B_K, R\,Hom^{\cdot}_{\mathfrak{g}}(M, N))$.

2.4.10. Lemma. (i) This arrow is isomorphism.

(ii) A natural morphism $O\ R\,Hom_{\mathfrak{g}}(M^{\cdot}, N^{\cdot})$ ——

$\to R\,\mathrm{Hom}(O_{K\Lambda}M^{\cdot}, O_{K\Lambda}N^{\cdot})$ is isomorphism.

2.4.11. Corollary. For $(\mathfrak{g}, K)$-modules M, N one has canonical spectral sequence that converges to $\mathrm{Ext}^n_{D^b(\mathfrak{g},K)}(M, N)$ with second term $E_2^{p,q} = H^p(B_K, \mathrm{Ext}^q_{D^b(\mathfrak{g})}(M, N))$; here $\mathrm{Ext}^q_{D^b(\mathfrak{g})}(M, N)$ are considered as K/K^o-modules by means of adjoint action of K.

Proof (i) follows since $\mathrm{Hom}^{\cdot}_{\mathfrak{g}}(M, \cdot)$ transforms injective complex Ind I to injective B_K-complex (see above formula). As for (ii) it suffice to verify it on generators $M = P(S)$, where S is finite dimensional K-module, and here it is clear. □

2.4.12. If $\mathfrak{g}$ is semisimple, then the functors Γ, Δ interchange the K-equivariant $\tilde{\mathcal{D}}_X$-complexes and equivariant Harish-Chandra complexes. If we bound ourselves with regular central characters the corresponding derived functors are equivalences of equivariant derived categories. This follows, say, from 2.3.3, 2.4.11 and the final lines of 1.3.

2.5. Lie algebra homology. Let M be a right $\tilde{\mathcal{D}}_X$-module, and N be a left one. Then $M \otimes_{\mathcal{O}_X} N$ has right $\tilde{\mathcal{D}}_X$-module structure (the element $\tau \in \tilde{\mathcal{T}}_X$ acts by the rule $(m \otimes n)\tau =$ $= m\tau \otimes n - m \otimes \tau n$), hence $M \odot N := M \otimes_{\mathcal{O}_X \otimes S(\mathfrak{f})} N =$ $= M \otimes_{\mathcal{O}_X} N/(M \otimes_{\mathcal{O}_X} N)\mathfrak{f}$ is right $\mathcal{D}_X$-module. Clearly $M \otimes_{\tilde{\mathcal{D}}_X} N$ coincides with $\mathcal{T}_X$ coinvariants on $M \odot N$. Note that if $M \in {}^{r}\tilde{M}(X)_{\chi,\infty}$, $N \in \tilde{M}(X)_{\chi,\infty}$, then $\pi^{o}(M \odot N)$ is the maximal quotient of $\tilde{\pi}^{\cdot} M \otimes_{\mathcal{O}_{\tilde{X}}} \tilde{\pi}^{\cdot} N$ that belongs to $\pi^{o}\, {}^{r}M(X)$

Consider now the derived bifunctors $\overset{L}{\odot} : D^{-r}\tilde{M}(X) \times$

$D^{-}\tilde{\mathcal{M}}(X) \to D^{-r}\tilde{\mathcal{M}}(X)$, $\overset{L}{\underset{\chi,n}{\odot}} : D^{-r}\tilde{\mathcal{M}}(X)_{\chi,n} \times D^{-}\tilde{\mathcal{M}}(X)_{\chi,n} \to$
$\to D^{-r}\mathcal{M}(X)$.

2.5.1. Lemma. For $A \in D^{-r}\mathcal{M}(U)$, $B \in D^{-}\mathcal{M}(U)$ one has canonical isomorphisms

$$A \overset{L}{\underset{U}{\otimes}} B = R\Gamma(X, (L\Delta\ A) \overset{L}{\underset{\tilde{\mathcal{D}}_X}{\otimes}} (L\Delta\ B)) = \int_X (L\Delta\ A) \overset{L}{\odot} (L\Delta\ B)$$

where $\int_X : D\,{}^r\mathcal{M}(X) \to D\ \mathrm{Vect}$ is direct image to the point functor. Same holds if one replaces U by $U_{\chi,n}$, by

Proof. The second isomorphism follows from the definition of $\int_X$. The first one comes from the obvious arrow $A \underset{U}{\otimes} B \to$
$\to \Gamma(X, \Delta A \underset{\tilde{\mathcal{D}}}{\otimes} \Delta B)$. If both A, B are free, then, by 2.2.2, this arrow is isomorphism. In this case also $\Delta = L\Delta$, $\underset{\tilde{\mathcal{D}}_X}{\otimes} = \overset{L}{\underset{\tilde{\mathcal{D}}_X}{\otimes}}$ and $\Gamma = R\Gamma$ (see 2.2.2). Now use a free resolvent to end up with 2.5.1. $\square$

2.5.2. Corollary. (i) For $A \in D^{-r}\mathcal{M}(U)_{\chi,\infty}$, $B \in D^{-}\mathcal{M}(U)_{\chi,\infty}$ one has

$$A \overset{L}{\underset{\dot{U}}{\otimes}} B = R\Gamma(\tilde{X}, (L\tilde{\Delta}\ A) \underset{\mathcal{D}_{\tilde{X}}}{\otimes} (L\tilde{\Delta}\ B)) = \int_{\tilde{X}} L\tilde{\Delta}\ A \overset{L}{\underset{\mathcal{O}_{\tilde{X}}}{\otimes}} L\tilde{\Delta}\ B$$

(ii) For $B \in \mathcal{M}(\dot{U})_{0,\infty}$ one has $H_i(\mathcal{G}, B) =$
$= H^{-i+\dim X} \int_{\tilde{X}} L\tilde{\Delta}\ B \cdot \det f^*$

Proof (i) is 2.5.1 joined with the following easy local formula: for $M \in D^{-r}\tilde{\mathcal{M}}(X)_{\chi,\infty}$, $N \in D^{-}\tilde{\mathcal{M}}(X)_{\chi,\infty}$ one has $M \overset{L}{\odot} N =$
$= R\pi_+(\tilde{\pi}^{\cdot}M \overset{L}{\underset{\mathcal{O}_{\tilde{X}}}{\otimes}} \tilde{\pi}^{\cdot}N)$; here, again, $\cdot \underset{\mathcal{O}_{\tilde{X}}}{\otimes} \cdot$ is right $\mathcal{D}_{\tilde{X}}$-module, see [9] VI 3.4.

(ii) is (i) applied to $A = \mathbb{C}$ with trivial right $\mathcal{G} \times f$ -

action: note that $\tilde{\Delta}(A) = \omega_{\tilde{X}} \cdot \det \mathcal{J}\,[\dim X]$. $\square$

2.5.3. We will need a variant of 2.5.1 for an action of correspondences. If A_1, A_2 are $\mathbb{C}$-algebras denote by $\mathcal{M}(A_1 - A_2)$ the category of $A_1 - A_2$-bimodules, i.e. $\mathcal{M}(A_1 - A_2) := \mathcal{M}(A_1 \underset{\mathbb{C}}{\otimes} A_2^{o})$, where A_2^{o} is A_2 with reversed multiplication. One has bifunctor $\cdot \underset{A_2}{\otimes} \cdot : \mathcal{M}(A_1 - A_2) \times \mathcal{M}(A_2) \to \mathcal{M}(A_1)$ and the corresponding derived functor $\cdot \overset{L}{\underset{A_2}{\otimes}} \cdot : D^- \mathcal{M}(A_1 - A_2) \times D^- \mathcal{M}(A_2) \to D^- \mathcal{M}(A_1)$; for $F \in D^- \mathcal{M}(A_1 - A_2)$, $M \in D^- \mathcal{M}(A_2)$ put $F(M) := F \overset{L}{\underset{A_2}{\otimes}} M$. Similarly, for good algebras A_i on varieties X_i we have the category $\mathcal{M}(A_1 - A_2) = \mathcal{M}(A_1 \boxtimes A_2^{o})$ of $A_1 \boxtimes A_2^{o}$-modules on $X_1 \times X_2$. Now let $X = X_{\mathfrak{g}}$, and $p_i : X \times X \to X$ be projections. We have the exact functor $p_{1*} : D\mathcal{M}(\tilde{\mathcal{D}}_X - \mathcal{D}_X) \to D\mathcal{M}(\tilde{\mathcal{D}}_X)$ (the integration along the second variable, defined just as for usual D-modules), and the bifunctor $\overset{L}{\odot} : D^-\mathcal{M}(\check{D}_X - \tilde{D}_X) \times D^-\mathcal{M}(\tilde{\mathcal{D}}_X) \to$ $\to D^-\mathcal{M}(\tilde{D}_X - \mathcal{D}_X)$: the derived functor for $F, M \mapsto F \underset{\mathcal{O}_X S(\mathfrak{f})}{\otimes} M$. Denote by $(F, M) \mapsto F(M) := p_{i*}\,(F \overset{L}{\odot} M)$ the composition of these functors.

The localisation functor obviously extends to bimodules: one has the functors $\Delta : \mathcal{M}(\dot{U} - \dot{U}) \to \mathcal{M}(\tilde{D}_X - \tilde{\mathcal{D}}_X)$, $L\Delta : D^- \mathcal{M}(U - \dot{U}) \to D^- \mathcal{M}(\tilde{\mathcal{D}}_X - \tilde{\mathcal{D}}_X)$, and the

2.5.4. Lemma. For $F \in D^- \mathcal{M}(\dot{U} - U)$, $B \in D^- \mathcal{M}(\dot{U})$ one has

$$L\Delta\, F(B) = L\Delta(F)(L\Delta(B)), \quad F(B) = R\Gamma[L\Delta(F)(L\Delta(B))]$$

The proof is similar to 2.5.1. Just as in 2.5.1 we have the obvious variant of 2.5.4 for $U_{\chi,n}$-modules (for $F \in D^-\mathcal{M}(\tilde{D}_{X\chi,n} - \tilde{D}_{X\chi,n})$, $M \in D^-\mathcal{M}(\tilde{D}_{X\chi,n})$ we put $F_{(n)}(M) :=$ $:= p_{1*}\,(F \underset{\chi,n}{\overset{L}{\odot}} M), \ldots)$

2.6. Verma modules and equivariant correspondences. For a point $x \in X$ and $w \in W$ consider the universal left and right Verma modules M_w^x, ${}^rM_w^x$ normalized as follows. The module $M_w^x \in \mathcal{M}(U)$ is generated by a vector $|vac\rangle \in M_w^x$ subject to only relation $b\,|vac\rangle = (2\rho(\bar{b}) + w^{-1}(\bar{b}))\,|vac\rangle$, where $b \in b_x \subset \mathfrak{g} \subset U$ and $\bar{b} := b \bmod n_x \in \mathfrak{h} \subset S(\mathfrak{h}) \subset U$. Similarly ${}^rM_w^x \in {}^r\mathcal{M}(U)$ is generated by $\langle vac| \in {}^rM_w^x$ such that $\langle vac|\, b = \langle vac|\, w^{-1}(\bar{b})$. The module M_w^x is just M_1^x with U-action turned by $w \in \mathrm{Aut}\, U$, same holds for ${}^rM_w^x$; the involution ^{t} of U transforms ${}^rM_w^x$ to M_w^x. For $\chi \in \mathfrak{h}^*$ put $M_{w\chi,n}^x := M_w^x/m_\chi^n M_w^x$, ${}^rM_{w\chi,n}^x := {}^rM_w^x/{}^rM_w^x m_\chi^n$. One knows that χ is dominant iff $M_{1\chi,1}^x$ is irreducible.

Let A be an U-module. Then the homology $H_i(n_x, A)$ carries 2 natural actions of $\mathfrak{h}$: the first one is the natural action of b_x/n_x (since b_x is normalizer of n_x), the second one comes from $\hat{\mathfrak{z}} \subset$ center $U \to \mathrm{End}\, A$. If A is an U^{reg}-module then $H^i({}^rM_w^x \overset{L}{\underset{\hat{U}}{\otimes}} A)$ is the component of $H_{-i}(n_x, A)$ on which the first action coincides with the second one turned by w; according to Casselman-Osborne theorem (see e.g. [34]) one has $H_{-i}(n_x, A) = \bigoplus_{w \in W} H^i({}^rM_w^x \overset{L}{\underset{U}{\otimes}} A)$. The similar fact holds if one replaces A by right U-module and ${}^rM_w^x$ by M_w^x.

In fact Verma modules are Harish-Chandra modules with respect to N_x. The orbits of N_x on X are in 1-1 correspondence with elements of Weyl group: I_{X,N_x} is W with Bruhat ordering. For $w \in W$ let $j_w : Q_w = N_x wx \hookrightarrow X$ be the w^{th} Schubert cell, $X_w = \bar{Q}_w$ be its closure.

2.6.1. Lemma. (i) $L\Delta\,(M_w^x)$ is a single $\tilde{D}_X$-module (i.e. $H^iL\Delta\,(M_w^x) = 0$ for $i \neq 0$) supported on $\bar{X}_w$. Same holds for

${}^{r}M^{\chi}_{w}$, $M^{\chi}_{w\chi,n}$, ${}^{r}M^{\chi}_{w\chi,n}$.

(ii) If χ is regular dominant then $\Delta\, M^{\chi}_{w\chi,n}$ is !-standard $(\tilde{D}_{\chi,n}, N)$-module for the orbit Q_w: namely $\Delta\, M^{\chi}_{w\chi,n} = j_{w!}(S(\mathfrak{f})/ m^n_\chi)$. If χ is regular antidominant, then $\Delta\, M^{\chi}_{w\chi,n} = j_{w*}(S(\mathfrak{f})/ m^n_\chi)$.

(iii) If A is a left U-module then one has canonical isomorphisms ${}^{r}M^{\chi}_{1} \overset{L}{\underset{\hat{U}}{\otimes}} A = \mathcal{O}_x/m_x \overset{L}{\underset{\mathcal{O}_X}{\otimes}} L\Delta A = \int_X \Delta({}^{r}M^{\chi}_{1}) \overset{L}{\odot} L\Delta A = i^{!}_{\tilde{x}}\, L\tilde{\Delta} A\,[\dim \tilde{X}]$, where $\tilde{x} \in \tilde{X}$ is any point over x.

Sketch of the proof. (i) Consider the morphism of algebras $\varphi : \mathcal{U}(b_X) \to \tilde{D}_X$ defined by formula $\varphi(b) = \tilde{\delta}(b - w^{-1}(\bar{b}))$ for $b \in b_X$ (here $w^{-1}(\bar{b}) \in \mathfrak{f} \subset \hat{U}$). Let $\mathbb{C}$ be a trivial b_X-module; one has $L\Delta\,(M^{\chi}_{w}) = \tilde{D}_X \overset{L}{\underset{\mathcal{U}(b_x)}{\otimes}} \mathbb{C}$. So it suffice to show that $H_i(b_X, \tilde{D}_X) = 0$ for $i > 0$ and $H_0(b_X, \tilde{D}_X)$ is supported on X_w. Consider canonical filtrations on $\mathcal{U}(b_X)$ and $\tilde{D}_X$, and the corresponding morphism of graded commutative algebras $\mathrm{gr}\,\varphi : S^{\cdot}(b_X) \to \mathrm{gr}^{\cdot}\tilde{D}_X$. An easy local calculations shows that $\mathrm{Tor}_i^{S^{\cdot}(b_x)}(\mathbb{C}, \mathrm{gr}^{\cdot}\tilde{D}_X) = 0$ for $i > 0$ and $\mathrm{Tor}_0^{S^{\cdot}(b_x)}(\mathbb{C}, \mathrm{gr}^{\cdot}D_X) = \mathrm{gr}^{\cdot}D_X/b_X\, \mathrm{gr}^{\cdot}D_X$ is supported on X_w. This implies the desired fact for $H_i(b_X, \tilde{D}_X)$ (say by a spectral sequence that relates this groups).

(ii) If χ is dominant, then $M^{\chi}_{w\chi,n}$ is projective object in $\mathcal{M}(U, N_X)_{\chi,n\,\overline{w}}$ (since χ is the lowest weight on this subcategory), hence is !-standard. The second statement is equivalent to the following: if χ is regular dominant, then $\Delta\,({}^{r}M^{\chi}_{w\chi,n}) = j_{w*}(S(\mathfrak{f})/ m^n_\chi)$. This is equivalent to the fact that $j^{!}_{w'}\, \Delta\, {}^{r}M^{\chi}_{w\chi,n} = 0$ for any $w' < w$, or, equivalently, that $\int_X \Delta\, {}^{r}M^{\chi}_{w\chi,n} \odot A = 0$ for any $A \in D^b\tilde{M}(X, N_X)$ supported on $X_w \setminus Q_w$.

It suffice to consider only the generating objects, namely $A = M^{x}_{w'\chi,n}$, $w' < w$. Then $\int M^{x}_{w\chi,n} \overset{L}{\odot} M_{w'\chi,n}$ is weight $w\chi$ part of $H_\bullet(\, n_x, M^{x}_{w'\chi,n})$ which is zero.

(iii) follows from 2.5.1. □

Now consider the Harish-Chandra pair $(\mathfrak{g}\times\mathfrak{g}, G^c)$ with $i : \mathfrak{g} = \mathrm{Lie}\, G^c \hookrightarrow \mathfrak{g}\times\mathfrak{g}$ the diagonal embedding and adjoint action (Ad, Ad). One has $U(\mathfrak{g}\times\mathfrak{g}) = U^{\otimes 2}$, $\tilde{X}_{\mathfrak{g}\times\mathfrak{g}} = \tilde{X}^2$, $\mathfrak{h}_{\mathfrak{g}\times\mathfrak{g}} = \mathfrak{h}^2$ etc. We have the categories $\tilde{M}(X^2)_{\chi_1,n_1\,\chi_2,n_2}$ and so on. This Harish-Chandra pair is finite; its orbits Y_w, $w \in W$, are numbered by elements of Weyl group with Bruhat ordering. According to 2.4.3 the categories $\mathcal{M}(U^{\otimes 2}, G^\circ)_{\chi_1\chi_2}$, $\tilde{M}(X^2, G^\circ)_{\chi_1\chi_2}$ are non=zero iff $\chi_1 - w\chi_2 \in \mathfrak{h}^*_{\mathbb{Z}}$ for some $w \in W$.

The $(\mathfrak{g}\times\mathfrak{g}, G^\circ)$-modules are almost the same as $(\mathfrak{g}, N)$-modules. One may see this using $\mathcal{D}$-modules as follows. For a point $\tilde{x}_0 \in \tilde{X}$, $x_0 = \pi(\tilde{x}_0) \in X$ consider the subvarieties $\tilde{X} = \tilde{X}\times\{\tilde{x}_0\} \subset \tilde{X}\times H = \tilde{X}\times\pi^{-1}(x_0)$ of $\tilde{X}\times\tilde{X}$: the first one is H-monodromic, the second one is $H\times H$-monodromic. Clearly $\tilde{X}$ is N-invariant, $\tilde{X}\times H$ is B-invariant, and $\tilde{X}^2 = G^\circ \times_N \tilde{X} = G^\circ \times_B (X\times H)$ (here $N := N_{x_0}$, $B := B_{x_0}$). Hence, by 1.2 (ii) we have canonical equivalences of categories $\mathcal{M}(\tilde{X}^2, G^\circ)^m = \mathcal{M}(\tilde{X}, N)^m = \mathcal{M}(\tilde{X}\times H, B)^m$.

Using 1.3.1 we may reformulate these equivalences in the language of $\tilde{\mathcal{D}}\times\tilde{\mathcal{D}}$-modules. Namely, for $\chi_1, \chi_2 \in \mathfrak{h}^*$ put $W_{\bar\chi_1\bar\chi_2} := \{w \in W : w\chi_2 - \chi_1 \in \mathfrak{h}^*_{\mathbb{Z}}\}$ and let $\tilde{M}(X, N)^{(\chi_2)}_{\chi_1,n}$ be the Serre subcategory of $\tilde{M}(X, N)_{\chi_1,n}$ generated by irreducibles supported at Schubert cells X_w for $w \in W_{\bar\chi_1\bar\chi_2}$ (it coincides with the whole $\tilde{M}(X, N)_{\chi_1,n}$, if $\chi_1 - \chi_2 \in \mathfrak{h}^*_{\mathbb{Z}}$). Then one has canonical equivalence of categories $\tilde{M}(X^2, G^\circ)_{\chi_1,n\,\chi_2,\infty} = \tilde{M}(X, N)^{(\chi_2)}_{\chi_1,n}$ that transforms to the above equivalence via the functor $\widetilde{\pi}^{\cdot}$.

2.6.2. At this point we may explain another equivalence, namely the one between Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [7] and $(\mathfrak{g}, N)$-modules; this n° will not be used in a sequel. Let $\chi_1, \chi_2 \in \mathfrak{h}^*$ be regular dominant weights such that $\chi_1 - \chi_2 \in \mathfrak{h}^*_{\mathbb{Z}}$. Then the equivalence 2.6.1 joined with 2.4.2 gives rise to canonical equivalence $\mathcal{M}(U^{\otimes 2}, G^{\circ})_{\chi_1,n\ \chi_2,\infty} = \mathcal{M}(U, N\ )_{\chi_1,n}$ (in fact, we may assume there that χ is good dominant only). Now let us interchange the multiples, and consider the equivalence of categories $\mathcal{M}(H \times \widetilde{X}, B\ )_{\bar{\chi}_1\infty\,\bar{\chi}_2\infty} = \mathcal{M}(\widetilde{X}, N\ )_{\bar{\chi}_2\infty}$. It corresponds to the equivalence of categories $\mathcal{M}(\mathbf{U}(\mathfrak{h} \times \mathfrak{g}), B\ )_{\chi_1\infty\chi_2\infty} = \mathcal{M}(U, N\ )_{\chi_2\infty}$ which is just the the forgetting functor for the obvious embedding $U \hookrightarrow \mathbf{U}(\mathfrak{h} \times \mathfrak{g})$. Note that this equivalence transforms the subcategory $\mathcal{M}(\mathbf{U}(\mathfrak{h} \times \mathfrak{g}), B^{\cdot}\ )_{\chi_1,1\ \chi_2,\infty}$ just to $(\mathfrak{g}, N\ )$-modules which are diagonalisable with respect to the action of a Cartan subalgebra of b_{x_0}, i.e. to Bernstein-Gelfand-Gelfand category $\mathcal{O}$. The composition of this with previous equivalences $\mathcal{M}(U, N\ )_{\chi_1,1} = \mathcal{M}(U^{\otimes 2}, G^{\circ})_{\chi_1,1\ \chi_2,\infty} = \mathcal{M}(\mathbf{U}(\mathfrak{h} \times \mathfrak{g}), B^{\cdot}\ )_{\chi_1,1\ \chi_2,\infty}$ is canonical equivalence between $\mathcal{M}(U, N\ )_{\chi_1,1}$ and $\mathcal{O}_{\chi_2}$.

2.6.3. We may use the above equivalences in $\widetilde{\mathcal{D}}_X$-bimodules case, since ^{t} along the second variable identifies them with $\widetilde{\mathcal{D}}_{X \times X}$-modules. Hence one has the equivalence $\mathcal{M}(\widetilde{\mathcal{D}}_X - \widetilde{\mathcal{D}}_X, G^{\circ})_{\chi_1,n\ \chi_2,\infty} = \widetilde{\mathcal{M}}(X, N\ )^{(\chi_2)}_{\chi_1,n}$. It transforms the correspondence F to $(\widetilde{\mathcal{D}}_X, N)$-module $F(\delta_{x_0})$, where $\delta_{x_0} = \Delta M_1^{x_0}$ is a universal $\widetilde{\mathcal{D}}_X$-module supported at x_0. We will denote the inverse equivalence $M \mapsto [M]^{(\chi_2)}$, so $[M]^{(\chi_2)}(\delta^{t}_{x_0}) = M$. The point is that the total functor F is completely determined by the single value

$F(\check{\delta}_{X_0})$.

2.6.4. Remarks. (i) If $F \in D^- M(\tilde{D}_X - \check{D}_X)$ is a complex with G-equivariant cohomology, then F is a single $\tilde{D}_X$- $\check{D}_X$-bimodule iff $F(\check{\delta}_{X_0}) \in \tilde{M}(X) \subset D^-\tilde{M}(X)$.

(ii) Usually one encounters with functors with values in $\tilde{M}(X)_{\chi,\infty}$ (instead of $\tilde{M}(X)_{\chi,n}$, n finite, as above) that come as follows. Put $\tilde{M}(X)_{\hat{\chi}} := \varprojlim \tilde{M}(X)^{fl}_{\chi,n}$, where $\tilde{M}(X)^{fl}_{\chi,n} \subset \tilde{M}(X)_{\chi,n}$ is the subcategory of $S(\mathfrak{f})/m^n_\chi$ -flat modules, and $\varprojlim$ is taken with respect to $M_n \in \tilde{M}_{\chi,n} \longmapsto M_{n-1} = M_n/m_\chi^{n-1}M_n \in \tilde{M}_{\chi,n-1}$; similarly one defines $M(U)_{\hat{\chi}}$ etc. Then any object F of $M(\tilde{D}_X - \tilde{D}_X, G^o)_{\hat{\chi}_1\hat{\chi}_2}$ (which coincides with $M(\tilde{D}_X - \tilde{D}_X, G^o)_{\hat{\chi}_1\chi_2,\infty} =$ $= M(\tilde{D}_X - \tilde{D}_X, G^o)_{\chi_1,\infty\hat{\chi}_2}$) defines the exact functor $F : D^-\tilde{M}(X)_{\chi_2,\infty} \longrightarrow D^-\tilde{M}(X)_{\chi_1,\infty}$ (by the formula from 2.5.3). $\square$

Let us consider the basic examples of functors that come from equivariant correspondences (see [4] for details).

(i) Translation functor. It was defined in 1.3.1; namely, for any $\varphi \in \mathfrak{f}^*_{\mathbb{Z}}$ we have canonical autoequivalence $T_\varphi : \tilde{M}(X) \longrightarrow \tilde{M}(X)$, $T_\varphi(\tilde{M}(X)_\chi) = \tilde{M}(X)_{\chi+\varphi}$. It comes from the correspondence supported on the diagonal. The corresponding functor for representations is coherent continuation functor. Recall its construction (see [8], [15], [34] for details). Take an irreducible finite dimensional representation V_φ with boundary weight φ. For $\chi \in \mathfrak{f}^*$ the coherent continuation functor $\Psi_{\chi+\varphi,\chi} : M(\mathfrak{g})_{\gamma(\chi),\infty} \longrightarrow M(\mathfrak{g})_{\gamma(\chi+\varphi),\infty}$ transforms $\mathfrak{g}$-module P to $\gamma(\chi+\varphi)$-component of $V_\varphi \otimes P$.

2.6.5. Lemma. If both χ, $\chi+\varphi$ are dominant and χ is regular (so $M(\mathfrak{g})_{\gamma(\chi),\infty} = M(U)_{\chi,\infty}$) then

$$\Gamma_{\chi+\varphi} T_\varphi = \Psi_{\chi+\varphi,\chi} \Gamma_\chi$$

Proof. Consider the sheaf $V_\varphi \underset{\mathbb{C}}{\otimes} \mathcal{O}_X$ equipped with obvious G°-action. It admits a G°-invariant filtration F^i such that F_i/F_{i-1} are line bundles $\mathcal{L}_{\varphi_i}$, $\{\varphi_i\}$ is the set of weights of V_φ. For a $\tilde{D}_{\chi,n}$-module M the sheaf $V_\varphi \underset{\mathbb{C}}{\otimes} M = (V_\varphi \otimes \mathcal{O}_X) \underset{\mathcal{O}_X}{\otimes} M$ is $\tilde{\mathfrak{g}}$-module filtered by $F_i \underset{\mathcal{O}_X}{\otimes} M$, the successive quotients are $\tilde{\mathcal{D}}_{\chi+\varphi_i,n}$-modules $T_{\varphi_i}(M)$. Hence $V_\varphi \otimes M$ as $\mathcal{Z}$-module is supported at $\{\gamma(\chi+\varphi_i)\} \subset \operatorname{Spec} \mathcal{Z}$. The condition of lemma implies that $T_\varphi(M)$ splits off as $\gamma(\chi+\varphi)$-component of $V_\varphi \underset{\mathbb{C}}{\otimes} M$. Since $\Gamma(X, V_\varphi \otimes M) = V_\varphi \otimes \Gamma(M)$, we are done. □

(ii) Intertwining functors. The action of $w \in W$ on U defines an obvious autoequivalence of $\mathcal{M}(U)$ which transforms $\mathcal{M}(U)_\chi$ to $\mathcal{M}(U)_{w\chi}$, M^χ_w, to $M^\chi_{ww'}$. It comes from $U-U$-bimodule U_w which is U with bimodule structure given by formula $a \cdot u \cdot b := w(a)u\,b$. Its localised version is $L\Delta(U_w) \in D^-\mathcal{M}(\tilde{D}_X - \tilde{D}_X)$ which transforms $D^b\tilde{\mathcal{M}}(X)_{\chi,n}$ to $D^b\tilde{\mathcal{M}}(X)_{w\chi,n}$. In fact 2.6.1, 2.6.4 imply that $L\Delta U_w = \Delta U_w \in \mathcal{M}(\tilde{\mathcal{D}}_X - \tilde{\mathcal{D}}_X, G^\circ)$ is a single bimodule supported on $\bar{Y}_w$.

Here is a geometric version for these correspondences. First let us construct the monodromic data over Y_w. Namely, assume we have fixed a type of Chevalley basis. Say that $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$, $\tilde{x}_i = (b_i, \{a_i^\alpha\})$ are in relative position w if $(x_1, x_2) \in Y_w$ and for some (or any) Cartan subalgebra $\mathfrak{h}_{12} \subset b_1 \cap b_2$ the elements a_1^α, a_2^α of root spaces of $\mathfrak{h}_{12}$ are a part of certain common Chevalley basis of $\mathfrak{g}$ (with respect to $\mathfrak{h}_{12}$) of the type we fixed. It is easy to see that the space $\tilde{Y}_w$ of such pairs is H-monodromic over Y_w with H-action given by formula $h(\tilde{x}_1, \tilde{x}_2) = ((w\,h)\tilde{x}_1, h\tilde{x}_2)$. If we change the type of basis, then Y_w will change into $(\varepsilon, 1)\tilde{Y}_w \subset \tilde{X} \times \tilde{X}$, where $\varepsilon \in H$ is certain element of order 2. Hence the category $\tilde{M}(Y_w)$ actually does not depends on the type of basis.

Consider the H-monodromic projections $\tilde{p}_{wi} : \tilde{Y}_w \to \tilde{X}$. They define the functor $w_* := \tilde{p}_{w1*}\tilde{p}^{\circ}_{w2} : D^b(\tilde{X})^m \to D^b(\tilde{X})^m$. It is easy to see that it transforms coherent modules to coherent ones, hence we have the functor $w_! := D_{\tilde{X}}\, w_* D_{\tilde{X}} = \tilde{p}_{w1!}\tilde{p}^{c}_{w2} :$ $: D^b(\tilde{X})^m \to D^b(\tilde{X})^m$. Note that these functors actually does not depend on the above choice of type of Chevalley basis (since they differ by H-translation, and we consider monodromic modules).

2.6.6. <u>Lemma</u>. (i) If $\ell(w) = \ell(w_1) + \ell(w_2)$, then $w_* =$ $= w_{1*}\, w_{2*}$, $w_! = w_{1!}\, w_{2!}$.

(ii) $w_!(w^{-1})_* = (w^{-1})_* w_! = \mathrm{Id}_{D^b(\tilde{X})^m}$

(iii) If χ is dominant, then the following diagram of functors commutes

$$\begin{array}{ccccc}
D^b(\tilde{X})^m & \xrightarrow{w_!} & D^b(\tilde{X})^m & \xrightarrow{w_*} & D^b(\tilde{X})^m \\
\uparrow \tilde{\pi}^{\cdot} & & \uparrow \tilde{\pi}^{\cdot} & & \uparrow \tilde{\pi}^{\cdot} \\
D^b\tilde{M}(X)_{w^{-1}\chi,n} & \xrightarrow{\Delta U_w} & D^b\tilde{M}(X)_{\chi,n} & \xrightarrow{\Delta U_w} & D^b\tilde{M}(X)_{w\chi,n}
\end{array}$$ □

((i) is immediate, (ii) follows from (i) and an easy calculation for a simple reflection, (iii) follows from 2.6.1 (ii)).

In particular (i), (ii) imply that $w_!$, w_* define the action of the braid group on $D^b(\tilde{X})^m$.

The next important example is

2.7. <u>Wall crossing</u> (see e.g. [15],[34]). Let $S \subset \Sigma(\Delta^+)$ be a set of simple roots and $\chi \in \mathfrak{h}^*$ be a weight with $W_\chi = W_S$. Let φ be a positive integral weight (say, $\varphi = \rho$), so $\nu :=$

$:= \chi + \varphi$ is regular dominant. Consider the commutative diagram of exact functors

$$\begin{tikzcd} \tilde{M}(X)_{\nu,\infty} \arrow[r, "T_{-\varphi}", "\sim"'] \arrow[d, "\Gamma", "\wr"'] & \tilde{M}(X)_{\chi,\infty} \arrow[r, "\Delta\tilde{\Psi}_{\nu,\chi}"] \arrow[d, "\Gamma"] & \tilde{M}(X) \arrow[d, "\Gamma", "\wr"'] \\ \mathcal{M}(U)_{\nu,\infty} \arrow[r, "\Psi_{\chi,\nu}"] & \mathcal{M}(\mathfrak{U})_{\gamma(\chi),\infty} \arrow[r, "\Psi_{\nu,\chi}"] & \mathcal{M}(U)_{\nu,\infty} \end{tikzcd}$$

Here the left square is 2.6.5, and $\Delta\tilde{\Psi}_{\nu,\chi} := \Delta_\nu \Psi_{\nu\chi} \Gamma$. In fact, $\Delta\tilde{\Psi}_{\nu,\chi}$ is an equivariant correspondence, $\Delta\tilde{\Psi}_{\nu,\chi} \in \mathcal{M}(\tilde{\mathcal{D}}_X - \tilde{\mathcal{D}}_X, G)$. To see this, consider the $\mathcal{U}\mathfrak{g}$-$U_{\chi,n}$-bimodule $V_\varphi \otimes U_{\chi,n}$ (the bimodule structure is $g(v\otimes \ell)u := gv\otimes \ell u + v\otimes g\ell u$, $g \in \mathfrak{g}$, $u \in U_{\chi,n}$). Let $\tilde{\Psi}_{\nu,\chi,n}$ be the $\gamma(\nu)$-component of $V_\varphi \otimes U_{\chi,n}$ with respect to left $\mathcal{Z}$-action; this is $\mathcal{U}\mathfrak{g}_{\gamma(\nu),\infty}$- $U_{\chi,n}$-bimodule, hence $U_{\nu,\infty}$ - $U_{\chi,n}$-bimodule (since ν is regular). We get an object $\tilde{\Psi}_{\nu,\chi} := \varprojlim \tilde{\Psi}_{\nu,\chi,n} \in \mathcal{M}(U_{\nu\infty} - U_{\hat{\chi}})$. By 2.5.5 the functor $\Delta\tilde{\Psi}_{\nu,\chi}$ above comes from the complex $L\Delta\tilde{\Psi}_{\nu,\chi} \in D^- \mathcal{M}(\tilde{D}_X - \tilde{D}_X)$. By 2.6.4 (i) $L\Delta\tilde{\Psi}_{\nu,\chi}$ is a single bimodule: $L\Delta\tilde{\Psi}_{\nu,\chi} = \Delta\tilde{\Psi}_{\nu,\chi} \in \mathcal{M}(\tilde{\mathcal{D}}_X - \tilde{\mathcal{D}}_X, G^0)_{\hat{\nu}\hat{\chi}}$

The functor $R_S := \Psi_{\nu\chi}\Psi_{\chi,\nu} : \mathcal{M}(U)_{\nu,\infty} \to \mathcal{M}(U)_{\nu,\infty}$ is S-reflection functor; this a projective functor [8]. It comes from the same-noted $U_{\hat{\nu}}$ - $U_{\hat{\nu}}$ -bimodule R_S, and $L\Delta(R_S) = \Delta(R_S)$ defines the functor $\Delta\tilde{\Psi}_{\nu,\chi} T_{-\varphi}$. We will look at R_S through the equivalence 2.6.3. Denote by $M_{w\hat{\nu}} := M^{x_0}_{w\hat{\nu}} := \varprojlim M^{x_0}_w / m^n_\nu \in \mathcal{M}(U, N)_{\hat{\nu}}$ the universal Verma module with weight in formal neighbourhood of ν .

2.7.1. Lemma. $R_S(M_{1\hat{\nu}}) = \Gamma(\Delta(R_S)(\hat{\delta}_{x_0\hat{\nu}}))$ has a filtration $F_{w_1} \subset F_{w_2} \subset \dots \subset F_{w_n} = F$, terms numbered by the elements of W_S (so $n = |W_S|$), such that

(i) $F_{w_i}/F_{w_{i-1}}$ is isomorphic to $M_{w_i\hat{\nu}}$

(ii) If $w_a < w_b$ by Bruhat order, then $F_{w_a} \supset F_{w_b}$.

Proof.

One has $\psi_{\chi,\nu}(M_{1\hat{\nu}}) = M_{1\hat{\chi}}$, hence $R_S(M_{1\hat{\nu}})$ is $\chi(\nu)$-component of $V_\varphi \otimes M_{1\hat{\chi}}$. Consider any B_X-filtration $\Phi_1 \subset \ldots \subset \Phi_{\dim V_\varphi} = V_\varphi$ on V_φ with 1-dimensional successive quotients; denote by φ_i the weight of Φ_i/Φ_{i-1}, so $\varphi_{\dim V_\varphi} = \varphi$. The filtration $\widetilde{\Phi}_i := \mathcal{U}(\mathfrak{g})\cdot(\Phi_i \otimes \text{(vacuum vector)})$ on $\mathfrak{g}$-module $V_\varphi \otimes M_{1\hat{\chi}}$ has successive quotients isomorphic to $\mathfrak{g}$-modules $M_{1\widehat{\chi+\varphi_i}}$. It is easy to see such a quotient is supported at $\chi(\nu)$ (with respect to $\mathcal{Z}$-action) iff φ_i is weight W_S-conjugate to φ, i.e. $\varphi_i + \chi \in W_S(\varphi + \chi)$. Our filtration is the one induced by $\widetilde{\Phi}_i$: namely $F_w = \widetilde{\Phi}_j \cap R_S(M_{1\hat{\nu}})$ where $w(\varphi+\chi) = \varphi_j + \chi$. Clearly it has properties (i), (ii) above. $\square$

2.7.2. Remarks. (i) The functor R_S is not $S(\mathfrak{f})$-linear, though is linear with respect to subring of W_S-invariant functions (translated by ν). In particular, $R_S(M_{1\nu,1})$ does not belongs to $\mathcal{M}(U, N)_{\nu,1}$.

(ii) According to 2.6.3, 2.7.1 implies that $\Delta(R_S) \in \mathcal{M}(\widetilde{D}_X - \widetilde{D}_X, \overset{o}{G})_{\hat{\nu}\hat{\nu}}$ has a filtration with successive quotients isomorphic to $[\Delta M_{w\hat{\nu}}]^{(\nu)}$, $w \in W_S$, which is a !-standard module for $\overset{o}{G}$-orbit Y_w.

(iii) Clearly 2.7.1 remains valid if we replace $M_{1\hat{\nu}}$ by $M_{1\nu,n}$, $M_{w\hat{\nu}}$ by $M_{w\nu,n}$.

2.7.3. Proof of 2.3.6 (iii). The statements for $\widetilde{M}(X)$ and $\mathcal{M}(\widetilde{X})$ are equivalent; we will prove $\widetilde{M}(X)$-version. Put $A_n := T_\rho \widetilde{M}(X)^{o}_{\chi,n}$, $B_n := \widetilde{M}(X)^{S}_{\nu,n}$ where $\nu = \chi + \rho$. These are Serre subcategories of $\widetilde{M}(X)_{\nu,n}$; our aim is to show that $A_n = B_n$.

(i) Let us prove that $A_n \supset B_n$. It suffice to show $\Gamma T_{-\rho}$

vanishes on $\tilde{M}(X)^{\{s\}}_{\nu,1}$, $s \in S$. But the sheaf $\mathcal{O}_X(-\rho)$ is isomorphic to $\mathcal{O}(-1)$ along each fiber of projection $\pi_{\{s\}} : X \to X_{\{s\}}$. Hence for any $\mathcal{O}_{X_{\{s\}}}$-module N one has $\Gamma(X, \mathcal{O}_X(-\rho) \otimes \pi^*_{\{s\}} N) = 0$, and we are done.

(ii) Let $L_{w\nu}$ be the irreducible quotient of $M_{w\nu,1}$. Then $\Delta(L_{w\nu}) \in \tilde{M}(X)^S_{\nu,1}$ if $w \in W_S$, $w \neq 1$ (in fact, $\Delta(L_{w\nu}) \in \tilde{M}(X)^{\{s\}}_\nu$ if $\ell(\sigma_s w) < \ell(w)$).

(iii) For $w \in W_S$ $L_{1\nu}$ $(= M_{1\nu,1})$ occurs in $M_{w\nu}$ with multiplicity one. To see this note that by [7] it siffice to find a projective object P of category $\mathcal{O}$ that maps onto L_1 and contains each $M_{w\nu,1}$, $w \in W_S$, with multiplicity one. To construct P choose a dominant weight μ such that $\nu - \mu$ is integral and $M_{1\mu,1} = L_{1\mu}$ is projective (it means, also, say, by [7], that the values of $\mu+\rho$ on each coroot are either 0 or non-integral). Then $P = \Psi_{\nu,\mu}(L_{1,\mu})$ is also projective, and it has the desired properties by 2.7.1, 2.7.2 (iii) (in fact, P is projective covering of $L_{1,\nu}$).

(iv) For $w \in W_S$ consider a (unique) embedding $M_{1\nu,1} = L_{1\nu} \subset M_{w\nu,1}$. Then, by (ii), (iii), one has $\Delta(M_{w\nu,1}/M_{1\nu,1}) \in \tilde{M}(X)^S_{\nu,1}$. Hence, by 2.6.3 for any $I \in \tilde{M}(X)_{\nu,1}$ the induced morphism $I = [\Delta M_{1\nu,1}]^{(\nu)}(I) \to [M_{w\nu,1}]^{(\nu)}(I)$ is isomorphism in the quotient category $D^b(\tilde{M}(X)_\nu / \tilde{M}(X)^S_\nu)$.

(v) It remains to show that $A_n \subset B_n$. We may assume that $n = 1$ (since $n \neq \infty$).

Take a module $I \in \tilde{M}(X)_{\nu,1}$. By 2.7.2 (ii), 2.7.3 (iv) $(\Delta R_S)(I)$, considered as object of $\tilde{M}(X)_{\nu,m}/B_m$ (for some m), has a filtration of length $|W_S|$ with successive quotients isomorphic to I. If $I \in A_1$, then $(\Delta R_S)(I) = 0$, hence $I = 0$ modulo

B_m, and we are done. $\square$

2.8. Vogan's conjecture. Consider the case when S reduces to a single element α. The short exact sequence $0 \to M_{\sigma_\alpha \nu,1} \to R_{\{\alpha\}}(M_{1\nu,1}) \to M_{1\nu,1} \to 0$ (see 2.7.1) and $0 \to M_{1\nu,1} \to M_{\sigma_\alpha \nu,1} \to L_{\sigma_\alpha \nu} \to 0$ define a complex $0 \to M_{1\nu,1} \to R_{\{\alpha\}}(M_{1\nu,1}) \to M_{1\nu,1} \to 0$ with only cohomology equal to $L_{\sigma_\alpha \nu}$. By 2.6.3 we get a complex of exact functors $Id_{\tilde{M}(X)_{\nu,1}} \to \Delta R_{\{\alpha\}} \to Id_{\tilde{M}(X)_{\nu,1}}$ on $\tilde{M}(X)_{\nu,1}$ with values in $\tilde{M}(X)_{\nu,2}$, and the similar complex for $\mathfrak{g}$-modules $Id_{M(U)_{\nu,1}} \to R_{\{\alpha\}} \to Id_{M(U)_{\nu,1}}$. For any complex $M^\cdot$ of $\tilde{\mathcal{D}}_{X\nu,1}$-modules the complex associated with the bicomplex $M^\cdot \to \Delta R_{\{\alpha\}}(M^\cdot) \to M^\cdot$ is canonically isomorphic to $[\Delta L_{\sigma_\alpha \nu}]^{(\nu)}_{(1)}(M^\cdot)$ in $D^b \tilde{M}(X)_{\nu,2}$. The geometric version of the above is as follows. Let $H^\alpha \subset H$ be the subtorus generated by coroot h_α, $H_\alpha = H/H^\alpha$. We have the proper map $\tilde{r}_\alpha : H^\alpha\backslash\tilde{X} \to \tilde{X}_\alpha$ of H_α-monodromic spaces (see 2.3.5). Since ν takes integral value on the coroot h_α, we have $\tilde{\tilde{\pi}}^\cdot \tilde{M}(X)_{\nu,1} \subset M(H^\alpha\backslash\tilde{X})^m \subset M(\tilde{X})^m$, and there is commutative diagram of functors

$$\begin{array}{ccc}
D^b M(H^\alpha\backslash\tilde{X})^m & \xrightarrow{\tilde{r}_{\alpha *}\tilde{r}^{\circ}_{\alpha}} & D^b M(H^\alpha\backslash\tilde{X})^m \\
\uparrow \tilde{\tilde{\pi}}^\cdot & & \uparrow \tilde{\tilde{\pi}}^\cdot \\
D^b \tilde{M}(X)_{\nu,1} & \xrightarrow{[\Delta L_{\sigma_\alpha \nu}]^{(\nu)}_{(1)}} & D^b \tilde{M}(X)_{\nu,1}
\end{array}$$

Now assume that ν is rational weight, and $(\mathfrak{g}, K)$ is any finite Harish-Chandra pair. Then, by 1.4.4 (v), for any irreducible $(U, K)_\nu$-module V the $\mathcal{D}_{\tilde{X}}$-module $\tilde{\Delta}(V) = \tilde{\pi}^\cdot \Delta(V)$ is irreducible $\mathcal{D}_{\tilde{X}}$-module of geometric origin. Hence decomposition theorem implies that $r_{\alpha *} r^c_\alpha \tilde{\Delta}(V)$ has semisimple cohomologies. Now $\tilde{\Gamma}_\nu$ translates everything back to $\mathfrak{g}$-modules, and we get

2.8.1. Corollary. The complex $V \to R_{\{\alpha\}}(V) \to V$ has semisimple cohomology. □

When $(\mathfrak{g}, K)$ is symmetric this is just the basic Vogan's conjecture that keeps his Kazhdan-Lusztig algorithm going. Note that we prove 2.8.1 assuming that ν is rational. Perhaps 2.8.1 for arbitrary ν may be deduced from rational case by some formal algebraic arguments.

§3. Symmetric pairs

In this § we give some preliminaries necessary for the construction of Jantzen filtration; the lengthy n° 3.3 is needed for the comparison of definitions of § 4 with classical Jantzen ones and may be omitted.

3.1. Admissible orbits. Let $\tilde{X}$ be an H-monodromic K-variety. For $x \in X$ consider the pair $(\mathfrak{f}, K_{(x)})$ (see 1.5.1). Put $\mathfrak{f}^*(x) := \{\varphi \in \mathfrak{f}^* : Ad_{K_{(x)}}(\varphi) = \varphi,\ \varphi(i(k_x)) = 0\}$ (this are just the morphisms of $(\mathfrak{f}, K_{(x)})$ to the trivial Harish-Chandra pair $(\mathbb{C}, \{1\}))$, $\mathfrak{f}^*_{\mathbb{Z}}(x) := \mathfrak{f}^*_{\mathbb{Z}} \cap \mathfrak{f}^*(x)$. Clearly one has $\mathfrak{f}^*(kx) = Ad_k(\mathfrak{f}^*(x))$ for $k \in K$. So we may use Ad_k to identify canonically $\mathfrak{f}^*(x)$ with x in a fixed orbit Q; denote this group $\mathfrak{f}^*(Q)$, same for $\mathfrak{f}^*_{\mathbb{Z}}(Q)$. It is easy to see that $\varphi \in \mathfrak{f}^*_{\mathbb{Z}}$ belongs to $\mathfrak{f}^*_{\mathbb{Z}}(x)$ iff there exists a non-zero K-invariant function f_φ on $\tilde{Q} = K\pi^{-1}(x)$ such that $f_\varphi(h\tilde{x}) = (\exp\varphi)(h)f_\varphi(\tilde{x})$ for $\tilde{x} \in \pi^{-1}(x)$. Such f_φ is determined by φ uniquely up to multiplication by non-zero constant. Say that f_φ is $\bar{Q}$-regular if $f_\varphi \in \mathcal{O}(\bar{Q}) \subset \mathcal{O}(Q)$; f_φ

is $\bar{Q}$-invertible if $f_\varphi \in \mathcal{O}^*(\bar{Q})$; and f_φ is $\bar{Q}$-positive if f_φ is $\bar{Q}$-regular and $f_\varphi^{-1}(0) = \bar{Q} \setminus Q$. Put $\mathfrak{f}_{\mathbb{Z}}^{*0}(Q) = \{\varphi \in \mathfrak{f}_{\mathbb{Z}}^*(Q) :$ $: f_\varphi$ is $\bar{Q}$-invertible$\}$, $\mathfrak{f}_{\mathbb{Z}}^{*+}(Q) = \{\varphi \in \mathfrak{f}_{\mathbb{Z}}^*(Q) : f_\varphi$ is $\bar{Q}$-positive$\}$.

3.1.1. Definition. (i) An orbit Q is admissible if $\mathfrak{f}_{\mathbb{Z}}^{*+}(Q)$ is not empty.

(ii) The K-action on $\tilde{X}$ is admissible if it has finitely many orbits on X and any orbit is admissible. □

3.1.2. Lemma. (i) For any admissible orbit Q the embeddings $Q \hookrightarrow X$, $\tilde{Q} \hookrightarrow \tilde{X}$ are affine.

(ii) $\mathfrak{f}_{\mathbb{Z}}^{*0}(Q)$ is the subgroup of $\mathfrak{f}_{\mathbb{Z}}^*(Q)$, and $\mathfrak{f}_{\mathbb{Z}}^*(Q)/\mathfrak{f}_{\mathbb{Z}}^{*0}(Q)$ has no torsion.

(iii) If Q is admissible, then $\mathfrak{f}_{\mathbb{Z}}^{*+}(Q)$ is a subsemigroup of $\mathfrak{f}_{\mathbb{Z}}^*(Q)$ that generates $\mathfrak{f}_{\mathbb{Z}}^*(Q)$ and invariant under $\mathfrak{f}_{\mathbb{Z}}^{*0}(Q)$-translations. If $\varphi \in \mathfrak{f}_{\mathbb{Z}}^*(Q)$ and $n\varphi \in \mathfrak{f}_{\mathbb{Z}}^{*+}(Q)$ for certain $n > 0$, then $\varphi \in \mathfrak{f}_{\mathbb{Z}}^{*+}(Q)$. The quotient $\mathfrak{f}_{\mathbb{Z}}^{*+}(Q)/\mathfrak{f}_{\mathbb{Z}}^{*0}(Q)$ is isomorphic to Z_+^a.

(iv) An orbit is admissible iff (some, or any) its connected component is admissible with respect to K°-action. Hence a K-action is admissible iff such is K°-action.

(v) Assume we are in the induced situation, i.e. we have $K' \subset K$, a monodromic K'-space $\tilde{X}'$ and $X = K \times_{K'} \tilde{X}'$ (see 1.2.5). Let Q' be a K'-orbit in X', $Q = KQ' \subset X$. Then $\mathfrak{f}_{\mathbb{Z}}^*(Q) = \mathfrak{f}_{\mathbb{Z}}^*(Q')$, same for $\mathfrak{f}^{*0}$, $\mathfrak{f}^{*+}$. Hence $\tilde{X}'$ is admissible K'-variety iff such is K-variety $\tilde{X}$. □

3.2. Case of a flag variety. Assume we are in a situation 2.4 , so $(\mathfrak{g}, K)$ is a Harish-Chandra pair, $\mathfrak{g}$ is semisimple. Say that $(\mathfrak{g}, K)$ is admissible if such is K-action on $\tilde{X}_{\mathfrak{g}}$.

3.2.1. Lemma. The pair $(\mathfrak{g}, N)$, where N is maximal nilpotent subgroup, is admissible.

Proof. Consider Schubert cell Q_w, $w \in W$. Let $\mathfrak{h}_{\mathbb{Z}}^{*+}$ be the cone of positive regular integral characters. We will see that for any $w \in W$ one has $\mathfrak{h}_{\mathbb{Z}}^{*+}(Q_w) \supset \rho + \mathfrak{h}_{\mathbb{Z}}^{*+}$, hence Q_w is admissible. For $\chi \in \rho + \mathfrak{h}_{\mathbb{Z}}^{*+}$ take an irreducible $G^{\mathbb{C}}$-module V with highest weight χ; let $v \in V^N \setminus \{0\}$ be a lowest weight vector. Consider the map $q_v : \tilde{X} = G^{\mathbb{C}}/N \to V \setminus \{0\}$, $q_v(g) = gv$. So if $\ell \in V^*$ is a linear function on V, then ℓq_v is a χ-homogenuous function on X. Choose $\ell \in V^*$ such that $\ell(wv) \neq 0$, $\ell(\mathfrak{n}wv) = 0$ (here $\mathfrak{n} = \text{Lie } N$). One has $wv \in q_v(\tilde{Q}_w)$, the image $q_v(\tilde{Q}_w)$ lies in the linear N-invariant subspace generated by wv, and $q_v(\check{X}_w - \tilde{Q}_w) \subset \mathfrak{n} \cdot wv$. Hence ℓq_v is the desired χ-homogenuous N-invariant function that vanishes on $\tilde{X}_w \setminus \check{Q}_w$. □

Remark. The B-action is not admissible.

Now assume that $(\mathfrak{g}, K)$ is symmetric pair which means that $k = \mathfrak{g}^{\Theta}$ for an involution Θ of $\mathfrak{g}$. Note that Θ is uniquely determined by k (its -1 eigenspace coincides with Killing orthogonal complement to k), in particular Θ commutes with $\text{Ad } K$. For $x \in X$ denote by $\mu_\Theta(x) \in W$ the relative position of $(b_x, \Theta b_x)$. Clearly $\mu_\theta(kx) = \text{Ad}(k)(\mu_\theta(x))$ for $k \in K$. In particular μ_Θ is constant along the connected components of K-orbits; if Q^o is such a component we will write $\mu_\Theta(Q^o) = \mu_\Theta(x)$, $x \in Q^o$.

√ Remark. The following points are equivalent: (i) An orbit Q is closed, (ii) $\mu(Q) = 1$, (iii) For $x \in Q$ one has $\dim K \cap N_x = \dim Q$, (iv) $\dim Q = \dim X_k$.

3.2.2. Lemma. The symmetric pair is admissible.

Proof. According to 3.1.2 (iv) we may assume that K is connected

(i) Let us consider the particular case: $\mathfrak{g} = k \times k$, $i : k \to \mathfrak{g}$ is diagonal embedding; hence Θ is transposition, $\tilde{X}_{\mathfrak{g}} = \tilde{X}_k \times \tilde{X}_k$. If $\tilde{x} \in \tilde{X}_k$, then the K-space $\tilde{X}_{\mathfrak{g}}$ is induced from N_x-space $\tilde{X}_k = \tilde{X}_k \times \{\tilde{x}\} \hookrightarrow \tilde{X}_{\mathfrak{g}}$, and the K-orbits on $X_{\mathfrak{g}}$ are the same as N_x-orbits on X_k: these are $Y_w := K(Q_w \times x)$, $w \in W_k$. One has $\mathfrak{f}^*_{\mathfrak{g}\mathbb{Z}}(Y_w) = \{(\chi, -w\chi), \chi \in \mathfrak{f}^*_{k\mathbb{Z}}\} \subset \mathfrak{f}^*_{k\mathbb{Z}} \times \mathfrak{f}^*_{k\mathbb{Z}} = \mathfrak{f}^*_{\mathfrak{g}\mathbb{Z}}$ since $K \backslash \tilde{Y}_w$ is isomorphic to H_k with $H_{\mathfrak{g}} = H_k \times H_k$ -action given by formula $(h_1, h_2)h = h_1 w(h_2^{-1})h$. Then clearly $\mathfrak{f}^{*+}_{\mathfrak{g}\mathbb{Z}}(Y_w) = \{(\chi, -w\chi), \chi \in \mathfrak{f}^{*+}_{k\mathbb{Z}}(Q_w)$, since $f_{(\chi, -w\chi)} | \tilde{X}_w \times \{\tilde{x}\} = f_\chi$, hence we are done by 3.2.1.

(ii) The general case. Consider the embeddings $m_\Theta : X \hookrightarrow X \times X$, $\tilde{m}_\Theta : \tilde{X} \hookrightarrow \tilde{X} \times \tilde{X}$, defined by formula $m_\Theta(x) = (x, \Theta(x))$, same for $\tilde{m}_\Theta$. These maps are equivariant with respect to K-action on X and the diagonal action of K on $X \times X$ (via $Ad : K \to G$); one has $m_\Theta(x) \in Y_{\mu_\Theta(x)}$. For $w \in W$ consider the locally closed K-invariant subvariety $X_{\Theta w} := \mu_\Theta^{-1}(w) = m_\Theta^{-1}(Y_w) \subset X$. The number of K-orbits on X is finite since one has

(*) $X_{\Theta w}$ is finite disjoint union of K-orbits

This follows from the corresponding infinitezimal statement: namely, for $x \in X_{\Theta w}$ the tangent space $\mathcal{T}_{Kx,x}$ to the K-orbit coincides with $dm_\Theta^{-1}(\mathcal{T}_{Y_w, m_\theta(x)})$ (proof: one has $\mathcal{T}_{Y_w, m_\theta(x)} = \{(g \bmod b_x, g \bmod \Theta\, b_x), g \in \mathfrak{g}\} = \{(g \bmod b_x, \Theta(\Theta g \bmod b_x))\}$, $dm_\Theta(\mathcal{T}_{X,x}) = \{(g \bmod b_x, \Theta(g \bmod b_x)\}$; hence

$\mathcal{T}_{Y_w, m_\theta(x)} \cap dm_\Theta(\mathcal{T}_{X,x}) = \{(g \bmod b_x, \Theta(g \bmod b_x)) : g - \Theta(g) \in b_x\} =$
$= \{(g \bmod b_x, \Theta(g \bmod b_x)), g \in \mathfrak{g}^\Theta\}$, and we are done). In particular (*) implies that for any K-orbit Q on X one has $\mu_\Theta(Q) \notin \mu_\Theta(\tilde{Q} \setminus Q)$. So for any homogenuous $\overset{\circ}{G}$-invariant function f on $\widetilde{Y}_{\mu_\theta(Q)}$ the function $f \circ \tilde{m}_\Theta$ is homogenuous K-invariant on $\tilde{Q}$, and if f was positive, then $f \circ m_\Theta$ is also positive, and (i) above finishes the proof. □

3.3. Contravariant duality for standard modules. If $(\mathfrak{g}, K)$ is a finite pair, then we have the Verdier duality on the category of $S(\mathfrak{h})$-finite coherent $(\tilde{D}_X, K)$-modules. This duality is local with respect to X and transforms to Verdier duality on perverse constructible sheaves via Riemann-Hilbert correspondence. On the other hand, if $(\mathfrak{g}, K)$ is symmetric, or $K = N$, then one has the usual contravariant duality for $(\mathfrak{g}, K)$-modules. In this section we will see how this duality act on standard and irreducible modules (in terms of their geometric Langlands data); a similar description in a different terms may be found in [34].

Consider the involution C_U on U defined by formula $C_U(u) = w_{max}\,{}^t u$ (here $w_{max} \in W$ is the element of maximal length). It coincides with -1 on $\mathfrak{g}$, and induces on $S(\mathfrak{h})$ the involution $C_{\mathfrak{h}}$ such that $C_{\mathfrak{h}}(\chi) = W_{max}(-2\rho - \chi)$ for $\chi \in \mathfrak{h}^*$. This $C_{\mathfrak{h}}$ comes from the involution C_H of torus H; one has $C_{\mathfrak{h}}(\Delta^+) = \Delta^+$. For a left $\dot{U}$-module V let V° be the dual vector space to V with left $\dot{U}$-module structure defined by formula $\langle uv^*, v\rangle = \langle v^*, C_U(u)v\rangle$, $u \in U$, $v \in V$, $v^* \in V^\circ$. As $\mathfrak{g}$-module V° is just the dual module to V; we use C_U instead of ^{t} since C_U transforms dominant weights to dominant ones which is handy for localisation (note that $V \in M(U)_\chi$ iff $V^\circ \in M(U)_{C_g \chi}$).

Here is a $\mathcal{D}$-module interpretation of V°. Consider the open G-orbit $Y = Y_{w_{max}} \subset X \times X$ and the variety $\tilde{Y} := \tilde{Y}_{w_{max}}$ defined in 2.6.6; $\tilde{Y}$ is an H-monodromic variety over Y with H-action $h(x_1, x_2) = (hx_1, C_H(h^{-1})x_2)$, the projections $\tilde{p}_1, \tilde{p}_2 : \tilde{Y} \to \tilde{X}$ are monodromic maps. Hence we have the algebra $\tilde{\mathcal{D}}_Y$ on Y; if M is $\tilde{\mathcal{D}}_X$-module then $p_i^\circ M$ are $\tilde{\mathcal{D}}_Y$-modules. For a pair M_1, M_2 of $\tilde{\mathcal{D}}_X$-modules put $M_1 \boxdot M_2 :=$
$:= p_1^\circ M_1 \underset{\mathcal{O}_Y}{\otimes} p_2^\circ M_2 / \mathfrak{h}(p_1^\circ M_1 \underset{\mathcal{O}_Y}{\otimes} p_2^\circ M_2)$: this is a $\mathcal{D}_Y$-module (since $p_1^\circ M_1 \underset{\mathcal{O}_Y}{\otimes} p_2^\circ M_2$ is $\tilde{\mathcal{D}}_Y$-module). Note that if $M_i \in \tilde{M}(X)_{\chi_i}$ then $M_1 \boxdot M_2 = 0$ if $\chi_1 \neq C_g(\chi_2)$.

The tangent space to $(x_1, x_2) \in Y$ coincides with $\mathfrak{g}/b_{x_1} \cap b_{x_2} = \mathfrak{g}/\mathfrak{h}_{x_1,x_2}$, hence ω_Y is isomorphic canonically to $\varepsilon\mathcal{O}_Y := \varepsilon \underset{\mathbb{C}}{\otimes} \mathcal{O}_Y$, where $\varepsilon := \det \mathfrak{h} \cdot \det^{-1} \mathfrak{g}$. If N is a left $\mathcal{D}_Y$-module, then the sheaf $\varepsilon \cdot N$ is just the corresponding right $\mathcal{D}$-module. The variety Y is affine, hence $\int_Y$ (:= zeros direct image to the point functor) is right exact and we have $\int_Y \varepsilon N = \varepsilon\left(\Gamma(Y, N)/\mathfrak{g}\,\Gamma(Y, N)\right)$; here $\mathfrak{g}$ acts on sections of N via the vector fields that correspond to G-action on Y. In particular for $\tilde{\mathcal{D}}_X$-modules M_1, M_2 we have the pairing $\Gamma(M_1) \times \Gamma(M_2) \to \Gamma(Y, M_1 \boxdot M_2) \to \varepsilon^{-1} \int_Y \varepsilon\, M_1 \boxdot M_2$, or for U-modules V_1, V_2 the one $(,) : V_1 \times V_2 \to \varepsilon^{-1} \int_Y \varepsilon \Delta(V_1) \boxdot \Delta(V_2)$. This pairing is U-invariant: one has $(uv_1, v_2) = (v_1, C_U(u)v_2)$, i.e. we have the map $\psi : \varepsilon\left(\int_Y \varepsilon\, \Delta(V_1) \boxdot \Delta(V_2)\right)^* \longrightarrow \mathrm{Hom}_U(V_1, V_2^\circ)$. Now 2.5.1, 2.6.6 imply

2.3.1. Lemma. Assume that $V_i \in \mathcal{M}(U)_{\chi_i, n}$ where χ_i are dominant regular. Then ψ is isomorphism. $\square$

3.3.2. Let $(\mathfrak{g}, K_1)$, $(\mathfrak{g}, K_2)$ be finite Harish-Chandra pairs; assume for simplicity that Ad K_i act trivially on $\mathfrak{z}$. Let $V \in \mathcal{M}(U, K_1)^{f}_{\chi, n}$ (here f means "finitely generated" = "of finite length"). Put $C_{K_2 K_1}(V) := \{v^* \in V^c : \dim \mathrm{Lie}\, K_2 v^* < \infty\}$
V
Assume that

(3.3.2) (i) For any V $C_{K_2 K_1}(V)$ is finitely generated U-module and $C_{K_2 K_1}(V)$ carries in a canonical way the structure of (U, K_2)-module (i.e. Lie K_2-action on $C_{K_2 K_1}(V)$ "integrates" to K_2-action), hence $C_{K_2 K_1}(V) \in \mathcal{M}(U, K_2)^{f}_{c_{\mathfrak{z}}\chi, n}$

(ii) The condition (i) also holds for $C_{K_1 K_2}$(and $c_{\mathfrak{z}}\chi$), and the canonical maps $V_1 \to C_{K_1 K_2} C_{K_2 K_1} V_1$, $V_2 \to C_{K_2 K_1} C_{K_1 K_2}(V_2)$ are isomorphisms.

Then $C_{K_2 K_1}(V) \in \mathcal{M}(U, K_2)_{c_{\mathfrak{z}}\chi, n}$ is called contragradient dual to V; the functors $C_{K_2 K_1}$, $C_{K_1 K_2}$ are mutually inverse equivalences of categories.

The above condition holds in either of the following cases:

-- $K_1 = N = N_x$ is maximal nilpotent subgroup, $K_2 = N^c = N_y$ is a complementary subgroup.

-- $(\mathfrak{g}, K_1) = (\mathfrak{g}, K_2)$ is symmetric pair (here $K = K_i$ is reductive, for a finitely generated V the K-isotypic components of V are finite dimensional, hence $C_{KK}(V)$ carries the dual algebraic action of K and $C_{KK}C_{KK}(V) = V$).

From now on assume that χ is dominant and regular. One wishes to compute C $= C_{K_2 K_1}$ geometrically. In principle 3.3.1 does this job but in a rather implicit way. In the rest of the § we will compute the contragradient duals to irreducible and standard modules. Namely we will see that C is C_I-stratified equivalence for certain $C_I : I_{XK_1} \to I_{XK_2}$, and we will compute explicitly the corresponding strata $C_\alpha : M(U, K_1)^{\xi}_{\chi,n\alpha} \longrightarrow M(U, K_2)_{c_\xi\chi, n\, c_I(\alpha)}$. According to 1.5.6 (iv), (v) this implies that one has natural isomorphisms

$$(3.3.3) \quad Cj_{\alpha !*}(V) = j_{c_I(\alpha)!*}C_\alpha(V_\alpha)$$

$$Cj_{\alpha !}(V_\alpha) = j_{c_I(\alpha)*}C_\alpha(V_\alpha)$$

$$Cj_{\alpha *}(V_\alpha) = j_{c_I(\alpha)!}C_\alpha(V_\alpha)$$

for $V_\alpha \in M(U, K_1)^{\xi}_{\chi,n\alpha} \xrightarrow[\sim]{F_{x_\alpha}} M(\xi, K_{1(x_\alpha)})^{\xi}_{\chi,n}$

3.3.4. Remarks. (i) Let $\mathrm{Irr}_{\chi\alpha}$ be the set of isomorphism classes of irreducible objects in $M(U, K_1)_{\chi\alpha}$ and $[C_\alpha]: \mathrm{Irr}_{\chi\alpha} \longrightarrow \mathrm{Irr}_{c_\xi\chi\, c_I(\alpha)}$ be the isomorphism that comes from C_α. Then C_α is unique, up to non-canonical isomorphism of functors, $S(\mathfrak{z})$-linear (with respect to $c_{\mathfrak{z}} : S(\mathfrak{z}) \to S(\mathfrak{z})$) equivalence of categories that induces the map $[C_\alpha]$ on irreducibles. This follows from the last ligne of 1.5.2 (since we assumed that $\mathrm{Ad}\, K$ acts trivially on $\mathfrak{z}$). Hence the computation of C_α up to isomorphism reduces to the one of $[C_\alpha]$

(ii) To verify that C is C_I-stratified equivalence it suffice to see that C_I preserves the ordering of I and for any irreducible V from α stratum $C(V)$ belongs to $C_I(\alpha)$-stratum.

3.3.5. Consider the simplest Verma modules case first, so $K_1 = N = N_x$, $K_2 = N^c = N_y$ are opposite maximal nilpotent subgroups, and $C = C_{N^c N} : M(U, N)^{\xi o}_{\chi,n} \longrightarrow M(U, N^c)^{\xi}_{c_\xi\chi,n}$ One has $I_{XN} = I_{XN^\bullet} = W$ with Bruhat ordering. For $w \in W$

the subcategory $\mathcal{M}(U, N)^{f}_{\chi,n\,\bar{w}}$ is generated by Verma modules $M^{x}_{w'\chi,1}$, $w' \leq w$. The Verma module $M^{x}_{w\chi,n}$ is projective in this subcategory and the fiber functor $F_{wx} : \mathcal{M}(U, N)^{f}_{\chi,n\,\bar{w}} \longrightarrow \mathcal{M}(U, N)^{f}_{\chi,n\,w} \xrightarrow{\sim} \mathcal{M}(\mathfrak{h})^{f}_{\chi,n}$ is $F_{wx}(L) = \mathrm{Hom}(M^{x}_{w\chi,n}, L)$ = lowest weight vectors of L = the χ -component of the space L^N of singular vectors equipped with $\mathfrak{h}$ -action that comes from b_x -action on L^N via the isomorphism $S(\mathfrak{h}) \to S(\mathfrak{h}) : \chi \longmapsto w^{-1}(\chi - 2\rho)$. For $w \in W$ put $C_I(w) = {}^{c}w := w_{max} w w^{-1}_{max}$; then $C_I : W \to W$ is an involution that preserves Bruhat ordering. An easy computation with singular vector shows that c trancforms the irreducible quotient $L_{w\chi}$ of $M_{w\chi}$ to $L_{c_w\chi}$, hence C is C_I-stratified equivalence. By 3.3.3, 3.3.4 one has $C\, j_{w!}(V) = j_{c_w *}(V^{o})$, $C\, j_{w*}(V) = j_{c_w !}(V^{o})$ for any $V \in \mathcal{M}(\mathfrak{h})^{f}_{\chi,n}$. The above description of the fiber functor F_{wx} shows that a canonical map $j_{w!}(V) \to j_{w*}(V)$ identifies this way with $j_{w!}(V) \to C\, j_{c_{w!}}(V^{o})$ which is exactly the usual contravariant form $\langle , \rangle : j_{w!}(V) \times j_{c_{w!}}(V^{c}) \to \mathbb{C}$ i.e. the $\mathfrak{g}$ -invariant pairing that coincides with the obvious pairing $V \times V^{o} \to \mathbb{C}$ on the space of lowest weight vectors.

3.3.6. Now let us turn to the symmetric case. So $(\mathfrak{g}, K)$ is symmetric pair and the contragradient duality is $C = C_{KK} : \mathcal{M}(U, K)^{fo}_{\chi,n} \to \mathcal{M}(U, K)^{f}_{c_{\mathfrak{h}}\chi,n}$. For a K-orbit Q_α put

$$\hat{Q}_\alpha := \{(x_1, x_2) \in Y : x_1 \in Q_\alpha, \mathfrak{h}_{x_1 x_2} := b_{x_1} \cap b_{x_2} \text{ is } \theta\text{-stable Cartan subalgebra}\}$$

Clearly $(x_1, x_2) \mapsto (x_1, \mathfrak{h}_{x_1 x_2})$ is 1-1 correspondence between $\hat{Q}_\alpha$ and the set of pairs $(x, \mathfrak{h}_x)$, $x \in Q_\alpha$, $\mathfrak{h}_x$ is θ-stable Cartan subalgebra in b_x. One has (see, e.g. [17] (A.2.3)):

3.3.7. Lemma. (i) For $x \in Q_\alpha$ the x-fiber of $p_{\alpha 1} : \hat{Q}_\alpha \longrightarrow Q_\alpha$ is non-empty and the nilpotent group $N_x \cap K$ acts on $p_{\alpha 1}^{-1}(x)$ in a simply transitive way

(ii) $\hat{Q}_\alpha$ is a single K-orbit closed in Y. □

Put $Q_{C_I(\alpha)} := p_{\alpha 2}(\hat{Q}_\alpha)$: this is a single K-orbit; clearly C_I is an involution of the set $I = I_{XK}$ of orbits.

3.3.8. Let $\tilde{\hat{Q}}$ be $\check{Y}$ restricted to $\hat{Q}$; consider the K-equivariant monodromic projections $\tilde{Q}_\alpha \xleftarrow{\tilde{p}_{\alpha 1}} \tilde{\hat{Q}}_\alpha \xrightarrow{\tilde{p}_{\alpha 2}} \tilde{Q}_{C_I(\alpha)}$. Now 3.3.7 (i), 1.5.2 imply that $\tilde{p}^c_{\alpha i}$ are equivalences of categories: $\tilde{M}(Q_\alpha, K) \xrightarrow[\sim]{\tilde{p}^c_{\alpha 1}} \tilde{M}(\hat{Q}_\alpha, K) \xleftarrow[\sim]{\tilde{p}^c_{\alpha 2}} \tilde{M}(Q_{C_I(\alpha)}, K)$. Hence we have the equivalence $S := \tilde{p}^{c-1}_{\alpha 2}\tilde{p}^c_{\alpha 1} : \tilde{M}(Q_\alpha, K) \longrightarrow \tilde{M}(Q_{C_I(\alpha)}, K)$. One may define S in terms of fibers: for $(x_1, x_2) \in \hat{Q}_\alpha$ we have the isomorphisms of Harish-Chandra pairs $(\mathfrak{g}, K_{(x_1)}) \xleftarrow{\sim} (\mathfrak{g}, K_{(x_1, x_2)}) \xrightarrow{\sim} (\mathfrak{g}, K_{(x_2)})$, hence the composition $S_{x_1 x_2} : (\mathfrak{g}, K_{(x_1)}) \xrightarrow{\sim} (\mathfrak{g}, K_{(x_2)})$ which is $-C_{\mathfrak{g}}$ on $\mathfrak{g}$, and fiber functors F_{xi} identify S with equivalence $M(\mathfrak{g}, K_{(x_1)}) \longrightarrow M(\mathfrak{g}, K_{(x_2)})$ which comes from $S_{x_1 x_2}$. Now let $\bar{C} : M(\mathfrak{g}, K_{(x_1)})^{fc}_{\chi, n} \longrightarrow M(\mathfrak{g}, K_{(x_2)})^{f}_{C_{\mathfrak{g}}\chi, n}$ be the duality defined by formula $\bar{C}(V) := S(V^*) = S(V)^*$. Our aim is to prove the following

3.3.9. Theorem. (i) C_I preserves the partial ordering on I and dimensions of orbits.

(ii) The contravariant duality $C : M(U, K)^{f\sigma}_{\chi, n} \longrightarrow M(U, K)^{f}_{C_{\mathfrak{g}}\chi, n}$ is C_I-stratified.

(iii) The fiber functors F_x identify the strata of C with $\bar{C}$.

3.3.10. Corollary. C interchanges ! - and * -standard modules. □

This is 3.3.3 above. In particular the main result of [19] means that ! -standard modules are just the ones obtained by Zuckerman's induction.

3.3.11. Remark. Let V be an $(\mathfrak{f}, K_{(x_1)})$-module. Then 3.3.9 identifies canonical morphism $j_{\alpha !}(V) \to j_{\alpha *}(V)$ with $j_{\alpha !}(V) \to C j_{C_I(\alpha)!}(\bar{C}\, V)$ i.e. with the $(\mathfrak{g}, K)$-invariant pairing $\langle , \rangle : j_{\alpha !}(V) \times j_{C_I(\alpha)!}(\bar{C}\, V) \to \mathbb{C}$. This pairing is compatible with respect to morphisms of V's. Hence it induces the non-trivial pairing between $j_{\alpha !}(V/m_\chi V) = j_{\alpha !} V/m_\chi j_{\alpha !} V$ and $j_{C_I(\alpha)!}\bar{C}\,(V/m_\chi V) = [j_{C_I(\alpha)!}(\bar{C}\, V)]^{m_\chi} \hookrightarrow j_{C_I(\alpha)!}(\bar{C}\, V)$. If V is indecomposable module then $\langle\ \rangle$ is unique, up to isomorphism (i.e. up to multiplication by an invertible element of $S(\mathfrak{f}/\mathrm{Lie}\, K_{(x_1)})/m_\chi^{\wedge}$), pairing with this property. □

Proof of 3.3.9. Unfortunately it is long and messy. We will start with some preliminaries on Weyl group action on orbits and Θ action on roots.

3.3.12. The Weyl group W acts on Y: for $(x_1, x_2) \in Y$, $w \in W$ the element $w(x_1, x_2) = (y_1, y_2)$ Y is defined by properties $\hat{\mathfrak{f}}_{x_1 x_2} := b_{x_1} \cap b_{x_2} = \hat{\mathfrak{f}}_{y_1 y_2}$, $(x_1, y_1) \in Y_w$. This action lifts to H-monodromic action on $\breve{Y}$: namely $\tilde{w}(\tilde{x}_1, \tilde{x}_2) = (\tilde{y}_1, \tilde{y}_2)$, where $(\tilde{x}_1, \tilde{y}_1) \in \tilde{Y}_w$. One has $\tilde{w}\, h(\tilde{x}_1, \tilde{x}_2) = \mathrm{ad}_w(h)\tilde{w}(\tilde{x}_1, \tilde{x}_2)$, $\tilde{w} g(\tilde{x}_1, \tilde{x}_2) = g\tilde{w}(\tilde{x}_1, \tilde{x}_2)$ for $h \in H$, $g \in G^{\circ}$.

Remark. Strictly speaking this H-monodromic lifting $\tilde{w}$ of the action of $w \in W$ on Y depends on the choice of the type of Chevalley basis that defines $\tilde{Y}_w$ (see 2.6.6). The various choices differ by the multiplication by an order 2 element of H, hence it is the extension $\widetilde{W}$ of W by order 2 elements of H that acts naturally on $\check{Y}$. This will make no difficulties in what follows. □

Clearly W preserves the close K-invariant subvariety $\bigcup_{\alpha \in I_{X,K}} \dot{Q}_\alpha \subset Y$. Put $\hat{Q}_{w\alpha} := w\hat{Q}_\alpha$; then $\alpha \longmapsto w\alpha$ is the action of W on the set of orbits I_{XK} (this action usually breaks the ordering of I_{XK}). Let $\tilde{w}$ denotes the composition $\tilde{M}(Q_\alpha, K) \xrightarrow[\sim]{\tilde{p}^{o}_{\alpha 1}} \tilde{M}(\hat{Q}_\alpha, K) \xrightarrow[\sim]{\tilde{w}} M(\hat{Q}_{w\alpha}, K) \xrightarrow[\sim]{\tilde{p}^{o-1}_{\alpha 1}} \tilde{M}(Q_{w\alpha}, K)$; this equivalence corresponds to the isomorphism $(\mathfrak{g}, K_{(x_1)}) \xrightarrow{\sim} (\mathfrak{g}, K_{(x_1,x_2)}) \xrightarrow[\sim]{w} (\mathfrak{g}, K_{(y_1,y_2)}) \xrightarrow{\sim} (\mathfrak{g}, K_{(y_1)})$ of Harish-Chandra pairs where $(x_1,x_2) \in \dot{Q}_\alpha$, $(y_1, y_2) = w(x_1, x_2)$. This way the action of W on I_{XK} lifts to the action on corresponding strata categories. Clearly the action of $w_{max} \in W$ on I_{XK} is just C_I, and $\tilde{w}_{max} = S$ (see 3.3.8). Hence one has canonical isomorphism of functors $\bar{C}\,\tilde{w} = {}^{c}w\,\bar{C} : \tilde{M}(Q_\alpha, K)^{f\circ} \longrightarrow \tilde{M}(Q_{w_{max}\, w\alpha}, K)^{f}$.

3.3.13. For the details about the action of Θ on roots in convenient form see [27] A3, or [34]. For $\alpha \in I$ take $(x_1,x_2) \in \hat{Q}_\alpha$ and consider the action of Θ on $\mathfrak{g}_{x_1x_2}$; the projection $\mathfrak{g}_{x_1x_2} \xrightarrow{\sim} b_{x_1}/n_{x_1} = \mathfrak{f}$ transmits it to the automorphism Θ_α of $\mathfrak{f}$ that depends on α only. This Θ_α decomposes the set Δ^+ of positive roots into the disjoint union of the following subsets: $\Delta^1_\alpha := \{\delta \in \Delta^+ : \Theta_\alpha\delta = -\delta\}$, $\Delta_{1\alpha} := \{\delta \in \Delta^+ : \Theta_\alpha\delta = \delta,\ \Theta_\alpha$ acts on δ root space as $-1\}$, $\Delta^2_\alpha := \{\delta \in \Delta^+ : \Theta_\alpha\delta \in -\Delta^+,\ \Theta_\alpha\delta \neq -\delta\}$, $\Delta_{2\alpha} := \{\delta \in \Delta^+ : \Theta_\alpha\delta \in \Delta^+,\ \Theta_\alpha\delta \neq \delta\}$, $\Delta^3 := \{\delta \in \Delta^+ : \Theta_\alpha\,\delta = \delta$, Θ_α acts on δ root space as $+1\}$. The set $\Sigma = \Sigma(\Delta^+)$ of simple roots gets the induced decomposition: $\Sigma^1_\alpha := \Sigma \cap \Delta^1_\alpha$ etc.

Now for $\hat{\delta} \in \Sigma$ consider the projection $r_\delta : X \to X_\delta$ (see 2.3.5) with fiber $\mathbb{P}^1_{\bar{x}} = r_\delta^{-1}\bar{x}$, $\bar{x} \in X_\delta$. For $\bar{x} \in r_\delta(Q_\alpha)$ one has $\hat{\delta} \in \Sigma^1_\alpha \iff \mathbb{P}^1_{\bar{x}} \cap Q_\alpha$ is torus; $\hat{\delta} \in \Sigma^2_\alpha \iff \mathbb{P}^1_{\bar{x}} \cap Q_\alpha$ is affine line, $\hat{\delta} \in \Sigma^3_\alpha \iff \mathbb{P}^1_{\bar{x}} \cap Q_\alpha = \mathbb{P}^1_{\bar{x}}$, $\hat{\delta} \in \Sigma_{1\alpha} \Rightarrow \mathbb{P}^1_{\bar{x}} \cap Q_\alpha$ consists of 1 or 2 points, $\hat{\delta} \in \Sigma_{2\alpha} \Rightarrow \mathbb{P}^1_{\bar{x}} \cap Q_\alpha$ is one point. For $\hat{\delta} \in \Sigma_{1\alpha} \cup \Sigma_{2\alpha}$ let $Q_{\sigma^+_\delta(\alpha)}$ be the unique open orbit in $r_\delta^{-1} r_\delta(Q_\alpha)$: one has $\dim Q_{\sigma^+_\delta(\alpha)} = \dim Q_\alpha + 1$ and $\hat{\delta} \in \Sigma_{i\alpha} \iff \hat{\delta} \in \Sigma^i_{\sigma^+_\delta(\alpha)}$. For $\hat{\delta} \in \Sigma^2_\alpha$ let $Q_{\sigma^-_\delta(\alpha)}$ be the unique closed orbit in $r_\delta^{-1} r_\delta(Q_\alpha)$: one has $\dim Q_{\sigma^-_\delta(\alpha)} = \dim Q_\alpha - 1$, $\hat{\delta} \in \Sigma_{2\sigma^-_\delta(\alpha)}$ and $\alpha = \sigma^+_\delta \sigma^-_\delta(\alpha)$.

3.3.14. <u>Lemma</u>. (i) $C_\delta : \Delta^+ \to \Delta^+$ transforms α-decomposition of Δ^+ to $C_I(\alpha)$-decomposition: $C_\delta(\Delta^i_\alpha) = \Delta^i_{C_I(\alpha)}$, etc.

(ii) $\dim Q_\alpha = |\Delta^1_\alpha \cup \Delta^2_\alpha \cup \Delta^3_\alpha| + \frac{1}{2}|\Delta_{2\alpha}|$

(iii) For $\hat{\delta} \in \Sigma^2_\alpha$ one has $\sigma^-_\delta(\alpha) = \sigma_\delta(\alpha)$ (here $\sigma_\delta \in W$ is $\hat{\delta}$-reflection); for $\hat{\delta} \in \Sigma_{2\alpha}$ one has $\sigma^+_\delta(\alpha) = \sigma_\delta(\alpha)$.

(iv) C_I commutes with $\sigma^\pm_\delta$, i.e. for $\hat{\delta} \in \Sigma_{1\alpha} \cup \Sigma_{2\alpha}$ one has $C_I\sigma^+_\delta(\alpha) = \sigma^+_{C_\delta\delta} C_I(\alpha)$, for $\hat{\delta} \in \Sigma^2_\alpha$ one has $C_I\sigma^-_\delta(\alpha) = \sigma^-_{C_\delta\delta} C_I(\alpha)$. $\square$

3.3.15. Now we may prove 3.3.9 (i). One has $\dim Q_\alpha = \dim Q_{C_I(\alpha)}$ by 3.3.14 (i), (ii). It remains to show that $C_I(\bar{\alpha}) \subset \overline{C_I(\alpha)}$. The proof goes by induction by $\dim \alpha := \dim Q_\alpha$. The fact is obvious for orbits of minimal dimension = closed orbits. If Q_α is not closed, then $\Sigma^1_\alpha \cup \Sigma^2_\alpha \neq \emptyset$. Choose $\hat{\delta} \in \Sigma^1_\alpha \cup \Sigma^2_\alpha$; then $\alpha = \sigma^+_\delta(\beta_i)$ for either 1 or 2 elements $\beta_i \in I_{X,K}$, $\dim \beta_i = \dim \alpha - 1$. One has $\bar{\alpha} =$

$= \bar\beta_1 \cup \bar\beta_2 \cup \{\sigma_\delta^+(\beta')\}$, where β' runs the elements of $\bar\beta_i$ with $\delta \in \Sigma_{1\beta'} \cup \Sigma_{2\beta'}$. Since $c_I(\bar\beta_i) \subset \overline{c_I(\beta_i)}$ by induction and σ_δ^+ commutes with c_I by 3.3.14 (iv), we are done.

For an orbit $\alpha \in I_{XK}$ consider the following condition:

3.3.16. For $V \in M(\mathfrak{g}, K_{(x_\alpha)})_{\chi,n}$ one has canonical morphism $\Phi_\alpha : j_{\alpha!}\bar c V \to c\, j_{c_I(\alpha)*}V$ (i.e. Φ is morphism of $M(U, K)_{\chi,n}$-valued functors on $M(\mathfrak{g}, K_{(x_\alpha)})_{\chi,n}$) such that Φ_α is non-zero for any $V \neq 0$.

Recall that $\alpha \in \bar I_{X,K}$ is called Θ-compatible if $\Sigma_\alpha^2 = \emptyset$.

3.3.17. <u>Lemma</u>. Assume that 3.3.16 holds for any Θ-compatible $\alpha \in \bar I_{X,K}$. Then 3.3.9 (ii), (iii) are valid.

<u>Proof</u>. We will prove 3.3.9 (ii), (iii) using induction by dimension n of orbits. So assume that for any $\beta \in I_{XK}$ with $\dim\beta < n$ one has

$(i)_\beta$ For any irreducible module L_β that belongs to β the irreducible module $c(L_\beta)$ belongs to $c_I(\beta)$ (this implies that c transforms $\mathcal{M}_n := \mathcal{M}_{\cup\beta,\ \dim\beta<n}$ to $\mathcal{M}_{n-1}$ and induces c_I-stratified equivalence on this subcategory).

$(ii)_\beta$ The β-stratum of c coincides with $\bar c$.

We have to prove the statements $(i)_\alpha$, $(ii)_\alpha$ for α with $\dim\alpha = n$. Consider 2 possibilities.

a. α is Θ-compatible. Let V_α be an irreducible $(\mathfrak{g}, K_{(x_\alpha)})_\chi$-module. Then all the components of $j_{\alpha*}V_\alpha$ but $j_{\alpha!*}V_\alpha$ lie over strata of dimension $< n$, hence all the components of $c\, j_{\alpha*}V_\alpha$ but $c\, j_{\alpha!*}V$ have the same property by induction hypothesis. On the other hand $j_{c_I(\alpha)!*}\bar c V_\alpha$ is the only irreducible quotient of $j_{c_I(\alpha)!}\bar c V_\alpha$, hence Φ_α identifies it with non-zero subquotient of $c\, j_{\alpha*}V_\alpha$. Since it is supported on Q_α, one should have $j_{c_j(\alpha)!*}\breve{\tau}V_\alpha = c\, j_{\alpha!*}V_\alpha$ hence $(i)_\alpha$. Now $(ii)_\alpha$ follows since $c\, j_{c_j(\alpha)*}V$ belongs to $M_{\bar\alpha}$ and Φ_α induces isomorphism in the quotient category M_α.

b. α is not Θ-compatible. Choose $\delta \in \Sigma'^{\,2}_\alpha$ and put $\beta = \sigma_\delta^-(\alpha)$. Consider the functor $R_\delta : M(U, K)_{\chi,n} \to M(U, K)_{\chi,\infty}$ (see 2.8). Then $cR_\delta = R_{c_j(\delta)}c$, $R_\delta(M_{\bar\beta}) \subset M_{\bar\alpha}$ and R_δ induces the functor on quotient categories $M_\beta \to M_\alpha$ which coincides with $\tilde\sigma_\delta$ from 3.3.12, 3.3.14 (iii) (the first property follows from definition of R_δ, the second one comes from geometric interpretation (2.8.1) of R_δ). Both $(i)_\alpha$, $(ii)_\alpha$ are now clear since R_δ transforms them into $(i)_\beta$ $(ii)_\beta$. $\Box$

It remains to prove the condition 3.3.16 for a Θ-compatible $\alpha \in I_{X,K}$.

3.3.18. Consider first the case when $Q_\alpha = Q$ is open orbit and $\Sigma' = \Sigma^1_\alpha$. Then the stabilizer K_x of a point $x \in Q$ is finite, hence Q is affine, and $(\mathfrak{h}, K_{(x)})$-modules are just the spaces with commuting $\mathfrak{h}$- and K_x-actions. Similarly $(\tilde{D}_Q, K)$-modules are the same as (D_Q, K)-modules with $\mathfrak{h}$-action, since over Q we have canonical splitting $s_K : \mathcal{T}_Q \to \tilde{\mathcal{T}}_Q$: the one that maps the field $m_y(k) \in \mathcal{T}_Q$ to $\tilde{m}_y(k) \in \tilde{\mathcal{T}}_Q$ for any $k \in \mathfrak{k}$. Explicitly, if M is (D_Q, K)-module with $\mathfrak{h}$-action, then the corresponding $(\tilde{D}_Q, K)$-module $\tilde{M}$ is M considered as $(\mathcal{O}_Q, K)$-module with $\tilde{\mathcal{T}}_Q$-action given by formula $\tilde\tau(m) = \tau(m) + h(\tilde\tau)m$, where $h : \tilde{\mathcal{T}}_Q \to \tilde{\mathfrak{h}} = \mathfrak{h}\otimes\mathcal{O}_Q$ is projection $h(\tilde\tau) := \tilde\tau - s_K(\tau)$. For an $(\mathfrak{h}, K_x)$-module V we will denote by $\mathcal{F}_V$ the corresponding (D_Q, K)-module with $\mathfrak{h}$-action, hence $\tilde{\mathcal{F}}_V = F_x^{-1}(V)$. Clearly $j_*(V) := j_{\alpha*}(V) = \tilde{\mathcal{F}}_V(Q)$, so $*$-standard modules are just the principal series representations.

Denote by ε_K the rank 1 K_x-module which coincides with $\det^{-1}k$ with adjoint action; one also has $\varepsilon_K = H_{DR}^{\dim Q}(Q)$. Consider the $(\mathfrak{h}, K_x)$-module $\varepsilon_K(-2\rho)$; since $\tilde{\mathcal{F}}_{\varepsilon_K(-2\rho)} = \omega_Q$ we have canonical projection $\int : \tilde{\mathcal{F}}_{\varepsilon_K(-2\rho)}(Q) = \omega_Q(Q) \to H_{DR}^{\dim Q}(Q) = \varepsilon_K$ which is morphism of $(\mathfrak{g}, K)$-modules (ε has trivial $\mathfrak{g}$-action). This way for any $(\mathfrak{h}, K_x)$-module V one gets the $(\mathfrak{g}, K)$-invariant pairing $\langle\, ,\rangle : \tilde{\mathcal{F}}_V(Q) \times \tilde{\mathcal{F}}_{{}^tV}(Q) \to \mathbb{C}$, $\langle f, g\rangle := \int fg$, where ${}^tV = V^*(-2\rho)$. If V has finite rank, then $\langle\, ,\rangle$ identifies $\tilde{\mathcal{F}}_{{}^tV}(Q)$ with contravariant dual to $\tilde{\mathcal{F}}_V(Q)$ as $(\mathfrak{g}, K)$-module. To get the contravariant dual as (U, K)-module one should turn the U-action by w_{max} (see the beginning of 3.3). Finally we join this to 2.6.6 (iii) and get

3.3.19. <u>Lemma</u>. Assume that χ is regular dominant and conditions 3.3.17 hold. Then for any $V \in \mathcal{M}(\mathfrak{h}, K_x)^f_{\chi,n}$ one has $c\, j_*(V) = w_{max!}\, j_*({}^tV)$. $\square$

Now Φ_α from 3.3.16 come from the usual intertwinning operators. These may be constructed geometrically as follows. First note that $\hat{Q}$ projects isomorphically onto Q and $\hat{Q}$ is W-invariant, hence one gets the W-action on Q and $\tilde{Q}$ such that $wkx = kwx$, $whx = ad_w(h)wx$ for $k \in K$, $h \in H$, $w \in W$, $x \in Q$. So W acts on categories $\tilde{\mathcal{M}}(Q, K)$, $\mathcal{M}(\tilde{Q}, K)^m$; for $w \in W$ denote the corresponding functor $\tilde{w} : \tilde{\mathcal{M}}(Q, K) \to \tilde{\mathcal{M}}(Q, K)$; it transforms $\tilde{\mathcal{M}}(Q, K)_\chi$ to $\tilde{\mathcal{M}}(Q, K)_{w\chi}$.

3.3.20. <u>Lemma</u>. For a simple reflection $\sigma \in W$ and $M \in \tilde{\mathcal{M}}(Q, K)^f$ one has canonical isomorphism of $(\mathfrak{h}, K_x)$-modules $F_x \sigma_! j_* M = F_x \tilde{\sigma} M$. $\square$

(this is a matter of a simple calculation; note that if $M = \widetilde{L}$, where L is $(\mathcal{D}_Q, K)$-module with $\mathfrak{f}$ -action, then over Q one has $\sigma_! j_* M = \widetilde{\sigma_! j_* L}$, so one may work with ordinary local systems on $P^1 \setminus \{0, \infty\}$)

Clearly 3.3.20 means that one has a canonical morphism $\varphi_\sigma : \sigma_! j_* M \to j_* \tilde{\sigma} M$ which is isomorphism on Q. For $w \in W$ consider a reduced decomposition $w = \sigma_1 \ldots \sigma_{\ell(w)}$; for a dominant regular $\chi \in \mathfrak{f}^*$ and $M \in \widetilde{M}(Q, K)_{\chi, n}$ define $\varphi_w : w_! j_* \tilde{w}^{-1} M \longrightarrow j_* M$ to be the composition $w_! j_* \tilde{w}^{-1} M \to \ldots \longrightarrow \sigma_{1!} \sigma_{2!} j_* \tilde{\sigma}_2^{-1} \tilde{\sigma}_1^{-1} M \xrightarrow{\sigma_{1!} \varphi_{\sigma_2}} \sigma_{1!} j_* \tilde{\sigma}_1^{-1} M \longrightarrow j_* M$. This is a classical intertwinning operator; one may prove that φ_w is isomorphism being restricted to Q. Now note that $\bar{c}(V) = \tilde{w}_{max}\, {}^t V$. Hence 3.3.19 gives us $\varphi_{w_{max}} : c\, j_* V \to j_* \bar{c}\, V$ which is isomorphism over Q; this gives the map from 3.3.16 (the one that coincides with $\varphi^{-1}_{w_{max}}$ on Q).

So we have proven that for this open orbit Q_α the duality c preserves $\mathcal{M}_{I \setminus \alpha}$ and induces $\bar{c}$ on the quotient $\mathcal{M}_\alpha$. As usually, the definition of $j_{\alpha *}, j_{\alpha !}$ as adjoint functors to the projection implies that $c\, j_{\alpha *} = j_{\alpha !} \bar{c}$, $c\, j_{\alpha !} = j_{\alpha *} \bar{c}$. We may interpret this using 3.3.1: namely it means that for $V \in \mathcal{M}(\mathfrak{f}, K_{(x)})$ one has a canonical K-invariant element $\varphi_V \in \varepsilon\, (\int_Y \varepsilon\; j_{\alpha !}(V) \boxtimes j_{\alpha *}(\bar{c}\, V))^*$ which is non-zero if V is non-zero. To finish the proof of 3.3.16 we have to construct the similar elements for any Θ-compatible $\alpha \in I_{XK}$. This may be done as follows. Put $S = \Sigma^1_\alpha$ and consider the map $r_S : X \to X_S$. For $\bar{x} \in r_S(Q_\alpha)$ let $\mathfrak{p}_{\bar{x}}$ be the corresponding parabolic subalgebra, $\mathfrak{g}_{\bar{x}}$ be its semisimple Levi quotient. Then $X_{\bar{x}} := r_S^{-1}(\bar{x})$ is flag space for $\mathfrak{g}_{\bar{x}}$. Let $K^!_{\bar{x}} \subset K_{\bar{x}}$ be the largest connected subgroup that acts trivially on $X_{\bar{x}}$ and $K_{(\bar{x})} := K_{\bar{x}}/K^!_{\bar{x}}$. Then (see [31], [27] A3) $Q_{\alpha S} := r_S(Q_\alpha)$ is a closed K-orbit, Q_α is open suborbit in $r_S^{-1}r_S(Q_\alpha)$, $(\mathfrak{g}_{\bar{x}}, K_{(\bar{x})})$ is symmetric Harish-Chandra pair, and $Q_\alpha \cap X_{\bar{x}}$ is open $K_{(\bar{x})}$ orbit such that 3.3.18 holds.

Put $Z := (r_S \times r_S)(\hat{Q}_\alpha)$: this is an open K-orbit in $Q_{\alpha S} \times Q_{c_I(\alpha)S}$, and $T := (r_S \times r_S)^{-1}(Z) \cap Y$ is a connected component of $(\bar{Q}_\alpha \times \bar{Q}_{c_I(\alpha)}) \cap Y$. The projection $Z \xrightarrow{p_1} Q_{\alpha S}$ has affine fibers and $Q_{\alpha S}$ is a compact K-orbit, hence Z is simply connected and $\dim H^{\dim Z}_{DR}(Z)^K = 1$. For $z := (\bar{x}_1, \bar{x}_2) \in Z$ one has $K_z = K_{(\bar{x}_1)} = K_{(\bar{x}_2)}$ and $\mathfrak{g}_{\bar{x}_1} = P_{\bar{x}_1} \cap P_{\bar{x}_2} = \mathfrak{g}_{\bar{x}_2} := \mathfrak{g}_z$. This identifies $(X \times X)_z := (r_S \times r_S)^{-1}(z)$ with $X_{\mathfrak{g}_z} \times X_{\mathfrak{g}_z}$, T_z with $Y_{\mathfrak{g}_z} \subset X_{\mathfrak{g}_z} \times X_{\mathfrak{g}_z}$, $(Q_\alpha \times Q_{c_I(\alpha)}) \cap (X \times X)_z$ with the product of open K_z-orbits in $X_{\mathfrak{g}_z}$. This way we see that

$$\varepsilon\int_Y \varepsilon\, j_{\alpha !}(V) \boxtimes j_{c_I(\alpha)*}(\bar{c}V) = \int_{Y \times \bar{Q}_\alpha \times \bar{Q}_{c_I(\alpha)}} (..) \supset \int_T (..) \supset H^{\dim Z}_{DR}(Z) \otimes \int_{Y_{\mathfrak{g}_z}}$$

where $\int_{Y_{\mathfrak{g}_z}}$ is similar integral for same V, considered as $(\mathfrak{f}_{\mathfrak{g}_z}, K_x)$-module that corresponds to the open orbit. This way we are reduced to the case 3.3.18, and we are done.

§4. Jantzen filtration

In this § we will define Jantzen filtration on standard modules; the main point is its relation with monodromy filtration on nearby cycles.

4.1. <u>The monodromy filtration</u> (see [12] 1.6). For an object Q of an abelian category and a nilpotent endomorphism

$s \in \operatorname{End} Q$ let $\mu. = \mu.^{Q}$ denotes the monodromy filtration on Q [12] (1.6.1). Let $P_i^Q = P_i := \operatorname{Ker}(\operatorname{Gr}_i^{\mu} \xrightarrow{s} \operatorname{Gr}_{i-2}^{\mu})$ be the primitive part [12] (1.6.3); one has the primitive decomposition [12] (1.6.4) - a canonical isomorphism of graded $\mathbb{Z}[s]$-modules $\operatorname{Gr}^{\mu}_{\cdot} \simeq \bigoplus_{j \leq 0} P_j \otimes \mathbb{Z}[s]/s^{-j}$, $\deg s = -2$, $\deg P_j = -j$.

Put $J_{!i} := \operatorname{Ker} s \cap \operatorname{Im} s^{-i}$ for $i \leq 0$, $J_{!i} = \operatorname{Ker} s$ for $i \geq 0$: this is an increasing filtration on Ker s; dually one has the filtration $J_{*i} := \operatorname{Ker} s^{i} + \operatorname{Im} s / \operatorname{Im} s$ on $\operatorname{Coker} s$. Call $J_{!.}$, $J_{*.}$ the Jantzen filtrations (for reasons to be seen below). The filtrations $J_{!.}$, $J_{*.}$ are just the filtrations induced by $\mu.$ on Ker s, Coker s, and $\operatorname{Gr}_i^{J_!} = P_i$, $\operatorname{Gr}_i^{J_*} = P_{-i}$ [12] (1.6.6). More precisely, let $\bar{Q}$ be $Q/\operatorname{Ker} s \xrightarrow[\sim]{s} \operatorname{Im} s$ with nilpotent endomorphism s induced by s, and $\bar{\mu}.$ be the corresponding monodromy filtration. Then one has

4.1.1. <u>Lemma</u>. (i) The exact sequences

$$0 \to (\operatorname{Ker} s, J_{!.}) \to (Q, \mu.) \to (\bar{Q}, \bar{\mu}. \text{ shifted by } -1) \to 0$$

$$0 \to (\bar{Q}, \bar{\mu}. \text{ shifted by } 1) \to (Q, u.) \to (\operatorname{Coker} s, J_{*.}) \to 0$$

are strictly compatible with filtrations

(ii) Conversely, $\mu.$ is unique filtration on Q such that $s\,\mu. \subset \mu._{-2}$ and either of two above sequences is strictly compatible with filtrations.

<u>Proof</u>. (i) is [12] 1.6.5; (ii): let $\mu!$ be another such filtration strictly compatible with, say, the first exact sequence. It suffice to show that $\mu'_i \supset \mu_i$. But $\mu'_i = \mu_i$ for $i \geq 0$ (since $J_{!0} = \operatorname{Ker} s$). For $i \leq 0$ $\mu'_i \supset J_{!i} + s(\mu'_{i-2})$, and $s(\mu'_{i-2}) = s(u_{i-2})$. So we are done by downward induction by i. □

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Assume now that our categories are over a field k of characteristic 0, and let $\boxtimes$ be an exact k-bilinear bifunctor. Let (R, t) be another object with nilpotent endomorphism, and $\mu^R_\cdot$ be its monodromy filtration. Consider the tensor product filtration $\mu_i^{Q,R} := \sum_{a+b=i} \mu_a^Q \boxtimes \mu_b^R$ on $Q \boxtimes R$. We have $Gr_\cdot^{\mu^{Q,R}} = Gr_\cdot^{\mu^Q} \boxtimes Gr_\cdot^{\mu^R}$, and the primitive decomposition together with [12](1.6.11, 1.6.12) implies

4.1.2. <u>Lemma</u>. (i) $\mu_\cdot^{Q,R}$ is monodromy filtration with respect to $s \boxtimes id_R + id_Q \boxtimes t$.

? (ii) One has an "almost canonical" isomorphism $P^{Q\boxtimes R}_{-j} \simeq \simeq \oplus P^Q_{-j'} \boxtimes P^R_{-j''}$ where (j', j'') run the set of pairs $\{(j', j'') : |j' - j''| \leq j \leq |j' + j''|, \ j \equiv j' + j'' \mod 2\}$. □

4.2. <u>The Jantzen and monodromy filtrations in geometric setting</u>. Recall the construction of nearby cycles for $\mathcal{D}$-modules [2], [21], [26], [33]; we follow mainly [2]. Let Y be a smooth variety, $f : Y \to \mathbb{A}^1$ be a function, $Z :=$?
$:= f^{-1}(0) \xhookrightarrow{i} Y \xhookleftarrow{j} U := f^{-1}(\mathbb{A}^1 \setminus \{0\})$. For $n > 0$ consider the lisse $\mathcal{D}_{\mathbb{A}^1\setminus\{0\}}$-module $I^{(n)}$ with $\mathbb{C}[s]/s^n$-action which is free rank 1 $\mathcal{O}_{\mathbb{A}^1\setminus\{0\}} \otimes \mathbb{C}[s]/s^n$-module with generator "$t^s$" such that $t\partial_t(t^s) = st^{s-1}$ (here t is the parameter on $\mathbb{A}^1$); we have the obvious projections $I^{(n)} \to I^{(n)}/s^{n-1} = I^{(n-1)}$.

For a $\mathcal{D}_U$-module M_U put $f^s M_U^{(n)} := f^\circ I^{(n)} \otimes_{\mathcal{O}_U} M_U$: this is $\mathcal{D}_U \otimes \mathbb{C}[s]/s^n$-module, $f^s M_U^{(1)} = M_U$, and $f^s M_U^{(a)} = f^s M_U^{(n)}/s^a$. Assume now that M_U is holonomic. Fix some $a \geq 0$. Consider the morphism $\mathfrak{z}^{a(n)} : j_! f^s M_U^{(n)} \to j_* f^s M_U^{(n)}$ of $\mathcal{D}_Y \otimes \mathbb{C}[s]/s^n$-modules that coincides with s^a on U; one has $\mathfrak{z}^{a(n)} \bmod s^{n-1} =$

$= \varsigma^{a(n-1)}$. The lemma on b-functions implies that the projective system Coker $\varsigma^{a(n)}$ stabilizes, so we may put $\Pi_f^a(M_U) :=$ Coker $\varsigma^{a(n)}$ for $n \gg 0$. This is a holonomic $\mathcal{D}_Y$-module with nilpotent endomorphism s; one has $\Pi_f^a(M_U)/\varsigma^i =$ Coker $\varsigma^{a+i)}$? and $j^\circ \Pi_f^a(M_U) = f^s M_U^{(a)}$. The most important Π's are $\Pi_f^0 =: \Psi_f^{un}$ - the part of nearby cycles functor with unipotent action of monodromy $(j^\circ \Pi_f^0 = 0)$, and $\Pi_f^1 =: \Xi_f$ - the maximal extension functor $(j^\circ \Pi_f^1 = \mathrm{Id})$. Here is a list of properties of Π_f^a:

4.2.1. <u>Lemma</u> (see [2]). (i) $\Pi_f^a : \mathcal{M}(U)_{hol} \to \mathcal{M}(Y)_{hol}$ is exact functor.

(ii) For $a, b \gg 0$ one has canonical exact sequences

$$0 \to j_!(f^s M_U^{(a)}) \to \Pi_f^{a+b}(M_U) \to \Pi_f^b(M_U) \to 0$$

$$0 \to \Pi_f^b(M_U) \to \Pi^{a+b}(M_U) \to j_*(f^s M_U^{(a)}) \to 0$$

and $\quad \mathrm{Im}(s^a : \Pi_f^{a+b} \to \Pi_f^{a+b}) = \Xi_f^b$

(ii)' In particular one has exact sequences

$$0 - j_!(M_U) \to \Xi_f(M_U) \to \Psi_f^{un}(M_U) - 0$$

$$0 - \Psi_f^{un}(M_U) - \Xi_f(M_U) \to j_*(M_U) - 0$$

with $j_! = \mathrm{Ker}(s : \Xi_f \to \Xi_f)$, $j_* = \mathrm{Coker}(s : \Xi_f \to \Xi_f)$

(iii) Π_f^a commutes with duality. $\square$

Now 4.1 gives us the monodromy filtration $\mu_\cdot^{(a)}$ on Π_f^a. On U the term $\mu_i^{(a)}$ coincides with

$s^{[\frac{a-i}{2}]} \cdot f^s M_U^{(a)}$ (here $[\ ] :=$ integral part). In particular we have the monodromy filtrations on Ψ_f^{un} and Ξ_f, and the

Jantzen filtrations $J_{f!\cdot}$, $J_{f*\cdot}$ on $j_!$, j_* (via Ξ_f and (ii)' above).

4.2.2. Remarks. (i) 4.1 implies that, up to shift, we will get the same Jantzen filtration if we will use 4.2.1 (ii) $j_! \simeq \mathrm{Ker}(s : \Pi^a \to \Pi^a)$ for any $a \geq 1$; same for j_*.

(ii) One has $J_{f!0} = j_!$, $J_{f!-1} = \mathrm{Ker}(j_! \to j_{!*})$, and the embedding ((ii)' above) $\Psi_f^{un} \to \Xi_f$ identifies $\mathrm{Ker}(s : \Psi_f^{un} \to \Psi_f^{un})$ with $J_{f!-1}$ with corresponding Jantzen filtration shifted by one. Dually, $J_{f*-1} = 0$ $J_{f*0} = j_{!*} \subset j_*$ etc.

(iii) Let $Q_U \subset U$ be a closed subvariety, and Q be the closure of Q_U in Y. Let $\mathcal{M}(Q) \subset \mathcal{M}(Y)$, $\mathcal{M}(Q_U) \subset \mathcal{M}(U)$? be the subcategories of $\mathcal{D}$-modules supported on Q. The above functors Π_f^a transform $\mathcal{M}(Q_U)$ to $\mathcal{M}(Q)$, and, being restricted to $\mathcal{M}(Q_U)$, they depend on $f|_Q$ only. Since everything is local, we get the functors $\Pi_f^a : \mathcal{M}(Q_U) \to \mathcal{M}(Q)$ and all the stuff above for any regular functions f on Q with $Q_U = Q \setminus f^{-1}(0)$.

(iv) The above functors will not change if we multiply ? f by non-zero constant $c \in \mathbb{C}$ since one has the isomorphism of $\mathcal{D}_{\mathbb{A}^1\setminus\{0\}}[s]/s^n$-modules $I^n \simeq c^* I^n$, $t^s \longmapsto (ct)^s$ (here $c : t \mapsto ct$ is multiplication by c automorphism of $\mathbb{A}^1 \setminus \{0\}$).

(v) The above constructions have obvious counterpart for constructible perverse sheaves compatible with Riemann-Hilbert correspondence (see [9]). One identifies canonically Ψ_f^{un} with the part of nearby cycles functor $R\Psi_{\bar\eta}[-1]$ on which the geometric monodromy acts unipotently and s corresponds to logarithm of monodromy; here $\bar\eta$ is the generic geometric point $\mathrm{Spec} \bigcup_N \mathbb{C}((t^{1/N}))$ of $\mathbb{C}((t))$.

4.3. Case of standard modules. Assume we are in a situation 3.1. Let $Q \subset X$ be an admissible orbit. For $\varphi \in \mathfrak{h}_{\mathbb{Z}}^{*+}(Q)$ consider the corresponding function f_φ on $\bar{\tilde{Q}}$ and the corresponding functors $\Pi^a_{f_\varphi} : M(\tilde{Q}) \to M(\bar{\tilde{Q}})$, see 4.2.2 (iii); since, by 4.2.2 (iv), they depend on φ only, we will write $\Pi^a_\varphi := \Pi^a_{f_\varphi}$.

These functors preserve K-equivariance and monodromicity (by construction), so we have the functors $\Pi^a_\varphi : M(\mathfrak{h}, K_{(x)})_{\chi,\infty} = M(\tilde{Q}, K)_{\tilde{\chi},\infty} \to M(\bar{\tilde{Q}}, K)_{\tilde{\chi},\infty} \hookrightarrow M(\tilde{X}, K)_{\tilde{\chi},\infty}$ (here $x \in Q$, see 1.5.4) and the Jantzen filtrations $J_{!\cdot}$, $J_{*\cdot}$ on the functors $j_{Q!}, j_{Q*} : M(\mathfrak{h}, K_{(x)})_{\chi,n} \to M(\tilde{X}, K)_{\tilde{\chi},n}$. In particular we have the Jantzen filtrations on standard modules $j_{Q!}(V)$, $j_{Q*}(V)$, where V is irreducible $(\mathfrak{h}, K_{(x)})_\chi$-module. A priori these filtrations depend on the choice of $\varphi \in \mathfrak{h}_{\mathbb{Z}}^{*+}(Q)$.

Note that these constructions may be done directly in terms of I-stratification pattern (see 1.5.4). Namely, for an orbit Q_α, $x \in Q_\alpha$ and $\varphi \in \mathfrak{h}_{\mathbb{Z}}^{*+}(Q_\alpha)$ let $I_\varphi^{(n)}$ be the $(\mathfrak{h}, K_{(x)})$-module $\mathbb{C}[s]/s^n$ such that $h \in \mathfrak{h}$ acts as $\varphi(h)s$ and $K_{(x)}$ acts trivially. The equivalence of categories $F_x : \tilde{M}(Q_\alpha, K) \xrightarrow{\sim} M(\mathfrak{h}, K_{(x)})$ (see 1.5.2) transforms $M \otimes f_\varphi^o(I^{(n)})$ to $F_x(M) \otimes I_\varphi^{(n)}$, and we may repeat the constructions of 4.2 using the functors $j_{\alpha!}$, $j_{\alpha*}$ and $\otimes I_\varphi^{(n)}$.

If $(\mathfrak{g}, K)$ is admissible Harish-Chandra pair, we get the Jantsen filtrations on !- and * -standard $(U, K)_\chi$ -modules ($\chi \in \mathfrak{h}^*$ is dominant regular weight) using the equivalence 2.4.2. If

$K = N$ or K is symmetric than $j_!$-extension is contravariant conjugate to j_*-extension (see 3.3) hence the morphism $j_{\lambda !}(V \otimes I_\varphi^{(n)}) \to j_{\lambda *}(V \otimes I_\varphi^{(n)})$ is just the contravariant form. This shows that here our definition of $J_{!*}$ coincides with the original Jantzen's one.

Remark. In Verma modules case one may define the Jantzen filtration using the deformations of central character in arbitrary non-degenerate direction φ, not necessary in the positive one. According to Barbash [1] the result does not depends on the choice of φ. In the geometric situation we may do, in principle, the same constructions and consider for any non=zero meromorphic function f on X the morphism $j_!(M_U \otimes f^\circ(I^{(n)})) \to j_*(M_U \otimes f^\circ(I^{(n)}))$, where $U := X \setminus \mathrm{div}(f)$. To define the vanishing cycles one needs the stabilization of cokernel when $n \to \infty$. It would be very nice if this fact would be true for any f, just as in the case when f (or f^{-1}) is regular on X, but I have no idea how to prove it.

§5. Weight filtrations

5.1. Weights of nearby cycles. The Gabber's theorem, which is our main tool, seems not to be published yet. Below we reproduce the proof following Gabber's report at IHES in spring 1981.

5.1.1. Kunneth formula for nearby cycles. Let S be spectrum of a strictly localHenselian ring; o, η be closed and generic points of S respectively, $\bar\eta$ be a geometric point localised at η. Let $X \to S$ be an S-scheme, $X_o \xhookrightarrow{i} X \xleftarrow{j} X_\eta \xleftarrow{k} X_{\bar\eta}$

be the fibers. In what follows $D^b(Y)$ will denote either the bounded derived category of étale constructible $\mathbb{Z}/\ell^n$ -sheaves on Y (where ℓ is prime to char c) or its $\mathbb{Q}_\ell$ -counterpart. According to [11] (3.2) one has the nearby cycles functors $\Psi_{\bar\eta} = \Psi_{\bar\eta X}$: $: D^b(X_\eta) \to D^b(X_s)$, $\Psi_{\bar\eta X} := i^* R j_* k_* k^*$. Let $Y \to S$ be another S-scheme, and $Z = X \times_S Y \to S$ be the fiber product. Then for $F \in D^b(X_\eta)$, $G \in D^b(Y_\eta)$ one has a canonical morphism in $D^b(Z_s)$:

$$(*) \quad \Psi_{\bar\eta X}(F) \boxtimes \Psi_{\bar\eta Y}(G) \longrightarrow \Psi_{\bar\eta Z}(F \boxtimes G)$$

<u>Lemma</u> (*) is isomorphism.

<u>Remark</u>. The transcendental version (and, hence, characteristic 0 case) is almost obvious by ordinary Kunneth applied to local varieties of vanishing cycles. This, joined with Riemann-Hilbert correspondence, implies the similar fact for tame $\mathcal{D}$-modules. To have this formula for arbitrary holonomic $\mathcal{D}$-modules one should use the total nearby cycles functor of Deligne (letter to Malgrange)

<u>Proof of lemma</u>. We may assume that the coefficients are $\mathbb{Z}/\ell$ (the $\mathbb{Z}/\ell^n$ and $\mathbb{Q}_\ell$ version follow in a moment). Put $m = \dim X$, $n = \dim Y$. The proof goes by simultaneous induction by m, n. The induction assumption plus the trick of Deligne [11] (3.3) show that cohomology sheaves of a cone of (*) are supported at finite set of points. So cone (*) = 0 is equivalent to $R\Gamma(\text{Cone } (*)) = 0$. The problem is local, hence we may assume X, Y to be affine, then projective, and the usual Kunnet formula for X, Y implies $R\Gamma(\text{Cone } (*)) = 0$ (since $R\Gamma\, \Psi_{\bar\eta}(F) = R\Gamma(F_{\bar\eta})$ in projective case). $\square$

5.1.2. Gabber's theorem. Assume we are in a mixed situation, so our schemes are over the finite field $\mathbb{F}_q$. Let $M(X)_{\text{mixed}} \subset D^b(X)_{\text{mixed}}$ be the category of mixed perverse sheaves on X and the corresponding derived category. Let T be a curve, $o \in T$ be a closed point, $U := T \setminus \{o\}$, S be the strict localisation of T at o, $\bar{\eta}$ be the generic geometric point of S. For a T-scheme $f : X \to T$ put $X_o = f^{-1}(o)$, $X_U = f^{-1}(U)$. One has a nearby cycles functor $\Psi_{\bar{\eta}X} : D^b(X_U)_{\text{mixed}} \longrightarrow D^b(X_o)_{\text{mixed}}$ (see [12]). It is convenient to use the shift $\Psi_f := \Psi_{\bar{\eta}X}[-1]$. This functor is t-exact, i.e. $\Psi_f(M(X_U)) \subset M(X_o)$, and commutes with Verdier duality as follows: $\Psi_f D = D\Psi_f(1)$ (here $(1) = \otimes \mathbb{Q}_\ell(1)$ is Tate twist). The monodromy group acts on Ψ_f; for a perverse sheaf M let $s \in \operatorname{End} \Psi_f(M_U)$ be the logarithm of the unipotent part of geometric monodromy, and $\mu.$ be the corresponding monodromy filtration on $\Psi_f(M_U)$.

Theorem. If M_U is pure of weight w, then $\mu_{\cdot+w-1}$ coincides with weight filtration $W.$.

Proof. The case when f is identity (or finite map) is Deligne's theorem [12] (1.8.4). The proof in the general case follows the similar lines.

(i) We may assume that M_U is irreducible.

(ii) Replacing T by a finite cover, we may assume that the geometric monodromy is unipotent.

(iii) The weights on Ker s (= invariants of monodromy action) are $\leq$ w-1. Proof. Consider a canonical isomorphism $\operatorname{Ker} s = \operatorname{Ker}(j_! M_U \to j_* M_U)$. Since $j_!$ does not increases weights, the weights of $j_! M_U$ are $\leq$ w. But the only irreducible quotient of $j_! M_U$ is $j_{!*} M_U$, hence $\operatorname{Ker} s = W_{w-1}(j_! M_U)$.

Dually, the weights of Coker s are $\geq w-1$ (since Coker s = Coker $(j_! M_U \to j_* M_U)(1)$).

(iv) Since the weight of s is -2, to prove the theorem it suffice to show that the primitive part P_{-i} is pure of weight $w-1-i$. We have $Gr_i^{J_!} = P_i$, $Gr_i^{J_*} = P_{-i}(-i)$ (see 4.1), so (iii) implies the inequalities for weights $\{w_i\}$ of P_i: $w_i \leq w-1$, $w_i + 2i \geq w-1$, i.e. $w-1-2i \leq w_i \leq w-1$. In particular for $i = 0$ we are done.

(v) Consider the fiber square $M_U^{\boxtimes 2}[-1]$: this is a perverse sheaf on $X \times_T X$ (at least over the generic point of T - the only thing we need) of weight $2w-1$. Since $\Psi_{f \times f}(M_U^{\boxtimes 2}[-1]) = \Psi_f(M_U)^{\boxtimes 2}$ by 5.1.1, the lemma 4.1.2 (ii) implies that $P_{-i} \boxtimes P_{-i}(-i)$ occurs in $P_0(\Psi_{f \times f} M_U^{\boxtimes 2}[-1])$. Hence, by (iv), one has $2w_i + 2i = 2w - 2$, or $w_i = w-i-1$. $\square$

Assume we have a parameter t at 0, $t \in \mathcal{O}_T(T)$. We may define the functors of 4.2 in the mixed situation (see [2]): one has $\Pi_f^a : \mathcal{M}(U)_{\text{mixed}} \to M(X)_{\text{mixed}}$, $(\Pi_f^a M_U)|_U$ is successive extension of twists $M_U, M_U(1), \ldots, M_U(a-1)$. Now 5.1.2 joined with 4.1.1, 4.2.1 and 4.2.2 (v) gives

5.1.3. <u>Corollary</u>. (i) If M_U is pure of weight w then the filtrations $J_{f!}$, $W_{\cdot + w}$ on $j_{U!}(M_U)$ coincide. Same for filtrations J_{f*}, $W_{\cdot + w}$ on $j_{U*}(M_U)$.

(ii) The monodromy filtration $\mu.$ on $\Pi_f^a(M_U)$ coincides with $W_{\cdot + w + a - 1}$. In particular on $\Xi_f(M_U)$ one has $\mu. = W_{\cdot + w}$.

5.2. <u>Pointwise purity and socle property of weight filtration</u>. A mixed complex $F^\cdot$ on X is $*$-pointwise pure of weight w if for any closed point $x \in X$ the complex $i_x^*(F^\cdot)$ is pure of weight w (i.e. $H^i i_x^* \mathcal{F}^\cdot$ is pure of weight $i+w$; here $i_x : x \hookrightarrow X$). One defines ! -pointwise purity similarly using $i_x^!$ instead of i_x^*; the Verfier duality interchanges $*$- and !-purity. Note that if a pure perverse sheaf is $*$-pointwise pure of weight w, then w coincides with its weight.

Now let X be a finite monodromic K-variety. Recall that any pure monodromic sheaf has finite geometric monodromy along the fibers of $\tilde{X} \to X$ hence is $\bar{\chi}$ -monodromic for $\bar{\chi} \in \mathfrak{f}_{\mathbb{Q}}^*/\mathfrak{f}_{\mathbb{Z}}^*$ (by local monodromy theorem; note that restriction of monodromic sheaf to the fiber is tame [32]). We will say that $\tilde{X}$ is $(K, \bar{\chi})$-pointwise pure (here $\bar{\chi} \in \mathfrak{f}_{\mathbb{Q}}^*/\mathfrak{f}_{\mathbb{Z}}^*$) if any pure K-equivariant $\bar{\chi}$ -monodromic sheaf is $*$- and !-pointwise pure, and $\tilde{X}$ is K-pointwise pure if this holds for any $\bar{\chi}$.

5.2.1. <u>Examples</u>. (i) Here is a simple sufficient condition for $*$-pointwise purity. Let M be a pure perverse sheaf. Assume that for any $x \in X$ there exists an etale neighbourhood U of x such that a canonical map $H^\cdot(U, M) \to H^\cdot i_x^* M$ is surjective. Then M is $*$-pointwise pure (since the weights on $H^\cdot(U, M)$ are $\geq \cdot +w$ by Deligne's Weil II, and weights on $H^\cdot i_x^* M$ are $\leq \cdot + w$ by definition). In particular this implies that "toric" irreducible perverse sheaves on a toric variety are pointwise pure, which leads to the explicit formula for Goreski-Macpherson Betti numbers of toric varieties (J.Bernstein, summer 1981).

(ii) According to Kazhdan-Lusztig [23] and Lusztig [24] ch.1 the flag variety $\tilde{X}_{\mathfrak{g}}$ is N-pointwise pure. Lusztig and Vogan [25] have shown that $X_{\mathfrak{g}}$ is K-pointwise pure if K is a fixed point subgroup of an involution; it seems that their method, joined with decomposition theorem, should prove the K-pointwise purity of $\widetilde{X}_{\mathfrak{g}}$ for any symmetric pair $(\mathfrak{g}, K)$.

Recall that one defines a socle filtration $S_\cdot(M)$ on an object M of an abelian category by induction: $S_{-1} = 0$, $S_0 :=$ maximal semisimple subobject of M, $S_i(M)/S_{i-1}(M) := S_0(M/$ $/S_{i-1}(M))$. One defines a cosocle filtration $M = C_0(M) \supset$ $C_{-1}(M) \supset \ldots$ in a dual manner.

If M is a mixed perverse sheaf, then $S_\cdot(M)$, $C_\cdot(M)$ will denote socle and cosocle filtrations on M considered as geometric sheaf (Frobenius forgotten). Clearly both $S_\cdot$ and $C_\cdot$ are (being functorial) Frobenius invariant, hence $S_\cdot(M)$, $C_\cdot(M)$ are mixed subsheaves of M.

5.2.2. Lemma. Let $i_Y : Y \hookrightarrow X$ be a locally closed subscheme, and M_Y be a weight w pure perverse sheaf on Y. Let $N \subset {}^pH^0 i_{Y*} M_Y$ be a mixed subsheaf such that any irreducible subquotient of N is $!$-pointwise pure. Then $S_\cdot(N) = W_{\cdot+w}(N)$.

Proof. We have $S_{-1}(N) = 0 = W_{w-1}(N)$ (since i_{Y*} increases weights), $S_0(N) = W_w(N) = i_{Y!*}(i_Y^* N)$ (since, by adjunction property of i_{Y*}, $S_0({}^pH i_{Y*} M_Y) = i_{!*} M_Y$). Since Gr^W is geometrically semisimple one has $S_i(N) \supset W_{i+w}(N)$, so it remains to prove that $S_i(N) \subset W_{i+w}(N)$ for $i \geq 1$. We will do this by the double induction: first by $\dim Y$, then by i. So assume that 5.2.2 is known for any $(Y', M_{Y'})$ with $\dim Y' < \dim Y$, and that $S_j(N) = W_{j+w}(N)$ for $j < i$. Suppose that $S_i(N) \not\subset W_{i+w}(N)$. Then $S_i/S_{i-1} = S_i/W_{w+i-1}$ contains a pure geometrically irreducible subsheaf A of weight $a > i+w$ (possibly, after a finite extension of the finite base field). Note that $\operatorname{Supp} A \subset \bar{Y} \setminus Y$.

(i) Assume that A is supported at closed point x. Consider the extension $0 \to W_{w+i-1}(N)/W_{w+i-2}(N) \to B \to A \to 0$ defined by N. Since $B \notin S_{i-1}(N)$, this extension is geometrically non-trivial, hence it corresponds to non-zero element in $\mathrm{Hom}_{F_{\ell o g}}(A, H^1 i_x^! W_{w+i-1}(N)/W_{w+i-2}(N))$. By !-pointwise purity condition $H^1 i_x^!(W_{w+i-1}(N)/W_{w+i-2}(N))$ has weight $w+i$; but $a > w+i$, hence contradiction

(ii) If $\dim \operatorname{supp} A > 0$ we will use the induction by $\dim Y$. The conditions of lemma are local, so we may assume that X is affine. Choose a generic hyperplane section $Z \subset X$, namely such one that for any irreducible subquotient L of ${}^p H^\circ i_{Y*} M_Y$ a canonical morphism $i_Z^! L(1)[2] \to i_Z^* L$ is isomorphism. Then $M_{Y\cap Z} := i^!_{Y\cap Z}(M_Y)[1]$ is a pure weight $w+1$ perverse sheaf on $Y \cap Z$. Consider the complex $i_{Y\cap Z *}(M_{Y\cap Z}) = i_Z^![1]\ i_{Y*} M_Y$; one has $W_a\ {}^p H^0 i_{Y\cap Z *} M_{Y\cap Z} = i_Z^![1]\ W_{a-1}\ H^0 i_{Y*} M_Y$. A subsheaf $N_Z := i_Z^![1](N)$ of ${}^p H^0 i_{Y\cap Z *} M_{Y\cap Z}$ satisfies the conditions of lemma, hence, by induction hypothesis, $i_Z^![1](A)$ has weight $i+1$. Since $i_Z^![1](A) \neq 0$ (since $\dim \operatorname{supp} A > 0$) our A has weight i, and we are done. □

5.2.3. <u>Corollary</u>. Let M_1, M_2 be pure perverse sheaves of weights w_1, w_2 that are both *- and ! -pointwise pure. Suppose that $\mathrm{Ext}^1_{\mathcal{M}_{mixed}}(M_1, M_2) \neq 0$. Then either of the following conditions holds (here $Y_i := \operatorname{supp} M_i$)

(i) $Y_1 \subset Y_2$, $Y_1 \neq Y_2$, $w_1 = w_2 + 1$

(ii) $Y_2 \subset Y_1$, $Y_1 \neq Y_2$, $w_1 = w_2 + 1$

(iii) $Y_1 = Y_2$.

<u>Proof</u>. Clearly either $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$ (otherwise

$\mathrm{Ext}^1 = 0$). Let $0 \to M_2 \to N \to M_1 \to 0$ be the non-split mixed extension. If $Y_1 \neq Y_2$ and $Y_1 \subset Y_2$, then a canonical morphism $N \to {}^pH^0\, i_{Y_2\setminus Y_1 *} i^*_{Y_2\setminus Y_1} N = {}^pH^0\, i_{Y_2\setminus Y_1 *}(M_2|_{Y_2\setminus Y_1})$ is injective (since N is non-split), so we are in situation (i) by 5.2.2. If $Y_1 \neq Y_2$ and $Y_2 \subset Y_1$, then N is a quotient of $i_{Y_1\setminus Y_2 !}(M_1|_{Y_1\setminus Y_2})$, and (ii) holds by the Verdier dual to 5.2.2. □

Let $\check{X}$ be a finite monodromic K-variety. Note that if M_1, M_2 are irreducible objects in $\mathcal{M}(\check{X}, K)_{\bar{\chi},1}$ such that $\mathrm{Ext}^1_{\mathcal{M}(\check{X},K)_{\bar{\chi},1}}(M_1, M_2) \neq 0$ then $\mathrm{supp}\, M_1 \neq \mathrm{supp}\, M_2$ (this follows, using the functor $i_{!*}$, from the fact that the category $\mathcal{M}(\tilde{Q}, K)_{\bar{\chi},1}$ is semisimple if Q is a single orbit).

5.2.4. <u>Corollary</u>. Assume that $\hat{X}$ is K-pointwise pure. Let M be an object in $\mathcal{M}(\check{X}, K)_{\bar{\chi},1\,\mathrm{mixed}}$ (so $\chi \in \mathcal{f}^*_{\mathbb{Q}}$) such that $W_{a-1}(M) = 0$ and $W_a(M) = S_0(M)$. Then $W_{a+i}(M) = S_i(M)$ for any i.

<u>Proof</u>. Induction by i, using 5.2.2, the previous remark and also the fact that any subquotient of M is K^0-equivariant. □

5.2.5. <u>Example</u>. Consider an irreducible K-equivariant sheaf M. Let $I(M)$ be an injective envelope of M in $\mathcal{M}(\check{X},K)_{\bar{\chi},1}$. Then $I(M)$ admits a mixed structure (possibly after a finite extension of base field), and for any one the weight filtration coincides with socle filtration up to a shift.

<u>Proof</u>. The only problem is existence of mixed structure. But M clearly has one (being a middle extension of a lisse sheaf with finite monodromy). Any extension of Frobenius action $M \to \mathrm{Frob}^* M$ to $I(M) \to \mathrm{Frob}^* I(M)$ defines the mixed structure

on I(M) (since any irreducible subquotient of I(M) admits mixed structure, and any Frobenius action on irreducible is unique up to twist).

5.3. Jantzen conjectures. Let us apply the above considerations to ($\mathfrak{g}$, K)-modules. Let ($\mathfrak{g}$, K) be an admissible Harish-Chandra pair (see 3.2), and $\chi \in \mathfrak{h}^*_{\mathbb{Q}}$ be a fixed rational dominant regular weight. The irreducible objects of $\mathcal{M}(\tilde{X}, K)_{\tilde{\chi}}$ have geometric origin (see 1.5.4 (v)), hence the corresponding standard objects carry the weight filtration defined up to a shift. According to 5.1.3 (i) it coincides with the Jantzen filtration. So a bunch of weight filtration properties also holds for the Jantzen counterpart via the equivalence $\mathcal{M}(U, K)_{\chi,\infty} \xrightarrow{\tilde{\Delta}_\chi} \mathcal{M}(\tilde{X}, K)_{\tilde{\chi},\infty}$ (below we use freely the road from $\mathbb{F}$ to $\mathbb{C}$, see [5], §6):

5.3.1. Corollary. The Jantzen filtration on standard $(U, K)_{\chi,1}$-modules has semisimple successive quotients and does not depends on the choice of positive deformation direction φ (see 4.3). □

5.3.2. Corollary. Assume that $\tilde{X}$ is $(K, \tilde{\chi})$-pointwise pure (see 5.2.1 (ii)).

(i) The Jantzen filtration $J_{*}.$ on $*$-standard $(U,K)_{\chi,1}$-module coincides with the socle filtration; the one $J_{!}.$ on !-standard module coincides with cosocle filtration.

(ii) If $K = N$, then $J_{*}.$ also coincides, up to a shift, with cosocle filtration, and $J_{!}.$ - with socle one.

Proof (i) is, say, 5.2.2 plus the Verdier dual statement. (ii) follows from 5.2.4 and the fact that Verma module contains a unique irreducible submodule. □

5.3.3. Remarks. (i) The statement (ii) above was proven in [1] by purely algebraic means. One may conjecture that it remains valid in case of arbitrary symmetric pair.

(ii) In fact, in [1] the socle property of $J_{!\cdot}$ for Verma modules was proven for Jantzen filtration defined by means of deformations of central character in arbitrary non-degenerate direction, and we in §4 used the deformations in positive directions only. I do not know whether one may use such arbitrary deformations in the definition of $J_{!\cdot}$ for any symmetric pair. □

For a regular $\chi \in \mathfrak{h}^*$ put $\Delta^{(\chi)} := \{\alpha \in \Delta : \chi(h_\alpha) \in \mathbb{Z}\}$. It is well known that $\Delta^{(\chi)}$ is root system with Weyl group $W^{(\chi)} = \{w \in W : w\chi - \chi \in \mathfrak{h}^*_{\mathbb{Z}}\}$ (recall that $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\Delta$). The orbit $W^{(\chi)}\chi$ contains unique dominant weight, and for $\chi' \notin W^{(\chi)}\chi$ one has $\mathrm{Hom}(M_{\chi'}, M_\chi) = 0$ and $[M_\chi : L_{\chi'}] = 0$ (here $M_\chi \in \mathcal{M}(\mathfrak{g}, N)_{\theta(\chi),1}$ is Verma module with lower weight $2\rho + \chi$, L_χ is its irreducible quotient).

Let $\chi_1, \chi_2 \in \mathfrak{h}^*_{\mathbb{Q}}$ be regular weights such that $M_{\chi_1} \subset M_{\chi_2}$. Then for (unique) dominant χ one has $\chi_i = w_i \chi$, where $w_i \in W^{(\chi)}$ and $w_1 \leq w_2$ with respect to usual order on $W^{(\chi)}$.

5.3.4. Corollary. One has $J_{!i}(M_{\chi_1}) = M_{\chi_1} \cap J_{!i+\ell(w_2)-\ell(w_1)}(M_{\chi_2})$ (here ℓ is length function on $W^{(\chi)}$).

Proof. Since $\dim \mathrm{Hom}(M_{\chi_1}, M_{\chi_2}) = 1$ the embedding of the corresponding standard mixed sheaves is pure of certain weight a. Turning back to representations we see that $J_{!\cdot}(M_{\chi_1}) = M_{\chi_1} \cap J_{!\cdot+a}(M_{\chi_2})$. It remains to show that $a = \ell(w_2) - \ell(w_1)$. We may assume that $\ell(w_2) - \ell(w_1) = 1$ (if not, choose a chain $M_{\chi_1} \subset M_{\psi_1} \subset \dots \subset M_{\psi_{\ell(w_2)-\ell(w_1)-1}} \subset M_{\chi_2}$ of Verma submodules such that each successive M's has this property, and descend along it). Then the Shapovalov's formula for the determinant of contravariant form implies that the vacuum vector of M_{χ_1}

lies in $J_{!-1}(M_{\chi_2}) \setminus J_{!-2}(M_{\chi_2})$. Hence $a = 1$ q.e.d. □

Let $\chi \in \mathfrak{h}^*_{\mathbb{Q}}$ be dominant regular, $w_1, w_2 \in W^{(\chi)}$. Put $P_{w_1, w_2} := \sum_i \mathrm{Gr}^{J_!}_{-i} [M_{w_2\chi} : L_{w_1\chi}]\, t^i$.

5.3.5. Corollary. This polynomial equals to Kazhdan-Lusztig polynomial for the group $W^{(\chi)}$.

Proof. According to [24] ch.1 the Kazhdan-Lusztig polynomials form the matrix coefficients of the matrix that transforms the basis $j_{w!}(\mathbb{Q}_\ell)$ of K-group of the category $\mathcal{M}(\check{X})_{\tilde{\chi}}$ mixed to the basis $j_{w!*}(\mathbb{Q}_\ell)$. Since Jantzen filtration coincides with weight one, our polynomials correspond to the entries of the inverse matrix. Since these matrices coincides up to standard changes of signs of the coefficients [22], we are done.

5.3.6. Remarks. (i) 5.3.4 is Jantzen's conjecture [20] (5.18), see also [15] (4.2), and 5.3.5 was conjectured in [15], [17]; in [15] it was shown that 5.3.4 implies 5.3.5 by purely algebraic arguments.

(ii) It would be nice to get the analogs of 5.3.4, 5.3.5 for arbitrary symmetric pair. The only problem is to compute the weights in the space of Hom's between standard modules. Also one wishes to know the weights in all the Ext's; I am ignorant of this even in Verma modules case. May be $\mathrm{Ext}^i(M_{w_1\chi}, M_{w_2\chi})$ is pure of weight $2i + \ell(w_2) - \ell(w_1)$?

(iii) Denote by $\mathcal{M}(\mathfrak{g}, N)^{(\chi)}$ the Serre subcategory of $\mathcal{M}(\mathfrak{g}, N)_{\gamma(\chi)}$ generated by $L_{\chi'}$, $\chi' \in W^{(\chi)}\chi$. one knows that $\mathcal{M}(\mathfrak{g}, N)_{\gamma(\chi)}$ splits into the direct product of such subcategories (with χ runs the set of dominant weights in W-orbit). It seems that the category $\mathcal{M}(\mathfrak{g}, N)^{(\chi)}$ depends on the root system $\Delta^{(\chi)}$ only (i.e. that $M(\mathfrak{g}, N)^{(\chi)}$'s with isomorphic root systems $\Delta^{(\chi)}$'s are equivalent). In particular, $M(\mathfrak{g}, N)^{(\chi_1)}$ is equivalent to $M(\mathfrak{g}, N)^{(\chi_2)}$ if $\gamma(\chi_1) = \gamma(\chi_2)$. Also this will imply the above corollaries for arbitrary (not necessary rational, as we supposed) weights by reduction to rational case. I do not know any proof.

(iv) Consider the category $M(\tilde{X}, N)_{\bar{\chi} \text{ mixed}}$ which may be called the category of mixed representations. One knows that it is equivalent to the category of graded modules over certain graded finite dimensional Koszul quadratic algebra possibly Koszul self-dual [6]. Whether one may ascend the Kazhdan-Lusztig algorithm to get the construction of this category (or this algebra) directly in terms of root system or Weyl group?

References

1. D.Barbash. Filtrations on Verma modules. Ann. Sci. ENS, 4 serie, t.16, 1983, 489-494.

2. A.Beilinson. How to glue perverse sheaves, LN 1289, Springer, 1987, 42-51.

3. A.Beilinson, J.Bernstein. Localisation de $\mathfrak{g}$-modules. C.R.A.S. Paris, Ser. I 292, 1981, 15-18.

4. A.Beilinson, J.Bernstein. A generalisation of Casselman's submodule theorem, Progress in Mathematics, Boston: Birkhauser, vol. 40, 1983, 35-52.

5. A.Beilinson, J.Bernstein, P.Deligne. Faisceax pervers, Asterisque 100 (1982).

6. A.Beilinson, V.Ginsburg, V.Schechtman. Koszul duality and representations, to appear ~~in Journal of Geometry and Physics~~

7. J.Bernstein, I.Gelfand, S.Gelfand. Category of $\mathfrak{g}$-modules. Funct. Anal. Priloz., vol.10, 1976, pp.1-8.

8. J.Bernstein, S.Gelfand. Tensor products of finite and infinite dimensional representations of semisimple Lie algebras Compos. Math., vol.41, 1980, pp.245-285.

9. A.Borel et al. Algebraic $\mathcal{D}$-modules, Academic press 1987.

10. J.L.Brylinski. Transformations canoniques, duality projective, theorie de Lefschetz ... Asterisque

11. P.Deligne. Theoremes de finitude en cohomologie ℓ-adique, in SGA 4½, L.N. 569, Springer, 1977.

12. P.Deligne. La conjecture de Weil II, Publ. math. IHES 52, 1980.

13. M.Duflo, M.Vergne. Sur le foucteur de Zuckerman C.R.A.S. Paris t. 304, Serie I, n 16, 1987, 467-469.

14. R.Elkik. Desingularisation des adherences d'orbites polarisables et des nappes daus les algebras de Lie reductives, preprint, Paris 1978.

15. O.Gabber, A.Joseph. Towards the Kazhdan-Lusztig conjecture. Ann. Sci. ENS, 4 serie, t.14, 1981, pp.261-302.

16. O.Gabber, A.Joseph. The Bernstein-Gelfand-Gelfand resolution and the Duflo sum formula. Compos. Math., vol.43, 1981, pp.107-131.

17. S.Gelfand, R.Macpherson. Verma modules and Schubert cells; a dictionary, LN 924, Springer, 1982, 1-50

18. V.Ginsburg. Equivariant cohomology and Kahler geometry. Funct. Anal. Prilòz., vol 21, 1987, 19-34 (Russian)

19. H.Hecht, D.Milicic, W.Schmid, J.A.Wolf. Localization and standard modules for real semisimple Lie groups I: The duality theorem. Invent. Math. vol.90, fasc. 2, 1987, pp.297-332.

20. J.C.Jantzen. Moduln mit einem höhsten Gewicht, LN 750, Springer, 1980.

21. M.Kashiwara. Vanishing cycle sheaves and holonomic systems of differential equations, LN 1016, Springer, 1986, 134-142.

22. D.Kazhdan, G.Lusztig. Representations of Coxetez groups and Hecke algebras, Invent. Math. 53, 1979, 165-184.

23. D.Kazhdan, G.Lusztig. Schubert variety and Poincare duality.

24. G.Lusztig. Characters of reductive groups over a finite field. Princeton, Ann. of Math. studies, vol.107, 1984.

25. G.Lusztig, D.Vogan. Singularities of closures of K-orbits on flag manifolds, Invent. Math. 71, 1983, 365-379.

26. B.Malgrange. Polynomes de Bernstein-Sato et evanescence, Asterisque 101-102, 1983, 243-267.

27. I.Mirkovic. Classification of irreducible tempered representations of semisimple groups, preprint, 1986.

28. I.Mirkovic, K.Villonen. Characteristic varieties of character sheaves, preprint, 1987.

29. W.Schmid. Recent developments in representation theory, Springer Lect. Notes Math., vol. 1111, 1985, pp.135-153.

30. T.Springer. Quelques applications de la cohomologie d'intersection, Seminaire Bourbaki, exp. 589, Fevr. 1982.

31. T.Springer. The orbits in the flag variety under the action of subgroups defined by involutions, preprint.

32. J.-L.Verdier. Specialisation de faisceaux et monodromie modere, Asterisque 101-102, 1983, 332-364.

33. J.-L.Verdier. Prolongement des faisceaux pervers monodromiques, Asterisque 130, 1985, 218-236.

34. D.Vogan. Representations of real reductive groups, Progress in Math., vol.15, Boston: Birkhauser 1981.