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A unitarity criterion for p-adic groups

Dan Barbasch and Allen Moy
Rutgers University, Department of Mathematics, New Brunswick, NJ 08903, USA

1. Introduction

Let $G = G(\mathbb{F})$ be a split reductive group with connected center over a p-adic field $\mathbb{F}$. In [KL2] (see also [G]), the classification of the irreducible representations with Iwahori fixed vectors is obtained. This is a special case of a more general conjecture of Langlands [Ln] and is achieved as follows. Let $\mathscr{I}$ be an Iwahori subgroup of G and $\mathscr{H} = \mathscr{H}(G//\mathscr{I})$ the algebra of $\mathscr{I}$-biinvariant compactly supported functions on G. The category $\mathscr{C}(\mathscr{I})$ of admissible modules generated by their Iwahori fixed vectors is naturally equivalent to the category $\mathscr{C}(\mathscr{H})$ of finite dimensional $\mathscr{H}$-modules. The equivalence from $\mathscr{C}(\mathscr{I})$ to $\mathscr{C}(\mathscr{H})$ is $V \mapsto V^{\mathscr{I}}$. Under this equivalence V is hermitian precisely when $V^{\mathscr{I}}$ is hermitian.

A long standing question is whether the unitarity of V can be detected on $V^{\mathscr{I}}$ (see [B, C]). The main result in this paper is the following.

Theorem 1.1. *An irreducible real hermitian module $V \in \mathscr{C}(\mathscr{I})$ is unitary if and only if $V^{\mathscr{I}} \in \mathscr{C}(\mathscr{H})$ is unitary.*

The proof of Theorem 1.1 relies heavily on the classification of irreducible $\mathscr{H}$-modules in [KL2] and the idea of the signature of a hermitian module in [V2].

The methods employed are valid for arbitrary infinitesimal character. However Corollary 4.8 is false for non-real infinitesimal character. Therefore the assumption that V be real hermitian in Theorem 1.1 is essential. We hope to treat the general case in a subsequent paper using different techniques.

Let $K \supset \mathscr{I}$ be a special maximal compact subgroup of G. As a corollary to our methods, we obtain a characterization of the irreducible $\mathscr{H}$-modules in terms of their $\mathscr{H}_W = \mathscr{H}(K//\mathscr{I})$ structure. This can be thought an analogue of the classification of $(\mathfrak{g}, K)$ modules over $\mathbb{R}$ in terms of their lowest K-types [V1]. This is the content of Theorems 4.6 and 6.3.

Iwahori-Matsumoto have defined an involution $'$ on $\mathscr{C}(\mathscr{H})$ hence $\mathscr{C}(\mathscr{I})$, which preserves hermitian representations. The involution $'$ preserves unitarity on $\mathscr{C}(\mathscr{H})$. As an immediate corollary of Theorem 1.1, we have

Theorem 1.2. *The involution $'$ on real representations of $\mathscr{C}(\mathscr{I})$ preserves unitarity.*

As an application of Theorem 1.2, we obtain a large class of unitary isolated K-spherical representations. They can be described as follows. Let ${}^LG={}^LG(\mathbb{C})$ be the complex group dual to G. Every map

$$\phi\colon \mathrm{SL}(2,\mathbb{C})\to {}^LG \tag{1.1}$$

gives rise to a real tempered irreducible representation V_ϕ. In Theorem 8.1 we show that $W_\phi=V_\phi'$ is K-spherical. By Theorem 1.2, the W_ϕ's are unitary. They can be viewed as a special case of the more general conjecture in [A].

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2. Basic results

We establish notation and review some basic results. Fix a Borel subgroup $B=A_0N_0$ of G. In the notation of [HC] and [BW], let

$$\mathfrak{a}_0^*=X(A_0)\otimes\mathbb{R} \quad\text{and}\quad \mathfrak{a}_0=\mathrm{Hom}(X(A_0),\mathbb{R}). \tag{2.1}$$

Let (,) denote a Weyl group invariant inner product on the complexification $(\mathfrak{a}_0^*)_c$ of $\mathfrak{a}_0^*$. For $P=MN\supset B$, a parabolic subgroup, let

$$\begin{aligned}&\mathfrak{a}^*=X(M)\otimes\mathbb{R}\,(\subset\mathfrak{a}_0^*),\qquad \mathfrak{a}=\mathrm{Hom}(X(M),\mathbb{R})\,(\subset\mathfrak{a}_0),\\ &\text{and}\quad H\colon M\to\mathfrak{a}\text{ be the Harish-Chandra map.}\end{aligned} \tag{2.2}$$

If A is a maximal split torus in the center of M, view $X(A)$ as a subset of $\mathfrak{a}^*$. If σ is an admissible representation of M and $\nu\in\mathfrak{a}_c^*$, let $I(P,\sigma,\nu)$ denote the induced representation of right translation on the space of functions

$$\begin{aligned}\{f\colon G\to V_\sigma\,|\,&f\text{ locally constant and}\\ &f(mng)=\delta^{1/2}(m)\,\sigma(m)\,q^{\nu(H(m))}f(g)\}.\end{aligned} \tag{2.3}$$

We call the representation $I(\nu)=I(P,\sigma,\nu)$ where $P=B$ and σ is the trivial representation of A_0, an unramified principal series and any subquotient of it an $\mathscr{I}$-spherical representation. The following well known theorem justifies the term $\mathscr{I}$-spherical.

Theorem 2.1. *If σ is a subquotient of $I(\nu)$, then σ has a nonzero Iwahori fixed vector. Conversely, if X is an irreducible admissible representation of G with a nonzero Iwahori fixed vector, then X occurs as a subquotient of some $I(\nu)$.*

Let $\mathscr{C}(\mathscr{I})$ be the category of (equivalence classes of) admissible G-modules whose subquotients are generated by their $\mathscr{I}$-fixed vectors. Let

$$\mathscr{H}=\mathscr{H}(G//\mathscr{I})=\text{compactly supported } \mathscr{I}\text{-biinvariant functions on } G. \tag{2.4}$$

This is an algebra under convolution. Moreover, $\mathscr{H}$ acts on the set $V^{\mathscr{I}}$ of $\mathscr{I}$-fixed vectors of any admissible representation (π, V) by

$$\pi(f)\,v=\int_G f(g)\,\pi(g)\,v\,dg\,(v\in V^{\mathscr{I}}, f\in\mathscr{H}). \tag{2.5}$$

Denote by $\mathscr{C}(\mathscr{H})$ the category of finite dimensional $\mathscr{H}$-modules.

Theorem 2.2. *The map*

$$V\mapsto V^{\mathscr{I}}$$

is an equivalence of categories between $\mathscr{C}(\mathscr{I})$ and $\mathscr{C}(\mathscr{H})$.

Borel has shown in [B] that the inverse functor is

$$V^{\mathscr{I}}\mapsto C_c^\infty(G/\mathscr{I})\otimes_{\mathscr{H}} V^{\mathscr{I}}. \tag{2.6}$$

Kazhdan and Lusztig classify the irreducible objects in $\mathscr{C}(\mathscr{H})$.

An admissible representation (π, V) of G is hermitian if there is a hermitian form $\langle\,,\rangle$ on V such that

$$\langle\pi(g)\,v, w\rangle=\langle v, \pi(g^{-1})\,w\rangle\;(v, w\in V, g\in G). \tag{2.7}$$

If (π^h, V^h) denotes the hermitian dual of (π, V), then (π, V) is hermitian if and only if $(\pi, V)\approx(\pi^h, V^h)$. One can also define the notions of hermitian and unitarity for $\mathscr{H}$. Indeed, $\mathscr{H}$ is a * algebra with

$$f^*(g)=\overline{f((g)^{-1})}\,(f\in\mathscr{H}, g\in G). \tag{2.8}$$

An $\mathscr{H}$-module E is hermitian if there is a hermitian form $\langle\,,\rangle$ on E such that

$$\langle\pi(f)\,v, w\rangle=\langle v, \pi(f^*)\,w\rangle\;(v, w\in E, f\in\mathscr{H}). \tag{2.9}$$

In the equivalence of categories between $\mathscr{C}(\mathscr{I})$ and $\mathscr{C}(\mathscr{H})$, it follows from (2.6) that the property of being hermitian is preserved. In one direction, we merely restrict the form on V to $V^{\mathscr{I}}$. In the other direction the form is given by

$$\langle f_1\otimes v_1, f_2\otimes v_2\rangle_V=\langle\pi(f_2^* * f_1)\,v_1, v_2\rangle_{V^{\mathscr{I}}}. \tag{2.10}$$

Clearly, if V is unitary, then so is $V^{\mathscr{I}}$.

3. Standard and simple modules

We review the Langlands conjecture. Let $W_{\mathbb{F}}$ be the Weil group of $\mathbb{F}$. The basic facts about $W_{\mathbb{F}}$ can be found in [T]. A homomorphism

$$\phi\colon\ W_{\mathbb{F}} \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G \tag{3.1}$$

is admissible if the closure of $\phi(W_{\mathbb{F}})$ is reductive. According to a conjecture of Langlands [Ln], refined later by Langlands, Deligne and Lusztig, to each admissible homomorphism ϕ there should correspond a finite packet $\Pi(\phi)$ of representations of $G(\mathbb{F})$. Furthermore, as one varies over all ${}^L G$ conjugacy classes of ϕ's, the packets $\Pi(\phi)$ should partition the representations of G.

Langlands and Deligne have envisioned a description of those ϕ's for which the representations $\Pi(\phi)$ are $\mathscr{I}$-spherical. Let $I_{\mathbb{F}}$ be the inertia subgroup of $W_{\mathbb{F}}$ so that $W_{\mathbb{F}}/I_{\mathbb{F}} = \mathbb{Z}$. Denote the Frobenius generator by ϖ. Then, conjecturally, the $\mathscr{I}$-spherical packets are parameterized by admissible ϕ's satisfying

$$\phi\colon\ W_{\mathbb{F}}/I_{\mathbb{F}} \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G. \tag{3.2}$$

The condition of admissibility is that $s = \phi(\varpi)$ be semisimple. Call these ϕ's $I_{\mathbb{F}}$-spherical. This is often reformulated with the aid of the Jacobson-Morozov Theorem. Let

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{3.3}$$

be the standard triple in the Lie algebra $\mathrm{sl}(2, \mathbb{C})$. Set $s = \phi(\varpi)$, a semisimple element and $u = \phi(\exp(e))$, a unipotent element. Then, the $\mathscr{I}$-spherical packet parameters are pairs of commuting elements

$$(s, u)\colon\ s \text{ semisimple}, u \text{ unipotent}. \tag{3.4}$$

Denote by q the order of the residue field of $\mathbb{F}$. The element s is often replaced by

$$\sigma = s \cdot \phi(\exp(\log(q)/2)\, h). \tag{3.5}$$

The $\mathscr{I}$-spherical packets are then parametrized by pairs

$$(\sigma, u)\colon\ \sigma \text{ semisimple}, u \text{ unipotent with } \sigma u \sigma^{-1} = u^q. \tag{3.6}$$

The members of a packet $\Pi(\phi)$ have a conjectural parametrization in terms of a component group. Let $\mathscr{Z}$ be the center of ${}^L G$ and $C(\sigma, u)$ the centralizer of σ and u in ${}^L G$. Then, conjecturally, the members of $\Pi(\phi)$ are parametrized by representations of the component group

$$A(\phi, u) = C(\phi, u)/[C(\phi, u)^0\, \mathscr{Z}]. \tag{3.7}$$

The data (u, σ, q, ρ), $\rho \in A(\phi, u)$ is called the L-group data of an $\mathscr{I}$-spherical representation. In their proof of the Langlands conjecture, Kazhdan and Lusztig construct a packet of $\mathscr{H}$-modules for each pair (ϕ, u). Their construction is in fact more general. To describe their work we need to review the generic affine Hecke algebra.

Let $B=A_0 N_0$ be the Borel subgroup in Sect. 2. Denote by (W, S) the Weyl group of G. Let $\mathscr{P}$ denote the weight lattice of a maximal torus in ${}^LG$. Observe that $\mathscr{P}$ can be viewed as the group $\mathrm{Hom}[X(A_0):\mathbb{Z}]$. There is a canonical action $L \mapsto w(L)$, $(w \in W, L \in \mathscr{P})$ of W on $\mathscr{P}$. For $s \in S$, let $\alpha_s \in \mathrm{Hom}[\mathscr{P}:\mathbb{Z}]$ be the coweight corresponding to s. The generic affine Hecke algebra is the abstract algebra $\mathscr{H}(X)$ over $\mathbb{C}[X, X^{-1}]$ with generators T_s, $(s \in S)$, θ_L, $(L \in \mathscr{P})$ and the relations

$$\begin{aligned}
&T_s^2 = X + (X-1)\,T_s \quad \text{for } s \in S,\\
&T_s\,T_{s'}\,T_s \ldots = T_{s'}\,T_s\,T_{s'} \ldots \; s \neq s', \quad m \text{ factors on both sides,}\\
&\qquad \text{where } m \text{ is the order of } s s' \text{ in } W,\\
&T_s\,\theta_{s(L)} = \theta_L\,T_s - (X-1)\,\theta_L \text{ when } s \in S, L \in \mathscr{P} \text{ and } \alpha_s(L) = 1,\\
&T_s\,\theta_{s(L)} = \theta_L\,T_s \text{ when } s \in S, L \in \mathscr{P} \text{ and } \alpha_s(L) = 0,\\
&\theta_L\,\theta_{L'} = \theta_{L \otimes L'}, \quad L, L' \in \mathscr{P},\\
&\theta_{L'} = 1 \text{ when } L \text{ is the trivial element of } \mathscr{P}.
\end{aligned} \tag{3.8}$$

The subalgebra $\mathscr{H}_W(X)$ generated by the T_s's is the generic Hecke algebra of (W, S). Denote by $\mathscr{H}(1)$, the specialization of $\mathscr{H}(X)$ to $X=1$. It is clear that $\mathscr{H}(1)$ is the group algebra $\mathbb{C}[W\mathscr{P}]$, while $\mathscr{H}_W(1)$ is $\mathbb{C}[W]$. When X is specialized to q, Bernstein has shown that

$$\mathscr{H}(q) \quad \text{and} \quad \mathscr{H} \text{ are naturally isomorphic.} \tag{3.9}$$

We are now ready to explain the work of Kazhdan and Lusztig. For ϕ satisfying (3.2), let $\mathscr{B}_u$ denote the complex variety of Borel subgroups of ${}^LG$ containing $u=\exp(e)$ (notation 3.3). Decompose

$$s = \phi(\varpi) = s_e\,s_h \tag{3.10}$$

into its compact and hyperbolic parts. Let $\langle s_e \rangle$ (resp. $\langle s_h \rangle$) denote the smallest diagonalizable algebraic subgroup ${}^LG$ containing s_e (resp. s_h). Define $M \subset {}^LG \times \mathbb{C}^\times$ by

$$M = \{(g, 1) | g \in \langle s_e \rangle\} \times \{(g, 1) | g \in \langle s_h \rangle\} \times \{(\exp(yh), e^{2y}) | y \in \mathbb{C}\}. \tag{3.11}$$

The group M acts on $\mathscr{B}_u$ by

$$(g, t) \cdot B = g B g^{-1} ((g, t) \in M, B \in \mathscr{B}_u). \tag{3.12}$$

Kazhdan-Lusztig define an action of $\mathscr{H}(X)$ on the M-equivariant K-homology group $K^M(\mathscr{B}_u)$. This K-group is a free $R(M)$-module. Any $m \in M$, determines, by evaluation at m, an algebra homomorphism μ_m: $R(M) \to \mathbb{C}$. Set

$$K^M(\mathscr{B}_u)_m = \mathbb{C} \otimes_{\mu_m} K^M(\mathscr{B}_u) \tag{3.13}$$

where K^M is K-homology as in [KL2], Sect. 1.3.

Let p_2: ${}^LG \times \mathbb{C}^\times \to \mathbb{C}^\times$ be the projection map to the second factor, and set $t = p_2(m)$. Then, $K^M(\mathscr{B}_u)_m$ is a module for $\mathscr{H}(t)$. The group $A(s, u)$ acts on $K^M(\mathscr{B}_u)$

in a fashion which commutes with $\mathscr{H}(X)$ and specialization. For $\rho \in \hat{A}(u, s)$, define a standard module

$$\mathscr{M}_{u,m,\rho} = \mathrm{Hom}_{A(u,s)}[\rho\colon K^M(\mathscr{B}_u)_m] \tag{3.14}$$

whenever it is nonzero.

Theorem 3.1. ([KL2] 7.11) *Fix $t \in \mathbb{C}^\times$ which is not a root of unity. For each map*

$$\phi\colon\ W_{\mathbb{F}}/I_{\mathbb{F}} \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G$$

let

$$m = (\phi(\varpi \exp((\log(t)/2)\, h)), t). \tag{3.15}$$

Then

(1) *The standard module $\mathscr{M}_{u,m,\rho}$ has a unique simple quotient $\mathscr{L}_{u,m,\rho}$, which we call the Langlands quotient.*

(2) *Every simple module of $\mathscr{H}(t)$ is obtained in this fashion.*

The map ϕ is said to be

$$\begin{aligned} &\text{real if } s_e = 1,\\ &\text{tempered if } s_h = 1,\\ &L^2 \text{ if tempered, and no nontrivial torus } T \text{ centralizes both } s \text{ and } u. \end{aligned} \tag{3.16}$$

When $s = 1$, the ρ's defining standard modules coincide with those occuring in the Springer representation.

Theorem 3.2. ([KL2] 8.2, 8.3) *Assume $t = q$. Identify $\mathscr{H}(q)$ with $\mathscr{H}$. Let $\pi(\phi, \rho)$ be the representation of G whose Iwahori fixed vectors afford $\mathscr{L}_{u,s,q,\rho}$. Then*

(1) *$\pi(\phi, \rho)$ is tempered if and only if ϕ is tempered (here, $\mathscr{M}_{u,s,q,\rho} = \mathscr{L}_{u,s,q,\rho}$),*

(2) *$\pi(\phi, \rho)$ is square integrable if and only if ϕ is L^2.*

The packet $\Pi(\phi)$ is defined to be the set

$$\Pi(\phi) = \{\mathscr{L}_{u,s,q,\rho} \mid \rho \in A(u, s) \text{ and } \mathscr{M}_{u,s,q,\rho} \neq 0\}. \tag{3.17}$$

In the next result we specialize X to 1. We focus on a real packet. Here, the group M is connected. Choose m to be 1. Let $K(\mathscr{B}_u)$ be the K-homology group (over $\mathbb{C}$) of $\mathscr{B}_u$. As an immediate consequence of Proposition 5.11 in [KL2], we have the following fundamental result, important in determining the $\mathscr{H}_W(t)$ structure of $\mathscr{M}_{u,s,t,\rho}$.

Theorem 3.3. *The natural map $K^M(\mathscr{B}_u)_1 \to K(\mathscr{B}_u)$ is an isomorphism.*

A real tempered packet is a ϕ which is trivial on $W_{\mathbb{F}}$. It is thus a homomorphism

$$\phi\colon\ \mathrm{SL}(2, \mathbb{C}) \to {}^L G. \tag{3.18}$$

By the Jacobson-Morozov Theorem, the ${}^L G$ conjugacy classes of ϕ's correspond naturally to unipotent conjugate classes.

We close this section with a review of the center of $\mathscr{H}(X)$. Let $\mathscr{A}$ be the group algebra $\mathbb{C}\mathscr{P}$. The $\mathbb{C}[X, X^{-1}]$-subalgebra of $\mathscr{H}(X)$ generated by the θ_L's is $\mathscr{A}[X, X^{-1}]$. The following result of Bernstein describes the center of $\mathscr{H}(X)$.

Theorem 3.4. *The center of $\mathscr{H}(X)$ is $\mathscr{A}^W[X, X^{-1}]$.*

The center acts as scalars on any simple module. In analogy with real groups the character by which the center acts is called the infinitesimal character. A character of $\mathscr{A}^W$ is a Weyl orbit in $X(A_0)\otimes_{\mathbb{Z}}\mathbb{C}$. The exponential map

$$\exp\colon\ X(A_0)\otimes_{\mathbb{Z}}\mathbb{C}\to {}^LG \tag{3.19}$$

maps $X(A_0)\otimes_{\mathbb{Z}}\mathbb{C}$ to a maximal torus. It maps the infinitesimal character of $\mathscr{L}_{u,s,q,\rho}$ multiplied by $\log(q)$ to a conjugate of s.

4. $\mathscr{H}_W$ Structure of tempered modules

In this section, we determine the structure of real tempered modules. To accomplish this, we need to review Springer's work on the representations of W.

Given a unipotent element $u\in {}^LG(\mathbb{C})$, consider the cohomology groups $H^*(\mathscr{B}_u, \mathbb{C})$. For brevity, denote these groups by $H^*(\mathscr{B}_u)$. The centralizer $C(u)$ of u in ${}^LG(\mathbb{C})$ acts by conjugation on $\mathscr{B}_u$; therefore $C(u)$ acts on $H^*(\mathscr{B}_u)$. The action is easily seen to factor to the component group

$$A(u)=C(u)/[C(u)^0\mathscr{Z}]. \tag{4.1}$$

For $\rho\in\hat{A}(u)$, let $H^*(\mathscr{B}_u)^\rho$ be the ρ-component of $H^*(\mathscr{B}_u)$, i.e.

$$H^*(\mathscr{B}_u)^\rho=\operatorname{Hom}_{A(u)}[\rho\colon H^*(\mathscr{B}_u)]. \tag{4.2}$$

Springer has defined an action of W on $H^*(\mathscr{B}_u)$ and proved the following fundamental result.

Theorem 4.1.

(1) *The actions of W and $A(u)$ commute. Thus, W acts on each $H^*(\mathscr{B}_u)^\rho$, $\rho\in\hat{A}(u)$.*

(2) *The natural map*

$$H^*(\mathscr{B})\to H^*(\mathscr{B}_u)$$

induced by the inclusion $\mathscr{B}_u\subset\mathscr{B}$ is W-equivariant.

(3) *For $\rho\in\hat{A}(u)$, the representation $\sigma(u,\rho)$ of W on $H^{2d_u}(\mathscr{B}_u)$ ($d_u=\dim_{\mathbb{C}}(\mathscr{B}_u)$) is irreducible or zero.*

(4) $\sigma(u,1)\neq 0$.

(5) *Every $\sigma\in\hat{W}$ occurs as some $\sigma(u,\rho)$ with $\mathscr{O}_u$ and ρ uniquely determined.*

The map

$$(u,\rho)\mapsto\sigma(u,\rho) \tag{4.3}$$

is called the Springer correspondence, and the data (u, ρ) Springer data. The Springer correspondence has been calculated explicitly. Let sgn denote the sign character of W.

Proposition 4.2. *When u is the principal unipotent element, $A(u)$ is trivial and*

$$\sigma(u, 1) = \mathrm{sgn}.$$

Corollary 4.3. *For any unipotent element u, the action of W on $H^0(\mathscr{B}_u)^1$ is the* sgn *representation.*

Proof. The variety $\mathscr{B}_u$ is connected; hence $H^0(\mathscr{B}_u)^1 = H^0(\mathscr{B}_u) = \mathbb{C}$. The action of W on this one dimensional space now follows from the Proposition 4.2 and Theorem 4.1 part (2).

For suggestive notational convenience, call the representation of W on $H^*(\mathscr{B}_u)^\rho$ a standard module for W. Borho and MacPherson have shown

Theorem 4.4.
(1) $\mathrm{Hom}_W[\sigma(u,\rho) : H^i(\mathscr{B}_u)] = 0$ *if* $i \neq 2d_u$.
(2) $\mathrm{Hom}_W[\sigma(x,\rho) : H^*(\mathscr{B}_u)] = 0$ *for* $u \notin \bar{\mathcal{O}}_x$.

Theorem 4.4 says that the standard modules of W form a basis for $R(W)$, and the change of basis matrix between the standard and irreducible modules bases of $R(W)$ is (with a suitable ordering) upper triangular.

Let $H_*(\mathscr{B}_u)$ be the direct sum of the homology groups with coefficients in $\mathbb{C}$. This group is also dual to $H^*(\mathscr{B}_u)$ and therefore carries a natural action of $\mathbb{C}W$ induced by the Springer action in cohomology.

We now define an action of $\mathbb{C}W$ on the standard modules $\mathscr{M}_{u,s,t,\rho}$. Consider the group algebra $\mathbb{Q}[X^{1/2}, X^{-1/2}]\, W$ and the Hecke algebra $\mathscr{H}_{\mathbb{Q}[X^{1/2}, X^{-1/2}]}$ over $\mathbb{Q}[X^{1/2}, X^{-1/2}]$ defined by generators $T_s (s \in S)$, $\theta_L (L \in \mathscr{P})$ and the relations (3.8).

Theorem 4.5. [Ls1] *There is a natural map*

$$\begin{aligned} \iota &: \mathbb{Q}[X^{1/2}, X^{-1/2}]\, W \to \mathscr{H}_{\mathbb{Q}[X^{1/2}, X^{-1/2}]}, \\ \iota(w') &= \sum_w f_{w',w} T_w, \qquad f_{w',w} \in \mathbb{Q}[X^{1/2}, X^{-1/2}] \end{aligned} \tag{4.4}$$

which becomes an isomorphism when $\mathbb{Q}[X^{1/2}, X^{-1/2}]$ is extended to $\mathbb{Q}(X^{1/2})$.

If X is specialized to $t \in \mathbb{C}^\times$ which is not a nontrivial root of unity, we obtain an isomorphism

$$\iota_t : \mathbb{C}W \to \mathscr{H}_W(t).$$

This allows us to view $\mathscr{M}_{u,s,t,\rho}$ as a $\mathbb{C}W$-module.

We are ready to determine the $\mathscr{H}_W(X)$ structure of real modules. Fix a real $I_{\mathbb{F}}$-spherical map ϕ (notation (3.2)). In this case, the group M of (3.11) is

$$M = \{(g, 1) \mid g \in \langle s_h \rangle\} \times \{(\exp(yh), e^{2y}) \mid y \in \mathbb{C}\}. \tag{4.5}$$

The algebra $R(M)$ is a Laurent polynomial algebra in several variables.

Theorem 4.6. *For ϕ real and $m \in M$, $K^M(\mathscr{B}_u)_m$ and $H_*(\mathscr{B}_u)$ are isomorphic as $\mathbb{C}W$ modules.*

This is a rephrasing of Theorem 6.4 in [Ls3]. Its proof will appear in [Ls4].

To give some idea of the proof, we show how $K^M(\mathscr{B}_u)$ and $K(\mathscr{B}_u)_1$ are equivalent as $\mathbb{C}W$ modules.

The K-group $K^M(\mathscr{B}_u)$ is a finite rank free module over $R(M)$. Pick a basis $e_1, e_2, \ldots, e_d$ of $K^M(\mathscr{B}_u)$ as an $R(M)$-module. For $s \in S$ and $L \in \mathscr{P}$, write

$$T_s(e_i) = \sum_i f^s_{i,j} e_j, \quad f^s_{i,j} \in R(M), \tag{4.6a}$$

$$\theta_L(e_i) = \sum_i f^\theta_{i,j} e_j, \quad f^\theta_{i,j} \in R(M). \tag{4.6b}$$

Let

$$\begin{aligned} &\Theta_t\colon\ M \times W \to \mathbb{C} \\ &(m, w) \mapsto \operatorname{trace}(w, K^M(\mathscr{B}_u)_m). \end{aligned} \tag{4.7}$$

We conclude from Theorem 4.5 and (4.6) that Θ_m is a Laurent polynomial in several variables. Fix an irreducible representation E of W. Let Θ_E be the character of E. The multiplicity of E in $K^M(\mathscr{B}_u)_m$ is of course

$$d_E(m) = |W|^{-1} \sum \Theta_m(w)\, \overline{\chi_E(w)}. \tag{4.8}$$

This is a Laurent polynomial, which is integer valued. Thus d_E is constant, indeed equal to $d_E((1,1))$, the multiplicity of E in $H_*(\mathscr{B}_u)$. Hence $K^M(\mathscr{B}_u)_m$ and $K^M(\mathscr{B}_u)_1 \approx K(\mathscr{B}_u)$ are equivalent modules of W.

Remark. By property (1.3.m2) in [KL2], there is a natural isomorphism

$$C\colon\ K(\mathscr{B}_u) \to H_*(\mathscr{B}_u) \tag{4.9}$$

which is $A(u)$-equivariant. The difficulty is to show that this map is also W-equivariant. The fact that these are equivalent as $\mathbb{C}W$ modules follows from the proof in [Ls4]. The stronger equivariance statement is not known.

We say more about the real tempered case. Here,

$$M = \{(\exp(yh), e^{2y}) \mid y \in \mathbb{C}\}, \tag{4.10}$$

so $R(M) = \mathbb{C}[X, X^{-1}]$. The groups $A(u, s)$ and $A(u)$ coincide. They act on the K-group $K^M(\mathscr{B}_u)$. For $\rho \in \hat{A}(u)$, set

$$\mathscr{M}_{u,\rho} = \operatorname{Hom}_{A(u)}[\rho\colon K^M(\mathscr{B}_u)]. \tag{4.11}$$

The group $\mathscr{M}_{u,\rho}$ is a finite rank free module over $R(M) = \mathbb{C}[X, X^{-1}]$. When X is specialized to t, the resulting module is $\mathscr{M}_{u,s,t,\rho}$. The above proof applies, to yield

Theorem 4.7. *For ϕ real tempered and m as in* (3.15), *$\mathscr{M}_{u,m,\rho}$ and $H_*(\mathscr{B}_u)^\rho$ are equivalent $\mathbb{C}W$-modules.*

Corollary 4.8. *As $\mathscr{H}_W(q)$-modules, the real tempered representations of $\mathscr{H}$ are linearly independent.*

Proof. This is an easy consequence of Theorem 4.7 and the remark after Theorem 4.4.

5. Signature theorem

In this section, we establish the p-adic analogue of a fundamental result of Vogan in [V2]. It says the signature character of an $\mathcal{I}$-spherical hermitian module V can be expressed in terms of K-characters of tempered modules. The methods employed are Vogan's ideas transposed to the p-adic setting. We begin by recalling the Langlands classification (see [BW, S]) of representations in terms of tempered representations. In the notation of section 2, fix a Borel subgroup $B=A_0N_0$ of G and a parabolic subgroup $P=MN\supset B$. Let σ be a tempered representation of M, and $\nu\in\mathfrak{a}^*$ satisfying $(\nu,\alpha)>0$ for all roots α of A occuring in N. Let $\bar{N}$ denote the unipotent radical opposite to N and $\bar{P}=M\bar{N}$. Then

Theorem 5.1.
(1) *There is an absolutely convergent integral intertwining operator*

$$\mathcal{L}\colon\ I(P,\sigma,\nu)=\mathrm{Ind}_P^G(\sigma\otimes\nu)\to I(\bar{P},\sigma,\nu)=\mathrm{Ind}_{\bar{P}}^G(\sigma\otimes\nu)$$
$$(\mathcal{L}f)(x)=\int_{\bar{N}} f(nx)\,dn$$

so that

$$J(\sigma\otimes\nu)=I(P,\sigma,\nu)/\ker\mathcal{L}$$

is the unique irreducible quotient of $I(P,\sigma,\nu)$.

(2) *Any irreducible admissible representation π of G occurs uniquely as some $J(\sigma\otimes\nu)$. The parameter ν is called the Langlands parameter and denoted by λ_π.*

(3) *If τ is an irreducible constituent of $I(P,\sigma,\nu)$, then $\lambda_\tau\leqq\nu$ with equality occuring if and only if $\tau=J(\sigma\otimes\nu)$.*

The irreducible hermitian representations are those $J(\sigma\otimes\nu)$, for which there is an $w\in\mathrm{Weyl}(G,A)$ satisfying

$$-\nu=w(\nu)\quad\text{and}\quad\sigma\approx w(\sigma). \tag{5.1}$$

In this situation, $I(P,\sigma,\nu)$ admits a possibly degenerate hermitian form as follows. Pair the two spaces $I(P,\sigma,\nu)$ and $I(P,\sigma,-\nu)$ via the natural pairing

$$(f_1,f_2)=\int_G\langle f_1(g),f_2(g)\rangle_\sigma\,dg\qquad(f_1\in I(P,\sigma,\nu),f_2\in I(P,\sigma,-\nu)). \tag{5.2}$$

Because of (5.1), there is a choice of an isomorphism γ between $I(P,\sigma,-\nu)$ and $I(\bar{P},\sigma,\nu)$ such that the hermitian form on $I(P,\sigma,\nu)$ is given by

$$\langle f_1,f_2\rangle=(f_1,\gamma^{-1}\mathcal{L}f_2). \tag{5.3}$$

The choice of γ depends on the isomorphism $\sigma\approx w(\sigma)$ and not on ν. The radical of $\langle,\rangle$ is precisely $\ker\mathcal{L}$. The quotient form on $J(\sigma\otimes\nu)$ is a nonzero multiple of the unique hermitian form.

Suppose $\pi = J(\sigma \otimes \nu)$ is $\mathscr{I}$-spherical. Let (u, τ, q, ρ) be the L-group data for π. The infinitesimal character of π is

$$\chi_\pi = \log_q(\tau). \tag{5.4}$$

Let s_q denote $\phi(\exp(\log(q)/2)\, h)$. Write τ as $\tau = s_e\, s_h\, s_q$, where s_e and s_h are the elliptic and hyperbolic parts of $\phi(\varpi)$ respectively. Then

$$\nu = \log_q(s_h) \quad \text{and} \quad \operatorname{Re}(\chi_\pi) = \log_q(s_h\, s_q). \tag{5.5}$$

Fix a special maximal compact subgroup K containing $\mathscr{I}$. Under the appropriate conventions, the group K is $G(\mathscr{R})$ the $\mathscr{R}$-rational points of G, where $\mathscr{R}$ is the ring of integers in $\mathbb{F}$. The Hecke algebra $\mathscr{H}(K//\mathscr{I})$ is a subalgebra of $\mathscr{H}(G//\mathscr{I})$. In fact,

$$\mathscr{H}(K//\mathscr{I}) \text{ is canonically isomorphic to } \mathscr{H}_W(q). \tag{5.6}$$

For ease of notation, we write $\mathscr{H}_W$ for $\mathscr{H}(K//\mathscr{I})$.

Let (π, V) be an irreducible representation of G with a nonzero hermitian form $\langle , \rangle$. For each irreducible representation δ of K fix a positive definite hermitian form. Let $V(\delta)$ be the δ-isotypical component of V. The finite dimensional space $F(\delta) = \operatorname{Hom}_K[\delta \colon V(\delta)]$ acquires a nondegenerate form from $\langle , \rangle$. Denote the dimension and signature of $F(\delta)$ by $m(\delta)$ and $(p(\delta), q(\delta))$ respectively. Trivially, $m(\delta) = p(\delta) + q(\delta)$. Define the formal K-character of V to be

$$\theta_K(V) = \sum_\delta m(\delta)\, \delta. \tag{5.7}$$

Define the signature character of $\langle , \rangle$ to be the pair of formal sums

$$\Big(\sum_\delta p(\delta)\, \delta, \sum_\delta q(\delta)\, \delta\Big). \tag{5.8}$$

Theorem 5.2. *Suppose V is an irreducible $\mathscr{I}$-spherical representation of G admitting a nonzero hermitian form $\langle , \rangle$. There are finitely many irreducible tempered $\mathscr{I}$-spherical modules $V_1, \ldots, V_n$ and integers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ so that the signature of $\langle , \rangle$ is*

$$(a_1\, \theta_K(V_1) + \ldots + a_n\, \theta_K(V_n), b_1\, \theta_K(V_1) + \ldots + b_n\, \theta_K(V_n)).$$

The proof of the signature theorem follows along the same lines as Vogan's proof [V2] in the real case. We need a few preliminary results.

Theorem 5.3. *The K-character of an $\mathscr{I}$-spherical representation is a linear combination of K-characters of tempered representations.*

Proof. It is enough to prove the assertion for an irreducible $\mathscr{I}$-spherical representation V. Let (u, σ, q, ρ) be the L-group data for V, and let σ_e be the compact part of σ. Denote the set of tempered modules $\mathscr{L}_{u,\theta,q,\rho}$ with $\theta_e = \sigma_e$ by $\mathscr{T}(\sigma_e)$. This is a finite set. Write V in the Langlands classification as $J(\tau \otimes \nu)$. The K-character of $I(P, \tau, \nu)$ is that of the tempered representation $I(P, \tau, 0)$. This means the assertion is true when $J(\tau\otimes\nu)$ equals $I(P,\tau,\nu)$. Suppose they are not equal. Let π be an irreducible composition factor of $I(P,\tau,\nu)$ different from $J(\tau\otimes\nu)$. Write π in the Langlands classification as $J(\tau'\otimes\lambda_\pi)$, where $\lambda_\pi<\nu$. Denote the infinitesimal characters of V, σ and σ' by χ_V, χ_σ and χ'_σ respectively. It follows from

(1) $\sigma'\in\mathscr{T}(s_e)$

(2) $\lambda_\pi+\mathrm{Re}(\chi_{\tau'})=\mathrm{Re}(\chi_V)=\nu+\mathrm{Re}(\chi_\tau)$

that the length of $\nu-\lambda_\pi$ is bounded below by a positive number. The theorem follows by an induction on the length of the Langlands parameter.

The next result is Lemma 3.9 in [V2].

Lemma 5.4. *Suppose V is an $\mathscr{I}$-spherical representation admitting a nondegenerate hermitian form $\langle,\rangle$. Let*

$$\Big(\sum_\delta p(\delta)\,\delta,\sum_\delta q(\delta)\,\delta\Big)$$

be the signature of $\langle,\rangle$. Then, there exist irreducible modules $Y_1,\ldots,Y_n$ and irreducible hermitian modules $X_1,\ldots,X_m$ such that (in the Grothendieck group)

$$V=\sum_i X_i+\sum_j (Y_j+Y_j^h).$$

Furthermore, if m_j is the K-character of Y_j and (p_i,q_i) is the signature character of X_i, then

$$p(\delta)=\sum_i p_i(\delta)+\sum_j m_j(\delta),$$

$$q(\delta)=\sum_i q_i(\delta)+\sum_j m_j(\delta).$$

Suppose V is an irreducible hermitian $\mathscr{I}$-spherical representation. Let $J(\sigma\otimes\nu)$ be the Langlands classification realization of V. Since V is hermitian, the standard module $I(P,\sigma,\nu)$ carries the hermitian form (5.3). This standard module is a member of the one parameter family

$$X_t=I(P,\sigma,t\nu)\qquad(t\in\mathbb{R}).\tag{5.9}$$

The representation X_0 is tempered while X_1 is the standard representation $I(P,\sigma,\nu)$. Because of the decomposition $G=PK$, the spaces X_t, which are all clearly hermitian, can be realized on the fixed space of functions

$$\begin{aligned}X=\{f\colon K\to V_\sigma\,|\,&f\text{ locally constant and}\\ &f(mnk)=\sigma(m)f(g)\ m\in M\cap K,\ n\in N\cap K\}.\end{aligned}\tag{5.10}$$

View $\mathscr{L}_t$ as an operator on X. The function $t\mapsto\gamma^{-1}\mathscr{L}_t$ can be continued to a meromorphic operator. The operators $\gamma^{-1}\mathscr{L}_t$ can be multiplied by a meromorphic function in t which yields analytic intertwining operators $t\mapsto\mathscr{A}_t$ on X. From this, we can construct an analytic family of hermitian forms $\langle,\rangle_t$ on X, namely

$$\langle f_1,f_2\rangle_t=(f_1,\mathscr{A}_t f_2),\tag{5.11}$$

with the property

$$\operatorname{rad}(\langle , \rangle_t) = \ker(\mathscr{A}_t) = \ker(\mathscr{L}_t) \quad \text{for } t > 0. \tag{5.12}$$

We need to know how the signature changes from 0 to 1. It does not change over intervals where the form $\langle , \rangle$ is nondegenerate. To determine how the signature changes at a reducibility point we review the Jantzen filtration of X at an arbitrary point t_0. This filtration is the sequence of subspaces

$$X = X^0 \supset X^1 \supset \dots \supset X^N = \{0\} \tag{5.13}$$

defined as follows. The space X^h is the set of vectors $v \in X$ for which there is a neighborhood U of t_0 and an analytic function

$$f_v \colon U \to X \tag{5.14}$$

satisfying

(1) f_v takes values in a finite dimensional K-subspace of X

(2) $f_v(t_0) = v$

(3) $\forall v' \in X$ the function $t \mapsto \langle f_v(t), v' \rangle_t$ vanishes at t_0 to order at least n.

Define a hermitian form $\langle , \rangle^n$ on X^n by the formula

$$\langle v, v' \rangle^n = \lim_{t \to t_0} \frac{1}{(t - t_0)^n} \langle f_v(t), f_{v'}(t) \rangle_t, \tag{5.15}$$

where f_v and $f_{v'}$ are choosen as above. The limit depends only on v and v' and not on the particular choice of f_v and $f_{v'}$. Jantzen has shown (see Theorem 3.2 in [V2])

Theorem 5.5. *The form $\langle , \rangle^n$ on X^n has radical exactly X^{n+1}.*

Let (p_n, q_n) be the signature character of $\langle , \rangle^n$. The following theorem is Proposition 3.3 in [V2].

Theorem 5.6.

(1) *For $t - t_0$ small positive, $\langle , \rangle_t$ has signature character*

$$\left(\sum_n p_n, \sum_n q_n\right).$$

(2) *For $t - t_0$ small negative, $\langle , \rangle_t$ has signature character*

$$\left(\sum_{n \text{ even}} p_n + \sum_{n \text{ odd}} q_n, \sum_{n \text{ odd}} p_n + \sum_{n \text{ even}} q_n\right).$$

Proof of Theorem 5.2. By Theorem 5.3 and Lemma 5.4, we may assume V is an irreducible hermitian representation, say $V = J(\sigma \otimes \nu)$. Let $t_1 < \dots < t_{r-1}$ be the points in $(0, 1)$ where A_t is not an isomorphism. They are the points where $I(P, \sigma, t\nu)$ is reducible. Set $t_0 = 0$ and $t_r = 1$. Let

$$(p^j, q^j) \text{ be the signature of } \langle , \rangle \text{ on the open interval } (t_{j-1}, t_j). \tag{5.16}$$

Denote the Jantzen filtration of X_{t_j} by

$$X_{t_j} = X^0_{t_j} \supset X^1_{t_j} \supset \dots, \tag{5.17}$$

and the signature of $X^n_{t_j}/X^{n+1}_{t_j}$ by (p^j_n, q^j_n). The representation X_0 is tempered. Consequently $X^n_{t_0}/X^{n+1}_{t_0}$ is tempered and therefore p^0_n and q^0_n are K-characters of tempered representations. By Theorem 5.6, both p^1 and q^1 are linear combinations of tempered K-characters. Exactly as in [V2], we have

$$\text{signature of } V = (p^1, q^1) + \sum_{l=1}^{r-1} \sum_m (p^l_{2m+1}, q^l_{2m+1})$$
$$- \sum_{l=1}^{r} \sum_m (q^l_{2m+1}, p^l_{2m+1}) - \sum_{m>0} (p^r_{2m}, q^r_{2m}) \tag{5.18}$$

By induction on the length of the Langlands parameter as explained in Theorem 5.3, we can assume the result for all terms on the right; hence V also satisfies the hypothesis. This completes the proof. Note that, if V is real then the V_i's can also be taken real.

To be able to apply Theorem 5.2 to the proof of Theorem 1.1, we need to relate the K-character and signature character of a hermitian representation V to analogous characters of $V^{\mathscr{I}}$. Let E be a hermitian $\mathscr{H}$-module. If δ is an irreducible hermitian module of $\mathscr{H}$, let $(p(\delta), q(\delta))$ be the signature of $\operatorname{Hom}_{\mathscr{H}_W}[\delta: E]$. In analogy with (5.6) and (5.7), the formal $\mathscr{H}_W$-character and signature character are

$$\theta_{\mathscr{H}_W}(E) = \sum_\delta (p(\delta) + q(\delta))\, \delta, \quad \text{and} \quad \Big(\sum_\delta p(\delta)\, \delta, \sum_\delta q(\delta)\, \delta\Big) \tag{5.19}$$

respectively. Let $\mathscr{S}$ be the set of irreducible representations of K occuring in the induced representation $\operatorname{Ind}^K_{\mathscr{I}} 1$. The map

$$\delta \mapsto \delta^{\mathscr{I}} \tag{5.20}$$

is a bijection between $\mathscr{S}$ and the simple $\mathscr{H}_W$-modules. Suppose $E = V^{\mathscr{I}}$. Write the K-character of V as the sum

$$\theta_K(V) = \sum_{\delta \in \mathscr{S}} m(\delta)\, \delta + \sum_{\delta \notin \mathscr{S}} m(\delta)\, \delta. \tag{5.21}$$

The $\mathscr{H}_W$-character of E is

$$\theta_{\mathscr{H}_W}(E) = \sum_{\delta \in \mathscr{S}} m(\delta)\, \delta^{\mathscr{I}}. \tag{5.22}$$

The signature character of E is gotten from that of V in the same fashion.

Theorem 5.7. *Suppose $V_1, \ldots, V_m$ are $\mathscr{I}$-spherical real tempered representations of G. Then, a $\mathbb{Z}$ combination*

$$r_1\,\theta_K(V_1)+\ldots+r_m\,\theta_K(V_m)=0$$

if and only if

$$r_1\,\theta_{\mathscr{H}_W}(V_1^{\mathscr{I}})+\ldots+r_m\,\theta_{\mathscr{H}_W}(V_m^{\mathscr{I}})=0.$$

Proof. This is a consequence of Corollary 4.8.

As an application of Theorems 5.2 and 5.7 we prove Theorem 1.1.

Proof of Theorem 1.1. Denote by $\langle,\rangle$ the hermitian form on V, and by $\langle,\rangle^{\mathscr{I}}$ its restriction to $V^{\mathscr{I}}$. Assume $\langle,\rangle^{\mathscr{I}}$ is positive. We need to show $\langle,\rangle$ is also positive. Let $\mathscr{T}$ denote the set of $\mathscr{I}$-spherical real tempered representations of G. By Theorem 5.2, the signature of $\langle,\rangle$ can be written as a $\mathbb{Z}$ combination of tempered K-characters

$$\text{signature of}\langle,\rangle=\Big(\sum_{\pi\in\mathscr{T}}a_\pi\,\theta_K(\pi),\ \sum_{\pi\in\mathscr{T}}b_\pi\,\theta_K(\pi)\Big). \tag{5.23}$$

Then

$$\text{signature of}\langle,\rangle^{\mathscr{I}}=\Big(\sum_{\pi\in\mathscr{T}}a_\pi\,\theta_{\mathscr{H}_W}(\pi),\ \sum_{\pi\in\mathscr{T}}b_\pi\,\theta_{\mathscr{H}_W}(\pi)\Big). \tag{5.24}$$

From

(1) the linear independence of the $\theta_{\mathscr{H}_W}(\pi)$'s

(2) and the fact that $\langle,\rangle^{\mathscr{I}}$ is positive definite,

we conclude the b_π's are zero. Thus $\langle,\rangle$ is positive.

Iwahori and Matsumoto have defined an algebra involution $'$ on $\mathscr{H}$. It is the specialization of the involution of $\mathscr{H}(X)$ given by

$$T_s\mapsto -X\,T_s^{-1}\ s\in S\quad\text{and}\quad \theta_L\mapsto\theta_{L^{-1}}\ L\in\mathscr{P}. \tag{5.25}$$

An easy verification shows $'$ commutes with the $*$ operation (see 2.7) on $\mathscr{H}$. This means $'$ takes a hermitian module of $\mathscr{H}$ to a hermitian module. The same statement is also true for unitary modules. As a corollary to this observation and Theorem 1.1, we obtain Theorem 1.2.

We observe here that

$$'\ \text{takes the infinitesimal character to its inverse.} \tag{5.27}$$

This follows from Theorem 3.4 and (5.25).

6. Lowest K-types

We explain the connection between the standard modules in the Langlands classification and the standard modules of Kazhdan-Lusztig. To do this we need to recall the interpretation of parabolic induction in terms of Hecke algebras. Fix a parabolic subgroup $P=LU$ containing B. The center of L is connected. In particular, the methods of Kazhdan-Lusztig apply to L. We take the Iwahori subgroup of L to be the intersection

$$\mathcal{I}_L = L \cap \mathcal{I}. \tag{6.1}$$

The Hecke algebra $\mathcal{H}_L = \mathcal{H}(L//\mathcal{I}_L)$ can be embedded into $\mathcal{H} = \mathcal{H}(G//\mathcal{I})$ as follows. Denote by S_L the set of simple reflections contained in $\mathrm{Weyl}(L, A_0)$. Let $\mathcal{H}_L(X)$ be the $\mathbb{C}[X, X^{-1}]$-subalgebra of $\mathcal{H}(X)$ generated by $\mathcal{A}$ and $T_s \in S_L$. Let $\mathcal{H}_L(q)$ be the specialization of $\mathcal{H}_L(X)$ to $X = q$. By (3.8), $\mathcal{H}_L = \mathcal{H}_L(q)$. Hence, it is a subalgebra of $\mathcal{H} = \mathcal{H}(q)$. Suppose V is an $\mathcal{I}_L$-spherical representation of L. Bernstein has shown

Theorem 6.1.

$$(\mathrm{Ind}_P^G(V))^{\mathcal{I}} = \mathcal{H} \otimes_{\mathcal{H}_L} V^{\mathcal{I}_L}.$$

Let ${}^L L$ be the Levi subgroup of ${}^L G$ dual to L. Fix an $I_{\mathbb{F}}$-spherical map ϕ

$$\phi\colon\ W_{\mathbb{F}}/I_{\mathbb{F}} \times \mathrm{SL}(2, \mathbb{C}) \to {}^L L. \tag{6.2}$$

Let $A_L(u)$ denote the component group of $u = \phi(\exp(e))$ in ${}^L L$, and $\hat{\mathcal{B}}_u$, the variety of Borel subgroups in ${}^L L$ containing u. The K-group $K^M(\hat{\mathcal{B}}_u)_m$ is a module for $\mathcal{H}_L$.

Theorem 6.2. ([KL2], 7.8) *As $\mathcal{H}$-modules*

$$K^M(\mathcal{B}_u)_m = \mathcal{H} \otimes_{\mathcal{H}_L} K^M(\hat{\mathcal{B}}_u)_m.$$

We are ready to compare the standard modules of Langlands and those of Kazhdan-Lusztig. Fix an admissible homomorphism ϕ, and let $\phi(\varpi) = s_e s_h$ be the decomposition (3.10). The centralizer ${}^L L$ of s_h in ${}^L G$ is a Levi subgroup. Define

$$\phi_{\mathrm{temp}}\colon\ W_{\mathbb{F}}/I_{\mathbb{F}} \times \mathrm{SL}(2, \mathbb{C}) \to {}^L L \tag{6.3}$$

by $\phi_{\mathrm{temp}}(g) = \phi(g)$, $(g \in \mathrm{SL}(2, \mathbb{C}))$ and $\phi_{\mathrm{temp}}(\varpi) = s_e$. The representation τ of L corresponding to a nonzero $K^M(\hat{\mathcal{B}}_u)_{m,\rho}$ is tempered. The element $v = \log_q(s_h)$ determines a positive chamber in $\mathfrak{a}^*$. Let $P = LU$ be the parabolic subgroup determined by this chamber. Then

$$\mathcal{M}_{u,m,\rho} = I(P, \tau, v)^{\mathcal{I}} \quad \text{and} \quad \mathcal{L}_{u,m,\rho} = J(\tau \otimes v)^{\mathcal{I}}. \tag{6.4}$$

We describe how to determine the simple quotient of a standard module in terms of K-types. Our description applies to the Kazhdan-Lusztig standard modules. This description can be transferred to the Langlands standard modules via (6.4) and (5.20).

Define a partial ordering on the representations of W by

$$\sigma(u, \rho) \leqq \sigma(u', \rho') \quad \text{when} \quad u \in \overline{\mathcal{O}_{u'}}. \tag{6.5}$$

In this partial ordering, the sgn (resp. trivial) representation of W is a maximal (resp. minimal) element. Fix a ϕ which is real. The group $A(u, s)$ is naturally a subgroup of $A(u)$ because in a connected algebraic group, the centralizer of a torus is connected. For each $\rho\in\hat{A}(u,s)$ we define a subset $\Lambda(\phi,\rho)$ of $\hat{W}$ by

$$\Lambda(\phi,\rho)=\{\sigma(u,\psi)|\psi \textit{ a representation of } A(u) \textit{ containing } \rho\}. \tag{6.6}$$

In the special case when ϕ is real tempered, $A(u)=A(u,s)$ and $\Lambda(\phi,\rho)=\{\sigma(u,\rho)\}$.

Theorem 6.3. *Fix ϕ real and $\rho\in\hat{A}(u,s)$.*

(1) *Each $\sigma\in\Lambda(\phi,\rho)$ occurs in $\mathcal{M}_{u,m,\rho}$. Among the representations of W occuring in $\mathcal{M}_{u,m,\rho}$, they are minimal for the partial order $\leqq$.*

(2) *For any $\sigma\in\Lambda(\phi,\rho)$, the smallest subquotient of $\mathcal{M}_{u,m,\rho}$ containing $\sigma(u,\psi)$ is $\mathcal{L}_{u,m,\rho}$.*

Proof. Assertion (1) is a consequence of Theorems 4.4 and 4.2. To prove (2), we do an induction on the dimension of $\mathcal{B}_u$. When u is a principal unipotent element the assertion is obvious. Assume the assertion is true for all u' satisfying $\dim(\mathcal{B}_{u'})<\dim(\mathcal{B}_u)$. By Theorem 4.2 and (1), a necessary condition for $\mathcal{L}_{u',m',\rho'}$ to appear as a constituent of $\mathcal{M}_{u,m,\rho}$ is

$$u\in\overline{\mathcal{O}_{u'}}. \tag{6.7}$$

For those u' satisfying (6.9) and $\mathcal{O}_{u'}\neq\mathcal{O}_u$, the $\mathbb{C}W$-module $\mathcal{L}_{u',m',\rho'}$ does not contain any representation in $\Lambda(\phi,\rho)$. The assertion follows.

Theorem 6.3 was obtained earlier for $\mathrm{GL}(n)$ by Rogawski [R].

7. Reducibility

We state here explicitly some results on reducibility of tempered representations which can be found implicity in [KL2]. Suppose (u,σ,q,ρ) are L-group data for a real module. Assume further that the associated map ϕ maps $\mathrm{SL}(2,\mathbb{C})$ into a Levi subgroup ${}^LL$ of ${}^LG$. By Theorems 4.1 and 6.1 (under the assumptions of Theorem 6.2, [KL2]),

$$\begin{aligned}\bigoplus_{\rho\in\hat{A}(u)}(K^M(\mathcal{B}_u)_{m,\rho}\otimes\rho)&=K^M(\mathcal{B}_u)_m\\&=\mathcal{H}\otimes_{\mathcal{H}_L}K^M(\hat{\mathcal{B}}_u)_m\\&=\mathcal{H}\otimes_{\mathcal{H}_L}\Big(\bigoplus_{\psi\in\hat{A}_L(u)}K^M(\hat{\mathcal{B}}_u)_{m,\psi}\otimes\psi\Big)\\&=\bigoplus_{\psi\in\hat{A}_L(u)}(\mathcal{H}\otimes_{\mathcal{H}_L}K^M(\hat{\mathcal{B}}_u)_{m,\psi})\otimes\psi.\end{aligned} \tag{7.1}$$

This enables us to calculate the multiplicity of the irreducible tempered module $K^M(\mathcal{B}_u)_{m,\rho}$ in the module induced from $K^M(\hat{\mathcal{B}}_u)_{m,\psi}$. Indeed, if $m(\rho,\psi)$ is the multiplicity of ψ in the restriction of ρ to $A_L(u)$, then

$$\mathcal{H}\otimes_{\mathcal{H}_L}K^M(\hat{\mathcal{B}}_u)_{m,\psi}=\bigoplus_{\rho\in\hat{A}(u)}\dim(\rho)\,m(\rho,\psi)\,K^M(\mathcal{B}_u)_{m,\rho}. \tag{7.2}$$

We give two examples where the induced module is not multiplicity free.

Example 7.1. Take ${}^LG = E_r(\mathbb{C})$, with $r = 6, 7$ or 8. There is a Levi subgroup ${}^LL$ of type $D_4(\mathbb{C}) \times (\mathbb{C}^\times)^{r-4}$. This group is unique up to conjugacy. Let u be a subregular unipotent element in ${}^LL$. The component group $A_L(u)$ (resp. $A(u)$) in ${}^LL$ (resp. ${}^LG$) is trivial (resp. S_3). The former means

$$K^M(\hat{\mathscr{B}}_u)_m \text{ is irreducible tempered,} \tag{7.3}$$

while the latter means

$$\mathscr{H} \otimes_{\mathscr{H}_L} K^M(\hat{\mathscr{B}}_u)_m = \bigoplus_{\rho \in \hat{S}_3} (K^M(\mathscr{B}_u)_{m,\rho} \otimes \rho). \tag{7.4}$$

In particular, the multiplicity of $K^M(\mathscr{B}_u)_{m,\theta}$, θ the reflection representation of S_3, in the induced representation is 2.

Example 7.2. Take ${}^LG = E_8(\mathbb{C})$. There is an unique conjugacy class of Levi subgroups of type $D_4(\mathbb{C}) \times A_2(\mathbb{C})$. Fix such an ${}^LL$. Let u be an unipotent element in ${}^LL$ which projects to a subregular (resp. principal) unipotent element in $D_4(\mathbb{C})$ (resp. $A_2(\mathbb{C})$). As in Example 7.1, $A_L(u)$ is trivial, while $A(u)$ is S_3. Formulas (7.3) and (7.4) hold, so the induced representation is again not multiplicity free.

If ${}^LG$ is simple, the above two examples give all instances in which an induced tempered representation is not multiplicity free.

8. Unitary K-spherical representations

The trivial and sgn representations of $\mathscr{H}_W$ are the one dimensional representations given on the generators $T_s (s \in S)$ by

$$T_s \mapsto q, \quad \text{and} \quad T_s \mapsto -1 \tag{8.1}$$

respectively. An $\mathscr{I}$-spherical module V is K-spherical precisely when $V^{\mathscr{I}}$ contains the trivial representation of $\mathscr{H}_W$. The Iwahori-Matsumoto involution $'$ of $\mathscr{H}$ restricts to an algebra involution of $\mathscr{H}_W$. The trivial and sgn representation are exchanged by $'$. More generally, under the Lusztig isomorphism between $\mathscr{H}_W$ and $\mathbb{C}W$, the trivial and sgn representation in (8.1) correspond respectively to the trivial and sgn representation of W. The deformation arguments in Sect. 4 can be applied to show that the effect of $'$ on a representation τ of W is

$$\tau \mapsto \tau \otimes \mathrm{sgn}. \tag{8.2}$$

Suppose (u, σ, q, ρ) is L-group data for a real tempered module $\mathscr{M}_{u,\sigma,q,\rho}$. We know from Theorem 4.7 that the action of $\mathbb{C}W$ on $\mathscr{M}_{u,\sigma,q,\rho}$ and $H^*(\mathscr{B})^\rho$ are equivalent. Consider the case when ρ is trivial. By Corollary 4.3, sgn occurs in $H^*(\mathscr{B})^1$. This means $\mathscr{M}_{u,\sigma,q,1}$ contains the sgn representation of $\mathscr{H}_W$. Apply the involution $'$. We conclude

Theorem 8.1. *The module $\mathscr{M}'_{u,\sigma,q,1}$ is*

(1) *K-spherical,*

(2) *unitary and*
(3) *has the same infinitesimal character as the tempered module* $\mathcal{M}_{u,\sigma,q,1}$.

This verifies the analogue of conjecture 1.3.2 in [A] in this case.

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