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# UNITARY REPRESENTATIONS WITH HALF-INTEGER PARAMETER

## 1. Parametrization via $\check{\mathcal{O}}$

1.1. There is an exact sequence

$$1 \longrightarrow \kappa \longrightarrow A({}^L\phi) \longrightarrow \overline{A({}^L\phi)} \longrightarrow 1. \tag{1.1.1}$$

We attach a *unipotent* infinitesimal character to each conjugacy class in $\overline{A({}^L\phi)}$.

1.2. **Type B.** The dual group is type C. A nilpotent orbit $\check{\mathcal{O}}$ is given by a partition

$$0 \le 2n_0 < 2n_1 \le \cdots < 2n_{2i-1} \le 2n_{2i} < \cdots \le 2n_{2p}. \tag{1.2.1}$$

The centralizer is $G({}^L\phi)$ is a product of $O(2)$'s, one for each pair $n_{2i-1} = n_{2i}$ and $O(1)$'s one for each remaining $n_i > 0$.

An element $s \in G({}^L\phi)$ which gives rise to a halfinteger parameter can be written as

$$s = (\epsilon_1, \dots, \epsilon_p), \qquad \epsilon_i = \pm 1 \tag{1.2.2}$$

with $\epsilon_0 = 1$ if $n_0 = 0$. An element $s$ is in $\kappa$ if and only if

$$s \in \kappa \leftrightarrow \epsilon_{2i-1}\epsilon_{2i} = 1. \tag{1.2.3}$$

The infinitesimal character of the unipotent representation is as follows

(1) For each $\epsilon_i = 1$, add $(n_i - 1/2, \dots, 1/2)$.
(2) For $\epsilon_i = -1$ order the $n_i$ in increasing order and name them $m_0, \dots, m_{2q}$. Then add

$$\begin{aligned} &(m_j, \dots, 1) && \text{if } j \text{ is even,}\\ &(m_j - 1, \dots, 1, 0) && \text{if } j \text{ is odd.} \end{aligned} \tag{1.2.4}$$

Every coset has a representative of the form

$$\begin{aligned} &(\eta_0, \eta_0, \dots, \eta p - 1, \eta_{p-1}, 1) && \text{if } n_0 \ne 0,\\ &(1, \eta_0, \eta_1, \eta_1, \dots, \eta_{p-1}, \eta_{p-1}, 1) && \text{if } n_0 = 0. \end{aligned} \tag{1.2.5}$$

The infinitesimal character corresponding to this representative is called unipotent.

1.3. **Type C.** The dual group is type B. A nilpotent orbit $\check{\mathcal{O}}$ is given by a partition

$$0 < 2n_0+1 \leq 2n_1+1 < \cdots < 2n_{2i}+1 \leq 2n_{2i+1}+1 < \cdots < 2n_{2p}+1. \quad (1.3.1)$$

The centralizer is $G({}^L\phi)$ is a product of $O(2)$'s, one for each pair $n_{2i} = n_{2i+1}$ and $O(1)$'s one for each remaining $n_i > 0$.

An element $s \in G({}^L\phi)$ which gives rise to a halfinteger parameter can be written as

$$s = (\epsilon_0, \dots, \epsilon_{2p}), \qquad \epsilon_i = \pm 1, \qquad \epsilon_0 \cdot \cdots \cdot \epsilon_{2p} = 1. \quad (1.3.2)$$

An element $s$ is in $\kappa$ if and only if

$$s \in \kappa \leftrightarrow \epsilon_{2i}\epsilon_{2i+1} = 1. \quad (1.3.3)$$

The infinitesimal character of the unipotent representation is as follows

(1) For $\epsilon_i = 1$, order the $n_i$ in increasing order and name them $m_0, \dots, m_{2r}$. Then add

$$\begin{aligned} &(m_j, \dots, 1) && \text{if } j \text{ is even,} \\ &(m_j, \dots, 1, 0) && \text{if } j \text{ is odd.} \end{aligned} \quad (1.3.4)$$

(2) For $\epsilon_i = -1$, order the $n_i$ in increasing order and name them $m_0, \dots, m_{2q+1}$. Then add

$$\begin{aligned} &(m_j + 1/2, \dots, 1/2) && \text{if } j \text{ is odd,} \\ &(m_j - 1/2, \dots, 1/2) && \text{if } j \text{ is even.} \end{aligned} \quad (1.3.5)$$

Every coset has a representative of the form

$$(1, \eta_1, \eta_1, \dots, \eta_p, \eta_p). \quad (1.3.6)$$

The infinitesimal character corresponding to this representative is called unipotent.

1.4. **Type D.** The dual group is type D. A nilpotent orbit $\check{\mathcal{O}}$ is given by a partition

$$0 < 2n_0 + 1 \le 2n_1 + 1 < \cdots < 2n_{2i} + 1 \le 2n_{2i+1} + 1 < \cdots < 2n_{2p+1} + 1. \tag{1.4.1}$$

The centralizer is $G(^L\phi)$ is a product of $O(2)$'s, one for each pair $n_{2i} = n_{2i+1}$ and $O(1)$'s one for each remaining $n_i > 0$.

An element $s \in G(^L\phi)$ which gives rise to a halfinteger parameter can be written as

$$s = (\epsilon_1, \dots, \epsilon_{2p+1}), \qquad \epsilon_i = \pm 1. \tag{1.4.2}$$

An element $s$ is in $\kappa$ if and only if

$$s \in \kappa \leftrightarrow \epsilon_{2i}\epsilon_{2i+1} = 1. \tag{1.4.3}$$

The infinitesimal character of the unipotent representation is as follows

(1) For $\epsilon_i = 1$, order the $n_i$ in increasing order and name them $m_0, \dots, m_{2r+1}$. Then add

$$\begin{aligned} &(m_j, \dots, 1) && \text{if } j \text{ is even,} \\ &(m_j, \dots, 1, 0) && \text{if } j \text{ is odd.} \end{aligned} \tag{1.4.4}$$

(2) For $\epsilon_i = -1$, order the $n_i$ in increasing order and name them $m_0, \dots, m_{2q+1}$. Then add

$$\begin{aligned} &(m_j + 1/2, \dots, 1/2) && \text{if } j \text{ is odd,} \\ &(m_j - 1/2, \dots, 1/2) && \text{if } j \text{ is even.} \end{aligned} \tag{1.4.5}$$

Every coset has a representative of the form

$$(1, \eta_1, \eta_1, \dots, \eta_p, \eta_p, 1). \tag{1.4.6}$$

The infinitesimal character corresponding to this representative is called unipotent.

## 2. Parametrization via $\mathcal{O}$

We rely on [BV2] and [B1]. First, we prescribe the infinitesimal character $\lambda_{\mathcal{O}}$. The main property will be that the unipotent representations (irreducible $(\mathfrak{g}, K)$ modules whose annihilator in the universal enveloping algebra is maximal with the given infinitesimal character) are unitary and in 1-1 correspondence with the irreducible characters of the component group $A(\mathcal{O})$. The notation is as in [B1]. An orbit is called cuspidal if it is not induced from any proper Levi component. For special orbits whose dual is even the infinitesimal character is one half the semisimple element of the Lie triple corressponding to the dual orbit. For the other orbits we need the case-by-case analysis.

**Type A.** A nilpotent orbit is determined by its Jordan canonical form. It is given by a partition *i.e.* a sequence of numbers in decreasing order $(n_1, \dots, n_k)$ that add up to $n$. Let $(m_1, \dots, m_l)$ be the dual partition. Then the infinitesimal character is

$$(\frac{m_1-1}{2}, \dots, -\frac{m_1-1}{2}, \dots, \frac{m_l-1}{2}, \dots, -\frac{m_l-1}{2})$$

The orbit is induced from the trivial orbit on the Levi component $GL(m_1) \times \cdots \times GL(m_l)$. The corresponding unipotent representation is spherical and induced irreducible from the trivial representation on the same Levi component. All orbits are *stably trivial.*

**Type B.** A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \dots, n_k)$ of $2n+1$ such that every even part occurs an even number of times. Let $(m'_0, \dots, m'_{2p'})$ be the dual partition (add an $m'_{2p'} = 0$ if necessary, in order to have an odd number of terms). If there are any $m'_{2j} = m'_{2j+1}$ then pair them together and remove them from the partition. Then relabel the remaining columns and pair them up, the rest of the columns $(m_0)(m_1, m_2)\dots(m_{2p-1}m_{2p})$. The members of each pair have the same parity and $m_0$ is odd. Then form a parameter

$$\begin{aligned}(m_0) &\leftrightarrow (\frac{m_0-2}{2}, \dots, 1/2),\\ (m'_{2j} = m'_{2j+1}) &\leftrightarrow (\frac{m_{2j}-1}{2}, \dots, -\frac{m_{2j}-1}{2})\\ (m_{2i-1}m_{2i}) &\leftrightarrow (\frac{m_{2i-1}}{2}, \dots, -\frac{m_{2i}-2}{2})\end{aligned}$$

In case $m'_{2j} = m'_{2j+1}$, the nilpotent orbit is induced from a parabolic subalgebra $\mathfrak{p}$ with Levi component $so(*) \times gl(m'_{2j})$ with the trivial nilpotent on the $gl$ factor. The component groups in $G$ and $P$ are equal. The unipotent representations are unitarily induced irreducible from similar parameters on the Levi component. Similarly if some $m_{2i-1} = m_{2i}$, then the nilpotent is induced irreducible from a nilpotent on an $so(*) \times gl(\frac{m_{2i-1}+m_{2i}}{2})$ with the trivial nilpotent on the $gl$ factor. The component groups of the centralizers in $G$ and $P$ coincide. The unipotent representations are again induced irreducible from the nilpotent orbit on $so(*)$ with partition the one for $\mathcal{O}$ but with $m_{2i-1}, m_{2i}$ removed. The *stably trivial* orbits are the ones such that every odd sized part appears an even number of times except for the largest size. An orbit is trinagular if it has partition $(1, 1, 3, 3, \dots, 2m-1, 2m-1, 2m+1)$. It is induced from the trivial nilpotent orbit on $\mathfrak{m} = gl(2) \times gl(4) \dots gl(2m)$. The component group $A_P$ is trivial.

**Type C.** A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \dots, n_k)$ of $2n$ such that every odd part occurs an even number of times. Let $(m'_0, \dots, m'_{2p'})$ be the dual partition (add a $m'_{2p'} = 0$ if necessary in order to have an odd number of terms). If there are any $m'_{2j-1} = m'_{2j}$ pair them up and remove them from the partition. Then relabel and pair up the remaining columns $(m_0 m_1) \dots (m_{2p-2} m_{2p-1})(m_{2p})$. The members of each pair have the same parity. The last one, $m_{2p}$, is always even. Then form a parameter

$$
\begin{aligned}
(m'_{2j-1} = m'_{2j}) &\leftrightarrow (\frac{m_{2j}-1}{2}, \dots, -\frac{m_{2j}-1}{2}) \\
(m_{2i} m_{2i+1}) &\leftrightarrow (\frac{m_{2i}}{2}, \dots, -\frac{m_{2i+1}-2}{2}), \\
m_{2p} &\leftrightarrow (\frac{m_{2p}}{2}, \dots, 1).
\end{aligned}
$$

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B. The *stably trivial* orbits are the ones such that every even sized part appears an even number of times. An orbit is called triangular if it corresponds to the partition $(2, 2, 4, 4, \dots, 2m, 2m)$. It is induced from the trivial orbit on $\mathfrak{m} = sp(2m) \times gl(1) \times \dots \times gl(2m-1)$. The component group $A_P$ is trivial.
**Type D.** A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \dots, n_k)$ of $2n$ such that every even part occurs an even number of times. Let $(m'_0, \dots, m'_{2p'-1})$ be the dual partition (add a $m'_{2p-1} = 0$ if necessary). If there are any $m'_{2j} = m'_{2j+1}$ pair them up and remove from the partition. Then pair up the remaining columns $(m_0 m_{2p})(m_1, m_2) \dots (m_{2p-2} m_{2p-1})$. The members of each pair have the same parity and $m_0, m_{2p-1}$ are even. Then form a parameter

$$
\begin{aligned}
(m'_{2j} = m'_{2j+1}) &\leftrightarrow (\frac{m'_{2j}-1}{2} \dots, -\frac{m'_{2j}-1}{2}) \\
(m_0 m_{2p-1}) &\leftrightarrow (\frac{m_0-2}{2}, \dots, -\frac{m_{2p-1}}{2}), \\
(m_{2i-1} m_{2i}) &\leftrightarrow (\frac{m_{2i-1}}{2} \dots, -\frac{m_{2i}-2}{2})
\end{aligned}
$$

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B. An exception occurs when the partition is formed of pairs $(m'_{2j} = m'_{2j+1})$ only. In this case there are two nilpotent orbits corresponding to the partition. There are also two nonconjugate Levi components of the form $gl(m'_0) \times gl(m'_2) \times \dots gl(m'_{2p'-2})$ of parabolic subalgebras. There are two unipotent representations each induced irreducible from the trivial representation on the corresponding Levi component. The *stably trivial* orbits are the ones such that every even sized part appears an even number of times. A nilpotent orbit is triangular if it corresponds to the partition $(1, 1, 3, 3, \dots, 2m-1, 2m-1)$. It is induced from the trivial orbit in the Levi component $\mathfrak{m} = gl(2) \times \dots \times gl(2m-2)$. The component group $A_P$ is trivial.

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