CERTAIN COMPONENTS OF THE SPRINGER FIBER AND ASSOCIATED CYCLES FOR DISCRETE SERIES REPRESENTATIONS OF $SU(p, q)$

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1. Introduction

Suppose G is a connected complex simple Lie group with Lie algebra $\mathfrak{g}$, and let $G_{\mathbf{R}}$ be a real form of G. Let X denote the flag variety of G. A Cartan involution of the Lie algebra of $G_{\mathbf{R}}$ extends (complex linearly) to an involution of $\mathfrak{g}$. Write the complexified Cartan decomposition as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The moment map for the natural action of G on the cotangent bundle T^*X plays an important role in the theory of the associated cycle of Harish-Chandra modules. Viewing the cotangent bundle as $\{(\mathfrak{b}, \xi) : \mathfrak{b} \in X, \xi \in (\mathfrak{g}/\mathfrak{b})^*\}$, the moment map is given by $\mu((\mathfrak{b}, \xi)) = \xi$. It follows that μ maps T^*X into the nilpotent cone $\mathcal{N}^*$ in $\mathfrak{g}^*$. For $f \in \mathcal{N}^*$, $\mu^{-1}(f)$ is an interesting subvariety of T^*X, which is called the *Springer fiber* over f. Of particular importance in the representation theory of $G_{\mathbf{R}}$ is the Springer fiber when $f \in (\mathfrak{g}/\mathfrak{k})^*$. In this case the irreducible components of $\mu^{-1}(f)$ may be described as follows. Let K be the fixed point group of the lift to G of the complexified Cartan involution of $\mathfrak{g}$. Then, at least when K is connected, the irreducible components of $\mu^{-1}(f)$ are all of the form $\overline{T_Z^*X} \cap \mu^{-1}(f)$, where Z is a K-orbit in X and T_Z^*X is the conormal bundle to Z in T^*X. The purpose of this article is to give an explicit description of the components of the Springer fiber that correspond to the *closed* orbits Z when $G_{\mathbf{R}}$ is the real group $SU(p, q)$. The main result is contained in Theorem 4.8. This result is then used to give an algorithm that computes the associated cycles of discrete series representations.

We now consider the group $G_{\mathbf{R}} = SU(p, q)$. To describe the statement of Theorem 4.8 let us fix a closed K-orbit Z in X. There is a positive system of roots Δ^+ (with respect to the diagonal compact Cartan subalgebra) that is naturally associated to Z. The first point is that one needs to obtain a useful description of the image of T_Z^*X under μ. For this it is convenient to use the Killing form to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $(\mathfrak{g}/\mathfrak{k})^*$ with $\mathfrak{p}$. It is a fact that $\mu(T_Z^*X)$ is the closure of a single K-orbit $K \cdot f$ in $\mathcal{N}_\theta$, the cone of nilpotent elements in $\mathfrak{p}$. The details of a procedure for finding such a nilpotent element f, which we will call generic, are contained in Section 3. We mention here that $f = f_0 + f_1 + f_2 + \dots$, where f_0 is specified first (as a sum of certain root vectors), then there is a reduction to a smaller rank group G_1 where f_1 is specified, and so on. At each stage of the procedure a reductive subgroup L_i of $K \cap G_i$ (where $G_0 = G$) is defined. The groups L_i, $i = 0, 1, 2, \dots$, are easy

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to describe; the Lie algebra of L_i has root system generated by the simple compact roots in $\mathfrak{g}_i$. Theorem 4.8 states that the corresponding component of the Springer fiber is

$$T_Z^*X \cap \mu^{-1}(f) \simeq L_m \cdots L_2 L_1 L_0 \cdot \mathfrak{b}. \tag{1.1}$$

(We remark that we are identifying the Springer fiber with a subvariety of the flag variety $X = G \cdot \mathfrak{b}$, as described in (2.3).) The proof of this theorem is given in Section 4. In Section 5 we give some geometric consequences.

Our motivation for understanding these components of the Springer fiber was the problem of computing the associated cycles of discrete series representations. It is convenient to write γ for $\mu|_{T_Z^*X}$; the component of the Springer fiber corresponding to Z is therefore $\gamma^{-1}(f)$. J.-T. Chang ([2]) has given a formula for the associated cycle in terms of a sheaf cohomology space on $\gamma^{-1}(f)$. It says that the associated cycle of a discrete series representation associated to Z is $m \cdot \gamma(T_Z^*X)$ and the multiplicity m is the dimension of a cohomology space. For each discrete series representation there is a parameter λ (the infinitesimal character) and a line bundle $\mathcal{L}_\tau \to Z$ ($\tau = \lambda + \rho - 2\rho_c$). Then for the sheaf of local sections $\mathcal{O}(\tau)$ of $\mathcal{L}_\tau$ restricted to $\gamma^{-1}(f)$, Chang's theorem states that

$$m = \dim H^0(\gamma^{-1}(f), \mathcal{O}(\tau)).$$

The important point is that the description of $\gamma^{-1}(f)$ given in (1.1) allows one to use the Borel-Weil Theorem (and a simple branching law) to compute the cohomology space. This is carried out in Section 6.

An algorithm to find image γ, i.e., the orbit closure $\overline{K \cdot f}$, has been given by P. Trapa ([12]). He describes the orbit in terms of signed tableaux. His inductive procedure is quite different from ours. A. Yamamoto ([15]) has described the image of γ in terms of matrices. The significance of our procedure for producing the generic element f is that the method allows us to compute $\gamma^{-1}(f)$. We believe that our method will compute $\gamma^{-1}(f)$ for the other classical groups. Chang ([2], [4]) has used his formula to determine the associated cycles for holomorphic discrete series representations and for the discrete series for rank one groups. From a different point of view, D. King has computed character polynomials (which are the multiplicities in the associated cycles) for holomorphic discrete series and for discrete series of $SU(n,1)$. In the appendix.....

2. Preliminaries

Let $G_{\mathbf{R}}$ be a real form of a connected complex semisimple Lie group G. The Lie algebra of G will be denoted by $\mathfrak{g}$, and similar notation will be used for the Lie algebras of other Lie groups. Fix a Cartan involution of the Lie algebra of $G_{\mathbf{R}}$ and let θ denote its complex linear extension to $\mathfrak{g}$. Then θ lifts to an involution of G, which we will also denote by θ. Define K to be the fixed points of θ. The Cartan decomposition of $\mathfrak{g}$ is written as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The variety of all Borel subalgebras of $\mathfrak{g}$, the flag variety, is denoted by X. As mentioned in the introduction, our main interest is in the restriction of the moment map of T^*X

to the closures of the conormal bundles to certain K-orbits in X. Therefore we need to carefully define these objects and express them in a useful way. For any point $\mathfrak{b}$ in X, letting $B = N_G(\mathfrak{b})$, X is the homogeneous space G/B. The tangent space to X at a point $\mathfrak{b} \in X$ is naturally identified with $\mathfrak{g}/\mathfrak{b}$. Therefore the cotangent bundle is the homogeneous bundle built on the B-representation $(\mathfrak{g}/\mathfrak{b})^*$:

$$T^*X = G \underset{B}{\times} (\mathfrak{g}/\mathfrak{b})^*.$$

This is the space of equivalence classes in $G \times (\mathfrak{g}/\mathfrak{b})^*$ with respect to the equivalence relation defined by $(gb, \lambda) \sim (g, \mathrm{Ad}^*(b)\lambda)$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is contained in $\mathfrak{b}$ and write the Levi decomposition of $\mathfrak{b}$ as $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$. The Killing form allows us to identify the G-representations $\mathfrak{g}^*$ and $\mathfrak{g}$. Since $(\mathfrak{g}/\mathfrak{b})^*$ is the space of linear functionals that vanish on $\mathfrak{b}$ we have

$$(\mathfrak{g}/\mathfrak{b})^* \hookrightarrow \mathfrak{g}^*,$$

which, via the Killing form, is the inclusion

$$\mathfrak{n}^- \hookrightarrow \mathfrak{g}.$$

We will therefore take the cotangent bundle to be

$$T^*X = G \underset{B}{\times} \mathfrak{n}^-.$$

The moment map associated to the G-action on T^*X is, after identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Killing form, denoted by $\mu : G \underset{B}{\times} \mathfrak{n}^- \to \mathfrak{g}$ and is given by the formula

$$\mu(g, \xi) = \mathrm{Ad}(g)\xi, \qquad \text{for } g \in G, \xi \in \mathfrak{n}^-.$$

We consider the action of the complex group K on X and let Z be a K-orbit. The base point $\mathfrak{b}$ may be chosen so that $Z = K \cdot \mathfrak{b}$. The conormal bundle to Z in T^*X is the set of cotangent vectors at points of Z that vanish on the tangent space of Z. This is therefore the homogeneous vector bundle $K \underset{B\cap K}{\times} (\mathfrak{g}/(\mathfrak{b}+\mathfrak{k}))^*$, since the tangent space (at $\mathfrak{b}$) is $\mathfrak{k}/\mathfrak{b} \cap \mathfrak{k} \simeq (\mathfrak{k} + \mathfrak{b})/\mathfrak{b} \subset \mathfrak{g}/\mathfrak{b}$. We use the Killing form to identify the conormal bundle with

$$T^*_Z X = K \underset{B\cap K}{\times} (\mathfrak{n}^- \cap \mathfrak{p}).$$

Definition 2.1. The map γ is defined to be the restriction of the moment map μ to the closure of $T^*_Z X$ in T^*X.

Note that γ depends on the orbit Z. Since we will be considering just one K-orbit at any given time, there will be no need to include Z in the notation for γ.

Writing $g \cdot \xi = \mathrm{Ad}(g)\xi$, for $g \in G, \xi \in \mathfrak{g}$, we have

$$\gamma(k, \xi) = k \cdot \xi \in K \cdot (\mathfrak{n}^- \cap \mathfrak{p}).$$

In particular, the image of γ is the closure of $K \cdot (\mathfrak{n}^- \cap \mathfrak{p})$, which lies in the nilpotent cone

$$\mathcal{N}_\theta \equiv \mathcal{N} \cap \mathfrak{p}, \quad \mathcal{N} = \{\xi \in \mathfrak{g} : \xi \text{ is nilpotent}\}.$$

The image of γ is therefore a union of K-orbits in $\mathcal{N}_\theta$; it is in fact the closure of a single K-orbit. Therefore, there exists an $f \in \mathfrak{n}^- \cap \mathfrak{p}$ so that $\gamma(\overline{T_Z^*X}) = \overline{K \cdot f}$.

Definition 2.2. We say that $f \in \mathfrak{n}^- \cap \mathfrak{p}$ is *generic in* $\mathfrak{n}^- \cap \mathfrak{p}$ whenever $\gamma(\overline{T_Z^*X}) = \overline{K \cdot f}$.

It follows that f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$ if and only if $\overline{K \cdot f}$ contains every K-orbit in $\mathcal{N}_\theta$ that meets $\mathfrak{n}^- \cap \mathfrak{p}$. In particular, $K \cdot f$ is the K-orbit of greatest dimension meeting $\mathfrak{n}^- \cap \mathfrak{p}$.

Now let us specialize to the situation where Z is a closed K-orbit in X. Then Z is a flag variety for K. Since $\overline{T_Z^*X} = T_Z^*X$, the domain of γ is T_Z^*X and the image is $K \cdot (\mathfrak{n}^- \cap \mathfrak{p})$. For any $f \in \mathfrak{n}^- \cap \mathfrak{p}$,

$$\begin{aligned}\gamma^{-1}(f) &= \{(k,\xi) \in T_Z^*X \,:\, k \cdot \xi = f\} \\ &= \{(k, k^{-1} \cdot f) \,:\, k^{-1} \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}\}.\end{aligned}$$

Defining $N(f, \mathfrak{n}^- \cap \mathfrak{p}) \equiv \{k \in K \,:\, k \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}\}$, it follows (by restricting the natural projection $T^*X \to X$ to $\gamma^{-1}(f)$) that

$$\begin{aligned}\gamma^{-1}(f) &\simeq \{k \cdot \mathfrak{b} \,:\, k^{-1} \in N(f, \mathfrak{n}^- \cap \mathfrak{p})\} \\ &= N(f, \mathfrak{n}^- \cap \mathfrak{p})^{-1} \cdot \mathfrak{b} \subset Z.\end{aligned} \tag{2.3}$$

Thus, the fiber $\gamma^{-1}(f)$ may be identified with a subvariety of the flag variety Z.

Since the remainder of this article deals with closed K-orbits in X, we will need to describe them. It suffices for our purposes to assume that $G_{\mathbf{R}}$ has a compact Cartan subgroup. We may therefore fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is contained in $\mathfrak{k}$. Let $\Delta(\mathfrak{h}, \mathfrak{g})$ (resp., $\Delta(\mathfrak{h}, \mathfrak{k})$) be the system of roots of $\mathfrak{h}$ in $\mathfrak{g}$ (resp., in $\mathfrak{k}$), and let W and W_c be the corresponding Weyl groups. Then it is well known that the closed K-orbits in X are in one-to-one correspondence with W/W_c. One way to express such a one-to-one correspondence is as follows. Fix a positive system Δ_c^+ in $\Delta(\mathfrak{h}, \mathfrak{k})$. Then for each positive system $\Delta^+ \subset \Delta(\mathfrak{h}, \mathfrak{g})$ containing Δ_c^+ define a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ by specifying that $\mathfrak{n}^-$ is the sum of all root spaces for roots in $-\Delta^+$. Since $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra in $\mathfrak{k}$, $Z = K \cdot \mathfrak{b}$ is a closed K-orbit in X. All closed orbits occur exactly once in this manner. Thus, we have a one-to-one correspondence between the set of closed K-orbits and the set of positive systems of $\Delta(\mathfrak{h}, \mathfrak{g})$ containing Δ_c^+, which is in bijection with W/W_c.

We now restrict our attention to the group $G_{\mathbf{R}} = SU(p,q)$, where p and q are nonnegative integers. Let $n = p + q$. The realization we use is

$$G_{\mathbf{R}} = \{g \in M_{n\times n}(\mathbf{C}) \,:\, \overline{g}^t I_{p,q} g = I_{p,q}, \det(g) = 1\}, \text{ with } I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

The Cartan involution is chosen to be $\theta = \mathrm{Ad}(I_{p,q})$. Then

$$G = GL(n) \text{ and } K = \Big\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,:\, a \in GL(p), d \in GL(d) \text{ and } \det(a)\det(d) = 1 \Big\}.$$

Let $\mathfrak{h} \subset \mathfrak{k}$ be the Cartan subalgebra of all diagonal matrices of trace zero. Define $\epsilon_j \in \mathfrak{h}^*$ to be the linear functional giving the j^{th} diagonal entry. Then

$$\Delta(\mathfrak{h}, \mathfrak{g}) = \{\epsilon_j - \epsilon_k \,:\, 1 \le j, k \le n, j \ne k\}.$$

We fix once and for all the positive system of compact roots

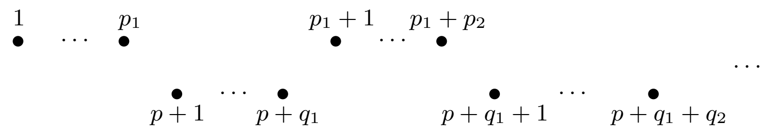
$$\Delta_c^+ = \{\epsilon_j - \epsilon_k \,:\, 1 \le j < k \le p \text{ or } p+1 \le j < k \le n\}. \tag{2.4}$$

As discussed earlier, the closed K-orbits in X are parameterized by the positive systems $\Delta^+ \subset \Delta(\mathfrak{h}, \mathfrak{g})$ that contain Δ_c^+. They are in one-to-one correspondence with sequences of integers $(p_1, q_1, p_2, \dots, p_r, q_r)$ so that

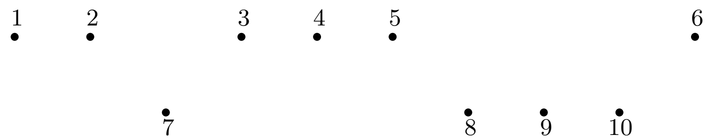
$$\begin{aligned} &\Sigma p_i = p \text{ and } \Sigma q_i = q, \\ &p_i, q_i \text{ are non-negative integers and} \\ &p_i > 0 \text{ for } i = 2, 3, \dots, r \text{ and } q_j > 0 \text{ for } j = 1, 2, \dots, r-1. \end{aligned} \tag{2.5}$$

Note that p_1 and q_r may be zero.

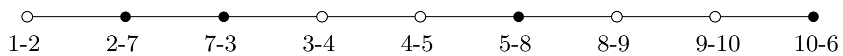
It will be useful to describe such a sequence in terms of an array



We call a sequence of consecutive labelled dots in the array a *block*. Therefore, the i^{th} block (counting from left to right) in the upper row has p_i dots and the i^{th} block on the lower row has q_i dots. The simple compact roots are the roots $\epsilon_i - \epsilon_{i+1}$ with $(i, i+1)$ indices of dots that belong to the same block. The simple non-compact roots are the roots $\epsilon_i - \epsilon_j$ with i, j indices of consecutive dots that lie in different rows, and so that i precedes j when reading the array from left to right. Thus, the simple non-compact roots correspond to the "jumps" between the rows. Here is an example. The array



determines the Dynkin diagram



where 'i-j' means the root $\epsilon_i - \epsilon_j$ (and the blackened nodes correspond to non-compact simple roots).

The final bit of preliminary information is the parametrization of the K-orbits in $\mathcal{N}_\theta$. We will describe these in terms of signed tableaux. This information is well-known and can be found in the present form in [1]. Suppose that $\{e, h, f\} \subset \mathfrak{g}$ spans a copy of $\mathfrak{sl}(2)$. Let $SL(2)$ be the corresponding complex subgroup of G. Suppose also that e, h and f satisfy the relations

$$\begin{aligned} &[e, f] = h, [h, e] = 2e \text{ and } [h, f] = -2f \\ &\theta(h) = h, \theta(e) = -e \text{ and } \theta(f) = -f. \end{aligned} \tag{2.6}$$

Form the semidirect product $\mathbf{Z}_2 \ltimes SL(2)$ where the non-trivial element of $\mathbf{Z}_2$ acts by θ. Irreducible finite dimensional representations of $SL(2)$ extend to representations of the semidirect product in two distinct ways. These are distinguished by the action of the non-trivial element of $\mathbf{Z}_2$ being $+1$ or -1 on the lowest weight space. Define the signature of a (possible reducible) representation π of $\mathbf{Z}_2 \ltimes SL(2)$ to be the pair (a_+, a_-), where $a_\pm$ is the dimension of the ± 1 eigenspace of θ in the kernel of $\pi(f)$ (= the lowest weight space).

Now suppose that $f \in \mathcal{N}_\theta$. Then f fits into a triple $\{e, h, f\}$ satisfying (2.6); see [5]. This gives a copy of $SL(2)$ inside $G = SL(n)$, thus a representation of $SL(2)$ on $\mathbf{C}^n, n = p + q$. Extend this representation to a representation π of $Z_2 \ltimes SL(2)$ so that the action of the non-trivial element of Z_2 is by $I_{p,q}$. Define $a_\pm(f^j)$ to be the dimension of the ± 1 eigenspace of $I_{p,q}$ on the kernel of $\pi(f^j)$. Write $a(f^j) = a_+(f^j) + a_-(f^j)$ for the dimension of the kernel of $\pi(f^j)$. Decompose $\mathbf{C}^n = \oplus V_i$ into irreducible $Z_2 \ltimes SL(2)$-representations and let δ_i be the eigenvalue of θ on the lowest weight vector of V_i.

Theorem 2.7. ([6]) Two nilpotent elements f and f' are K-conjugate if and only $a_\pm(f^j) = a_\pm(f'^j)$, for every $j = 1, 2, \ldots$. The inclusion $\mathcal{O}(f') \subset \overline{\mathcal{O}(f)}$ holds if and only if for every j

$$a_+(f'^j) \geq a_+(f^j) \quad \text{and} \quad a_-(f'^j) \geq a_-(f^j).$$

To each nilpotent orbit we associate a signed tableau as follows. The tableau has a row for each irreducible constituent V_i; the number of boxes in the i^{th} row is the dimension of the representation V_i. Signs are inserted in each box by beginning the i^{th} row with the sign of δ_i, then alternating the signs along each row. Then two such signed tableaux correspond to the same orbit if and only if they are the same up to a permutation of the rows.

Lemma 2.8. *A nilpotent element f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$ if and only if for all j*

$$\begin{aligned} a_+(f^j) &= \min\{a_+(f'^j) : f' \in \gamma(T^*_Z(X))\} \text{ and} \\ a_-(f^j) &= \min\{a_-(f'^j) : f' \in \gamma(T^*_Z(X))\}. \end{aligned}$$

Proof. An element f is generic if and only if $\gamma(T^*_Z(X)) = \overline{K \cdot f}$. Thus, f is generic if and only if $K \cdot f' \subset \overline{K \cdot f}$ for any other $f' \in \gamma(T^*_Z(X))$. The lemma now follows from Theorem 2.7. $\square$

3. Generic elements

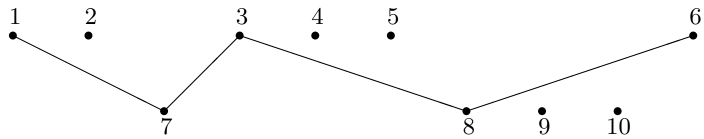
Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$, $\mathfrak{h}$ the diagonal Cartan subalgebra be a Borel subalgebra. An algorithm will now be given for finding a generic element in $\mathfrak{n}^- \cap \mathfrak{p}$.

For the remainder of this section we fix a closed K-orbit Z in X. As in Section 2, there is therefore a positive system $\Delta^+ \subset \Delta(\mathfrak{h}, \mathfrak{g})$ containing Δ^+_c and a corresponding Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ so that $Z = K \cdot \mathfrak{b}$. Let $\{p_1, q_1, p_2, \ldots, q_r\}$ be the sequence of non-negative integers as in (2.5) that determines Δ^+ (and hence Z). The algorithm of this section will produce $f \in \mathfrak{n}^- \cap \mathfrak{p}$ so that $K \cdot f$ is dense in the image of $\gamma : T^*_Z X \to \mathfrak{g}$.

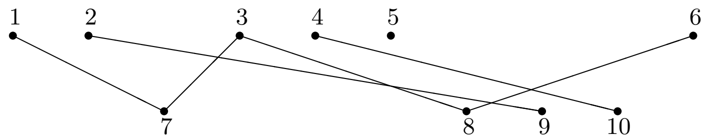
The algorithm is as follows. From the sequence $\{p_1, q_1, \ldots, p_r, q_r\}$, first form an array as in the paragraph following (2.5). Second, form a *string* consisting of diagonal lines connecting the first dots in each pair of consecutive blocks. Define a nilpotent element f_0 of $\mathfrak{n}^- \cap \mathfrak{p}$ as follows. Let $A_0 = \{i_1, i_2, \ldots, i_N\}$ be the set of indices of dots that the string passes through, ordered from left to right. Then

$$f_0 = \sum_{s=1}^{N-1} X_{i_{s+1}, i_s}, \tag{3.1}$$

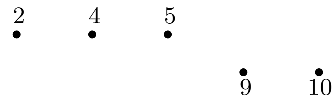
where $X_{i,j}$ is the matrix that is a root vector for $\epsilon_i - \epsilon_j$ with a one in the (i,j) place. In the example following (2.5), we have



Third, omit the dots that the string passes through and repeat the procedure with the smaller array to obtain an f_1 and an A_1. The procedure is continued until no more diagonals can be drawn. In the example, we have



Note that as the dots in the most recent string are omitted a new array is formed. For example, to choose the second string in the example we omit the dots numbered 1, 7, 3, 8 and 6 to get



Each string corresponds to a sum of root vectors in $\mathfrak{n}^- \cap \mathfrak{p}$. In the example we have

$$f_0 = X_{6,1} + X_{3,6} + X_{8,3} + X_{6,8}, f_1 = X_{9,2} \text{ and } f_2 = X_{10,4}.$$

Set

$$f = f_0 + f_1 + \ldots + f_{m-1}, \text{ with } m \text{ equal to the number of strings.}$$

Theorem 3.2. *Let Z be a closed K-orbit in X, and let the Borel subalgebra $\mathfrak{b}$ and the sequence $\{p_1, q_1, p_2, \ldots, p_r, q_r\}$ be as described above. Then the element f built by the algorithm is generic in $\mathfrak{n}^- \cap \mathfrak{p}$, i.e., $\overline{K \cdot f} = \gamma(T_Z^* X)$.*

The remainder of this section is devoted to a proof of this Theorem. It should be emphasized that the method of proof allows us to describe the relevant components of the Springer fiber. This will be done in Section 4; the crucial ingredient is isolated in Proposition 3.14.

Observe that for each string, f_j is a principal nilpotent element in a subalgebra $\mathfrak{sl}(d_j)$ where d_j is the number of dots in the corresponding string. Starting with f_j it is possible to form an $\mathfrak{sl}(2)$-triple $\{f_j, h_j, \hat{e}_j\}$ so that $h_j \in \mathfrak{h}$ and

$$\hat{e}_j = \sum_{\{(k,l):\ X_{l,k} \text{ occurs in } f_j\}} a_{k,l}\, X_{k,l}$$

with non-zero coefficients $a_{k,l}$. Since the $\mathfrak{sl}(d_j)$'s commute, $\{f, h = \sum h_j, e = \sum \hat{e}_j\}$ spans a copy of $\mathfrak{sl}(2)$. Let $SL(2)_f$ be the Lie subgroup of $SL(n)$ whose Lie algebra is this copy of $\mathfrak{sl}(2)$. It is clear that the standard basis vectors $e_l \in \mathbf{C}^n$ are weight vectors for the action π of $\mathbf{Z}_2 \ltimes SL(2)_f$ on $\mathbf{C}^n$. Consider the decompose $\mathbf{C}^n$ under π. We may conclude

(1) the dimension of the non-trivial irreducible subrepresentations of $\mathbf{C}^n$ are given by the numbers of dots in the various strings,
(2) the lowest weight vector of a non-trivial irreducible subrepresentation is the standard basis vector e_k where k is the label of the last dot (that is, the dot farthest to the right) in the corresponding string, and
(3) the trivial subrepresentations are spanned by the t vectors e_k for which no dot contained in any string has label k.

This information translates into the following description of the signed tableau for f.

Fact 3.3. *The signed tableau corresponding to the nilpotent K-orbit $K \cdot f$ has $m + t$ rows. If $1 \leq i \leq m$, then the length of the i^{th} row in the tableau is the number of dots occurring in the i^{th} string. If the i^{th} string ends at a dot in the top row of the array, then the i^{th}row of the tableau has alternating signs starting with $+$. Otherwise, the i^{th}row of the tableau has alternating signs starting with $-$. The remaining t rows have length one and their corresponding signs are so that the total number of $+$ signs in the tableau is p and the total number of $-$ signs is q.*

In our example the tableau corresponding to $K \cdot f$ is

$$\begin{array}{|c|c|c|c|c|}
\hline
+ & - & + & - & + \\
\hline
- & + \\
\cline{1-2}
- & + \\
\cline{1-2}
+ \\
\cline{1-1}
\end{array}$$

The subgroups defined below are crucial to both our description of the Springer fiber and to the inductive proofs in the rest of the paper.

Let S be the set of simple compact roots in Δ^+ and $\langle S \rangle$ the set of roots generated by S.

Definition 3.4. (a) Define $\mathfrak{q}$ to be the parabolic subalgebra of $\mathfrak{g}$ defined by the simple roots S, i.e.,

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}^-, \quad \text{with } \mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \langle S \rangle} \mathfrak{g}^{(\alpha)} \text{ and } \mathfrak{u}^- = \sum_{\alpha \in \Delta^+ \setminus \langle S \rangle} \mathfrak{g}^{(\alpha)}.$$

The connected subgroup of G with Lie algebra $\mathfrak{q}$ (resp., $\mathfrak{l}$) will be denoted by Q (resp., L). Set $Q_K = Q \cap K$.
(b) Consider the array that is the result of omitting all dots that are passed though by any one of the first i strings. Then $\Delta_i \cong \{\epsilon_j - \epsilon_k \,:\, j, k \text{ are indices of remaining dots}\}$ is a root system of type $A_{n'}, n' < n$. The Lie subalgebra generated by root spaces for roots in Δ_i is denoted by $\mathfrak{g}_i$. The corresponding subgroup of G is denoted by G_i. We set $K_i = K \cap G_i$.
(c) Let S_i be the set of simple compact roots in Δ_i^+, then S_i determines a parabolic subalgebra $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i^-$ of $\mathfrak{g}_i$ as in (a). Let Q_i be the subgroup of G_i with Lie algebra $\mathfrak{q}_i$; we set $Q_{i,K} = Q_i \cap K$.

The subalgebra $\mathfrak{g}_i$ is θ-stable and is the complexification of a smaller indefinite unitary Lie algebra. Furthermore, $\mathfrak{h}_i \cong \mathfrak{h} \cap \mathfrak{g}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$ and $\mathfrak{b}_i = \mathfrak{b} \cap \mathfrak{g}_i = \mathfrak{h}_i + \mathfrak{n}_i^-$ is a Borel subalgebra so that the negative root vectors with respect to Δ_i span $\mathfrak{n}_i^-$. This positive system corresponds to the array with the first i strings omitted. There is a corresponding closed K_i-orbit $Z_i = K_i \cdot \mathfrak{b}_i$ in the flag variety for G_i.

Note that $\mathfrak{g}_1$, for example, is the subalgebra of $\mathfrak{sl}(n)$ consisting of matrices having 0's in the j^{th} row and column for each index j occurring as index of a dot in the first string.

Remark 3.5. The following properties follow easily.

(1) Q_K normalizes $\mathfrak{n}^- \cap \mathfrak{p}$ and $\mathfrak{n}^- \cap \mathfrak{p} = \mathfrak{u}^- \cap \mathfrak{p}$
(2) $L_i \subset K$ and $\mathfrak{u}_i^- \cap \mathfrak{p} = \mathfrak{g}_i \cap (\mathfrak{n}^- \cap \mathfrak{p})$.
(3) $\mathfrak{u}_i^- \subset \mathfrak{u}_{i-1}^-$.
(4) $Q_i \cdot f_k = f_k$ for all $k = 0, 1, \ldots, i-1$.

One should be aware that it is *not* always the case that $\mathfrak{q} \cap \mathfrak{g}_i = \mathfrak{q}_i$. Our example in $SL(9)$ illustrates this; when a string is omitted, several blocks 'collapse' to one block in the smaller array.

We next describe the parabolic subgroup Q as the subgroup of G consisting of all linear transformations preserving a flag in $\mathbf{C}^n$. The following definition specifies the correct flag. Let N be the number of blocks in the array.

Definition 3.6. Define F_j to be the span of the e_i for all i occurring in any one of the $N - j + 1$ blocks farthest to the right. Set $F_{N+1} = \{0\}$.

Lemma 3.7. *The following hold.*

(1) *If* $Y \in \mathfrak{n}^- \cap \mathfrak{p}$, *then* $Y(F_k) \subset F_{k+1}$ *and* $Y^j(F_k) \subset F_{k+j}$.
(2) *If* $Y \in \mathfrak{n}^- \cap \mathfrak{p}$, *then* $Y^{N-k+1}(F_k) = 0$. *In particular* $Y^N = 0$.
(3) *The spaces* F_k *are preserved by the* Q_K*-action.*
(4) *The stabilizer of the flag* $\mathbf{C}^n = F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \ldots F_N \supsetneq F_{N+1} = \{0\}$ *is* Q.

We are now in position to begin the proof of Theorem 3.2. Continue with our fixed positive system Δ^+ containing Δ_c^+, and resulting sequence $\{p_1, q_1, p_2 \ldots, q_r\}$ as in (2.5) and

$f = f_0 + \cdots f_{m-1}$ built by the algorithm. Set $f = f_0 + f', f' = f_1 + \cdots + f_{m-1}$. Let e, h be chosen as in the paragraph preceding Facts 3.3. Then $SL(2)_f$ denotes the corresponding subgroup of G. Let $(\pi, \mathbf{C}^n)$ be the representation of $\mathbf{Z}_2 \ltimes SL(2)_f$ for which the non-trivial element of $\mathbf{Z}_2$ acts by $I_{p,q}$ and $SL(2)_f$ acts by it embedding in G. Let A_0 be the indices labelling dots in the array that the string for f_0 passes through. Thus, $\#A_0 = N$, which is the length of the first string as well as the length of the flag (F_j) that defines the parabolic subgroup Q. The proof of the following lemma is immediate from the definitions.

Lemma 3.8. *Let $V_0 = \text{span}_{\mathbf{C}}\{e_i : i \in A_0\}$ and $W_0 = \text{span}_{\mathbf{C}}\{e_k : k \notin A_0\}$. Under the action of π, $\mathbf{C}^n$ decomposes as $\mathbf{C}^n = V_0 \oplus W_0$ and*

$$\pi(f_0)|_{W_0} = 0 \text{ and } \pi(f_0)V_0 \subset V_0$$
$$\pi(f')|_{V_0} = 0 \text{ and } \pi(f')W_0 \subset W_0.$$

Now let $Y \in \mathfrak{n}^- \cap \mathfrak{p}$. Form a triple $\{X, H, Y\}$ spanning a copy of $\mathfrak{sl}(2)$ with $X \in \mathfrak{n} \cap \mathfrak{p}$ and $H \in \mathfrak{k}$ and let $SL(2)_Y$ be the subgroup of G with Lie algebra $\text{span}_{\mathbf{C}}\{X, H, Y\}$. Then $\mathbf{Z}_2 \ltimes SL(2)_Y$ acts on $\mathbf{C}^n$.

Lemma 3.9. *If $K \cdot f \subset \overline{K \cdot Y}$, then $\mathbf{C}^n$ has a $\mathbf{Z}_2 \ltimes SL(2)_Y$-irreducible constituent of dimension N.*

Proof. By Proposition 3.7, $Y^N = 0$. Hence, $\mathbf{C}^n$ admits no irreducible constituent of dimension greater than N. Assume that no $Z_2 \ltimes SL(2)_Y$-constituent is of dimension N. Write $\mathbf{C}^n = R_1 \oplus \ldots \oplus R_t$ where R_i are $Z_2 \ltimes SL(2)_Y$-irreducible subrepresentations. Then $\max_i\{\dim(R_i)\} = N'$ with $N' < N$, so $\dim(\text{Ker } (Y^{N'})) = p + q$. On the other hand, since $Z_2 \ltimes SL(2)_f$ admits an irreducible subrepresentation of $\mathbf{C}^n$ of dimension N, $\dim(\text{Ker}(f^{N'})) < p + q$. Then Theorem 2.7 gives a contradiction to our hypothesis that $K \cdot f \subset \overline{K \cdot Y}$. □

Continue with Y as in Lemma 3.9. Decompose $\mathbf{C}^n$ under the $Z_2 \ltimes SL(2)_Y$-action as $\mathbf{C}^n = V_N \oplus W$ with V_N irreducible of dimension N. Denote by v_0 the highest weight vector of V_N. The set $\{v_0, Yv_0, \ldots, Y^{N-1}v_0\}$ is therefore a basis for V_N. Note that $Y^{k-1}v_0 \in F_k$.

Lemma 3.10. *For each k, $F_k = (F_k \cap V_N) \oplus (F_k \cap W)$.*

Proof. Write $v \in F_k$ as $v = v_N + w$ with $v_N = \sum_{j=0}^{N-1} a_j Y^j v_0 \in V_N$ and $w \in W$. We need to show that v_N and w belong to F_k. It is enough to show that $v_N \in F_k$.

Observe that $0 = Y^{N-k+1}v = Y^{N-k+1}v_N + Y^{N-k+1}w$, so

$$0 = Y^{N-k+1}v_N = \sum_{j=0}^{k-2} a_j Y^{N-k+1+j} v_0.$$

Since the vectors $\{v_0, Yv_0, \ldots, Y^{N-1}v_0\}$ are linearly independent, $a_j = 0$ for all $j \leq k-2$. Thus, $v_N = \sum_{j=k-1}^{N-1} a_j Y^j v_0$ lies in F_k, by Proposition 3.7 (1). □

Since $F_k \cap V_N = \mathbf{C} \cdot Y^{k-1}v_0 + F_{k+1} \cap V_N$, we have the following corollary.

Corollary 3.11. *With W as above*

$$\dim(F_k \cap W)/(F_{k+1} \cap W) = \dim(F_k/F_{k+1}) - 1.$$

The following lemma implies the existence of a basis of $\mathbf{C}^n$ that is well-behaved with respect to the action of Y.

Lemma 3.12. *There is a basis $\beta = \beta_1 \cup \cdots \cup \beta_N$ of $\mathbf{C}^n$ with the following properties.*

(a) *β_j is contained in either $F_j \cap (\mathbf{C}^p \times \{0\})$ or $F_j \cap (\{0\} \times \mathbf{C}^q)$.*
(b) *the cardinality of β_j is $\dim(F_j/F_{j+1}), j = 1, \dots, N$.*
(c) *$Y^{j-1}v_0$ is in β_j.*
(d) *$\beta_j \setminus \{Y^{j-1}v_0\} \subset F_j \cap W$.*

Proof. Each β_j may be defined as follows. Put $Y^{j-1}v_0$ in β_j. If the j^{th} block in the array is up, then, by the definition of the flag and the fact that $F_j \cap W$ is $I_{p,q}$-stable, the natural map

$$(F_j \cap W) \cap (\mathbf{C}^p \times \{0\}) \to F_j \cap W/F_{j+1} \cap W$$

is an isomorphism. If the j^{th} block is down then we have an isomorphism

$$(F_j \cap W) \cap (\{0\} \times \mathbf{C}^q) \to F_j \cap W/F_{j+1} \cap W.$$

Fill out the remainder of β_j by pulling back a basis of $F_j \cap W/F_{j+1} \cap W$. □

A basis as in the Lemma may be ordered by (1) putting $Y^{j-1}v_0$ first in each β_j, and (2) by choosing the β_j's in the order

$\beta_1, \beta_3, \dots, \beta_2, \beta_4, \dots$ (odd indices first), if the first block is up

and

$\beta_2, \beta_4, \dots, \beta_1, \beta_3, \dots$ (even indices first), if the first block is down.

Let q be the matrix with the basis vectors of β inserted as columns, ordered as above. Then q preserves the flag (F_j), so lies in Q. It follows from (a) that q is also in K. Then, writing $A_0 = \{i_1, \dots, i_N\}$ for the indices of the dots passed through by the first string, ordered from left to right, we have

$$\begin{aligned} q^{-1}Yqe_{i_j} &= q^{-1}YY^{j-1}v_0 \\ &= q^{-1}Y^j v_0 \\ &= e_{i_{j+1}}. \end{aligned}$$

For $i \notin A_0$,

$$q^{-1}Yqe_i \in \operatorname{span}_{\mathbf{C}}\{e_j \,:\, j \notin A_0\} = W_0$$

by (d). Therefore $f_0 - q^{-1} \cdot Y \in (\mathfrak{n}^- \cap \mathfrak{p}) \cap \mathfrak{g}_1 = \mathfrak{u}_1^- \cap \mathfrak{p}$. The following lemma is now proved.

Lemma 3.13. *There exists $q \in Q_K$ so that $q \cdot Y = f_0 + Y_1$, with $Y_1 \in \mathfrak{u}_1^- \cap \mathfrak{p}$.*

Proposition 3.14. *If $Y \in \mathfrak{n}^- \cap \mathfrak{p}$ with $K \cdot f \subset \overline{K \cdot Y}$, then there exists $q \in Q_K$ and $q_i \in Q_{i,K}$ so that $q_{m-1} \cdots q_2 q_1 q \cdot Y = f$.*

Proof. We use induction on the complex rank of $\mathfrak{g}$. Lemma 3.13 tells us that there exists $q \in Q$ so that $q \cdot Y = f_0 + Y_1, Y_1 \in \mathfrak{u}_1^- \cap \mathfrak{p}$. Recall that we have written $f = f_0 + f'$. We claim that for $K_1 = K \cap G_1$, $K_1 \cdot g \subset \overline{K_1 \cdot Y_1}$. Once this claim is proved the inductive hypothesis gives $q_{m-1} \cdots q_1 \cdot Y_1 = f'$. Since $q_i \cdot f_0 = f_0$, for all $i = 1, \dots, m-1$ (as observed in Remark 3.5), $q_{m-1} \dots q_1 q \cdot Y = f_0 + f' = f$.

Now we turn to the proof of the claim. Write $\mathbf{C}^n = V_0 \oplus W_0$ as earlier. Then by Lemma 3.8

$$\begin{aligned} a_\pm(f^j) &= a_\pm((f_0|_{V_0})^j) + a_\pm((f'|_{W_0})^j) \\ a_\pm(Y^j) &= a_\pm((q \cdot Y)^j) = a_\pm((f_0|_{V_0})^j) + a_\pm((Y_1|_{W_0})^j). \end{aligned}$$

Since $K \cdot f \subset \overline{K \cdot Y}$ we conclude from Theorem 2.7 that

$$a_\pm((Y_1|_{W_0})^j) \le a_\pm((f'|_{W_0})^j),$$

for each $j = 1, 2, \dots$. Now Theorem 2.7 (applied in G_1) proves the claim. □

Proof of Theorem 3.2. Assume that $Y \in \mathfrak{n}^- \cap \mathfrak{p}$ is generic. Then $K \cdot Y$ is dense in $K \cdot \mathfrak{n}^- \cap \mathfrak{p}$. Hence, $K \cdot f \subset \overline{K \cdot Y}$. By Proposition 3.14, there exist $k_0 = q_{m-1} \dots q_1 q_k \in K$ so that $k_0 \cdot Y = f$. Hence, $K \cdot Y = K \cdot f$, i.e., f is also generic. □

4. The Springer Fiber

In this section Proposition 3.14 of Section 3 is used to determine the structure of the fiber of γ over a generic element. We continue with the setup of Section 3. In particular, a closed K-orbit in X, which determines a sequence $(p_1, q_1, p_2, \dots, q_r)$ and a corresponding array, has been fixed. We write $f = f_0 + \dots + f_{m-1}$ for the given element of $\mathfrak{n}^- \cap \mathfrak{p}$ built by the algorithm. We denote the centralizer of f in K by $Z_K(f)$.

Theorem 4.1. *The following expression for the fiber of γ holds.*

$$\gamma^{-1}(f) = Z_K(f) L_{m-1} L_{m-2} \dots L_1 L_K \cdot \mathfrak{b} \subset K/K \cap B. \tag{4.2}$$

Proof. We begin by showing that

$$\gamma^{-1}(f) = Z_K(f) Q_{m-1,K} Q_{m-2,K} \dots Q_{1,K} Q_K \cdot \mathfrak{b} \subset K/K \cap B. \tag{4.3}$$

By equation (2.3), $\gamma^{-1}(f) = (N_K(f, \mathfrak{n}^- \cap \mathfrak{p}))^{-1} \cdot \mathfrak{b}$ where

$$N_K(f, \mathfrak{n}^- \cap \mathfrak{p}) = \{k \in K : k \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}\}.$$

To prove (4.3) it is therefore enough to show that

$$N_K(f, \mathfrak{n}^- \cap \mathfrak{p}) = Q_K Q_{1,K} \dots Q_{m-1,K} Z_K(f).$$

To show $N_K(f, \mathfrak{n}^- \cap \mathfrak{p}) \subset Q_K Q_{1,K} \dots Q_{m-1,K} Z_K(f)$, take $k \in N_K(f, \mathfrak{n}^- \cap \mathfrak{p})$. Then, $k \cdot f$ lies in $\mathfrak{n}^- \cap \mathfrak{p}$ and is generic. Therefore by Proposition 3.14 there exist $q_i \in Q_{i,K}$ and $q \in Q_K$

so that $q_{m-1}q_{m-2}\dots q_1q\cdot(k\cdot f)=f$. Thus, $q_{m-1}q_{m-2}\dots q_1qk\in Z_K(f)$. The inclusion follows.

For the other inclusion observe that Q_K normalizes $\mathfrak{n}^-\cap\mathfrak{p}$ $(=\mathfrak{u}^-\cap\mathfrak{p})$ and $Z_K(f)$ fixes f. Hence, it is enough to show that $Q_{1,K}Q_{2,K}\dots Q_{m-1,K}\subset N_K(f,\mathfrak{n}^-\cap\mathfrak{p})$. Recall that in the expression $f=f_0+f_1+\ldots+f_i+\ldots+f_{m-1}$, we have $f_0\in\mathfrak{n}^-\cap\mathfrak{p}$ and $f_i\in\mathfrak{u}_i^-\cap\mathfrak{p}$ for $i\geq 1$. By Remark 3.5, $Q_{i,K}$ normalizes $\mathfrak{u}_i^-\cap\mathfrak{p}$, $\mathfrak{u}_i^-\cap\mathfrak{p}\subset\mathfrak{u}_{i-1}^-\cap\mathfrak{p}$ and $Q_{i,K}$ stabilizes all f_j with $j<i$. Therefore,

$$\begin{aligned}Q_{m-1,K}(f)&\subset f_0+f_1+\ldots+f_{m-2}+Q_{m-1,K}(f_{m-1})\\&\subset f_0+f_1+\ldots+f_{m-2}+(\mathfrak{u}_{m-1}^-\cap\mathfrak{p}).\end{aligned}$$

Proceeding by (downward) induction on i, assume that

$$Q_{i,K}Q_{i+1,K}\dots Q_{m-1,K}(f)\subset f_0+f_1+\ldots+f_{i-1}+(\mathfrak{u}_i^-\cap\mathfrak{p}).$$

holds. Then,

$$\begin{aligned}&Q_{i-1,K}Q_{i,K}Q_{i+1,K}\dots Q_{m-1,K}(f)\\&\qquad\subset f_0+f_1+\ldots+f_{i-2}+Q_{i-1,K}\big(f_{i-1}+(\mathfrak{u}_i\cap\mathfrak{p})\big)\\&\qquad\subset f_0+f_1+\ldots+f_{i-2}+Q_{i-1,K}\big(f_{i-1}+(\mathfrak{u}_{i-1}\cap\mathfrak{p})\big)\\&\qquad\subset f_0+f_1+\ldots+f_{i-2}+(\mathfrak{u}_{i-1}\cap\mathfrak{p}).\end{aligned}$$

Therefore we conclude that $Q_{1,K}Q_{2,K}\dots Q_{m-1,K}(f)\subset f_0+(\mathfrak{u}_1\cap\mathfrak{p})\subset\mathfrak{n}^-\cap\mathfrak{p}$ and (4.3) holds.

Now we check that each $Q_{i,K}$ may be replaced by L_i. Since $\mathfrak{u}^-\cap\mathfrak{k}\subset\mathfrak{b}$ it is clear that $Q_K\cdot\mathfrak{b}=L\cdot\mathfrak{b}$, so Q_K may be replaced by L. We show by induction that

$$Q_{j,K}\cdots Q_{1,K}Q_K\cdot\mathfrak{b}=L_j\cdots L_1L\cdot\mathfrak{b}. \tag{4.4}$$

Since $\mathfrak{u}_1^-\cap\mathfrak{k}\subset\mathfrak{u}^-\cap\mathfrak{k}$, we have $Q_{1,K}Q_K\cdot\mathfrak{b}=L_1Q_K\cdot\mathfrak{b}=L_1L\cdot\mathfrak{b}$, proving the $j=1$ case. By Remark 3.5 $\mathfrak{u}_i^-\subset\mathfrak{u}_{i-1}^-$, so $[\mathfrak{l}_{i-1},\mathfrak{u}_i^-\cap\mathfrak{k}]\subset[\mathfrak{l}_{i-1},\mathfrak{u}_{i-1}^-\cap\mathfrak{k}]\subset\mathfrak{q}_{i-1}\cap\mathfrak{k}$. Therefore, $Q_{i,K}Q_{i-1,K}=L_iQ_{i-1,K}$. Assuming (4.4) holds for $j=i-1$,

$$Q_{i,K}Q_{i-1,K}\cdots Q_{1,K}Q_K\cdot\mathfrak{b}=L_iQ_{i-1,K}\cdots Q_{1,K}Q_K\cdot\mathfrak{b}=L_iL_{i-1}\cdots L_1L\cdot\mathfrak{b}.$$

The proposition is now proved. ☐

Theorem 4.8 below makes the structure of the fiber of γ much more tractable. It essentially says that the centralizer may be dropped from the expression for the fiber in the above theorem. We must however include L_m, which is formed in the algorithm for the generic element f after the last string is formed. Note that $\Delta(\mathfrak{l}_m)$ consists of roots with indices not in any of the strings, therefore L_m is contained in the centralizer of f (which is why L_m is not needed in (4.2)).

The proof will use a fairly explicit description of the centralizer of f, and this will require the introduction of some (temporary) notation. Recall that m is the number of strings. These strings give $f=f_0+\cdots+f_{m-1}$. For $a=0,1,\ldots,m-1$ define A_a to be the set of all indices of dots in the string from which f_a is formed. In other words A_a is the set of indices occurring in the root vectors in the expression for f_a. Let A_m be the set of indices not occurring in any of the strings. Now set

$$V_{a,b} = \mathrm{span}_{\mathbf{C}}\{X_{i,j} \,:\, i \in A_a, j \in A_b\}.$$

Recall that $X_{i,j}$ is the root vector with a 1 in the (i,j)-place and zeros elsewhere. Let $\mathfrak{z} = \mathfrak{z}_{\mathfrak{k}}$, the Lie algebra of $Z_K(f)$, and set

$$\mathfrak{z}_{a,b} = \mathfrak{z} \cap V_{a,b}.$$

Since $V_{a,b}$ is $\mathrm{ad}(f)$-invariant

$$\mathfrak{z} = \oplus \mathfrak{z}_{a,b}.$$

In fact, $V_{a,b}$ is invariant under the $\mathfrak{sl}(2)$ corresponding to f.

Consider one of the A_a's. Write $A_a = \{i_1, \dots, i_R\}$ ordered so that each i_r occurs to the left of i_{r+1} in the array. Therefore,

$$f_a = \sum_{r=2}^{R} X_{i_r, i_{r-1}}.$$

Similarly, write

$$f_b = \sum_{t=2}^{T} X_{i_t, i_{t-1}}.$$

We now find a basis of $\mathfrak{z}$ by finding a basis for each $\mathfrak{z}_{a,b}$. There are 5 different cases which must be considered.

Case (1) $a \neq b$ and $a, b \neq m$. Let $X = \sum a_{ij} X_{i,j} \in V_{a,b}$. We see when X commutes with f.

$$\begin{aligned}
[f, X] &= [f_a, X] + [f_b, X] \\
&= f_a X - X f_b \\
&= \sum_{r=2}^{R} \sum_{i \in A_a} \sum_{j \in A_b} a_{ij} X_{i_r, i_{r-1}} X_{i,j} - \sum_{t=2}^{T} \sum_{i \in A_a} \sum_{j \in A_b} a_{ij} X_{i,j} X_{j_t, j_{t-1}} \\
&= \sum_{r=2} \sum_{j \in A_b} a_{i_{r-1}, j} X_{i_r, j} - \sum_{t=2}^{T} \sum_{i \in A_a} a_{i, j_t} X_{i, j_{t-1}} \\
&= \sum_{r=2}^{R} \sum_{t=1}^{T-1} (a_{i_{r-1}, j_t} - a_{i_r, j_{t+1}}) X_{i_r, j_t} + \sum_{r=2}^{R} a_{i_{r-1}, j_T} X_{i_r, j_T} - \sum_{t=2}^{T} a_{i_1, j_t} X_{i_1, j_t}.
\end{aligned}$$

This is 0 precisely when

$$\begin{aligned}
a_{i_r, j_T} &= 0, \text{for } r = 1, \dots, R-1, \\
a_{i_1, j_t} &= 0, \text{ for } t = 2, \dots, T \text{ and} \\
a_{i_r, j_t} &= a_{i_{r+1}, j_{t+1}}, \text{ for } r = 1, \dots, R-1,\ t = 1, \dots, T-1.
\end{aligned}$$

Therefore, the centralizer of f in $V_{a,b}$ is spanned by

$$\sum_{s=1}^{n} X_{i_{R-n+s}, j_s}, \text{ for } n = 1, \dots, R, \text{ when } R \leq T \tag{4.5}$$

and by

$$\sum_{s=1}^{n} X_{i_{R-n+s},j_s}, \text{ for } n=1,\dots,T, \text{ when } R \geq T. \tag{4.6}$$

Case (2) $a = b \neq m$. Essentially the same calculation as in Case (1) gives a basis for the centralizer of f in $V_{a,a}$ as

$$\sum_{s=n}^{R} X_{i_{R-n+s},j_s}, \text{ for } n=1,\dots,R\,(=T). \tag{4.7}$$

Case (3) $a \neq b$, $b = m$. A similar calculation shows that$\{X_{i_R,j},\, j \in A_m\}$ is a basis of $\mathfrak{z}_{a,m}$.

Case (4) $a \neq b$, $a = m$. Similarly, $\{X_{i,j_1},\, i \in A_m\}$ is a basis for $\mathfrak{z}_{m,b}$.

Case (5) $a = b = m$. Then $V_{a,b}$ commutes with f by the construction of f, so $\mathfrak{z}_{a,b} = V_{a,b}$.

Theorem 4.8. *If f is the generic element constructed by the algorithm then*

$$\gamma^{-1}(f) = L_m \cdots L_2 L_1 L \cdot \mathfrak{b} \subset K/K \cap B.$$

Proof. The proof is by induction on m, the number of strings in the array. Since Z_K is connected (a special fact for the indefinite unitary groups), Z_K is generated by $\exp(tZ)$ with $t \in \mathbf{R}$ and Z in the basis described above. Therefore, by Theorem 4.1 it suffices to show that for such Z

$$\exp(tZ) L_m \cdots L_2 L_1 Q \subset L_m \cdots L_2 L_1 Q. \tag{4.9}$$

There four cases.

Case (1) $Z \in \mathfrak{z}_{a,b}, 1 \leq a, b \leq m$. This puts us in the situation of $f' = f - f_0$ ($m-1$ strings) inside G_1. By induction

$$\exp(tZ) L_m \cdots L_2 Q_1 \subset L_m \cdots L_2 Q_1.$$

Therefore,

$$\begin{aligned} \exp(tZ) L_m \cdots L_2 L_1 Q &= \exp(tZ) L_m \cdots L_2 Q_1 Q \\ &\subset L_m \cdots L_2 Q_1 Q \\ &= L_m \cdots L_2 L_1 Q. \end{aligned}$$

Case (2) $Z \in \mathfrak{z}_{0,0}$. Each of the root vectors occurring in Z is in $\mathfrak{q} \cap \mathfrak{k}$ by (4.7). Also, Z commutes with each L_k, therefore (4.9) holds.

The final two cases are $\mathfrak{z}_{a,0}$ and $\mathfrak{z}_{0,a}$, $a > 0$. The proofs of (4.9) in these two cases require some preparation. For this recall that the array consists of a number of blocks and the string defining f_0 passes through each block. Now consider the strings defining f_c for $c = 1, 2, \dots, m-1$. Define an equivalence relation on the set $\{1, 2, \dots, p+q\}$ of indices by $i \sim j$ if and only if either (i) $1 \leq i, j \leq p$ and there exists no $\ell \in S_c$ so that $p+1 \leq \ell \leq p+q$ and $\epsilon_i - \epsilon_\ell$ and $\epsilon_\ell - \epsilon_j$ are both positive or both negative, or (ii) $p+1 \leq i, j \leq p+q$ and there exists no $\ell \in S_c$ so that $1 \leq \ell \leq p$ and $\epsilon_i - \epsilon_\ell$ and $\epsilon_\ell - \epsilon_j$ are both positive or both negative. We call the equivalence classes *c-blocks*.

Now define a Levi subalgebra of $\mathfrak{k}_1$ by specifying its roots: $\Delta(\mathfrak{m}_c)$ contains $\epsilon_i - \epsilon_j$ if and only if $i, j \notin S_0$ and i, j are in the same c-block. Let M_c be the connected subgroup of K_1 with Lie algebra $\mathfrak{m}_c$. Note that for $k = 1, 2, \dots, c$, $\Delta(\mathfrak{l}_k) \subset \Delta(\mathfrak{m}_c)$. Therefore,

$$L_c \cdots L_2 L_1 \subset M_c.$$

In the remaining two cases we will show that $[\mathfrak{m}_a, \mathfrak{z}_{a,0}] \subset \mathfrak{q} \cap \mathfrak{k}$ and $[\mathfrak{m}_a, \mathfrak{z}_{0,a}] \subset \mathfrak{q} \cap \mathfrak{k}$. Then (4.9) will follow.

Case (3) $Z \in \mathfrak{z}_{a,0}$, $a \geq 1$. First suppose that $a \neq m$. Then, as in (4.5), Z is a linear combination of root vectors $X_{i_{R+s-n}, j_s}, n = 1, \dots, R$. Since $j_s \in S_0$ and f_0 passes through each block in the array, j_s is the label of the first dot in the s^{th} block. It follows that for each $s = 1, \dots, R$, j_s is to the left of i_s in the array, and therefore j_s is also to the left of i_{R-n+s}. With this observation and the equivalence relation defining the a-blocks we will show that

$$[\mathfrak{m}_a, X_{i_{R-n+s}}] \in \mathfrak{q} \cap \mathfrak{k}. \tag{4.10}$$

Let Y be a root vector in $\mathfrak{m}_a$. Then

$$[Y, X_{i_{R-n+s}, j_s}] \in \mathbf{C} X_{i', j_s} \tag{4.11}$$

with $i' \sim i_{R-n+s}$ (i.e., i' and i_{R+s-n} in the same a-block. If $s = 1$, then $j_s = j_1$ is the dot farthest to the left in the array, so $X_{i_{R+s-n}, j_s} \in \mathfrak{n}^- \cap \mathfrak{k} \subset \mathfrak{q} \cap \mathfrak{k}$. When $s > 1$, consider $\epsilon_{i'} - \epsilon_{j_s}$. Suppose $\epsilon_{i'} - \epsilon_{j_s}$ were positive. Then in the array i' would be to the left of j_s, so also to the left of j_{s-1}. But j_{s-1} is to the left of $i_{R+s-n-1}$ (by the above observation). Therefore $\epsilon_{i'} - \epsilon_{i_{R+s-n-1}} > 0$ and $\epsilon_{i_{R+s-n-1}} - \epsilon_{i_{R+s-n}} > 0$, and we have a contradiction to $i' \sim i_{R+s-n}$. We therefore have that $X_{i', j_s} \in \mathfrak{n}^- \cap \mathfrak{k} \subset \mathfrak{q} \cap \mathfrak{k}$.

From (4.11), it follows that $\mathrm{ad}(Y)^k (X_{i_{R-n+s}, j_s})$ is contained in the span of X_{i, j_s} with $i \sim i_{R-n+s}$, so is in $\mathfrak{q} \cap \mathfrak{k}$. Therefore, $\mathrm{Ad}(\exp(Y))(X_{i_{R-n+s}, j_s}) \subset \mathfrak{q} \cap \mathfrak{k}$, and so $\mathrm{Ad}(M_a)(\exp(tZ)) \subset Q \cap K$, for Z in the basis for $\mathfrak{z}_{a,0}$. In particular, for $\ell_k \in L_k, k = 1, 2, \dots, a$,

$$\exp(tZ) \ell_a \cdots \ell_1 \in L_a \cdots L_1 Q \cap K.$$

Now, $\mathfrak{z}_{a,0}$ commutes with $L_m, \dots, L_{a+1}$ (since these $\mathfrak{l}_k$ have no root vectors involving indices from A_a and S_0). Therefore,

$$\begin{aligned} \exp(tZ) L_m \cdots L_1 Q_K &= L_m \cdots L_{a-1} \exp(tZ) L_a \cdots L_1 Q_K \\ &\subset L_m \cdots L_1 Q_K. \end{aligned}$$

Now if $a = m$, then Z is a linear combination of root vectors X_{i, j_1}, $i \in A_m$. For any root vector Y in $\mathfrak{k}_1$, $\mathrm{ad}(Y)^k (X_{i, j_1}) \in \mathfrak{q} \cap \mathfrak{k}$. So, $\mathrm{Ad}(K_1)(\exp(tZ)) \subset Q \cap K$. So (4.9) follows.

Case (4) $Z \in \mathfrak{z}_{0,b}$. This case is very similar the previous case. Here, Z is a sum of root vectors X_{i_{R-n+s}, j_s}, with $n = 1, \dots, T$, as in (4.6). $\square$

5. Q_K-ORBITS IN $\mathfrak{u}^- \cap \mathfrak{p}$

In this section we continue our study of the fibers of γ. In light of Richardson's Theorem [9] it is reasonable to ask the following question. Is there a dense $Q_K = Q \cap K$-orbit in $\mathfrak{u}^- \cap \mathfrak{p}$? There are examples in the literature for which $B \cap K$ does not have a dense orbit in $\mathfrak{n}^- \cap \mathfrak{p}$. See [11] for an example in $SO(4,4)$. We give criteria for Q_K to be transitive on the generic elements in $\mathfrak{u}^- \cap \mathfrak{p}$ (Theorem 5.9) and for Q_K to have a dense orbit in the generic elements in $\mathfrak{u}^- \cap \mathfrak{p}$ (Corollary 5.16). These criteria are in terms of the algorithm for the construction of the generic element f. At the end of this section an example in $SU(7,7)$ is given for which there is no dense Q_K-orbit in $\mathfrak{u}^- \cap \mathfrak{p}$.

We continue with our fixed closed K-orbit $Z = K \cdot \mathfrak{b}$ in the flag variety X and the corresponding sequence $(p_1, q_1, p_2, \dots, q_r)$ and array. We also continue with the parabolic subgroup Q defined by the set of compact simple roots.

We begin this section with a proposition, which we learned through discussions with H. Ochiai, that indicates one reason it is of interest to understand the Q_K-orbit structure of $\mathfrak{u}^- \cap \mathfrak{p}$.

Let $\tilde{\mu} : T^*(G/Q) \to \mathfrak{g}$ be the moment map for the cotangent bundle of the generalized flag variety G/Q. Let $\tilde{Z}$ be the closed orbit $K \cdot \mathfrak{q}$. Let $\tilde{\gamma}$ be the restriction of the moment map to the conormal bundle to $\tilde{Z}$. Thus

$$\tilde{\gamma} : K \underset{Q_K}{\times} (\mathfrak{u}^- \cap \mathfrak{p}) \to \mathfrak{g}$$

is given by the formula $\tilde{\gamma}(k, \xi) = k \cdot \xi$. For an arbitrary $Y \in \mathfrak{u}^- \cap \mathfrak{p}$

$$\tilde{\gamma}^{-1}(Y) = N(Y, \mathfrak{u}^- \cap \mathfrak{p})^{-1} \cdot \mathfrak{q}$$

as described in Section 2. Note that $(K \cdot Y) \cap (\mathfrak{u}^- \cap \mathfrak{p}) = \{k \cdot Y : k \in N(Y, \mathfrak{u}^- \cap \mathfrak{p})\}$. We write $Z_K(Y)$ for the centralizer in K of Y.

Proposition 5.1. *For arbitrary $Y \in \mathfrak{u}^- \cap \mathfrak{p}$, there is a bijection*

$$\begin{aligned} \{Z_K(Y)\text{-orbits in } \tilde{\gamma}^{-1}(Y)\} &\leftrightarrow \{Q_K\text{-orbits in } (K \cdot Y) \cap (\mathfrak{u}^- \cap \mathfrak{p})\} \\ Z_K(Y)k \cdot \mathfrak{q} &\leftrightarrow Q_K k^{-1} \cdot Y,\ k \in N(Y, \mathfrak{u}^- \cap \mathfrak{p}). \end{aligned}$$

Moreover, if Y is generic in $\mathfrak{u}^- \cap \mathfrak{p}$, then $Z_K(Y) \cdot \mathfrak{q}$ is open in $\tilde{\gamma}^{-1}(Y)$ if and only if $Q_K \cdot Y$ is open in $\mathfrak{u}^- \cap \mathfrak{p}$.

Proof. For the first statement, notice that for $k_1, k_2 \in N(Y, \mathfrak{u}^- \cap \mathfrak{p})$

$$\begin{aligned} &Z_K(Y)k_1 \cdot \mathfrak{q} = Z_K(Y)k_2 \cdot \mathfrak{q} \text{ if and only if} \\ &k_1 = zk_2q, \text{ for some } q \in Q_K, z \in Z_K(Y) \text{ if and only if} \\ &k_1^{-1}Y = q^{-1}k_2^{-1}Y, \text{ for some } q \in Q_K \text{ if and only if} \\ &Q_K k_1^{-1}Y = Q_K k_2^{-1}Y. \end{aligned}$$

For the second statement we prove the following formula for the dimension of the fiber of γ. If Y is generic then,

$$\dim \tilde{\gamma}^{-1}(Y) = \operatorname{codim}_{\mathfrak{u}^-\cap\mathfrak{p}}(Q_K\cdot Y) + \dim Z_K(Y) - \dim Z_{Q_K}(Y). \tag{5.2}$$

The proof is a simple computation:

$$\begin{aligned}\dim \tilde{\gamma}^{-1}(Y) &= \dim \mathfrak{u}^- - \dim(K\cdot Y)\\ &= \dim \mathfrak{u}^- - \dim \mathfrak{k} + \dim Z_K(Y)\\ &= \dim(\mathfrak{u}^-\cap\mathfrak{p}) - \dim(\mathfrak{q}\cap\mathfrak{k}) + \dim Z_K(Y)\end{aligned}$$

(since $\dim\mathfrak{k} = \dim\mathfrak{q}_k + \dim(\mathfrak{u}^-\cap\mathfrak{k})$ and $\dim\mathfrak{u}^- = \dim(\mathfrak{u}^-\cap\mathfrak{p}) + \dim(\mathfrak{u}^-\cap\mathfrak{k})$)

$$\begin{aligned}&= (\dim(\mathfrak{u}^-\cap\mathfrak{p}) - \dim Q_K + \dim Z_{Q_K}(Y)) + (\dim Z_K(Y) - \dim Z_{Q_K}(Y))\\ &= (\operatorname{codim}_{\mathfrak{u}^-\cap\mathfrak{p}} Q_K\cdot Y) + (\dim Z_K(Y) - \dim Z_{Q_K}(Y)).\end{aligned}$$

□

For $Y\in\mathfrak{u}^-\cap\mathfrak{p}$ generic, we write a formula for the $\dim(\tilde{\gamma}^{-1}(Y))$ in terms of data produced by the algorithm in Section 3. This formula will be used later in this section to study the structure of Q_K-orbits in $\mathfrak{u}^-\cap\mathfrak{p}$.

Let N_p (resp., N_q) stand for the number of nonzero p_i (resp., q_i) occurring in our sequence $(p_1,q_1,p_2,\dots,q_r)$. Then $N = N_p + N_q$. Write $Q_{0,K} = Q_K = L\exp(\mathfrak{u}^-\cap\mathfrak{k})$ and $Q_{i,K} = L_i\exp(\mathfrak{u}_i^-\cap\mathfrak{k})$. We will obtain a formula for $\dim(\tilde{\gamma}^{-1}(Y))$ as a corollary of the following theorem.

Proposition 5.3. *Let $Y\in\mathfrak{u}^-\cap\mathfrak{p}$ be a generic element. Then,*

$$\begin{aligned}\dim Z_G(Y) &= \sum_1^{N_p} p_i^2 + \sum_1^{N_q} q_j^2 + 2\sum_1^m \dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K}) - 1\\ &= \dim\mathfrak{l} + 2\sum_1^m \dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K}).\end{aligned}$$

Corollary 5.4. *If $Y\in\mathfrak{u}^-\cap\mathfrak{p}$ is generic, then*

$$\dim\tilde{\gamma}^{-1}(Y) = \sum_1^m \dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K}).$$

Proof. On the one hand

$$\begin{aligned}\dim\tilde{\gamma}^{-1}(Y) &= \dim(K/Q_K) + \dim(\mathfrak{u}^-\cap\mathfrak{p}) - \dim(\mathcal{O})\\ &= \dim(\mathfrak{u}^-) - \dim(\mathcal{O}).\end{aligned} \tag{5.5}$$

On the other hand the dimension of the nilpotent K-orbit $\mathcal{O} = K \cdot Y$ is half the dimension of $G \cdot Y$. Hence,

$$\begin{aligned}\dim(\mathcal{O}) &= \frac{1}{2}(\dim(\mathfrak{g}) - \dim(\mathfrak{z}_{\mathfrak{g}}(Y))) \\ &= \frac{\dim(\mathfrak{l})}{2} - \frac{\dim(\mathfrak{z}_{\mathfrak{g}}(Y))}{2} + \dim(\mathfrak{u}^-). \end{aligned} \tag{5.6}$$

Combining formulas (5.5) and (5.6), we get $\dim \tilde{\gamma}^{-1}(Y) = \frac{1}{2}\dim(\mathfrak{z}_{\mathfrak{g}}(Y)) - \frac{1}{2}\dim(\mathfrak{l})$. Now, the formula in Proposition implied the formula of the corollary.

□

We begin the proof of Proposition 5.3 with two preliminary lemmas.

Lemma 5.7. *Write* $\mathfrak{q}_K = \mathfrak{l} + (\mathfrak{u}^- \cap \mathfrak{k})$, $\mathfrak{q}_{1,K} = \mathfrak{l}_1 + (\mathfrak{u}_1^- \cap \mathfrak{k})$. *Then,*

$$\dim(\mathfrak{l}) = \dim(\mathfrak{l} \cap \mathfrak{l}_1) + 2\,(p+q) - N.$$

Proof. By construction, $\dim(\mathfrak{l}) = \sum_1^{N_p} p_i^2 + \sum_1^{N_q} q_j^2 - 1$, while $\dim(\mathfrak{l} \cap \mathfrak{l}_1) = \sum_1^{N_p}(p_i - 1)^2 + \sum_1^{N_q}(q_j - 1)^2 - 1$. Hence,

$$\begin{aligned}\dim(\mathfrak{l} \cap \mathfrak{l}_1) &= \sum_1^{N_p} p_i^2 + \sum_1^{N_q} q_j^2 - 2\,(\sum p_i + \sum q_j) + N - 1 \\ &= \dim(\mathfrak{l}) - 2\,(p+q) + N.\end{aligned}$$

□

Lemma 5.8. *For* $f = f_0 + f_1 + \ldots + f_{m-1}$ *as in the algorithm in Section 3, set* $f' = f - f_0 = f_1 + f_2 + \ldots + f_{m-1}$. *Then,*

$$\dim Z_G(f) = \dim Z_{G_1}(f') + 2\,(p+q) - N.$$

Proof. Associate to f the tableau that parameterizes the nilpotent K-orbit through f. Let a_i stand for the number of rows in the tableau having at least i blocks. Then, by [5, Thm 6.1.], we know that $\dim Z_G(f) = \sum a_i^2 - 1$. Similarly, since the tableau corresponding to the nilpotent orbit $K_1(f')$ is obtained from that of f by removing a longest row, we have $\dim Z_{G_1}(f') = \sum (a_i - 1)^2 - 1$. Thus,

$$\dim Z_G(f) - \dim Z_{G_1}(f') = \sum a_i^2 - \sum (a_i - 1)^2 = 2\,\sum_1^N a_i - N = 2\,(p+q) - N.$$

□

Proof of Proposition 5.3. We proceed by induction on the number of strings produced by the algorithm.

Assume that the dimension formula holds for $f' = f_1 + f_2 + \ldots + f_{m-1}$ with $m \geq 1$. By Lemma 5.8, we know that

$$\begin{aligned}
\dim Z_G(f) &= \dim Z_{G_1}(f') + 2\,(p+q) - N \\
&= \dim(\mathfrak{l}_1) + 2\sum_2^m \dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K}) + 2\,(p+q) - N \\
&\qquad \text{(by the inductive hypothesis)} \\
&= \dim(\mathfrak{l}_1\cap\mathfrak{l}) + 2\,\dim(\mathfrak{l}_1\cap\mathfrak{u}^-) + 2\sum_2^m \dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K}) + 2\,(p+q) - N \\
&= \dim(\mathfrak{l}_1\cap\mathfrak{l}) + 2\sum_1^m \dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K}) + 2\,(p+q) - N \\
&\qquad \text{(since } \dim(Q_{1,K}/Q_{1,K}\cap Q_K) = \dim(L_1/L_1\cap Q_K) = \dim(\mathfrak{l}_1\cap\mathfrak{u}^-)\text{)} \\
&= \dim(\mathfrak{l}) + 2\sum_1^m \dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K}) \\
&\qquad \text{(by Lemma 5.7).}
\end{aligned}$$

Begin the induction with the case of no strings (so $f = 0$). Then either $p = 0$ or $q = 0$ and $L = G$, making the formula clearly true. □

In Theorem 5.9 we give a condition for Q_K to be transitive on the generic elements in $\mathfrak{u}^-\cap\mathfrak{p}$. Let $\mathcal{O} = K\cdot f$ be the K-orbit of generic elements in $\mathfrak{u}^-\cap\mathfrak{p}$.

Theorem 5.9. *Q_K acts transitively on $\mathcal{O}\cap(\mathfrak{u}^-\cap\mathfrak{p})$ if and only if $Q_K\cap Q_{1,K}$ acts transitively on the set of generic elements in $\mathfrak{u}_1^-\cap\mathfrak{p}$.*

Proof. Assume that Q_K acts transitively on $\mathcal{O}\cap(\mathfrak{u}^-\cap\mathfrak{p})$. Let $Y' \in \mathfrak{u}_1^-\cap\mathfrak{p}$ be a generic element and form $Y = f_0 + Y'$. By the proof of Proposition 3.14 we know that $Y \in \mathfrak{u}^-\cap\mathfrak{p}$ is generic. Since Q_K is assumed to act transitively on $\mathcal{O}\cap(\mathfrak{u}^-\cap\mathfrak{p})$, we conclude that $Q_K\cdot Y = Q_K\cdot(f_0+Y')$ is open in $\mathfrak{u}^-\cap\mathfrak{p}$. Hence, the tangent space to the orbit $Q_K\cdot(f_0+Y')$ at the base point f_0+Y' coincides with $\mathfrak{u}^-\cap\mathfrak{p}$. This implies that

$$[\mathfrak{q}\cap\mathfrak{k}, f_0+Y'] = T_{f_0+Y'}(Q_K\cdot(f_0+Y')) = \mathfrak{u}^-\cap\mathfrak{p}. \tag{5.10}$$

We show that $Q_K\cap Q_{1,K}\cdot Y'$ is open in $\mathfrak{u}_1^-\cap\mathfrak{p}$.

The Borel subalgebra $\mathfrak{b} = \mathfrak{h}+\mathfrak{n}^- \subset \mathfrak{q}$ is determined by an array of numbered dots. The first step of our algorithm determines f_0 by choosing a first string. Recall that A_0 is the set of labels of dots occurring in the first string. In particular, notice that f_0 is a sum of root vectors for roots $\epsilon_i - \epsilon_j$ where i and j belong to the set A_0. Moreover, $\Delta(\mathfrak{g}_1,\mathfrak{h}) = \{\epsilon_i-\epsilon_j : i,j\notin A_0\}$. The set A_0 determines a decomposition

$$\mathfrak{q}\cap\mathfrak{k} = \mathfrak{q}\cap\mathfrak{g}_1\cap\mathfrak{k} + \tilde{\mathfrak{h}} + \mathfrak{v}_o + \mathfrak{v}_1$$

where

$$\begin{aligned}\Delta(\mathfrak{q}\cap\mathfrak{g}_1\cap\mathfrak{k}) &= \{\epsilon_i-\epsilon_j : i,j\notin A_0\}\cap\Delta(\mathfrak{q}\cap\mathfrak{k})\\ \Delta(\mathfrak{v}_0) &= \{\epsilon_i-\epsilon_j : i,j\in A_0\}\cap\Delta(\mathfrak{q}\cap\mathfrak{k})\\ \Delta(\mathfrak{v}_1) &= \{\epsilon_i-\epsilon_j : \text{ exactly one of } i,j \text{ belongs to } A_0\}\cap\Delta(\mathfrak{q}\cap\mathfrak{k})\end{aligned}$$

and $\tilde{\mathfrak{h}}$ is the part of $\mathfrak{h}$ consisting of matrices with 0 in the i^{th} diagonal entry when $i\in A_0$. Observe that,

$$\begin{aligned}&[\mathfrak{q}\cap\mathfrak{g}_1\cap\mathfrak{k}, f_0+Y']\subset[\mathfrak{q}\cap\mathfrak{g}_1\cap\mathfrak{k}, Y']\\ &[\tilde{\mathfrak{h}}, f_0+Y']=[\tilde{\mathfrak{h}}, f_0]\subset\mathfrak{v}_0\\ &[\mathfrak{v}_0, f_0+Y']\subset[\mathfrak{v}_0, f_0]\subset\mathfrak{v}_0\\ &[\mathfrak{v}_1, f_0+Y']\subset\mathfrak{v}_0+\mathfrak{v}_1.\end{aligned}\tag{5.11}$$

We claim that $T_{Y'}\big((Q_{1,K}\cap Q_1)\cdot Y'\big)=\mathfrak{u}_1^-\cap\mathfrak{p}$. This is equivalent to $[\mathfrak{q}\cap\mathfrak{q}_1\cap\mathfrak{k}, Y']=\mathfrak{u}_1^-\cap\mathfrak{p}$. Since $Y'\in\mathfrak{u}_1\cap\mathfrak{p}$ the inclusion '$\subset$' is clear. For the other inclusion, let $X_\beta\in\mathfrak{u}_1^-\cap\mathfrak{p}$. Then

$$\begin{aligned}X_\beta &\in[\mathfrak{q}\cap\mathfrak{k}, f_0+Y']\cap\mathfrak{g}_1, \text{ by (5.10)},\\ &=\big([\mathfrak{q}_1\cap\mathfrak{q}\cap\mathfrak{k}+\tilde{\mathfrak{h}}+\mathfrak{v}_0+\mathfrak{v}_1, f_0+Y']\big)\cap\mathfrak{g}_1\\ &\subset[\mathfrak{q}_1\cap\mathfrak{q}\cap\mathfrak{k}, Y'], \text{ by (5.11)},.\end{aligned}$$

The claim is now proved. Therefore $(Q_{1,K}\cap Q_K)\cdot Y'$ is open in $\mathfrak{u}_1^-\cap\mathfrak{p}$.

Since Y' is an arbitrary generic element in $\mathfrak{u}_1^-\cap\mathfrak{p}$, we conclude that $Q_{1,K}\cap Q_K$ acts transitively on the set of generic elements in $\mathfrak{u}_1^-\cap\mathfrak{p}$.

For the converse, let Y be generic in $\mathfrak{u}^-\cap\mathfrak{p}$ and let $f=f+f'$, $f'=\sum_{i=1}^{m-1}f_i$ as in Section 3. By Proposition 3.14 there exist $q\in Q_K$ and $q_i\in Q_{i,K}$ so that $Y=qq_1\cdots q_{m-1}\cdot(f_0+f')$. Since each q_i commutes with f_0, $q^{-1}Y=f_0+Y'$, where $Y'=q_1\cdots q_{m-1}\cdot f'$, a generic element of $\mathfrak{u}_1^-\cap\mathfrak{p}$. Now assume $Q_K\cap Q_{1,K}$ is transitive on the generic elements of $\mathfrak{u}_1^-\cap\mathfrak{p}$. Then,

$$\dim(Q_{1,K}\cdot Y')=\dim((Q_K\cap Q_{1,K})\cdot Y')=\dim(\mathfrak{u}_1^-\cap\mathfrak{p}).\tag{5.12}$$

Therefore it suffices to show that $Q_K\cdot Y=Q_K\cdot(f+Y')$ has codimension zero in $\mathfrak{u}^-\cap\mathfrak{p}$.

By Lemma 5.2 and Corollary 5.4 applied to $Y'\in\mathfrak{u}_1^-\cap\mathfrak{p}$, along with (5.12),

$$\begin{aligned}0&=\operatorname{codim}_{\mathfrak{u}_1^-\cap\mathfrak{p}}(Q_{1,K}(Y'))\\ &=\sum_{i=2}^m\dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K})-\Big(\dim Z_{K_1}(Y')-\dim Z_{Q_{1,K}}(Y')\Big).\end{aligned}\tag{5.13}$$

Also, by (5.12),

$$\begin{aligned}\dim(Q_{1,K}/Q_{1,K}\cap Q_K)&=\dim Q_{1,K}-\dim Q_{1,K}\cap Q_K\\ &=\dim Z_{Q_{1,K}}(Y')-\dim Z_{Q_K\cap Q_{1,K}}(Y').\end{aligned}\tag{5.14}$$

Applying formula 5.2 and Corollary 5.4 for the first equality and (5.13) and (5.14) for the second, we have

$$\begin{aligned}
&\operatorname{codim}_{\mathfrak{u}^-\cap\mathfrak{p}}\big(Q_K(f+Y')\big)\\
&\quad=\sum_1^m \dim(Q_{i,K}/Q_{i,K}\cap Q_{i-1,K})-\Big(\dim Z_K(Y)-\dim Z_{Q_K}(Y)\Big)\\
&\quad=\Big(Z_{K_1}(Y')-\dim Z_{Q_{1,K}\cap Q_K}(Y')\Big)-\Big(\dim Z_K(Y)-\dim Z_{Q_K}(Y)\Big).
\end{aligned}\tag{5.15}$$

Since

$$Z_{K_1}(Y')/Z_{Q_{1,K}\cap Q_K}(Y')\to Z_K(f_0+Y')/Z_{Q_K}(f_0+Y')$$

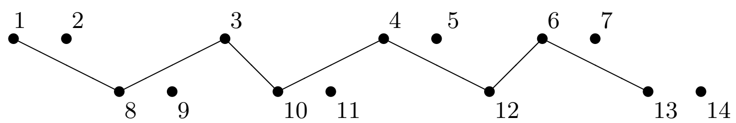
is injective, we may conclude that the right hand side of (5.15) is less than or equal to zero. Therefore, $\operatorname{codim}_{\mathfrak{u}^-\cap\mathfrak{p}}(Q_K(f+Y'))=0$, and the proof is complete.

□

Corollary 5.16. *Q_K has an open orbit in $\mathfrak{u}^-\cap\mathfrak{p}$ if and only if $Q_{1,K}\cap Q_K$ has an open orbit in $\mathfrak{u}_1^-\cap\mathfrak{p}$.*

We conclude this section with an example of how Corollary 5.16 identifies a situation where Q_K does not have an open orbit in $\mathfrak{u}^-\cap\mathfrak{p}$.

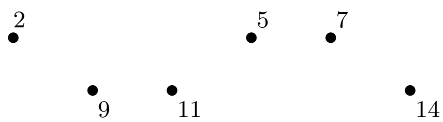
Example 5.17. Let $G_{\mathbf{R}}=SU(7,7)$. Consider the positive root system $\Delta^+=\Delta^+(\mathfrak{g},\mathfrak{h})$ determined by the following numbered array. The first string formed by the algorithm is shown.



Equivalently, Δ^+ is the system of positive roots having positive inner product with

$$(14,13,10,7,6,4,3\,|\,12,11,9,8,5,2,1).$$

After deleting the first string the resulting array is



Thus, $\mathfrak{g}_1=\mathfrak{sl}(6)$ and $Q_{1,K}\cap Q_K=B_1$ is a Borel subgroup of K_1. Moreover, $\dim(\mathfrak{k}\cap\mathfrak{b}_1)=11$, while $\dim(\mathfrak{u}_1^-\cap\mathfrak{p})=9$. An arbitrary element X in $\mathfrak{u}_1^-\cap\mathfrak{p}$ is of the form

$$X=aX_{9,2}+bX_{11,2}+cX_{14,2}+dX_{14,7}+eX_{5,11}+fX_{14,5}+gX_{5,9}+hX_{7,9}+iX_{11,7}.$$

We claim that $B_1 \cdot X$ is not dense in $\mathfrak{u}_1^- \cap \mathfrak{p}$ for any $X \in \mathfrak{u}_1^- \cap \mathfrak{p}$. Indeed, when $a \neq 0$ $\mathfrak{z}_{\mathfrak{b}_1}(X)$ contains

$$aX_{5,2} + fX_{14,9},\ aX_{7,2} + dX_{14,9}, \text{ and } aX_{14,11} - bX_{14,9}.$$

We then conclude that $\dim(B_1 \cdot X) \leq 8 < \dim(\mathfrak{u}_1^- \cap \mathfrak{p}) = 9$. When $a = 0$, the argument is slightly different: $X_{9,2} \notin [\mathfrak{b}_1, X]$ (as is easily checked). But, $[\mathfrak{b}_1, X]$ is the tangent space to $B_1 \cdot X$ at X, so $\dim(B_1 \cdot X) < \dim(\mathfrak{u}^- \cap \mathfrak{p})$. This proves the claim. Now, Proposition 5.9, implies that Q_K has no open orbit in $\mathfrak{u}^- \cap \mathfrak{p}$.

The orbit structure of Q_K on the generic elements in $\mathfrak{u}^- \cap \mathfrak{p}$ may be described as follows. Assume that $Y \in \mathfrak{u}^- \cap \mathfrak{p}$ is generic. By Corollary 5.4, we know that $\dim \tilde{\gamma}^{-1}(Y) = \dim L_1/(L_1 \cap Q_K) = 2$. Hence, by Lemma 5.2, the only possible dimensions of the Q_K-orbits in $K(Y) \cap \mathfrak{p}$ are 47, 48 and 49. We have just argued that no orbit has dimension 49.

Observe that $L_1 = L_1^1 \times L_1^2$ is the product of two commuting copies of $GL(2)$ (generated by the roots $\pm\{\epsilon_7 - \epsilon_5, \epsilon_{11} - \epsilon_9\}$). By using the Bruhat decompositions of the subgroups $L_1^i, i = 1, 2$ one sees that the Q_K-orbits in the generic elements of $\mathfrak{u}^- \cap \mathfrak{p}$ are as follows. Orbits of dimension 47:

$$Q_K \cdot f,\ Q_K(\exp(X_{9,11}) \cdot f),\ Q_K(\sigma_{5,7} \exp(X_{9,11} \cdot f),$$

and orbits of dimension 48:

$$\begin{aligned}
&Q_K(\sigma_{5,7} \cdot f),\ Q_K(\sigma_{9,11} \cdot f),\\
&Q_K(\sigma_{5,7}\sigma_{9,11} \cdot f),\ Q_K(\exp(X_{5,7}) \cdot f),\ Q_K(\exp(X_{5,7})\sigma_{9,11} \cdot f),\\
&Q_K(\exp(X_{5,7} \exp(sX_{9,11}), s \in \mathbf{C} \text{ (an infinite family)}.
\end{aligned}$$

We have the following orbit decomposition

$$\begin{aligned}
K(Y) \cap \mathfrak{p} = {} & Q_K \cdot f \cup Q_K(\exp\ (X_{9,11})f) \cup Q_K(s_{9,11}f) \cup Q_K(s_{9,11}\exp\ (X_{5,7})f)\\
& \cup Q_K(s_{5,7}f) \cup Q_K(\exp\ (X_{9,11})f) \cup Q_K(s_{9,11}s_{5,7}f)\\
& \cup \bigcup_{s \in \mathbf{C}} Q_K(\exp\ (X_{5,7})\ \exp\ (sX_{9,11}) \cdot f)
\end{aligned}$$

with $s_{9,11}$ and $s_{5,7}$ the non-trivial Weyl group elements of the respective copies of $GL(2)$.

6. MULTIPLICITY POLYNOMIALS FOR DISCRETE SERIES REPRESENTATIONS

An important invariant of a Harish-Chandra module V is its associated variety. In general, the associated variety, denoted by $AV(V)$, is the union of the closures of several K-orbits in $\mathcal{N}_\theta$. The associated cycle is a formal integer combination of the orbit closures $\overline{\mathcal{O}}$ occurring in $AV(V)$. The integer attached to $\overline{\mathcal{O}}$ is referred to as the *multiplicity* of $\overline{\mathcal{O}}$ in the associated cycle of V. In this section we will use Theorem 4.8 to give a simple algorithm for computing the multiplicities for discrete series representations of $G_{\mathbf{R}} = SU(p, q)$. Our starting point is a formula of J.T. Chang that gives a formula for the multiplicities in terms of a sheaf cohomology space on $\gamma^{-1}(f)$. For generalities on the associated cycle and multiplicities see, for example, [3] and [13]. See [10] for the proof of a conjecture of D.

Barbasch and D. Vogan that relates the associated cycle of a Harish-Chandra module to its global character.

We begin by giving a parameterization of the discrete series. For each closed K-orbit in the flag variety X there is a family of discrete series representations. So let us fix such a closed orbit Z in X. Then, as in earlier parts of this article, there is a positive system $\Delta^+ \subset \Delta(\mathfrak{h}, \mathfrak{g})$ containing Δ_c^+ so that $Z = K \cdot \mathfrak{b}$, with $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$, $\Delta(\mathfrak{n}^-) = -\Delta^+$. The discrete series representations corresponding to Z are parameterized by $\lambda \in \mathfrak{h}^*$ satisfying

$$\begin{aligned} &\text{(i) } \lambda \text{ is } \Delta^+\text{-dominant regular and} \\ &\text{(ii) } \tau = \lambda + \rho - 2\rho_c \text{ is analytically integral.} \end{aligned} \tag{6.1}$$

Here we are using the standard notation ρ (resp., ρ_c) for one half the sum of the roots in Δ^+ (resp. Δ_c^+). The discrete series representation corresponding to λ will be denoted by V_λ; it has infinitesimal character λ and has lowest K-type of highest weight τ.

It is well-known that $AV(V_\lambda)$ is the image of $\gamma : T_Z^* X \to \mathcal{N}_\theta$. Therefore $AV(V_\lambda) = \overline{K \cdot f}$, where f is the generic element constructed in Section 3. It is also known that the multiplicity of $\overline{K \cdot f}$ in the associated cycle of V_λ is a polynomial in λ. (Note that we have fixed an arbitrary closed K-orbit in X; there is one multiplicity polynomial for each such K-orbit Z.)

Let $\mathcal{O}_Z$ (resp., $\mathcal{O}_{\gamma^{-1}(f)}$) be the structure sheaf of Z (resp., $\gamma^{-1}(f)$). Extend $\tau \in \mathfrak{h}^*$ to a representation of $\mathfrak{b}$ by requiring that $\tau|_{\mathfrak{n}^- \cap \mathfrak{k}} = 0$. By (ii) of (6.1) τ lifts to a character χ_τ of $B \cap K$. This defines a homogeneous line bundle $\mathcal{L}_\tau \to Z$. The sheaf of local regular sections $\mathcal{O}(\tau)$ is described as follows. Let $p : K \to K \cdot \mathfrak{b}$ be the natural quotient map. Then for an open set $U \subset Z$ a section on U is a regular function φ on $p^{-1}(U)$ so that

$$\varphi(kb) = \chi_\tau(b^{-1})\varphi(k), \text{ for } k \in K, b \in B \cap K.$$

Let

$$\mathcal{O}_{\gamma^{-1}(f)}(\tau) = \mathcal{O}(\tau) \underset{\mathcal{O}_Z}{\otimes} \mathcal{O}_{\gamma^{-1}(f)}.$$

We now may state J.T. Chang's theorem ([2]).

Theorem 6.2. *If V_λ is the Harish-Chandra module of a discrete series representation parameterized by a closed K-orbit Z in X and $\lambda \in \mathfrak{h}^*$, as described above, then the multiplicity of $\overline{K \cdot f}$ in the associated cycle of V_λ is*

$$\dim\big(H^0(\gamma^{-1}(f), \mathcal{O}_{\gamma^{-1}(f)}(\tau))\big).$$

This cohomology space may be described by the Borel-Weil Theorem as follows. Let $W_{-\tau}$ be the irreducible finite dimensional K-representation of lowest weight $-\tau$ and let $w_{-\tau}$ be a lowest weight vector. The Borel-Weil Theorem states that $W_{-\tau}^* \simeq H^0(Z, \mathcal{O}(\tau))$. Note that τ satisfies (ii) of (6.1), so is Δ_c^+ dominant. This isomorphism is implemented by

$$\begin{aligned} &v \mapsto \varphi_v,\ v \in W_{-\tau}^* \\ &\varphi_v(k) = \langle v, k w_{-\tau} \rangle. \end{aligned}$$

As stated (and attributed to J. Bernstein) in [7, §6.1-6.3], for λ sufficiently dominant

$$\dim\big(H^0(\gamma^{-1}(f), \mathcal{O}_{\gamma^{-1}(f)})\big) = \dim\big(\mathrm{span}_{\mathbf{C}}\{k^{-1}w_{-\tau} \,:\, k \in N(f, \mathfrak{n}^- \cap \mathfrak{k})\}\big). \tag{6.3}$$

This, along with our description of $\gamma^{-1}(f)$ given in Theorem 4.8, implies the following proposition.

Proposition 6.4. *The multiplicity of $\overline{K \cdot f}$ in the associated cycle of V_λ is*

$$\dim\big(\mathrm{span}_{\mathbf{C}}\{k \cdot w_{-\tau} \,:\, k \in L_m \dots L_2 L_1 L\}\big),$$

provided λ is sufficiently dominant.

For any $\lambda \in \mathfrak{h}^*$ satisfying (6.1), and $\tau = \lambda + \rho - 2\rho_c$, we define

$$q_Z(\lambda) = \dim\big(\mathrm{span}_{\mathbf{C}}\{k \cdot w_{-\tau} \,:\, k \in L_m \dots L_2 L_1 L\}\big). \tag{6.5}$$

We show that $q_Z(\lambda)$ extends to a polynomial on all of $\mathfrak{h}^*$. Since the multiplicity (for the part of the discrete series corresponding to Z) is also a polynomial in λ, we may then conclude that $q_Z(\lambda)$ equals the multiplicity polynomial.

Theorem 6.6. *For all $\lambda \in \mathfrak{h}^*$ satisfying (6.1) the multiplicity of $\overline{K \cdot f}$ in the associated cycle of V_λ is $q_Z(\lambda)$.*

Proof. The notation will be slightly less burdensome if we define $p(\tau)$ to be the right-hand side of 6.5 for any dominant integral τ. By the relation $\tau = \lambda + \rho - 2\rho_c$ it will be enough to show that $p(\tau)$ extends to a polynomial in τ. We will do this by induction on m, the number of strings making up f.

If $m = 0$ the group $G_{\mathbf{R}}$ is compact ($p = 0$ or $q = 0$). Then $L = K = G$ and $f = 0$, and the Springer fiber is $Z = X$ and $p(\tau)$ is given by the Weyl dimension formula (for $\mathfrak{g}$). Therefore, $p(\tau)$ extends to a polynomial.

Now consider $m > 1$. Write $U_{-\tau}$ for $\mathrm{span}_{\mathbf{C}}\{Lw_{-\tau}\}$, the irreducible representation of L having lowest weight $-\tau$. Decompose $U_{-\tau}$ as a representation of $L_1 \cap L$. Write this decomposition as $\sum E_{-\tau_i}$ and write the lowest weight vectors as $w_{-\tau_i}$.

Claim: Each $w_{-\tau_i}$ is annihilated by $\mathfrak{n}^- \cap \mathfrak{g}_1 \cap \mathfrak{k}$.

To verify the claim, note that since L normalizes $\mathfrak{u}^- \cap \mathfrak{k}$ and $w_{-\tau}$ is annihilated by $\mathfrak{u}^- \cap \mathfrak{k}$, each $w_{-\tau_i}$ (in fact all of $U_{-\tau}$) is annihilated by $\mathfrak{u}^- \cap \mathfrak{k}$. Now each $w_{-\tau_i}$ is annihilated by $\mathfrak{n}^- \cap \mathfrak{l}_1 \cap \mathfrak{l}$. But, $\mathfrak{n}^- \cap \mathfrak{g}_1 \cap \mathfrak{k} \subset \mathfrak{u}^- \cap \mathfrak{k} + \mathfrak{n}^- \cap \mathfrak{l}_1 \cap \mathfrak{l}$.

The claim tells us that $F_{-\tau_i} \equiv \mathrm{span}_{\mathbf{C}}\{K_1 w_{-\tau_i}\}$ is the irreducible K_1-representation of lowest weight $-\tau_i$. Therefore,

$$p(\tau) = \sum_i \dim\big(\mathrm{span}_{\mathbf{C}}\{L_m \cdots L_1 w_{-\tau_i}\}\big). \tag{6.7}$$

By induction on m, each $p_1(\tau_i) \equiv \dim\big(\mathrm{span}_{\mathbf{C}}\{L_m \cdots L_1 w_{-\tau_i}\}\big)$ extends to a polynomial in τ_i.

There are two observations to make. First, L is a product of a number of groups isomorphic to a $GL(r)$ for various r. Furthermore, $L_1 \cap L$ is a product of various groups isomorphic to $GL(r')$, where r' is r or $r-1$. The standard branching law for the restriction of representations of $GL(r)$ to $GL(r-1)$ is as follows. Let V_{-a} be the irreducible $GL(r)$-representation of lowest weight $-a = -(a_1, \dots, a_r)$, $a_1 \geq a_2 \geq \cdots \geq a_r$. Similarly, let U_{-b} be the irreducible $GL(r-1)$ representation of lowest weight $-b = -(b_1, \dots, b_{r-1})$. The the restriction of V_{-a} to $GL(r-1)$ is $\sum U_{-b}$, with the sum being over all $b \in Z^{r-1}$ so that $a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq b_{r-1} \geq a_r$. Each occurs with multiplicity one.

The second observation is stated as an elementary Lemma.

Lemma 6.8. *If $P_1(b)$, $b \in \mathbf{C}^{r-1}$ is a polynomial, then for $a \in \mathbf{Z}^r$*

$$P(a) \equiv \sum_{a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq b_{r-1} \geq a_r, b_j \in \mathbf{Z}} P_1(b_1, \dots, b_{r-1})$$

extends to a polynomial on $\mathbf{C}^r$.

Proof of lemma. For $a \in \mathbf{Z}^r$,

$$P(a) = \sum_{b_1 = a_2}^{a_1} \cdots \sum_{b_{n-1} = a_r}^{a_{r-1}} P_1(b_1, \dots, b_r).$$

It follows easily, from the fact that $\sum_{n=1}^{N} n^k$ is polynomial in N, that $P(a)$ extends to a polynomial in $a \in \mathbf{C}^r$. □

We now conclude the proof of the theorem by noting that the τ_i's occurring in (6.7) come from the branching rule mentioned above (for the various factors of L), and the Lemma along with (6.7) says that $p_Z(\tau)$ extends to a polynomial in τ. □

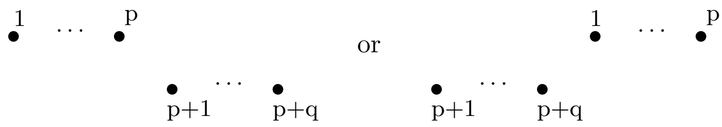
The proof of the theorem contains an algorithm for computing the multiplicity of $\overline{K \cdot f}$ in V_λ. We describe an algorithm for computing $p_Z(\tau)$ for any τ which is a Δ_c^+-dominant integral weight. Given a closed K-orbit and corresponding positive system Δ^+ containing Δ_c^+, form the sequence as in (2.5) and the corresponding array. Form the first string and f_0 as in (3.1), also form G_1 and $Q_{1,K}$ (as at the end of Section 3).

(1) Decompose the L-representation $U_{-\tau} = \mathrm{span}_{\mathbf{C}}\{L(w_{-\tau})\}$ under $L \cap L_1$ using the branching law for restricting $GL(r)$-representations to $GL(r-1)$. Call the constituents $E_{-\tau_i}$.
(2) As shown in the proof of the theorem (see the 'Claim'), each τ_i is dominant for $\Delta_c^+ \cap \Delta(\mathfrak{l}_1)$ and
$$p(\tau) = \sum_i p_1(\tau_i).$$
(3) Now repeat the procedure to find the $p_1(\tau_i)$.

The procedure ends after m (the number of strings) iterations.

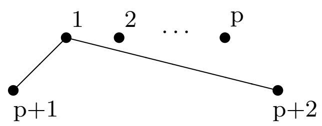
We now give several examples of computations of the multiplicities of discrete series representations using the algorithm described above. The result of the first example is now well-known ([8] and [2]), and the second follows from [2].

Example 6.9. (Holomorphic Discrete Series) This is the case where there is a unique simple noncompact root. The array is therefore one of the following:



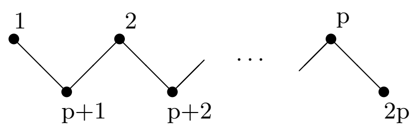
and (assuming $p \leq q$) $f = \pm \sum_{i=1}^{p} (\epsilon_i - \epsilon_{p+i})$. Therefore, $L = K$, so $L_m \cdots L_1 L = K$ and the multiplicity of V_λ is the dimension of the lowest K-type of V_λ.

Example 6.10. (Quaternionic Discrete series of $SU(p, 2)$) Consider the positive system determined by the following diagram:



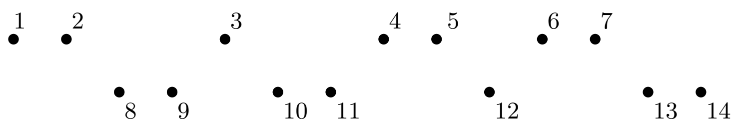
The reductive part of Q_K is $L = S(GL(p) \times GL(1) \times GL(1))$ and $L_1 \subset L$. Therefore, the multiplicity is $\dim(L \cdot w_{-\tau})$, i.e., the dimension of the irreducible representation of L with lowest weight $-\tau = -(\lambda + \rho - 2\rho_c)$.

Example 6.11. Consider the group $G = SU(p, p)$ with the positive system given by a Dynkin diagram with the maximum number of simple noncompact roots. The array is



Here $L = L_1 =$ the torus and the multiplicity is one.

Example 6.12. We consider $G = SU(7, 7)$ and the positive system determined by the following array



(See also Example 5.17.) Then $\text{span}_{\mathbf{C}}\{L\cdot w_{-\tau}\}$ is the irreducible L-representation of lowest weight $-\tau$, call it $U_{-\tau}$. Then L is a product of six copies of $SL(2)$ (and a torus) and $U_{-\tau}$ is the tensor product of representations of these $SL(2)$'s. Since $L_1\cap L$ is the torus, the decomposition of $U_{-\tau}|_{L_1\cap L}$ is given by the weights

$$-\tau+a(\epsilon_1-\epsilon_2)+a(\epsilon_4-\epsilon_5)+c(\epsilon_6-\epsilon_7)+d(\epsilon_8-\epsilon_9)++e(\epsilon_{10}-\epsilon_{11})+f(\epsilon_{13}-\epsilon_{14}).$$

with $a=0,\dots,\tau_1-\tau_2, b=0,\dots,\tau_4-\tau_5, c=0,\dots,\tau_6-\tau_7, d=0,\dots,\tau_8-\tau_9, e=0,\dots,\tau_{10}-\tau_{11}$ and $f=0,\dots,\tau_{13}-\tau_{14}$. L_1 is the product of two copies of $SL(2)$ (and a torus). The roots in $\mathfrak{l}_1$ are $\pm\{\epsilon_5-\epsilon_7,\epsilon_9-\epsilon_{11}\}$. Using the formula $\displaystyle\sum_{n=0}^{N} n=\frac{N(N+1)}{2}$, the dimension of $\text{span}_{\mathbf{C}}\{L_1L\cdot w_{-\tau}\}$ is therefore

$$\begin{aligned}\sum_{a,\dots,f}&(\tau_5-\tau_7+b-c+1)(\tau_9-\tau_{11}+d-e+1)\\ &=(\tau_1-\tau_2+1)(\tau_4-\tau_5+1)(\tau_6-\tau_7+1)(\tau_5-\tau_7+1+\frac{\tau_4-\tau_5-\tau_6+\tau_7}{2})\\ &\quad(\tau_8-\tau_9+1)(\tau_{10}-\tau_{11}+1)(\tau_9-\tau_{11}+1+\frac{\tau_8-\tau_9-\tau_{10}+\tau_{11}}{2})(\tau_{13}-\tau_{14}+1).\end{aligned}$$

Writing this in terms of λ (using $\tau=\lambda+\rho-2\rho_c$) the formula for multiplicity is

$$\frac{1}{4}(\lambda_1-\lambda_2)(\lambda_4-\lambda_5)(\lambda_6-\lambda_7)(\lambda_8-\lambda_9)(\lambda_{10}-\lambda_{11})(\lambda_{13}-\lambda_{14})(\lambda_4+\lambda_5-\lambda_6-\lambda_7)(\lambda_8+\lambda_9-\lambda_{10}-\lambda_{11}).$$

We end with two remarks.

Remark 6.13. As a consequence of Proposition 5.1 and the above discussion we have an alternative formula for the multiplicity.

Proposition 6.14. *If there exists $Y\in\mathfrak{u}^-\cap\mathfrak{p}$ so that $Q_K\cdot Y$ is dense in the generic elements, then*

$$q_Z(\lambda)=\dim\{Z_K(Y)L\cdot w_{-\tau}\}.$$

Proof. This follows from (6.3), since $Z_K(Y)L\cdot\mathfrak{b}$ is dense in $\gamma^{-1}(f)$. □

Remark 6.15. In [14] H. Yamashita constructs a $Z_K(f)$-representation which is contained in the isotropy representation ([13]). The description of $H^0(\gamma^{-1}(f),\mathcal{O}_{\gamma^{-1}(f)})$ given here shows that Yamashita's $Z_K(f)$-representation is *equal* to $\text{span}_{\mathbf{C}}\{L_m\dots L_1L(w_{-\tau})\}$.

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