CUSPIDAL REPRESENTATIONS OF REDUCTIVE GROUPS

DAN BARBASCH AND BIRGIT SPEH[†]

ABSTRACT. The goal of this paper is to prove the existence of cuspidal automorphic representations of a reductive group $\mathbb{G}$ which are invariant under an (outer) automorphism τ of finite order. In particular we focus on the well known examples are $\mathbb{G} = GL(n)$ with $\tau(x) := {}^t x^{-1}$ and in the even case the inner twist with fixed points $Sp(2n)$. Our main tool is the twisted Arthur trace formula, and a local analysis of orbital integrals and Lefschetz numbers of representations.

Outline

I. Introduction

In classical analytic number theory automorphic functions are holomorphic functions on the upper half plane $\mathcal{H} = SL(2,\mathbb{R})/SO(2)$ with a prescribed transformation rule under a subgroup of finite index Γ of $SL(2,\mathbb{Z})$. Automorphic functions lift to square integrable functions on $L^2(\Gamma\backslash G)$ with respect to an invariant measure, and under the right action of $SL(2,\mathbb{R})$ theses functions generate a subspace of $L^2(\Gamma\backslash G)$ which

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decomposes into a direct sum of irreducible automorphic cuspidal representations of $SL(2,\mathbb{R})$.

A well known generalization is as follows. Let G be a semisimple non compact Lie group and Γ a discrete subgroup of finite covolume with respect to a some right invariant measure dg, and let $L^2(\Gamma\backslash G)$ be the space of square integrable functions with respect to dg. When $\Gamma\backslash G$ is compact, $L^2(\Gamma\backslash G)$ decomposes into a direct sum of irreducible representations occuring with finite mulitplicity by results of Gelfand and Piatetsky-Shapiro. When $\Gamma\backslash G$ is not compact, an irreducible (necessarily unitary) representation Π is said to be automorphic with respect to Γ if it occurs discretely with finite multiplicity in $L^2(\Gamma\backslash G)$. These representations are also referred to as the *discrete spectrum* of $L^2(\Gamma\backslash G)$. The discrete spectrum contains a $G-$invariant subspace denoted $L^2_0(\Gamma\backslash G)$. Functions in $L^2_0(\Gamma\backslash G)$ are called cuspidal, and are characterized by the property that they decay very rapidly at the cusps of $\Gamma\backslash G$. For $G = SL(2,\mathbb{R})$ the representation generated by a given automorphic function is in $L^2_0(\Gamma\backslash G)$. The complement of the cuspidal spectrum in the discrete spectrum is called the residual spectrum. One of the major unsolved problems in the theory of automorphic forms is to determine the multiplicities of the irreducible representations occuring in $L^2_0(\Gamma\backslash G)$.

The techniques used to show that certain representations Π occur with nonzero multiplicity either exploit the connection with the geometry of the corresponding locally symmetric space (see for example [Rohlfs-Speh] and references therein), or use the Arthur trace formula (see [B-L-S] and the references therein).

For the construction of representations in the residual spectrum, it is also very important to know the existence of cuspidal representations invariant under an automorphism τ of $\mathbb{G}$. For example $GL(n,\mathbb{R})$ is the Levi component of a parabolic subgroup of the split real form of $\mathbb{G} = SO(2n)$ or $SO(2n+1)$, as well as $\mathbf{G} = Sp(2n)$. Let π be a cuspidal automorphic representation of $GL(n,\mathbb{R})$. The residual spectrum of the Eisenstein series associated to the induced modules $\mathrm{Ind}_{GL(n)}^{G}[\chi_s \otimes \pi]$, where χ_s is a character of $GL(n,\mathbb{R})$, is tied to the nature of the poles of the $L-$functions $L(s,\pi,S^2\mathbb{C}^n)$ and $L(s,\pi,\bigwedge^2\mathbb{C}^n)$. With the appropriate normalization, the product of these two functions has a simple pole at $s=1$ precisely when π is invariant under the outer automorphism of $GL(n)$. A detailed discussion of results and conjectures about the poles of these $L-$functions can be found in [Bump-Ginsburg] and [Bump-Friedberg].

I.1. The main goal of this paper is to prove the existence of cuspidal automorphic representations of reductive groups G. We are in particular interested in those representations with integral nonsingular infinitesimal character which are also invariant under an automorphism of the group G. Our main tool is the Arthur trace formula together with local harmonic analysis. All the local results hold for arbitrary fields. But the global techniques mostly apply to the case of a totally real number field $\mathbf{K}$. In the interest of clarity in the global situation we present the case of $\mathbf{K} = \mathbb{Q}$ only.

So let $\mathbb{G}/\mathbb{Q}$ be a connected reductive algebraic group so that $\mathbb{G}(\mathbb{R})$ is noncompact. Let $\tau : \mathbb{G} \to \mathbb{G}$ be a $\mathbb{Q}-$rational automorphism of finite order. The automorphism acts on the cuspidal automorphic functions on $\mathbb{G}(\mathbb{A})$. If F is a finite dimensional representation of $\mathbb{G}(\mathbb{R}) \rtimes \{1, \tau\}$, then $\mathrm{tr}F(\tau)$ is well defined. Note that if $\mathrm{tr}F(\tau) \neq 0$, then the restriction of F to $\mathbb{G}(\mathbb{R})$ must be irreducible.

The main result is the following theorem.

Theorem (theorem XI.1)**.** *Let $\mathbb{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$, and assume that $G(\mathbb{R})$ has no compact factors. Let F be a finite dimensional irreducible representation of $\mathbb{G}(\mathbb{R}) \ltimes \{1, \tau\}$, and assume that the centralizer of τ in $\mathfrak{g}(\mathbb{R})$ is of equal rank. If $\mathrm{tr}F(\tau) \neq 0$, then there exists a cuspidal automorphic representation $\pi_{\mathbb{A}}$ of $\mathbb{G}(\mathbf{A})$ stable under τ, with the same infinitesimal character as F.*

In addition

$$\mathrm{H}^*(\mathfrak{g}(\mathbb{R}), K_\infty, \pi_{\mathbb{A}} \otimes F) \neq 0. \tag{I.1.1}$$

If τ is an involution, $\mathrm{tr}F(\tau)$ is computed in [Rohlfs-Speh] (see V.3 for a more uniform proof). In general we show that there exist infinitely many irreducible representations F with $\mathrm{tr}F(\tau) \neq 0$.

This theorem is a generalization of the results of A.Borel, J.P. Labesse and J.Schwermer [B-L-S]. They prove such a result for an almost absolutely simple, connected, algebaic group $\mathbb{G}$ and a Cartan-like involution τ.

I.2. In the special case of $\mathbb{G} = GL_n$ we consider the involution τ_c with fixed points $SO(n)$, and if $n = 2m$ also the the symplectic involution τ_s with fixed points $Sp(n)$. The following theorem summarizes our results for these special cases.

Theorem (theorems (1) and (2) in XI.2)**.** *There exist cuspidal representations $\pi_{\mathbb{A}}$ of GL(n,$\mathbb{A}$) with trivial infinitesimal character invariant under the Cartan involution τ_c. If n= 2m there also exist cuspidal*

representations $\pi_{\mathbb{A}}$ of GL(n,$\mathbb{A}$) with trivial infinitesimal character invariant under τ_s.

A suitable generalization can be proved for the case when the infinitesimal character of $\pi_{\mathbb{A}}$ coincides with that of a finite dimensional representation F.

Using base change results of J. Arthur and L. Clozel [AC], we obtain in theorem (3) in section XI.2 cuspidal representations for GL(n) over number fields which are towers of cyclic extensions of prime order of $\mathbb{Q}$.

I.3. Let K_f be an open compact subgroup of $\mathbb{G}(\mathbb{A}_f)$, and A_G the split component of the center of $\mathbb{G}(\mathbb{A})$. Then

$$S(K_f) := (K_\infty K_f)\backslash \mathbb{G}(\mathbb{A})/A_G\mathbb{G}(\mathbb{Q})$$

is a locally symmetric space.

Theorem (theorem XI.3)**.** *Let $\mathbb{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$ which admits a Cartan like involution. Then for K_f small enough*

$$H^*_{cusp}(S(K_f), \mathbb{C}) \neq 0.$$

Previously, nonvanishing results for the cohomology of locally symmetric spaces were proved by [B-L-S] for the case of semisimple groups and $S-$arithmetic groups also using L^2-Lefschetz numbers. Using geometric techniques results of this type for an anisotropic form of $\mathbb{G}$ were proved in [Rohlfs-Speh] and in the special case of the SO(n,1). (For a more detailed history of the problem see section XI.3)

I.4. Our main tool is the twisted Arthur trace formula. We construct a function $f_{\mathbb{A}}$ which satisfies the conditions for the simple trace formula of Kottwitz/Labesse to hold. The major part of the article is devoted to analyzing the twisted orbital integrals of this function.

In the real case the main result is the following. We first prove in theorem VI.3 a formula for the Lefschetz numbers of the automorphism τ on the $(\mathfrak{g}, K_\infty)-$cohomology of standard representations and define a Lefschetz function f_F. We use this to find an explicit formula for the twisted orbital integral $O_\gamma(f_F)$ of an arbitrary elliptic element $\gamma = \delta\tau$ in VII.5.

Theorem (theorem VII.5). *Let f_F be the Lefschetz function corresponding to a $\tau-$stable finite dimensional representation F and $\gamma = \delta\tau$ be an elliptic element. Then*

$$O_\gamma(f_F) := \int_{\mathbb{G}(\mathbb{R})/\mathbb{G}(\mathbb{R})(\gamma)} f_F(g\gamma g^{-1})\, dg = (-1)^{q(\gamma)} e(\tau) \mathrm{tr} F^*(\gamma)$$

The undefined notation is as in section VII.

At the finite places, the main result is a slight generalization of a result of Kottwitz for the value of the orbital integral of an elliptic element $\gamma = \delta\tau$ on a Lefschetz function $f_{\mathcal{L}}$.

Theorem (theorem VIII.5). *The orbital integrals of $f_{\mathcal{L}}$ are*

$$O_\gamma(f_{\mathcal{L}}) = \begin{cases} 1 & \text{if } \gamma \text{ is elliptic,} \\ 0 & \text{otherwise.} \end{cases}$$

In the last section we plug the function $f_{\mathbb{A}}$ into the trace formula, and prove that under the assumption of theorem XI.1 we get a nonzero cuspidal contribution on the spectral side of the trace formula.

Throughout the article we illustrate the results in the example of $GL(n)$.

I.5. The paper is organized as follows. In sections II and III we introduce notation and review basic facts about twisted conjugacy classes. In section IV we introduce orbital integrals, in particular we specify the normalization of the invariant measures we use. In section V and VI and VII we deal with finite dimensional representations and Lefschetz numbers in the real case. The main idea is well known; for a finite dimensional representation F, we construct a *Lefschetz function* f_F which has the property that for any representation π, $\mathrm{tr}\pi(f_F)$ equals the Lefschetz number. We rely heavily on the work of Knapp-Vogan, [Knapp-Vogan] and Labesse [Labesse1]. Representations of a disconnected group, $\widetilde{\mathbb{G}} := \mathbb{G} \ltimes \langle\tau\rangle$, where τ acts on $\mathbb{G}$ by an automorphism of finite order, are described by Mackey theory. We use the version of the classification of irreducible $(\mathfrak{g}, K)$ modules of Vogan, where a standard module is cohomologically induced, *i.e.* of the form $\mathcal{R}^s_{\mathfrak{b}}(\chi)$, where $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra. This makes it convenient to extend modules of $\mathbb{G}(\mathbb{R})$ to $\widetilde{\mathbb{G}}(\mathbb{R})$ in a uniform way by using only the extension of the action of τ to Verma modules. The main result is theorem VI.3 which computes the Lefschetz number of a standard module. Section VII computes orbital integrals $O_\gamma(f_F)$. These formulas are used in the trace formula. Since our methods are global, we have to construct

Lefschetz functions at the finite places as well, and compute orbital integrals; for this we rely heavily on [Kottwitz2]. In section IX we describe the twisted trace formula, and in X we give the *simple form* that we actually use. This relies on not only choosing Lefschetz functions at the infinite places and two of the finite places, but also choosing the components of $f_{\mathbb{A}}$ to have very small support at a finite set of places, lemmas IX.4 and XI.1.

The main result is in section XI proposition XI.1 and theorem XI.1. Section XI.2 is devoted to the standard example of $GL(n)$ with the automorphism transpose inverse, and other applications where τ does not necessarily have to be an outer automorphism.

II. **Assumptions and Notation**

II.1. Let $\mathbf{K}$ be an arbitrary number field with Galois group Γ. Its adeles are denoted by $\mathbb{A}$ and the finite adeles by $\mathbb{A}_f$. If the adeles refer to a field other than $\mathbf{K}$, this will be indicated by a subscript.

A localization $\mathbf{K}_v$ at a place v will be abbreviated $\mathbf{k}$. Denote by $\overline{\mathbf{k}}$ its algebraic closure, and $\Gamma_{\mathbf{k}}$ the Galois group. Since $\mathbf{K}$ is totally real, $\mathbf{k}$ is $\mathbb{R}$ at an infinite place, and a finite extension of $\mathbb{Q}_p$ for some finite prime p at a finite place.

Let $\mathbb{G}$ be a connected reductive linear algebraic group defined over $\mathbf{K}$. Since $\mathbf{K} \subset \mathbf{k}$, $\mathbb{G}$ is also defined over $\mathbf{k}$, *i.e.* there is a group homomorphism

$$\Gamma_{\mathbf{k}} \longrightarrow Aut(\mathbb{G}(\overline{\mathbf{k}})) \tag{II.1.1}$$

which takes regular functions to regular functions in the sense that if f is regular, then so is $[\sigma \cdot f](x) := \sigma^{-1}[f(\sigma(x))]$ for any $\sigma \in \Gamma_{\mathbf{k}}$.

To simplify notation we write G for $\mathbb{G}(\mathbf{k})$ and $\mathbf{G}$ for $\mathbb{G}(\overline{\mathbf{k}})$. The connected component of the identity of a group is denoted by subscript 0. The Lie algebra of a subgroup is always denoted by the corresponding gothic letter.

We assume that the derived group $\mathbb{G}_{der}$ is simply connected so that we have to deal with fewer technicalities. This is certainly true for the main example we have in mind which is $\mathbb{G} = GL(n)$ whose derived group is $SL(n)$. But in general we will also consider groups of the form $\mathbf{G}(\tau)$, the centralizer of an element $\tau \in \mathbf{G}$. Such a group may not be simply connected, and in fact may not even be connected. A result of Steinberg states that if $\mathbb{G}$ is simply connected, then $\mathbb{G}(\tau)$ is connected.

II.2. We will fix an automorphism τ of finite order d *defined* over $\mathbf{K}$ (*i.e.* $\tau\sigma = \sigma\tau$ for all $\sigma \in \Gamma$).

Denote by $\widetilde{G}$ the group

$$\widetilde{\mathbb{G}} := \mathbb{G} \ltimes \{1, \tau\}, \qquad [g_1, a_1] \cdot [g_2, a_2] = [g_1 \tau(g_2), a_1 a_2], \tag{II.2.1}$$

and write $\mathbb{G}^* := \mathbb{G}\tau$ for the connected component containing τ. The center of $\mathbb{G}$ is denoted $\mathcal{Z}$, and the center of $\mathbf{G}$ by $\mathbf{Z}$.

For $\gamma \in G^*$ we denote its centralizer in G by $G(\gamma)$.

II.3. If $\gamma \in \tilde{G}$ is arbitrary, then it has a Jordan decomposition $\gamma = \gamma_{ss}\gamma_n$. Then $\gamma_n = e^N$ where $N \in \mathfrak{g}(\gamma)$ is nilpotent. In particular $\gamma_n \in G$, and $\gamma_s \in G^*$.

Definition. *An element γ is called semisimple if* $\mathrm{Ad}\gamma$ *is semisimple. An element $\gamma \in \widetilde{G}$ is called almost semisimple if it stabilizes a pair $(\mathfrak{b}, \mathfrak{h})$ where $\mathfrak{b}$ is a Borel subgroup of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ and $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra. For such elements,* $\mathrm{Ad}\gamma$ *is semisimple on $[\mathfrak{g}, \mathfrak{g}]$, but the action on the center of $\mathfrak{g}$ need not be semisimple.*

A semisimple $\gamma \in \tilde{G}$ is called

1. *elliptic if $G(\gamma)$ contains a maximal anisotropic torus,*
2. *compact if the closure of the group $< \gamma >$ generated by γ is compact*
3. *regular if it is in G and its centralizer in G is a torus,*
4. *superregular if it is of the form $s\tau$ with $s \in G$, and $s\tau(s)\tau^2(s)\dots\tau^{d-1}(s) = (s\tau)^d$ is regular.*

The set of superregular points is denoted by $\widetilde{G}_{sr}$.
The map

$$N : s\tau \to \ s\tau(s)\tau^2(s)\dots\tau^{d-1}(s) \tag{II.3.1}$$

is called the norm map.

Remarks.

1. In the $p-$adic case the definition of elliptic is equivalent to the closure of the group $< \gamma >$ generated by γ being compact (see [Kottwitz1] section 9.1 or [Kott-Shel] section I), but not in the real case.
2. In section 3.2 of [Kott-Shel] the norm map is defined for an abelian group H as the canonical quotient map

$$H \longrightarrow H/(1-\tau)H. \tag{II.3.2}$$

where $(1-\tau)H := \{x\tau(x^{-1}) \ : \ x \in H\}$. See lemma III.1 for a comparison between H^τ and the image of the map in II.3.1 over a closed field. □

The set $\widetilde{G}_{sr}$ is open and dense in G^*.

For $\gamma \in G^*$ its G-conjugacy class is denoted by $O(\gamma)$. Its $\mathbf{G}$-conjugacy class is denoted by $\mathbf{O}(\gamma)$.

III. **Twisted conjugacy classes**

In this section we discuss the following:

(1) describe the conjugacy classes of elements $\gamma \in G^*$ under the adjoint action of G. These are called twisted conjugacy classes.
(2) set up a map from twisted conjugacy classes in $\mathbf{G}^*$ to usual conjugacy classes in $G(\tau)$. This is called a norm class correspondence. We also discuss this map in detail for $\mathbb{G} = GL(n)$.

For the real case many or most of these results are well known from the work of Bouaziz. We provide proofs that are uniform for both the real and the p-adic case.

III.1. We first work in $\mathbf{G} := \mathbb{G}(\overline{\mathbf{k}})$. Suppose that $\gamma \in \mathbf{G}^*$ is semisimple. According to [Steinberg1], there is a pair $(\mathfrak{b}, \mathfrak{h})$, where $\mathfrak{b}$ is a Borel subgroup and $\mathfrak{h} \subset \mathfrak{b}$ is a *Cartan subalgebra*, which is stabilized by γ. Since τ is semisimple ($\tau^d = 1$), fix a τ-stable pair $(\mathfrak{b}_0, \mathfrak{h}_0)$. Then there is $g \in \mathbf{G}$ such that $\mathfrak{b}_0 = g\mathfrak{b}g^{-1},\ \mathfrak{h}_0 = g\mathfrak{h}g^{-1}$. Thus replacing γ by $g\gamma g^{-1}$ we may assume that γ stabilizes a τ-stable pair $(\mathfrak{b}_0, \mathfrak{h}_0)$, *i.e.* there is a pair $(\mathfrak{b}_0, \mathfrak{h}_0)$ such that

$$\gamma(\mathfrak{b}_0) = \mathfrak{b}_0,\ \gamma(\mathfrak{h}_0) = \mathfrak{h}_0 \quad \text{and} \quad \tau(\mathfrak{b}_0) = \mathfrak{b}_0,\ \tau(\mathfrak{h}_0) = \mathfrak{h}_0. \tag{III.1.1}$$

It follows that if we write $\gamma = h\tau$, then $h \in \mathbf{H}$, the *Cartan subgroup* corresponding to $\mathfrak{h}$. Write $\mathbf{T}$ for the fixed points of τ in $\mathbf{H}$.

Lemma. *Let*

$$\mathbf{H}^{\perp} := \{h \in \mathbf{H} \ :\ h\tau(h) \cdot \ \dots \ \cdot \tau^{d-1}(h) = 1\}.$$

Then

$$\mathbf{H}^{\perp} = (1-\tau)\mathbf{H} := \{h\tau(h)^{-1} \ :\ h \in \mathbf{H}\},$$

and the map

$$\Psi : \mathbf{T} \times \mathbf{H}^{\perp} \longrightarrow \mathbf{H}, \qquad \Psi(t,h) := th^{-1}\tau(h) \tag{III.1.2}$$

is onto and has finite kernel.

Proof. Let $\mathcal{X}_*$ be the weight lattice. Since $\mathbf{H}$ is abelian and connected, $\mathbf{H} = \mathcal{X}_* \otimes_{\mathbb{Z}} \overline{\mathbf{k}}^{\times}$, and $\mathbf{T} = \mathcal{X}_*^{\tau} \otimes_{\mathbb{Z}} \overline{\mathbf{k}}^{\times}$. To show that Ψ is onto, observe that the polynomial relation

$$T^{d-1} + \cdots + 1 = [T-1] \cdot [T^{d-2} + 2T^{d-3} + \cdots + (d-1)] + d \tag{III.1.3}$$

holds. Thus any $x \in \mathcal{X}_*$ can be written as

$$x = \frac{x + \tau(x) + \cdots + \tau^{d-1}(x)}{d} + \frac{y - \tau(y)}{d}, \tag{III.1.4}$$

where

$$y := -(d-1)x - (d-2)\tau(x) - \cdots - 2\tau^{d-3}(x) - \tau^{d-2}(x) \in \mathcal{X}_*,$$

Write

$$z := x + \tau(x) + \cdots + \tau^{d-1}(x) \in \mathcal{X}_*^\tau.$$

Passing to $\mathbf{H}$, let $h := x \otimes a$, with $x \in \mathcal{X}_*$ and $a \in \overline{\mathbf{k}}^\times$, and let $\alpha \in \overline{\mathbf{k}}^\times$ be such that $\alpha^d = a$. Then

$$x \otimes a = (z \otimes \alpha)(y \otimes \alpha)\tau(y \otimes \alpha)^{-1} \tag{III.1.5}$$

where $z \otimes \alpha \in \mathbf{T}$, and $(y \otimes \alpha)\tau(y \otimes \alpha)^{-1} \in \mathbf{H}^\perp$. The fact that the map has finite kernel now follows from computing the differential of Ψ, and observing that it is nondegenerate. □

Proposition. *Every semisimple $\gamma \in \mathbf{G}^*$ is conjugate under $\mathbf{G}$ to an element of the form $t\tau$ with $t \in \mathbf{T}$.*

Proof. By the discussion at the beginning of section III.1, there is an element $g \in G$ such that $g\gamma g^{-1}$ stabilizes the pair $(\mathfrak{b}_0, \mathfrak{h}_0)$. Thus $\widetilde{\gamma} := g\gamma g^{-1}$ is of the form $h\tau$ with $h \in \mathbf{H}$. By lemma III.1, $h\tau = h^\perp t\tau = x\tau(x)^{-1}t\tau = xt\tau x^{-1}$. □

Corollary. *Suppose that $\gamma \in \mathbf{G}^*$ is semisimple and $\gamma^d = 1$. Then γ is conjugate by $\mathbf{G}$ to τ.*

Proof. By the above discussion, γ is conjugate to an element of the form $t\tau$ with t in a torus $\mathbf{T}$ fixed under τ. On the other hand, the condition $\gamma^d = 1$ implies that $t \in \mathbf{H}^\perp$. Thus $t = h^{-1}\tau(h)$ so

$$t\tau = h^{-1}\tau(h)\tau = h^{-1}\tau h.$$

□

III.2. We now consider the case of $\mathbf{k}$ which is not necessarily closed. Recall $\mathbf{H} = \mathbf{T}\mathbf{H}^\perp \subset \mathbf{G}$, a τ-invariant Cartan subgroup.

Proposition (1)**.**

(1) $\mathbf{T}$ *contains regular elements as well as superregular elements.*
(2) *If $\mathbf{T}$ is defined over $\mathbf{k}$, then $T := \mathbb{T}(k)$ contains regular elements as well as superregular elements.*

Proof. We first show the assertions for $\mathbb{T}$. Because we are working over $\overline{\mathbf{k}}$, it is enough to consider the case of the Lie algebra; then it is the same as over $\mathbb{C}$ and it is due to F. Gantmacher (See theorem 5 and 28 in [Gantmacher]). Thus the set of regular elements as well as the set of superregular elements is dense in $\mathbf{T}$. For T, the assertions follow from Rosenlicht's density theorem, which says that for a perfect field k, $\mathbb{G}(\mathbf{k})$ is dense in $\mathbb{G}(\overline{\mathbf{k}})$ for any $\mathbb{G}$ defined over $\mathbf{k}$, [Rosenlicht]. □

Remark. In case $\mathbf{k} = \mathbb{R}$, a similar result is proved by A. Bouaziz (1.3.1 in [Bouaziz]) for a more general type of group in essentially in the same way.

Proposition (2)**.** *There exist finitely many superregular elements $\gamma_1, \dots, \gamma_k \in G^*$ such that every superregular element is conjugate under G to an element in $T_i\gamma_i$, where $T_i := G(\gamma_i)$.*

Proof. By proposition (1) of III.2, there are superregular elements defined over $\mathbf{k}$; in fact the set of such elements is open and dense in G^*. Fix a superregular $\gamma \in G^*$, and let $(\mathfrak{b}, \mathfrak{h})$ be a pair stabilized by γ. Let $\mathbb{T}$ be the centralizer of γ. Then $\mathbf{T}$ is stabilized by any $\sigma \in \Gamma$ because γ is stabilized by such a σ. Since $\mathbf{H}$ is the centralizer of $\mathbf{T}$, the same holds for $\mathbf{H}$. Thus there is a Cartan subgroup $H \subset G$ defined over $\mathbf{k}$ which is normalized by γ. Two superregular elements γ_1 and γ_2 which stabilize the pair $(\mathfrak{b}, \mathfrak{h})$ differ by an automorphism of the pair; there are only finitely many such automorphisms. Suppose γ_1 and γ_2 induce the same automorphism. Then $H' := T\{h^{-1}\gamma(h)\}_{h\in H}$ has finite index in H. Let $a_1, \dots, a_l$ be representatives of H/H'. It follows that γ_2 is conjugate to $ta_j\gamma_1$ for some $1 \leq j \leq l$. The claim now follows from the fact that the number of $G-$conjugacy classes of *Cartan subgroups* and pairs $(\mathfrak{b}, \mathfrak{h})$ with the same $\mathfrak{h}$ is finite.

If a superregular γ' normalizes the Cartan subgroup H, then $\mathrm{Ad}\gamma'$ is an automorphism of H which stabilizes the root system, and there are only finitely many such automorphisms.

Then $\gamma_2 = h\gamma_1$ with $h \in H$. Let T be the fixed points of γ_1 (and γ_2 as well). □

III.3. Recall (II.3.1) the *norm map*, $N : \mathbf{G}^* \longrightarrow \mathbf{G}$, given by $N(\gamma) := \gamma^d$. It induces a map from twisted $\mathbf{G}$-orbits in $\mathbf{G}^*$ to usual orbits in $\mathbf{G}$. The discussion above shows that for γ semisimple, the orbit of $N(\gamma)$ intersects the fixed points $\mathbf{G}(\tau)$. Thus N induces a a *norm class correspondence* $\mathcal{N}$, which takes twisted semisimple orbits in $\mathbf{G}^*$ to unions of semisimple orbits in $\mathbf{G}(\tau)$:

$$\mathcal{N} : \mathbf{O}(\gamma) \mapsto \mathbf{O}(\gamma^d) \cap \mathbf{G}(\tau). \tag{III.3.1}$$

Proposition. *If γ is semisimple, then $\mathbf{O}(\gamma^d) \cap \mathbf{G}(\tau)$ is a finite union of orbits.*

Proof. From earlier, γ is conjugate to $t\tau$ with $t \in \mathbf{T}$ showing that the intersection is nonempty. The orbit $\mathbf{O}(\gamma^d)$ coincides with $\mathbf{O}(t^d)$. Thus it suffices to show that $\mathbf{O}(t^d)$ intersects $\mathbf{G}(\tau)$ in finitely many orbits. But this is clear since $\mathbf{O}(t^d) \cap \mathbf{H}$ is finite. □

The map N and correspondence $\mathcal{N}$ make sense for G^* and G. It is still true that $O(\gamma^d) \cap G(\tau)$ consists of finitely many orbits, but the image might be empty.

III.4. We illustrate this for $\mathbf{G} = GL(n, \overline{\mathbf{k}})$. The automorphism τ will be of order two related to Cartan involutions of real groups. We write it as

$$\tau(g) := w_0({}^t x^{-1}) w_0^{-1}, \qquad w_0 = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 1 \\ 0 & \dots & \dots & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & \dots & \dots & 0 \\ 1 & \dots & \dots & \dots & \dots & 0 \end{bmatrix} \tag{III.4.1}$$

If n is odd τ is up to conjugation the unique outer automorphism of order two. If n is even, there is another conjugacy class of automorphisms of order two with representative $\mathrm{Ad}(t_o) \circ \tau$ where

$$t_0 = \mathrm{diag}[i, \dots, i, -i, \dots, -i].$$

The element $t_0\tau$ has order 4, and is conjugate to the one in more familiar form where w_0 in (III.4.1) is replaced by

$$w_0 = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 1 \\ 0 & \dots & \dots & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & \dots & \dots & \dots & 0 \\ -1 & \dots & \dots & \dots & \dots & 0 \end{bmatrix}. \tag{III.4.2}$$

These elements preserve the upper triangular group which gives a τ stable pair $(\mathfrak{b}_0, \mathfrak{h}_0)$ defined over $\mathbf{k}$. The centralizers are $\mathbf{G}(\tau) = O(n)$ for (III.4.1) and $\mathbf{G}(\tau) = Sp(n)$ for (III.4.2). In case (III.4.1) with n even, the automorphism τ does not preserve the root vectors. In other words it does not preserve a splitting.

More generally, outer automorphisms of finite order of $GL(n, \overline{\mathbf{k}})$ are (conjugate to elements) of the form $t\tau$ with $t \in \mathbf{H}$ fixed by τ. The centralizer of such an element is a product of $GL(m)'$s and possibly an

orthogonal or a symplectic group. This follows from the fact that the centralizer of t is a product of $GL(m)'$s.

Conjugacy classes of semisimple elements in $GL(n, \mathbf{k})$ are determined by the characteristic polynomial

$$p_A(\lambda) = \det(\lambda I - A).$$

The minimal polynomial $m_A(\lambda)$ of A is the polynomial of minimal degree with leading coefficient 1 satisfying $m_A(A) = 0$. The matrix A is regular if $m_A = p_A$, where m_A is the minimal polynomial of A. Two regular semisimple matrices A and B are similar if and only if $m_A = m_B$.

Suppose $\gamma = x\tau \in \widetilde{\mathbf{G}}_{sr}$ *is superregular.* From III.1 we know that we can assume $x \in \mathbf{T}$, so γ^2 can be conjugated into $\mathbf{G}(\tau)$.

Proposition (1)**.** *Suppose* $\gamma \in \widetilde{\mathbf{G}}_{sr}$*. Then* $\mathbf{O}(\gamma^2) \cap \mathbf{G}(\tau)$ *is a single orbit.*

Proof. Two regular semisimple elements $A,\ B \in \mathbf{G}(\tau)$ are conjugate if and only if $p_A = p_B$. This follows by using an explicit realization of a *Cartan subgroup* of $\mathbf{G}(\tau)$. (Note however that this fact is **not** true for $SO(\overline{\mathbf{k}})$.) Thus all the elements in $\mathbf{O}(\gamma^{\mathbf{2}}) \cap \mathbf{G}(\tau)$ have the same minimal polynomial and the claim follows. □

We will now show that $\mathbf{O}(\gamma^2)$ has points defined over $\mathbf{k}$. For this we will use the cross section introduced by Steinberg. Let $\mathbf{G}$ be any connected linear algebraic semisimple group, and $(\mathbf{B}, \mathbf{H})$ be as in section III.1. Let Δ^+ be the system of positive roots attached to $(\mathbf{B}, \mathbf{H})$ and Π be the simple roots. Let σ_i be representatives for the simple root reflections, and X_i be root vectors. Let $Z_i(t) = exp(tX_i)$ be the corresponding 1-parameter subgroup. In section 1.4 of [Steinberg2]) it is proved that the set

$$\mathcal{M} := Z_1(t_1)\sigma_1 \cdot \ \ldots \ \cdot Z_n(t_n)\sigma_n \tag{III.4.3}$$

is a cross section for the set of regular elements of G.
If G is defined over $\mathbf{k}$ and is quasiplit, then $\mathcal{M}$ can be defined over $\mathbf{k}$.

Remarks.

(1) There is a restriction on n for type A for the fact that $\mathcal{M}$ is defined over $\mathbf{k}$. This restriction applies to the unitary groups, not the general linear group.

(2) For $SL(n,\mathbf{k})$, this cross section is the well known canonical form

$$\mathcal{M} = \{ \begin{bmatrix} c_{n-1} & -c_{n-2} & \dots & (-1)^{n-1}c_1 & (-1)^n \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \}. \qquad \text{(III.4.4)}$$

The characteristic polynomial is $p(\lambda) = \lambda^n - c_{n-1}\lambda^{n-1} + \dots + (-1)^n$.

Since γ^2 can be conjugated into $\mathbf{G}(\tau)$ the characteristic polynomial of $\gamma^2 = x\tau(x)$ satisfies

$$\lambda^n p_A(\lambda^{-1}) = p_A(\lambda). \qquad \text{(III.4.5)}$$

Proposition (2)**.** *Suppose that either n is odd and $G(\tau) = O(n)$ or n is even and $G(\tau) = Sp(n)$. For any polynomial $p(\lambda)$ satisfying (III.4.5) with coefficients in $\mathbf{k}$, there is a regular element A in the split form of $G(\tau)$ with characteristic polynomial p.*

Proof. Write $n = 2m + \epsilon$. We can choose a labelling of the simple roots so that the one parameter subgroups $Y_i(t)$ of the simple root vectors in $G(\tau)$ are

$$Y_1(t_1) := Z_1(t_1)Z_2(t_1), \dots, Y_{m-1}(t_{m-1}) := Z_{m-1}(t_{m-1})Z_m(t_{m-1}),$$
$$Y_\epsilon(t_\epsilon) := Z_\epsilon(t_\epsilon)$$

and the σ_i have similar expressions. The proof follows from a direct calculation. Any polynomial $p(t) = t^n - c_1 t^{m-1} + c_2 t^{m-2} + \dots$ satisfying III.4.5 is the characteristic polynomial of the matrix obtained by setting $t_1 = c_1, \dots, t_{m-1} = c_{m-1},\ t_\epsilon = c_m$. □

Now assume $n = 2m$ and that $G(\tau) = O(n)$. Consider the split form of $O(n)$. In this case we can choose a labelling of the simple roots so that

$$Y_1(t_1) := Z_1(t_1)Z_2(t_1), \dots, Y_{m-1}(t_{m-1}) := Z_{m-1}(t_{m-1})Z_m(t_{m-1})$$

are the 1-parameter subgroups corresponding to $m-1$ of the simple roots of type D. The last simple root vector $Y_m(t_m)$ cannot be written in terms of simple root vectors of $GL(2n,\mathbf{k})$. Let t_m be the parameter for the last simple root. The characteristic polynomial of an element in this cross section is

$$x^{2m} + a_m x^{2m-1} + (a_{m-1} - 1)x^{2m-2} + (a_{m-2} - a_m)x^{2m-3} + \dots$$
$$- (a_4 + a_2 a_1)x^{m+1} - (2a_3 + a_1^2 + a_2^2)x^m + \dots \qquad \text{(III.4.6)}$$

All but the last two equations are linear so we can solve for $a_3, \dots, a_m$ in terms of the coefficients of the polynomial for A. For the last two we need that

$$c_m - 2a_3 \pm (2c_{m-1} - 2a_4) \tag{III.4.7}$$

be squares.

Proposition (3)**.** *If $n = 2m$, the orbit $O(\gamma^2)$ intersects a unique quasi split form of $O(n)$.*

Proof. Let $\zeta_1,\ \zeta_2 \in \mathbf{k}$. Consider the orthogonal group which preserves the form

$$Q(x) := x_1 x_{2m} + \cdots + x_{m-1}x_{m+2} + \zeta_1 x_m^2 + \zeta_2 x_{m+1}^2. \tag{III.4.8}$$

These groups are quasisplit. Precisely, the most split *Cartan subgroup* is

$$H := \{\ diag(a_1, \dots, a_{m-1}, \begin{bmatrix} a & b \\ -b\zeta & a \end{bmatrix}, a_{m-1}^{-1}, \dots, a_1^{-1}), \} \tag{III.4.9}$$

where $\zeta = \zeta_1/\zeta_2,\ a^2 + \zeta b^2 = 1$.

A similar calculation in the proof of the previous theorem shows that there is always a choice of $\zeta_1,\ \zeta_2$ so that the equations have a solution. □

If a regular semisimple $t \in GL(n, \mathbf{k})$ is such that its orbit is $\tau-$stable, then t is conjugate to a $\tau-$stable element by a $g \in GL(n, \mathbf{k})$ not just $GL(n, \overline{\mathbf{k}})$. This is again because the characteristic polynomial determines the conjugacy class over $GL(n, \mathbf{k})$.

Corollary. *Every semisimple $\gamma \in G^*$ is conjugate by an element of $GL(n, \mathbf{k})$ to one of the form $x\tau$, wth x in a τ-stable Cartan subalgebra H.*

Proof. The centralizer $C(\gamma, \overline{\mathbf{k}})$ contains a superregular point whose G-orbit is τ-stable, because by section III.1 γ is conjugate to an element $h\tau$ with $h \in H_0$. Because the variety of such elements is invariant under Γ, it has regular rational points in $C(\gamma, \mathbf{k})$. Let t be such a point. Then conjugate t via $GL(n, \mathbf{k})$ into a τ-fixed point. So we may as well assume that $C(\gamma, \mathbf{k})$ has a τ-fixed regular semisimple point t. Let H be the (necessarily τ-stable) *Cartan subgroup* corresponding to t. Then since $\gamma = x\tau$ centralizes it, $x \in H$. □

In particular, all the conclusions about the norm map and regular elements, extend to the case of semisimple elements.

III.5. **Orbits in the Real Case.** Let $G := \mathbf{G}(\mathbb{R})$ be the real points of $\mathbb{G}$. We assume that τ is defined over $\mathbb{R}$, so it induces an automorphism of G. We denote by subscript 0 a real algebra; an absence of a subscript indicates a complex algebra or vector space. So let $\mathfrak{g}_0$ be the Lie algebra of G. It is well known ([Helgason]) that we may fix a maximal compact subgroup K, and a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ with Cartan involution θ so that θ commutes with τ. Then the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ is invariant under τ. Let $\widetilde{K} := K \ltimes \{\tau\}$.

If $\gamma \in G^*$ is semisimple, it has a decomposition into its compact and hyperbolic parts $\gamma = \gamma_e\gamma_h$. Here $\gamma_h = expY$ where Y is hyperbolic.

Suppose γ is compact. There is a Cartan involution θ' which commutes with γ ([Helgason]). Then let $g \in G$ be such that $g^{-1}\theta g = \theta'$. Then $g\gamma g^{-1}$ is fixed by θ. So if we write $\gamma = x\tau$, then $x \in K$.

So in general, if $\gamma = \gamma_e\gamma_h$ is arbitrary, we may assume (by possibly conjugating γ by G) that $\theta(\gamma_e) = \gamma_e$ and $\theta(Y) = -Y$. We will do so without further mention.

The classification of compact elliptic elements reduces to the corresponding problem for the compact group. But since this group may be disconnected, we need some modification of our previous results.

We assume in this section that K is arbitrary compact with identity component K_0, and Lie algebra $\mathfrak{k}_0$. Denote by K_c the *connected* group with Lie algebra $\mathfrak{k}$, the complexification of $\mathfrak{k}_0$. We say that a pair $(\mathfrak{b}, \mathfrak{h})$ in $\mathfrak{k}$ is rational if $\mathfrak{h}$ is rational (or equivalently $\mathfrak{b} \cap \theta\mathfrak{b} = \mathfrak{h}$).

Proposition. *Suppose that $\gamma \in K\tau$. There is a rational pair $(\mathfrak{b}, \mathfrak{h})$ stable under γ.*

Proof. Let $(\mathfrak{b}, \mathfrak{h})$ be a fixed rational pair. There is $k \in K_0$ such that $k\gamma$ stabilizes $(\mathfrak{b}, \mathfrak{h})$. Let $B \subset K_c$ be the Borel subgroup with Lie algebra $\mathfrak{b}$. The map

$$\psi : K_0 \times B \longrightarrow K_c, \qquad \psi(x, b) := xbk\gamma(x^{-1})k^{-1} \tag{III.5.1}$$

is onto. The pair $(x\mathfrak{b}, x\mathfrak{h})$ satisfies the required properties. □

Recall that for an arbitrary compact group, a *Cartan subgroup $H \subset K$* is defined to be the normalizer of a rational pair $(\mathfrak{b}, \mathfrak{h})$. Similarly $\widetilde{H}$ is the normalizer of $(\mathfrak{b}, \mathfrak{h})$ in $\widetilde{K}$.

Corollary. *Suppose that $\gamma \in K\tau$. Then γ is conjugate via K_0 to an element which leaves a τ-stable rational pair $(\mathfrak{b}_\tau, \mathfrak{h}_\tau)$ invariant. Thus γ is conjugate to an element of the form $\gamma = h\tau$ with $h \in H$. Any element of $\widetilde{K}$ is conjugate via K_0 to an element in $\widetilde{H}$, and $\widetilde{H}$ meets every connected component of $\widetilde{K}$.*

Proof. Let $(\mathfrak{b}, \mathfrak{h})$ be a rational pair stable under γ, and $(\mathfrak{b}_\tau, \mathfrak{h}_\tau)$ a rational pair stable under τ. There is $k \in K_0$ such that $\mathrm{Ad}k(\mathfrak{b}, \mathfrak{h}) = (\mathfrak{b}_\tau, \mathfrak{h}_\tau)$. Then $k\gamma k^{-1} = h\tau$. It is clear that $h \in H$, and therefore $k\gamma k^{-1} \in \widetilde{H}$. The claims of the corollary follow. □

III.6. Results about twisted orbits are often expressed in terms of group cohomology. Let $\mathcal{G}$ be a group acting on another group $\mathcal{A}$. A cocycle is a map

$$\psi : \mathcal{G} \longrightarrow \mathcal{A}, \quad \text{satisfying} \quad \psi(st) = \psi(s) \cdot s(\psi(t)) \tag{III.6.1}$$

Two cocycles $\psi,\ \psi'$ are called equivalent if there is $g \in \mathcal{A}$ such that

$$\psi'(s) = g\psi(s)s(g)^{-1}. \tag{III.6.2}$$

The quotient space of cocycles under this relation is the cohomology group $\mathrm{H}^1(\mathcal{G}, \mathcal{A})$.

There are two instances where this construction arises. In the first case, let $\mathcal{G} = \langle \tau \rangle$, the group generated by an automorphism τ of $\mathcal{A}$. Then a cocycle ψ is determined by its value $\psi(\gamma) := a$ in $\mathcal{A}$. If τ is of finite order d, then

$$a \cdot \tau(a) \cdot \ \ldots \ \cdot \tau^{d-1}(a) = 1.$$

Proposition (1)**.** *The map $a \mapsto a\tau$ is a bijection*

$$\mathrm{H}^1(\mathcal{G}, \mathcal{A}) \longleftrightarrow \{x \in \mathcal{A}^*\}/\mathcal{A}.$$

In words, H^1 *parametrizes twisted conjugacy classes of elements in* $\mathcal{A}^*$.

Proof. We omit the details which are standard. □

In the second instance, let $\mathcal{G} = \Gamma$, the Galois group of $\overline{\mathbf{k}}/\mathbf{k}$. In this case, H^1 is denoted $\mathrm{H}^1(\mathbf{k}, \mathbf{G})$ and is called Galois cohomology. Recall that if $\mathbb{G}$ is reductive connected simply connected or $GL(n)$, and $\mathbf{k}$ is a p-adic field, these groups are trivial.

Proposition (2)**.** *Let $\mathbf{O}(\gamma)$ be the $\mathbf{G}$ orbit of $\gamma \in G^*$. Then*

$$[\mathbf{O}(\gamma) \cap G^*]/G \longleftrightarrow \ker[\mathrm{H}^1(\mathbf{k}, \mathbf{G}(\gamma)) \longrightarrow \mathrm{H}^1(\mathbf{k}, \mathbf{G})].$$

Proof. This is well known. An element $x\gamma x^{-1}$ is Γ-stable $s(x\gamma x^{-1}) = x\gamma x^{-1}$ for all $s \in \Gamma$ which is equivalent to $x^{-1}s(x) \in \mathbf{G}(\gamma)$. It is clear that $\psi(s) := x^{-1}s(x)$ is a cocycle with trivial image in $\mathrm{H}^1(\Gamma, G)$. This cocyle depends only on the G coset of x. Conversely any cocycle in the kernel must be of the form $\psi(x) = x^{-1}s(x)$ for some $x \in \mathbf{G}$. □

IV. **Orbital Integrals**

In this section we discuss twisted orbital integrals. These results will be used in section VI.

IV.1. Recall from section III that if $\gamma \in G^*$ is superregular, it can conjugated (by $\mathbf{G}$) into an element of the form $\gamma = t\tau$, where $t \in \mathbf{T}$ is a semisimple element of a τ-invariant rational maximal torus of $\mathbf{G}(\tau)$. As before, let $\mathbf{H}$ be the centralizer of $\mathbf{T}$, a maximal torus of $\mathbf{G}$.

Proposition. *For any compact set $\omega \subset G^*$, there exists a compact set $\Omega \subset G/G(\gamma)$ satisfying the condition that if $g\gamma g^{-1} \in \omega$, then $g \in \Omega G(\gamma)$.*

Proof. There is a field $\mathbf{k} \subset \mathbf{k}'$ such that $\mathbb{H}(\mathbf{k}')$ is split. Since

$$\mathbb{G}(\mathbf{k})/\mathbb{G}(\mathbf{k})(\gamma) \subset \mathbb{G}(\mathbf{k}')/\mathbb{G}(\mathbf{k}')(\gamma)$$

is a closed embedding it is enough to show the claim for the case when H is split. Write $G = KB$ for a maximal compact K and $B = NH$ a Borel subgroup so that $G = BK$. Then decompose $g = knh$. The claim follows by applying the following lemma and the observation that

$$\psi : T \times H^{\perp} \longrightarrow \gamma H, \qquad (t, h) \mapsto \gamma h^{-1}\gamma(h) \tag{IV.1.1}$$

has finite fiber and its image has finite index in H. More details can be found in [Arthur] or in the untwisted case in [Harish-Chandra3] Part I. □

Assume $P = MN \subset G$ is a rational parabolic subgroup and γ is rational semisimple such that $G(\gamma) \subset M$. Let $O_M(\gamma)$ be the orbit of γ under M.

Lemma. *The map*

$$\Psi : M/G(\gamma) \times N \longrightarrow O_M(\gamma)N,$$
$$\Psi(m, n) := m\gamma m^{-1}[Ad(m\gamma m^{-1})^{-1}(n)n^{-1}]$$

is an isomorphism.

Proof. The proof is identical to the similar result proved by Harish-Chandra. The statements are straightforward consequences of the fact that $d\Psi$ is an isomorphism. We omit the details which for the untwisted case can be found for example in [Warner] section 8.1.3, particularly lemma 8.1.3.6 and corollary 8.1.3.7. □

The proposition shows that for γ semisimple the *orbital integrals*

$$\begin{aligned} O_\gamma(f) &:= \int_{G(\gamma)_0\backslash G} f(g^{-1}\gamma g)\ dg \\ I_\gamma(f) &:= \int_{G(\gamma)\backslash G} f(g^{-1}\gamma g)\ dg \end{aligned} \tag{IV.1.2}$$

are well defined. Following [Kottwitz2], for a connected reductive group H we use the Euler-Poincare measure. If the group is disconnected, we use the unique invariant measure which restricts to the Euler-Poincare measure on H_0.

IV.2. Harish-Chandra ([Harish-Chandra3] Part II) considered the integrals

$$F_f^G(\gamma) := \nabla(\gamma)^{1/2} \int_{G(\gamma)_0\backslash G} f(g^{-1}\gamma g)\ dg \tag{IV.2.1}$$

where $\nabla(\gamma) := |\det(I - \mathrm{Ad}\gamma)|_{\mathfrak{g}/\mathfrak{g}(\gamma)}|$. We suppress the superscript G when it is clear what group is involved.

In the real case we use the following variant of IV.2.1. Let $\gamma \in G^*$ be semisimple, and $(\mathfrak{b}, \mathfrak{h})$ a $\gamma-$stable pair. Let $\mathfrak{t} := \mathfrak{g}(\gamma) \cap \mathfrak{h}$. Then $\mathfrak{t}$ is a $\Gamma-$stable Cartan subalgebra, and by theorem 1.1A in [Kott-Shel] its centralizer is $\mathfrak{h}$. Define

$$'F_f^G(\gamma) := {}'D(\gamma)O_\gamma(f), \qquad F_f^G(\gamma) := D(\gamma)O_\gamma(f) \tag{IV.2.2}$$

with

$${}'D(\gamma) := \prod_{\alpha\in\Delta_\gamma^+} [1 - e^\beta(\gamma)], \qquad D(\gamma) := \prod_{\alpha\in\Delta_\gamma^+} [e^{\beta/2} - e^{-\beta/2}](\gamma). \tag{IV.2.3}$$

The quantities $'D$ and D do not depend on the choice $(\mathfrak{b}, \mathfrak{h})$, and the formula for D only makes sense for a cover for which e^ρ exists, see section V. The first asertion follows from the fact that γ is conjugate by an element in $\mathbf{G}$ to one of the form $t\tau$ as in proposition III.1.

Let $\gamma_1, \dots, \gamma_k$ be a set of representatives of conjugacy classes satisfying the conclusion of proposition (2) of section III.2.

Proposition. *Suppose $f \in C_c^\infty(G^*)$ and Θ is a locally L^1 invariant function analytic on the regular set. Then*

$$\int_{G^*} f(g^*)\Theta(g^*)\ dg^* = \sum_i \int_{T_i} D'(t_i\gamma_i)\Theta(t_i\gamma_i)F_f(t_i\gamma_i)\ dt_i \tag{IV.2.4}$$

where $D' := \nabla/D$.

Proof. The proof is the same as for the untwisted case. It follows from the fact that the differentials of the maps

$$\Psi_i : G/T_i \times T_i \longrightarrow G^*, \qquad \Psi_i(g,t) := gt\gamma_i g^{-1} \tag{IV.2.5}$$

are $D'(t_i\gamma_i)$ (so are isomorphisms when restricted to the regular set), and proposition (2) of section III.2. For the formula in the untwisted case see for example (IIA) in section 8.1.2 in [Warner]. □

IV.3. Assume $\mathbf{k}$ is real and that $\gamma \in G^*$ is semisimple. Let $\mathfrak{t}$ be a fundamental *Cartan subalgebra* in $\mathfrak{g}(\gamma)$ and write T for the corresponding group. Let $(\mathfrak{b}, \mathfrak{h})$ be a pair which is stable under γ such that $\mathfrak{t} \subset \mathfrak{h}$. Then in fact $\mathfrak{h}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. The roots Δ^+ of $\mathfrak{b}$ are stable under γ. Decompose $\Delta^+ = \Delta^+_\gamma \cup \Delta^\perp$, and so $D = D_\gamma \cdot D^\perp$ where D is defined in (IV.2.3). Let

$$\varpi_\gamma := \prod_{\beta \in \Delta^+_\gamma} \beta, \tag{IV.3.1}$$

and write $\partial(\omega_\gamma)$ for the corresponding differential operator. Let $h\gamma$ be superregular, with $h \in T$. Then $F_f(\gamma; \partial(\varpi_\gamma))$ is defined to be the derivative of $F_f(h\gamma)$ in h, and then setting $h = 1$.

Theorem. *There is a nonzero constant $c(\gamma)$ such that*

$$F^G_f(\gamma; \partial(\varpi_\gamma)) = c(\gamma) F^G_f(\gamma).$$

Proof. Let $t \in T$ be such that $t\gamma$ is regular. First observe that

$$\partial(\varpi) \circ D^\perp|_{t=1} = D^\perp(\gamma)\partial(\varpi)|_{t=1}, \tag{IV.3.2}$$

because the left hand side is, on the one hand skew invariant under W_γ, on the other hand a linear combination of constant coefficient differential operators of degree less than or equal to $\deg \partial(\omega_\gamma)$. Then only the leading term survives, which is the right hand side of IV.3.2.

On the other hand,

$$F_f(t\gamma) = \int_{G/G(\gamma)} D^\perp(t\gamma) F^{G(\gamma)}_{\mathrm{Ad}(g^{-1})f}(t\gamma) dg. \tag{IV.3.3}$$

The result now follows from Harish-Chandra's formula [Harish-Chandra2]

$$F^{G(\gamma)}_f(1; \partial(\varpi)) = c(\gamma) f(1). \tag{IV.3.4}$$

□

V. **Finite dimensional representations**

Let $\widetilde{K}$ be an arbitrary compact group with identity component K_0, and Lie algebra $\mathfrak{k}_0$ with complexification $\mathfrak{k}$. Let $(\mathfrak{b}, \mathfrak{h})$ be a pair of a Borel subgroup $\mathfrak{b} \subset \mathfrak{k}$ and a (rational) *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{b}$. Denote by Δ^+ the roots of $\mathfrak{h}$ in $\mathfrak{b}$, and by δ all the roots of $\mathfrak{h}$ in $\mathfrak{k}$. Let $\widetilde{H}$ be the Cartan subgroup corresponding to this pair and recall its properties from section III.5. The irreducible finite dimensional representations of $\widetilde{K}$ are parametrized by *highest weights* μ which are irreducible representations of $\widetilde{H}$ with differential dominant for Δ^+. For details about the Cartan Weyl theory of disconnected groups see [Knapp-Vogan]. In this section we obtain a formula for the character $\mathrm{tr}\pi_\mu(\gamma)$ for π_μ the irreducible representation with highest weight μ and $\gamma \in \widetilde{H}$. We also evaluate the character on γ of order 2.

The results of this section are known, but since our proofs are different we include them here.

V.1. Let H_0 be the connected component of $\widetilde{H}$ and write H for the centralizer of $\mathfrak{h}$ in $\widetilde{H}$. Then

$$H_0 \subset H \subset \widetilde{H}. \tag{V.1.1}$$

The Weyl group is defined as $W := N(\widetilde{H})/H$. If $w \in W$, then $l(w) := \dim \mathfrak{b}/(w\mathfrak{b} \cap \mathfrak{b})$.

We first extend the roots to the group generated by H_0 and γ. Let Δ^+_γ be the orbits of the action of $< \gamma >$ (the group generated by γ) on Δ^+. Fix root vectors $\{E_\beta\}_{\beta \in \Delta}$. Let $d(\beta)$ be the size of the orbit of $\beta \in \Delta$. The vector

$$X_\beta = E_\beta \cdot E_{\gamma\beta} \cdot \ \dots \ \cdot E_{\gamma^{d(\beta)-1}\beta} \in S^{d(\beta)}(\mathfrak{n}) \ \beta \in \Delta^+ \tag{V.1.2}$$

is independent of the choice of β in its $\gamma-$orbit. Define e^β via

$$\mathrm{Ad}(\gamma)X_\beta = e^\beta(\gamma)X_\beta. \tag{V.1.3}$$

This is independent of the choice of root vectors E_β as well. Similarly let

$$Y_\beta := E_\beta \wedge E_{\gamma\beta} \wedge \ \dots \ \wedge E_{\gamma^{d(\beta)-1}\beta} \in \bigwedge^{d(\beta)} \mathfrak{n}. \tag{V.1.4}$$

Then

$$\mathrm{Ad}(\gamma)Y_\beta = (-1)^{d(\beta)-1}e^\beta(\gamma)Y_\beta \tag{V.1.5}$$

Recall that an element $x \in \widetilde{H}$ is regular if $[\det \mathrm{Ad}(x) - I]|_{\mathfrak{k}/\mathfrak{h}} \neq 0$. In particular $e^\beta(x) \neq 1$ for any $\beta \in \Delta^+$.

If $w \in W$, then $\widetilde{H}$ stabilizes $\mathfrak{b} \cap w\mathfrak{b}$ and therefore also $\mathfrak{b}/(\mathfrak{b} \cap w\mathfrak{b})$. This is because if $\beta = w\alpha$ with $\beta, \alpha \in \Delta^+$, then $\gamma\beta \in \Delta^+$, and $\gamma\beta =$

$ww^{-1}(\gamma)\alpha$; since $w^{-1}(\gamma) \in \widetilde{H}$, it stabilizes $(\mathfrak{b}, \mathfrak{h})$, so $w^{-1}(\gamma)\alpha \in \Delta^+$. We will identify $\mathfrak{b}/(\mathfrak{b} \cap w\mathfrak{b})$ with

$$Q_w := \{\beta \in \Delta^+ \ : \ \beta = -w\alpha \text{ with } \alpha \in \Delta^+\}. \tag{V.1.6}$$

Write $Q_{w,\gamma}$ for the γ-orbits in Q_γ. Then $\widetilde{H}$ acts on $\Lambda^{\ell(w)}[\mathfrak{b}/(\mathfrak{b} \cap w\mathfrak{b})]$ by $(-1)^{|Q_w|-|Q_{w,\gamma}|}e^{\rho-w\rho}$. We write $\epsilon(w) := (-1)^{|Q_w|}$, and $\epsilon_\gamma(w) := (-1)^{|Q_{w,\gamma}|}$.

Lemma. *For $h \in H_0$,*

$$\mathrm{tr}[h\gamma : \sum (-1)^i \bigwedge^i \mathfrak{n}] = \prod_{\beta \in \Delta^+_\gamma} (1 - e^\beta(h\gamma))$$

Proof. We have

$$\mathrm{tr}[h\gamma : \bigwedge^i \mathfrak{n}] = \sum_{Q \subset \Delta^+,\ \gamma Q = Q} (-1)^{i-|Q\cap\Delta^+_\gamma|} e^{<Q>}(h\gamma) \tag{V.1.7}$$

where $Q \cap \Delta^+_\gamma$ are the γ-orbits in Q, and $e^{<Q>}$ is the product of the e^β with $\beta \in \Delta^+_\gamma \cap Q$. The claim follows from the fact that

$$\sum_{Q \subset \Delta^+,\ \gamma Q = Q} (-1)^{|Q\cap\Delta^+_\gamma|} e^{<Q>}(h\gamma) = \prod_{\beta \in \Delta^+_\gamma} (1 - e^\beta(h\gamma)). \tag{V.1.8}$$

□

Let (χ, V_χ) be an irreducible representation of $\widetilde{H}$, and consider the Verma module

$$M_\chi := U(\mathfrak{k}_c) \otimes_{U(\mathfrak{b})} V_\chi. \tag{V.1.9}$$

This is a $(\mathfrak{k}, \widetilde{H})$ module.

Proposition. *Let $h \in H_0$ be such that $h\gamma$ is regular. Then*

$$\mathrm{tr}M_\chi(h\gamma) = \sum_{\substack{Q = \sum m_\alpha \alpha, \\ \gamma(Q) = Q}} \mathrm{tr}\chi(h\gamma) e^{-Q}(h\gamma) = \frac{\mathrm{tr}\chi(h\gamma)}{\prod_{\beta \in \Delta^+_\gamma}(1 - e^{-\beta}(h\gamma))}.$$

Proof. As an $\widetilde{H}$-module M_χ is $S(\mathfrak{n}^-) \otimes V$. The weights of $S(\mathfrak{n}^-)$ are all of the form e^{-Q} with $Q = \sum_{\alpha \in \Delta^+} m_\alpha \alpha$. If the weight is not fixed by γ, it contributes zero to the trace. If it is, it contributes the corresponding product of characters e^β defined in (V.1.3). The formula then follows in the usual manner. □

Theorem. *Let π_μ be an irreducible representation of $\widetilde{K}$. Then*

$$\mathrm{tr}\pi_\mu(h\gamma) = \frac{\sum_{w \in W} \epsilon_\gamma(w) \mathrm{tr} e^{w\mu}(h\gamma) e^{w(\rho)-\rho}(h\gamma)}{\prod_{\beta \in \Delta^+_\gamma}(1 - e^{-\beta}(h\gamma))}.$$

Proof. The trace can be computed as in the untwisted case by establishing a BGG type resolution of the representation (π_μ, V_μ), [BGG]. Then the formula follows from proposition V.1.

We will sketch a different approach. The cohomology is computed from the complex

$$\ldots \longrightarrow V \otimes \bigwedge^i \mathfrak{n}^* \xrightarrow{d^i} V \otimes \bigwedge^{i+1} \mathfrak{n}^* \longrightarrow \ldots \tag{V.1.10}$$

Then

$$\begin{aligned} &\sum (-1)^i \mathrm{tr}[\gamma; H^i(\mathfrak{n}, V)] = \sum (-1)^i \mathrm{tr}[h\gamma; V \otimes \bigwedge^i \mathfrak{n}^*] = \\ &\mathrm{tr}\pi_\mu(h\gamma) \prod_{\beta \in \Delta_\gamma^+} (1 - e^{-\beta}(h\gamma)), \end{aligned} \tag{V.1.11}$$

To prove the theorem it suffices to prove that as an $\widetilde{H}$-module,

$$\mathrm{H}^i(\mathfrak{n}, \pi_\mu) = \bigoplus_{l(w)=i} V_{w\mu} \otimes V_{w\rho - \rho} \tag{V.1.12}$$

We follow [GS]. Let $\omega^{-\alpha}$ be the basis dual to the root vectors, and $\epsilon(\omega^{-\alpha})$ the exterior wedge, and $\iota(\omega^{-\alpha})$ contraction with $\omega^{-\alpha}$. Then $d^i = \partial + T$, where

$$\begin{aligned} \partial(f \otimes \omega) &= \sum \pi(E_{-\alpha}) f \otimes \epsilon(\omega^{-\alpha})\omega, \\ T(f \otimes \omega) &= \frac{1}{2} \sum f \otimes \epsilon(\omega^{-\alpha}) E_{-\alpha}\omega. \end{aligned} \tag{V.1.13}$$

So $(d^i)^* = \partial^* + T^*$ is given by

$$\begin{aligned} \partial^*(f \otimes \omega) &= -\sum \pi(E_\alpha) f \otimes \iota(\omega^{-\alpha})\omega, \\ T^*(f \otimes \omega) &= \frac{1}{2} \sum f \otimes \iota(\omega^{-\alpha}) E_\alpha \omega. \end{aligned} \tag{V.1.14}$$

The basis $\{E_\alpha\}_{\alpha \in \pm\Delta^+}$ is in Weyl normal form; if we write $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$, then

$$\begin{aligned} E_\alpha \omega^{-\beta} &= N_{\alpha,\beta} \omega^{-\alpha-\beta}, \\ E_{-\alpha}\omega^{-\beta} &= \begin{cases} N_{-\alpha,\beta}\omega^{\alpha-\beta} & \text{if } \beta - \alpha \in \Delta^+, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{V.1.15}$$

Then the cohomology is given by harmonic forms, *i.e.* forms annihilated by d^i and $(d^i)^*$. Section 6 of [GS] then proves the result.

□

Remark: A similar formula has also been obtained by B. Kostant in [Kostant].

V.2. We can construct a cover $\widetilde{\widetilde{H}}$ such that the square root of the character $e^{2\rho}$ (same as $\Lambda^{top}(\mathfrak{n})$) makes sense; namely take

$$\widetilde{\widetilde{H}} := \{\ (h,z) \in \widetilde{H} \times \mathbb{C}^* \mid e^{2\rho}(h) = z^2\ \}. \tag{V.2.1}$$

The character e^{ρ} is defined as the projection onto the second component of $\widetilde{\widetilde{H}}$. Then $\mu \otimes e^{\rho}$ makes sense on $\widetilde{\widetilde{H}}$ and equals -1 on $(1,-1)$ and has differential $d\mu + \rho$. The formula in the proposition becomes

$$\mathrm{tr}\pi_\mu(h\gamma) = \frac{\sum_{w\in W} \epsilon_\gamma(w)\mathrm{tr}\ e^{w(\mu+\rho)}(h\gamma)}{\prod_{\beta\in\Delta_\gamma^+}(e^{\beta/2}(h\gamma) - e^{-\beta/2}(h\gamma))}. \tag{V.2.2}$$

Separately, the numerator and denominator only make sense on $\widetilde{\widetilde{H}}$, but the ratio makes sense on $\widetilde{H}$.

V.3. The value at a singular γ can be computed by taking the limit $h \to 1$. In this section we consider the special case when $\gamma = \tau$ is such that $\mathrm{Ad}\tau^2 = Id$ on K. We will assume (as we may) that $\widetilde{K}$ is generated by K and τ. Choose a pair $(\mathfrak{b},\mathfrak{h})$ which is stable under τ. Decompose

$$\Delta^+ = \Delta_{im} \cup \Delta_{cx} = \Delta_c \cup \Delta_{nc} \cup \Delta_{cx}$$

where Δ_{cx} are the roots such that $\tau\alpha \neq \alpha$, Δ_{im} are the roots satisfying $\tau\alpha = \alpha$, Δ_{nc} are the roots so that τ acts by -1 on the root vector and Δ_c are the roots such that τ acts by 1 on the root vectors. In other words, $e^\beta(\tau) = 1$ for $\beta \in \Delta_c$, and $e^\beta(\tau) = -1$ for $\beta \in \Delta_{nc}$. Then $\mathfrak{k}(\tau)$ is spanned by $\mathfrak{h}^\tau$, $\{E_\alpha\}_{\alpha\in\pm\Delta_c}$, and $\{E_\alpha + c_\tau E_{\tau\alpha}\}_{\alpha\in\pm\Delta_{cx}}$. The restrictions of the roots in Δ^+ to $\mathfrak{h}(\tau)$ form a positive system $\Delta(\tau)^+$ of a reduced root system. The Weyl subgroup W_τ of $\widetilde{K}(\tau)$ corresponding to $(\mathfrak{b}(\tau),\mathfrak{h}(\tau))$ is generated by the s_α for $\alpha \in \Delta_c$, $s_\alpha s_{\tau\alpha}$ for $\alpha \in \Delta_{cx}$ such that $\langle\alpha,\tau\alpha\rangle = 0$, and $s_{\alpha+\tau\alpha}$ if $\langle\alpha,\tau\alpha\rangle > 0$, or $s_{\alpha-\tau\alpha}$ if $\langle\alpha,\tau\alpha\rangle < 0$. We write $s_{\alpha,\tau}$ for these reflections. Let $W^+ := W/W_\tau$. Each coset has a unique representative w such that if μ is dominant, then $w\mu$ is dominant for $\Delta(\tau)^+$. Then

Theorem ([Rohlfs-Speh], section 3)**.** *Suppose F is τ-stable finite dimensional with highest weight μ satisfying $\tau\mu = \mu$ and $\langle\mu,\check{\alpha}\rangle \in 2\mathbb{Z}$ for all roots α. Then*

$$\mathrm{tr}F(\tau) \neq 0.$$

Proof. A typical term in the numerator of (V.2.2) is of the form

$$\epsilon_\gamma(xw)e^{xw(\mu+\rho(\tau))}e^{xw\rho - xw\rho(\tau)} \tag{V.3.1}$$

When evaluated at τ,

$$e^{xw\rho - xw\rho(\tau)}(\tau) = \epsilon_\gamma(xw)\epsilon(x). \tag{V.3.2}$$

It follows that

$$\mathrm{tr}F(\tau) = \frac{\sum_{w\in W^+}\mathrm{tr}F_w(\tau)}{\prod_{\beta\in\Delta_{nc}}(e^{\beta/2}(\tau)-e^{-\beta/2}(\tau))} \tag{V.3.3}$$

where F_w is the representation of $\widetilde{K}(\tau)$ with highest weight $w(\mu+\rho(\tau))-\rho(\tau)$. Because of the condition $\langle\mu,\check{\alpha}\rangle\in 2\mathbb{Z}$, $\mathrm{tr}F_w(\tau)=\dim F_w$. We conclude that

$$\mathrm{tr}F(\tau) = \frac{\sum_{w\in W^+}\dim F_w}{2^{|\Delta_{nc}|}}. \tag{V.3.4}$$

VI. **Lefschetz Numbers for Real Groups**

Recall the notation in II.1, $\widetilde{G}=G\ltimes\{1,\tau\}$ for an element τ of finite order and let θ be a Cartan involution which commutes with τ. Then τ stabilizes the maximal compact subgroup $\widetilde{K}$. Its Lie algebra $\mathfrak{k}_0$ has a τ-stable Cartan subalgebra $\mathfrak{t}_0$.

If π is an admissible representation of $\widetilde{G}$, then τ induces an automorphism of $\mathrm{H}^i(\mathfrak{g},K,\pi)$ (see [Knapp-Vogan] chapter II, section 6 for a definition of $(\mathfrak{g},K)$-cohomology). The Lefschetz number of τ, $L(\tau,\pi)$, is by definition the Euler characteristic of the trace of τ on $\mathrm{H}^i(\mathfrak{g},K,\pi)$. More general, let F be a finite dimensional representation of $\widetilde{G}$ whose restriction to G is irreducible. Then we define the Lefschetz number of τ with respect to F and π,

$$L(\tau,F,\pi) := L(\tau,F^*\otimes\pi). \tag{VI.0.5}$$

In this section we determine the Lefschetz numbers of τ.

VI.1. **Cohomology.** Recall that we do not necessarily assume that G is connected, rather that $G=\mathbb{G}(\mathbb{R})$, the real points of a connected reductive group $\mathbb{G}$. To determine the Lefschetz numbers we follow [Labesse1] and [B-L-S]. The module $\bigwedge^*(\mathfrak{g}/\mathfrak{k})\cong\bigwedge^*\mathfrak{s}$ is a representation of $\widetilde{K}$. Write

$$\chi_{\mathfrak{s}}(k\tau) := \chi(k\tau;\sum(-1)^i\Lambda^i\mathfrak{s}) = \det[\mathbb{1}-\mathrm{Ad}(k\tau):\mathfrak{s}]. \tag{VI.1.1}$$

The Lefschetz number is given by the formula

$$L(\tau,\pi) := \sum(-1)^i\mathrm{tr}(\tau,\mathrm{H}^i(\mathfrak{g},K,\pi)) = \int_K \chi_\pi(k\tau)\overline{\chi_{\mathfrak{s}}(k\tau)}\,dk. \tag{VI.1.2}$$

The following is well known, [Labesse1], [Rohlfs-Speh], and [Borel-Wallach].

Proposition. *Assume π is irreducible. Then $L(\tau, F, \pi) = 0$ unless the restriction of π to G is irreducible and the infinitesimal character of π coincides with that of F. It only depends on the left coset of τ under K. In other words if $\beta = k\tau$ with $k \in K$, then $L(\beta) = L(\tau)$.*

□

VI.2. **Standard Representations.** In this section we construct the irreducible representations with nonzero Lefschetz numbers.

Let $P^0 = M^0A^0N^0$ be a minimal parabolic subgroup of G such that $\mathfrak{a}^0 \subset \mathfrak{s}_0$, $M^0 \subset K$. Then $\tau(P^0) = \tau(M^0)\tau(A^0)\tau(N^0)$, and because τ commutes with θ, $\tau\mathfrak{a}^0 \subset \mathfrak{s}_0$, and $\tau M^0 \subset K$.

Let K_0 be the connected component of the identity of K. Since any two minimal parabolic subgroups are conjugate by K_0, there is an element $k_0 \in K_0$ such that $k_0\tau$ fixes M^0, A^0 and N^0. We will show that in fact there is a minimal parabolic subgroup P^0 which is stable under τ.

Lemma. *Let τ_0 be an isomorphism of G which commutes with the Cartan involution θ. Assume P^0 is a minimal τ_0-stable parabolic subgroup. Then the map*

$$\alpha : K_0 \times M^0 \longrightarrow K_0, \qquad \alpha(k, m) = km\tau_0(k^{-1})$$

is onto.

Proof. Because K_0 and M^0 are compact, the image of α is closed. The set of (k_0, m_0) for which $d\alpha$ is *not* onto is given by algebraic equations with real coefficients. Thus its complement is either empty or Zarisky dense. It is enough to show it is not empty. We show that there is a point (k_0, m_0) such that $d\alpha_{k_0,m_0}$ is onto. Take $k_0 = Id$. Then

$$\exp X m_0 \exp \tau_0(-X) = m_0 \exp(\mathrm{Ad}(m_0)^{-1}X) \exp -\tau_0(X),$$

and so the differential equals

$$\begin{aligned} d\alpha_{k_0,m_0}(X, Y) &= Y + \mathrm{Ad}(m_0)^{-1}X - \tau_0(X) \\ &= \mathrm{Ad}(m_0)^{-1}[Y + X - \mathrm{Ad}(m_0)\tau_0(X).] \end{aligned}$$

It is therefore enough to show that the map

$$\alpha' : X \mapsto X - \mathrm{Ad}(m_0)\tau_0(X) \tag{VI.2.1}$$

is onto the complement of $\mathfrak{m}_0$. Let $\Delta_0 := \Delta(\mathfrak{g}, \mathfrak{a}^0)$ be the restricted roots, and $\{X_\alpha\}_{\alpha\in\Delta_0}$ a basis of root vectors. Then the complement of $\mathfrak{m}_0$ in $\mathfrak{k}$ has as basis $\{X_\alpha + \theta X_\alpha\}_{\alpha\in\Delta_0}$. The claim that α' in (VI.2.1) is onto follows from the fact that we can choose m_0 such that $Id - \mathrm{Ad}(m_0)\tau_0$ has no fixed points on the aforementioned basis. □

Proposition (1)**.** *There exists a minimal parabolic* $P^0 = M^0A^0N^0$ *satisfying* $\theta(\mathfrak{a}^0) = \mathfrak{a}^0, M^0 \subset K$ *and* $\tau P^0 = P^0$.

Proof. Since P^0 and τP^0 are both minimal parabolic subgroups, there is $k \in K_0$ such that $\mathrm{Ad}(k)\tau P^0 = P^0$. Then $\tau' := \mathrm{Ad}k \circ \tau$ does fix P^0, so we can apply lemma VI.2 whose proof applies to any isomorphism τ_0 not just ones of finite order. We conclude that $k^{-1} = xmk\tau(x^{-1})k^{-1}$ or $k = mx^{-1}\tau(x)$. Then $\mathrm{Ad}(x)P^0$ is $\tau-$stable. □

We fix a minimal $\tau-$stable parabolic subgroup P^0 with properties as in corollary (1). Since N^0 is the radical of P^0, we conclude $\tau N^0 = N^0$ as well.

Let now P be a standard parabolic subgroup, *i.e.* $P \supset P^0$. Then $\tau(P)$ is also a standard parabolic subgroup. If it has decomposition $P = MAN$, and it is τ-stable, then $\tau M = M,\ \tau A = A$ and $\tau N = N$.

Proposition (2)**.** *If* P *and* τP *are conjugate under* K_0*, then there is a conjugate of* P *which is* τ*-stable.*

Proof. The analogue of lemma VI.2 holds by essentially the same proof, and then the reasoning in the proof of corollary 1 in VI.2 applies. □

Remark. The arguments in lemma VI.2 and proposition (1) and (2) are adapted from [Steinberg1]. Note however that in proposition (2) we do need the assumption that P and τP are conjugate under the *connected* group K_0. To see that this is necessary, consider the example $G = GL(3)$ with $\tau(x) = {}^tx^{-1}$, and P a proper maximal parabolic subgroup.

Since $G = G(\mathbb{R})$ we do not assume in this paper that K is connected and so special care is needed in the arguments starting with the notion of $\tau-$stable data in definition VI.2. □

Let $X(P, \mathcal{W}, \nu)$ be a standard module for G, where $P = MAN$ is a standard cuspidal parabolic subgroup (*i.e.* $P^0 \subset P$), $\mathcal{W}$ is a tempered irreducible module for M and $< Re\ \nu, \alpha >> 0$ for all $\alpha \in \Delta(\mathfrak{n})$. We will always denote by $L(P, \mathcal{W}, \nu)$ the irreducible Langlands quotient of $X(P, \mathcal{W}, \nu)$.

Suppose an irreducible module π of $\widetilde{G}$ has nonzero Lefschetz number. By proposition VI.1 its restriction to G is irreducible; it is of the form $L(P, \mathcal{W}, \nu)$. It must be isomorphic to $L(\tau P, \tau\mathcal{W}, \tau\nu)$. The modules $L(P, \mathcal{W}, \nu)$ and $L(\tau P, \tau\mathcal{W}, \tau\nu)$ are equivalent if and only if there exists

$k \in K$ such that

$$k\tau M = M, \quad k\tau A = A, \quad k\tau N = N, \quad k\tau \mathcal{W} \cong \mathcal{W}, \quad k\tau\nu = \nu. \tag{VI.2.2}$$

Definition. *We say that the data $(P, \mathcal{W}, \nu)$ are τ-stable, if they satisfy equation VI.2.2.*

It follows that there is an intertwining operator $a_\tau : \mathcal{W} \longrightarrow \mathcal{W}$, satisfying

$$a_\tau \pi_{\mathcal{W}}(m) = \pi_{\mathcal{W}}(k\tau(m))a_\tau. \tag{VI.2.3}$$

Two choices of a_τ differ by a scalar multiple. Then a_τ induces an intertwining operator

$$A_\tau : X(P, \mathcal{W}, \nu) \to X(\tau P, \tau\mathcal{W}, \tau\nu)$$

by the formula

$$A_\tau(f)(g) = a_\tau f((k\tau)^{-1}(g)), \quad \text{for} \quad f(gm) = \pi_{\mathcal{W}}(m^{-1})f(g), \tag{VI.2.4}$$

satisfying $A_\tau^d = Id$ (because a_τ does). Since ν is strictly dominant for $\Delta(\mathfrak{n})$, $\tau\nu$ is strictly dominant for $\tau\Delta(\mathfrak{n})$. Thus $L(P, \mathcal{W}, \nu)$ is the unique irreducible quotient of $X(P, \mathcal{W}, \nu)$, and $L(\tau P, \tau\mathcal{W}, \tau\nu)$ is the unique irreducible quotient of $X(\tau P, \tau\mathcal{W}, \tau\nu)$. Thus A_τ induces a nonzero operator $\mathcal{A}_\tau$ from $L(P, \mathcal{W}, \nu)$ to $L(\tau P, \tau\mathcal{W}, \tau\nu)$, and both are equivalent to π. Any two such operators are scalar multiples of each other. We normalize A_τ so that $\mathcal{A}_\tau$ coincides with $\pi(k\tau)$. Thus the action of G on $X(P, \mathcal{W}, \nu)$ extends to $\widetilde{G}$, in such a way that π is its unique irreducible quotient.

Theorem. *Suppose the data $(P, \mathcal{W}, \nu)$ are τ-stable so that $X(P, \mathcal{W}, \nu)$ has an action of $\widetilde{G}$. If dim $A \geq 1$, then*

$$L(\tau, X(P, \mathcal{W}, \nu)) = 0.$$

Proof. We can replace τ by $k\tau$ and apply formula (VI.1.2). By Frobenius reciprocity, χ_π is supported on M. Thus we need to calculate $\chi_{\mathfrak{s}}(m\tau)$ with $m \in M \cap K$. Because $m\tau$ has fixed points (namely ν) on $\mathfrak{a}$, formula (VI.1.1) equals 0. See also [B-L-S]. □

Thus if π satisfies $L(\tau, \pi) \neq 0$, then π is a tempered representation.

VI.3. **Tempered Representations.** According to the refinements of the Langlands classification due to Knapp and Zuckermann an irreducible tempered representation of G is of the form

$$\mathcal{W} = Ind_P^G[\mathcal{W}_0 \otimes \mathbb{C}_{i\nu}]$$

with $\mathcal{W}_0$ a *Discrete Series* or *Limit of Discrete Series* representation, and $< \nu, \alpha >> 0$ for all $\alpha \in \Delta(\mathfrak{n})$ [KnZ] .

Suppose that the tempered representation $\mathcal{W}$ is τ-stable and that $H^*(\mathfrak{g}, K, F^* \otimes \mathcal{W}) \neq 0$ for some finite dimensional F. Because the infinitesimal character of $\mathcal{W}$ coincides with that of a finite dimensional representation, $\nu = 0$ and $\mathcal{W}$ is associated to a pair $(\mathfrak{b}, \mathfrak{h})$ where $\mathfrak{b}$ is a θ-stable Borel subalgebra and $\mathfrak{h} \subset \mathfrak{b}$ a θ-stable *Cartan subalgebra*. In particular, $\mathfrak{h}$ is a fundamental *Cartan subalgebra*.

Now let H be the stabilizer in G of a θ-stable pair $(\mathfrak{b}, \mathfrak{h})$; then $H = H_I \cdot H_R$ is a θ-stable fundamental *Cartan subgroup*. Let $\mathfrak{h}_I$ be the Lie algebra of $H_I := H \cap K$, and $\mathfrak{h}_R := \mathfrak{h} \cap \mathfrak{s}$. Since G is the real points of a connected reductive linear algebraic group, H is abelian. Write $s = \dim(\mathfrak{n} \cap \mathfrak{k})$, $r = \dim(\mathfrak{n} \cap \mathfrak{s})$ and let $\mathfrak{h}_R$ be the complexification of the Lie algebra of H_R. Then

$$\mathcal{W} = \mathcal{R}^s_{\mathfrak{b}}(\chi) \tag{VI.3.1}$$

where $\chi \in \widehat{H}_I$ is a character such that $d\chi + \rho$ is dominant for $\mathfrak{b}$. (See chapter V in [Knapp-Vogan] for the definition of the functor $\mathcal{R}^s_{\mathfrak{b}}$.) We will write $\mathcal{R}_{\widetilde{G},\mathfrak{b}}(\chi)$ or $\mathcal{R}_{G,\mathfrak{b}}(\chi)$ or $\mathcal{R}_{G_0,\mathfrak{b}}(\chi)$ when we need to emphasize whether the derived functor module is a $(\mathfrak{g}, \widetilde{K})$, or $(\mathfrak{g}, K)$ or $(\mathfrak{g}, K_0)$ module. The $(\mathfrak{g}, K)$-module $\mathcal{W}$ is τ-stable if and only if there is $k \in K$ such that $\gamma := k\tau$ stabilizes the data $(\mathfrak{b}, \mathfrak{h}, \chi)$. By proposition VI.1 when we compute Lefschetz numbers we can replace τ by $\gamma = k\tau$ and thus the data $(\mathfrak{b}, \mathfrak{h}, \chi)$ is $\gamma-$stable.

Theorem. *Let $\mathcal{W} = \mathcal{R}^s_{G,\mathfrak{b}}(\chi)$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be an irreducible tempered τ-stable $(\mathfrak{g}, K)-$module. Let γ be as before. Assume that F is an irreducible finite dimensional γ-stable $(\mathfrak{g}, K)-$module and that $F^{\mathfrak{n}} = \chi$ as an $\mathfrak{h}$–module. The Lefschetz number $L(\tau, F^* \otimes \mathcal{W})$ equals*

$$(-1)^r \sum (-1)^i \text{ tr}(\ \gamma \ : \ \bigwedge^i \mathfrak{h}^*_R).$$

It is zero if and only if γ has fixed points in $\mathfrak{h}^*_R$.

Proof. Denote by $\mathbb{C}$ the trivial representation of H. Note that

$$\mathrm{H}^i(\mathfrak{g}, K, F^* \otimes \mathcal{R}^s_{\mathfrak{b}}(\chi)) \cong \mathrm{Ext}^i_{\mathfrak{g},K}[F, \mathcal{R}^s_{\mathfrak{b}}(\chi)]. \tag{VI.3.2}$$

Corollary 5.121 in [Knapp-Vogan] applies, and there is a first quadrant spectral sequence

$$E_r^{p,q} \Longrightarrow \mathrm{Ext}^{p+q-s}_{\mathfrak{g},K}[F, \mathcal{R}^s_{\mathfrak{b}}(\chi)] \tag{VI.3.3}$$

with differential of bidegree $(r, 1-r)$ and with E_2 term

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathfrak{h},H_I}[H_q(\mathfrak{n}, F), \chi]$$

The E_2 term is nonzero only for $q = \dim \mathfrak{n}$. The conclusion is

$$\mathrm{H}^i(\mathfrak{g}, K, F^* \otimes \mathcal{R}^s_{\mathfrak{b}}(\chi)) \cong \mathrm{H}^{i-r}[\mathfrak{h}, H_I, \mathbb{C}]. \tag{VI.3.4}$$

In view of this, the Lefschetz number is

$$\begin{aligned} L(\tau, F^* \otimes \mathcal{W}) &= \sum (-1)^i \mathrm{tr}(\gamma : \mathrm{H}^{i-r}[\mathfrak{h}, H_I, \mathbb{C}]) = \\ &= (-1)^r \sum (-1)^i \mathrm{tr}(\gamma, \textstyle\bigwedge^i \mathfrak{h}_R). \end{aligned} \tag{VI.3.5}$$

Finally if $\zeta_1, \dots, \zeta_l$ are the eigenvalues of γ on $\mathrm{Hom}_{H_I}[\mathfrak{h}_R, \mathbb{C}]$ (with multiplicities), then

$$\sum (-1)^i tr[\ \gamma\ :\ \textstyle\bigwedge^i \mathfrak{h}^*_R] = \prod (1 - \zeta_j), \tag{VI.3.6}$$

which is zero if and only if one of the ζ_j equals 1. □

Example. Consider the case $G = GL(n)$ with the standard τ, transpose inverse. Then $\lambda = (\lambda_1, \dots, \lambda_n)$ is the highest weight of a self dual finite dimensional representation F if and only if $\lambda_i = -\lambda_{n+1-i}$. There is exactly one irreducible tempered $(\mathfrak{g}, K)$–module $\mathcal{W}$ with nontrivial $(\mathfrak{g}, K)$–cohomology with the same infinitesimal character as F [Speh]. If $n = 2m$, the Lefschetz number $L(\tau, F^* \otimes \mathcal{W})$ is $(-1)^m 2^m$, if $n = 2m + 1$, then it is equal to $(-1)^{m+1} 2^{m+1}$.

VII. **Lefschetz functions in the real case**

By [Bouaziz], the distribution character of a $(\mathfrak{g}, \widetilde{K})$ module π is given by a function Θ_π which is analytic on the set of regular semisimple elements in G^*. We want to compute Θ_π on the regular elliptic set $H_{reg}\gamma$ for the tempered representations $\mathcal{W} = \mathcal{R}^s_{\mathfrak{b}}(\chi)$ considered in section VI.3 This is known [Bouaziz], but we sketch a different treatment here based on derived functors.

VII.1. Let π be an admissible $(\mathfrak{g}, \widetilde{K})$ module. The formal sum

$$\Theta_{\pi, \widetilde{K}} = \sum_{\mu \in \widehat{\widetilde{K}}} m[V_\mu, \pi] V_\mu. \tag{VII.1.1}$$

is a distribution on $\widetilde{K}$ in the sense that we can replace V_μ by its character and evaluate on $\widetilde{K}$-finite functions.

For a vector space V, define a formal sum

$$e(V) := \sum (-1)^i \textstyle\bigwedge^i V. \tag{VII.1.2}$$

If V is a representation of some group, view this sum in the corresponding Grothendieck group as a formal sum of characters.

VII.2. Recall $\mathcal{W} = \mathcal{R}^s_{G,\mathfrak{b}}(\chi)$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, $s = \dim(\mathfrak{n} \cap \mathfrak{k})$, $r = \dim(\mathfrak{n} \cap \mathfrak{s})$. We assume that $\mathcal{W}$ is τ-stable, and choose H and γ as in (VI.3.1) *i.e.* so that $(\mathfrak{b}, \mathfrak{h}, \chi)$ is $\gamma-$stable. We lift χ to a double cover as in section (V.2) and tensor with e^ρ. Since the group H_I is the stabilizer of $(\mathfrak{b}, \mathfrak{h})$ in K, it is a normal subgroup of the stabilizer H_K in K of $(\mathfrak{b} \cap \mathfrak{k}, \mathfrak{h} \cap \mathfrak{k})$. So we have $H_0 \subset H_I \subset H_K$, where H_0 is the connected component of the identity in H_I, same as the connected component of the identity in H_K. Let $\mathfrak{b}_j$ the conjugates of $\mathfrak{b}$ under H_K. Then the restriction of $\mathcal{R}_{G,\mathfrak{b}}(\chi)$ to G_0 is a direct sum of $\mathcal{R}^s_{G_0,\mathfrak{b}_j}(\chi_j)$ one for each $\mathfrak{b}_j$. Suppose $\gamma \in \widetilde{H}_K$. Then γ permutes the $\mathfrak{b}_j$, and in particular fixes $\mathfrak{b}$. Let T be the fixed points of γ in H_K. Then $t\gamma$ permutes the $\mathfrak{b}_j$. If it does not fix any $\mathfrak{b}_j$, then $\mathrm{tr}\mathcal{W}(t\gamma) = 0$. Thus we only need to compute $\mathrm{tr}\mathcal{W}(t\gamma)$ for $t\gamma$ which fix a $\mathfrak{b}_j$. Then $t\gamma$ is conjugate to an element in $\widetilde{H}_I$. Thus we only need to compute the character for elements in $\widetilde{H}_I$.

Remark. The results and proofs in section III.1 are for the case of an algebraically closed field, but they also hold for compact connected groups. But since we do not assume that K is connected, it is not necessarily true that $H_K = H^\perp \cdot T$. □

Let Δ^+_γ be the roots in $\mathfrak{b}$ which are not 1 on γ. Similarly $\Delta_\gamma(\mathfrak{s})^+$ is the set of roots in $\mathfrak{b} \cap \mathfrak{s}$ which are not 1 on γ. The $\tilde{K}$ character $e(V)$ for $V = \mathfrak{s}$ equals

$$e(\mathfrak{s})(h\gamma) = e(\mathfrak{h}_R)(\gamma) \prod_{\beta \in \Delta_\gamma(\mathfrak{s})^+} (1 - e^{-\beta}(h\gamma))(1 - e^{\beta}(h\gamma)). \qquad \text{(VII.2.1)}$$

Assume that $\mathrm{rk}\,\mathfrak{g} = \mathrm{rk}\,\mathfrak{k}$. Then the spin representation decomposes into a sum of two representations denoted $S^\pm$. They extend to $\widetilde{K}$. The formula

$$\begin{aligned} (\mathrm{tr}S^+ - \mathrm{tr}S^-)(h\gamma) &= \prod_{\beta \in \Delta_\gamma(\mathfrak{s})^+} (e^{\beta/2} - e^{-\beta/2})(h\gamma) = & \text{(VII.2.2)} \\ &= (-1)^r e^{-\rho(\mathfrak{n}) + \rho(\mathfrak{n} \cap \mathfrak{k})} e(\mathfrak{n} \cap \mathfrak{s})(h\gamma) & \text{(VII.2.3)} \end{aligned}$$

holds. When $\mathrm{rk}\,\mathfrak{k} < \mathrm{rk}\,\mathfrak{g}$, the expression

$$e(S)^2 = \prod_{\beta \in \Delta_\gamma(\mathfrak{s})^+} (e^{\beta/2} - e^{-\beta/2})^2 \qquad \text{(VII.2.4)}$$

is a virtual character of $\widetilde{K}$.

Proposition (1)**.** *Assume* $\mathrm{rk}\,\mathfrak{k} = \mathrm{rk}\,\mathfrak{g}$, *and let* $\mathcal{W} = \mathcal{R}^s_{\widetilde{G},\mathfrak{b}}(\chi)$ *with* $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ *and* $s = \dim(\mathfrak{n} \cap \mathfrak{k})$. *The formal combination* $(S^+ - S^-) \otimes \Theta_{\mathcal{W},\tilde{K}}$

is a finite linear combination of irreducible $\widetilde{K}$ representations. It equals the irreducible module with highest weight $\chi \otimes e^{\rho(\mathfrak{n})}$.

Proof. We use the notation and results in [Knapp-Vogan] chapter V. Write $\chi^{\#}$ for the representation $\chi \otimes \wedge^r(\mathfrak{n})$. If we denote by W an arbitrary $\widetilde{K}$ module, then

$$(-1)^r \dim \mathrm{Hom}_{\widetilde{K}}(W, \mathcal{R}^s(\chi)) = \\ = \sum_{j=0}^{s} (-1)^j \sum_{n=0}^{\infty} \dim \mathrm{Hom}_{\widetilde{H}_I}(\mathrm{H}_j(\mathfrak{n} \cap \mathfrak{k}, W), S^n(\mathfrak{n} \cap \mathfrak{s}) \otimes_{\mathbb{C}} \chi^{\#}). \tag{VII.2.5}$$

Tensoring $\mathcal{R}^s(\chi)$ with $(S^+ - S^-)$ in (VII.2.5), and using the formula

$$\sum S^n(\mathfrak{n} \cap \mathfrak{s}) \cdot e(\mathfrak{n} \cap \mathfrak{s}) = 1,$$

we get

$$\dim \mathrm{Hom}_{\widetilde{K}}(W, \mathcal{R}^s(\chi) \otimes (S^+ - S^-)) = \\ = \sum_{j=0}^{s} (-1)^j \dim \mathrm{Hom}_{\widetilde{H}_I}(\mathrm{H}_j(\mathfrak{n} \cap \mathfrak{k}, W), \chi^{\#} \otimes e^{-\rho(\mathfrak{n}) + \rho(\mathfrak{n} \cap \mathfrak{k})}). \tag{VII.2.6}$$

Recall that $\chi^{\#} = \chi \cdot e^{2\rho(\mathfrak{n})}$, and the weights of $H_j(\mathfrak{n} \cap \mathfrak{k}, W)$ are of the form $w(w_0\mu - \rho(\mathfrak{n} \cap \mathfrak{k})) + \rho(\mathfrak{n} \cap \mathfrak{k})$ with w_0 the longest element in W_K. We get the equation

$$w(w_0 d\mu - \rho(\mathfrak{n} \cap \mathfrak{k})) + \rho(\mathfrak{n} \cap \mathfrak{k}) = d\chi + \rho(\mathfrak{n}) + \rho(\mathfrak{n} \cap \mathfrak{k}). \tag{VII.2.7}$$

Since $d\chi + \rho(\mathfrak{n})$ is dominant, it follows that $w = 1$, and

$$\mu \cdot e^{\rho(\mathfrak{n} \cap \mathfrak{k})} = \chi \cdot e^{\rho(\mathfrak{n})}.$$

□

Now assume that $\mathrm{rk}\, \mathfrak{k} < \mathrm{rk}\, \mathfrak{g}$.

Proposition (2)**.** *The formal combination $e(S)^2 \otimes \Theta_{\widetilde{K}}$ is a finite linear combination of irreducible representations of $\widetilde{K}$. Assume $d\chi + \rho(\mathfrak{n})$ is very dominant. For a subset $B \subset \mathfrak{n} \cap \mathfrak{s}$, let $< B >$ be the sum of roots in B, and denote by $V(\chi \cdot e^{-<B>})$ the finite dimensional module of $\widetilde{K}$ with this highest weight. Then*

$$e(S)^2 \otimes \Theta_{\widetilde{K}} = \sum_{i, |B| = i} (-1)^i V(\chi \cdot e^{-<B>}).$$

Proof. We tensor $\mathcal{R}^s(\chi)$ with $e(S)^2$ as in the proof of VII.2:

$$\dim \mathrm{Hom}_{\widetilde{K}}(W, \mathcal{R}^s(\chi) \otimes e(S)^2) =$$
$$= \sum_{j=0}^{s} (-1)^j \dim \mathrm{Hom}_{\widetilde{H}_I}(\mathrm{H}_j(\mathfrak{n} \cap \mathfrak{k}, W), \chi^{\#} \otimes e^{-\rho(\mathfrak{n})+\rho(\mathfrak{n}\cap\mathfrak{k})} \otimes e(\overline{\mathfrak{n}} \cap \mathfrak{s})). \tag{VII.2.8}$$

Because $d\chi+\rho(\mathfrak{n})$ is very dominant, $d\chi+\rho(\mathfrak{n})- < B >$ is dominant, and the reasoning in the proof of proposition VII.2 after (VII.2.7) applies. The claim follows. □

The distribution character of $\mathcal{W}$ denoted $\Theta_{\mathcal{W}}$ is given by integration against a locally L^1 analytic function on the regular set [Bouaziz].

Let $\epsilon(w)$ for $w \in W_K$ be defined by

$$\prod_{\beta \in \Delta_\gamma(\mathfrak{s})^+} (e^{w\beta/2} - e^{-w\beta/2}) = \epsilon(w) \prod_{\beta \in \Delta_\gamma(\mathfrak{s})^+} (e^{\beta/2} - e^{-\beta/2}). \tag{VII.2.9}$$

Theorem. *Let $(\mathfrak{b}, \mathfrak{h}, \chi)$ be τ-stable data for a tempered module $\mathcal{W} = \mathcal{R}^r_{\mathfrak{b}}(\chi)$. Then*

$$\Theta_{\mathcal{W}}(h\gamma) = (-1)^r \frac{\sum_{w \in W_K} \epsilon(w) w\chi(h\gamma) e^{w(\rho)-\rho}(h\gamma)}{\prod_{\beta \in \Delta_\gamma}(1 - e^{-\beta}(h\gamma))}$$

for any $h\gamma \in \widetilde{H}_{reg}$ stabilizing $\mathfrak{b}$.

Proof. By results of Harish-Chandra, [Harish-Chandra1] sections 11 and 12, the distribution $\Theta_{\mathcal{W},\widetilde{K}}$ coincides with $\Theta_{\mathcal{W}}$ when restricted to the regular set intersected with $\widetilde{G}_{reg}$. The formula now follows from proposition (1) for $\mathrm{rk}\,\mathfrak{k} = \mathrm{rk}\,\mathfrak{g}$. If $\mathrm{rk}\,\mathfrak{k} \neq \mathrm{rk}\,\mathfrak{g}$ and $d\chi + \rho(\mathfrak{n})$ is very dominant, it follows from proposition (2) of VII.2 and in general by using translation functors as given in chapter VII of [Knapp-Vogan]. □

As in section V.2 we can twist χ by e^ρ and change the definition of $\mathcal{R}$ accordingly. We refer to [Knapp-Vogan] for details. The formula is rewritten as

$$\Theta_{\mathcal{W}}(h\gamma) = (-1)^r \frac{\sum_{w \in W_K} \epsilon(w) w\chi(h\gamma) e^{w\rho}(h\gamma)}{\prod_{\beta \in \Delta_\gamma(\mathfrak{s})}(e^{\beta/2}(h\gamma) - e^{-\beta/2}(h\gamma))} \tag{VII.2.10}$$

for any $h\gamma \in \widetilde{H}_{I,reg}$ stabilizing $\mathfrak{b}$.

Corollary. *Let F be finite dimensional τ invariant irreducible representation and $\gamma \in \widetilde{H}_K$. Then*

$$\Theta_F(h\gamma) = \sum \Theta_{\mathcal{W}}(h\gamma)$$

where the sum is over all the $\mathcal{W}$ corresponding to the $\widetilde{H}_K$ conjugacy classes of γ-stable $(\mathfrak{b}, \mathfrak{h}, \chi)$.

VII.3. Assume γ is arbitrary semisimple. Recall that $\gamma = \gamma_{ell} e^Y$ with $Y \in \mathfrak{g}(\gamma)$ hyperbolic. We can conjugate γ so that Y is in a θ-stable *Cartan subalgebra* $\mathfrak{h} = \mathfrak{h}_I + \mathfrak{h}_R$, in fact $Y \in \mathfrak{h}_R$. Let $P = MN$ be the parabolic subgroup defined by Y; the roots $\Delta(\mathfrak{m})$ are the ones that are zero on Y, the roots $\Delta(\mathfrak{n})$ are the ones that are positive on Y. Then $G(\gamma) \subset M$.

If $f \in C_c^\infty(G\tau)$ and γ normalizes M, then define

$$f^P(m\gamma) = \delta(m\gamma)^{1/2} \int_K \int_N f(km\gamma nk^{-1}) dn. \tag{VII.3.1}$$

Then

$$F_f^G(\gamma) = F_{f^P}^M(\gamma). \tag{VII.3.2}$$

is well defined.

In [B-L-S], a function $f_F \in C_c^\infty(G\tau)$ is defined which has the property that

(1) $f_F(kxk^{-1}) = f_F(x)$,
(2) $f_F^P = 0$ for P a real parabolic whose conjugacy class is stable under τ (this means P and $\tau(P)$ are conjugate under G),
(3) $\Theta_\pi(f_F) = L(\tau, \pi \otimes F)$.

We refer to f_F as the *Lefschetz function for F, τ.*

For the next results, keep in mind also that orbits of semisimple elements are closed.

Theorem. *Let f_F be a Lefschetz function for F, τ. Suppose that γ has nontrivial hyperbolic part. Then*

$$F_{f_F}^G(\gamma) = 0.$$

Proof. Apply formula (VII.3.2). □

VII.4. In this section we compute the orbital integrals on Lefschetz functions. For general results see [Renard1] and [Renard2].

We use the conventions and notation of section IV. Assume that $\gamma = k\tau$ with $k \in K$ is compact, and let $\mathfrak{t}$ be a fundamental *Cartan subalgebra* in $\mathfrak{g}(\gamma)$. The centralizer of $\mathfrak{t}$ is a fundamental *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$. Fix $(\mathfrak{b}, \mathfrak{h})$ a γ-stable pair, $H = H_I \cdot H_R$ the corresponding *Cartan subgroup* in G and $\widetilde{H}$ the *Cartan subgroup* in $\widetilde{G}$. If $\mathfrak{b}'$ is another γ-stable Borel subgroup, let $\epsilon(\mathfrak{b}') := (-1)^{\dim[\mathfrak{b}'/(\mathfrak{b} \cap \mathfrak{b}')]}$.

Theorem. *Suppose $f \in C_c^\infty(G^*)$ is such that $f^P = 0$ for all P. Then there is a constant $c(\gamma)$ depending on the Haar measures on G and $G(\gamma)$ such that*

$$F_f(\gamma) = c(\gamma) \sum |W(\chi)|^{-1} \overline{\chi(\gamma)} \sum \epsilon(\mathfrak{b}') \Theta_{\mathcal{W}(\mathfrak{b}',\chi)}(f).$$

The first sum is over χ such that $d\chi$ is dominant for $\mathfrak{b} \cap \mathfrak{k}$ and the second sum over $\mathfrak{b}' \supset \mathfrak{b} \cap \mathfrak{k}$.

Proof. The idea of the proof originates in the work of Sally, Warner and Herb.

Suppose f is $C_c^\infty(G^*)$ supported on the regular elliptic set. Then the function $F_f(h\gamma)$ is a well defined function ϕ which is C_c^∞ on $\widetilde{H}_{I,reg}$. Its Fourier transform is

$$\widehat{\phi}(\chi) = \int_{\widetilde{H}_I} \chi(x)\phi(x)\ dx. \tag{VII.4.1}$$

Assume for the moment that the support of f is contained in $\mathrm{Ad}G(T\gamma)$. Since ϕ is invariant (*i.e.* we assume as we may by averaging that $f(\mathrm{Ad}\gamma(x)) = f(x)$ for $x \in G^*$), Fourier inversion gives

$$\phi(h\gamma) = \sum \mathrm{tr}\overline{\chi(h\gamma)}\ \widehat{\phi}(\chi). \tag{VII.4.2}$$

If the restriction of χ to H is not irreducible, then $\mathrm{tr}\chi(h\gamma) = 0$. In other words we may assume that χ is γ-stable so 1-dimensional and so we can suppress tr from the notation.

We now compute $\widehat{\phi}$. On the one hand, because $F_f(ht\gamma h^{-1}) = F_f(t\gamma)$, we have

$$\widehat{\phi}(\chi) = \mathrm{vol}(H_I/T) \int_T |e(\mathfrak{h}/\mathfrak{t})(t\gamma)| \theta_\chi(t\gamma) F_f(t\gamma)\ dt \tag{VII.4.3}$$

where $e(\mathfrak{h}/\mathfrak{t})$ is defined in (VII.1.2), and

$$\theta_\chi(t\gamma) = \sum_{w \in W} \epsilon(w)\ w\chi(t\gamma). \tag{VII.4.4}$$

On the other hand, for the tempered module corresponding to $\mathfrak{b}'$ and $\chi - \rho$, we can group the terms in the sum in (VII.4.2) according to the χ such that $d\chi$ is dominant for $\mathfrak{b} \cap \mathfrak{k}$. Fix a γ-stable $\mathfrak{b}'$ which is dominant for $d\chi$. Then

$$\Theta_{\mathcal{W}(\mathfrak{b}',\chi)}(f) = \epsilon(\mathfrak{b}')\mathrm{vol}(H_I/T) \int_T |e(\mathfrak{h}/\mathfrak{t})(t\gamma)|\ \theta_\chi(t\gamma) F_f(t\gamma) + \tag{VII.4.5}$$

$+$ (integrals of f coming from Cartan subgroups of higher real rank).

So (VII.4.3) is equal to the first term of (VII.4.5). By the continuity of the F_f, the equality holds for all C_c^∞ functions. In particular for a cuspidal function f_F the integrals coming from the more split Cartan subgroups vanish and we get the claimed formula. □

VII.5. Fix a Haar measure on G. There is a canonical normalization of measures on the $G(\gamma)$, namely the ones where $c(\gamma) = 1$. Equivalently, when $G(\gamma)$ is elliptic this measure is the one so that the formal dimension of the discrete series with infinitesimal character equal to the one of the trivial representation, is 1. These choices induce invariant measures on the elliptic orbits. With this normalization, the formulas in the previous sections simplify so that there are no $c(\gamma)$. Furthermore note that r coincides with the number

$$q(\gamma) = \frac{1}{2}(\dim \mathfrak{g}(\gamma) - \dim \mathfrak{k}(\gamma))$$

associated to a real form of $G(\gamma)$ by Kottwitz, so that

$$(-1)^r = (-1)^{q(\gamma)}.$$

Theorem. *(1) Let f_F be the Lefschetz function corresponding to a τ-stable finite dimensional representation F and suppose that γ is elliptic. With the normalizations above,*

$$O_\gamma(f_F) = (-1)^{q(\gamma)} e(\tau) \mathrm{tr} F^*(\gamma)$$

where $e(\tau) = \sum_i (-1)^i \mathrm{tr}(\tau \ : \ \bigwedge^i \mathfrak{h}_R^*)$ as in proposition VI.3. In particular $O_\gamma(f_F) = 0$ unless $\mathfrak{g}(\tau)$ is equal rank.

Proof. The formula follows from the above discussion and the fact that the Lefschetz number is independent of the choice of γ. □

Theorem. *(2) Fix an elliptic γ. The stable combination of orbital integrals associated to γ satisfies*

$$\sum (-1)^{q(\gamma')} O_{\gamma'}(f_F) = e(\tau) |\ker[\mathrm{H}^1(\Gamma, I(\gamma)) \longrightarrow \mathrm{H}^1(\Gamma, G)]| \mathrm{tr} F^*(\gamma).$$

The sum on the left is over the stable conjugacy class of γ.

Proof. If γ and γ' are elliptic stably conjugate (definition VIII.3), then $\mathrm{tr} F^*(\gamma) = \mathrm{tr} F^*(\gamma')$. The proof follows from the fact that

$$|\ker[\mathrm{H}^1(\Gamma, I(\gamma)) \longrightarrow \mathrm{H}^1(\Gamma, G)]|$$

is the number of stable conjugacy classes, (see proposition III.6. □

VII.6. Suppose that γ stabilizes $(\mathfrak{b}, \mathfrak{h}, \chi)$ with $\mathfrak{h}$ fundamental as before. If γ' stabilize the data as well, then $\gamma'\gamma^{-1}$ is in the Cartan subgroup attached to $(\mathfrak{b}, \mathfrak{h})$ which is abelian. Thus $e(\gamma) = e(\gamma')$.

Now consider the restriction of $\mathcal{R}_{\mathfrak{b}}(\chi)$ to G_0 :

$$\mathcal{R}_{\mathfrak{b}}(\chi) = \sum \mathcal{R}_{\mathfrak{b}_i}(\chi_i). \tag{VII.6.1}$$

If none of the modules on the right are stabilized by τ, (with an element $k_0\tau$ with $k_0 \in K_0$) then the Lefschetz number is zero.

So let $(\mathfrak{b}_i, \mathfrak{h}, \chi_i)$ be $\tau-$stable data of a summand in VII.6.1. We can use it instead of the original $(\mathfrak{b}, \mathfrak{h}, \chi)$. Thus we can assume that $\gamma \in G_0\tau$. Recalling the assumption that τ itself is elliptic, corollary III.5 shows that γ is conjugate to $h\tau$ and we can assume h is in the Cartan subgroup in K_0. It follows that $e(\gamma) = e(\tau)$ because H is abelian.

VIII. **Lefschetz functions in the p-adic case**

Recall the *twisted* orbital integral of a function $f \in C_c^\infty(G\tau)$,

$$O_\gamma(f) := \int_{G(\gamma)\backslash G} f(g^{-1}\gamma g)\, dg.$$

In this section we compute the orbital integrals for Lefschetz functions in the $p-$adic case. The results and techniques are essentially in [Kottwitz1]. There are minor modifications due to the fact that G is reductive and possibly disconnected rather than semisimple. The definition of the Lefschetz function $f_{\mathcal{L}}$ follows [B-L-S].

VIII.1. In this section $\mathbb{G}$ is a linear algebraic reductive group, and τ an automorphism of finite order, both defined over a nonarchimedean local field $\mathbf{k}$ of characteristic zero. Let $G := \mathbb{G}(\mathbf{k})$.

Now consider the building $\mathcal{B}$ associated to $\mathbb{G}$. Recall that G acts transitively on the chambers, and τ permutes them. Thus fix a chamber C and let $\beta = b \ltimes \tau$ be in the stabilizer of C. Denote by $\mathcal{F}(\mathcal{B})$ the set of facets of $\mathcal{B}$, and by $\mathcal{F}(C)$ the facets of C. These are permuted by β. Let ${}^0G$ be the intersection of the kernels of the absolute values of all characters of G. This is a normal open compact subgroup of G. Let P_σ be the stabilizer in ${}^0G$ of the facet σ. Then P_σ is an open compact group which we will call a parahoric subgroup. An element x which stabilizes the facet σ, permutes its vertices. Let $sgn_\sigma(x)$ be the sign of this permutation. Fix a Haar measure m of G. The Lefschetz function

is defined as

$$f_{\mathcal{L}}(x) = \sum_{\substack{\sigma\in\mathcal{F}(C)\\ \beta(\sigma)=\sigma}} (-1)^{dim\sigma}\frac{1}{m[P_\sigma]} sign_\sigma(x)\delta_{P\beta}(x). \tag{VIII.1.1}$$

VIII.2. Let $\gamma = \delta'\tau = \delta\beta \in \widetilde{G}$ with $\delta \in G$ be a fixed *almost semisimple*element. See section II.3 for the definition. We want to evaluate $O_\gamma(f_{\mathcal{L}})$. Fix P a parahoric subgroup of G corresponding to a β-stable facet σ of C. Write $\mathcal{P}$ for the normalizer of P in G and $X_P := {}^0G/P$. Then X_P is equivalent to the set of facets of type P; the left action of G corresponds to the standard action of G on $\mathcal{B}$. Let

$$f_{P\beta} := \frac{1}{m(P)}\delta_{P\beta}. \tag{VIII.2.1}$$

Then $f_{P\beta}(g^{-1}\gamma g) \neq 0$ if and only if $g^{-1}\gamma g\beta^{-1} \in P$, equivalently, $g^{-1}\delta\beta(g) \in P$. Thus

$$O_\gamma(f_{P\beta}) = \frac{1}{m(P)}\, m\big[G(\gamma)\backslash\{G(\gamma)g \ :\ g^{-1}\delta\beta(g)\in P\}\big]. \tag{VIII.2.2}$$

In this formula m refers to the quotient measure on $G(\gamma)\backslash G$.

By possibly using a conjugate we may as well assume that $\delta \in P$, or else all integrals are zero anyway. If g satisfies $g^{-1}\delta\beta(g) \in P$, then so does gn for any $n \in P$. Thus

$$O_\gamma(f) = \frac{1}{m(P)} \sum_{\substack{g\in G(\gamma)\backslash G/P\\ g^{-1}\delta\beta(g)\in P}} m[G(\gamma)\backslash G(\gamma)gP]. \tag{VIII.2.3}$$

The group G equals ${}^0G\cdot A$ where A is the split component of the center. Then $A \cong (\mathbb{F}^\times)^r = GL(1,\mathbb{F})^r$. The lattice of coroots is $X^*(A) \cong \mathbb{Z}^r$. Then the automorphism β induces a linear isomorphism on this lattice, also denoted β, satisfying $\beta^m = Id$ for some m. There is a basis in which β is block diagonal with blocks corresponding to irreducible factors of $t^m - 1$. Precisely, let

$$t^s + b_{s-1}t^{s-1} + \cdots + b_0 \tag{VIII.2.4}$$

be such a factor. On the basis of this block,

$$\beta(a_0,\dots,a_{s-1}) = (a_2,\dots,a_{s-2}, a_0^{-b_0}\dots a_{s-1}^{-b_{s-1}}). \tag{VIII.2.5}$$

Let ${}^0A := A \cap {}^0G$. Suppose $a \in A$ is such that $a^{-1}\beta(a) \in {}^0A$. Using the block decomposition of (VIII.2.5), we conclude that $a = a'x$ where $\beta(a') = a'$ and $x \in {}^0A$. It follows that we can replace G by ${}^0G$ and $G(\gamma)$ by $G_\#(\gamma) :=^0 G \cap G(\gamma)$. The condition $g^{-1}\delta\beta(g) \in P$ is equivalent to

$$Ad(\gamma)(gPg^{-1}) = gPg^{-1}. \tag{VIII.2.6}$$

Let $R := gPg^{-1}$. Then

$$G(\gamma)\backslash G(\gamma)gP \cong [G(\gamma)\cap R]\backslash R \cong [G_{\#}(\gamma)\cap R]\backslash R.$$

Then (VIII.2.2) becomes

$$O_\gamma(f_{\mathcal{P}\beta}) = \sum_{\sigma\in G^0(\gamma)\backslash X_P(\gamma)} \frac{1}{m[G_{\#}(\gamma)_\sigma]}. \tag{VIII.2.7}$$

We conclude that

$$O_\gamma(f_{\mathcal{L}}) = \sum_{\rho\in G^0(\gamma)\backslash X_P(\gamma)} (-1)^{dim\rho}\frac{1}{m[G_{\#}(\gamma)_\rho]}. \tag{VIII.2.8}$$

VIII.3. Suppose $\mathbb{H}$ is a unimodular group acting in a *cell-wise* fashion on a CW–complex (or more generally on a polysimplicial complex) $\mathcal{T}$. Assume the following hold:

(i): $\mathcal{T}$ is contractible.
(ii): $\mathcal{T}$ is locally compact.
(iii): The stabilizer $\mathbb{H}_\sigma$ of any cell σ is an open compact subgroup of $\mathbb{H}$.
(iv): Any compact subgroup of $\mathbb{H}$ is contained in a $\mathbb{H}_\sigma$.
(v): The number of cells are finite modulo the action of $\mathbb{H}$.

Denote by Σ the set of orbits of the cells. Let m be an invariant measure. Then write

$$\chi(m) := \sum_{\sigma\in\Sigma}(-1)^{dim\sigma}\frac{1}{m[\mathbb{H}_\sigma]} \tag{VIII.3.1}$$

Theorem ([Serre])**.** *The measure $\mu = \chi(m)m$ is independent of m and is an Euler-Poincarémeasure. If $\mathbb{H}$ is semisimple (or reductive but has a totally anisotropic torus) then this measure is nonzero.*

VIII.4. We show that conditions (i)-(v) are satisfied for $\mathbb{H} = G_{\#}(\gamma)$ and $\mathcal{T} = \mathcal{B}(\gamma)$. Items (i)-(iv) are straightforward. For (v), suppose that P is stabilized by γ. There is $x \in^0 G$ such that $P = xP_\sigma x^{-1}$. It follows that $x\gamma x^{-1}$ stabilizes P_σ, *i.e.* $x^{-1}\delta\beta(x)$ is in the normalizer $\mathcal{P}_\sigma$ of P_σ. This is an open compact group. The orbit $O(\gamma)$ is closed, so the set $\Gamma := \{x^{-1}\delta\beta(x)\}$ is also closed. Thus the intersection $\Gamma\cap\mathcal{P}_\sigma$ is compact. Thus there are $x_1,\dots,x_n$ and a neighborhood $\mathcal{U}\subset\mathcal{P}$ such that

$$\Gamma\cap\mathcal{P}_\sigma = \bigcup_{i,u\subset\mathcal{U}} ux_i^{-1}\delta\beta(x_iu^{-1}) \tag{VIII.4.1}$$

The claim follows.

VIII.5. We say that γ is elliptic if $G(\gamma)$ contains a maximal anisotropic torus.

Theorem. *The orbital integrals of $f_{\mathcal{L}}$ are*

$$O_\gamma(f_{\mathcal{L}}) = \begin{cases} 1 & \text{if } \gamma \text{ is elliptic,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as in [Kottwitz1]. The necessary modification were discussed in sections VIII.1-VIII.4. □

VIII.6.

Theorem ([B-L-S] section 8.2 and 8.4, and [Kottwitz2])**.** *Assume G is simple. The only irreducible unitary representations for which* $\mathrm{tr}\pi(f_{\mathcal{L}}) \neq 0$, *are the Trivial and the Steinberg representations. In these cases,*

$$\mathrm{tr}\pi(f_{\mathcal{L}}) = \begin{cases} 1 & \text{if } \pi = Trivial, \\ (-1)^{q(G)} & \text{if } \pi = Steinberg, \end{cases}$$

where $q(G)$ is the $\mathbf{k}$ *rank of G.*

IX. **The twisted trace formula**

In this section we describe the trace formula and the effect of plugging in a function which has local components as in sections II-VIII. The formulation of the simple version of the trace formula we use can be found in [B-L-S] and in [Kottwitz2]. In turn it is based on [Arthur].

The assumptions on the group will be as in section II.

IX.1. **Generalities.** Recall that $\mathbf{K}$ is a totally real number field. Let χ be a unitary character of $G(\mathbb{A})$ trivial on $G(\mathbf{K})$. We assume that it satisfies $\chi \cong \chi \circ \tau$ so that it has an extension to $\tilde{G}(\mathbb{A})$. If $\mathbb{U}$ is unipotent, we normalize the Haar measure $du_{\mathbb{A}}$ so that $meas(\mathbb{U}(\mathbb{A})/\mathbb{U}(\mathbf{K})) = 1$ where $\mathbb{U}(\mathbf{K})$ has the counting measure. We fix a Haar measure $dk_{\mathbb{A}}$ on the maximal compact subgroup $K_{\mathbb{A}}$ so that $\mathrm{meas}(K_{\mathbb{A}}) = 1$. Fix a minimal parabolic subgroup $\mathbb{P}_0 = \mathbb{M}_0\mathbb{U}_0$ defined over $\mathbf{K}$. Fix a Haar measure $dm_{\mathbb{A}}$ on $\mathbb{M}_0(\mathbb{A})$. Then

$$f \mapsto \int_{\mathbb{U}(\mathbb{A})\mathbb{M}_0(\mathbb{A})K_{\mathbb{A}}} f(u_{\mathbb{A}}m_{\mathbb{A}}k_{\mathbb{A}})m^{-2\rho_{P_0}}\,dk_{\mathbb{A}}dm_{\mathbb{A}}du_{\mathbb{A}}$$

defines a Haar measure $dg_{\mathbb{A}}$ on $\mathbb{G}(\mathbb{A})$. We also fix a Haar measure $dz_{\mathbb{A}}$ on $\mathcal{Z}(\mathbb{A})$.

Let $L^2(\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K}),\chi)$ be the space of square integrable functions on $\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K})$ so that $f(gz)=\chi(z)f(g)$ for $g\in\mathbb{G}(\mathbb{A})$ and $z\in\mathcal{Z}(\mathbb{A})$. The group $\widetilde{\mathbb{G}}(\mathbb{A})$ acts unitarily on the space $L^2(\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K}),\chi)$.

Let $\mathbb{A}_G$ be the split component of the center of $\mathbb{G}$, and $\mathcal{X}^*(\mathbb{A}_G)$ be its rational characters. Let $\mathfrak{a}_{G,\mathbb{C}}:=\mathrm{Hom}[\mathcal{X}(\mathbb{A}_G),\mathbb{C}]$, and $\mathfrak{a}_G:=\mathrm{Hom}[\mathcal{X}(\mathbb{A}_G),\mathbb{R}]$. The function H_G is defined as

$$H_G: G(\mathbb{A})\longrightarrow\mathbb{R},\qquad H_G: a\mapsto(\chi\mapsto|\chi(a)|_{\mathbb{A}}). \tag{IX.1.1}$$

Let ${}^0\mathbb{G}(\mathbb{A})$ be the intersection of the absolute values of the kernels of the rational characters of $\mathbb{G}(\mathbb{A})$. The group $(\mathbb{A}_G)_\infty$ has a subgroup A_G^+ such that $\mathbb{G}(\mathbb{A})={}^0\mathbb{G}(\mathbb{A})\cdot A_G^+$.

The above discussion allows us to work with $L^2({}^0\mathbb{G}(\mathbb{A})/{}^0\mathbb{G}(\mathbf{K}))$ instead of of $L^2(\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K}),\chi)$. By abuse of notation we write $L^2(\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K}))$ for $L^2({}^0\mathbb{G}(\mathbb{A})/{}^0\mathbb{G}(\mathbf{K}))$.

As reminder, the goal of this article is to show that there are irreducible representations $\pi_{\mathbb{A}}$ of $\mathbb{G}(\mathbb{A})$ in $L^2_{cusp}(\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K}),\chi)$ so that $H^*(\mathfrak{g},K,\pi_{\mathbb{A}}\otimes F)\neq 0$ for some finite dimensional representation F such that $\pi_{\mathbb{A}}\cong\pi_{\mathbb{A}}\circ\theta$. For this we will use the twisted Arthur trace formula on $\mathbb{G}^*(\mathbb{A})$.

We define a function $f_{\mathbb{A}}=\prod_\nu f_\nu$ on $\widetilde{\mathbb{G}}(\mathbb{A})$ as follows. We fix a finite dimensional θ-stable representation F of $\widetilde{\mathbb{G}}(\mathbb{C})$ with infinitesimal character λ. For each infinite place ν_∞ choose $f_{\nu_\infty}=f_F\in C_c^\infty(\mathbb{G}^*(\mathbb{R}))$, the Lefschetz function in section VII.2 attached to F. For the finite number S of places where χ is not trivial on $\mathbb{G}(\mathcal{O}_\nu)$, choose f_ν to have support in a small enough open set K'_ν on which χ_ν is trivial. We fix two finite places $\nu_0,\ \nu_1\notin S$ where we assume (as we may) that $\mathbf{K}_{\nu_i}=\mathbb{G}(\mathcal{O}_{\nu_i})$. At these places we let $f_{\nu_i}=f_{\mathcal{L}}$ be the Lefschetz functions constructed in section VIII.6. For all other places let h_ν be the characteristic function of a maximal compact subgroup $K_\nu\subset\mathbb{G}(\mathbf{K}_\nu)$. We summarize the properties of the function $f_{\mathbb{A}}$.

- **a:** $\mathrm{tr}\pi(f_F)$ is $L(\tau,F,\pi)=e(\tau,\mathfrak{h}_R)$ if π is a τ stable representation of the form $\mathcal{W}=\mathcal{R}_{\mathfrak{b}}(\chi)$ of $\widetilde{\mathbb{G}}(\mathbb{R})$ with the same infinitesimal character λ (section VI.3.1). For other tempered representations, $\mathrm{tr}\pi(f_F)=0$. Furthermore f_F is very cuspidal in the sense of [Labesse1].
- **b:** $\mathrm{tr}\pi_{\nu_i}(f_{\mathcal{L}})$ is equal to 1 if π_{ν_i} is the trivial or the Steinberg representation. The trace is zero on any other irreducible representation.

c: Suppose $\gamma \in \mathbb{G}^*(\mathbb{R})$. The orbital integral

$$O_\gamma(f_\lambda) = \int_{\mathbb{G}(\gamma)(\mathbb{R})\backslash\mathbb{G}(\mathbb{R})} f_\lambda(g^{-1}\gamma g)dg$$

is 0 if γ is regular semisimple but not elliptic.

d: Suppose $\gamma \in \mathbb{G}(\mathbf{k}_{\nu_i})$. The orbital integral

$$O_\gamma(f_\mathcal{L}) = \int_{\mathbb{G}(\gamma)(\mathbf{k}_{\nu_i})\backslash\mathbb{G}(\mathbf{k}_{\nu_i})} f_\mathcal{L}(g^{-1}\gamma g)dg$$

is 1 if γ is elliptic and zero otherwise, for $i = 0, 1$.

The twisted trace formula is an identity of distributions

$$\text{LHS} = \text{RHS}$$

on ${}^0\mathbb{G}^*(\mathbb{A})/{}^0\mathbb{G}^*(\mathbf{K})$, where the right hand side is parameterized by harmonic, i.e representation theoretic data, whereas the left hand side is parameterized by geometric data.

IX.2. **The harmonic side.** Following the notation in [Arthur]we write $R_{d,t}$ for the representations in the discrete spectrum of the right regular representation of $\widetilde{\mathbb{G}}(\mathbb{A})$ on $L^2(\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K}))$ whose infinitesimal character has length t. Let $m_{disc}(\pi_\mathbb{A})$ be the multiplicity of a representation $\pi_\mathbb{A}$ of $\widetilde{\mathbb{G}}(\mathbb{A})$ in the discrete spectrum of $L^2(\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K}))$. We also write $R_{d,\lambda}$ for the discrete spectrum with infinitesimal character λ.

Proposition. *Let $f_\mathbb{A}$ be as above. Then*

$$\text{LHS}(f_\mathbb{A}) = \sum_{\pi_\mathbb{A} \in R_{d,\lambda}} m_{disc}(\pi_\mathbb{A})\text{tr}\pi_\mathbb{A}(f_\mathbb{A})$$

Proof. The function $f_\mathbb{A}$ satisfies assumption a) and b) of 9.2 in [B-L-S]. Furthermore, since at the local place ν_0 the Lefschetz function $f_\mathcal{L}$ is a factor of $f_\mathbb{A}$, the assumption c) of 9.5 in [B-L-S] is satisfied. Thus by the formula 9.2 in [B-L-S] $a^L_{disc}(\pi_\mathbb{A}) = 0$ for $L \neq G$ and $a^G_{disc}(\pi_\mathbb{A}) = m_{disc}(\pi_\mathbb{A})$ (see proof of Corollary 7.3 in [Arthur]). So

$$\text{LHS}(f_\mathbb{A}) = \sum_{t \geq 0} \sum_{\pi_\mathbb{A} \in R_{d,t}} m_{disc}(\pi_\mathbb{A})\text{tr}\pi_\mathbb{A}(f_\mathbb{A}).$$

Taking into account that tr $\pi_\infty(f_\lambda) \neq 0$ only if the infinitesimal character of π_∞ is equal to λ, the sum over t disappears, and $R_{d,t}$ is replaced by $R_{d,\lambda}$. □

IX.3. Let $\mathbb{P} = \mathbb{MN}$ be a parabolic subgroup defined over $\mathbf{K}$ and $K_{\mathbb{A}}$ a maximal compact group so that $\mathbb{G}(\mathbb{A}) = \mathbb{P}(\mathbb{A})K_{\mathbb{A}}$. Let $\mathbb{A}_P$ be the split component of the center of $\mathbb{M}$ and $\mathcal{X}^*(\mathbb{A}_P)$ be its rational characters. The complexified Lie algebra $\mathfrak{a}_P$ of A_P is isomorphic to $\mathcal{X}(A_P) \otimes_{\mathbb{Z}} \mathbb{C}$. Let $\Delta(\mathfrak{a}_P, P)$ be the simple roots of P and write ρ_P for half the sum of positive roots.

The group $(A_P)_\infty$ has a subgroup A_P^+ so that $\mathbb{M}(\mathbb{A}) = {}^0\mathbb{M}(\mathbb{A}) \cdot A_P^+$, where ${}^0\mathbb{M}(\mathbb{A})$ is the set of all $m_{\mathbb{A}} \in \mathbb{M}(\mathbb{A})$ so that $|\chi(m_{\mathbb{A}})|_{\mathbb{A}} = 1$ for all rational characters χ of $\mathcal{X}^*(M)$. The function $H_{\mathbb{P}}(\ )$ on A_P^+ is defined by the condition

$$e^{\langle H_{\mathbb{P}}(a), \chi\rangle} = |\chi(a)|. \tag{IX.3.1}$$

for all $\chi \in \mathcal{X}(A_P)$.

Let $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$ be the global Hecke algebra. If X is an $\mathcal{H}$–invariant space of automorphic forms on $\mathbb{G}(\mathbb{A})$, then the constant term $f_{\mathbb{P}}$ of any $f \in X$ along $\mathbb{P}$ has an expression

$$f_{\mathbb{P}}(k_{\mathbb{A}} m_{\mathbb{A}} a\, n_{\mathbb{A}}) = \sum_i P_i(H_{\mathbb{P}}(a)) a^{\mu_i + \rho_P}([\sum \phi_{i,j}(m_{\mathbb{A}}) f_{i,j}(k_{\mathbb{A}})]. \tag{IX.3.2}$$

The P_i are polynomials, the $\phi_{i,j}$ are automorphic forms of ${}^0\mathbb{M}(\mathbb{A})$ and the $f_{i,j}$ are $K_{\mathbb{A}}$-finite functions. The μ_i are distinct and the ones with nonzero contribution are called the automorphic exponents of f along $\mathbb{P}$ and we call the set of all μ_i's which appear as we vary f over X the automorphic exponents of X along $\mathbb{P}$.

The *local exponents* of f at a place ν along $\mathbb{P}$ are defined as follows: If ν is finite, then the Jacquet module of the $\mathcal{H}_\nu$ module $(\mathcal{H}_\nu * f_\nu)$ associated to $\mathbb{N}(\mathbf{K}_\nu)$ is a finitely generated admissible $\mathbb{M}(\mathbf{K}_\nu)$–module. The exponents at the place ν are the absolute values of the characters of $\mathbb{A}(\mathbf{K}_\nu)$ that occur in the Jacquet module.

An automorphic function is called concentrated along $\mathbb{P}$ if $f_{\mathbb{Q}} = 0$ for any $\mathbb{Q}$ which is not associate to $\mathbb{P}$.

Theorem (1)**.** *(*[Kudla-Rallis-Soudry]*, 6.9) Suppose f that the automorphic form is concentrated along $\mathbb{P}$. Let μ be an automorphic exponent of f.*

(1) *For any finite ν there is an exponent η along $\mathbb{P}$ so that*

$$Re(\mu) = \eta.$$

(2) *Suppose that ν is an infinite place. The generalized eigenspace $H_0(\mathbf{n}_\nu, X)_\mu$ is non zero. Here $\mathbf{n}_\nu$ is the Lie algebra of $\mathbb{N}(K_\nu)$.*

Proposition (Rallis)**.** *If an automorphic form is tempered at one place, then it is cuspidal.*

Proof. We may as well assume that f is concentrated along $\mathbb{P}$. The condition for the local component to be tempered is that the exponents should be of the form

$$Re(\mu) = \rho_P + \sum x_\alpha \alpha, \qquad x_\alpha \geq 0,\ \alpha \in \Delta(\mathfrak{a}_P, \mathfrak{n}).$$

The condition for $f \in L^2(G(\mathbb{A})/G(\mathbf{K}))$ is

$$Re(\mu) = \rho_P - \sum y_\alpha \alpha^*, \qquad y_\alpha \geq 0,\ \alpha \in \Delta(\mathfrak{a}_P, \mathfrak{n}),$$

where α^* is the dual basis to the simple roots. These two conditions are incompatible unless the $\phi_{i,j}$ in (IX.3.2) are all zero. □

Remark: In the nonadelic context, this proposition is an earlier result of Wallach [Wallach]. The above adelic version already appears in [Clozel]. □

Lemma. *Suppose $\pi = \otimes \pi_\nu$ is such that π_ν is 1-dimensional for some ν. Then π is 1-dimensional.*

Proof. A 1-dimensional representation has a single exponent η, and this exponent satisfies $Re(\eta) = \rho_P$. By theorem IX.3, all automorphic exponents satisfy $Re(\mu) = \rho_P$, and therefore for any place ν there is an exponent ν_ν satisfying $Re(\nu_\nu) = \rho_P$. By theorem 6.1 of [Howe-Moore], a unitary representation with this property has to be a unitary character. □

The discrete spectrum of the regular representation of $\widetilde{G}(\mathbb{A})$ on $L^2(\mathbb{G}(\mathbb{A})/\mathbb{G}(\mathbf{K}))$ decomposes into a cuspidal part and a residual part. Recall that $R_{d,\lambda}$ is the set of representations in the discrete spectrum with infinitesimal character λ and write $R_{c,\lambda}$ for the subset of representations in the the cuspidal part.

Theorem (2)**.** *Every representation which contributes to* RHS($f_{\mathbb{A}}$) *is either one dimensional or in the cuspidal spectrum.*

Proof. The Steinberg representation is tempered. So the theorem follows from the previous propositions. □

IX.4. **The geometric side.** Recall that for $\gamma = \{\gamma_\nu\} \in \mathbb{G}^*(\mathbb{A})$

$$\begin{aligned} J_{\mathbb{G}}(\gamma, f_{\mathbb{A}}) &= \int_{\mathbb{G}(\gamma)^0(\mathbb{A})\backslash\mathbb{G}(\mathbb{A})} f_{\mathbb{A}}(g^{-1}\gamma g)dg_{\mathbb{A}} \\ &= \prod_\nu \int_{\mathbb{G}(\gamma_\nu)^0(\mathbf{k}_\nu)\backslash\mathbb{G}(\mathbf{k}_\nu)} f_\nu(g_\nu^{-1}\gamma_\nu g)dg_\nu. \end{aligned}$$

In the previous sections we have computed orbital integrals of the form

$$\int_{G(\gamma)\backslash G} f(g^{-1}\gamma g)dg. \tag{IX.4.1}$$

In what follows we will use

$$\int_{G(\gamma)^0\backslash G} f(g^{-1}\gamma g)dg. \tag{IX.4.2}$$

where $G(\gamma)^0$ is the connected component of the centralizer of γ. The relation between the two is a factor $|G(\gamma)/G(\gamma)^0|$.

The results in 9.2 of [B-L-S] combined with section 5 of [Kottwitz2] show that

$$\mathrm{LHS}(f_{\mathbb{A}}) = \sum_{\gamma\in(\mathbb{G}^*(\mathbf{K}))_{elliptic}} a^G(\gamma) J_{\mathbb{G}}(\gamma, f_{\mathbb{A}}) \tag{IX.4.3}$$

where

$$a^G(\gamma) = \mathrm{vol}\left|\frac{\mathbb{G}(\gamma)_0(\mathbb{A})}{\mathbb{G}(\gamma)_0(\mathbf{K})}\right| \cdot \left|\frac{\mathbb{G}(\gamma)}{\mathbb{G}(\gamma)_0}\right|. \tag{IX.4.4}$$

We note that the argument in [Kottwitz2] about the geometric side of the trace formula depends only on the fact that at one place v, the component f_v of $f_{\mathbb{A}}$ is an Euler-Poincaré function which in turn relies on results of Arthur for a connected component of a reductive group.

Lemma ([Clozel])**.** *Let $\mathcal{K} \subset \mathbf{G}_\infty$ be a fixed compact set. There is a set S_1 of finite places with $\nu_0, \nu_1 \notin S_1$ with the following property. There is a choice of compact open subgroups $K_\nu,\ \nu \in S_1$ so that if*

$$\gamma \in G(\mathbf{K}) \cap \mathcal{K} \prod_{\nu\notin S_1} G(\mathcal{O}_\nu) \prod_{\nu\in S_1} K_\nu$$

then γ is unipotent. The set $\mathcal{K}\prod_{\nu\in S_1} K_\nu \prod_{\nu\notin S_1} G(\mathcal{O}_\nu)$ can be chosen so that it is τ-stable.

Proof. (included for completeness) Choose any set S_1 of finite places that does not contain ν_0 and ν_1. Let $\rho : G \to GL(m)$ be a faithful representation. Let

$$p(x,t) := \det(t - 1 + \rho(x)) = t^m + a_{m-1}(x)t^{m-1} + \cdots + a_0(x). \tag{IX.4.5}$$

The a_i are polynomials which extend to $G(\mathbb{A})$ and equal

$$a_i(x) = a_{i,\infty}(x) \prod_{\nu \notin S_1} a_{i,\nu}(x) \prod_{\nu \in S_1} a_{i,\nu}(x). \tag{IX.4.6}$$

If all the $a_i(x) = 0$, then x is unipotent. The first two factors of the product are bounded. The last part can be made arbitrarily small for $x_\nu \in K_\nu$ by making K_ν small enough. The claim follows from the fact that for $x \in G(\mathbf{K})$, $|a_i(x)|_{\mathbb{A}}$ is either 1 or 0. □

Theorem. *There is a choice of $f_{\mathbb{A}}$ so that*

$$\sum_{\substack{\gamma \in (\widetilde{G}(\mathbf{K}))_{elipptic} \\ N(\gamma)=1}} a^G(\gamma) J_{\widetilde{G}}(\gamma, f_{\mathbb{A}}) = \sum_{\pi_{\mathbb{A}} \in R_{d,\lambda}} m_{disc}(\pi_{\mathbb{A}}) \mathrm{tr}\pi_{\mathbb{A}}(f_{\mathbb{A}})$$

As before, the sum is over (representatives of) conjugacy classes. All representations contributing are either one dimensional or in the cuspidal spectrum.

Proof. Recall that f_∞ is a Lefschetz function, and has compact support contained in a set $\tau\mathcal{K}$. Modify $f_{\mathbb{A}}$ so that f_ν is the delta function of K_ν for $\nu \in S_1$. Then apply lemma IX.4 with $\mathcal{K}$ as above to γ^d to conclude it must be the identity. Thus (IX.4.3) simplifies to the formula in the proposition. See also [Rohlfs-Speh]. □

X. **A simplification of theorem IX.4**

In this section we combine the terms in RHS($f_{\mathbb{A}}$) in proposition (IX.4) along stable conjugacy classes. The references are [Kott-Shel], [Labesse2] and [Kottwitz1]. Most of section is a summary of those results.

We consider in this section a connected reductive algebraic group $\mathbb{G}$. This will be either the group considered in section I with an automorphism τ of finite order or the connected component $\mathbb{I}(\gamma) := \mathbb{G}(\gamma)^0$ of the centralizer of an elliptic element γ in $\mathbb{G}^*$. Denote by $\mathbb{G}_{der}$ the derived group and by $\mathbb{G}_{SC}$ its simply connected cover.

X.1. Let F be a global or local field. For σ in the Galois group of F and $g \in \mathbb{G}(\bar{F})$ we define a cocycle by

$$v_g(\sigma) := g^{-1}\sigma(g). \tag{X.1.1}$$

Fix a semisimple element $\gamma = \delta\tau$ in $\mathbb{G}^*(F)$ and let $\gamma' = g\gamma g^{-1} \in \mathbb{G}^*(F)$ with $g \in \mathbb{G}(\bar{F})$. The cocycle v_g takes values in $\mathbb{G}(\gamma)(\bar{F})$ but not necessarily in $\mathbb{I}(\gamma)(\bar{F})$.

Definition. *We say that two elements γ, and $\gamma' = g\gamma g^{-1} \in \mathbb{G}^*(F)$ are stably conjugate if the cocycle v_g of (X.1.1) takes values in $\mathbb{I}(\gamma)(\bar{F})$ for all σ in the Galois group of F.*

Conversely if $v_g(\sigma)$ is in $\mathbb{I}(\gamma)(\bar{F})$ for all σ in the Galois group of F then $g\gamma g^{-1} \in \mathbb{G}(F)$.

If γ is stably conjugate to $\gamma' = g\gamma g^{-1}$, then the cocycle $v_g(\sigma)$ in $H^1(F, \mathbb{I}(\gamma))$ belongs to

$$\mathcal{D}(\mathbb{I}/F) = \ker[H^1(F, \mathbb{I}(\gamma)) \to H^1(F, \mathbb{G})].$$

See also 2.6.

Remarks:

(1) If γ and γ' are stably conjugate then $I(\gamma')$ is an inner twist of $I(\gamma)$.

(2) Assume that F is a number field, that G is a simply connected semisimple group. Then Kneser, Harder, Springer and Chernousov [Chernousov] show that the Hasse principle holds, *i.e.*

$$H^1(F, \mathbb{G}(F)) \hookrightarrow \prod_v H^1(F, \mathbb{G}(F_v)).$$

This implies that $\gamma, \gamma' \in \widetilde{\mathbb{G}}(F)$ are conjugate by an element in $\mathbb{G}(\bar{F})$ if and only if the components in $\widetilde{\mathbb{G}}(F_v)$ are conjugate by elements in $\mathbb{G}(\bar{F}_v)$.

(3) Suppose that F is a number field and that the Hasse principle holds for the derived group $\mathbb{G}_{SC}$. Let $\gamma \in \mathbb{G}(\mathbb{A})$. In 6.6 of [Kottwitz1], R. Kottwitz defines an invariant $\mathrm{obs}(\gamma)$ which is trivial if and only if γ conjugate under $\mathbb{G}(\mathbb{A})$ to an element in $\mathbb{G}(F)$.

X.2. **A local example.** Suppose $\mathbf{k}$ is a local field and $\mathbb{G} = GL(n)$. We consider the automorphism $\tau(x) := {}^t x^{-1}$. An element $\gamma = \delta\tau \in \widetilde{\mathbb{G}}(\mathbf{k})$ is conjugate to τ if and only if $\delta = g\tau(g^{-1}) = gg^t$, *i.e.* it is a symmetric matrix. Equivalently, we can think of the γ's as quadratic forms and the problem is then to classify them according to usual conjugacy under $GL(n, \mathbf{k})$ and $GL(n, \overline{\mathbf{k}})$.

Proposition. *An element $\gamma = \delta\tau \in \widetilde{\mathbb{G}}(\mathbf{k})$ is stably conjugate to τ if and only if the determinant of δ is a square in $\mathbf{k}^*$. The stable conjugacy classes satisfying $N(\gamma) = 1$ are parametrized by $\mathbf{k}^*/(\mathbf{k}^*)^2$.*

Proof. The centralizer of τ is the orthogonal group $O(n)$ which has two connected components corresponding to $\det = \pm 1$. Let $\mathbb{H}$ be the diagonal Cartan subgroup which is both τ and γ stable. The fact that a quadratic form over any field $\mathbf{k}$ can be diagonalized is equivalent to the fact that any γ is conjugate by $SL(n, \mathbf{k})$ to an element $\delta\tau$ with $\delta \in \mathbb{H}(\mathbf{k})$. It is clear that there is $h \in \mathbb{H}(\overline{\mathbf{k}})$ such that $\delta = hh^t$. The element h can be chosen so that $\det h = \det \sigma(h)$ for any $\sigma \in \Gamma$ precisely when $\det \delta \in (\mathbf{k}^*)^2$.

The proof follows by recalling that γ and γ' viewed as symmetric forms are conjugate by an element in $\mathbb{G}(\mathbf{k})$ if and only if the discriminant and determinant of δ and δ' are equal modulo squares in $\mathbf{k}$. □

Remark. By corollary III.1, the condition $N(\gamma) = 1$ in the proposition is equivalent to the fact that γ is conjugate via $\mathbb{G}(\bar{F})$ to the automorphism τ.

X.3. Recall the formulas in section IX.4. Fix Tamagawa measures on $\mathbb{G}(\mathbb{A})$ and $\mathbb{I}(\gamma)(\mathbb{A})$. Then the first factor in $a^G(\gamma)$ in (IX.4.4) is the Tamagawa number of $I(\gamma)(\mathbb{A})$ which we denote $\tau(\gamma)$. By [Kottwitz2], if γ is stably conjugate to γ'

$$\tau(\gamma) = \tau(\gamma'). \tag{X.3.1}$$

Thus IX.4.3 becomes

Theorem.

$$\mathrm{RHS}(f_{\mathbb{A}}) = \sum_{\gamma \in \Delta} \tau(\gamma) \left| \frac{\mathbb{G}(\gamma)_0}{\mathbb{G}(\gamma)} \right| \sum_{\gamma' \in \mathcal{D}(\mathbb{I}/\mathbf{K})} J_\gamma(f_{\mathbb{A}}) \tag{X.3.2}$$

where Δ is a set of representatives of stable conjugacy classes of elliptic semisimple elements γ in $\widetilde{\mathbb{G}}(\mathbf{K})$ satisfying $N(\gamma) = 1$, and $\mathcal{D}(\mathbb{I}/\mathbf{K})$ parametrizes the stable conjugacy class of γ as in X.1.

If $\mathbb{I}(\gamma)$ is simply connected, then $\tau(\gamma) = 1$.

XI. **The main theorems**

We assume $\mathbf{K} = \mathbb{Q}$. We prove that $RHS(f_{\mathbb{A}}) \neq 0$ and use this to show that there exist $\tau-$invariant cuspidal automorphic forms, and prove nonvanishing theorems for cuspidal cohomology. In particular we illustrate these results in the case of $\mathbb{G} = GL(n)$.

XI.1. Since Tamagawa numbers are volumes, the coefficient of each integral in X.3.2 is positive. We need a function $f_{\mathbb{A}}$ such that the orbital integrals all have the same sign and at least one is nonzero.

The orbital integrals have a product formula

$$J_\gamma(f_{\mathbb{A}}) = [\prod_{\nu \ infinite} J_{\gamma_\nu}(f_\nu)]\cdot[J_{\gamma_{\nu_0}}(f_{\nu_0})]\cdot[J_{\gamma_{\nu_1}}(f_{\nu_1})]\cdot[\prod_{\nu \ finite, \nu\neq\nu_0,\nu_1} J_{\gamma_\nu}(f_\nu)]. \tag{XI.1.1}$$

By VIII.5 $J_{\gamma_{\nu_i}}(f_{\nu_i}) = 1$ for i = 0, 1 and $J_\gamma(f_\gamma) \geq 0$ for ν finite, $\nu \neq \nu_i$. In addition, if ν is finite and $\gamma = \tau$, $J_\tau(f_\nu) > 0$. Recall that F is a fixed irreducible finite dimensional representation of $\mathbb{G}$ and that for each finite place ν f_ν is a Lefschetz function f_F and that by VI.3

$$O_\gamma(f_F) = (-1)^{q(\gamma)} e(\gamma) \mathrm{tr} F^*(\gamma)$$

where

$$e(\gamma) = \sum_i (-1)^i \mathrm{tr}(\gamma \ : \ \bigwedge^i \mathfrak{h}_R^*)$$

and

$$q(\gamma) = \frac{1}{2}(\dim \mathfrak{g}(\gamma) - \dim \mathfrak{k}(\gamma))$$

is the number associated to a real form $\mathbb{G}(\gamma)$ by Kottwitz. Therefore,

$$J_\gamma = (-1)^{q(\gamma)} e(\gamma) \mathrm{tr} F^*(\gamma) \big| \frac{G(\gamma)}{G(\gamma)^0} \big| \tag{XI.1.2}$$

We will restrict the support of the function $f_{\mathbb{A}}$ at a finite number of finite places such that the contribution of only one γ in theorem X.3.2 is nonzero.

Let $\Gamma = G(\mathbb{Q}) \cap G_\infty K_{fin}$ where K_{fin} is a product of compact open subgroups as in lemma IX.4. This choice depends on the function f_∞ only. A theorem of Borel-Serre [Borel-Serre], section 3.8, states that $\mathrm{H}^1(\langle\tau\rangle, \Gamma)$ (notation III.6) is finite dimensional, *i.e.* that the intersection of the set of elements satisfying $N(\gamma) = 1$ with Γ breaks up into finitely many orbits under Γ. Let $\tau_1 = \tau, \ldots, \tau_k$ be representatives of these Γ-orbits.

Lemma. *There is an open compact subgroup $K_f = \prod K_\nu \subset K_{fin}$ with $K_\nu = G(\mathcal{O}_\nu)$ for all but finitely many places S_1 such that $K_f\tau_i \cap K_f\tau_j = \emptyset$ for all $i \neq j$.*

Proof. The elements $(\tau_i)_\nu$ are semisimple. So for each $\nu \in S_1$ replace K_ν by a smaller K'_ν so that the orbit of $(\tau_i)_\nu$ does not intersect $K'_\nu(\tau_1)_\nu$. □

Recall that the set S was defined in section IX.1 as the finite set of finite places ν where the character χ is not trivial on $\mathbb{G}(\mathcal{O}_\nu)$. The set S_1 is defined in lemma IX.4.

We also recall that according to theorem VI.3, $e(\tau)$ is nonzero precisely when the centralizer of of τ in $\mathfrak{g}(\mathbb{R})$ is equal rank.

Proposition. *Let $f_{\mathbb{A}} = \prod_\nu f_\nu$ be a function on $\widetilde{\mathbb{G}}(\mathbb{A})$ satisfying the following properties.*

1. *$f_{\nu_\infty} = f_F \in C_c^\infty(\mathbb{G}^*(\mathbb{R}))$ is the Lefschetz function defined in section VII.2.*
2. *For the finite places $\nu_0,\ \nu_1$ $f_{\nu_i} = f_{\mathcal{L}}$ is the Lefschetz functions constructed in section VIII.6.*
3. *For the places $\nu \in S \cup S_1$ let f_ν be the characteristic function of a compact subgroup $K_\nu \subset \mathbb{G}(\mathbb{Q}_\nu)$ which satisfies the assumptions in IX.1, XI.1 and IX.4.*
4. *For all places $\nu \notin S \cup S_1$ let f_ν be the characteristic function of a maximal compact subgroup $\tau_\nu K_\nu \subset \mathbb{G}(\mathbb{Q}_\nu)$.*

If $e(\tau) \neq 0$, then

$$RHS(f_{\mathbb{A}}) \neq 0.$$

Proof. The proof follows from the lemma and the discussion above. □

In conclusion we have proved the following theorem.

Theorem. *Let $\mathbb{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$ and F a finite dimensional irreducible representation of $\mathbb{G}(\mathbb{R})$. If $\mathrm{tr}F(\tau) \neq 0$ and the centralizer of τ in $\mathfrak{g}(\mathbb{R})$ is equal rank, then there exist cuspidal automorphic representations of $\mathbb{G}(\mathbf{A})$ stable under τ.*

Proof. Let $f_{\mathbb{A}}$ be the function in proposition XI.1. Theorem IX.4 and the previous proposition imply that

$$\sum_{\pi_{\mathbb{A}} \in R_{d,\lambda}} m_{disc}(\pi_{\mathbb{A}})\mathrm{tr}\pi_{\mathbb{A}}(f_{\mathbb{A}}) \neq 0.$$

If $\dim F > 1$, then all the representations contributing to the sum are in the cuspidal spectrum by the results in IX.3.

Suppose now that the representation F is one dimensional. Denote the contribution of the one dimensional representations of $\mathbb{G}(\mathbb{A})$ by $\mathbf{I}_{\mathbb{A}}$. Then

$$\mathbf{I}_{\mathbb{A}}(f_{\mathbb{A}}) + \sum_{\pi_{\mathbb{A}} \in R_{d,\lambda}} m_{cusp}(\pi_{\mathbb{A}}) \mathrm{tr}\pi_{\mathbb{A}}(f_{\mathbb{A}}) = a^G(\tau) J_\tau(f_{\mathbb{A}}).$$

We make the simplifying assumption that the center of $\mathbb{G}(\mathbb{A})$ is isomorphic to $(\mathbb{A}^\times)^r$. A character of $\mathbb{G}(\mathbb{A})$ is determined by its values on the center. Let χ be a character of $(\mathbb{A}^\times)^r$ trivial on $(\mathbf{K}^\times)^r$ is of the form

$$\chi(a_1, \dots, a_r) = |a_1|^{s_1} \cdot \dots \cdot |a_r|^{s_r} \chi_1(a_1, \dots, a_r) \tag{XI.1.3}$$

where χ_1 is unitary. By the discussion in section IX.1 we can work with ${}^0\mathbb{G}(\mathbb{A})$, the intersection of the kernels of the absolute values of the characters of $\mathbb{G}(\mathbb{A})$, we may as well assume $s_1 = \dots s_r = 0$. The character χ_1 must be trivial at the infinite places as well as $\nu_0,\ \nu_1$. There are finitely many places v such that $(\chi_1)_v \neq triv$. If such a place is not in $S \cup S_1$, then since $f_v = \mathbb{1}_{\mathcal{O}_v}$, we have $\chi_1(f_v) = 0$. Thus $\mathbb{I}_{\mathbb{A}}$ consists of finitely many characters.

Following the idea in [B-L-S] we fix a finite place $w \notin S \cup S_1 \cup \{\nu_0, \nu_1\}$. Choose a sequence of compact open subgroups $K_w(i)$ (congruence subgroups $1 + \varpi^i \mathbb{G}(\mathcal{O})$) with characteristic functions h_i^w. Then

$$J_{\tau_w}(h_i^w) = c_w q_w^{-i(\dim \mathbb{G} - \dim \mathbb{G}(\tau))} \longrightarrow 0 \text{ for } i \longrightarrow \infty,$$

and $c_w \neq 0$ independent of i. Similarly,

$$\mathrm{vol}(K_w(i)) = d_w q_w^{-i \dim \mathbb{G}} \to 0 \text{ for } i \to \infty.$$

A character χ_w satisfies $\chi_w(h_i^w) = 0$ unless it is trivial when restricted to $K_w(i)$. But there are at most $q_w^{i \dim \mathcal{Z}(\mathbb{G}(F_w))^\tau}$ such characters. Thus if

$$\dim \mathbb{G} - \dim \mathbb{G}(\tau) < \dim \mathbb{G} - \dim \mathcal{Z}(\mathbb{G})^\tau, \tag{XI.1.4}$$

$a^G(\tau) J_\tau(f_{\mathbb{A}}(i))$ goes to zero strictly slower than $\mathbb{I}_{\mathbb{A}}(f_{\mathbb{A}}(i))$. It follows that there must be a nonzero contribution from the cuspidal part. $\square$

Remark: The number $\mathrm{tr}F^*(\theta)$ for $\tau = \theta$ an automorphism of order 2 is computed in section IV. In the case when τ has order $d > 2$, we can see that there are infinitely many finite dimensional representations satisfying $\mathrm{tr}F(\tau) \neq 0$ as follows. It is enough to prove this for the case of finite dimensional representations of the compact group $\widetilde{K}$. Let $\widetilde{K}(\tau)$ be the centralizer of τ. The Fourier expansion of the delta function δ_τ is

$$\delta_\tau = \sum_{V \in \widehat{\widetilde{K}}(\tau)} \mathrm{tr}V(\tau) \dim V. \tag{XI.1.5}$$

Since δ_τ is not smooth, there are infinitely many nonzero terms in the right hand side. The claim that there are infinitely many finite dimensional representations F of $\widetilde{K}$ satisfying $\mathrm{tr}F(\tau) \neq 0$ follows from the fact that the restrictions of the representations of $\widetilde{K}$ span the Grothendieck group of $\widetilde{K}(\tau)$.

XI.2. **An example.** For $\mathbb{G} = GL(n)$ we consider the automorphism τ_c which is transpose inverse. Then $\tau_c(g_\mathbb{A}) = g_\mathbb{A}^{-1}$ for all $g_\mathbb{A} \in \mathbb{Z}(\mathbb{A})$. Therefore the set $X(\tilde{\mathbb{G}})_\mathbb{Q}$ of $\mathbb{Q}-$rational characters of $\widetilde{\mathbb{G}}$ is trivial, $a_{\tilde{G}} = Hom(X(\tilde{\mathbb{G}})_\mathbb{Q}, \mathbb{R}) = 0$ and Arthur's function $H_{\tilde{G}}$ equals zero. Thus in Arthur's notation

$$\tilde{\mathbb{G}}(\mathbb{A})^1 = \tilde{\mathbb{G}}(\mathbb{A}).$$

For $GL(n)$ the local measures and all the normalization factors are as in [Cassels-Fröhlich] page 261. We call a discrete series representation D of GL(2,$\mathbb{R}$) even if

$$D(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) = Id$$

and odd otherwise. A tempered representation of GL(n,$\mathbb{R}$) induced from a maximal cuspidal parabolic subgroup $P = MAN$ is even if it is induced from an even discrete series representation of every factor of M. We call it odd if it is induced from an odd representation on every factor of M.

Recall that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the highest weight of a self dual finite dimensional representation F if and only if $\lambda_i = -\lambda_{n+1-i}$. The conditions of V.3 are satisfied if

(1) n is odd and all λ_i are even,
(2) n is even and for all $i, j,\ \lambda_i = \lambda_j\ (mod\ 2)$

For GL(n,$\mathbb{A}$) all representations in the cuspidal spectrum are tempered [Shalika]. There is exactly one irreducible tempered $(\mathfrak{g}, K)$–module $\mathcal{W}$ with nontrivial $(\mathfrak{g}, K)$–cohomology with the same infinitesimal character as F [Speh] and

(1) if n is odd then $\mathcal{W}$ is even
(2) if n is even then $\mathcal{W}$ is odd

$\mathcal{W}$ is invariant under τ_c and has nontrivial Lefschetz number.

Theorem (1)**.** *There exist tempered cuspidal representations $\pi_\mathbb{A} = \prod \pi_\nu$ of GL(n,$\mathbb{A}$) with the following properties:*

(1) *$\pi_\mathbb{A}$ is invariant under the Cartan involution τ_c.*

(2) *π_∞ has an integral nonsingular infinitesimal character satisfying the conditions of V.3.*
(3) *If n is even then π_∞ an odd representation.*
(4) *If n is odd then π_∞ is an even representation.*

Proof. This is essentially theorem XI.1 combined with the results in section V.3. □

For $\mathbb{G} = GL(2m)$ we also consider the symplectic automorphism τ_s with fix point set Sp(2m). The irreducible finite dimensional representation F with highest weight $(\lambda_1, \dots, \lambda_{2m})$ is invariant under τ_s if $\lambda_j = \lambda_1$ for $i = 2, \dots, m$ and $\lambda_j = -\lambda_1$ for $i = m+1, \dots, 2m$. The conditions of V.3 are satisfied if $\lambda_1 \in \mathbb{N}$.

Theorem (2)**.** *There exist tempered cuspidal representations $\pi_{\mathbb{A}} = \prod \pi_\nu$ of GL(2m,$\mathbb{A}$) with the following properties:*

(1) *$\pi_{\mathbb{A}}$ is invariant under the symplectic automorphism τ_c.*
(2) *π_∞ has an integral nonsingular infinitesimal character satisfying the conditions of V.3.*

The following is a generalization of the theorems (1) and (2) using base change.

Theorem (3)**.** *Let $\mathbf{K}/\mathbb{Q}$ be an extension of $\mathbb{Q}$ such that there is tower*

$$\mathbb{Q} \subset \mathbf{K}_1 \subset \mathbf{K}_2 \subset \cdots \subset \mathbf{K}_r = \mathbf{K}$$

of cyclic extensions of prime degree. There exist tempered cuspidal representations $\Pi_{\mathbb{A}_\mathbf{K}}$ of GL(n,$\mathbb{A}_\mathbf{K}$) with nontrivial cohomology.

Proof. Let $\mathbf{K}/\mathbb{Q}$ be a cyclic extension of prime degree, and $\pi_{\mathbb{A}}$ a cuspidal representation of GL(n,$\mathbb{A}$) with nontrivial cohomology constructed in theorem (1). J. Arthur and L. Clozel proved that there exists an automorphic representation $\Pi_{\mathbb{A}_\mathbf{K}}$ of GL(n,$\mathbb{A}_\mathbf{K}$) lifting π ([AC], chap.3, theorem 4.2). This representation has a Steinberg representation at 2 finite places ([AC], chap.1, lemma 6.2) and is therefore also cuspidal. Furthermore this representation has nontrivial cohomology. □

Remark: In the proof of theorem (2) no use is made of property (2) of $\pi_{\mathbb{A}}$ in theorem (1).

XI.3. In this section we assume again that $\mathbb{G}$ is defined over $\mathbb{Q}$ and satisfies of section II.1. Consider the locally symmetric space

$$S(K_f) := K_\infty \; K_f \backslash \mathbb{G}(\mathbb{A})/A_G\mathbb{G}(\mathbb{Q})$$

with K_f *small enough* as in section XI.1. The DeRham cohomology

$$\mathrm{H}^*(S(K_f), F)$$

with coefficients in the sheaf defined by a finite dimensional representation F is isomorphic to

$$\mathrm{H}^*(\mathfrak{g}, K_\infty, \mathcal{A}(\mathbb{G}(\mathbb{A})/A_G\mathbb{G}(\mathbb{Q})) \otimes F)^{K_f}$$

where $\mathcal{A}(\mathbb{G}(\mathbb{A})/\mathbb{G}A_G(\mathbb{Q}))$ is the space of automorphic forms [Franke] and the upper index denotes the invariants under K_f . Denote by

$$\mathcal{A}_{cusp}(\mathbb{G}(\mathbb{A})/A_G\mathbb{G}(\mathbb{Q}))$$

the space of cusp forms. Then by [Borel]

$$\begin{aligned}\mathrm{H}^*(\mathfrak{g}, K_\infty, \mathcal{A}_{cusp}(\mathbb{G}(\mathbb{A})/A_G\mathbb{G}(\mathbb{Q})) \otimes F))^{K_f} \\ \hookrightarrow \mathrm{H}^*(\mathfrak{g}, K_\infty, \mathcal{A}(\mathbb{G}(\mathbb{Q})A_G\backslash\mathbb{G}(\mathbb{A})) \otimes F))^{K_f}.\end{aligned}$$

The image is denoted by $\mathrm{H}^*_{cusp}(S(K_f), F)$.

Let F be a finite dimensional irreducible representation of $\mathbb{G}$ which is invariant under an automorphism τ of $\mathbb{G}$. Then τ acts on $H^*(S(K_f), F)$.

Let U be a maximal normal compact subgroup of $\mathbb{G}(\mathbb{R})$. An involution τ is called Cartan like if it defines an involution on $\mathbb{G}(\mathbb{R})/U$ which is conjugate to a Cartan involution. The assumptions of nonvanishing theorem XI.1 are satisfied for a Cartan like involution τ and the trivial representation F. Thus theorem XI.1 implies

Theorem. *Suppose that K_f satisfies the condition of proposition XI.1. Let $\mathbb{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$. Then*

$$\mathrm{H}^*_{cusp}(S(K_f), \mathbb{C}) \neq 0.$$

Remarks: In the equal rank case the nonvanishing of the cuspidal cohomology was first proved by L.Clozel and bounds on the cuspidal cohomology were obtained in [Rohlfs-Speh2] and in [Savin]. In these cases the ordinary trace formula respectively the Euler-Poincaré characteristic was used and no twisting by an automorphism was necessary.

For $S(K_f)$ is compact and F nontrivial a nonvanishing theorem was proved in [Rohlfs-Speh] using geometric Lefschetz numbers for Cartan like involutions. For subgroups $\Gamma \subset SO(n, 1)$ and $S(K_f)$ this theorem was proved in [Rohlfs-Speh3] also using Lefschetz numbers.

For semisimple $\mathbb{G}$ and $S-$arithmetic groups it proved in [B-L-S] using L^2-Lefschetz numbers and a twisted trace Arthur trace formula.

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Dan Barbasch, Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, NY 14853-4201, U.S.A.

E-mail address: barbasch@math.cornell.edu

Birgit Speh, Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, NY 14853-4201, U.S.A.

E-mail address: speh@math.cornell.edu