

A reduction theorem for the unitary dual of $U(p, q)$

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^{1 2}

July 21, 2004

1 Introduction

This paper investigates the unitary dual of the groups $U(p, q)$, in particular the spherical case.

The general philosophy for parametrizing the unitary dual is that each reductive group G should have a set of *basic* unitary irreducible representations which generate the unitary dual in the following sense. To any unitary irreducible representation π , there should be attached a pair (Q, π_L) , where $Q = LU$ is a real or θ stable parabolic subgroup, and π_L is a basic representation of the Levi component L . Then π should be a factor of the unitarily or cohomologically induced module from π_L , or it should be in a complementary series from such a module.

A basic representation should have the properties that

- (1) it is not obtained by induction or complementary series from a proper Levi component,
- (2) its restriction to the semisimple part of L is a *unipotent* representation.

Unipotent representations are parametrized by nilpotent orbits. Let $\check{\mathcal{O}}_c \subset \check{\mathfrak{g}}$ be a nilpotent orbit in the Lie algebra of the dual group $\check{G}$. A representation π is called unipotent if its annihilator in the universal enveloping algebra of G is the maximal primitive ideal with infinitesimal character one of a finite set $\chi_{1, \check{\mathcal{O}}_c}, \dots, \chi_{k, \check{\mathcal{O}}_c}$ associated to $\check{\mathcal{O}}_c$. It is not completely settled what these infinitesimal characters should be. One of them was introduced by Arthur, and is as follows. Let $\check{e}, \check{h}, \check{f}$ be a Lie triple attached to $\check{\mathcal{O}}_c$. Then $\check{h}/2$ is one of the $\chi_{j, \check{\mathcal{O}}_c}$. The unipotent representations with this infinitesimal character are

¹**2000 Mathematics Subject Classification:** Primary 22E46; Secondary 17B10, 20G20.

²**Key words:** unitary dual, unitary groups, spherical representations.

called *special unipotent*. For classical complex groups, a larger suggested set of infinitesimal characters is in [B2].

For a unipotent representation, condition (1) is implied by the requirement that $\check{\mathcal{O}}_c$ should not meet any proper Levi component of $\check{\mathfrak{g}}_c$. In type A, this forces $\check{\mathcal{O}}_c$ to be the principal nilpotent orbit. The only unipotent representation attached to the principal nilpotent is the trivial one. Thus for a group of type A, it is reasonable to expect that the basic representations can only be unitary characters. In [V5] it is shown that $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ conform to the aforementioned philosophy. The results in this paper present further evidence that the spherical unitary dual of $U(p, q)$, conforms as well. A more detailed and precise discussion of these ideas can be found in [V4].

In the case of the quasisplit groups, $U(n, n)$ and $U(n + 1, n)$, we determine the *full* spherical unitary spectrum. The main result is theorem 5.

The necessary conditions for unitarity are obtained in sections 4 and 5 of [B1]. The connection is as follows. Let M be the Levi component of the minimal parabolic subgroup of G . To establish whether a spherical module π is unitary, one has to check that for any K type (V, μ) occurring in π , a certain form on $(V^*)^M$ is positive definite. To get necessary conditions for unitarity we compute the signature of this hermitian form on a certain set of K-types which we call *relevant* (section 4). The Weyl group W acts on $(V^*)^M$, and for a relevant K-type, the hermitian form is completely determined by the action of W . The W representations that come from relevant K types are called *relevant W-types*. The hermitian form is the same as the one for the affine Hecke algebras of type B and C considered in [B1].

To show that the necessary conditions for unitarity obtained from the relevant K-types are also sufficient, we have to prove certain irreducibility and unitarity results for unipotent representations. For the unitary groups these representations are cohomologically induced, so we use [KV] and the references therein. The conclusion is that the unitary spherical dual for $U(n, n)$, coincides with the unitary spherical dual of the affine Hecke algebra of type C, while the spherical unitary dual of $U(n + 1, n)$ coincides with the spherical dual of the affine Hecke algebra of type B.

Section 6 deals with Langlands parameters containing nontrivial fine K-types. These occur in $U(n, n)$ only. The same techniques as for the spherical case imply that the unitary dual for such parameters is contained in the spherical unitary dual of the affine Hecke algebra of type D. The main result is theorem 6.2.

For the case $p - q > 1$, the results in [KS] show that the relevant K-types are not sufficient to determine the entire spherical unitary dual. For these cases we only consider integral infinitesimal character. The following theorem is consistent with the philosophy outlined earlier, and holds for all values of p, q . This result is independent of [B1].

Theorem. *Let $X(\gamma)$ be a spherical principal series with integral infinitesimal character and Langlands subquotient $\bar{X}(\gamma)$. Then $\bar{X}(\gamma)$ is unitary if and only if*

there is a θ -stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and an irreducible 1-dimensional unitary representation W on L such that

$$\mathcal{R}_{\mathfrak{q}}^i(W) = \begin{cases} 0 & i \neq \dim(\mathfrak{u} \cap \mathfrak{k}) \\ \text{has } \overline{X}(\gamma) \text{ as a subquotient for} & i = s := \dim(\mathfrak{u} \cap \mathfrak{k}) \end{cases}$$

The sharper statement that $\mathcal{R}_{\mathfrak{q}}^s(W) = \overline{X}(\gamma)$ should hold. I expect it can be proved by using the techniques in [KV] and [T].

The title of the paper refers to the way the 1-dimensional representation W is obtained. If the form is negative on $\mu_n^{\pm}(1)$ (notation 4.4), then $\overline{X}(\gamma)$ is not unitary. If the form is positive on $\mu_n^{\pm}(1)$, then we find a θ -stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ with the following properties. The Levi component is of the form

$$L = U(k - x + 1 + R, k - x + 1) \times U(n - k + x - 1, n - k + x - 1),$$

and there exists a unitary representation W on $U(n - k + x - 1, n - k + x - 1)$ such that $\overline{X}(\gamma)$ is a factor of $\mathcal{R}_{\mathfrak{q}}^s(\text{triv} \otimes W)$. The question of whether $\overline{X}(\gamma)$ is unitary is then *reduced* to the same question for W . The representation W is not necessarily spherical. Its Langlands parameter is induced from an antiholomorphic discrete series. If the form is negative on a certain K-type analogous to $\mu_n^{\pm}(1)$, (section 7.7), then the form is negative on the isotypic component $\mu_c^{\pm}$ of $\overline{X}(\gamma)$. On the other hand if the form is positive on this K-type, we find a proper parabolic subalgebra $\mathfrak{q}_1$ of the same kind as $\mathfrak{q}$ such that W occurs in $\mathcal{R}_{\mathfrak{q}_1}^{s_1}(\chi_1 \otimes W_1)$. In finitely many steps we conclude that either $\overline{X}(\gamma)$ is not unitary or else it is of the form claimed by the theorem.

In the paper we don't quite follow this outline. Instead we find necessary conditions for the form to be positive definite on $\mu_n^{\pm}(1)$, $\mu_c^{\pm}(1)$. We then show that these parameters are as in the theorem. The necessary conditions for unitarity implied by positivity of the form on these K types should be proved in the same spirit as [B1]. There we find certain representations induced from Levi components so that we can deform the parameter until we can determine the signature on the relevant K-types. The definition of these parabolic subgroups is very involved combinatorially. So instead, we use the technique employed in [B2]. We induce the parameter unitarily up to a bigger group and deform it until the spherical module is a subquotient of a module unitarily induced from a smaller rank Levi component than the original group. Keeping track of signatures in this way is a lot more efficient. The drawback is that this does not generalize to exceptional groups. Similar techniques were also used in [B-J2].

I first proved a version of theorem 1 in 1984 while I was supported by a CNRS position in the mathematics department at Luminy. In particular I had the opportunity to get to know J. Carmona more closely, and I benefitted from many mathematical conversations with him. I would also like to thank P. Delorme and the rest of the department for their hospitality. Another version of

these results was presented at MSRI in the spring of 1988. I would also like to thank J. Bang-Jensen, S. Riba, D. Vogan as well as the referee for their input.

2 Notation and Preliminary Results

2.1

Let $G = U(p, q)$ with $p \geq q$, $p + q = n$ be the group of $n \times n$ matrices which leave the form $\sum_{i \leq p} |x_i|^2 - \sum_{i > p} |x_i|^2$ invariant. Then its Lie algebra is given by complex $n \times n$ matrices which are skew hermitian with respect to this form. In general, for any real Lie subgroup, we will denote its Lie algebra by the corresponding gothic german letter with subscript 0 and will drop the subscript for the complexification.

We fix the Cartan involution $\theta(x) := Jx^*J$ where x^* is the conjugate transpose and

$$J = \begin{bmatrix} I_{p \times p} & 0 \\ 0 & -I_{q \times q} \end{bmatrix}. \quad (2.1.1)$$

Write $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$, $G = K \cdot S$ for the corresponding Cartan decomposition.

2.2

We parametrize conjugacy classes of Cartan subgroups in the following way. For each $r \leq q$ let $H^r = T^r \cdot A^r$ be the Cartan subgroup such that

$$\begin{aligned} T^r &= \text{diag}(e(i\varphi_1), \dots, e(i\varphi_{p-r}), e(i\psi_1), \dots, e(i\psi_r), e(i\psi_r), \\ &\quad \dots, e(i\psi_1), e(i\varphi_{p+r+1}), \dots, e(i\varphi_n)) \\ A^r &= \text{diag}(t(x_1), \dots, t(x_r)), \end{aligned} \quad (2.2.1)$$

where $e(i\varphi) = \exp(i\varphi)$, $t(x_j) = \exp[x_j(E_{p-j, p+j} + E_{p+j, p-j})]$, and $E_{j,k}$ is the matrix with a 1 in the (j, k) entry, 0 otherwise. In this notation the compact Cartan subgroup is H^0 and the most split Cartan subgroup is H^q .

According to [BV2], the real Weyl group $W(H^r)$ is identified with

$$W(H^r) \cong S_{p-r} \times [(\mathbb{Z}/2\mathbb{Z})^r \rtimes S_r] \times S_{q-r},$$

where

- (1) S_{p-r} acts by permutations on $(\varphi_1, \dots, \varphi_{p-r})$,
- (2) S_r permutes the pairs (x_i, φ_i) ,
- (3) $(\mathbb{Z}/2\mathbb{Z})^r$ changes the sign of φ_i ,
- (4) S_{q-r} permutes $(\varphi_{p+r+1}, \dots, \varphi_n)$.

Let $\Delta(\mathfrak{g}, \mathfrak{h}^r)$ be the root system. Then a root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^r)$ is called

$$\begin{aligned} &\text{real, if} && \alpha|_{\mathfrak{h}^r} = 0, \\ &\text{imaginary, if} && \alpha|_{\mathfrak{a}^r} = 0, \\ &\text{complex otherwise.} \end{aligned} \tag{2.2.2}$$

An imaginary root is called compact if the root vector is in $\mathfrak{k}$ and noncompact if the root vector is in $\mathfrak{s}$. Given any subset $\Phi \subseteq \Delta(\mathfrak{g}, \mathfrak{h}^r)$ we will write

$$\rho(\Phi) = \frac{1}{2} \sum_{\alpha \in \Phi} \alpha.$$

2.3

The irreducible representations of K will be parametrized by their highest weights, written as

$$\mu = (\alpha_1, \dots, \alpha_p \mid \beta_1, \dots, \beta_q) = \sum_{i=1}^p \alpha_i \varepsilon_i + \sum_{j=1}^q \beta_j \varepsilon_{p+j},$$

such that $\alpha_1 \geq \dots \geq \alpha_p, \beta_1 \geq \dots \geq \beta_q$. We will denote by Φ_c the corresponding positive root system in $\Delta(\mathfrak{k}, \mathfrak{h}^0)$.

Let $\rho_c = \rho(\Phi_c)$. Then according to [V1], to each μ one can attach a Cartan subalgebra $\mathfrak{h}^r$ and a $\lambda^G \in (\mathfrak{h}^r)^*$ given by

$$\lambda^G = \mu + 2\rho_c - \rho(\Phi) + \frac{1}{2}v \tag{2.3.1}$$

where Φ is a positive system such that $\mu + 2\rho_c$ is dominant and v a certain sum of noncompact roots (see Chapter 5 in [V1]). We will assume familiarity with calculations involving μ and λ^G .

Given $\nu \in (\mathfrak{a}^r)^*$ we denote by $X(\lambda^G, \nu)$ the standard generalized principal series and by $\overline{X}(\lambda^G, \nu; \mu)$ the unique irreducible subquotient containing the lowest K type μ . We call (λ^G, ν) a Langlands parameter.

For $U(p, q)$, a Langlands parameter is going to be written out in coordinates. The coordinates of λ^G on the ϕ_i will be denoted by

$$\lambda_1^+, \dots, \lambda_{p-r}^+, \lambda_{p+r+1}^-, \dots, \lambda_n^-, \tag{2.3.2}$$

while the coordinates of (λ^G, ν) on (ψ_i, x_i) will be written

$$\underline{\eta_1 + \nu_1}, \underline{\eta_1 - \nu_1}, \dots, \underline{\eta_r + \nu_r}, \underline{\eta_r - \nu_r}. \tag{2.3.3}$$

The relation between the coordinates of μ and the λ 's and η 's come from (2.3.1). The coordinates of $\mu + 2\rho_c$ are

$$\alpha_1 + p - 1, \alpha_2 + p - 3, \dots, \alpha_p - p + 1, \beta_1 + q - 1, \beta_2 + q - 3, \dots, \beta_q - q + 1. \tag{2.3.4}$$

Reorder the entries in (2.3.4) in decreasing order and subtract the entries of $\rho(\Phi)$ which are

$$\left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-3}{2}, \frac{n-1}{2}\right),$$

to get

$$(r_1 - \frac{n-1}{2}, \dots, r_n + \frac{n-1}{2}). \quad (2.3.5)$$

The entries of (2.3.5) satisfy one of the relations

$$\begin{aligned} r_i - \frac{n-2i+1}{2} &> r_{i+1} - \frac{n-2i-1}{2}, \quad \text{or} \\ r_i - \frac{n-2i+1}{2} &= r_{i+1} - \frac{n-2i-1}{2}, \quad \text{or} \\ r_i - \frac{n-2i+1}{2} + 1 &= r_{i+1} - \frac{n-2i-1}{2}. \end{aligned}$$

Form pairs of entries in (2.3.5) that are equal and come one from an α the other from a β . These form η_i 's. For the last relation, add $1/2$ to one of them, subtract $1/2$ from the other to make them equal. The resulting coordinates are η_i 's as well.

In practice, we will follow the procedure in the example.

Example

Let

$$\mu = (2, 2, 2, 2, 2, 2, 2 \mid 0, -3, -3, -4).$$

Then

$$\mu + 2\rho_c = (8, 6, 4, 2, 0, -2, -4 \mid 3, -2, -4, -7)$$

The coordinates satisfy

$$\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_8 > \epsilon_4 > \epsilon_5 > \epsilon_9 \geq \epsilon_6 > \epsilon_7 \geq \epsilon_{10} > \epsilon_{11}.$$

Instead of permuting the r_i , we reorder the entries of ρ and subtract from $\mu + 2\rho_c$:

$$\begin{aligned} (8, 6, 4, 2, 0, -2, -4 \mid 3, -2, -4, -7) - \\ (5, 4, 3, 1, 0, -2, -3 \mid 2, -1, -4, -5) = \\ (3, 2, 1, 1, 0, \quad 0, -1 \mid 1, -1, \quad 0, -2). \end{aligned}$$

In $\mu + 2\rho_c$, the coordinates were in order $\epsilon_9 \geq \epsilon_6$ and $\epsilon_7 \geq \epsilon_{10}$, but now they are in opposite order. So we add $1/2$ to the coordinates $\epsilon_9, \epsilon_{10}$ and subtract $1/2$ from the coordinates ϵ_6, ϵ_7 to get

$$\lambda^G = (3, 2, 1, 1, 0, -1/2, -1/2 \mid 1, -1/2, -1/2, -2).$$

So we have paired up the coordinates (ϵ_3, ϵ_8) , (ϵ_6, ϵ_9) and $(\epsilon_7, \epsilon_{10})$ to form η_1 , η_2 and η_3 . The coordinates $\epsilon_1, \epsilon_2, \epsilon_3$ are $\lambda_1^+, \lambda_2^+, \lambda_3^+$ while ϵ_{11} is a λ_{11}^- . A typical Langlands parameter with lowest K-type μ is written as

$$(3^+, 2^+, 1^+, 0^+, \underline{1 + \nu_1, 1 - \nu_1}, \underline{1/2 + \nu_2, 1/2 - \nu_2}, \underline{1/2 + \nu_3, 1/2 - \nu_3}, -2^-)$$

or

$$(3, 2, 1, 1 + \nu_1, 0, 1/2 + \nu_2, 1/2 + \nu_3 \mid -1 - \nu_1, -1/2 - \nu_2, -1/2 - \nu_3, 2).$$

2.4

Let $\xi \in (\mathfrak{h}^0)^*$. Then ξ defines a complex θ -stable parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$$

such that

$$\begin{aligned} \Delta(\mathfrak{l}, \mathfrak{h}^0) &= \{\alpha : (\xi, \alpha) = 0\} \\ \Delta(\mathfrak{u}, \mathfrak{h}^0) &= \{\alpha : (\xi, \alpha) > 0\}. \end{aligned} \tag{2.4.1}$$

Given an $(\mathfrak{l}, L \cap K)$ module X_L , one can define, following chapter 6 in [V1], or [KV] chapter V, functors $\mathcal{R}_q^i X_L$ such that for any $(\mathfrak{q}, K)$ module Y ,

$$\text{Hom}_{(\mathfrak{g}, K)}(Y, \mathcal{R}^i X_L) \cong \text{Hom}_{(\mathfrak{l}, L \cap K)}(H^i(\mathfrak{u}, Y), X_L).$$

(Here $L = \text{Cent}(\xi, G)$.)

Since our classifications of unitary representations is in terms of such functors we will rely on the algebraic properties of the $\mathcal{R}_q^i$ as developed in [V1] Chapter 6, or [KV].

2.5

A real form $\mathfrak{g}_0$ of a complex reductive algebra $\mathfrak{g}$ defines a conjugation $-$ on $\mathfrak{g}$ by the formula $X^* := -\overline{X}$. This extends to an antiautomorphism of $U(\mathfrak{g})$ which we denote by $*$ as well. A bilinear form $(\ , \)$ on a $(\mathfrak{g}, K)$ module (π, X) is called hermitian invariant if

- a) $(v, w) = \overline{(w, v)}$,
- b) $(cv, w) = c(v, w)$, $(v, cw) = \bar{c}(v, w)$, $c \in \mathbb{C}$
- c) $(\pi(X)v, w) = (v, \pi(X^*)w)$.

Let $\gamma = (\lambda^G, \nu) \in (\mathfrak{h}^r)^*$ be a Langlands parameter. Let Σ, Σ' be two positive root systems for $\Delta(\mathfrak{g}_0, \mathfrak{a}_0^r)$. Assume $\nu \in (\mathfrak{a}^r)^*$ is such that $\text{Re } \nu$ is strictly dominant for Σ . Let $P(\Sigma)$ and $P(\Sigma')$ be the real parabolic subalgebras corresponding to Σ and Σ' as in [SV], Chapter 3. Let $X_\Sigma(\lambda^G, \nu)$ and $X_{\Sigma'}(\lambda^G, \nu)$ be the induced representations. Then there is an intertwining operator

$$I(P(\Sigma), P(\Sigma'), \gamma) : X_\Sigma(\lambda^G, \nu) \rightarrow X_{\Sigma'}(\lambda^G, \nu)$$

(which we sometimes denote by $I(\Sigma, \Sigma', \gamma)$). If $\Sigma' = -\Sigma$, $I(\Sigma, -\Sigma, \gamma)$ is called the long intertwining operator and its image is a sum $\oplus \bar{X}(\lambda^G, \nu; \mu)$ with μ a lowest K -type for $X(\lambda^G, \nu)$. Every irreducible $(\mathfrak{g}, K)$ module appears as a $\bar{X}(\lambda^G, \nu; \mu)$ in this fashion. Fix μ a lowest K -type. Then since it appears with multiplicity 1 in $X(\lambda^G, \nu)$ we can normalize $I(\Sigma, -\Sigma, \gamma)$ to be the identity on $X(\lambda^G, \nu)^\mu$.

Theorem (KZ).

- (1) $\bar{X}(\lambda^G, \nu; \mu)$ admits a hermitian form if and only if there is $w \in W(H^r)$ such that $w(\lambda^G, \nu) = (\lambda^G, -\bar{\nu})$.
- (2) Suppose (1) is satisfied for some w and assume Σ is dominant for $\text{Re } \nu$. Then $\bar{X}(\lambda^G, \nu; \mu)$ is unitary if and only if the form

$$\langle v, w \rangle = (I(\Sigma, -\Sigma, \gamma)v, w)$$

is positive definite.

For (2), recall that $X(\lambda^G, \nu)$ and $X(\lambda^G, -\bar{\nu})$ are hermitian dual. The form $(\ , \)$ is the hermitian pairing between these two modules.

Let (V, μ) be a K -type. Fix a positive definite hermitian form on it. Then identifying $X(\Sigma, \gamma)$ with $I(\sigma(\lambda^G)) := \text{Ind}_{K \cap M}^K[\sigma(\lambda^G)]$, we get a fixed positive definite hermitian form on

$$\text{Hom}_K[V_\mu, X(\Sigma, \gamma)], \quad (2.5.1)$$

independent of ν and a map

$$I(\mu, \gamma) : \text{Hom}_K[V_\mu, I(\sigma(\lambda^G))] \longrightarrow \text{Hom}_K[V_\mu, I(\sigma(\lambda^G))]. \quad (2.5.2)$$

This map is hermitian symmetric and depends analytically on ν for $\text{Re } \nu$ dominant. Part (2) of the theorem can be rephrased as saying that $\bar{X}(\lambda^G, \nu)$ is unitary if and only if $I(\mu, \gamma)$ is positive semidefinite for all μ .

In the spherical case, $\sigma(\lambda^G)$ is trivial and the map $I(\mu, \nu)$ can be viewed via Frobenius reciprocity as

$$I(\mu, \nu) : (V_\mu^*)^M \longrightarrow (V_\mu^*)^M. \quad (2.5.3)$$

2.6

The following theorem reduces the problem of determining the unitary dual to the case when $\text{Im } \nu = 0$. Let $P' = M'N'$ be the parabolic subgroup determined by $\text{Im } \nu$ in the sense that M is the centralizer of $\text{Im } \nu$ and the roots of N are $\alpha \in \Delta(\mathfrak{h}^r, \mathfrak{a}^r)$ for which

$$(\alpha, \text{Im } \nu) > 0. \quad (2.6.1)$$

Then (λ^G, ν) defines a Langlands parameter for both M and G ; denote the standard modules and their Langlands quotients by subscript G and M respectively.

Theorem ([K], theorem 16.10).

$$\overline{X}_G(\lambda^G, \nu) = \text{Ind}_P^G[\overline{X}_M(\lambda^G, \nu)].$$

When using normalized Harish-Chandra induction, $\overline{X}(\lambda^G, \nu)$ is unitary if and only if $\overline{X}_M(\lambda^G, \nu)$ is unitary.

Because of this, we will assume that ν is always real, i.e. $\text{Im } \nu = 0$.

2.7

Given any two positive systems Σ and Σ' as in 2.5 there is a chain $\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_k = \Sigma'$ such that the span of each $\Sigma_j \setminus (\Sigma_j \cap \Sigma_{j+1})$ is 1-dimensional. Let P_j be the smallest subgroup containing both $P(\Sigma_j)$ and $P(\Sigma_{j+1})$. If we pick $\alpha_j \in \Sigma_j$, $\alpha_j \in \Sigma_{j+1}$ we denote by $G(\alpha_j) = M_j A_j$ the Levi component of P_j . Then

$$I(\Sigma, \Sigma') = I_0 \circ \dots \circ I_{k-1}$$

where

$$I_j = I(\Sigma_j, \Sigma_{j+1}) = \text{Ind}_{P_j}^G(I(P(\Sigma_j) \cap G(\alpha_j), P(\Sigma_{j+1}) \cap G(\alpha_j), \gamma)).$$

3 Cells in $U(p, q)$

3.1

Let $\gamma = (\lambda^G, \nu)$ be such that γ is regular integral. Let

$$\mathcal{R}^+(\gamma) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^r) : (\alpha, \gamma) > 0\}.$$

Suppose $\alpha \in \mathcal{R}^+(\gamma)$ is a real root. Let

$$\mathfrak{a}_0^\alpha = \{x \in \mathfrak{a}_0^r : \alpha(X) = 0\}$$

and

$$M^\alpha = \text{Cent}(A^\alpha, G).$$

Then there are maps

$$\begin{aligned} d\varphi_\alpha &: \mathfrak{sl}(2, R) \rightarrow \mathfrak{m}_0^\alpha \\ \varphi_\alpha &: \text{SL}(2, R) \rightarrow M^\alpha \end{aligned} \tag{3.1.1}$$

such that,

$$\begin{aligned} d\varphi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &\in \alpha_0^r, & d\varphi_\alpha(-X^t) &= \theta(\varphi_\alpha(X)), \\ d\varphi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &\in \mathfrak{g}_\alpha. & (\mathfrak{g}_\alpha &\text{ the root space of } \alpha). \end{aligned} \tag{3.1.2}$$

The map $d\varphi_\alpha$ is the differential of φ . Let

$$m_\alpha = \varphi_\alpha \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition. Let α be real. We say that α satisfies the parity condition if

$$\lambda^G(m_\alpha) = -(-1)^{(\nu, \alpha)}.$$

In practice suppose

$$(\lambda^G, \nu) = (\eta_1, \dots, \eta_{p-r}, \eta_{p-r+1} + \nu_1, \dots, \eta_p + \nu_r, \eta_p - \nu_r, \dots, \eta_{p-r+1} - \nu_1, \eta_{p+r+1}, \dots, \eta_n) \quad (3.1.3)$$

describes the infinitesimal character. The real roots are $\alpha_1, \dots, \alpha_r$ such that

$$\alpha_j(\nu) = 2\nu_j,$$

and

$$\lambda^G(m_{\alpha_j}) = e(2\pi i \eta_{p-r+j}) = \pm 1.$$

So the parity condition is

$$e(2\pi i \eta_{p-r+j}) = -(-1)^{2\nu_r}.$$

(λ^G, ν) is integral means that $\eta_1, \dots, \eta_{p-r}, \eta_{p-r+j} + \nu_j, \eta_{p+r+1}, \dots, \eta_n$ are all either integers or half-integers.

3.2

Definition. Suppose $\alpha \in \mathcal{R}^+(\gamma)$ is simple. We say $\alpha \in \tau(\gamma)$ (α is in the τ -invariant) if one of the following holds:

- a) α is compact imaginary,
- b) α is complex, $\theta\alpha \in \mathcal{R}^+(\gamma)$,
- c) α is real and satisfies the parity condition.

This is definition 7.3.8 of [V1] for our case.

3.3

Fix a regular integral infinitesimal character χ . Let

$$\mathfrak{G}(\chi) = \{ \Sigma c_\gamma X(\gamma) : X(\gamma) \text{ has infinitesimal character } \chi \}$$

be the Grothendieck group of virtual characters with coefficients $c_\gamma \in \mathbb{C}$. Then W , the complex Weyl group, acts on $\mathfrak{G}(\chi)$ via the coherent continuation action ([V3]). This action decomposes into blocks which in turn can be decomposed into cones and cells.

Definition. We say that $\gamma_1 \leq \gamma_2$ if there is F_2 finite dimensional, so that $\overline{X}(\gamma_2) \otimes F_2$ contains $\overline{X}(\gamma_1)$ as a factor. The relation $\sim$, called block equivalence, is the equivalence relation generated by $\leq$. For π irreducible we write

$$\begin{aligned}\overline{B}(\pi) &= \{\pi' \text{ irreducible} : \pi' \sim \pi\} \\ B(\pi) &= \text{linear span}\{\pi' : \pi' \in \overline{B}(\pi)\} \\ \overline{C}(\pi) &= \{\pi' \text{ irreducible} : \pi' \leq \pi\} \\ C(\pi) &= \text{linear span}\{\pi' : \pi' \in \overline{C}(\pi)\}.\end{aligned}\tag{3.3.1}$$

$B(\pi)$ is called the block of π , $C(\pi)$ is called the cone of π .

3.4

Definition. We say $\pi_1 \approx \pi_2$ where π_1, π_2 are irreducible if $\pi_1 \leq \pi_2$ and $\pi_2 \leq \pi_1$. We write

$$\begin{aligned}\overline{V}(\pi) &= \{\pi' \text{ irreducible} : \pi' \approx \pi\} \\ V(\pi) &= C(\pi) / \text{linear span}\{\pi' : \pi' \leq \pi, \pi' \notin \overline{V}(\pi)\}.\end{aligned}\tag{3.4.1}$$

Then $B(\pi)$, $C(\pi)$ and $V(\pi)$ are Weyl group representations and

$$\mathfrak{G}(\chi) \cong \bigoplus B(\pi), \quad B \simeq \bigoplus_{\pi \in \overline{B}} V(\pi).\tag{3.4.2}$$

We now describe these objects for $U(p, q)$. We note that χ need be defined only up to translation functors; so we use $\chi = \rho$.

- a) For $p \neq q$ there is one block, which we call $\overline{B}_0$. Its dual is formed of representations in $GL(n, \mathbb{R})$.
- b) For $p = q$, there are two blocks, the one containing the trivial representation which we call $\overline{B}_0$, dual to one in $GL(n, \mathbb{R})$, and the block whose dual is formed of representations in $U^*(2p)$ which we call $\overline{B}_*$.

3.5

We recall some facts about the wave front set of a representation for $U(p, q)$. For the basic notions on nilpotent orbits see [B1], [CMcG] and the references therein. Recall that nilpotent orbits in $U(p, q)$ are parametrized by signed partitions, *i.e.* partitions $(b_1, \dots, b_l)$ so that each row gets a $+$ or a $-$. We write the partition as $(b_1^{\epsilon_1}, \dots, b_l^{\epsilon_l})$ with $\epsilon_i = \pm$. Two entries of equal size are interchangeable. Usually the partition is pictured as a tableau and every row has alternating signs starting with the ϵ_i .

For any irreducible representation π , we will write $AV(\pi)$ for the union of orbits in the asymptotic support of π as defined in [BV3].

3.6

The block $\overline{B}_0$ was studied in [BV2]. Denote by $\sigma(\mathcal{O}_c)$ the irreducible Weyl group representation attached to $\mathcal{O}_c$ by Springer ([S]), tensored with sign so that the trivial orbit corresponds to the sign representation.

Proposition. *Let $\overline{X}(\gamma) \in \overline{B}_0$.*

a) $AV(X(\gamma))$ consists of exactly one nilpotent orbit in $\mathfrak{g}_0$.

b) $B_0 \simeq \sum_{\mathcal{O}_c \subseteq \mathfrak{gl}(n, \mathbb{C})} m(\mathcal{O}_c) \sigma(\mathcal{O}_c)$ where

$$m(\mathcal{O}_c) = |\{\text{orbits of } U(p, q) \text{ in } \mathcal{O}_c \cap \mathfrak{g}_0\}|.$$

c) Let π be irreducible such that $AV(\pi) = \mathcal{O}$. Then $V(\pi) = \sigma(\mathcal{O}_c)$.

3.7

Similar results hold for the block $\overline{B}_*$, dual to $U^*(2p)$. The techniques for proving them are the same as in [BV2] but much easier. For example, $U^*(2p)$ has only one conjugacy class of Cartan subgroups. If we write the Lie algebra in coordinates as

$$(a_1, \dots, a_{2p}), \tag{3.7.1}$$

then the Cartan involution interchanges a_i with $-a_{2p-i+1}$. We record the result in the next proposition, but omit the details.

Proposition. a) $\overline{X}(\gamma) \in B_*$, if and only if γ is the parameter of a principal series such that all real roots satisfy the parity condition. Furthermore,

b) $AV(\overline{X}(\gamma)) = \mathcal{O}_c \cap \mathfrak{g}_0$, where $\mathcal{O}_c$ is a complex orbit with Jordan decomposition such that each block is of even size.

c) $B_* = \bigoplus \sigma(\mathcal{O}_c)$ with $\mathcal{O}_c$ as in b).

d) $V(\pi) = \sigma(\mathcal{O}_c)$.

4 Some Results on Derived Functors

4.1

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subgroup defined by $\gamma \in (\mathfrak{h}^0)^*$. Let L be the real group with Lie algebra $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$ and W an $(\mathfrak{l}, L \cap K)$ module. Then we recall that ([V1])

$$\mathcal{R}_{\mathfrak{q}}^i(W) = \left(\Gamma_{\mathfrak{q}, L \cap K}^{\mathfrak{g}, L \cap K} \right)^i \circ \text{pro}_{\mathfrak{q}, L \cap K}^{\mathfrak{g}, L \cap K} (W \otimes \wedge^{\dim \mathfrak{u}} \mathfrak{u}) \tag{4.1.1}$$

where

$$\text{pro}_{\mathfrak{q}, L \cap K}^{\mathfrak{g}, L \cap K}(W \otimes \wedge^{\dim \mathfrak{u}} \mathfrak{u}) = \text{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), W \otimes \wedge^{\dim \mathfrak{u}} \mathfrak{u}). \quad (4.1.2)$$

If W has infinitesimal character λ , then $R_{\mathfrak{q}}^i(W)$ has infinitesimal character $\lambda + \rho(\mathfrak{u})$.

Let $\xi \in (\mathfrak{h}^0)^*$ be such that

$$\begin{aligned} (\xi, \alpha) &\in \mathbb{N}^+ & \alpha &\in \Delta(\mathfrak{u}) \\ (\xi, \alpha) &= 0 & \alpha &\in \Delta(\mathfrak{l}). \end{aligned} \quad (4.1.3)$$

Let (π, V) be a hermitian admissible $(\mathfrak{g}, K)$ module. For each K-type (μ, F) , the hermitian form is given by a hermitian matrix A_{μ} . It has a signature $(p_{\mu}, r_{\mu}, q_{\mu})$, where p_{μ} is the number of positive eigenvalues, r_{μ} is the dimension of the kernel and q_{μ} is the number of negative eigenvalues of A_{μ} . We form the formal combination of representations of K ,

$$[V]_+ = \sum p_{\mu} F_{\mu}, \quad [V]_0 = \sum r_{\mu} F_{\mu}, \quad [V]_- = \sum q_{\mu} F_{\mu}. \quad (4.1.4)$$

The signature is defined to be

$$[V]_+ - [V]_- = \sum (p_{\mu} - q_{\mu}) F_{\mu}.$$

Let (E, τ) be an $L \cap K$ type. We define

$$\begin{aligned} R_{\mathfrak{q}}(E) &:= \sum_{\mu} [R_{\mathfrak{q}}(E) : F_{\mu}] F_{\mu}, \text{ where } [R_{\mathfrak{q}}(E) : F_{\mu}] \text{ equals} \\ &\sum (-1)^j \sum_n \dim \text{Hom}_{L \cap K}[H_j(\mathfrak{u} \cap \mathfrak{k}, F_{\mu}) : S^n(\mathfrak{u} \cap \mathfrak{s}) \otimes E \otimes \wedge^{\dim \mathfrak{u}}(\mathfrak{u})], \end{aligned} \quad (4.1.5)$$

Proposition. *Let W be a hermitian irreducible $(\mathfrak{l}, L \cap K)$ module. Assume that there is t_0 such that*

$$V_t = \text{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), W \otimes \wedge^{\dim \mathfrak{u}} \mathfrak{u} \otimes \mathbb{C}_{t\xi})$$

is irreducible for $t \geq t_0$. Then $\mathcal{R}_{\mathfrak{q}}^i(W \otimes \mathbb{C}_{t\xi}) = 0$ for $i \neq s = \dim(\mathfrak{u} \cap \mathfrak{s})$ and $t \geq t_0$. The module $\mathcal{R}_{\mathfrak{q}}^s(W \otimes \mathbb{C}_{t\xi})$ is hermitian. Its signature satisfies

$$[\mathcal{R}_{\mathfrak{q}}^s(W \otimes \mathbb{C}_{t\xi})]_+ - [\mathcal{R}_{\mathfrak{q}}^s(W \otimes \mathbb{C}_{t\xi})]_- = R_{\mathfrak{q}}([W]_+) - R_{\mathfrak{q}}([W]_-).$$

In particular, if W is unitary, then $R_{\mathfrak{q}}^s(W)$ is either unitary or 0.

Proof. See [V2] or [W]. ■

4.2

In order to apply the results in section 4.1, we need some conditions that insure the irreducibility of V_t and $\mathcal{R}_q^s(W \otimes \mathbb{C}_{t_0\xi})$. Let

$$\xi = \sum_{i \leq p_1} \varepsilon_i + \sum_{j \leq q_1} \varepsilon_{p+j}. \quad (4.2.1)$$

The parabolic subalgebra determined by ξ has Levi component $L = L_1 \times L_2$ with $L_1 = U(p_1, q_1)$, $L_2 = U(p_2, q_2)$. Let $W = W_1 \otimes W_2$ be such that W_1 is a character. Assume that $W_1 \otimes W_2 \otimes \mathbb{C}^{-\rho(u)}$ has infinitesimal character in coordinates given by γ_1 for W_1 and γ_2 for W_2 . These are

$$\begin{aligned} \gamma_1 &= \left(\frac{p_1 + q_1 - 1}{2}, \dots, -\frac{p_1 + q_1 - 1}{2} \right) \\ \gamma_2 &= (\lambda_1, \lambda_2, \dots, \lambda_{p_2+q_2}, -\lambda_{p_2+q_2}, \dots, -\lambda_1), \end{aligned} \quad (4.2.2)$$

where $\dots \geq \lambda_i \geq \lambda_{i+1} \geq \dots \geq 0$.

Proposition. *If γ_2 is such that $\lambda_1 \leq \frac{p_1+q_1-1}{2}$, then V_t is irreducible for all $t \geq 1$.*

Proof. There are two cases to consider:

- a) t is such that the infinitesimal character is integral.
- b) t is such that the infinitesimal character is not integral.

In case a), any factor of V_t must have a parameter that is a γ' which is a permutation of $\gamma := (\gamma_1, \gamma_2)$ but such that it is a parameter for a finite dimensional on L_1 , and the difference is a sum of roots dominant for ξ . This is not possible in view of the assumption.

Case b) reduces to a similar question on the root system for which (γ_1, γ_2) is integral. It is then easy to see that V_t must be irreducible. ■

Because of the shift by $\rho(u)$ in the infinitesimal character, we will work with the functors

$$\mathcal{R}_q^i(W) := \left(\Gamma_{\mathfrak{q}, L \cap K}^{\mathfrak{g}, L \cap K} \right)^i \circ \text{pro}_{\mathfrak{q}, L \cap K}^{\mathfrak{g}, L \cap K} (W \otimes \mathbb{C}^{\rho(u)}) \quad (4.2.3)$$

instead of (4.1.1). Then $\mathcal{R}^i(W)$ and W have the same infinitesimal character. The formulas for K-types are adjusted by $2\rho(u \cap \mathfrak{s}) - \rho(u)$ instead of $2\rho(u \cap \mathfrak{s})$. The disadvantage is that we have to consider representations that are genuine for a double cover of $U(p, q)$. This is the cover where highest weights with half-integers in the coordinates give rise to representations of the corresponding maximal compact group.

4.3

Let $\pi = \overline{X}(\gamma; \mu)$ be an irreducible representation with integral singular infinitesimal character. Let A be the set of simple roots in $\mathcal{R}^+(\gamma)$ which are not zero on the infinitesimal character. Let $W(A^c) \subset W$ be the subgroup generated by the reflections from the roots in A^c .

Proposition. *Let $\mathcal{O}_c = AV(\pi)_c$ and $\mathcal{O}'_c \subset \overline{\mathcal{O}_c}$. Assume*

$$[\sigma(\mathcal{O}'_c)|_{W(A^c)} : \text{triv}] = \begin{cases} 1 & \text{if } \mathcal{O}'_c = \mathcal{O}_c, \\ 0 & \text{if } \mathcal{O}'_c \neq \mathcal{O}_c. \end{cases}$$

Then the only irreducible representation $\overline{X}$ with the same infinitesimal character as π satisfying

$$AV(\overline{X}) \subseteq AV(\pi)$$

is π .

Proof. This follows from section 3 and [BV1]. ■

This result will be applied in the following way. Let $X = \mathcal{R}_{\mathfrak{q}}^s(W)$ where $\mathfrak{q}$ is θ -stable, or $X = \text{Ind}_Q^G(W)$ where $Q = MN$ is a real parabolic subgroup. When $\mathfrak{q}$ is θ -stable, assume that W satisfies the vanishing conditions in sections 4.1-4.2 and X is nonzero. The infinitesimal character of X determines the set A^c in section 4.4. Any nilpotent orbit $\mathcal{O}'$ in $AV(\pi')$ must satisfy

$$[\sigma(\mathcal{O}'_c) |_{W(A^c)} : \text{triv}] \neq 0.$$

If the condition of proposition 4.4 is satisfied, then $\sigma(\mathcal{O}'_c) = \sigma(\mathcal{O}_c)$, so $\mathcal{O}'_c = \mathcal{O}_c$. Thus $AV(\pi') \subset AV(X)$. The results in sections 3.6 and 3.7 imply that X has only one possible factor.

4.4

Suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and γ_1, γ_2 are as in 4.1-4.2. We will need the multiplicities of certain K -types in $\mathcal{R}_{\mathfrak{q}}^s(\text{triv} \otimes W_2)$ with $\text{triv} \otimes W_2$ satisfying the conditions in 4.2. Denote

$$\begin{aligned} \mu_n^+(1) &:= (1, 0, \dots, 0 \mid 0, \dots, 0, -1) \\ \mu_n^-(1) &:= (0, \dots, 0, -1 \mid 1, 0, \dots, 0) \\ \mu_c^+(1) &:= (1, 0, \dots, 0, -1 \mid 0, \dots, 0) \\ \mu_c^-(1) &:= (0, \dots, 0 \mid 1, 0, \dots, 0, -1). \end{aligned} \tag{4.4.1}$$

Let $\chi := \mathbb{C}^{2\rho(\mathfrak{u} \cap \mathfrak{s})}$. For later reference, $2\rho(\mathfrak{u} \cap \mathfrak{s}) - \rho(\mathfrak{u})$ has coordinates,

$$2\rho(\mathfrak{u} \cap \mathfrak{s}) - \rho(\mathfrak{u}) = \left(\underbrace{\frac{p_2 - q_2}{2}}_{p_1}, \underbrace{-\frac{p_1 - q_1}{2}}_{p_2} \mid -\underbrace{\frac{p_2 - q_2}{2}}_{q_1}, \underbrace{\frac{p_1 - q_1}{2}}_{q_2} \right). \tag{4.4.2}$$

Proposition.

$$\begin{aligned} [\mu_n^\pm(1) : R_q^s(\text{triv} \otimes W_2)] &= [\mu_c^\mp(t) \otimes \chi : W_2] + 1, \\ [\mu_c^\pm(1) : R_q^s(\text{triv} \otimes W_2)] &= [\mu_n^\mp(t) \otimes \chi : W_2] \end{aligned}$$

Proof. The main tool is the Euler-Poincaré principle in section V.5 of [KV]:

$$\begin{aligned} \sum_{j=0}^S (-1)^j \dim \text{Hom}_K[V, R_q^j(Z)] &= \\ \sum_{j=0}^S (-1)^j \sum_{n=0}^{\infty} \dim \text{Hom}_{L \cap K}[H_j(\mathfrak{u} \cap \mathfrak{k}, V), S^n(\mathfrak{u} \cap \mathfrak{s}) \otimes Z^\#], \end{aligned} \quad (4.4.3)$$

where $Z^\# = Z \otimes \mathbb{C}^{2\rho(\mathfrak{u})}$.

As a representation of $L \cap K$,

$$\mathfrak{u} \cap \mathfrak{s} = \mathbb{C}^{p_1} \otimes (C^{q_2})^* \oplus \mathbb{C}^{p_2} \otimes (C^{q_1})^*.$$

Then $S^n(\mathfrak{u} \cap \mathfrak{s})$ is the sum of representations of the form

$$\begin{aligned} (\alpha_1, \dots, \alpha_k, 0, \dots, 0 ; 0, \dots, 0, -\beta_\ell, \dots, -\beta_1 \mid \\ \beta_1, \dots, \beta_\ell, 0, \dots, 0 ; 0, \dots, 0, -\alpha_k, \dots, -\alpha_1). \end{aligned} \quad (4.4.4)$$

Let $w_0 \in W_K$ be the long element. The weights of $H_j(\mathfrak{u} \cap \mathfrak{s}, V_\mu)$ are of the form $w(\mu + \rho_c) - \rho_c$ with $w \in W$ such that they are dominant for $\mathfrak{l} \cap \mathfrak{k}$. Since the first factor of Z is the trivial representation, it follows that the only j that contributes is such that $w = 1$. It follows that

$$\dim \text{Hom}_K[\mu(1) : R_q^s(\text{triv} \otimes W_2)] = \dim \text{Hom}_{L_2 \cap K}[\mu(1) \otimes \chi^* : W_2] \quad (4.4.5)$$

Consider the case of $\mu_n^+(1)$; the others are similar. Then $\ell = k$ and $\alpha_1 = \dots = \alpha_\ell = 1$. The $L \cap K$ -types μ_L such that $\mu_L \otimes (-\alpha)$ contain the factor of μ on the $U(q_2)$ are exactly the ones in the statement of the proposition; $(0, \dots, 0, -1)$ occurs with multiplicity 1 in this tensor product. ■

4.5

Let $\mu(\alpha, \beta) = (\alpha, \dots, \alpha \mid \beta, \dots, \beta)$ be a 1-dimensional K -type, and write $\eta := \frac{\alpha + \beta}{2}$. We will show that there is at most one irreducible representation π with a given infinitesimal character χ containing $\mu(\alpha, \beta)$, and determine its Langlands parameter.

Proposition. *Let X be an irreducible $(\mathfrak{g}, K)$ module such that $X^{\mu(\alpha, \beta)} \neq 0$, and recall $\eta = \frac{\alpha + \beta}{2}$. Then the infinitesimal character χ must be of the form*

$$\begin{aligned} \chi \approx \left(\alpha + \frac{R-1}{2}, \dots, \alpha - \frac{R-1}{2}, \right. \\ \left. \eta + \nu_1, \dots, \eta + \nu_1, \dots, \eta + \nu_k, \dots, \eta + \nu_k, \eta - \nu_k, \dots, \eta - \nu_1 \right). \end{aligned} \quad (4.5.1)$$

The parameter (λ_0^G, ν_0) of any principal series containing $\mu(\alpha, \beta)$ is of the form

$$(\alpha + \frac{R-1}{2})^+, \dots, (\alpha - \frac{R-1}{2})^+, \underline{\eta + \nu_1, \eta - \nu_1, \dots, \eta + \nu_k, \eta - \nu_k}. \quad (4.5.2)$$

Proof. By the subquotient theorem, X occurs in a principal series. We determine the principal series that contains $\mu(\alpha, \beta)$. Let $P^R = M^R A^R N^R$ be minimal cuspidal parabolic subgroup where $R = p - q \geq 0$, and

$$A^R := \exp \mathfrak{a}_0^R.$$

Then

$$M^R = M^R \cap K \simeq U(R) \times U(1)^{n-R}.$$

In the usual coordinates on $\mathfrak{t}_0^R$,

$$\mu(\alpha, \beta)|_{M^R} = (\alpha, \dots, \alpha) \otimes \left(\frac{\alpha + \beta}{2}, \dots, \frac{\alpha + \beta}{2} \mid \frac{\alpha + \beta}{2}, \dots, \frac{\alpha + \beta}{2} \right). \quad (4.5.3)$$

Thus a principal series $X(\lambda_0^G, \nu_0)$ will contain $\mu(\alpha, \beta)$ if and only if

$$\lambda_0^G = \sum_{i \leq R} \left(\alpha + \frac{R+1-2i}{2} \right) \varepsilon_i + \sum_{j \leq n} \frac{\alpha + \beta}{2} (\varepsilon_{R+j} + \varepsilon_{p+j}). \quad (4.5.4)$$

The proof of the proposition follow from this. ■

Corollary. *Given an infinitesimal character χ , there is at most one irreducible module X such that $X^{\mu(\alpha, \beta)} \neq 0$.*

Proof. The coordinates of χ must coincide with the coordinates in (4.5.2). The only choice is for the ν_i , and two such choices differ by a permutation of the ν_i . The principal series for such parameters have the same composition factors. The corollary now follows from the fact that $\mu(\alpha, \beta)$ occurs with multiplicity 1 in such a principal series. ■

4.6

Consider the case of $U(R+1, 1)$.

Proposition. *Assume $\alpha + \frac{R-1}{2} - \eta - \nu_i \in \mathbb{Z}$.*

1. *If $\eta - \nu_1 \leq \eta + \nu_1 \leq \alpha - \frac{R}{2}$, then the parameter containing $\mu(\alpha, \beta)$ is*

$$(\alpha + \frac{R-1}{2})^+, \dots, (\alpha - \frac{R-1}{2})^+, (\eta + \nu_1)^+, (\eta - \nu_1)^-.$$

2. If $\eta + \nu_1 \geq \eta - \nu_1 \geq \alpha + \frac{R}{2}$, then the parameter containing $\mu(\alpha, \beta)$ is

$$(\alpha + \frac{R-1}{2})^+, \dots, (\alpha - \frac{R-1}{2})^+, (\eta - \nu_1)^+, (\eta + \nu_1)^-.$$

In all other cases, the parameter is the same as the one for the principal series in proposition 4.5.

Proof. The group is rank 1, so there are only two conjugacy classes of Cartan subgroups, one of real rank 1 and a compact one. Thus all parameters of irreducible representations are Langlands subquotients of principal series or are discrete series or limits of discrete series. Thus the composition series of $X(\lambda_0^G, \nu_0)$ containing $\mu(\alpha, \beta)$ is formed of parameters for other principal series or limits of discrete series. Proposition 4.5 establishes that $\mu(\alpha, \beta)$ cannot belong to any other principal series. The limits of discrete series are all derived functor modules satisfying a Blattner type multiplicity formula. They will contain $\mu(\alpha, \beta)$ precisely in the two cases listed above. We omit further details. ■

4.7

The notation is as in 4.5. Write the $\mathfrak{a}$ parameter as

$$(\nu_1, \dots, \nu_1 \dots \nu_k, \dots, \nu_k, -\nu_k, \dots, -\nu_k, \dots, -\nu_1, \dots, -\nu_1) \quad (4.7.1)$$

with $\dots > \nu_i > \nu_{i+1} > \dots > \nu_k \geq 0$.

Proposition. *The representation $\overline{X}(\gamma)$ containing $\mu(\alpha, \beta)$ is obtained from the parameter of the principal series in proposition 4.5 as follows.*

1. For each ν_i such that $\alpha + \frac{R}{2} \leq \eta - \nu_i$ and $\alpha + \frac{R}{2} - \eta + \nu_i \in \mathbb{Z}$, change one pair $\underline{\eta + \nu_i}, \underline{\eta - \nu_i}$ to $(\eta - \nu_i)^+(\eta + \nu_i)^-$.
2. For each ν_i such that $\eta + \nu_i \leq \alpha - \frac{R}{2}$ and $\alpha + \frac{R}{2} - \eta + \nu_i \in \mathbb{Z}$, change one pair $\underline{\eta + \nu_i}, \underline{\eta - \nu_i}$ to $(\eta - \nu_i)^-(\eta + \nu_i)^+$.

In all other cases the parameter is unchanged from 4.5.

Proof. Fix a parameter (λ^G, ν) as in the proposition with $\delta = \pm 1$ such that $X(\lambda^G, \nu)^{\mu(\alpha, \beta)} \neq 0$. Let $\Sigma \subseteq \Delta[\mathfrak{g}_0, \mathfrak{a}_0^r]$ be a positive system such that ν is dominant. By 2.6, the long intertwining operator factors into

$$I = I_1 \circ \dots \circ I_m.$$

It is enough to show that, under the conditions of the proposition, each I_j is an isomorphism when restricted to the $\mu(\alpha, \beta)$ -isotypic component.

If α_j is a short root then $G(\alpha_j) \simeq GL(2, \mathbb{C})$ and I_j is easily seen to be an isomorphism on $\mu(\alpha, \beta)$.

If α_j is a long root, then $G(\alpha_j)$ is an $U(R+1, 1)$ or $U(1, R+1)$, so section 4.6 proves the claim. ■

Suppose $R = 0$ and $\alpha = -\beta = m > 0$. The principal series that contains $\mu(\alpha, \beta)$ contains a fine K-type. Let x_1 be the smallest integer such that $m - \nu_i \in \mathbb{N}$ and $\nu_i > 0$. Let x_2 be the next larger integer with this property and so on until x_r .

Corollary. *The Langlands parameter with infinitesimal character (4.7.1) and containing the K-type $\mu(m, -m)$ is*

$$(\nu_{x_1})^+(-\nu_{x_1})^-, \dots, (\nu_{x_r})^+(-\nu_{x_r})^-, \underline{\nu_1, -\nu_1}, \dots, \underline{\nu_k, -\nu_k}$$

Similarly for $\mu(-m, m)$, with the $\pm$ signs reversed, i.e.

$$(\nu_{x_1})^-(-\nu_{x_1})^+, \dots, (\nu_{x_r})^-(-\nu_{x_r})^+, \underline{\nu_1, -\nu_1}, \dots, \underline{\nu_k, -\nu_k}$$

4.8

We review the results we need from affine graded Hecke algebras. Consider the Weyl group $W := W(B_n)$. Let $\mathfrak{a}$ be a real vector space of dimension n , which we think of as the Cartan subalgebra of the Lie algebra of type B_n . The roots are the usual ones,

$$\Delta := \{\epsilon_i \pm \epsilon_j, \pm \epsilon_k\},$$

with simple roots $\{\epsilon_i - \epsilon_{i+1}, \epsilon_n\}$. Then denote by $\mathbb{H}$ the algebra which is $\mathbb{C}[W] \otimes S(\mathfrak{a})$ as a vector space, with $\mathbb{C}[W]$ embedded as a subalgebra. Let s_α be the reflection corresponding to α , and let t_w be the generators of $\mathbb{C}[W]$. Let $c_\alpha > 0$ be real numbers for $\alpha \in \Delta$ such that $c_\alpha = c_{w\alpha}$. The affine graded Hecke algebra with parameters c_α is $\mathbb{H}$, with the additional relations

$$t_{s_\alpha} \omega = s_\alpha(\omega) t_{s_\alpha} + c_\alpha \langle \omega, \alpha \rangle, \quad \omega \in \mathfrak{a}. \quad (4.8.1)$$

We assume that $c_\alpha = 1$ for the long roots $\pm \epsilon_i \pm \epsilon_j$, $c_\alpha = c$ for the short roots $\pm \epsilon_i$. The case $c = 1$ corresponds to the Iwahori Hecke algebra for the split group of type C, and $c = 1/2$ to the split group of type B. The algebra $\mathbb{H}$ also has a $*$ operation. According to [BM1] and [BM2], the unitary dual of $\mathbb{H}$ is in one-to-one correspondence with the Iwahori spherical unitary dual of the corresponding split p-adic group. The theory of intertwining operators and hermitian forms is parallel to the real case. The induced module corresponding to the spherical principal series is $X(\nu) := \mathbb{H} \otimes_{\mathbb{A}} \mathbb{1}_\nu$, where we write $\mathbb{A}$ for $S(\mathfrak{a})$.

The intertwining operator $I(w, \nu)$ is a product of operators I_{α_i} according to a reduced decomposition of $w = s_{\alpha_1} \cdots s_{\alpha_k}$. If α is a simple root,

$$r_\alpha := (t_\alpha \alpha - c_\alpha)(\alpha - c_\alpha)^{-1}, \quad I_\alpha : x \otimes \mathbb{1}_\nu \mapsto x r_\alpha \otimes \mathbb{1}_{s_\alpha \nu}. \quad (4.8.2)$$

The $I(w, \nu)$ have the same properties as in the real case. Since the r_α are multiplied on the right, we can replace α with $-\langle \nu, \alpha \rangle$ in the formulas. Because,

$$\mathbb{C}[W] = \sum_{\sigma \in \widehat{W}} V_\sigma \otimes V_\sigma^*,$$

r_α gives rise to an operator

$$r_\sigma(s_\alpha, \nu) : V_\sigma^* \longrightarrow V_\sigma^*.$$

Lemma. $r_\sigma(s_\alpha, \nu)$ on V_σ^* is of the form

$$r_\sigma(s_\alpha, \nu) = \begin{cases} Id & \text{on the } +1 \text{ eigenspace of } t_\alpha \\ \frac{c_\alpha - \langle \nu, \alpha \rangle}{c_\alpha + \langle \nu, \alpha \rangle} Id & \text{on the } -1 \text{ eigenspace of } t_\alpha \end{cases} \quad (4.8.3)$$

Proof. This is clear from the above formulas.

4.9

In the unitary group $U(p, q)$ with $p \geq q$, we can identify the Levi component of the minimal parabolic subgroup with

$$M \cong U(p - q) \times \underbrace{U(1) \times \cdots \times U(1)}_q \quad (4.9.1)$$

where each $U(1)$ is embedded diagonally on the $p - q + i$ and $p + i$ entry. The Weyl group is $W(B_q)$. It acts on M by permuting and changing the signs of the θ_k in the $U(1) = \{e^{i\theta_k}\}$.

For $U(m)$, write

$$\mu^+(k, m - k) := (\underbrace{1, \dots, 1}_k, 0, \dots, 0), \quad \mu^-(k, m - k) := (0, \dots, 0, \underbrace{-1, \dots, -1}_k). \quad (4.9.2)$$

These are realized as $\Lambda^r \mathbb{C}^m$ and $\Lambda^r(\mathbb{C}^m)^*$, and are often abbreviated as $\mu^\pm(k)$.

Definition. The following K -types in $U(p) \times U(q)$ will be called relevant:

$$\mu_n^+(r) = \mu^+(r, p - r) \otimes \mu^-(r, q - r) \quad r \leq q, \quad (4.9.3)$$

$$\mu_n^-(r) = \mu^-(r, p - r) \otimes \mu^+(r, q - r) \quad r \leq q, \quad (4.9.4)$$

$$\mu_c^-(r) = \mu(0, p) \otimes (\underbrace{1, \dots, 1}_r, 0, \dots, 0, \underbrace{-1, \dots, -1}_r) \quad r \leq \lfloor \frac{q}{2} \rfloor. \quad (4.9.5)$$

We will suppress the $\pm$ superscripts; the $\mu_n^\pm$ behave the same way, and there is only one μ_c that we will consider. Recall that if (V, μ) is a K type occurring in a spherical principal series, then $V^M \neq (0)$, and is a representation of the Weyl group W .

We will parametrize irreducible representations of S_n by partitions written as $(a_1, \dots, a_k)$ with $a_i \leq a_{i+1}$. The representations of $W(B_n) \cong W(C_n)$ are parametrized, as in [L1], by pairs of partitions

$$(a_1, \dots, a_k) \times (b_1, \dots, b_l), \quad a_i \leq a_{i+1}, \quad b_j \leq b_{j+1}, \quad \sum a_i + \sum b_j = n. \quad (4.9.6)$$

Precisely the representation parametrized by (4.9.6) is as follows. Let $k = \sum a_i$, $l = \sum b_j$. Recall that $W \cong S_n \rtimes \mathbb{Z}_2^n$. Let χ be the character of $\mathbb{Z}_2^n$ which is trivial on the first k $\mathbb{Z}_2$'s, sign on the rest. Its centralizer in S_n is $S_k \times S_l$. Let σ_1 and σ_2 be the representations of S_k , S_l corresponding to the partitions (a) and (b). Then σ , the representation parametrized by (4.9.6) is

$$Ind_{(S_k \times S_l) \times \mathbb{Z}_2^n}^W [(\sigma_1 \otimes \sigma_2) \otimes \chi].$$

Proposition. *The W -representation V^M for the relevant K -type is irreducible and is given by*

$$\mu_n(k) \longleftrightarrow (q - k) \times (k), \quad (4.9.7)$$

$$\mu_c(k) \longleftrightarrow (k, q - k) \times (0). \quad (4.9.8)$$

Proof. The restriction of a K -type $(\mu_1, \dots, \mu_m)$ with $\mu_i \geq \mu_{i+1}$ from $U(m)$ to $U(m-1) \times U(1)$ is given by all K -types with highest weights

$$\begin{aligned} & (a_1, \dots, a_{m-1}) \otimes (\sum \mu_i - \sum a_i), \\ & \mu_1 \geq a_1 \geq \dots \geq \mu_i \geq a_i \geq \mu_{i+1} \geq \dots \end{aligned} \quad (4.9.9)$$

The space V^M in $\mu_n^+(r, p-r) \otimes \mu_n^-(r, q-r)$ consists of the weights

$$(\underbrace{0, \dots, 0}_{p-q}, \epsilon_1, \dots, \epsilon_q \mid -\epsilon_1, \dots, -\epsilon_q) \quad (4.9.10)$$

with $\epsilon_i = 0, 1$ and exactly r 1's. The action of $W(B_q)$ is clear, it is the standard one on $\Lambda^k \mathbb{C}^q$. Assertion (4.9.7) follows from the fact that the representation $(q-k) \times (k)$ is realized as $\Lambda^k \mathbb{C}^q$.

The space V^M for $(0) \otimes \mu_c(r, q-r)$ can be identified with the invariants under $U(1) \times \dots \times U(1) \subset U(q)$ in the representation

$$(\underbrace{0, \dots, 0}_{p-q}) \otimes (\underbrace{1, \dots, 1}_r, 0, \dots, 0, \underbrace{-1, \dots, -1}_r). \quad (4.9.11)$$

We can ignore the first factor. We do an induction on q . The case $q = 0$ is clear. Similarly the case $r = 1$ is easy because this is the adjoint representation. Consider the restriction of $(\underbrace{1, \dots, 1}_r, 0, \dots, 0, \underbrace{-1, \dots, -1}_r)$ to $U(q-1) \times U(1)$.

The K -types that contain V^M are

$$\begin{aligned} & (\underbrace{1, \dots, 1}_r, 0, \dots, 0, \underbrace{-1, \dots, -1}_r) \otimes (0), \\ & (\underbrace{1, \dots, 1}_{r-1}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{r-1}) \otimes (0). \end{aligned} \quad (4.9.12)$$

The first one does not appear if $r = q/2$. By induction, these factors give the representations of W parametrized by

$$(r, R - 1 - r) \times (0) \quad (r - 1, R - r) \times (0). \quad (4.9.13)$$

Thus the reflections corresponding to the short roots must act trivially. The factors in (4.9.13) can only come from

$$(r, R - r) \times (0), \quad (1, r, R - r - 1) \times (0), \quad (1, r - 1, R - r) \times (0).$$

The second and third choice have extra representations in their restriction which do not match (4.9.13). The claim of the proposition follows in this case. $\blacksquare$

Definition. *The K-types and W-types in proposition 4.9 will be called relevant.*

4.10

We review the values of the intertwining operators for real rank one. There are two P_j (notation as in section 2.7) that are relevant for our case, their M_j are $U(R + 1, 1)$ and $GL(2, \mathbb{C})$. In the case of $GL(2, \mathbb{C})$, the maximal compact group is $U(2)$. The restriction of a relevant K-type is either $(0, 0)$ or $(1, -1)$. The Weyl group element which has a representative in $GL(2, \mathbb{C})$ which we call t_{12} has eigenvalue $+1$ on the portion of V^M corresponding to $(0, 0)$ and -1 on $(1, -1)$. The intertwining operator is

$$\begin{cases} Id & \text{on the } 1 \text{ eigenspace of } t_{12}, \\ \frac{1 - \langle \nu, \alpha \rangle}{1 + \langle \nu, \alpha \rangle} Id & \text{on the } -1 \text{ eigenspace of } t_{12}. \end{cases} \quad (4.10.1)$$

For $U(R + 1, 1)$, the maximal compact subgroup is $U(R + 1) \times U(1)$. Let t_n be the Weyl group reflection that has a representative in $U(R, 1)$. The restriction of $\mu_c^-(r)$ is the trivial K-type and so the intertwining operator is the identity. For $\mu_n^+(r) \otimes \mu_n^-(r)$, the restriction is formed of either $\mu_n^+(0) \otimes \mu_n^-(0)$ or $\mu_n^+(1) \otimes \mu_n^-(1)$. The intertwining operator corresponding to this simple root is the identity on the first one, the element t_n acts trivially. On the second one, t_n acts by -1 .

Lemma. (*[JW]*) *The intertwining operator on $U(R + 1, 1)$ acts by*

$$\frac{(R + 1)/2 - \langle \nu, \alpha \rangle}{(R + 1)/2 + \langle \nu, \alpha \rangle} Id$$

on the K-type $\mu_n^+(1) \otimes \mu_n^-(1)$.

Corollary. *The intertwining operators for the real case on relevant K-types coincide with the ones in the affine graded Hecke algebra with parameter $c = (R + 1)/2$ for the corresponding Weyl group representations.*

4.11

The case $c = 1$ corresponds to the Hecke algebra of type B with all $c_\alpha = 1$. The special case $c = 1/2$ also corresponds to the Hecke algebra of type C with parameter $c_\alpha = 1$ for all α . In these cases, the spherical dual of the affine graded Hecke algebras is computed in [B1]. A spherical irreducible representation is unitary if and only if the form is positive definite on the *relevant* W -types (proposition 4.11). In view of corollary 4.10, we can derive some consequences for the unitary groups $U(n, n)$ and $U(n + 1, n)$.

We recall the results from [B1]. The infinitesimal character is always real.

A nilpotent orbit $\check{\mathcal{O}}$ in a simple Lie algebra of type B, C is parametrized by a partition $(m_1, \dots, m_k)$ such that

Type B_n , $\sum m_i = 2n + 1$, and every even part occurs an even number of times,

Type C_n , $\sum m_i = 2n$, and every odd part occurs an even number of times.

The nilpotent orbit $\check{\mathcal{O}}$ determines a partition

$$\underbrace{(a_1, \dots, a_1)}_{r_1}, \dots, \underbrace{(a_k, \dots, a_k)}_{r_k}, \quad a_l < a_{l+1}. \quad (4.11.1)$$

Let $\{\check{e}, \check{h}, \check{f}\}$ be a Lie triple associated to the orbit $\check{\mathcal{O}}$. The centralizer of the Lie triple, $\mathfrak{z}(\check{\mathcal{O}})$, is a product of $sp(r_l, \mathbb{C})$ or $so(r_l, \mathbb{C})$ $1 \leq l \leq k$ according to the rule

Type B , $sp(r_l)$ for a_l even, $so(r_l)$ for a_l odd,

Type C , $sp(r_l)$ for a_l odd, $so(r_l)$ for a_l even.

A spherical parameter can be viewed as a W -conjugacy class of an element $\chi \in \mathfrak{a}^*$, where $\mathfrak{a}$ is a maximally split Cartan subalgebra. To each ν there is attached a unique nilpotent orbit $\check{\mathcal{O}}$ with the following properties:

- (1) $\chi = \frac{1}{2}\check{h} + \nu$, $\nu \in \mathfrak{z}(\check{\mathcal{O}})$,
- (2) If $\chi = \frac{1}{2}\check{h}' + \nu'$ as in (1) for a different $\check{\mathcal{O}}'$, then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}$.

The set of parameters is partitioned in this way into a disjoint union according to nilpotent orbits.

Definition. *The set of unitary parameters attached to a fixed nilpotent orbit $\check{\mathcal{O}}$ is called a complementary series attached to $\check{\mathcal{O}}$.*

Proposition ([B1]). *A representation for the Hecke algebra of type B_n with parameter $c = 1/2$ or 1 , is unitary if and only if the form is positive definite on the relevant W -types.*

Corollary. *The parameter for the spherical unitary dual for $U(n, n)$ is contained in the spherical unitary dual for the Hecke algebra of type B_n with parameter $c = 1/2$. The spherical unitary dual of $U(n+1, n)$ is contained in the spherical unitary dual for the Hecke algebra of type B_n with parameter $c = 1$.*

We write down the explicit description of the unitary representations. The parameter ν gives rise to a spherical parameter on each of the simple factors in $\mathfrak{z}(\check{O})$ (by restriction to that factor). Each of them is attached to the trivial orbit, otherwise (2) would not be satisfied. Write ν in coordinates on each factor as

$$(\nu_1, \dots, \nu_1, \nu_2, \dots, \nu_2, \dots, \nu_k, \dots, \nu_k), \quad (4.11.2)$$

with $0 \leq \nu_1 \leq \dots \leq \nu_i < \dots < \nu_k$.

Theorem ([B1]). *The complementary series attached to $\check{O}$ coincides with the one attached to the trivial orbit in $\mathfrak{z}(\check{O})$. These are:*

B: $0 \leq \nu_1, \dots, \nu_1 < \dots < \nu_k, \dots, \nu_k < 1/2$.

C: $0 \leq \nu_1, \dots, \nu_1 < \dots < \nu_k, \dots, \nu_k \leq 1/2 < \nu_{k+1} < \dots < \nu_{k+l} < 1$

so that $\nu_i + \nu_j \neq 1$ for $i \neq j$ and there are an even number of ν_i such that $1 - \nu_{k+1} < \nu_i \leq 1/2$ and an odd number of ν_i such that

$1 - \nu_{k+j+1} < \nu_i < 1 - \nu_{k+j}$.

5 Unipotent Representations

Corollary 4.11 and theorem 4.11 provide necessary conditions for unitarity. We prove that they are also sufficient.

Theorem. *The unitary spherical parameters of the groups $U(n, n)$ and $U(n+1, n)$ coincide with the unitary parameters for the Hecke algebra of type B_n with parameter $c = 1/2$ and $c = 1$ respectively.*

The proof is in the next sections.

5.1

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra determined by

$$\xi = (\underbrace{1, \dots, 1}_{p_1}, \underbrace{0, \dots, 0}_{p_2} \mid \underbrace{1, \dots, 1}_{q_1}, \underbrace{0, \dots, 0}_{q_2}). \quad (5.1.1)$$

Let W be a $(\mathfrak{g}, K)$ module for $u(p_2, q_2)$. Then $AV(\mathcal{R}_q^i(\text{triv} \otimes W))$ is obtained from $AV(W)$ as follows:

To each nilpotent orbit in $AV(W)$ add p_1 $+$'s to the largest possible rows starting with $-$ and q_1 $-$'s to the largest possible rows starting with $+$.

Let now $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be real so that

$$\mathfrak{l} = \mathfrak{gl}(m, \mathbb{C}) \times u(p_2, q_2), \quad (5.1.2)$$

and let W be as before. Let $Q = LU$ be the corresponding parabolic subgroup. Then $AV(Ind_Q[triv \otimes W])$ is obtained as follows:

Increase the largest m rows of each nilpotent orbit in $AV(W)$ by 2. If there is a choice, include all possible nilpotent orbits that can be obtained this way.

Example

Suppose $m = 2$ and $AV(W)$ is $(1^+1^-2^+)$. Then AV of the induced module is the union of $(1^+3^-4^+)$ and $(1^-3^+4^-)$.

5.2

We recall the notion of special unipotent representation. Let $\check{O}$ be a nilpotent orbit in the dual algebra $\check{\mathfrak{g}}$, and let

$$\chi(\check{O}) := \frac{1}{2}\check{h}, \quad (5.2.1)$$

where $\check{h}$ is the semisimple element in the Lie triple attached to $\check{O}$.

Definition. An irreducible $(\mathfrak{g}, K)$ module (π, V) is called special unipotent attached to $\check{O}$, if it has infinitesimal character $\chi(\check{O})$ and its annihilator in $U(\mathfrak{g})$ is the maximal primitive ideal.

For a unitary group of rank m , $\check{\mathfrak{g}} = gl(m, \mathbb{C})$. If $(a_1, \dots, a_k)$ is the partition of $\check{O}$, then we can write $\chi(\check{O})$ as a concatenation of *strings*,

$$\left(\frac{a_i - 1}{2}, \frac{a_i - 3}{2}, \dots, -\frac{a_i - 3}{2}, -\frac{a_i - 1}{2}\right), \quad (5.2.2)$$

one for each a_i .

5.3

We determine a set of special unipotent representations which are spherical for $U(n, n)$ and $U(n + 1, n)$. For $U(n, n)$, let

$$\check{O} \subset sp(2n, \mathbb{C}) \subset gl(2n, \mathbb{C})$$

be an *even* nilpotent orbit so that all the a_i (notation (4.11.1)) are even. For $U(n + 1, n)$, let

$$\check{O} \subset so(2n + 1, \mathbb{C}) \subset gl(2n + 1, \mathbb{C})$$

be an *even* nilpotent orbit so that all a_i are odd. According to the notation in 2.3.3, the spherical parameter will be also written as

$$\left(\dots, \frac{a_i - 1}{2}, -\frac{a_i - 1}{2}, \frac{a_i - 3}{2}, -\frac{a_i - 3}{2}, \dots\right) \quad (5.3.1)$$

Proposition. *Suppose the a_i have the same parity, even in the case of $U(n, n)$, odd in the case of $U(n+1, n)$. Then there is a θ -stable parabolic subalgebra $\mathfrak{q}$ such that*

$$\mathcal{R}_{\mathfrak{q}}^i(\text{triv}) = \begin{cases} L(\check{\mathcal{O}}) & \text{if } i = \dim(\mathfrak{u} \cap \mathfrak{s}), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the representation $L(\check{\mathcal{O}})$ is unitary.

Proof. For $U(n, n)$, let $\mathfrak{q}$ be the parabolic subalgebra determined by ξ of the form

$$(\underbrace{k, \dots, k}_{a_1/2}, \underbrace{k-1, \dots, k-1}_{a_2/2}, \dots \mid \underbrace{k, \dots, k}_{a_1/2}, \underbrace{k-1, \dots, k-1}_{a_2/2}, \dots). \quad (5.3.2)$$

For $U(n+1, n)$, let $\mathfrak{q}$ be the parabolic subalgebra determined by ξ of the form

$$(\underbrace{k, \dots, k}_{(a_1+1)/2}, \underbrace{k-1, \dots, k-1}_{(a_2-1)/2}, \dots \mid \underbrace{k, \dots, k}_{(a_1-1)/2}, \underbrace{k-1, \dots, k-1}_{(a_2+1)/2}, \dots). \quad (5.3.3)$$

The vanishing part of the proposition follows from the results in section 4.1-4.2, and so does the unitarity. Section 4.3 implies the irreducibility once we check the assumptions. The facts about the set A and the infinitesimal character follow from [BV1] because $\check{\mathcal{O}}$ is even and the infinitesimal character is $\frac{1}{2}h$. Let $(b_1, \dots, b_l)$ with $b_i \leq b_{i+1}$ be the transpose partition to $(a, \dots, a_k)$, and let $\mathcal{O}_c$ be the corresponding nilpotent orbit. We need to compute $AV(\mathcal{R}_{\mathfrak{q}}^i(\text{triv}))$. According to 5.1, the answer is as follows. Note that when all a_i are even, each row size in $\mathcal{O}$ appears an even number of times. When all the a_i are odd, each row size appears an even number of times except for the largest one. There is a unique $\mathcal{O}$ in the AV-set, with signs on the rows as follows. Each row size gets an even number of $+$ and $-$, except for the largest size which gets one more $+$ than a $-$ when the a_i are odd. ■

5.4

Let $\check{\mathcal{O}}$ be a nilpotent orbit in $\check{\mathfrak{g}}$. If $\check{\mathcal{O}}$ intersects a proper Levi component, there is a chance of a complementary series. Let $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ be a nilpotent orbit and r_i the number of a_i in its partition (notation 4.11.1). Write $\check{\mathfrak{g}}(n)$ for $sp(2n)$ or $so(2n+1)$ depending on the case being considered. Suppose that $r_i > 2$. The nilpotent orbit $\check{\mathcal{O}}$ meets a Levi component of the form $gl(a_i) \times \check{\mathfrak{g}}(n-a_i)$ in a nilpotent $(a_i) \times \check{\mathcal{O}}'$. Let $R = a_i \pmod{2}$.

Proposition.

$$L(\check{\mathcal{O}}) = \text{Ind}_{GL(a_i, \mathbb{C}) \times U(n-a_i+R, n-a_i)} [triv \otimes L(\check{\mathcal{O}}')]. \quad (5.4.1)$$

Proof. We need to show that the AV-set of the induced module coincides with the AV-set of $L(\check{\mathcal{O}})$. This follows from the algorithms in section 5.1. ■

5.5

Let $\check{\mathcal{O}}$ be arbitrary, $\check{\mathcal{O}} \subset sp(2n, \mathbb{C})$ for $U(n, n)$, $\check{\mathcal{O}} \subset so(2n+1)$ for $U(n+1, n)$. Denote by a_i the even entries in the case of $sp(2n)$, odd in the case of $so(2n+1)$. Let b_j be the remaining entries. The infinitesimal character $\chi(\check{\mathcal{O}}) = \frac{1}{2}h$ is as in (5.2.2), a string for each a_i, b_j . Let r_i be the number of entries equal to a_i , and s_j the number of entries equal to b_j . The s_j are all even. Let $a = \sum a_i, b = \sum b_j$.

By the Jacobson-Morozov theorem, $\check{\mathcal{O}}_c$ corresponds to a homomorphism

$$\Psi : sl(2, \mathbb{C}) \longrightarrow \check{\mathfrak{g}}. \quad (5.5.1)$$

By conjugating by $GL(n, \mathbb{C})$ if necessary, we can assume that Ψ factors through $gl(a, \mathbb{C}) \times gl(b, \mathbb{C})$ so that if we write $\Psi = (\Psi_1, \Psi_2)$, then Ψ_1 corresponds to the nilpotent orbit $\check{\mathcal{O}}_e$ with partition (a_i) and Ψ_2 corresponds to the nilpotent orbit $\check{\mathcal{O}}_o$ with partition (b_j) . The Kazhdan-Lusztig conjectures for non-integral infinitesimal character imply that the character theory for modules with infinitesimal character $\chi(\check{\mathcal{O}})$ exactly matches the character theory of representations on $U(p_e, q_e) \times U(p_o, q_o)$ where $p_e + q_e = a$ and $|p_e - q_e| \leq 1$ while $p_o + q_o = b$ with $p_o = q_o$. The block is $\mathcal{B}_0$ on both factors.

Similar to 5.4, when $s_j > 0$, $\check{\mathcal{O}}$ meets a Levi component $gl(b_j) \times \check{\mathfrak{g}}(n - b_j)$ in a nilpotent orbit $(b_j) \times \check{\mathcal{O}}'$. This is because s_j is even.

Theorem. (1) *If $r_j > 2$, then*

$$L(\check{\mathcal{O}}) = Ind_{GL(a_i, \mathbb{C}) \times U(n - a_i + R, n - a_i)}[triv \otimes L(\check{\mathcal{O}}')].$$

(2) *For any b_j ,*

$$L(\check{\mathcal{O}}) = Ind_{GL(b_j, \mathbb{C}) \times U(n - b_j + R, n - b_j)}[triv \otimes L(\check{\mathcal{O}}')].$$

Proof. In view of the discussion on the Kazhdan-Lusztig conjectures for nonintegral infinitesimal character, part (a) is a consequence of section 5.3. For part (b) we omit the details. They are similar to section 5.3, but easier. ■

5.6 Proof of theorem 5

As indicated at the beginning of the section, section 4 provides necessary conditions for unitarity which shows that the spherical unitary dual for the real unitary group $U(n, n)$ or $U(n+1, n)$ is contained in the set given by the theorem. To conclude the proof, we need to show that this set is unitary. Theorem 5.5 shows that when $r_i > 2$ or $s_j > 0$, then $L(\check{\mathcal{O}})$ is induced irreducible.

Remains to show that the sets in theorem 4.11 are complementary series. Consider the case of $U(n, n)$, the other one is similar. Let (c_i) be the partition for $\check{\mathcal{O}} \subset sp(2n, \mathbb{C})$, each c_i occurring n_i times. Let $n_i > 2$ if c_i is even, $n_i \geq 2$ if it is odd. The parameter for $L(\check{\mathcal{O}})$ contains strings

$$\chi(c_i, \nu_{ij}) := \left(\frac{c_i - 1}{2} + \nu_{ij}, \frac{c_i - 3}{2} + \nu_{ij}, \dots, -\frac{c_i - 3}{2} + \nu_{ij}, -\frac{c_i - 1}{2} + \nu_{ij} \right). \quad (5.6.1)$$

The ν_{ij} for fixed i are the ν_i in theorem 4.11. The remaining c_i contribute strings of the form

$$\left(\frac{c_i - 1}{2}, \dots, \frac{1}{2}\right). \quad (5.6.2)$$

The notation $\chi(c_i, \nu_{ij})$ will also be used to represent the character of $GL(c_i, \mathbb{C})$ with the infinitesimal character given by the coordinates in (5.6.1).

Section 5.5 combined with the Kazhdan-Lusztig conjectures for nonintegral infinitesimal character show that whenever

$$\frac{c_i - 1}{2} + \nu_{ij} \notin \frac{1}{2}\mathbb{Z},$$

$L(\check{\mathcal{O}})$ is induced irreducible from

$$Ind_{GL(c_i, \mathbb{C}) \times U(n - c_i, n - c_i)}^{U(n, n)} [\chi(c_i, \nu_{ij}) \otimes L(\check{\mathcal{O}}')].$$

Thus we can deform the parameter ν_{ij} ; say we write $\nu_{ij} + t_{ij}$ for the deformation. Reducibility will occur only if

$$\frac{c_i - 1}{2} + \nu_{ij} + t_{ij} \in \frac{1}{2}\mathbb{Z}, \quad (5.6.3)$$

or there exists another string $c_k + \nu_{kl}$ such that we can extract coordinates from

$$\begin{aligned} &\frac{c_i - 1}{2} + \nu_{ij} + t_{ij}, \dots, -\frac{c_i - 1}{2} + \nu_{ij} + t_{ij}, \\ &\frac{c_k - 1}{2} + \nu_{kl}, \dots, -\frac{c_k - 1}{2} + \nu_{kl} \end{aligned} \quad (5.6.4)$$

to form a string of coordinates decreasing by 1 which is strictly longer than either string in (5.6.4). These conditions are exactly the same as for reducibility of the corresponding module for the Hecke algebra. So the results in [B1] apply, and the parameters in theorem 4.11 are complementary series. We sketch the argument. The parameters ν_{ij} are deformed separately for each c_i . First, if $\nu_{ij} = \nu_{ij'}$, then the conditions say that $\nu_{ij} < 1/2$. Then $L(\check{\mathcal{O}})$ is unitarily induced irreducible from a representation $L((c_i, c_i)) \times L(\check{\mathcal{O}}')$ on a Levi component $GL(2c_i, \mathbb{C}) \times U(n - 2c_i, n - 2c_i)$ such that $L((c_i, c_i))$ is a Stein complementary series. So we are reduced to the case when the ν_{ij} (i fixed) are distinct. Order them so that $\nu_{ij} > \nu_{i, j+1}$. A $\nu_{ij} + t_{ij}$ is deformed downward so that it equals $\nu_{i, j+1}$ or downward so that (5.6.3) holds. No intermediate t_{ij} has the property that we can extract a strictly longer string from the pair

$$\begin{aligned} &\frac{c_i - 1}{2} + \nu_{ij} + t_{ij}, \dots, -\frac{c_i - 1}{2} + \nu_{ij} + t_{ij}, \\ &\frac{c_k - 1}{2} + \nu_{kl}, \dots, -\frac{c_k - 1}{2} + \nu_{kl}. \end{aligned} \quad (5.6.5)$$

When $\nu_{ij} + t_{ij} = \nu_{i, j+1}$, the representation is unitarily induced irreducible from a unitary representation on a $GL(2c_i) \times U(n - 2c_i, n - 2c_i)$ as before. When (5.6.3) holds, section 5.4 applies, and the module is unitarily induced irreducible from a unipotent representation on a proper Levi component. ■

6 Principal series

6.1

Suppose $G = U(n, n)$ or $U(n+1, n)$ and write it as $U(n+R, n)$ where $R = 0, 1$. Then consider the irreducible representations with lowest K-types of the form

$$\mu := (\underbrace{a_1, \dots, a_1}_{n_1 + \epsilon_1}, \dots, \underbrace{a_k, \dots, a_k}_{n_k + \epsilon_k} \mid \underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_k, \dots, a_k}_{n_k}) \quad (6.1.1)$$

where $\epsilon_i = 0, 1$ and $\epsilon_1 + \dots + \epsilon_k = R$. Then a straightforward calculation implies that

$$\lambda^G = (\underbrace{a_1, \dots, a_1}_{n_1 + \epsilon_1}, \dots, \underbrace{a_k, \dots, a_k}_{n_k + \epsilon_k} \mid \underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_k, \dots, a_k}_{n_k}) \quad (6.1.2)$$

as well. The centralizer of λ^G is

$$u(n_1 + \epsilon_1, n_1) \times \dots \times u(n_k + \epsilon_k, n_k). \quad (6.1.3)$$

A Langlands parameter for an irreducible representation with a lowest K-type of the form (6.1.1) is (λ^G, ν) . We write

$$\nu = (\nu_1, \dots, \nu_k) \quad (6.1.4)$$

where the ν_i are the coordinates corresponding to $u(n_i + \epsilon_i, n_i)$.

6.2

The K-types

$$\begin{aligned} \mu_0^+ &:= (1/2, \dots, 1/2 \mid -1/2, \dots, -1/2), \\ \mu_0^- &:= (-1/2, \dots, -1/2 \mid 1/2, \dots, 1/2) \end{aligned} \quad (6.2.1)$$

for the double cover of $U(n, n)$ are called *fine* in [V1]. They have the property that $\lambda^G = 0$. The parameter ν can also be viewed as a parameter of a spherical representation for the Hecke algebra of type D.

Theorem. *An irreducible representation with a fine lowest K-types is unitary only if the parameter is unitary for the Hecke algebra of type D with parameter $c_\alpha = 1$. These are the parameters for which the form is positive definite on the K-types*

$$\mu_0^\pm + \mu_n^\pm(r), \quad \mu_0^\pm + \mu_c^\pm(r).$$

If the infinitesimal character is that of a finite dimensional module, the representation is unitary if and only if it is a derived functor module from a unitary character on a Levi component.

Proof. The techniques in sections 4 and 5 apply to these K-types essentially without change. The reason for the Hecke algebra of type D is that the intertwining operators corresponding to long simple roots are isomorphisms. We omit the details. $\blacksquare$

6.3

Consider the case of μ as in (6.1.1). The parameter (λ^G, ν) determines parameters (λ_i, ν_i) on each of the $U(n_i + \epsilon_i, n_i)$ which (up to the center) are either spherical when $\epsilon_i = 1$ or have fine lowest K-types when $\epsilon_i = 0$. The results in sections 4.5 and 6.2 apply to them.

Theorem. *The parameter (λ^G, ν) is unitary only if each of the parameters (λ_i, ν_i) for $U(n_i + \epsilon_i, n_i)$ is unitary for the corresponding Hecke algebra of type D if the lowest K-type is fine but not trivial, type B with $c = 1$ if it is trivial.*

Proof. Recall the relevant K-types $\mu_{n,i}^\pm$ and $\mu_{c,i}^\pm$ for $U(n_i + \epsilon_i, n_i)$ from definition 4.9. Then $\mu + \mu_{n,i}^\pm$ and $\mu + \mu_{c,i}^\pm$ are bottom layer K-types in the sense of [SV]. By [KV], the signature on such a K-type in $\overline{X}(\lambda^G, \nu)$ coincides with the signature on the corresponding $\mu_{0,i} + \mu_{n,i}$, or $\mu_{0,i} + \mu_{c,i}$ respectively in $\overline{X}_{u(n_i + \epsilon_i, n_i)}(\lambda_i^G, \nu_i)$. Thus a parameter (λ^G, ν) is unitary only if all the (λ_i^G, ν_i) are unitary. ■

6.4 Remark

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic subalgebra determined by λ^G . Write

$$\overline{X}_i := \overline{X}_{U(n_i + \epsilon_i, n_i)}(\lambda_i, \nu_i). \quad (6.4.1)$$

The results in [KV] should be sufficient to prove

$$\mathcal{R}_{\mathfrak{q}}^i(\overline{X}_1 \otimes \cdots \otimes \overline{X}_k) = \begin{cases} \overline{X}(\lambda^G, \nu) & \text{if } i = s := \dim \mathfrak{u} \cap \mathfrak{k}, \\ 0 & \text{otherwise,} \end{cases} \quad (6.4.2)$$

so that the representation $\overline{X}(\lambda^G, \nu)$ is unitary whenever the $\overline{X}_i$ are unitary. It amounts to proving analogues of the irreducibility results in section 5.4. But I haven't checked the details.

7 The case of integral infinitesimal character

The relevant K-types are not sufficient to determine the full unitary dual in the case $p - q > 1$. We write

$$p = n + R, \quad q = n, \quad c := \frac{R+1}{2}. \quad (7.0.3)$$

For general p, q , we restrict to the case of integral infinitesimal character; this means that all the entries in the parameter are congruent to c modulo $\mathbb{Z}$. Let

$$\epsilon_R \equiv c \pmod{\mathbb{Z}}, \quad 0 \leq \epsilon \leq 1/2. \quad (7.0.4)$$

In this case, we will show that the K-types $\mu_c^-(1)$ and $\mu_n^-(1)$ provide the necessary conditions to prove theorem 1. We sometimes abbreviate them as $\mu_c(1)$ and $\mu_n(1)$.

7.1

We compute the multiplicities of $\mu_c(1)$, $\mu_n(1)$ in $\overline{X}$. The Langlands parameter is

$$\begin{aligned} \lambda^G &= (c-1, \dots, -c+1) \\ \nu &= (\underbrace{k, \dots, k}_{n_k}, \dots, \underbrace{\epsilon_R, \dots, \epsilon_R}_{n_\epsilon}, \dots, \underbrace{-k, \dots, -k}_{n_k}). \end{aligned} \quad (7.1.1)$$

Theorem. a) *The multiplicity $[\mu_n(1) : \overline{X}(\lambda^G, \nu)]$ equals*

$$n - \sum_{x>0} \min\{n_{x+1}, n_x\} - \left\lfloor \frac{n_{1/2} + 1}{2} \right\rfloor - \min\{n_1, \frac{n_0}{2}\} - \min\{n_c, 1\}.$$

b) *The multiplicity $[\mu_c(1) : \overline{X}(\lambda^G, \nu)]$ equals*

$$n - 1 - \sum_{x>0} \min\{n_{x+1}, n_x\} - \min\{n_1, \frac{n_0}{2}\} - \left\lfloor \frac{n_{1/2} + 1}{2} \right\rfloor.$$

Proof. The first statement is in [B-J1]. The second one is proved by similar techniques, only it is simpler because the intertwining operators coming from the long restricted roots are isomorphisms. We give details for case b). We do an induction on $2n + R$. Assume that $n_x = 0$ for $x \geq c$. Let $\mathfrak{q}$ be the θ -stable parabolic subalgebra determined by

$$\xi := (1, 0, \dots, 0, -1 \mid 0, \dots, 0). \quad (7.1.2)$$

Then $\mu_c(1)$ is bottom layer for $\mathfrak{q}$, so we can conclude that the formula is true by induction.

Assume that $n_k > 0$ for some $k \geq c$. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the real parabolic subalgebra with Levi component

$$gl(1, \mathbb{C}) \times gl(n - k + x, \mathbb{C}) \times u(n - k + x - 1 + R, n - k + x - 1). \quad (7.1.3)$$

We choose x as follows. Let a be the smallest such that $n_y > 0$ for all $y \geq a$.

- i) If $a \geq 1$, or $a = 1/2$ and $n_{1/2} = 1$, or $a = 0$ and $n_1 = 1$, let $x = a$.
- ii) If $a = 1/2$, and $n_{1/2} > 1$, let $-x$ be the largest so that $n_y > 1$ for all $y \leq -x$.
- iii) If $a = 0$ and $n_1 > 1$, let $-x$ be the largest so that $n_y > 1$ for all $0 < y \leq -x$.

The meaning of these conditions is that we can extract coordinates $k, k-1, \dots, x$ from ν and form a character on $gl(k-x+1, \mathbb{C})$

$$\chi \longleftrightarrow (-k, -k+1, \dots, -x). \quad (7.1.4)$$

Let χ_1 be the character on $GL(k-x, \mathbb{C})$ corresponding to

$$\chi_1 \longleftrightarrow (-k+1, \dots, -x), \quad (7.1.5)$$

and let χ_0 be the character of $GL(1, \mathbb{C})$ corresponding to (k) . Let ν_1 be the parameter obtained from ν by removing k , *i.e.*

$$\nu_1 = (\underbrace{k, \dots, k}_{n_k-1}, \underbrace{k-1, \dots, k-1}_{k-1}, \dots),$$

and let ν_2 be the parameter which is obtained from ν by removing all the coordinates in (7.1.4). In case i),

$$\nu_2 = (\underbrace{k, \dots, k}_{n_k-1}, \dots, \underbrace{a, \dots, a}_{n_a-1}, \underbrace{a-1, \dots, a-1}_{n_{a-1}}, \dots),$$

but in the other cases the appropriate n_y are decreased by 2. Let

$$\begin{aligned} M &= GL(k-x+1, \mathbb{C}) \otimes U(n-k+x-1+R, n-k+x-1), \\ M_1 &= GL(1, \mathbb{C}) \times U(n+R-1, n-1), \\ M_2 &= GL(k-x, \mathbb{C}) \times U(n-k+x-1+R, n-k+x-1). \end{aligned} \quad (7.1.6)$$

Then the long intertwining operator I factors into $I = I_2 \circ I_1$ such that

$$\begin{aligned} X(\lambda^G, \nu) &\xrightarrow{I_1} Ind_{M_1}^G[\chi_0 \otimes \bar{X}(\lambda^G, \nu_1)] \longrightarrow 0, \\ Ind_{M_1}^G[\chi_0 \otimes \bar{X}(\lambda^G, \nu_1)] &\xrightarrow{I_2} X(\lambda^G, -\nu) \longrightarrow 0. \end{aligned} \quad (7.1.7)$$

The image of I is $\bar{X}(\lambda^G, \nu)$. Thus I_2 maps the $\mu_c(1)$ isotypic component of the induced module *onto* the corresponding isotypic component of $\bar{X}(\lambda^G, \nu)$. Let $G_1 = U(n-1+R, n-1)$. By induction, we can assume that

$$[\mu_c(1) : Ind_{M_2}^{G_1}[\chi_1 \otimes \bar{X}(\lambda^G, \nu_2)]] = [\mu_c(1) : \bar{X}(\lambda^G, \nu_1)]. \quad (7.1.8)$$

By Frobenius reciprocity and (7.1.8),

$$0 \longrightarrow Ind_{M_1}^G[\chi_0 \otimes \bar{X}(\lambda^G, \nu_1)] \longrightarrow Ind_{GL(1) \times M_2}^G[\chi_0 \otimes \chi_1 \otimes \bar{X}(\lambda^G, \nu_2)] \quad (7.1.9)$$

is an isomorphism on the level of the $\mu_c(1)$ isotypic components. The intertwining operator

$$\begin{aligned} A : Ind_{GL(1) \times M_2}^G[\chi_0 \otimes \chi_1 \otimes \bar{X}(\lambda^G, \nu_2)] &\longrightarrow \\ Ind_{GL(1) \times M_2}^G[\chi_0^{-1} \otimes \chi_1 \otimes \bar{X}(\lambda^G, \nu_2)], & \end{aligned} \quad (7.1.10)$$

when restricted to the $\mu_c(1)$ isotypic component, has as image the $\mu_c(1)$ isotypic component of $Ind_M^G[\chi \otimes \bar{X}(\lambda^G, \nu_2)]$. This follows from the fact that in the factoring of A into intertwining operators that are induced from rank one parabolic subalgebras all but the last one is an isomorphism on the $\mu_c(1)$ isotypic component. This is because the value of the corresponding coroot is not 1. The last one is induced from one of the form $B \otimes Id$ where

$$B : Ind_{GL(k-x, \mathbb{C}) \times GL(1, \mathbb{C})}^{GL(k-x+1, \mathbb{C})}[\chi_1 \otimes \chi_0^{-1}] \longrightarrow Ind_{GL(1, \mathbb{C}) \times GL(k-x, \mathbb{C})}^{GL(k-x+1, \mathbb{C})}[\chi_0^{-1} \otimes \chi_1]. \quad (7.1.11)$$

The intertwining operator B is known on the restriction of $\mu_c(1)$. The $\mu_c(1)$ isotypic component of $Ind_M^G[\chi \otimes \overline{X}(\lambda^G, \nu_2)]$ is a homomorphic image of the $\mu_c(1)$ isotypic component of $X(\lambda^G, \nu)$.

Finally, the intertwining operator

$$Ind_M^G[\chi \otimes \overline{X}(\lambda^G, \nu_2)] \longrightarrow X(\lambda^G, -\nu) \quad (7.1.12)$$

is injective on the $\mu_c(1)$ isotypic component. It follows that the multiplicity of $\mu_c(1)$ in $\overline{X}(\lambda^G, \nu)$ is the same as the multiplicity in the induced module in (7.1.12). By induction, this give the formula in the theorem. ■

7.2

Assume that $k \geq c$, and we can extract a sequence

$$(k, k-1, \dots, -x+1, -x) \quad (7.2.1)$$

from ν with $x \geq c$. Then let

$$\begin{aligned} M &:= GL(k+x+1) \times U(n-k-x-1+R, n-k-x), \\ \chi &\longleftrightarrow (k, \dots, -x), \end{aligned} \quad (7.2.2)$$

and let ν' be the parameter obtained from ν by removing the coordinates in (7.2.1).

Corollary. *Assume that ν is such that we can extract a sequence as in (7.2.1). Then*

$$\begin{aligned} [\mu_c(1) : \overline{X}(\lambda^G, \nu)] &= [\mu_c(1) : Ind_M^G[\chi \otimes \overline{X}(\lambda^G, \nu')]], \\ [\mu_n(1) : \overline{X}(\lambda^G, \nu)] &= [\mu_n(1) : Ind_M^G[\chi \otimes \overline{X}(\lambda^G, \nu')]]. \end{aligned}$$

Proof. The results in section 7.1 show that

$$\begin{aligned} [\mu_c(1) : \overline{X}(\lambda^G, \nu)] &= [\mu_c(1) : \overline{X}(\lambda^G, \nu')], \\ [\mu_n(1) : \overline{X}(\lambda^G, \nu)] &= [\mu_n(1) : \overline{X}(\lambda^G, \nu')] + 1. \end{aligned} \quad (7.2.3)$$

The restriction formulas of $\mu_c(1)$, $\mu_n(1)$ to $M \cap K$ completes the proof. ■

7.3

Let

$$\begin{aligned} M^+ &:= GL(2y+1) \times U(n+R, n), \\ G^+ &:= U(n+2y+1+R, n+2y+1), \\ \chi^+ &\longleftrightarrow (y, \dots, -y). \end{aligned} \quad (7.3.1)$$

Let ν^+ be the parameter obtained from ν by adding the coordinates of χ^+ in (7.3.1). If the form is positive definite on the $\mu_n(1)$ or $\mu_c(1)$ isotypic component

of $\overline{X}(\lambda^G, \nu)$, then it is positive definite on the $\mu_n(1)$ or $\mu_c(1)$ isotypic component of $\text{Ind}_{M^+}^G[\chi^+ \otimes \overline{X}(\lambda^G, \nu)]$. Therefore the same holds for the spherical subquotient $\overline{X}(\lambda^G, \nu^+)$.

Assume $k \geq c$, and let $y = k - 1$. Then in G^+ , let

$$\begin{aligned} M^- &= GL(2k, \mathbb{C}) \times U(n-1+R, n-1+R), \\ \chi_t &\longleftrightarrow (k-1+t, \dots, -k+t). \end{aligned} \quad (7.3.2)$$

Let ν^- be the parameter obtained from ν by removing a single coordinate k . By corollary 7.2,

$$\begin{aligned} [\mu_c(1) : \overline{X}(\lambda^G, \nu^+)] &= [\mu_c(1) : \text{Ind}_{M^-}^{G^+}[\chi_t \otimes \overline{X}(\lambda^G, \nu^-)]], \\ [\mu_n(1) : \overline{X}(\lambda^G, \nu^+)] &= [\mu_n(1) : \text{Ind}_{M^-}^{G^+}[\chi_t \otimes \overline{X}(\lambda^G, \nu^-)]] \end{aligned} \quad (7.3.3)$$

for $0 \leq t \leq 1/2$. At $t = 1/2$, $\chi_{1/2}$ is the trivial character, so the representation is unitarily induced from $\chi_{1/2} \otimes \overline{X}(\lambda^G, \nu^-)$.

Proposition. *Assume $k \geq c$. The form is positive definite on the isotypic components $\mu_n(1)$, $\mu_c(1)$ of $\overline{X}(\lambda^G, \nu)$ only if it is so for $\overline{X}(\lambda^G, \nu^-)$.*

Proof. This follows from the above discussion. ■

7.4

Theorem. *The hermitian form on the $\mu_n(1)$, $\mu_c(1)$ isotypic component of $\overline{X}(\lambda^G, \nu)$ is positive definite only if ν satisfies the following conditions:*

- a) $n_{x+1} \leq n_x$ for all $x \geq c$,
- b) if $x \leq c$, and n_x is even, then $n_x \leq n_y$ for all $y < x$,
- c) if $x \leq c$, and n_x is odd, then $n_x \leq n_y - 1$ for all $y < x$.

The proof will be broken up over several sections. It is by induction on rank.

7.5

Assume that $n_x = 0$ for all $x \geq c$. Let $\mathfrak{q}$ be the θ -stable parabolic subalgebra determined by (7.1.2). Let λ' be the parameter obtained from λ^G by removing the entries $\pm \frac{R-1}{2}$. Then the following holds:

- (1) $\mathcal{R}_{\mathfrak{q}}^i[(\frac{R-1}{2}) \otimes \overline{X}(\lambda', \nu) \otimes (-\frac{R-1}{2})] = \begin{cases} \overline{X}(\lambda^G, \nu) & \text{for } i = s = \dim(\mathfrak{u} \cap \mathfrak{k}), \\ 0 & \text{otherwise.} \end{cases}$
- (2) $\overline{X}(\lambda^G, \nu)$ is unitary if and only if $\overline{X}(\lambda', \nu)$ is unitary,
- (3) if the form is negative on $\mu_n(1)$ or $\mu_c(1)$ on $\overline{X}(\lambda', \nu)$, the same holds for $\overline{X}(\lambda^G, \nu)$.

Parts (1) and (2) are in [V2] or [KV]. Such parameters are called in the *weakly good range*. Part (3) follows by a multiplicity calculation which shows that $\mu_n(1)$, $\mu_c(1)$ can only come from the same kind of K-types or the trivial one in W , combined with the fact that for signatures,

$$[\mathcal{R}_q^s[(\frac{R-1}{2}) \otimes W \otimes (-\frac{R-1}{2})]]_{\pm} = [W]_{\pm}. \quad (7.5.1)$$

We omit the details.

7.6

Assume $k \geq c$. Apply proposition 7.3. The parameter (λ^G, ν) can fail to satisfy a) only if $k > c$ and $n_k = n_{k-1} + 1$. We reduce to the case when $n_k = 1$, $n_{k-1} = 0$. Suppose $n_k > 1$. Let

$$\begin{aligned} M^+ &= GL(2k-2) \times U(n+R, n), \quad G^+ = U(n+2k-2+R, n+2k-2), \\ \chi_t &\longleftrightarrow (k-3/2+t, \dots, -k+3/2+t). \end{aligned} \quad (7.6.1)$$

If the form is positive on $\mu_c(1)$, $\mu_n(1)$ on $\overline{X}$, it is so for the spherical irreducible subquotient of the induced module

$$Ind_{M^+}^{G^+}[\chi_t \otimes \overline{X}(\lambda^G, \nu)]. \quad (7.6.2)$$

for $0 \leq t \leq 1/2$. At $t = 1/2$, the module is also the spherical subquotient of

$$Ind_{M^-}^{G^+}[\chi^- \otimes \overline{X}(\lambda^G, \nu^-)], \quad (7.6.3)$$

where

$$\begin{aligned} M^- &= GL(2k) \otimes U(n-2+R, n-2), \\ \chi^- &\longleftrightarrow (k, \dots, -k+1). \end{aligned} \quad (7.6.4)$$

The parameter ν^- has $n_k^- = n_k - 1$, $n_{k-1}^- = n_{k-1} - 1$. The argument in section 7.3 implies that if the form is positive on the $\mu_n(1)$, $\mu_c(1)$ isotypic components of $\overline{X}(\lambda^G, \nu)$ it is so for $\overline{X}(\lambda^G, \nu^-)$.

So assume that $n_k = 1$, $n_{k-1} = 0$. We show that the form is negative on $\mu_n(1)$. Let

$$\begin{aligned} M &= GL(1) \times U(n-1+R, n-1), \\ \chi &\longleftrightarrow (k) \end{aligned} \quad (7.6.5)$$

and ν' be the parameter obtained from ν by removing k . Then

$$[\mu_n(1) : \overline{X}(\lambda^G, \nu)] = [\mu_n(1) : Ind_M^G[\chi \otimes \overline{X}(\lambda^G, \nu')]]. \quad (7.6.6)$$

If ν' is empty, $\overline{X}(\lambda^G, \nu)$ is finite dimensional, and the claim is well known. If not, do an induction on the number of coordinates of ν . Let $x < k-1$ be the

largest coordinate so that $n_x > 0$. Then apply (again) the argument in section 7.3. Induce up by using

$$\begin{aligned} M^+ &= GL(2x-1) \times U(n+R, n), \\ \chi^+ &\longleftrightarrow (x-1, \dots, -x+1). \end{aligned} \quad (7.6.7)$$

The spherical subquotient is a subquotient of

$$Ind_{M^-}^{G^+}[\chi_t \otimes \overline{X}(\lambda^G, \nu^-)] \quad (7.6.8)$$

where

$$\begin{aligned} M^- &= GL(2x) \times U(n-1+x+R, n-1+x), \\ \chi_t &\longleftrightarrow (x-1+t, \dots, -x+t). \end{aligned} \quad (7.6.9)$$

We can deform t from 0 to $1/2$ without any change in the multiplicity of $\mu_n(1)$. At $t = 1/2$, the module is unitarily induced and the signature of the $\mu_n(1)$ isotypic component only depends on the signature of the corresponding isotypic component of $\overline{X}(\lambda^G, \nu^-)$, and ν' has strictly fewer coordinates than ν .

7.7

Assume that a) holds, but b) or c) fail to be satisfied. The main cases are when the largest entry of ν is c and

(1) $n_c = 2$, while $n_{c-1} = 1$,

(2) $n_c = 3$ and $n_{c-1} = 1$.

Case (2) can be reduced to case (1) by inducing up. Let

$$\begin{aligned} M^+ &= GL(2c) \times U(n+R, n), \quad G^+ = U(n+2c+R, n+2c) \\ \chi_t^+ &\longleftrightarrow (c-1/2+t, \dots, -c+1/2+t). \end{aligned} \quad (7.7.1)$$

Deform t to $1/2$. The spherical subquotient of $Ind_{M^+}^{G^+}[\chi_{1/2}^+ \otimes \overline{X}(\lambda^G, \nu)]$ is a subquotient of $Ind_{M^-}^{G^+}[\chi^- \otimes \overline{X}(\lambda^G, \nu^-)]$, where

$$\begin{aligned} M^- &= GL(2c+1, \mathbb{C}) \otimes U(n-1+R, n-1), \\ \chi^- &\longleftrightarrow (c, \dots, -c). \end{aligned} \quad (7.7.2)$$

If the form is positive on $\mu_c(1)$, $\mu_n(1)$ on $\overline{X}(\lambda^G, \nu)$, it must be so on $\overline{X}(\lambda^G, \nu^-)$. But ν^- is as in (1).

Now consider case (1). Let $\mathfrak{q}$ be the θ -stable parabolic subgroup determined by

$$\xi = (\underbrace{1, \dots, 1}_{R+1}, 0, \dots, 0 \mid \underbrace{1}_1, 0, \dots, 0). \quad (7.7.3)$$

Then $\mu_0 - 2\rho(\mathfrak{u} \cap \mathfrak{s}) + \rho(\mathfrak{u})$ equals

$$(\underbrace{0, \dots, 0}_{R+1}, -R/2, \dots, -R/2 \mid \underbrace{0}_1, R/2, \dots, R/2). \quad (7.7.4)$$

We claim that $\overline{X}(\lambda^G, \nu)$ is a subquotient of an $\mathcal{R}_q^s(triv \otimes W)$. The infinitesimal character of W has to be ν' , obtained from ν by removing the coordinate c . There is a unique irreducible module containing

$$\mu(R) := (-R/2, \dots, -R/2 \mid R/2, \dots, R/2) \quad (7.7.5)$$

and infinitesimal character ν' . We may as well assume that all coordinates strictly less than $c-1$ occur at least once in ν' (by inducing the parameter unitarily up to a larger group). Since

$$\lambda^G(\mu(R)) = (0, \dots, 0, -\epsilon_R, \dots, -c+1 \mid c-1, \dots, \epsilon_R, 0, \dots, 0), \quad (7.7.6)$$

the parameter of W is $(\lambda(\mu(R)), \tilde{\nu})$, where $\tilde{\nu}$ is obtained from ν by removing the coordinates $c, c-1, \dots, \epsilon_R$. The results in section 4.7 show that

$$\mathcal{R}_q^i(triv \otimes W) = \begin{cases} 0 & \text{for } i \neq s, \\ \text{contains } \overline{X}(\lambda^G, \nu) & \text{for } i = s. \end{cases} \quad (7.7.7)$$

The spherical module $\overline{X}(\tilde{\nu})$ does not satisfy a) of the theorem, so the form is negative on $\mu_n^\pm(1)$. Then $\mu(R) + \mu_n^+(1)$ is bottom layer for $X(\lambda(\mu(R)), \tilde{\nu})$, so the form is negative on it, and

$$[\mu_n^+(1) : \overline{X}(\tilde{\nu})] = [\mu(R) + \mu_n^+(1) : \overline{X}(\lambda(\mu(R)), \tilde{\nu})]. \quad (7.7.8)$$

Then by section 4.4,

$$[\mu_c(1) : \mathcal{R}_q^i(triv \otimes W)] = [\mu(R) + \mu_n^+(1) : W]. \quad (7.7.9)$$

By section 7.1, the multiplicity of $\mu_n^+(1)$ in $\overline{X}(\lambda^G, \nu)$ and the multiplicity of $\mu_n^+(1)$ in $\overline{X}(\tilde{\nu})$ are the same. It follows that

$$[\mu_c^-(1) : \overline{X}(\lambda^G, \nu)]_\pm = [\mu_n^+(1) : W]_\pm \quad (7.7.10)$$

as well. This completes the proof of the theorem. $\blacksquare$

7.8 Proof of theorem 1

We need to see that the parameters satisfying the conditions of theorem 7.4 are in fact derived functor modules from unitary characters. Let k be the largest so that $n_k > 0$. If $k \leq c-1$, we can use the technique in section 7.5, the parameter is derived functor induced in the weakly good range. If $k \geq c$, we use the θ stable parabolic subgroup determined by

$$\xi = (\underbrace{1, \dots, 1}_{k+c}, 0, \dots, 0 \mid \underbrace{1, \dots, 1}_{k-c-1}, 0, \dots, 0). \quad (7.8.1)$$

(Recall that $c = \frac{R+1}{2}$.) As for (7.7.5) in section 7.7, $\mathcal{R}_q^i(\text{triv} \otimes W)$ vanishes for $i \neq s$ and contains $\bar{X}(\lambda^G, \nu)$. The representation W is not spherical. If the parameter ν' contains a coordinate larger than c , condition a) insure that we can write $W = \mathcal{R}_q^s(\text{triv} \otimes W_1)$ from the parabolic subgroup determined by

$$\xi_1 = (\underbrace{1, \dots, 1}_{k_1 - c + 1}, 0, \dots, 0 \mid \underbrace{1, \dots, 1}_{k_1 + c}, 0, \dots, 0). \quad (7.8.2)$$

Then W_1 is spherical and satisfies the conditions of the theorem.

If on the other hand, there is no coordinate $\geq c$, we can write

$$W = \mathcal{R}_q^s(\chi \otimes W_1 \otimes \chi^{-1}) \quad (7.8.3)$$

for a character χ so that the parameters are in the weakly good range. The parabolic subgroup is of the type as in section 7.5. The character χ is determined in sections 4.5-4.7. In finitely many such steps we either run out of coordinates or get again a spherical module satisfying the conditions of theorem 7.4. The rank is strictly smaller, so the induction hypothesis applies.

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