# REMARKS ON THE CHARACTERISTIC CYCLES OF DISCRETE SERIES REPRESENTATIONS FOR $SU(p,q)$

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## 1. Inroduction

The characteristic cycle of an irreducible admissible representation is an important invariant of the representation. This invariant consists of a the closure of several nilpotent orbits (the associated variety) along with integers (the multiplicities). In this article we consider discrete series representations of the group $SU(p,q)$. It is known that the associated variety is the image of a moment map $\gamma$ (see (2.2)) and the multiplicity is the dimension of a cohomology space on the fiber of $\gamma$. We present an algorithm for computing the associated variety and, more importantly, for describing the fiber of the moment map in a simple way. This description of the fiber can be used to compute the multiplicities. An effective algorithm is given in Section 6 for computing the multiplicities.

Assume that $G$ is a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$ and maximally compact subgroup $K$. Write $K_{\mathbf{C}}$ and $\mathfrak{g}_{\mathbf{C}}$ for the complexifications of $K$ and $\mathfrak{g}$ respectively. Assume that $\pi$ is an irreducible admissible representation of $G$ on a Hilbert space $\mathcal{H}$. An important invariant attached to $\pi$ is its distributional character $\Theta_\pi$. In the early 1980's Barbasch and Vogan [1] showed that $\theta_\pi$ (the pullback of $\Theta_\pi$ by the exponential map to a neigborhood of the identity in $\mathfrak{g}$) has an asymptotic expansion at 0 which is a sum of homogeneous tempered distributions on $\mathfrak{g}$. Its leading term is of the form

$$\sum c_j \widehat{\mu_{\Omega_j}}, \tag{1.1}$$

where the $\Omega_j \subset \mathfrak{g}^*$ are nilpotent $G$-orbits and $\widehat{\mu_{\Omega_j}}$ is the Fourier transform of the Liouville measure on $\Omega_j$. Call $c_j$ the analytic multiplicity of $V$ at $\Omega_j$ and write $c_j = \text{mult}_{\text{analytic}}(\pi, \Omega_j)$. The leading term (1.1) can be called the 'wave front cycle' and is an invariant of $\pi$. This analytically defined multiplicity coincides with an algebraically defined multiplicity ([18]), which is defined in terms of $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$. The multiplicity is very difficult to compute from both the analytic and the algebraic points of view. See, for example, [12].

The starting point for our method is a formula of J.-T. Chang for the characteristic cycle of a discrete series representation. It is well-known in this case that the associated variety is the closure of a single nilpotent $K_{\mathbf{C}}$-orbit $K_{\mathbf{C}}(f)$ in $\mathfrak{p}_{\mathbf{C}}$. Chang's formula for the multiplicity is expressed as the dimension of a cohomology space

$$H^0(\gamma^{-1}(f), \mathcal{O}_\mu), \tag{1.2}$$

where $\gamma$ is the moment map. Explicit formulas, applying 1.2 were found for the holomorphic discrete series and for discrete series representations of groups of real rank one. See [4] and [6]. The key is an understanding of the fiber of the moment map, which, in these special cases is given by easily described homogeneous spaces. In general the structure of $\gamma^{-1}(f)$ is difficult to understand.

Our algorithm does the following. The discrete series are parameterized (up to $K$-conjugacy) by regular weights $\lambda \in \mathfrak{h}^*_{\mathbf{C}}$ satisfying an appropriate integrality condition. Such a regular weight determines a positive system of roots $\Delta^+_\lambda(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. This positive system determines a Borel subalgebra $\mathfrak{b}_\lambda = \mathfrak{h}_C + \mathfrak{n}$. The $K_{\mathbf{C}}$-orbit (having closure equal to the associated variety) has the property that $K_{\mathbf{C}}(f)\cap(\mathfrak{n}\cap\mathfrak{p}_{\mathbf{C}})$ is (Zariski) dense in $\mathfrak{n}\cap\mathfrak{p}_{\mathbf{C}}$. We give a simple algorithm which determines a convenient base point $f$ (in terms of root vectors) for the associated variety. The key feature of this algorithm is that a useful expression for $\gamma^{-1}(f)$ results. In particular reductive groups $L_1,\dots,L_m$ are specified so that

$$\gamma^{-1}(f)=L_m\dots L_2L_1(\mathfrak{b}_\lambda)\subset Z.$$

Here $Z=K_{\mathbf{C}}(\mathfrak{b}_\lambda)$, a closed $K_{\mathbf{C}}$-orbit in the flag variety for $G_{\mathbf{C}}$. Then the cohomology space (1.2) is given by

$$H^0(\gamma^{-1}(f),\mathcal{O}_\mu)=\mathrm{span}_{\mathbf{C}}\{L_m\dots L_2L_1L\cdot w_{-\mu}\}\subset W_{-\mu}, \tag{1.3}$$

where $W_{-\mu}$ is the irreducible representation of $K_{\mathbf{C}}$ (of lowest weight $-\mu$) which is isomorphic to the lowest $K$-type of the discrete series. As the groups $L_j$ are easily described in terms of the positive root system $\Delta_\lambda(\mathfrak{g}_{\mathbf{C}},\mathfrak{h}_{\mathbf{C}})$, there is an algorithm to compute the multiplicity from (1.3). The algorithm uses not much more than the branching law for restricting finite dimensional representations of $GL(n)$ to $GL(n-1)$. This algorithm is described at the end of Section 6 and examples are given in Section 7. Incidentally, one may conclude from (1.3) and the work of Yamashita ([19]) that (1.3) is the isotropy representation ([18, Section 2]). The action of the centralizer of $f$ is clear from Section 5.

In [16] an algorithm is given to compute the associated varieties of a $A_{\mathfrak{q}}(\lambda)$ representations of classical groups in terms of the tableaux describing the nilpotent orbits. In [19] a method for constructing the isotropy representation is developed. The polynomials giving multiplicities are studied in [11] and [12]; computation of the multiplicities is also discussed.

Our study of the moment map yields some interesting facts. As mentioned above, the characteristic variety is the closure of an orbit $K_{\mathbf{C}}(f)$ having the property that $K_{\mathbf{C}}(f)\cap(\mathfrak{n}\cap\mathfrak{p}_{\mathbf{C}})$ is dense in $\mathfrak{n}\cap\mathfrak{p}_{\mathbf{C}}$. It is natural to ask if the Borel subgroup $B\cap K_{\mathbf{C}}$ acts with an open orbit on $K_{\mathbf{C}}(f)\cap(\mathfrak{n}\cap\mathfrak{p}_{\mathbf{C}})$. The answer is no in general ([15]). However, one may also ask the following question. Let $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$ containing $\mathfrak{b}$ defined by $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ with $\Delta(\mathfrak{l})$ spanned by the simple compact roots in $\Delta^+_\lambda(\mathfrak{g}_{\mathbf{C}},\mathfrak{h}_{\mathbf{C}})$, and let $Q$ be the corresponding parabolic subgroup of $G_{\mathbf{C}}$. Then $Q\cap K_{\mathbf{C}}$ acts on $\mathfrak{n}\cap\mathfrak{p}_{\mathbf{C}}$. Does $Q\cap K_{\mathbf{C}}$ have a dense orbit in $K_{\mathbf{C}}(f)\cap(\mathfrak{n}\cap\mathfrak{p}_{\mathbf{C}})$? Again the answer is no in general. In Section 8 we give a condition, in terms of the algorithm of Section 3, for determining when there is a dense $Q\cap K_{\mathbf{C}}$-orbit in $K_{\mathbf{C}}(f)\cap(\mathfrak{n}\cap\mathfrak{p}_{\mathbf{C}})$. An example in $SU(7,7)$ is given for which there is no such dense orbit.

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## 2. The characteristic cycle

For this section let $G_{\mathbf{C}}$ be a connected complex semisimple Lie group and $G\subset G_{\mathbf{C}}$ a real form. Let the Lie algebra of $G_{\mathbf{C}}$ (resp. $G$) be denoted by $\mathfrak{g}_{\mathbf{C}}$ (resp. $\mathfrak{g}$). Choose a Cartan involution $\theta$ of $\mathfrak{g}$ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. The complexifications of $\mathfrak{k}$ and $\mathfrak{p}$ are denoted by $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$. The connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{k}_{\mathbf{C}}$ is denoted by $K_{\mathbf{C}}$. The nilpotent cone in $\mathfrak{g}_{\mathbf{C}}$ is denoted by $\mathcal{N}$. Now set $\mathcal{N}_{\mathbf{R}}=\mathcal{N}\cap\mathfrak{g}$ and $\mathcal{N}_\theta=\mathcal{N}\cap\mathfrak{p}_{\mathbf{C}}$.

The wave front cycle described in the introduction has an algebraic counterpart. If $(V,\pi)$ is an irreducible admissible representation, then the $K$-finite part of $V$ is naturally a $(U(\mathfrak{g}_{\mathbf{C}}),K_{\mathbf{C}})$-module, where $U(\mathfrak{g}_{\mathbf{C}})$ is the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. With respect to the usual filtration by degree, the graded algebra $\mathrm{gr}(U(\mathfrak{g}_{\mathbf{C}}))$ is just the symmetric algebra $S(\mathfrak{g}_{\mathbf{C}})$. In [18], Vogan defines a notion of "good filtration" on the Harish-Chandra module of $V$. The associated graded object $\mathrm{gr}(V)$ turns out to be a finitely generated module over $\mathrm{gr}(U(\mathfrak{g}_{\mathbf{C}}))/\mathfrak{k}_{\mathbf{C}}\mathrm{gr}(U(\mathfrak{g}_{\mathbf{C}}))\simeq S(\mathfrak{p}_{\mathbf{C}})$. Regarding $S(\mathfrak{p}_{\mathbf{C}})$ as the polynomial ring on $\mathfrak{p}^*_{\mathbf{C}}$, invariants of the Harish-Chandra module of $V$ are defined via commutative algebra theory. In particular, the characteristic cycle of $V$, $\mathrm{Ch}(V)$, is the support with multiplicity of $\mathrm{gr}(V)$ in $\mathfrak{p}_{\mathbf{C}}^* \simeq \mathfrak{p}_{\mathbf{C}}$. While $\mathrm{gr}(V)$ depends on the choice of the good filtration, $\mathrm{Ch}(V)$ is a well-defined invariant. The cycle $\mathrm{Ch}(V)$ is closed, $\mathrm{Ad}(K_{\mathbf{C}})$-invariant and lies $\mathcal{N}_\theta$. As $K_{\mathbf{C}}$ acts on $\mathcal{N}_\theta$ with a finite number of orbits, $\mathrm{Ch}(V)$ is a union of the closures of finitely many nilpotent $K_{\mathbf{C}}$-orbits (in fact, all having the same dimension) in $\mathcal{N}_\theta$. Then the characteristic variety is written as a formal linear combination of these orbits with integer coefficients:

$$\mathrm{Ch}(V) = \sum \ \mathrm{mult}_{\mathrm{algebraic}}(V, \mathcal{O}_i)\, \overline{\mathcal{O}_i},$$

where $\mathcal{O}_i$ are $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$. The integers $\mathrm{mult}_{\mathrm{algebraic}}(V, \mathcal{O}_i)$ are the *multiplicities* in the characteristic cycle of $V$ at $\mathcal{O}_i$.

Vogan conjectured that the wave front cycle and the characteristic cycle are related. Recall that there is a one-to-one correspondence, referred to as the Sekiguchi correspondence, between $G$-orbits in $\mathcal{N}_{\mathbf{R}}$ and $K_{\mathbf{C}}$-orbits in $\mathcal{N}_\theta$. The relationship between the two types of cycles was proved by Schmid and Vilonen. Their result is the following.

**Theorem 2.1** ([14])**.** *Let $(V, \pi)$ be an irreducible admissible Harish-Chandra module and let $\Omega \subset \mathcal{N}_{\mathbf{R}}$ and $\mathcal{O} \subset \mathcal{N}_\theta$ be nilpotent orbits which are paired under the Sekiguchi correspondence, then $mult_{analytic}(V, \Omega) = mult_{algebraic}(V, \mathcal{O})$.*

Therefore, the problem of computing the multiplicities can be addressed in the algebraic setting.

The representations we wish to consider are the representations in the discrete series. Therefore we assume that $\mathrm{rank}(G) = \mathrm{rank}(K)$. This means we may choose a Cartan subalgebra $\mathfrak{h}_{\mathbf{C}}$ of $\mathfrak{g}_{\mathbf{C}}$ which is a Cartan subalgebra of $\mathfrak{k}_{\mathbf{C}}$. Fix $\Delta_c^+ \equiv \Delta^+(\mathfrak{k}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, a positive system for the roots of $\mathfrak{h}_{\mathbf{C}}$ in $\mathfrak{k}_{\mathbf{C}}$. Write $\rho_c$ for one half the sum of the roots in $\Delta_c^+$.

Let $X$ be the flag variety of $\mathfrak{g}_{\mathbf{C}}$, i.e., $X = \{\mathfrak{b} : \ \mathfrak{b} \text{ is a Borel subalgebra of } \mathfrak{g}_{\mathbf{C}}\}$. The closed $K_{\mathbf{C}}$-orbits in $X$ are in one-to-one correspondence with with the positive root systems $\Delta^+ \equiv \Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ containing $\Delta_c^+$. In particular, given such a positive system there is a Borel subalgebra $\mathfrak{b} = \mathfrak{h}_{\mathbf{C}} + \mathfrak{n}^-$ with $-\Delta^+ = \Delta^+(\mathfrak{n}^-, \mathfrak{h}_{\mathbf{C}})$. Then the corresponding closed $K_{\mathbf{C}}$-orbit in $X$ is $Z \equiv K_{\mathbf{C}} \cdot \mathfrak{b}$. Write $\rho$ for one half the sum of the roots in $\Delta^+$. The family $\{\pi_\lambda\}$ of discrete series representations associated to $Z$ (equivalently, associated to $\Delta^+$) is parameterized by $\lambda \in \mathfrak{h}_{\mathbf{C}}^*$ satisfying (i) $\lambda$ is regular and $\Delta^+$-dominant and (ii) $\lambda - \rho$ is analytically integral. The infinitesimal character of $\pi_\lambda$ has Harish-Chandra parameter $\lambda$ and the lowest $K$-type has highest weight $\lambda + \rho_n - \rho_c$ (where $\rho_n = \rho - \rho_c$). Considering all of the closed $K_{\mathbf{C}}$-orbits $Z$ in $X$, this gives a parameterization of the representations in the discrete series.

The main result on characteristic cycles for the discrete series representations is stated in terms of the moment map as follows. Write the conormal bundle of $Z$ in $X$ as $T_Z^*(X) = K_{\mathbf{C}} \underset{K_{\mathbf{C}} \cap B}{\times} (\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}})$ and observe that the moment map $\gamma$ restricted to $T_Z^*(X)$ is

$$\begin{aligned} \gamma : K_{\mathbf{C}} \underset{K_{\mathbf{C}} \cap B}{\times} (\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}) &\to \mathfrak{g}_{\mathbf{C}} \\ \gamma((k, Y)) &= k \cdot Y. \end{aligned} \tag{2.2}$$

It is well-known that $\gamma(T_Z^*(X))$ is a closed irreducible subvariety of $\mathcal{N}_\theta$, Indeed, $\gamma(T_Z^*(X))$ is the closure of a single nilpotent $K_{\mathbf{C}}$-orbit in $\mathfrak{p}_{\mathbf{C}}$.

**Definition 2.3.** We say that an element $f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ is *generic in* $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ if and only if $\gamma(T_Z^*(X)) = \overline{K_{\mathbf{C}}(f)}$.

When $f$ is generic in $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$, the closure $\overline{\mathcal{O}} = \gamma(T_Z^*(X))$ of the orbit $\mathcal{O} = K_{\mathbf{C}}(f)$ is an invariant of $\pi_\lambda$ called the *associated variety* of $\pi_\lambda$. We write $\mathrm{Av}(\pi_\lambda) = \overline{\mathcal{O}}$.

**Theorem 2.4.** ([6]) Let $\pi_\lambda$ be a discrete series representation of $G$ corresponding to a closed $K_{\mathbf{C}}$-orbit $Z$ in the flag variety $X$. Let $f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ be generic. Then,

$$\mathrm{Ch}(\pi_\lambda) = \dim(H^0(\gamma^{-1}(f), \mathcal{L}_{\lambda-\rho_c+\rho_n}|_{\gamma^{-1}(f)})) \cdot \gamma(T^*_Z(X)).$$

An important observation is that $\gamma^{-1}(f)$ may be identified with a (closed) subvariety of $Z$. To see this, suppose $f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ is generic and define

$$N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}) = \{k \in K_{\mathbf{C}} : k \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}\}.$$

The fiber of the moment map is given by

$$\begin{aligned} \gamma^{-1}(f) &= \{(k, Y) \in T^*_Z(X) : k \cdot Y = f\} \\ &= \{(k, k^{-1} \cdot f) : k^{-1} \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}\} \\ &\cong \{k \cdot \mathfrak{b} : k^{-1} \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}\} \\ &= (N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}))^{-1} \cdot \mathfrak{b} \subset Z. \end{aligned} \tag{2.5}$$

## 3. Associated variety of $\pi_\lambda$ when $G = SU(p,q)$

In this section we give an algorithm for finding $\mathrm{Av}(\pi_\lambda)$ for the indefinite unitary groups. This algorithm is given in terms of $\Delta^+$, the positive system of roots determined by $\pi_\lambda$ as above. Other algorithms are available that associate to discrete series representations the nilpotent orbit $\mathcal{O}$ with $\mathrm{Av}(\pi_\lambda) = \overline{\mathcal{O}}$. See for example [2], [16] and [19]. However, it is important for us that our algorithm allows us to give a description of the fibers $\gamma^{-1}(f)$ of the moment map. An elementary proof that the algorithm does in fact result in a generic element in $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ is the contained in Section 4. The description of the fiber $\gamma^{-1}(f)$ is given in Section 5

From now on we assume that $G = SU(p,q)$. We use the realization

$$\begin{aligned} &G = \{g \in M_{(p+q)\times(p+q)}(\mathbf{C}) : gI_{p,q}g^{-t} = I_{p,q}, \det(g) = 1\} \text{ where} \\ &I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}. \end{aligned}$$

The Cartan involution is chosen to be $\theta = \mathrm{Ad}(I_{p,q})$. Let $\mathfrak{h}_{\mathbf{C}} \subset \mathfrak{k}_{\mathbf{C}}$ be the diagonal Cartan subalgebra and let $\epsilon_j \in \mathfrak{h}^*_{\mathbf{C}}$ be, as usual, so that the roots of $\mathfrak{h}_{\mathbf{C}}$ in $\mathfrak{g}_{\mathbf{C}}$ are given by $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) = \{\epsilon_i - \epsilon_j : \ i \neq j\}$. As in Section 2, fix once and for all $\Delta^+_c \equiv \Delta^+(\mathfrak{k}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) = \{\epsilon_i - \epsilon_j \ : 1 \le i < j \le p \text{ or } p+1 \le i < j \le p+q\}$. For each positive system $\Delta^+$ containing $\Delta^+_c$ there is a Borel subalgebra $\mathfrak{b} = \mathfrak{h}_{\mathbf{C}} + \mathfrak{n}^-$ having nilradical spanned by the negative root vectors. The closed $K_{\mathbf{C}}$-orbit $Z = K_{\mathbf{C}}(\mathfrak{b})$ is therefore determined, as is a family $\{\pi_\lambda\}$ of discrete series representations.

We use the following well-known properties of $\mathrm{Av}(\pi_\lambda) = \overline{\mathcal{O}}$:

(1) $\mathcal{O} \cap (\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}})$ is open and dense in $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ and
(2) $\mathcal{O}$ is the unique largest dimensional nilpotent $K_{\mathbf{C}}$-orbit that intersects $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ non-trivially.

In order to implement Theorem 2.4, we need an algorithm that picks a "convenient" base point $f$ in $\mathcal{O}$ from $\Delta^+$. We also want to identify $\mathcal{O}$ by means of a signed tableau.

Before describing the algorithm it is convenient to recall a parameterization of nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$ for the group $SU(p,q)$; see for example [2] and [3]. Let $\{e, h, f\}$ be an $\mathfrak{sl}(2)$-triple so that $\theta(e) = -e$, $\theta(f) = -f$ and $\theta(h) = h$. Consider the group $Z_2 \rtimes SL(2,\mathbf{C})$ where the non-trivial element of $Z_2$ acts on $\{e, h, f\}$ as $\theta$ does. Irreducible representations of $SL(2,\mathbf{C})$ extend in two inequivalent ways to representations of $Z_2 \rtimes SL(2,\mathbf{C})$ according to whether $\theta$ acts by $\pm 1$ on the lowest weight vector. Define the signature of a (not necessarily irreducible) representation $\pi$ of $Z_2 \rtimes SL(2,\mathbf{C})$ to be the pair of integers $(a_+, a_-)$ where $a_\pm$ is the dimension of the $\pm 1$ eigenspace of $\theta$ in the kernel of $\pi(f)$.

Extend the representation of $SL(2,\mathbf{C})$ on $\mathbf{C}^{p,q}$ to a representation of $Z_2 \rtimes SL(2,\mathbf{C})$ so that the action of the nontrivial element of $Z_2$ is by $I_{p,q}$. Define $a_\pm(f^j)$ to be the dimension of the $\pm 1$ eigenspace of $I_{p,q}$ on the kernel of $\pi(f^j)$. Write $a(f^j) = a_+(f^j) + a_-(f^j)$ for the dimension of the kernel of $\pi(f^j)$. Decompose $\mathbf{C}^{p+q} = \oplus V_i$ into irreducible $Z_2 \rtimes SL(2,\mathbf{C})$ representations and let $\delta_i$ the eigenvalue of $\theta$ on the lowest weight vector of $V_i$. The nilpotent orbit $K_{\mathbf{C}}(f)$ is parameterized by the tableau with rows having lengths equal to the dimensions of the irreducible representations $V_i$ and alternate signs $+'s$ and $-'s$ starting with the sign of $\delta_i$. It is then clear that the number of $\pm$ signs in the first column is $a_\pm(f)$.

**Theorem 3.1.** ([8]) Two nilpotent elements $f$ and $f'$ are $K_{\mathbf{C}}$-conjugate if and only $a_\pm(f^j) = a_\pm(f'^j)$, for every $j$. The relation $\mathcal{O}(f') \subset \overline{\mathcal{O}(f)}$ holds if and only if for every $j$

$$a_+(f'^j) \geq a_+(f^j) \text{ and } \quad a_-(f'^j) \geq a_-(f^j).$$

**Lemma 3.2.** *A nilpotent element $f$ is generic in $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ if and only if for all $j$*

$$a_+(f^j) = min\{a_+(f'^j) : f' \in \gamma(T^*_Z(X))\} \text{ and}$$
$$a_-(f^j) = min\{a_-(f'^j) : f' \in \gamma(T^*_Z(X))\}$$

*Proof.* An element $f$ is generic if and only if $\gamma(T^*_Z(X)) = \overline{K_{\mathbf{C}}(f)}$. Thus, $f$ is generic if and only if $K_{\mathbf{C}}(f') \subset \overline{K_{\mathbf{C}}(f)}$ for any other $f' \in \gamma(T^*_Z(X))$. The lemma now follows from Theorem 3.1. $\square$

We next describe an algorithm which specifies a convenient generic element $f$ in $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$.

**The algorithm.** There is a one-to-one correspondence between positive systems $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ containing $\Delta^+_c$ and ordered sequences of integers $(p_1, q_1, p_2, \ldots, p_r, q_r)$ so that

$$\begin{aligned} &\Sigma p_i = p \text{ and } \Sigma q_i = q, \\ (3.3) \qquad &p_i, q_i \text{ are non-negative integers and} \\ &p_i > 0 \text{ for } i = 2,3,\ldots,r \text{ and } q_j > 0 \text{ for } j = 1,2,\ldots,r-1. \end{aligned}$$

Starting with such a sequence we form an array

$$\begin{array}{cccccc} \bullet_1 \cdots \bullet_{p_1} & & \bullet_{p_1+1} \cdots \bullet_{p_1+p_2} & & & \cdots \\ & \bullet_{p+1} \cdots \bullet_{p+q_1} & & & \bullet_{p+q_1+1} \cdots \bullet_{p+q_1+q_2} & \end{array}$$

We call a sequence of consecutive labeled dots in the array a *block*. Therefore, the blocks in the upper row have $p_i$ dots and those in the lower row have $q_i$ dots. The simple compact roots are the roots $\epsilon_i - \epsilon_{i+1}$ with $(i, i+1)$ indexes of dots that belong to the same block. The simple non-compact roots are the roots $\epsilon_i - \epsilon_j$ with $i, j$ indices of consecutive dots that lie in different rows, and so that $i$ precedes $j$ when reading the array from left to right. Thus, the simple non-compact roots correspond to the "jumps" between the rows. Here is an example. The array

$$\begin{array}{ccccccccc} \bullet_1 & \bullet_2 & & \bullet_3 & \bullet_4 & & & & \\ & & \bullet_5 & & & \bullet_6 & \bullet_7 & \bullet_8 & \bullet_9 \end{array}$$

determines the Dynkin diagram

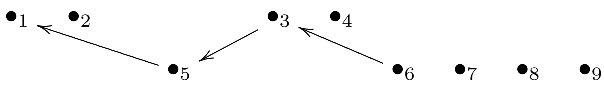
$$\begin{array}{ccccccccccccccc} \circ & — & \bullet & — & \bullet & — & \circ & — & \bullet & — & \circ & — & \circ & — & \circ \\ \text{1-2} & & \text{2-5} & & \text{5-3} & & \text{3-4} & & \text{4-6} & & \text{6-7} & & \text{7-8} & & \text{8-9} \end{array}$$

where $i - j$ means the root $\epsilon_i - \epsilon_j$ (and the blackened nodes correspond to noncompact simple roots).

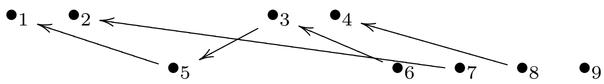
Our algorithm is as follows. From the sequence $\{p_1, q_1, \ldots, p_r, q_r\}$, form an array as above. Second, form a *string* consisting of diagonal lines connecting the first dots in each pair of consecutive blocks. Define a nilpotent element $f_0$ of $\mathfrak{n}^- \cap \mathfrak{p}$ as follows. Let $S_0 = \{i_1, i_2, \dots, i_N\}$ be the set of indices of dots which the string passes through, ordered from left to right. Then

$$f_0 = \sum_{s=2}^{N} X_{i_s, i_{s-1}}, \tag{3.4}$$

where $X_{i,j}$ is the matrix which is a root vector for $\epsilon_i - \epsilon_j$ with a one in the $(i,j)$ place. In the example, we have



Third, omit the dots that are vertices of the drawn string and repeat the procedure. The procedure is continued until no more diagonals can be drawn. In the example, we have



Note that as the dots in the most recent string are omitted, a new array is formed. For example, to choose the second string in the example we omit the dots numbered 1, 5, 3, and 6 to get

Each string corresponds to a sum of root vectors in $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$. In the example, we have

$$f_0 = (X_{5,1} + X_{3,5} + X_{6,3}), f_1 = X_{7,2} \text{ and } f_2 = X_{8,4}.$$

Set

$$f = f_0 + f_1 + \ldots + f_{m-1}, \text{ with m equal to the number of strings.}$$

**Theorem 3.5.** *Let $\{p_1, q_1, p_2, \dots, p_r, q_r\}$ be a sequence satisfying (3.3). Let $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be the positive system determined by the sequence $\{p_1, q_1, p_2, q_2, \dots, p_r, q_r\}$ and let $\mathfrak{b} = \mathfrak{h}_{\mathbf{C}} \oplus \mathfrak{n}^-$ be the corresponding Borel subalgebra. Set $Z = K_{\mathbf{C}}(\mathfrak{b})$ and let $f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ be the nilpotent element built by the algorithm. Then, $\overline{K_{\mathbf{C}}(f)} = \gamma(T^*_Z(X))$, i.e, $f$ is generic in $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$.*

We will prove this theorem in Section 5.

**Proposition 3.6.** *Let $\{p_1, q_1, p_2, q_2, \dots, p_r, q_r\}$ be a sequence satisfying (3.3) and $f = f_0 + f_1 + \ldots + f_{m-1}$ as constructed by the algorithm. Let $t$ be the number of dots that do not belong to any of the strings built by the algorithm. The signed tableau corresponding to the nilpotent $K_{\mathbf{C}}$-orbit $K_{\mathbf{C}}(f)$ has $m+t$ rows. If $1 \leq i \leq m$, then the length of the i-th row in the tableau is the number of dots occurring in the i-th string built by the algorithm. If the i-th string ends at a point in the top row of the array, then the i-row of the tableau has alternating signs starting with $+$. Otherwise, the i-row of the tableau has alternating signs starting with $-$. The remaining t rows have length one and their corresponding signs are so that the total number of $+$ signs in the tableau is p and the total number of $-$ signs is q.*

*Proof.* To prove the Proposition observe that for each string, $f_j$ is a principal nilpotent element in a subalgebra $\mathfrak{sl}(d_j, \mathbf{C})$ where $d_j$ is the number of dots in the corresponding string. Starting with $f_j$ it is possible to form an $\mathfrak{sl}(2)$-triple $\{f_j, h_j, e_j\}$ so that $h_j \in \mathfrak{h}_{\mathbf{C}}$ and $e_j = \sum_{\{(k,l):\ X_{l,k} \text{occurs in} f_j\}} a_{k,l} X_{k,l}$ with non-zero coefficients $a_{k,l}$. Since the $\mathfrak{sl}(d_j)$'s commute, $\{f, h = \sum h_j, e = \sum e_j\}$ spans a copy of $\mathfrak{sl}(2)$. Let $SL(2, \mathbf{C})_f$ be the Lie subgroup of $SU(p,q)$ whose Lie algebra is this copy of $\mathfrak{sl}(2)$. It is clear that the standard basis vectors $e_l \in \mathbf{C}^{p+q}$ are weight vectors for the action of $\mathbf{Z}_2 \rtimes SL(2,\mathbf{C})_f$ on $\mathbf{C}^{p+q}$. We may conclude

(1) the dimension of the non-trivial irreducible subrepresentations of $\mathbf{C}^{p+q}$ are given by the number of dots in the corresponding string,
(2) the lowest weight vector of an irreducible subrepresentation is $e_k$ where $k$ is the label of the last dot in the corresponding string,
(3) the trivial subrepresentations are spanned by the $t$ vectors $e_k$ so that no dot contained in any string has label $k$.

□

In our example the tableau corresponding to $K_{\mathbf{C}}(f)$ is

$$\begin{array}{|c|c|c|c|}\hline - & + & - & + \\ \hline - & + & \multicolumn{2}{c}{} \\ \cline{1-2} - & + & \multicolumn{2}{c}{} \\ \cline{1-2} - & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad .$$

In order to prove Theorem 3.5 we need some preliminary results on generic elements in $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$. The following definitions are important in what follows.

**Definition 3.7.** Starting with $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, let $S$ be the set of simple compact roots and let $\langle S\rangle$ be the set of roots generated by $S$. Define a parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$ by

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}^- \supset \mathfrak{b} \quad \text{with } \mathfrak{l} = \mathfrak{h} \oplus \sum_{\alpha \in \langle S\rangle} \mathfrak{g}^{\alpha} \text{ and } \mathfrak{u}^- = \sum_{\alpha \in \Delta^+ - \langle S\rangle} \mathfrak{g}^{-\alpha}.$$

Denote the connected subgroups of $G_{\mathbf{C}}$ corresponding to $\mathfrak{q}$, (resp., $\mathfrak{l}$) by $Q$ (resp., $L$). Observe that $L \subset K_{\mathbf{C}}$ and that $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}} = \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$. Let $Q_K$ be the parabolic subgroup $Q \cap K_{\mathbf{C}}$ of $K_{\mathbf{C}}$.

Start with a sequence $\{p_1, q_1, \ldots, p_r, q_r\}$ satisfying (3.3). This sequence determines an array and a positive system $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Use the algorithm to produce the first $i$ strings and delete the vertices of these strings from the array, keeping the numbering of the untouched dots. The resulting array determines a Dynkin diagram corresponding to a subalgebra $\mathfrak{g}_i$ isomorphic to some $\mathfrak{su}(p_i', q_i') \subset \mathfrak{su}(p,q)$. In particular, the new array defines a Borel subalgebra $\mathfrak{b}_i = \mathfrak{b} \cap \mathfrak{g}_{i,\mathbf{C}}$ of $\mathfrak{g}_{i,\mathbf{C}}$.

Let $S_i$ be the set of simple compact roots in the Dynkin diagram of $\mathfrak{g}_{i,\mathbf{C}} \subset \mathfrak{sl}(p+q,\mathbf{C})$ and let $\langle S_i\rangle$ be the set of roots generated by $S_i$. Define parabolic subalgebras of $\mathfrak{g}_{i,\mathbf{C}}$ by

$$\mathfrak{q}_i = \mathfrak{l}_i \oplus \mathfrak{u}_i^- \supset \mathfrak{b}_i, \quad \text{with } \mathfrak{l}_i = \mathfrak{h}_{\mathbf{C}} \oplus \sum_{\alpha \in \langle S_i\rangle} \mathfrak{g}^{\alpha}.$$

Denote the connected subgroups of $G_{\mathbf{C}}$ corresponding to $\mathfrak{q}_i$ (resp., $\mathfrak{l}_i$, $\mathfrak{u}_i^-$ resp.) by $Q_i$ (resp., $L_i$, $U_i$). Let $Q_{i,K} = Q_i \cap K_{\mathbf{C}}$.

*Remark* 3.8. The following properties follow easily.

(1) $Q_K$ normalizes $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$.
(2) $L_i \subset K_{\mathbf{C}}$ and $\mathfrak{u}_i^- \cap \mathfrak{p}_{\mathbf{C}} = \mathfrak{g}_{i,\mathbf{C}} \cap (\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}})$.
(3) It is not always the case that $\mathfrak{q} \cap \mathfrak{g}_{i,\mathbf{C}} = \mathfrak{q}_i$.
(4) Observe that, by construction, $\mathfrak{u}_i^- \subset \mathfrak{u}_{i-1}^-$.
(5) Writing $f = \Sigma_{j=0}^{m-1} f_j$ as in the algorithm, $Q_i \cdot f_k = f_k$ for all $k = 0, 1, \ldots, i-1$.

## 4. Generic elements.

Let $\{p_1, q_1, p_2, q_2, \dots, p_r, q_r\}$ be a sequence satisfying (3.3). From now on we assume without loss of generality that $p_1 \neq 0$. (See Remark 4.14.) Let $\{e_1, e_2, \dots e_{p+q}\}$ be the standard basis of $\mathbf{C}^{p+q}$. Let $f = f_0 + g$ with $g = f_1 + \dots + f_{m-1}$ be the nilpotent element built by the algorithm and form the $\mathfrak{sl}(2)$-triple $\{e, f, h\}$ with $e$ and $h$ as in the proof of Proposition 3.6. Let $(\pi, \mathbf{C}^{p+q})$ be the representation of $Z_2 \rtimes SL(p+q, \mathbf{C})$ for which the nontrivial element of $Z_2$ acts by $I_{p,q}$ and $SL(p+q, \mathbf{C})$ acts by the standard representation of $SL(p+q, \mathbf{C})$ on $\mathbf{C}^{p+q}$.

Each element $f_i$ corresponds to a string in the array given by the sequence $\{p_1, q_1, p_2, \dots, p_r, q_r\}$. If $A_1$ is the set of indexes labeling the vertices of the string corresponding to $f_0$, then we set $N = \#\{A_1\}$. An argument similar to that in the proof of Proposition 3.6 leads to the following Lemma.

**Lemma 4.1.** *Let $f = f_0 + g$ be as above. Let $V_0 = span_{\mathbf{C}}\{e_j : j \in A_1\} \subset \mathbf{C}^{p+q}$. Let $SL(2, \mathbf{C})_f$ correspond to the triple $\{f, e, h\}$. Under the action of $Z_2 \rtimes SL(2, \mathbf{C})_f$, $\mathbf{C}^{p+q}$ splits into invariant subspaces as $\mathbf{C}^{p+q} = V_0 \oplus W_0$ in such a way that*

$$\begin{aligned} \pi(f_0)|_{W_0} = 0 \quad &\textit{and} \quad \pi(g)|_{V_0} = 0 \\ \pi(f_0)V_0 \subset V_0 \quad &\textit{and} \quad \pi(g)W_0 \subset W_0. \end{aligned}$$

*Remark* 4.2. If $Y \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$, then we can write $Y = Y_1 + Y_2$ with $Y_1$ upper triangular and $Y_2$ lower triangular. In other words, $Y_1$ in the span of the roots spaces $\mathfrak{g}_{\epsilon_i - \epsilon_j}$ so that

$$\begin{aligned} (i,j) \in &\{(i,j) : p+1 \leq j \leq p + q_1 \text{ and } i \geq p_1 + 1\} \cup \\ &\cup_k \{(i,j) : p + q_1 + q_2 + \dots q_k + 1 \leq j \leq p + q_1 + q_2 + \dots q_{k+1} \text{ and } i \geq p_1 + \dots + p_{k+1} + 1\} \end{aligned}$$

and $Y_2$ is in the span of the roots spaces $\mathfrak{g}_{\epsilon_i - \epsilon_j}$ so that

$$\begin{aligned} (i,j) \in &\{(i,j) : 1 \leq j \leq p_1 \text{ and } i \geq p+1\} \cup \\ &\cup_k \{(i,j) : p_1 + \dots + p_k + 1 \leq j \leq p_1 + \dots + p_{k+1} \text{ and } i \geq p + q_1 + \dots + q_k + 1\}. \end{aligned}$$

Moreover, writing

$$Y_1 = \begin{pmatrix} 0 & Y_1' \\ 0 & 0 \end{pmatrix} \text{ and } Y_2 = \begin{pmatrix} 0 & 0 \\ Y_2' & 0 \end{pmatrix},$$

we have

$$\pi(Y) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} Y_1' \cdot v_2 \\ Y_2' \cdot v_1 \end{pmatrix}, \quad \text{where } \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbf{C}^{p+q}.$$

The following gives a flag which defines the parabolic subgroup $Q$.

**Definition 4.3.** For $j = 1, \dots, r$, define

$$\begin{aligned} \mathcal{U}_{2j-1} &= \text{span } \{e_i : (\Sigma_1^{j-1} p_k) + 1 \leq i \leq \Sigma_1^j p_k\} \\ \mathcal{U}_{2j} &= \text{span } \{e_i : p + (\Sigma_1^{j-1} q_k) + 1 \leq i \leq p + (\Sigma_1^j q_k)\} \\ F_k &= \bigoplus_{l=k}^{N} \mathcal{U}_l. \end{aligned}$$

**Proposition 4.4.** (1) *If $Y \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$, then $\pi(Y)(F_k) \subset F_{k+1}$ and $\pi(Y^s)(F_k) \subset F_{k+s}$.*
(2) *If $Y \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$, then $\pi(Y^{N-k+1})(F_k) = 0$. In particular $\pi(Y^N) = 0$.*
(3) *The spaces $F_k$ are preserved by the $Q_K$-action.*
(4) *The stabilizer of the flag $\mathbf{C}^{p+q} = F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \dots F_N \supsetneq F_{N+1} = \{0\}$ is $Q$.*

*Proof.* Part (1) of the proposition follows from the Remark 4.2 and the fact that $\pi(X_{l,t})e_j = \delta_{t,j}e_l$. Part (2) is obvious since $\pi(Y^{N-k+1})(F_k) \subset F_{N+1} = 0$. Part (3) is a consequence of the definition of the flag. To show part (4) observe on the one hand that the stabilizer of the flag is a parabolic subalgebra of $SL(p+q, \mathbf{C})$ with Levi component $S(\prod_i GL(p_i, \mathbf{C}) \times GL(q_i, \mathbf{C}))$. On the other hand, by (1) and (3), $Q$ is contained in the stabilizer. □

**Lemma 4.5.** *Assume that* $Y \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ *is so that* $K_{\mathbf{C}}(f) \subset \overline{K_{\mathbf{C}}(Y)}$. *Form an* $\mathfrak{sl}(2)$*-triple* $\{X, H, Y\}$ *with* $X \in \mathfrak{p}_{\mathbf{C}}$ *and* $H \in \mathfrak{k}_{\mathbf{C}}$ *and denote by* $SL(2, \mathbf{C})_Y$ *the copy of* $SL(2, \mathbf{C})$ *with Lie algebra* $\{X, H, Y\}$. *Then* $\mathbf{C}^{p+q}$ *has a* $Z_2 \rtimes SL(2, \mathbf{C})_Y$*-irreducible constituent of dimension* $N$.

*Proof.* By Proposition 4.4, $\pi(Y^N) \equiv 0$. Hence, $\mathbf{C}^{p+q}$ admits no constituent of dimension greater than $N$. Assume that $\mathbf{C}^{p+q} = R_1 \oplus \ldots \oplus R_t$ where $R_i$ are $Z_2 \rtimes SL(2, \mathbf{C})_Y$-irreducible subrepresentations. If $\max_i\{\dim(R_i)\} = N'$ with $N' < N$, then $\dim(\mathrm{Ker}\ (Y^{N'})) = p + q$. On the other hand, since $Z_2 \rtimes SL(2, \mathbf{C})_f$ admits an irreducible subrepresentation of $\mathbf{C}^{p+q}$ of dimension $N$, $\dim(\mathrm{Ker}(f^{N'})) < p + q$. This is a contradiction to the assumption that $K_{\mathbf{C}}(f) \subset \overline{K_{\mathbf{C}}(Y)}$, by Theorem 3.1. □

Continue with $Y$ as in the lemma. Decompose $\mathbf{C}^{p+q}$ under the $Z_2 \rtimes SL(2, \mathbf{C})_Y$-action as $\mathbf{C}^{p+q} = V_N \oplus W$ with $V_N$ irreducible of dimension $N$. Denote by $v_0$ the highest weight vector of $V_N$. The set $\{v_0, \pi(Y)v_0, \ldots, \pi(Y^{N-1})v_0\}$ is a basis for $V_N$.

**Lemma 4.6.** *For each* $k$, $F_k = (F_k \cap V_N) \oplus (F_k \cap W)$.

*Proof.* Write $v \in F_k$ as $v = v_N + w$ with $v_N = \sum_{j=0}^{N-1} a_j \pi(Y^j)v_0 \in V_N$ and $w \in W$. We need to show that $v_N$ and $w$ belong to $F_k$. It is enough to show that $v_N \in F_k$.

Observe that $0 = \pi(Y^{N-k+1})v = \pi(Y^{N-k+1})v_N + \pi(Y^{N-k+1})w$, so $0 = \pi(Y^{N-k+1})v_N = \sum_{j=0}^{k-2} a_j \pi(Y^{N-k+1+j})v_0$. Since the vectors $\{v_0, \pi(Y)v_0, \ldots, \pi(Y^{N-1})v_0\}$ are linearly independent, we have $a_j = 0$ for all $j \leq k-2$. Thus, $v_N = \sum_{j=k-1}^{N-1} a_j \pi(Y^j)v_0$ lies in $F_k$, by Proposition 4.4 (1). □

Since $(F_k \cap V_N) = \mathbf{C} \cdot \pi(Y^{k-1})v_0 + F_{k+1} \cap V_N$, we have the following corollary.

**Corollary 4.7.**

$$\dim (F_k \cap W)/(F_{k+1} \cap W) = \dim(F_k/F_{k+1}) - 1$$

**Definition 4.8.** Let $\mathbf{C}^{p+q} = F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \ldots F_N \supsetneq F_{N+1} = \{0\}$ be the flag introduced in Definition 4.3. Define,

$$P_i : F_i \to F_i/F_{i+1} \cong \mathcal{U}_i$$

to be the natural projections.

Write $\mathbf{C}^p \times \{0\}$ (resp., $\{0\} \times \mathbf{C}^q$) for the eigenspace of $I_{p,q}$ with eigenvalue $+1$ (resp., $-1$).

**Corollary 4.9.** *Assume that* $Y \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ *is so that* $K_{\mathbf{C}}(f) \subset \overline{K_{\mathbf{C}}(Y)}$. *Form* $Z_2 \rtimes SL(2, \mathbf{C})_Y$ *as above and decompose* $\mathbf{C}^{p+q}$ *as* $\mathbf{C}^{p+q} = V_N \oplus W$. *Then, there exists a basis* $\beta_1$ *of* $W$ *consisting of vectors in either* $\mathbf{C}^p \times \{0\}$ *or* $\{0\} \times \mathbf{C}^q$ *so that*

$$\#\{v \in \beta_1 : v \in F_k \cap W \text{ and } P_k(v) \neq 0\} = \dim(F_k/F_{k+1}) - 1. \tag{4.10}$$

*Proof.* Since the spaces $W \cap F_k$ are $I_{p,q}$-stable it is possible to find vectors $\{w_{2j+1,i} : w_{2j+1,i} \in W \cap \mathbf{C}^p \times \{0\} \cap F_{2j+1}\}$ so that $\{P_{2j+1}(w_{2j+1,i})\}$ forms a basis for $(F_{2j+1} \cap W)/(F_{2j+2} \cap W)$. Similarly, we choose vectors $\{w_{2j,i} : w_{2j,i} \in W \cap (\{0\} \times \mathbf{C}^q) \cap F_{2j}\}$ so that $\{P_{2j}(w_{2j,i})\}$ is a basis of $(F_{2j} \cap W)/(F_{2j+1} \cap W)$.

We claim that the set of vectors $\{w_{2j,i}\} \cup \{w_{2j+1,i}\}$ forms a basis for $W$ with the desired properties. First, we argue that the selected vectors are linearly independent. Indeed, if $\Sigma_{k,i} \lambda_{k,i} w_{k,i} = 0$, then

$P_1(\Sigma_{k,i}\lambda_{ki}w_{k,i}) = \Sigma_i\lambda_{1,i}P_1(w_{1,i}) = 0$. Since the vectors $\{P_1(w_{1,i})\}$ are linearly independent, it follows that $\lambda_{1,i} = 0$ for all $i$. Assume we have shown that $\lambda_{k,i} = 0$ whenever $k \le j$. Then, $0 = P_{j+1}(\Sigma_{k,i}\lambda_{ki}w_{k,i}) = \Sigma_i\lambda_{j+1,i}P_{j+1}(w_{j+1,i})$ and once again we get $\lambda_{j+1,i} = 0$ for all $i$.

To prove that the set $\{w_{2j,i}, w_{2j+1,i}\}$ is a maximal set of linearly independent vectors, observe that $\#\{w_{2j,i}, w_{2j+1,i}\} = \Sigma_1^N[\dim(F_k/F_{k+1}) - 1] = p + q - N = \dim(W)$. □

Define a basis of $\mathbf{C}^{p+q}$ as follows. Assume that $Y \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ is so that $K_{\mathbf{C}}(f) \subset \overline{K_{\mathbf{C}}(Y)}$. By Lemma 4.5, $\mathbf{C}^{p+q} = V_N \oplus W$ as a $Z_2 \rtimes SL(2,\mathbf{C})_Y$-representation, where $V_N$ is irreducible of dimension $N$. Let $v_0$ be the highest weight vector in $V_N$. Form $S_0 = \{v_0, \pi(Y)v_0, \ldots, \pi(Y^{N-1})v_0\}$, a basis for $V_N$. Let $\beta_1$ be the basis of $W$ built in Corollary 4.9. Construct an ordered basis $\beta$ of $\mathbf{C}^{p+q}$ from $S_0$ and $\beta_1$ in the following manner.

(1) The first vector in $\beta$ is $v_0$, the vector in position $p_1 + 1$ is $\pi(Y^2)v_0$, in position $p_1 + p_2 + 1$ is the vector $\pi(Y^4)v_0$ and so on so that in position $\Sigma_{k=1}^j p_k + 1$ is the vector $\pi(Y^{2j})v_0$; in position $p+1$ we place the vector $\pi(Y)v_0$, in position $p + (\Sigma_{k+1}^j q_k) + 1$ we place $\pi(Y^{2j+1})v_0$.
(2) Between $\pi(Y^{2j})v_0$ and $\pi(Y^{2j+2})v_0$ we place the vectors $\{w_{2j+1,i}\}$ in $\beta_1$.
(3) Between $\pi(Y^{2j-1})v_0$ and $\pi(Y^{2j+1})v_0$ we place the vectors $\{w_{2j,i}\}$ in $\beta_1$.

**Lemma 4.11.** *Let $\beta$ be the ordered basis of $\mathbf{C}^{p+q}$ just introduced. Let $\{e_1, e_2, \ldots, e_{p+q}\}$ be the ordered standard basis of $\mathbf{C}^{p+q}$. If $T : \mathbf{C}^{p+q} \to \mathbf{C}^{p+q}$ is a linear transformation that sends vectors of the standard ordered basis to vectors in the basis $\beta$ preserving the order, i.e first vector goes to first vector and so on, then there is a $q \in Q_K$ so that $T = \pi(q)$.*

*Proof.* Since $p_1 \neq 0$, $v_0 \in \mathbf{C}^p \times \{0\}$ and the linear transformation $T$ is an isomorphism so that $T : \mathbf{C}^p \times \{0\} \to \mathbf{C}^p \times \{0\}$ and $T : \{0\} \times \mathbf{C}^q \to \{0\} \times \mathbf{C}^q$. Hence, $T = \pi(k)$ for some $k \in K_{\mathbf{C}}$. On the other hand, by the construction of the basis $\beta$, such a $T$ preserves the flag $\mathbf{C}^{p+q} = F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \ldots F_N \supsetneq F_{N+1} = \{0\}$. Since the stabilizer of this flag is $Q$ it follows that $k \in Q \cap K_{\mathbf{C}} = Q_K$. □

**Proposition 4.12.** *Assume that $Y \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ is so that $K_{\mathbf{C}}(f) \subset \overline{K_{\mathbf{C}}(Y)}$, then there exists $q \in Q_K$ so that $q \cdot Y = f_0 + y_1$ with $f_0$ as in Lemma 4.1 and $y_1 \in \mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$.*

*Proof.* Without los of generality we assume that $\mathfrak{b}$ is determined by an array so that $p_1 \neq 0$. Form $Z_2 \rtimes SL(2,\mathbf{C})_Y$ and decompose $\mathbf{C}^{p+q}$ as $\mathbf{C}^{p+q} = V_N \oplus W$ where $V_N$ is $Z_2 \rtimes SL(2,\mathbf{C})_Y$-irreducible of dimension $N$. The existence of such decomposition is guaranteed by Lemma 4.5. Similarly, decompose $\mathbf{C}^{p+q}$ as $\mathbf{C}^{p+q} = V_0 \oplus W_0$ with $V_0 = \Sigma_{i\in A_1}\mathbf{C}e_i$ and $W_0 = \operatorname{span}\{e_i : i \notin A_1\}$. Let $q \in Q_K$ be as in Lemma 4.11. Then, the map $\pi(q)^{-1}\pi(Y)\pi(q) : \mathbf{C}^{p+q} = V_0 \oplus W_0 \to V_0 \oplus W_0$ is so that

(1) $\pi(q)^{-1}\pi(Y)\pi(q)W_0 \to W_0$,
(2) $[\pi(q)^{-1}\pi(Y)\pi(q)]|_{V_0} = \pi(f_0)|_{V_0}$.

Now, $\pi(f_0)|_{W_0} \equiv 0$, $\pi(q^{-1}Yq - f_0)$ preserves $W_0$ and $\pi(q^{-1}Yq - f_0)|_{V_0} \equiv 0$. Hence, $q^{-1}Yq - f_0 \in (\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}) \cap \mathfrak{g}_{1\mathbf{C}} = \mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$. □

**Proposition 4.13.** *Assume that $Y \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ is so that $K_{\mathbf{C}}(f) \subset \overline{K_{\mathbf{C}}(Y)}$, then there exist elements $q \in Q_K$ and $q_i \in Q_{i,K}$ so that $q_{m-1}q_{m-2}\ldots q_2q_1q \cdot Y = f$.*

*Proof.* By Proposition 4.12, there exists $q \in Q_K$ so that $q \cdot Y = f_0 + y_1$ with $y_1 \in \mathfrak{n}_1^- \cap \mathfrak{p}_{\mathbf{C}}$. Form $Z_2 \rtimes SL(2,\mathbf{C})_{q\cdot Y}$. By Proposition 4.12, we have $\mathbf{C}^{p+q} = V_0 \oplus W_0$ as $Z_2 \rtimes SL(2,\mathbf{C})_{q\cdot Y}$-representation and $\pi(f_0)W_0 \equiv 0$, $\pi(f_0)V_0 \subset V_0$, $\pi(y_1)V_0 \equiv 0$ and $\pi(y_1)W_0 \subset W_0$. Hence, $\operatorname{Ker}(\pi(q \cdot Y)^j) = \operatorname{Ker}(\pi(f_0^j)|_{V_0}) \oplus \operatorname{Ker}(\pi(y_1^j)|_{W_0})$ and

$$a_\pm(Y^j) = a_\pm((f_0|_{V_0})^j) + a_\pm((y_1|_{W_0})^j).$$

On the other hand, $f = f_0 + g$ with $g = f_1 + \ldots + f_{m-1}$ and

$$a_\pm(f^j) = a_\pm((f_0|_{V_0})^j) + a_\pm((g|_{W_0})^j).$$

The assumption $K_{\mathbf{C}}(f) \subset \overline{K_{\mathbf{C}}(Y)}$ and Theorem 3.1 imply that $a_\pm(g^j) \geq a_\pm(y_1^j)$. The vectors $g$ and $y_1$ belong to $\mathfrak{n}_1^- \cap \mathfrak{p}_{\mathbf{C}} = \mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}} \subset \mathfrak{q}_1$, where $\mathfrak{q}_1$ is a parabolic subalgebra of a smaller $\mathfrak{g}_1 \cong \mathfrak{su}(p'_1, q'_1)$. If $K'_{1,\mathbf{C}}$ is the complexification of the maximal compact subgroup of $G_1$, then $K'_{1,\mathbf{C}}(g) \subset \overline{K'_{1,\mathbf{C}}(y_1)}$. Applying Proposition 4.12 to $(g, y_1)$ we obtain $q_1 \in Q_{1,K}$ so that $q_1 \cdot y_1 = f_1 + y_2$ with $y_2 \in (\mathfrak{n}_2^- \cap \mathfrak{p}_{\mathbf{C}})$. The construction of $Q_{1,K}$ guarantees that $q_1 \cdot f_0 = f_0$. Thus, $q_1 q_K \cdot Y = q_1 \cdot f_0 + f_1 + y_2 = f_0 + f_1 + y_2$. The proof of the theorem now follows by induction on the complex rank. □

*Remark* 4.14. Assuming $p_1 \neq 0$ has no loss of generality. If $p_1 = 0$, then there is an isomorphism $\sigma : \mathfrak{su}(p, q) \to \mathfrak{su}(q, p)$ preserving the diagonal Cartan subalgebra and sending the positive system attached to $\{p_1 = 0, q_1, \ldots, q_r\}$ to a positive system $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ corresponding to a sequence $\{p'_1, q'_1, \ldots, q'_r\}$ with $p'_1 = q_1 \neq 0$. Now we may apply Proposition 4.13 in the case $p'_1 \neq 0$, $\mathfrak{g}' = \mathfrak{su}(q, p)$, to get the same result for $p_1 = 0$, $\mathfrak{g} = \mathfrak{su}(p, q)$.

*Proof of Theorem 3.5.* Assume that $Y \in \mathfrak{n}^- \cap \mathfrak{p}_C$ is generic. Then $K_{\mathbf{C}}(Y)$ is the unique maximal dimensional nilpotent $K_{\mathbf{C}}$-orbit that meets $\mathfrak{n}^- \cap \mathfrak{p}_C$. Hence, $K_{\mathbf{C}}(f) \subset \overline{K_{\mathbf{C}}(Y)}$. By Proposition 4.13, there exist $k_0 = q_{m-1} \ldots q_1 q_k \in K_{\mathbf{C}}$ so that $k_0 \cdot Y = f$. Hence, $K_{\mathbf{C}}(Y) = K_{\mathbf{C}}(f)$, i.e., $f$ is generic. □

## 5. The fiber of the moment map

In this section we use the algorithm developed in Section 4 to determine the structure of the fiber of the moment map.

**Theorem 5.1.** *With the notation of Section 4,*

$$\gamma^{-1}(f) = (Z_{K_{\mathbf{C}}}(f) Q_{m-1,K} Q_{m-2,K} \ldots Q_{1,K} Q_K) \cdot \mathfrak{b} \subset K_{\mathbf{C}}/K_{\mathbf{C}} \cap B.$$

*Proof.* By equation (2.5), $\gamma^{-1}(f) = (N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}))^{-1} \cdot \mathfrak{b}$ where

$$N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}) = \{k \in K_{\mathbf{C}} : k \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}\}.$$

To prove the theorem it is enough to show that $N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}) = Q_K Q_{1,K} \ldots Q_{m-1,K} Z_{K_{\mathbf{C}}}(f)$.

To show that $N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}) \subset Q_K Q_{1,K} \ldots Q_{m-1,K} Z_{K_{\mathbf{C}}}(f)$, take $k_0 \in N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}})$. Then, $k_0 \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ is generic and by Proposition 4.13, there exist $q_i \in Q_{i,K}$ and $q \in Q_K$ so that $q_{m-1} q_{m-2} \ldots q_1 q(k_0 \cdot f) = f$. Thus, $q_{m-1} q_{m-2} \ldots q_1 q k_0 \in Z_{K_{\mathbf{C}}}(f)$. The inclusion follows.

To show the other inclusion observe that $Q_K$ normalizes $\mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ and $Z_{K_{\mathbf{C}}}(f)$ fixes $f$. Hence, it is enough to show that $Q_{1,K} Q_{2,K} \ldots Q_{m-1,K} \subset N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}})$.

Write $f = f_0 + f_1 + \ldots + f_i + \ldots + f_{m-1}$ with $f_0 \in \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}}$ and $f_i \in \mathfrak{u}_i^- \cap \mathfrak{p}_{\mathbf{C}}$ for $i \geq 1$. Recall (Remark 3.8) that $Q_{i,K}$ normalizes $\mathfrak{u}_i^- \cap \mathfrak{p}_{\mathbf{C}}$, $\mathfrak{u}_i^- \cap \mathfrak{p}_{\mathbf{C}} \subset \mathfrak{u}_{i-1}^- \cap \mathfrak{p}_{\mathbf{C}}$ and $Q_{i,K}$ stabilizes all $f_j$ with $j < i$. Then,

$$\begin{aligned} Q_{m-1,K}(f) &\subset f_0 + f_1 + \ldots + f_{m-2} + Q_{m-1,K}(f_{m-1}) \\ &\subset f_0 + f_1 + \ldots + f_{m-2} + (\mathfrak{u}_{m-1}^- \cap \mathfrak{p}_{\mathbf{C}}). \end{aligned}$$

Assume we have shown that

$$Q_{i,K} Q_{i+1,K} \ldots Q_{m-1,K}(f) \subset f_0 + f_1 + \ldots + f_{i-1} + (\mathfrak{u}_i^- \cap \mathfrak{p}_{\mathbf{C}}).$$

Then,

$$\begin{aligned}Q_{i-1,K}Q_{i,K}Q_{i+1,K}\dots Q_{m-1,K}(f) &\subset f_0+f_1+\dots+f_{i-2}+Q_{i-1,K}f_{i-1}+(\mathfrak{u}_i\cap\mathfrak{p}_{\mathbf{C}})\\ &\subset f_0+f_1+\dots+f_{i-2}+Q_{i-1,K}\big(f_{i-1}+(\mathfrak{u}_{i-1}\cap\mathfrak{p}_{\mathbf{C}})\big)\\ &\subset f_0+f_1+\dots+f_{i-2}+(\mathfrak{u}_{i-1}\cap\mathfrak{p}_{\mathbf{C}})\end{aligned}$$

Thus, by induction we conclude that $Q_{1,K}Q_{2,K}\dots Q_{m-1,K}(f)\subset(f_0+(\mathfrak{u}_1\cap\mathfrak{p}_{\mathbf{C}}))\subset\mathfrak{n}^-\cap\mathfrak{p}_{\mathbf{C}}$ and the theorem follows. □

*Remark* 5.2. In Theorem 5.1, $Q_{m-1,K}Q_{m-2,K}\dots Q_{1,K}Q_K$ may be replaced by $L_{m-1}L_{m-2}\dots L_1L$.

The next theorem (Theorem 5.8) makes structure of the fiber of the moment map much more tractable. It essentially says that the centralizer may be dropped from the expression for the fiber in the above theorem. We must however include $L_m$, which is formed in the algorithm for the generic element $f$ after the last string is formed. Note that $L_m$ is contained in the centralizer of $f$ (thus not included previously) and $\Delta(\mathfrak{l}_m)$ consists of roots with indices not in any of the strings.

The proof will require a fairly explicit description of the centralizer of $f$, and this will require some (temporary) notation.

Recall that $m$ is the number of strings. These strings give $f=f_0+\dots+f_{m-1}$. For $a=0,1,\dots,m-1$ define $S_a=\{$indices in the string from which $f_a$ is formed$\}$. In other words $S_a$ is the set of indices occurring in the root vectors in the expression for $f_a$. Let $S_m$ be the set of indices not occurring in any of the strings. Now set

$$V_{a,b}=\mathrm{span}_{\mathbf{C}}\{X_{i,j}\,:\,i\in S_a,j\in S_b\}.$$

Recall that $X_{i,j}$ is the root vector with a 1 in the $(i,j)$-place and zeros elsewhere. Let $\mathfrak{z}=\mathfrak{z}_{\mathfrak{k}_{\mathbf{C}}}$ and set

$$\mathfrak{z}_{a,b}=\mathfrak{z}\cap V_{a,b}.$$

Since $V_{a,b}$ is $\mathrm{ad}(f)$-invariant

$$\mathfrak{z}=\oplus\mathfrak{z}_{a,b}.$$

In fact, $V_{a,b}$ is invariant under the $\mathfrak{sl}(2,\mathbf{C})$ corresponding to $f$.

Consider one of the $S_a$'s. Write $S_a=\{i_1,\dots,i_R\}$ ordered so that so that each $i_r$ occurs to the left of $i_{r+1}$ in the array. Therefore,

$$f_a=\sum_{r=2}^{R}X_{i_r,i_{r-1}}.$$

Similarly, write

$$f_b=\sum_{t=2}^{T}X_{i_t,i_{t-1}}.$$

We now find a basis of $\mathfrak{z}$ by finding a basis for each $\mathfrak{z}_{a,b}$. There are 5 different cases which must be considered.

Case (1), $a \neq b$ and $a, b \neq m$. Let $X = \sum a_{ij} X_{i,j} \in V_{a,b}$. We see when $X$ commutes with $f$.

$$\begin{aligned}
[f, X] &= [f_a, X] + [f_b, X] \\
&= f_a X - X f_b \\
&= \sum_{r=2}^{R} \sum_{i \in S_a} \sum_{j \in S_b} a_{ij} X_{i_r, i_{r-1}} X_{i,j} - \sum_{t=2}^{T} \sum_{i \in S_a} \sum_{j \in S_b} a_{ij} X_{i,j} X_{j_t, j_{t-1}} \\
&= \sum_{r=2} \sum_{j \in S_b} a_{i_{r-1}, j} X_{i_r, j} - \sum_{t=2}^{T} \sum_{i \in S_a} a_{i, j_t} X_{i, j_{t-1}} \\
&= \sum_{r=2}^{R} \sum_{t=1}^{T-1} (a_{i_{r-1}, j_t} - a_{i_r, j_{t+1}}) X_{i_r, j_t} + \sum_{r=2}^{R} a_{i_{r-1}, j_T} X_{i_r, j_T} - \sum_{t=2}^{T} a_{i_1, j_t} X_{i_1, j_t}.
\end{aligned}$$

This is 0 precisely when

$$\begin{aligned}
a_{i_r, j_T} &= 0, \text{for } r = 1, \dots, R-1, \\
a_{i_1, j_t} &= 0, \text{ for } t = 2, \dots, T \text{ and} \\
a_{i_r, j_t} &= a_{i_{r+1}, j_{t+1}}, \text{ for } r = 1, \dots, R-1,\ t = 1, \dots, T-1.
\end{aligned}$$

Therefore, the centralizer of $f$ in $V_{a,b}$ is spanned by

$$\sum_{s=1}^{n} X_{i_{R-n+s}, j_s}, \text{ for } n = 1, \dots, R, \text{ when } R \leq T \tag{5.3}$$

and by

$$\sum_{s=1}^{n} X_{i_{R-n+s}, j_s}, \text{ for } n = 1, \dots, T, \text{ when } R \geq T. \tag{5.4}$$

Case (2), $a = b \neq m$. Essentially the same calculation as in Case (1) (with $R = T$) gives a basis for the centralizer of $f$ in $V_{a,a}$ as

$$\sum_{s=n}^{R} X_{i_{R-n+s}, j_s}, \text{ for } n = 1, \dots, R\,(= T). \tag{5.5}$$

Case (3), $a \neq b$, $b = m$. A similar calculation gives a basis for the centralizer of $f$ in $V_{a,m}$ as

$$X_{i_R, j},\ j \in S_m. \tag{5.6}$$

Case (4), $a \neq b$, $a = m$. A basis for the centralizer of $f$ in $V_{m,b}$ is

$$X_{i, j_1},\ i \in S_m. \tag{5.7}$$

Case (5), $a = b = m$. Then $V_{a,b}$ commutes with $f$ by the construction construction of $f$.

**Theorem 5.8.** *If $f$ is the generic element constructed by the algorithm then*

$$\gamma^{-1}(f) = L_m \cdots L_2 L_1 L(\mathfrak{b}) \subset K_{\mathbf{C}} / K_{\mathbf{C}} \cap B.$$

*Proof.* The proof is by induction on $m$, the number of strings in the array determined by $\Delta^+$. We have given a basis for the centralizer $\mathfrak{z}_{\mathfrak{k}_{\mathbf{C}}}$ in (5.3-5.7). Since $Z_{K_{\mathbf{C}}}$ is connected (a special fact for the indefinite unitary groups), $Z_{K_{\mathbf{C}}}$ is generated by $\exp(tZ)$ with $t \in \mathbf{R}$ and $Z$ in the basis described in (5.3-5.7). Therefore, by Theorem 5.1 and Remark 5.2 it suffices to show that for such $Z$

$$\exp(tZ) L_m \cdots L_2 L_1 Q \subset L_m \cdots L_2 L_1 Q. \tag{5.9}$$

There four number of cases.

Case (1): $Z \in \mathfrak{z}_{a,b}, 1 \leq a, b \leq m$. This puts us in the situation of $f' = f - f_0$ ($m-1$ strings) inside $G_1$. By induction

$$\exp(tZ)L_m \cdots L_2 Q_1 \subset L_m \cdots L_2 Q_1.$$

Therefore,

$$\begin{aligned}\exp(tZ)L_m \cdots L_2 L_1 Q &= \exp(tZ)L_m \cdots L_2 Q_1 Q \\ &\subset L_m \cdots L_2 Q_1 Q \\ &= L_m \cdots L_2 L_1 Q.\end{aligned}$$

Case (2): $Z \in \mathfrak{z}_{0,0}$. Each of the root vectors occurring in $Z$ is in $\mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}}$ by (5.5). Also, note that $Z$ commutes with each $L_k$, therefore (5.9) holds.

The final two cases are $\mathfrak{z}_{a,0}$ and $\mathfrak{z}_{0,a}$, $a > 0$. The proofs of (5.9) in these two cases require some preparation.

Recall that the array consists of a number of blocks and the string defining $f_0$ passes through each block. Now consider the strings defining $f_c$ for $c = 1, 2, \ldots, m-1$. Define an equivalence relation on the set $\{1, 2, \ldots, p+q\}$ of indices by $i \sim j$ if and only if either (i) $1 \leq i, j \leq p$ and there exists no $\ell \in S_c$ so that $p+1 \leq \ell \leq p+q$ and $\epsilon_i - \epsilon_\ell$ and $\epsilon_\ell - \epsilon_j$ are both positive or both negative, or (ii) $p+1 \leq i, j \leq p+q$ and there exists no $\ell \in S_c$ so that $1 \leq \ell \leq p$ and $\epsilon_i - \epsilon_\ell$ and $\epsilon_\ell - \epsilon_j$ are both positive or both negative. We call the equivalence classes *$c$-blocks*.

Now define a Levi subalgebra of $\mathfrak{k}_{1,\mathbf{C}}$ by specifying its roots: $\Delta(\mathfrak{m}_c)$ contains $\epsilon_i - \epsilon_j$ if and only if $i, j \notin S_0$ and $i, j$ are in the same $c$-block. Let $M_c$ be the connected subgroup of $K_{1,\mathbf{C}}$ with Lie algebra $\mathfrak{m}_c$. Note that for $k = 1, 2, \ldots, c$, $\Delta(\mathfrak{l}_k) \subset \Delta(\mathfrak{m}_c)$. Therefore,

$$L_c \cdots L_2 L_1 \subset M_c.$$

In the remaining two cases we will show that $[\mathfrak{m}_a, \mathfrak{z}_{a,0}] \subset \mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}}$ and $[\mathfrak{m}_a, \mathfrak{z}_{0,a}] \subset \mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}}$. Then (5.9) will follow.

Case (3): $Z \in \mathfrak{z}_{a,0}$, $a \geq 1$. First suppose that $a \neq m$. Then, as in (5.3), $Z$ is a linear combination of root vectors $X_{i_{R+s-n}, j_s}, n = 1, \ldots, R$. Since $j_s \in S_0$ and $f_0$ passes through each block in the array, $j_s$ is the label of the first dot in the $s^{th}$ block. It follows that for each $s = 1, \ldots, R$, $j_s$ is to the left of $i_s$ in the array, and therefore $j_s$ is also to the left of $i_{R-n+s}$. With this observation and the equivalence relation defining the $a$-blocks we will show that

$$[\mathfrak{m}_a, X_{i_{R-n+s}}] \in \mathfrak{q} \cap \mathfrak{k}. \tag{5.10}$$

Let $Y$ be a root vector in $\mathfrak{m}_a$. Then

$$[Y, X_{i_{R-n+s}, j_s}] \in \mathbf{C} X_{i', j_s} \tag{5.11}$$

with $i' \sim i_{R-n+s}$ (i.e., $i'$ and $i_{R+s-n}$ in the same $a$-block. If $s = 1$, then $j_s = j_1$ si the dot farthest to the left in the array, so $X_{i_{R+s-n}, j_s} \in \mathfrak{n}^- \cap \mathfrak{k} \subset \mathfrak{q} \cap \mathfrak{k}$. When $s > 1$, consider $\epsilon_{i'} - \epsilon_{j_s}$. Suppose $\epsilon_{i'} - \epsilon_{j_s}$ were positive. Then in the array $i'$ would be to the left of $j_s$, so also to the left of $j_{s-1}$. But $j_{s-1}$ is to the left of $i_{R+s-n-1}$ (by the above observation). Therefore $\epsilon_{i'} - \epsilon_{i_{R+s-n-1}} > 0$ and $\epsilon_{i_{R+s-n-1}} - \epsilon_{i_{R+s-n}} > 0$, and we have a contradiction to $i' \sim i_{R+s-n}$. We therefore have that $X_{i', j_s} \in \mathfrak{n}^- \cap \mathfrak{k} \subset \mathfrak{q} \cap \mathfrak{k}$.

From (5.11), it follows that $\mathrm{ad}(Y)^k (X_{i_{R-n+s}, j_s})$ is contained in the span of $X_{i, j_s}$ with $i \sim i_{R-n+s}$, so is in $\mathfrak{q} \cap \mathfrak{k}$. Therefore, $\mathrm{Ad}(\exp(Y))(X_{i_{R-n+s}, j_s}) \subset \mathfrak{q} \cap \mathfrak{k}$, and so $\mathrm{Ad}(M_a)(\exp(tZ)) \subset Q \cap K$, for $Z$ in the basis for $\mathfrak{z}_{a,0}$. In particular, for $\ell_k \in L_k, k = 1, 2, \ldots, a$,

$$\exp(tZ)\ell_a \cdots \ell_1 \in \ell_a \cdots \ell_1 Q \cap K.$$

Now, $\mathfrak{z}_{a,0}$ commutes with $L_m, \ldots, L_{a+1}$ (since these $\mathfrak{l}_k$ have no root vectors involving indices from $S_a$ and $S_0$). Therefore,

$$\begin{aligned} \exp(tZ) L_m \cdots L_1 Q_K &= L_m \cdots L_{a-1} \exp(tZ) L_a \cdots L_1 Q_K \\ &\subset L_m \cdots L_1 Q_K. \end{aligned}$$

Now if $a = m$, then $Z$ is a linear combination of root vectors $X_{i,j_1}$, $i \in S_m$. For any root vector $Y$ in $\mathfrak{k}_1$, $\mathrm{ad}(Y)^k(X_{i,j_1}) \in \mathfrak{q} \cap \mathfrak{k}$. So, $\mathrm{Ad}(K_{1,\mathbf{C}})(\exp(tZ)) \subset Q \cap K$. So (5.9) follows.
Case (4): $Z \in \mathfrak{z}_{0,b}$. This case is very similar to Case 5. Here, $Z$ is a sum of root vectors $X_{i_{R-n+s}, j_s}$, with $n = 1, \ldots, T$, as in (5.4). □

## 6. An algorithm for computing the multiplicity of $\pi_\lambda$

Let $\pi_\lambda$ be a representation in the discrete series of $SU(p,q)$. Then, as described in Section 2, $\pi_\lambda$ is associated to a closed $K_{\mathbf{C}}$-orbit $Z$ in the flag variety $X$. Also, there is a corresponding positive system $\Delta^+ = \Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ containing $\Delta_c^+$. Then, by Theorem 2.4 ([6, Prop. 1.4]), the characteristic cycle of $\pi_\lambda$ is given by

$$Ch(\pi_\lambda) = \dim(H^0(\gamma^{-1}(f), \mathcal{L}_{\lambda-\rho_c+\rho_n}|_{\gamma^{-1}(f)})) \cdot \gamma(T_Z^*(X)). \tag{6.1}$$

The goal of this section is to use this formula for the multiplicity along with our description of the fiber of the moment map (Theorem 5.8) to give an algorithm for computing the the multiplicities of discrete series representations of $SU(p,q)$.

First, we give a description of the cohomology space $H^0(\gamma^{-1}(f), \mathcal{L}_{\lambda-\rho_c+\rho_n}|_{\gamma^{-1}(f)})$ in terms of representations. Identify $Z$ with the flag manifold $K_{\mathbf{C}}/B \cap K_{\mathbf{C}}$ for the complex group $K_{\mathbf{C}}$ and we interprete $\gamma^{-1}(f)$ as the closed subvariety $N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}})^{-1} \cdot \mathfrak{b}$ of $K_{\mathbf{C}}/B \cap K_{\mathbf{C}}$.

Let $\mu = \lambda - \rho_c + \rho_n$ and view $\mu$ as a linear functional on $\mathfrak{b} \cap \mathfrak{k}_{\mathbf{C}}$ by setting $\mu(\mathfrak{u}^- \cap \mathfrak{k}_{\mathbf{C}}) = 0$. Since $\mu$ is analytically integral, $\chi_\mu(b) = \mathrm{e}^{\mu}(log(b))$ is a well defined character of $B \cap K_{\mathbf{C}}$. We denote by $\mathcal{O}(\mu)$ the $K_{\mathbf{C}}$-equivariant sheaf over $Z$ defined by the local sections

$$\Gamma(U, \mathcal{O}(\mu)) = \{f : \text{ regular function on } p_o^{-1}(U) \text{ and } f(gb) = \chi_\mu^{-1}(b) f(g), \text{ for } b \in B\}.$$

Let $\mathcal{O}_{\gamma^{-1}(f)}$ (resp., $\mathcal{O}_Z$) be the structure sheaf over $\gamma^{-1}(f)$ (resp., $Z$) and let $\mathcal{O}(\mu) = \mathcal{O}(\mu) \otimes_{\mathcal{O}_Z} \mathcal{O}_{\gamma^{-1}(f)}$. Then, the cohomology in (6.1) is the sheaf cohomology $H^0(\gamma^{-1}(f), \mathcal{O}_{\gamma^{-1}(f)}(\mu))$.

Now take $W_{-\mu}$ to be the irreducible representation of $K$ with lowest weight $-\mu$ and let $w_{-\mu}$ be the lowest weight vector in $W_{-\mu}$. The Borel-Weil theorem gives an isomorphism

$$H^0(Z, \mathcal{O}(\mu)) = W_{-\mu}^* \tag{6.2}$$

which is implemented using matrix coefficients by

$$\begin{aligned} W_{-\mu}^* &\to \Gamma(Z, \mathcal{O}(\mu)) \\ v &\to \langle v, k w_{-\mu} \rangle. \end{aligned}$$

The description of such a space of sections, along with the following definitions are in [10, page 590] We set $W_{\gamma^{-1}(f)} \equiv \{z w_{-\mu} : z \in N_{K_{\mathbf{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}_{\mathbf{C}})^{-1}\}$, $q_{\gamma^{-1}(f)}(\mu) \equiv \dim(\mathrm{span}_{\mathbf{C}} W_{\gamma^{-1}(f)})$ and $W_{\gamma^{-1}(f)}^{\perp} \equiv \{f \in W_{-\mu}^* : f(a) = 0 \text{ for all } a \in W_{\gamma^{-1}(f)}\}$.

As in [10], we conclude from Serre's theorem [9, page 228], that for $\lambda$ sufficiently dominant

$$\begin{aligned} H^0(\gamma^{-1}(f), \mathcal{O}_{\gamma^{-1}(f)}(\mu)) &= \Gamma(Z, \mathcal{O}(-\mu)) / \{g \in \Gamma(Z, \mathcal{O}(-\mu)) : g(\gamma^{-1}(f)) = 0\} \\ &= W_{-\mu}^* / W_{\gamma^{-1}(f)}^{\perp}. \end{aligned}$$

Therefore, for sufficiently dominant $\lambda$

$$\dim H^0(\gamma^{-1}(f), \mathcal{O}_{\gamma^{-1}(f)}(\mu)) = q_{\gamma^{-1}(f)}(\mu). \tag{6.3}$$

Putting together the above observations and using our description of the fiber $\gamma^{-1}(f)$ in Section 5, we conclude the following.

**Proposition 6.4.** *The multiplicity of* $\overline{\mathcal{O}} = \overline{K_{\mathbf{C}}(f)}$ *in the associated cycle of the discrete series representation* $\pi_\lambda$ *is* $\dim_{\mathbf{C}}(span_{\mathbf{C}}(\{z \cdot w_\mu : z \in L_m L_{m-1} \dots L_1 L\})$*, provided* $\lambda$ *is very dominant.*

Recall $q_{\Delta^+}(\mu) = \dim(\mathrm{span}_{\mathbf{C}}\{L_m \cdots L_1 L(w_{-\mu})\})$. Then, by Joseph's argument, $q_{\Delta^+}(\mu)$ is the multiplicity of $\pi_\lambda$ (where $\mu = \lambda + \rho_n - \rho_c$) when $\mu$ is dominant enough. Since the multiplicity of $\pi_\lambda$ is a polynomial in $\lambda$ we will see that $q_{\Delta^+}(\mu)$ is the multiplicity for *all* $\lambda$ if we can show that $q_{\Delta^+}(\mu)$ extends to a polynomial in $\mu$. This is contained in the following theorem.

**Theorem 6.5.** $q_{\Delta^+}(\mu)$*, defined for* $\mu$ *integral and* $\Delta_c^+$*-dominant, extends to a polynomial on* $\mathfrak{h}^*$*. Therefore,* $q_{\Delta^+}(\lambda + \rho_n - \rho_c)$ *is the multiplicity of discrete series representations* $\pi_\lambda$ *corresponding to* $\Delta^+$*.*

*Proof.* We proceed by induction on $m$. If $m = 1$ there is just one string in the array determined by $\Delta^+$ and $L_1 \subset L$. Therefore, $U_{-\mu} \equiv \mathrm{span}_{\mathbf{C}}\{L(w_{-\mu})\}$ is the irreducible $L$-representation of lowest weight $-\mu$. The dimension extends to a polynomial on $\mathfrak{h}^*$ by the Weyl dimension formula.

Now consider $m > 1$. Decompose $U_{-\mu}$ as a representation of $L_1 \cap L$. Write this decomposition as $\sum E_{-\mu_i}$ and write the lowest weight vectors as $w_{-\mu_i}$.
Claim: each $w_{-\mu_i}$ is annihilated by $\mathfrak{n}^- \cap \mathfrak{g}_1 \cap \mathfrak{k}$. To see this, note that since $L$ normalizes $\mathfrak{u}^- \cap \mathfrak{k}$ and $w_{-\mu}$ is annihilated by $\mathfrak{u}^- \cap \mathfrak{k}$, each $w_{-\mu_i}$ (in fact all of $U_{-\mu}$) is annihilated by $\mathfrak{u}^-$. Now each $w_{-\mu_i}$ is annihilated by $\mathfrak{n}^- \cap \mathfrak{l}_1 \cap \mathfrak{l}$. But, $\mathfrak{n}^- \cap \mathfrak{g}_1 = \mathfrak{n}^- \cap \mathfrak{q} \subset \mathfrak{u}^- \cap \mathfrak{k} + \mathfrak{n}^- \cap \mathfrak{l}_1 \cap \mathfrak{l}$.

The claim tells us that $F_{-\mu_i} \equiv \mathrm{span}_{\mathbf{C}}\{G_1(w_{-\mu_i})\}$ is the irreducible $G_1$-representation of lowest weight $-\mu_i$. Therefore,

$$q_{\Delta^+}(\mu) = \sum_i \dim(\mathrm{span}_{\mathbf{C}}\{L_m \cdots L_1(w_{-\mu_i})\}) = \sum_i q_{\Delta_1^+}(\mu_i). \tag{6.6}$$

By induction each $q_{\Delta_1^+}(\mu_i)$ extends to a polynomial in $\mu_i$.

We now make two observations. First, $L$ is a product of a number of groups isomorphic to a $GL(r)$ for various $r$. Furthermore, $L_1 \cap L$ is a product of various groups isomorphic to $GL(r')$, and $r'$ is $r$ or $r-1$. The standard branching law for the restriction of representations of $GL(r)$ to $GL(r-1)$ is as follows. Let $V_{-a}$ be the irreducible $GL(r)$ representation of lowest weight $-a = -(a_1, \dots, a_r)$, $a_1 \geq a_2 \geq \cdots \geq a_r$. Similarly, let $U_{-b}$ be the irreducible $GL(r-1)$ representation of lowest weight $-b = -(b_1, \dots, b_{r-1})$. The the restriction of $V_{-a}$ to $GL(r-1)$ is $\sum U_{-b}$, with the sum being over all $b \in Z^{r-1}$ so that $a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq b_{r-1} \geq a_r$. Each occurs with multiplicity one.

We state the second observation as a Lemma

**Lemma 6.7.** *If* $p(b)$*,* $b \in \mathbf{C}^{r-1}$ *is a polynomial, then for* $a \in \mathbf{Z}^r$

$$P(a) \equiv \sum_{a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq b_{r-1} \geq a_r, b_j \in \mathbf{Z}} p(b_1, \dots, b_{r-1})$$

*extends to a polynomial on* $\mathbf{C}^r$*.*

Proof of lemma. For $a \in \mathbf{Z}^r$,

$$P(a) = \sum_{b_1 = a_2}^{a_1} \cdots \sum_{b_{n-1} = a_r}^{a_{r-1}} p(a_1, \dots, a_r)$$

extends to a polynomial in $a \in \mathbf{C}^r$. This is essentially because $\displaystyle\sum_{n=M-1}^{N} n^k = \sum_{n=1}^{N} n^k - \sum_{n=1}^{M} n^k$ is a polynomial in $M, N$.

We now conclude the proof of the theorem by noting that the $\mu_i$'s occurring in (6.6) come from the branching rule mentioned above (for the various factors of $L$), and the Lemma along with (6.6) says that $q_{\Delta^+}(\mu)$ extends to a polynomial in $\mu$.

$\square$

**Corollary 6.8.** *The multiplicity of $\overline{K_{\mathbf{C}}(f)}$ in $\pi_\lambda$ is $q_{\Delta^+}(\mu) = \dim(span_{\mathbf{C}}\{L_m \cdots L_1 L(w_{-\mu})\})$, for all $\lambda$ (dominant with $\lambda - \rho$ dominant integral for $\Delta^+$).*

The proof of the theorem contains an algorithm for computing the multiplicity $\overline{K_{\mathbf{C}}(f)}$ in $\pi_\lambda$. We describe an algorithm for computing

$$q_{\Delta^+}(\mu) = \dim(\mathrm{span}_{\mathbf{C}}\{L_m \cdots L_1 L(w_{-\mu})\}$$

for any $\mu$ which is a $\Delta_c^+$-dominant weight. This algorithm may be stated as follows. Given a positive system $\Delta^+$ containing $\Delta_c^+$, form the array as in (3). Form the first string and $f_0$ as in (3.4), also form and $G_1$ and $Q_{1,K}$ (as at the end of Section 3).

(1) Decompose the $L$-representation $U_{-\mu} = \mathrm{span}_{\mathbf{C}}\{L(w_{-\mu})\}$ into $L \cap L_1$-representations using the branching law for restricting $GL(r)$-representations to $GL(r-1)$-representations. Call the constituents $F_{-\mu_i}$.

(2) As shown in the proof of the theorem (see the 'Claim'), each $\mu_i$ is dominant for $\Delta_c^+ \cap \Delta(\mathfrak{l}_1)$ and
$$q_{\Delta^+}(\mu) = \sum_i q_{\Delta_1^+}(\mu_i).$$

(3) Now repeat the procedure to find the $q_{\Delta_1^+}(\mu_i)$, a computation on a smaller rank group.

The procedure ends after $m$ (the number of strings) iterations.

*Remark* 6.9. In [20] Yamashita constructs a $Z_{K_{\mathbf{C}}}(f)$-representation which is contained in the isotropy representation ([18, ]). The description of $H^0$ given here shows that Yamashita $Z_{K_{\mathbf{C}}}(f)$-representation is equal to $\mathrm{span}_{\mathbf{C}}\{L_m \ldots L_1 L(w_{-\mu})\}$. As we will see in Section 8, if $Q_K$ has a dense orbit in $\mathfrak{n}^- \cap \mathfrak{p}$ then the isotropy representation is $\mathrm{span}_{\mathbf{C}}\{Z(f)L(w_{-\mu})\}$.

## 7. Examples

We give several examples of computations of the multiplicities of discrete series representations using the algorithm of Section 6. The result of the first example is now well-known ([11] and [4]).

*Example* 7.1. (Holomorphic Discrete Series) This is the case where there is a unique simple non-compact root. The array is therefore one of the following:

$$\begin{array}{cccccc} \bullet_1 & \cdots & \bullet_p & & & \\ & & & \bullet_{p+1} & \cdots & \bullet_{p+q} \end{array} \qquad \text{or} \qquad \begin{array}{cccccc} & & & \bullet_{p+1} & \cdots & \bullet_{p+q} \\ \bullet_1 & \cdots & \bullet_p & & & \end{array}$$

and (assuming $p \le q$) $f = \pm \sum_{i=1}^{p} (\epsilon_i - \epsilon_{p+i})$. Therefore, $L = K_{\mathbf{C}}$, so $L_1 L = K_{\mathbf{C}}$ the multiplicity of $\pi_\lambda$ is the dimension of the lowest $K-$type of $\pi_\lambda$.

*Example* 7.2. (Quaternionic Discrete series of $SU(p,2)$) Consider the positive system determined by the following diagram:

$$\begin{tikzcd}[row sep=small] & \bullet_1 \arrow[dl] & \bullet_2 & \cdots & \bullet_p & \\ \bullet_{p+1} & & & & & \bullet_{p+2} \arrow[ullll] \end{tikzcd} \quad .$$

The reductive part of $Q_K$ is $L = S(GL(p,\mathbf{C}) \times \mathbf{C}^\times)$ and $L_1 \subset L$. Therefore, $m(K_{\mathbf{C}}(f), \pi_\lambda) = \dim_{\mathbf{C}}(L \cdot w_\mu)$, i.e., the dimension of the irreducible representation of $L$ with highest weight $\mu = \lambda + \rho_n - \rho_c$.

*Example* 7.3. Consider the group $G = SU(p, q)$ with $p \leq q$ and the positive system is given by a Dynkin diagram with the maximum number of simple roots noncompact. The array is

$$\begin{array}{ccccccccccc} \bullet_1 & & \bullet_2 & & \cdots & & \bullet_p & & & & \\ & \nwarrow \quad \swarrow & & \nwarrow & & & & \nwarrow & & & \\ & \bullet_{p+1} & & & \bullet_{p+2} & \cdots & & \bullet_{2p} & \bullet_{2p+1} & \cdots & \bullet_{p+q} \end{array} .$$

Here $L = L_1 =$ the torus and the multiplicity is one.

*Example* 7.4. We consider $G = SU(7, 7)$ and the positive system determined by the following array

$$\begin{array}{cccccccccccccc} \bullet_1 & \bullet_2 & & & \bullet_3 & & & \bullet_4 & \bullet_5 & & \bullet_6 & \bullet_7 & & \\ & & \bullet_8 & \bullet_9 & & \bullet_{10} & \bullet_{11} & & & \bullet_{12} & & & \bullet_{13} & \bullet_{14} \end{array} .$$

(See also Example 8.22.) Then $\mathrm{span}_{\mathbf{C}}\{L \cdot w_{-\mu}\}$ is the irreducible $L$-representation of lowest weight $-\mu$, call it $U_{-\mu}$. Then $L$ is a product of six copies of $SL(2)$ (and a torus) and $U_{-\mu}$ is the tensor product of representations of these $SL(2)$'s. Since $L_1 \cap L$ is the torus the decomposition of $U_{-\mu}|_{L_1 \cap L}$ is given by the weights

$$-\mu + a(\epsilon_1 - \epsilon_2) + a(\epsilon_4 - \epsilon_5) + c(\epsilon_6 - \epsilon_7) + d(\epsilon_8 - \epsilon_9) + +e(\epsilon_{10} - \epsilon_{11}) + f(\epsilon_{13} - \epsilon_{14}).$$

with $a = 0, \ldots, \mu_1 - \mu_2, b = 0, \ldots, \mu_4 - \mu_5, c = 0, \ldots, \mu_6 - \mu_7, d = 0, \ldots, \mu_8 - \mu_9, e = 0, \ldots, \mu_{10} - \mu_{11}$ and $f = 0, \ldots, \mu_{13} - \mu_{14}$. $L_1$ is the product of two copies of $SL(2)$ (and a torus). The roots in $\mathfrak{l}_1$ are $\pm\{\epsilon_5 - \epsilon_7, \epsilon_9 - \epsilon_{11}\}$. Using the formula $\displaystyle\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$, the dimension of $\mathrm{span}_{\mathbf{C}}\{L_1 L \cdot w_{-\mu}\}$ is therefore

$$\begin{aligned} &\sum_{a,\ldots,f} (\mu_5 - \mu_7 + b - c + 1)(\mu_9 - \mu_{11} + d - e + 1) \\ &\qquad = (\mu_1 - \mu_2 + 1)\big((\mu_4 - \mu_5 + 1)(\mu_6 - \mu_7 + 1)(\mu_5 - \mu_7 + 1 + \frac{\mu_4 - \mu_5 - \mu_6 + \mu_7}{2})\big) \\ &\qquad\quad \big((\mu_8 - \mu_9 + 1)(\mu_{10} - \mu_{11} + 1)(\mu_9 - \mu_{11} + 1 + \frac{\mu_8 - \mu_9 - \mu_{10} + \mu_{11}}{2})\big)(\mu_{13} - \mu_{14} + 1). \end{aligned}$$

Writing this in terms of $\lambda$ (using $\mu = \lambda + \rho_n - \rho_c$) the formula for multiplicity is

$$\frac{1}{4}(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_5)(\lambda_6 - \lambda_7)(\lambda_8 - \lambda_9)(\lambda_{10} - \lambda_{11})(\lambda_{13} - \lambda_{14})(\lambda_4 + \lambda_5 - \lambda_6 - \lambda_7)(\lambda_8 + \lambda_9 - \lambda_{10} - \lambda_{11}).$$

## 8. $Q_K$-orbits in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$

In this section we consider the $Q_K = Q \cap K$-orbits in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$. It is reasonable to ask whether the Borel subgroup $B \cap K_{\mathbf{C}}$ (with Lie algebra $(\mathfrak{t} + \mathfrak{n}^-) \cap \mathfrak{k}$) has a dense orbit in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$. There are examples for which the answer is no, see for example [15]. In fact, we will see that even $Q_K$ does not always have a dense orbit in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$. We give a criterion in terms of the algorithm of Section 3 for $Q_K$ to be transitive (or have a dense orbit) on $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$. At the end of the section an example is given, in $SU(7, 7)$, for which there is no such dense $Q_K$-orbit.

We begin this section with two propositions which indicate that it is of interest to understand the $Q_K$-orbits on $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$.

Let $\tilde{\gamma}$ be the moment map $\tilde{\gamma} : K_{\mathbf{C}} \underset{Q_K}{\times} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) \to \mathfrak{g}_{\mathbf{C}}$ Consider $Y = K_{\mathbf{C}}/Q_K$ as a closed $K_{\mathbf{C}}$-orbit in $G_{\mathbf{C}}/Q$, so $Y = \{k \cdot \mathfrak{q} : k \in K_{\mathbf{C}}\}$. Then the fiber of $\tilde{\gamma} : K_{\mathbf{C}} \underset{Q_K}{\times} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) \to \mathfrak{p}_{\mathbf{C}}$ may be described as follows. For $x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$ let $N(x, \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) = \{k \in K_{\mathbf{C}} : k \cdot x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}\}$ (as in Section 2). Then

$$\begin{aligned} \tilde{\gamma}^{-1}(x) &= \{\mathfrak{q}' \in Y : x \in \mathfrak{q}'\} \\ &= \{k^{-1} \cdot \mathfrak{q} : k \in N(x, \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}})\}. \end{aligned}$$

Note that $K_{\mathbf{C}}(x) \cap (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) = \{k \cdot x : k \in N(x, \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}})\}$.

**Proposition 8.1.** *For arbitrary $x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$, there is a bijection*

$$\{Z_{K_{\mathbf{C}}} - \text{orbits in } \tilde{\gamma}^{-1}(x)\} \leftrightarrow \{Q_K - \text{orbits in } K_{\mathbf{C}}(x) \cap (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}})\}$$
$$Z_{K_{\mathbf{C}}} \cdot (k\mathfrak{q}) \leftrightarrow Q_K(k^{-1} \cdot x), k \in N(x, \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}).$$

*Moreover, if $x$ is generic in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$, then $Z_{K_{\mathbf{C}}} \cdot \mathfrak{q}$ is open in $\tilde{\gamma}^{-1}(x)$ if and only if $Q_K(x)$ is open in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$.*

*Proof.* For the first statement, notice that for $k_1, k_2 \in N(x, \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}})$

$$\begin{aligned}
&Z_{K_{\mathbf{C}}} \cdot k_1\mathfrak{q} = Z_{K_{\mathbf{C}}} \cdot k_2\mathfrak{q} \text{ if and only if} \\
&\qquad k_1 = zk_2q, \text{ for some } q \in Q_K, z \in Z_{K_{\mathbf{C}}} \text{ if and only if} \\
&\quad k_1^{-1}x = q^{-1}k_2^{-1}, \text{ for some } q \in Q_K \text{ if and only if} \\
&Q_K(k_1^{-1}x) = Q_K(k_2^{-1}x).
\end{aligned}$$

For the second statement apply Lemma 8.13. □

Here is an alternative to the formula of Proposition 6.4 for the multiplicities.

**Proposition 8.2.** *If there exists $x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$ generic so that $Q_K(x) \subset K_{\mathbf{C}}(x) \cap (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}})$ is open and dense, then*

$$m(K_{\mathbf{C}}(x), \pi_\lambda) = \dim_{\mathbf{C}}\{Z_{K_{\mathbf{C}}}(x)L \cdot w_{-\mu}\}.$$

*Proof.* Indeed, under the assumptions in the Lemma we have

$$\begin{aligned}
m(K_{\mathbf{C}}(x), \pi_\lambda) &= \dim H^0(\gamma^{-1}(x), \mathcal{O}_{\gamma^{-1}(f)}(\mu)) \\
&= \dim H^0(\overline{Z_{K_{\mathbf{C}}}(x)Q_K \cdot \mathfrak{b}}, \mathcal{O}_{\gamma^{-1}(f)}(\mu)) \\
&= q_{\overline{Z_{K_{\mathbf{C}}}(x)Q_K \cdot \mathfrak{b}}}(\mu) = q_{Z_{K_{\mathbf{C}}}(x)Q_K \cdot \mathfrak{b}}(\mu).
\end{aligned}$$

□

Let $\tilde{\gamma}$ be the moment map $\tilde{\gamma} : K_{\mathbf{C}} \underset{Q_K}{\times} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) \to \mathfrak{g}_{\mathbf{C}}$ given by $\tilde{\gamma}(kQ_K, Y) = k \cdot Y$. For $x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$ generic, we write a formula for the $\dim_{\mathbf{C}}(\tilde{\gamma}^{-1}(x))$ in terms of data produced by the algorithm in Section 3. This formula will be used later in this section to study the structure of $Q_K$-orbits in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$.

Let $\{p_1, q_1, p_2, q_2, \ldots, p_r, q_r\}$ be a sequence satisfying (3.3) and let $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ be the positive system determined by the sequence. Let $N_p$ (resp., $N_q$) stand for the number of $p_i$ (resp., $q_i$) occurring in the sequence. Then $N = N_p + N_q$. Write, as in previous sections, $Q_K = Q_{0,K} = L \exp(\mathfrak{u}^- \cap \mathfrak{k}_{\mathbf{C}})$ and $Q_{i,K} = L_i \exp(\mathfrak{u}_i^- \cap \mathfrak{k}_{\mathbf{C}})$. We obtain a formula for $\dim_{\mathbf{C}}(\tilde{\gamma}^{-1}(x))$ as a corollary of the following theorem.

**Theorem 8.3.** *Let $x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$ be a generic element. Then,*

$$\begin{aligned}
\dim Z_{G_{\mathbf{C}}}(x) &= \sum_1^{N_p} p_i^2 + \sum_1^{N_q} q_j^2 + 2\sum_1^{N} \dim(Q_{K,i}/Q_{K,i} \cap Q_{K,i-1}) \\
&= \dim \mathfrak{l} + 2\sum_1^{N} \dim(Q_{K,i}/Q_{K,i} \cap Q_{K,i-1}). \qquad (8.4)
\end{aligned}$$

**Corollary 8.5.** *If $x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$ is generic, then $\dim \tilde{\gamma}^{-1}(x) = \sum_1^N \dim(Q_{K,i}/Q_{K,i} \cap Q_{K,i-1})$.*

*Proof.* On the one hand

$$\begin{aligned}
\dim \tilde{\gamma}^{-1}(x) &= \dim(K_{\mathbf{C}}/Q_K) + \dim(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) - \dim(\mathcal{O}) \\
&= \dim(\mathfrak{u}^-) - \dim(\mathcal{O}). \qquad (8.6)
\end{aligned}$$

On the other hand, the dimension of the nilpotent $K_C$-orbit $\mathcal{O} = K_{\mathbf{C}}(x)$ is half the dimension of $G_{\mathbf{C}}(x)$. Hence,

$$\begin{aligned}\dim(\mathcal{O}) &= \frac{1}{2}(\dim(\mathfrak{g}_{\mathbf{C}}) - \dim(\mathfrak{z}_{\mathfrak{g}_{\mathbf{C}}}(f))) \\ &= \frac{\dim(\mathfrak{l})}{2} - \frac{\dim(\mathfrak{z}_{\mathfrak{g}_{\mathbf{C}}}(f))}{2} + \dim(\mathfrak{u}^-).\end{aligned} \tag{8.7}$$

Combining formulas (8.6) and (8.7), we get $\dim \tilde{\gamma}^{-1}(f) = \frac{\dim(\mathfrak{z}_{\mathfrak{g}_{\mathbf{C}}}(f))}{2} - \frac{\dim(\mathfrak{l})}{2}$. Now, the formula in Theorem 8.3 gives $\dim \tilde{\gamma}^{-1}(f) = \sum_1^N \dim(Q_{K,i}/Q_{K,i} \cap Q_{K,i-1})$.

□

We begin the proof of Theorem 8.3 with two preliminary lemmas.

**Lemma 8.8.** *Write* $\mathfrak{q}_K = \mathfrak{l} + (\mathfrak{u}^- \cap \mathfrak{k}_{\mathbf{C}})$, $\mathfrak{q}_{1,K} = \mathfrak{l}_1 + (\mathfrak{u}_1^- \cap \mathfrak{k}_{\mathbf{C}})$. *Then,*

$$\dim(\mathfrak{l}) = \dim(\mathfrak{l} \cap \mathfrak{l}_1) + 2\,(p+q) - N.$$

*Proof.* By construction, $\dim(\mathfrak{l}) = \sum_1^{N_p} p_i^2 + \sum_1^{N_q} q_j^2$, while $\dim(\mathfrak{l} \cap \mathfrak{l}_1) = \sum_1^{N_p}(p_i-1)^2 + \sum_1^{N_q}(q_j-1)^2$. Hence,

$$\begin{aligned}\dim(\mathfrak{l} \cap \mathfrak{l}_1) &= \sum_1^{N_p} p_i^2 + \sum_1^{N_q} q_j^2 - 2\,(\sum p_i + \sum q_j) + N \\ &= \dim(\mathfrak{l}) - 2\,(p+q) + N.\end{aligned}$$

□

**Lemma 8.9.** *Starting from the partitions* $\{p_i\}$ *of* $p$ *and* $\{q_j\}$ *of* $q$, *form the nilpotent element* $f = f_0 + f_1 + \ldots + f_m$ *as in the algorithm in Section 3. Set* $f' = f - f_0 = f_1 + f_2 + \ldots + f_m$. *Then,*

$$\dim Z_{G_{\mathbf{C}}}(f) = \dim Z_{G_{1,\mathbf{C}}}(f') + 2\,(p+q) - N.$$

*Proof.* Associate to $f$ the tableau that parameterizes the nilpotent $K_{\mathbf{C}}$-orbit through $f$. Let $a_i$ stand for the number of rows in the tableau having at least $i$ blocks. Then, by [7, Thm 6.1.], we know that $\dim Z_{G_{\mathbf{C}}}(f) = \sum a_i^2$. Similarly, since the tableau corresponding to the nilpotent orbit $K_{1,\mathbf{C}}(f')$ is obtained from that of $f$ by removing a longest row, we have $\dim Z_{G_{1,\mathbf{C}}}(f') = \sum(a_i-1)^2$. Thus,

$$\dim Z_{G_{\mathbf{C}}}(f) - \dim Z_{G_{1,\mathbf{C}}}(f') = \sum a_i^2 - \sum(a_i-1)^2 = 2\sum_1^N a_i - N = 2\,(p+q) - N.$$

□

*Proof of Theorem 8.3.* We proceed by induction on the number of strings produced by the algorithm.

Assume first that our algorithm has produced only one string, i.e., $f = f_0$. Without loss of generality we can assume that $q_j = 1$ for $j = 1, 2, \ldots, N_q = q$. The tableau corresponding to the nilpotent orbit $K_{\mathbf{C}}(f)$ has a row of length $N$ and $p+q-N$ rows of length one. By [7, Thm 6.1.3] we have

$$\dim Z_{G_{\mathbf{C}}}(f) = (p+q-N+1)^2 + (N-1).$$

On the other hand, the right hand side of (8.4) is

$$
\begin{aligned}
&\sum_{1}^{N_p} p_i^2 + \sum_{1}^{N_q} q_j^2 + 2\ \dim(Q_{1,K}/Q_{1,K}\cap Q_K) \\
&= \sum_{1}^{N_p} p_i^2 + \sum_{1}^{N_q} q_j^2 + 2\ \dim(\mathfrak{u}^- \cap \mathfrak{l}_1) \\
&= \sum_{1}^{N_p} p_i^2 + q + 2\ \sum_{i<j}(p_i - 1)(p_j - 1).
\end{aligned} \tag{8.10}
$$

However,

$$
\begin{aligned}
\sum_{i<j}(p_i-1)(p_j-1) &= \sum_{i<j} p_i p_j - \sum_{i<j}(p_i+p_j) + \sum_{i<j} 1 \\
&= \sum_{i<j} p_i p_j - \sum_{1}^{N_p=N-q} (N-q-i)p_i - \sum_{1}^{N-q}(j-1)p_j + \frac{(N-q)(N-q-1)}{2} \\
&= \sum_{i<j} p_i p_j - (N-q-1) \sum_{1}^{N-q=N_p} p_i + \frac{(N-q)(N-q-1)}{2} \\
&= \sum_{i<j} p_i p_j - (N-q-1)p + \frac{(N-q)(N-q-1)}{2}.
\end{aligned} \tag{8.11}
$$

Combining equations (8.10) and (8.11) we write the right hand side of (8.4) as

$$
\begin{aligned}
&\sum_{1}^{N_p} p_i^2 + \sum_{1}^{N_q} q_j^2 + 2\sum_{1}^{N} \dim(Q_{K,i}/Q_{K,i}\cap Q_{K,i-1}) \\
&= \sum_{1}^{N_p} p_i^2 + q + 2\sum_{i<j} p_i p_j - 2(N-q-1)p + (N-q)(N-q-1) \\
&= (\sum p_i)^2 + q - 2(N-q-1)p + (N-q-1)^2 + (N-q-1) \\
&= [p^2 - 2(N-q-1)p + (N-q-1)^2] + N - 1 \\
&= (p+q-N+1)^2 + (N-1) = \dim Z_{G_{\mathbf{C}}}(f).
\end{aligned} \tag{8.12}
$$

Therefore, the proposed formula (8.4) holds when the algorithm produces exactly one string.

Next, assume that the dimension formula holds for $f' = f_1 + f_2 + \ldots + f_m$ with $m \geq 1$. By Lemma 8.9, we know that

$$\begin{aligned}
\dim Z_{G_{\mathbf{C}}}(f) &= \dim Z_{G_{1,\mathbf{C}}}(f') + 2\,(p+q) - N \\
&= \dim(\mathfrak{l}_1) + 2\,\sum_2^m \dim(Q_{K,i}/Q_{K,i} \cap Q_{K,i-1}) + 2\,(p+q) - N \\
&\qquad \text{(by the induction hypothesis)} \\
&= \dim(\mathfrak{l}_1 \cap \mathfrak{l}) + 2\,\dim(\mathfrak{l}_1 \cap \mathfrak{u}^-) + 2\,\sum_2^m \dim(Q_{K,i}/Q_{K,i} \cap Q_{K,i-1}) + 2\,(p+q) - N \\
&= \dim(\mathfrak{l}_1 \cap \mathfrak{l}) + 2\,\sum_1^m \dim(Q_{K,i}/Q_{K,i} \cap Q_{K,i-1}) + 2\,(p+q) - N \\
&\qquad \text{(since } \dim(Q_{1,K}/Q_{1,K} \cap Q_K) = \dim(L_1/L_1 \cap Q_K) = \dim(\mathfrak{l}_1 \cap \mathfrak{u}^-)) \\
&= \dim(\mathfrak{l}) + 2\,\sum_1^m \dim(Q_{K,i}/Q_{K,i} \cap Q_{K,i-1}) \\
&\qquad \text{(by Lemma 8.8).}
\end{aligned}$$

We give a condition for $Q_K$ to be transitive (or have a dense open orbit) on $\mathfrak{n}^- \cap \mathfrak{p}_C$. See Theorem 8.14 and Corollary 8.21.

**Lemma 8.13.** *Let $x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$ be generic. Then,*

$$\dim \tilde{\gamma}^{-1}(x) = \operatorname{codim}_{\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}} Q_K(x) + \dim Z_{K_{\mathbf{C}}}(x) - \dim Z_{Q_K}(x).$$

*Proof.* This is a simple computation:

$$\begin{aligned}
\dim \tilde{\gamma}^{-1}(x) &= \dim \mathfrak{u}^- - \dim \mathcal{O} \\
&= \dim \mathfrak{u}^- - \dim \mathfrak{k}_{\mathbf{C}} + \dim Z_{K_{\mathbf{C}}}(x) \\
&= \dim(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) - \dim \mathfrak{q}_k + \dim Z_{K_{\mathbf{C}}}(x)
\end{aligned}$$

(since $\dim \mathfrak{k}_{\mathbf{C}} = \dim \mathfrak{q}_k + \dim(\mathfrak{u}^- \cap \mathfrak{k}_{\mathbf{C}})$ and $\dim \mathfrak{u}^- = \dim(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) + \dim(\mathfrak{u}^- \cap \mathfrak{k}_{\mathbf{C}})$)

$$\begin{aligned}
&= (\dim(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}) - \dim Q_K + \dim Z_{Q_K}(x)) + (\dim Z_{K_{\mathbf{C}}}(x) - \dim Z_{Q_K}(x)) \\
&= (\text{codimension } Q_K(x)) + (\dim Z_{K_{\mathbf{C}}}(x) - \dim Z_{Q_K}(x)).
\end{aligned}$$

□

Let $f$ be constructed by the algorithm in Section 3 and $\mathcal{O} = K(f)$.

**Theorem 8.14.** *$Q_K$ acts transitively on $\mathcal{O} \cap (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}})$ if and only if $Q_K \cap Q_{1,K}$ acts transitively on the set of generic elements in $\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$.*

*Proof.* Assume that $Q_K$ acts transitively on $\mathcal{O} \cap (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}})$. Let $x' \in \mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$ be a generic element and form $x = f_0 + x'$. By Proposition 4.13 we know that $x \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$ is generic. Since $Q_K$ is assumed to act transitively on $\mathcal{O} \cap (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}})$, we conclude that $Q_K(x) = Q_K(f_0 + x')$ is open in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$. Hence, the tangent space to the orbit $Q_K(f_0 + x')$ at the base point $f_0 + x'$ coincides with $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$. This implies that

$$[\mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}}, f_0 + x'] = T_{f_0 + x'}(Q_K(f_0 + x')) = \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}} \supset \mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}. \tag{8.15}$$

We show that $Q_K \cap Q_{1,K}(x')$ is open in $\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$.

The Borel subalgebra $\mathfrak{b} = \mathfrak{h}_{\mathbf{C}} + \mathfrak{n}^- \subset \mathfrak{q}$ is determined by an array of numbered dots. The first step of our algorithm determines $f_0$ by choosing a first string. Let $S$ be the set of labels of the dots in the array that are left after deleting labels of the first string. In particular, notice that $f_0$ is a sum of root vectors for roots $\epsilon_i - \epsilon_j$ where neither $i$ nor $j$ belong to the set $S$. Moreover, $\Delta(\mathfrak{g}_{1,\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}) = \{\epsilon_i - \epsilon_j : i, j \in S\}$. The set $S$ determines a decomposition

$$\mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}} = \mathfrak{q} \cap \mathfrak{g}_{1,\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}} + \mathfrak{v}_o + \mathfrak{v}_1$$

where

$$\begin{aligned}\Delta(\mathfrak{q} \cap \mathfrak{g}_{1,\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}}) &= \{\epsilon_i - \epsilon_j : i, j \in S\} \cap \Delta(\mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}}) \\ \Delta(\mathfrak{v}_0) &= \{\epsilon_i - \epsilon_j : i, j \notin S\} \cap \Delta(\mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}}) \\ \Delta(\mathfrak{v}_1) &= \{\epsilon_i - \epsilon_j : \text{ exactly one of } i, j \text{ belongs to } S\} \cap \Delta(\mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}}).\end{aligned}$$

Observe that,

$$[\mathfrak{v}_0 + \mathfrak{v}_1, f_0 + x'] \subset \operatorname{span}\{X_\gamma : \gamma = \epsilon_i - \epsilon_j \text{ with at most one of } i \text{ and } j \in S\}. \tag{8.16}$$

If $X_\beta$ is a root vector in $\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$, then $\beta = \epsilon_i - \epsilon_j$ with $i, j \in S$ and, by inclusion (8.15),

$$X_\beta \in [\mathfrak{q} \cap \mathfrak{k}_{\mathbf{C}}, f_0 + x'] = [\mathfrak{q} \cap \mathfrak{g}_{1,\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}}, f_0 + x'] + [\mathfrak{v}_0 + \mathfrak{v}_1, f_0 + x'].$$

Now, the observation (8.16) and the description of the root $\beta$ imply that $X_\beta \in [\mathfrak{q} \cap \mathfrak{g}_{1,\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}}, f_0 + x']$. Hence,

$$\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}} \subset [\mathfrak{q} \cap \mathfrak{g}_{1,\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}}, f_0 + x'] \subset \mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}},$$

i.e $T_{x'}[Q_{1,K} \cap Q_K(x')] = \mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$ and $Q_{1,K} \cap Q_K(x')$ is open in $\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$. Since $x'$ is an arbitrary generic element in $\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$, we conclude that $Q_{1,K} \cap Q_K$ acts transitively on the set of generic elements in $\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$.

For the converse, let $x$ be generic in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$ and let $f = f + f'$, $f' = \sum_{i=1}^{m-1} f_i$ as constructed by the algorithm of Section 3. By Proposition 4.13 there exist $q \in Q_K$ and $q_i \in Q_{i,K}$ so that $x = qq_1 \cdots q_{m-1}(f_0 + f')$. Since each $q_i$ commutes with $f_0$), $q^{-1}x = f_0 + x'$, where $x' = q_1 \cdots q_{m-1}(f')$, a generic element of $\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$. Now assume $Q_K \cap Q_{1,K}$ is transitive on the generic elements of $\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}$. Then,

$$\dim(Q_{1,K}(x')) = \dim((Q_K \cap Q_{1,K})(x')) = \dim(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}). \tag{8.17}$$

Therefore it suffices to show that $Q_K(x) = Q_K(f + x')$ has codimension zero in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$.

By Lemma 8.13 and Corollary 8.5 applied to $x' \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}$, along with (8.17),

$$0 = \operatorname{codim}_{\mathfrak{u}_1^- \cap \mathfrak{p}_{\mathbf{C}}}(Q_{1,K}(x')) = \sum_{i=2}^{m-1} \dim(Q_{i,K}/Q_{i,K} \cap Q_{i-1,K}) - \Big(\dim Z_{K_{1,\mathbf{C}}}(x') - \dim Z_{Q_{1,K}}(x')\Big). \tag{8.18}$$

Also, by (8.17),

$$\begin{aligned}\dim(Q_{1,K}/(Q_{1,K} \cap Q_K)) &= \dim Q_{1,K} - \dim Q_{1,K} \cap Q_K \\ &= \dim Z_{Q_{1,K}}(x') - \dim Z_{Q_K \cap Q_{1,K}}(x').\end{aligned} \tag{8.19}$$

Applying Lemma 8.13 and Corollary 8.5 for the first equality and (8.18) and (8.19) for the second, we have

$$\begin{aligned}&\operatorname{codim}_{\mathfrak{u}^- \cap \mathfrak{p}_{\mathbf{C}}}\big(Q_K(f + x')\big) \\ &\quad = \sum_1^m \dim(Q_{i,K}/Q_{i,K} \cap Q_{i-1,K}) - \Big(\dim Z_{K_{\mathbf{C}}}(x) - \dim Z_{Q_K}(x)\Big) \\ &\quad = \Big(Z_{K_{1,\mathbf{C}}}(x') - \dim Z_{Q_{1,K} \cap Q_K}(x')\Big) - \Big(\dim Z_{K_{\mathbf{C}}}(x) - \dim Z_{Q_K}(x)\Big).\end{aligned} \tag{8.20}$$

Since

$$Z_{K_{1,\mathbf{C}}}(x')/Z_{Q_{1,K} \cap Q_K}(x') \to Z_{K_{\mathbf{C}}}(f_0 + x')/Z_{Q_K}(f_0 + x')$$

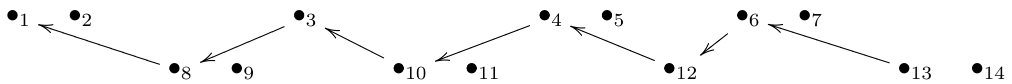
is injective, we may conclude that the right hand side of (8.20) is less than or equal to zero. Therefore, $\operatorname{codim}_{\mathfrak{u}^-\cap\mathfrak{p}_{\mathbf{C}}}(Q_K(f+x'))=0$, and the proof is complete.

$\square$

**Corollary 8.21.** *$Q_K$ has an open orbit in $\mathfrak{u}^-\cap\mathfrak{p}_{\mathbf{C}}$ if and only if $Q_{1,K}\cap Q_K$ has an open orbit in $\mathfrak{u}_1^-\cap\mathfrak{p}_{\mathbf{C}}$.*

We conclude this section with an example of how Corollary 8.21 produces a situation where $Q_K$ does not have an open orbit in $\mathfrak{u}^-\cap\mathfrak{p}_{\mathbf{C}}$.

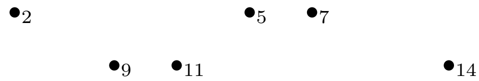
*Example* 8.22. Let $G=SU(7,7)$ and let $\mathfrak{h}_{\mathbf{C}}$ be the diagonal Cartan of $\mathfrak{g}_{\mathbf{C}}=\mathfrak{sl}(p+q,\mathbf{C})$ as in Section 3. Consider the positive root system $\Delta^+=\Delta^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{h}_{\mathbf{C}})$ determined by the following numbered array, and the first string formed by the algorithm,




Equivalently, $\Delta^+$ is the positive system of roots having positive inner product with

$$(14,13,10,7,6,4,3|12,11,9,8,5,2,1).$$

Apply the algorithm in Section 3 to produce the first string. After deleting the first string the resulting array is




Thus, $\mathfrak{g}_1=\mathfrak{su}(3,3)$ and $Q_K\cap K_{=}B_1$ is a Borel subgroup of $K_1$. Moreover, $\dim(\mathfrak{k}_{\mathbf{C}}\cap\mathfrak{b}_1)=11$ while $\dim(\mathfrak{u}_1^-\cap\mathfrak{p}_{\mathbf{C}})=9$. An arbitrary element $X$ in $\mathfrak{u}_1^-\cap\mathfrak{p}_{\mathbf{C}}$ is of the form

$$X=aX_{9,2}+bX_{11,2}+cX_{14,2}+dX_{14,7}+eX_{5,11}+fX_{14,5}+gX_{5,9}+hX_{7,9}+iX_{11,7}.$$

We claim that $K_{\mathbf{C}}\cap B_1(X)$ is not dense in $\mathfrak{u}_1^-\cap\mathfrak{p}_{\mathbf{C}}$ for any $X\in\mathfrak{u}_1^-\cap\mathfrak{p}_{\mathbf{C}}$. Indeed, when $a\neq 0$, then $\mathfrak{z}_{\mathfrak{k}_{\mathbf{C}}\cap\mathfrak{b}_1}(X)$ contains

$$aX_{5,2}+fX_{14,9},aX_{7,2}+dX_{14,9};aX_{14,11}-bX_{14,9}.$$

We then conclude that $\dim(K_{\mathbf{C}}\cap B_1)(X)\leq 8<\dim(\mathfrak{u}_1^-\cap\mathfrak{p}_{\mathbf{C}})=9$. When $a=0$, the argument is slightly different; $X_{9,2}\notin[\mathfrak{k}_{\mathbf{C}}\cap\mathfrak{b}_1,X]$ (as is easily checked). But, $[\mathfrak{k}_{\mathbf{C}}\cap\mathfrak{b}_1,X]$ is the tangent space to $(K_{\mathbf{C}}\cap B_1)(X)$ at $X$, so $\dim((K_{\mathbf{C}}\cap B_1)(X))<\dim(\mathfrak{u}^-\cap\mathfrak{p}_{\mathbf{C}})$. This proves the claim. Now, Proposition 8.14, implies that $Q_K$ has no open orbit in $\mathfrak{u}^-\cap\mathfrak{p}_{\mathbf{C}}$.

The orbit structure of $Q_K$ on the generic elements in $\mathfrak{u}^-\cap\mathfrak{p}_C$ may be described as follows. Assume that $x\in\mathfrak{u}^-\cap\mathfrak{p}_{\mathbf{C}}$ is generic. By Corollary 8.5, we know that $\dim\tilde{\gamma}^{-1}(x)=\dim L_1/(L_1\cap Q_K)=2$. Hence, by Lemma 8.13, the only possible dimensions of the $Q_K$-orbits in $K_{\mathbf{C}}(x)\cap\mathfrak{p}_{\mathbf{C}}$ are 47, 48 and 49. We have just argued that no orbit has dimension 49.

Observe that $L_1=L_1^1\times L_1^2$ is the product of two commuting copies of $GL(2,\mathbf{C})$ (generated by the roots $\pm\{\epsilon_7-\epsilon_5,\epsilon_{11}-\epsilon_9\}$). By using the Bruhat decompositions of the subgroups $L_1^i, i=1,2$ one sees that the $Q_K$-orbits in the generic elements of $\mathfrak{u}^-\cap\mathfrak{p}_{\mathbf{C}}$ are as follows. Orbits of dimension 47:

$$Q_K(f),Q_K(\exp(X_{9,11})\cdot f),Q_K(\sigma_{5,7}\exp(X_{9,11}\cdot f),$$

and orbits of dimension 48:

$$Q_K(\sigma_{5,7}\cdot f),Q_K(\sigma_{9,11}\cdot f),Q_K(\sigma_{5,7}\sigma_{9,11}\cdot f),Q_K(\exp(X_{5,7})\cdot f),Q_K(\exp(X_{5,7})\sigma_{9,11}\cdot f),$$
$$Q_K(\exp(X_{5,7}\exp(sX_{9,11}),s\in\mathbf{C}\text{ (an infinite family)}.$$

$$K_{\mathbf{C}}(x)\cap\mathfrak{p}_{\mathbf{C}} = Q_K(f)\cup Q_K(\exp\,(X_{9,11})f)\cup Q_K(s_{9,11}f)\cup Q_K(s_{9,11}\exp\,(X_{5,7})f)\cup Q_K(s_{5,7}f)$$
$$\cup\, Q_K(\exp\,(X_{9,11})f)\cup \cup_{s\in\mathbf{C}} Q_K(\exp\,(X_{5,7})\text{ epx }(sX_{9,11})\cdot f)\cup Q_K(s_{9,11}s_{5,7}f)$$

with $s_{9,11}$ and $s_{5,7}$ the non-trivial Weyl group elements of the respective copies of $GL(2,\mathbf{C})$.

## Appendix A. Real distinguished orbits

In this appendix we show that the associated variety of any discrete series representation of $SU(p,q)$ is the closure of distinguished real nilpotent orbit. This seems to be known by the experts, but no proof has appeared. We give a proof here because it follows very directly from our algorithm.

**Definition A.1.** A nilpotent element $E\in\mathfrak{g}$ is said to be $\mathbf{R}$-*distinguished* if $E$ does not lie in a Levi subalgebra of a (proper) parabolic subalgebra of $\mathfrak{g}$. A nilpotent orbit $\Omega$ in $\mathfrak{g}$ is called $\mathbf{R}$-*distinguished* if $\Omega$ does not meet the Levi component of a (proper) parabolic subalgebra of $\mathfrak{g}$. (So, $\Omega$ is $\mathbf{R}$-distinguished if and only if each element of $\Omega$ is $\mathbf{R}$-distinguished.)

We remark that an element in the complex Lie algebra $\mathfrak{g}_{\mathbf{C}}$ is distinguished means that it does not lie in a Levi component of a parabolic in $\mathfrak{g}_{\mathbf{C}}$ (see [7, page 121]). Thus $E\in\mathfrak{g}$ is distinguished implies $E$ is $\mathbf{R}$-distinguished. However, the converse fails.

**Lemma A.2.** *Let $E$ be a nilpotent element in $\mathfrak{g}$ and $\{E,H,F\}$ a standard basis for a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbf{R})$. Then $E$ is $\mathbf{R}$-distinguished if and only if $Z_G(E,H,F)$ is compact.*

*Proof.* We may assume that the triple $\{E,H,F\}$ satisfies $\theta(H)=-H$ and $\theta(E)=-F$. Then, since the triple is $\theta$-stable, the centralizer decomposes as

$$\mathfrak{z}_{\mathfrak{g}}(E,H,F)=\mathfrak{z}_{\mathfrak{k}}(E,H,F)\oplus\mathfrak{z}_{\mathfrak{p}}(E,H,F).$$

Therefore, $Z_G(E,H,F)$ is noncompact if and only if $\mathfrak{z}_{\mathfrak{p}}(E,H,F)$ is nonzero. However, the centralizer of a nonzero element of $\mathfrak{p}$ is a Levi subalgebra of a proper parabolic subalgebra of $\mathfrak{g}$. Therefore, if the centralizer is noncompact, then the orbit is not $\mathbf{R}$-distinguished. Conversely, if the orbit is not $\mathbf{R}$-distinguished then some element of $\Omega$ lies in a Levi subalgebra of a proper parabolic subalgebra. By conjugating in $G$ we see that some $E'$ in $\Omega$ lies in the centralizer of some nonzero $X\in\mathfrak{p}$. Then, inside the centralizer of $X$ (a reductive group), $E'$ is part of a triple $\{E',H',F'\}$. Therefore, $X\in\mathfrak{z}_{\mathfrak{p}}(E',H',F')$, so $Z_G(E,H,F)$ is noncompact. □

In order to see that the associated variety of a discrete series is the closure of an $\mathbf{R}$-distinguished orbit we will need to consider the Sekiguchi dual of a real orbit. For this let us write the triple $\{e,h,f\}$ for the Sekiguchi dual of $\{E,H,F\}$. We assume that $\theta(H)=-H$ and $\theta(E)=-F$. Then

$$e=\frac{1}{2}(H-i(E+F))$$
$$f=\frac{1}{2}(H+i(E+F))$$
$$h=i(E-F)$$

Observe that the two triples have the same centralizer $\mathfrak{z}$ in $\mathfrak{g}_{\mathbf{C}}$ (as they span the same complex subalgebra, which we will call $\mathfrak{s}$). Since $\mathfrak{z}$ is stable under both the complex conjugation and the Cartan involution, we have

$$\mathfrak{z}=\mathfrak{z}_{\mathfrak{k}}(\mathfrak{s})\oplus i\mathfrak{z}_{\mathfrak{k}}(\mathfrak{s})\oplus\mathfrak{z}_{\mathfrak{p}}(\mathfrak{s})\oplus i\mathfrak{z}_{\mathfrak{p}}(\mathfrak{s}).$$

Let $S$ be the subgroup of $GL(p+q,\mathbf{C})$ with Lie algebra $\mathfrak{s}=\text{span}_{\mathbf{C}}(E,H,F)=\text{span}_{\mathbf{C}}(e,h,f)$. Consider the the representation of $Z_2\rtimes S$ on $\mathbf{C}^{p+q}$. The algorithm has the property that for any isotypic subspace for the $S$-representation, all highest weight vectors (i.e., annihilated by $e$) lie in either $\mathbf{C}^p \times \{0\}$ or in $\{0\} \times \mathbf{C}^q$. If $F$ were not distinguished then there would be some nonzero $X \in \mathfrak{z}_{\mathfrak{p}}(\mathfrak{s})$. Since $X$ maps $\mathbf{C}^p \times \{0\}$ to $\{0\} \times \mathbf{C}^q$ (and $\{0\} \times \mathbf{C}^q$ to $\mathbf{C}^p \times \{0\}$), $X$ must be zero on each highest weight vector. Since $X$ commutes with $S$, $X$ must be zero. This is a contradiction. We have proved the first part of the following theorem.

**Theorem A.3.** *The associated varieties of the discrete series representations of $SU(p, q)$ are the closures of $\mathbf{R}$-distinguished orbits. Every $\mathbf{R}$-distinguished orbit is the associated variety of some discrete series representation.*

The second part follows by induction on the number of strings (or the number of rows in the tableaux). It is convenient to use the characterization of $\mathbf{R}$-distinguished in terms of the tableaux: a nilpotent orbit is $\mathbf{R}$-distinguished if and only if all rows of a given length begin with the same sign.

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