

October 24, 2003

THE LOCAL LANGLANDS CONJECTURES AND CHARACTERISTIC CYCLES

1. Introduction

The purpose of these talks is to present some aspects of the relation between geometry of singular spaces, and representation theory. Precisely, I want to look at

- admissible irreducible representations of real reductive groups
- admissible irreducible representations of p -adic groups admitting fixed vectors for an Iwahori subgroup.

I will follow [ABV] and [V?] quite closely.

These relations originate with the idea of Langlands that there is a close connection between irreducible admissible representations of the rational points of a connected reductive algebraic group G defined over a local field F and homomorphisms of the Weil–Deligne group into the (complex) dual group

$$W_F' \longrightarrow {}^L G. \tag{1.1}$$

Such a relationship is established by Langlands in [L] for $F = \mathbb{R}$ and $F = \mathbb{C}$ and a few other cases. The best thing would be that one has a bijection between equivalence classes of irreducible admissible representations and such homomorphisms up to conjugation by ${}^L G$. This is not true, so the notion of L -*packets* of representations was introduced. This partitions the set of irreducible representations into finite sets which are in 1–1 correspondence with homomorphisms as above. In particular the *stable* combination of characters formed from a packet is the natural object to consider when one investigates occurrence of representations in the space of automorphic forms via the trace formula.

Here are some refinements of these conjectures and related results:

- Knapp–Zuckerman and Shelstad, description of packets for tempered representations,
- Beilinson–Bernstein and Brylinski–Kashiwara for character formulas in the case $F = \mathbb{C}$,
- Lusztig–Vogan for character formulas in the case $F = \mathbb{R}$,
- work of Bernstein–Zelevinsky on the admissible dual of $GL(n, F)$ for a p -adic field,
- Kazhdan–Lusztig for representations admitting fixed vectors under an Iwahori subgroup.

These packets are very well suited for investigations involving *tempered* representations. Motivated by the problem of how non-tempered representations fit in the trace formula, Arthur introduced a different set of parameters; namely he considered maps

$$\psi : W_F' \times Sl(2, \mathbb{C}) \longrightarrow {}^L G^\Gamma. \tag{1.2}$$

(see §7 for the precise definition). He then conjectures that, attached to such a map there should be a *packet* of irreducible representations satisfying certain character identities with respect to endoscopic groups. A particular class of such maps is given by the case when $\psi|_{W_F}$ is essentially trivial. These are called *unipotent* representations. For example, in the case $F = \mathbb{R}$, $W_F = \mathbb{C} \times \{1, j\}$ satisfying the relation $jzj^{-1} = \bar{z}$. In this case, *essentially trivial* would mean $\psi|_{\mathbb{C}}$ is trivial. In [BV], the packet of *special unipotent* representations is defined as follows.

Definition. *An admissible representation of a real form $G(\mathbb{R})$ is called special unipotent if it has infinitesimal character*

$$\lambda_{\tilde{\sigma}} = d\psi \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$$

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and its annihilator in the universal enveloping algebra is maximal. The set of irreducible representations attached to ψ is called the representation theoretic packet.

Then formulas of the type conjectured by Arthur are proved for the case of complex groups.

In [ABV], a different more general approach is taken. The category of Harish–Chandra modules is related to constructible or perverse sheaves on an algebraic space Y equivariant with respect to the action of an algebraic group H . Then the packets can be defined in terms of characteristic cycles for sheaves. These are called the *microlocal* packets. In chapter 27 of [ABV] it is shown that for the case of special unipotent representations, the union of parameters with a fixed map of $Sl(2)$ coincides with the microlocal packet.

This second approach has the advantage that it is more general in that it applies to any Langlands parameter not just an Arthur parameter, but also the conjectured character identities relevant to endoscopic groups are a consequence of Lefschetz fixed point theorems of Goresky–MacPherson.

Another nice feature of this second approach is that a microlocal packet has a distinguished stable virtual combination of irreducible representations, namely the linear combination with coefficients equal to the dimensions of certain representations of the component group of the centralizer of the image of ψ . However these representations are not very easy to compute (see §? for more precise definitions).

A large portion of this talk will be devoted to describing the geometric setting that allows us to define the microlocal packets. This is a (very incomplete) summary of [ABV].

In §2 we describe the setting in which we do representation theory. In §3 we analyze weil group homomorphisms, and show how they parametrize irreducible admissible modules. In §4 we introduce the various geometric categories relevant for our analysis. In §5 we describe the notion of *geometrically* stable and its relation to the notion of *representation theoretically* stable introduced by Langlands–Shelstad. In §6 we introduce the *microlocal packets* and investigate the relation to the previously defined packets.

As already mentioned, one of the advantages for working with characteristic cycles is that the coefficients in the stable linear combinations of characters have a natural interpretation as dimensions of representations of certain component groups. It is not clear or easy to see what these representations are. In §7 we prove that in the complex case, these characters coincide with the results in [BV]. This is joint work with D. Vogan.

2. Rational Forms

We consider the case of a local field F of characteristic 0. Let $\overline{F}$ be its algebraic closure and $\Gamma := \text{Gal}(\overline{F}/F)$. A *rational form* is a map

$$\sigma : \Gamma \longrightarrow \text{Aut}(G(\overline{F})) \quad (2.1.1)$$

so that if f is a regular function on $G(\overline{F})$, then so is

$$\gamma \cdot_\sigma f(x) := \gamma \cdot (f(\sigma(\gamma^{-1}) \cdot x)). \quad (2.1.2)$$

The group of F -rational points is

$$G(F, \sigma) := G(\overline{F})^{\sigma(\Gamma)}.$$

Since $G(\overline{F})$ acts via conjugation, we can consider *equivalence classes* of rational forms. We say that σ is *inner* to σ' , if for each $\gamma \in \Gamma$, the automorphism $\sigma(\gamma)\sigma'(\gamma)^{-1}$ is an inner automorphism, say given by a $g_\gamma \in G(\overline{F})$.

A *representation* of a rational form, is an *admissible* representation of $G(F, \sigma)$. Generally we write (π, σ) , and the action of $G(\overline{F})$ is via

$$g \cdot (\pi, \sigma) := (\pi \circ \text{Ad}g^{-1}, g \cdot \sigma).$$

Suppose (π_1, σ) and (π_2, σ) are representations of the same rational form. Then it would be convenient to say that they are equivalent under the above action of $G(\overline{F})$ precisely when they are equivalent in the usual sense, *i.e.* there is an element $g \in G(F)$ such that $\pi_1 = \pi_2 \circ \text{Ad}(g^{-1})$. But this is just not true. For example, let $G = \text{Sl}(2, F)$. Suppose χ is a nontrivial character whose square is one. Then the induced principal series (normalized induction) decomposes as a direct sum of two inequivalent irreducible representations

$$\pi = \text{Ind}_B^G(\chi) = \pi_1 \oplus \pi_\chi.$$

There is a quadratic extension E of F and an element $g \in \text{Sl}(2, E)$ such that $g \cdot (\pi_1, \sigma) = (\pi_\chi, \sigma)$.

We can do one of two things:

- use only groups with trivial center; the (image of the) offending element is in $\text{PGL}(2, F)$,
- change the notions.

Here is the modification. Fix a σ . Define the (*weak extended*) group $G^\Gamma := G \rtimes \Gamma$ via σ *i.e.*

$$[g, \gamma][g', \gamma'] = [g\sigma(\gamma)g', \gamma\gamma'].$$

Then G^Γ is a group which contains G as a normal subgroup so that

$$1 \longrightarrow G \longrightarrow G^\Gamma \longrightarrow \Gamma \longrightarrow 1, \quad (2.1.3)$$

such that for every element $\gamma \in \Gamma$, the action of an inverse image g_γ is compatible with γ , *i.e.* if f is a regular function, then so is

$$g_\gamma \cdot f(x) := \gamma \cdot (f(g_\gamma^{-1}xg_\gamma)).$$

Definition. A rigid rational form of G is a continuous map

$$\delta : \Gamma \longrightarrow G^\Gamma$$

compatible with the map of G^Γ to Γ and such that the quotient to $G^\Gamma/Z(G)$ is a homomorphism. A pure rational form of G is a δ which is in fact a homomorphism.

The rational form attached to δ is the homomorphism

$$\sigma(\delta) : \Gamma \longrightarrow \text{Aut}(G(\overline{F})), \quad \sigma(\delta)(\gamma) \text{ is conjugation by } \delta(\gamma).$$

A *representation* is a pair (π, δ) where δ is a pure rational form and π a Harish–Chandra module for the fixed points $G(F, \sigma(\delta))$. We denote by $\Pi(G/F)$ the set of equivalence classes (under the action of $G(\overline{F})$) of irreducible representations.

There are some other conditions on σ ((2.3) in [V?]) which I won't bother to write down right now. Finally, two rigid rational forms are called equivalent if they are conjugate by G , *i.e.* there is $g \in G$ such that $\delta' = g\delta g^{-1}$.

2.2. The real case. Let G be a complex connected reductive algebraic group defined over $\mathbb{R}$, and let $\Gamma = \mathbb{Z}_2$ be the Galois group of $\mathbb{C}$ over $\mathbb{R}$. A weak extended group is attached to an antiholomorphic involutive automorphism $\sigma : G \longrightarrow G$. In this case it is better to use *strong* real forms of G ; this is a rigid form satisfying $\delta(1) = 1$, so it can be thought of as just an element $\delta \in G^\Gamma - G$ such that $\delta^2 \in Z(G)$. A pure real form satisfies $\delta^2 = 1$.

Example I. Suppose G_c is the complexification of a group G which is complex but viewed as a *real* group. Choose a conjugation which we denote by $\overline{}$ so that we can identify G_c with $G \times G$ and take

$$\sigma(x, y) = (\overline{y}, \overline{x}) \tag{2.2.1}$$

for such a real form. $\square$

Example II. Suppose $G = Sl(2, \mathbb{C})$. Then the conjugation

$$\sigma(g) = \overline{g} \quad \text{the usual conjugation on each entry} \tag{2.2.2}$$

gives the real form $Sl(2, \mathbb{R})$. $\square$

In **Example I**, set $\delta = [(a_1, a_2), \tau] \in G^\Gamma$ where

$$G^\Gamma = (G \times G) \times \mathbb{Z}_2, \quad \tau \cdot (g_1, g_2) = (\overline{g}_2, \overline{g}_1) \cdot \tau$$

The pure forms must satisfy $[(a_1, a_2), \tau]^2 = 1$, so $a_2 = \overline{a}_1^{-1}$. Furthermore, since

$$[(g_1^{-1}, g_2^{-1}), 1] \cdot [(a, \overline{a}^{-1}), \tau] \cdot [(g_1, g_2), 1] = [(g_1^{-1}a\overline{g}_2, g_2^{-1}\overline{a}^{-1}\overline{g}_1), \tau], \tag{2.2.3}$$

we can choose (g_1, g_2) such that $g_1^{-1}a\overline{g}_2 = 1$, to see that $[(a, \overline{a}^{-1}), \tau] \cong [(1, 1), \tau]$. Thus up to equivalence there is a unique pure form. The same calculation shows that (equivalence classes of) strong real forms are in 1–1 correspondence with elements $[(1, z), \tau]$ with $z \in Z(G)$.

In **Example II**, set $\delta = [a, \tau]$, so that the antiholomorphic action is $\sigma(\tau)(g) = \overline{g}$. Then there are two cases, $\delta^2 = Id$ and $\delta^2 = -Id$. The first case gives a single conjugacy class corresponding to the *pure* real form $Sl(2, \mathbb{R})$. The second case is a *strong* real form only, and has two equivalence classes corresponding to $Su(2, 0)$ and $Su(0, 2)$.

3. Langlands Parameters

3.1. Dual group. Fix a G^Γ . We form the L -group ${}^L G$ as follows. Let ${}^\vee G$ be the dual group. This is the usual object; if $G \leftrightarrow \Psi(G) := (X^*, X_*, R, {}^\vee R)$, then ${}^\vee G \leftrightarrow \Psi({}^\vee G) := (X_*, X^*, {}^\vee R, R)$. Then the automorphism σ induces a homomorphism

$${}^\vee \sigma : \Gamma \longrightarrow \text{Aut}(\Psi({}^\vee G)), \quad (3.1.1)$$

but this time we are twisting it so that it is *algebraic*.

Definition. Fix an inner class of F -rational forms corresponding to $\sigma : \Gamma \longrightarrow \text{Aut}(\Psi(G))$ with dual ${}^\vee \sigma$. Then we can define ${}^L G := {}^\vee G \rtimes \Gamma$.

This is a complex complex reductive algebraic group having the following structure.

- (1) There is a short exact sequence

$$1 \longrightarrow {}^\vee G \longrightarrow {}^L G \longrightarrow \Gamma \longrightarrow 1. \quad (3.1.2)$$

- (2) If $\gamma \in \Gamma$, then any preimage g_γ acts by ${}^\vee \sigma(\gamma)$.

- (3) There is an open subgroup $\Gamma_0 \subset \Gamma$ such that ${}^\vee G \times \Gamma_0$ is isomorphic to an open subgroup of ${}^L G$.

Typically, a Langlands parameter is a continuous homomorphism

$$\phi : W_F' \longrightarrow {}^L G \quad (3.1.3)$$

which is compatible with the canonical maps into Γ , *i.e.* the diagram

$$\begin{array}{ccc} W_F & \xrightarrow{\phi} & {}^L G \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{id} & \Gamma \end{array} \quad (3.1.4)$$

is commutative and has some other special properties as described later.

3.2. P-adic case. Write k for the residue field, $\bar{k}$ for its closure. Then there is a surjective homomorphism from Γ to $\hat{\mathbb{Z}} \cong \text{Gal}(\bar{k}/k)$ with kernel the *inertia* group I_F . Then $\mathbb{Z}$ is a dense subgroup of $\hat{\mathbb{Z}}$ generated by the *Frobenius* map $x \mapsto x^q$. The *Weil* group is defined to be the inverse image of $\mathbb{Z}$ in Γ . We get a sequence

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \mathbb{Z} \longrightarrow 1.$$

The topology is defined by requiring that I_F be compact and open. Then W_F acts on the additive group $G_a \cong \mathbb{C}$ by $w \cdot x = ||w||x$, where $||w|| = q^n$, with n the image of w in $\mathbb{Z}$. The *Weil-Deligne* group is then $\mathbb{C} \rtimes W_F$ with $w x w^{-1} = ||w||x$.

Definition. A Langlands parameter is a continuous map as in (3.1.3) satisfying

- (1) ϕ is compatible with Γ ,
- (2) the restriction to $\mathbb{C}$ is algebraic,
- (3) $\phi(W_F)$ consists of semisimple elements.

The collection of Langlands parameters is denoted $P({}^L G)$.

Suppose G is split. Then ${}^L G = {}^\vee G \times \Gamma$, and consider the special case of ϕ which satisfy $\phi(i) = (1, i)$. Then ϕ is determined by the ensuing map

$$\phi : \mathbb{Z} \times \mathbb{C} \longrightarrow {}^L G. \quad (3.2.1)$$

The restriction to $\mathbb{Z}$ is determined by (essentially the image of Fr), a semisimple element $s \in {}^\vee G$. The restriction to $\mathbb{C}$ yields a unipotent element u . The relation is $sus^{-1} = u^q$. We are supposed to consider ${}^\vee G$ conjugacy classes of such pairs. Let $\mathcal{O}$ be the orbit of s and suppose we restrict attention to the sets

$$\begin{aligned} P(s, {}^\vee G) &:= \{\phi \in P({}^L G) : \phi|_{W_F} \text{ is determined by } s\}, \\ P(\mathcal{O}, {}^\vee G) &:= \{\phi \in P({}^L G) : \phi|_{W_F} \text{ is determined by an } s' \in \mathcal{O}\}. \end{aligned} \quad (3.2.2)$$

For the first set, we observe that we can write $u = \exp(e)$ and e is nilpotent satisfying $Ad(s)e = qe$.

Remark. The element e can be embedded in a Lie triple $\{e, h, f\}$. We can write $s = s_0 s_h$, where $s_h := \exp(\frac{\log q}{2} h)$. Then s_0 commutes with e . Since $C_G(e) = L \cdot U$ where $L = C_G(e, h, f)$ and U is the nilradical, we can write $s = s_{00} s_h s_u$ where $s_{00} \in L$ and $s_u \in U$. But then observe that $AdU s_{00} s_h = s_{00} s_h U$, because the eigenvalues of s_h on the Lie algebra of U are strictly positive and s_h preserves the decomposition $C_G(e) = LU$.

Thus we may assume that $s_0 \in C_G(e, h, f)$.

Let ${}^\vee \mathfrak{g}(q)$ be the q -eigenspace of s . Then the space $P(s, {}^\vee G)$ can be identified with ${}^\vee \mathfrak{g}(q)$. The ${}^\vee G$ -orbits on $P(\mathcal{O}, {}^\vee G)$ are then in 1-1 correspondence with the orbits of $H := {}^\vee G(s)$ on $Y := {}^\vee \mathfrak{g}(q)$. It is well known that this space has finitely many orbits. In such a situation we always consider the component group

$$A(\phi) = C_{{}^\vee G}(\phi)/C_{{}^\vee G}(\phi)^0 \cong C_{{}^\vee G}(s, e)/C_{{}^\vee G}(s, e)^0. \quad (3.2.3)$$

An equivariant local system on the orbit of H on Y corresponding to ϕ identifies with an irreducible character of $A(\phi)$. If we want to stress the ${}^\vee G$ action, we use the space

$${}^\vee G \times_{{}^\vee G(s)} {}^\vee \mathfrak{g}(q). \quad (3.2.4)$$

For the geometric facts we will use, either version works.

The case of a general ϕ is about the same if we fix $\phi|_{W_F}$ and don't insist on saying too precisely what it is.

Example III. Suppose $H = Gl(2)$ acts on $Y = S(2, \mathbb{C})$, the space of 2×2 symmetric matrices. In this case the orbits are easy to determine, they are the ranks of the symmetric forms; there are exactly 3. The open orbit has $A(\phi) \cong \mathbb{Z}_2$. This example comes from $Sp(4)$ with $s = s_h$ for the subregular nilpotent orbit. $\square$

Example IV. Suppose $G = S[Gl(3) \times Gl(2)]$ acting on $[St_3 + St_3^*] \otimes St_2$ which has dimension 12; here St is the standard module of Gl and the subscript is the dimension. There are 11 orbits, the largest has component group S_4 and there are 4 others with component group S_2 . This example comes from F_4 with $s = s_h$ for a particular nilpotent orbit. $\square$

We can identify these parameters with representations as follows. Suppose G is split, so that ${}^L G = {}^\vee G \times \Gamma$. Then let $T \subset G$ be a maximally split torus contained in a Borel subgroup B . Its characters are given by $X(T) \otimes_{\mathbb{Z}} Hom(F^\times, \mathbb{C}^\times)$. The *unramified characters* are then identified with

$$X^*(T) \otimes_{\mathbb{Z}} Hom(F^\times / U_F, \mathbb{C}^\times) \cong X^*(T) \otimes_{\mathbb{Z}} (\mathbb{C}) \cong {}^\vee T(\mathbb{C}),$$

where U_F are the units in $F^\times$. The last isomorphism comes from $X^*(T) \cong X_*({}^\vee T)$. To an unramified character we can associate, on the one hand the unramified principal series

$$Ind_B^G(\chi), \tag{3.2.5}$$

and on the other hand the semisimple element $s \in {}^\vee T$ obtained from the above isomorphisms. The composition factors of the principal series are representations with Iwahori fixed vectors; and every such irreducible representation is a composition factor of some $I(\chi)$. The composition series of $Ind(\chi)$ only depends on the Weyl group orbit of χ , so the element s may be taken up to the action of the Weyl group. The next theorem makes this correspondence even sharper.

Theorem. (*Kazhdan–Lusztig*) *Let G be a split p -adic group. Then the factors of $Ind(\chi)$ can be put in 1–1 correspondence with equivariant local systems on the orbits of $H = C_{{}^\vee G}(s)$ on $Y = {}^\vee \mathfrak{g}(q)$; or equivalently pairs $(\mathcal{O}, \psi)$ where $\mathcal{O} = H \cdot x$ is an orbit and $\psi \in \widehat{H_x/H_x^0}$.*

Remarks. In **Example III**, take a ϕ corresponding to the maximal orbit $\mathcal{O}_2$ and an element $s = s_0 s_h$ where $s_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $A(\phi) \cong \mathbb{Z}_2$, but there is only one representation with nontrivial Iwahori–fixed vectors. Thus the correspondence in the theorem is not a bijection.

Similarly in **Example IV**, not all characters of the component group occur.

3.3. Real case. In this case $\Gamma = \mathbb{Z}_2$. Then a rational form is determined by the image of the nontrivial element $\tau \in \Gamma$. We assume (as we may) that this $\sigma = \sigma(\tau)$ is an antiholomorphic involutive automorphism of $G(\mathbb{C})$. A rigid form δ is determined by its values $\delta(1)$ and $\delta(\tau)$ in the weak extended group $G^\Gamma = G \rtimes \Gamma$. A strong real form must satisfy $\delta(1) = 1$, so it is determined by a single element $\delta = [a, \tau]$. The automorphism σ induces an automorphism of the dual root datum. This in turn extends to an automorphism of ${}^\vee G$ (up to conjugacy by an inner automorphism), and we can form the corresponding group ${}^L G$.

In **Example I**, we get ${}^L G = ({}^\vee G \times {}^\vee G) \times \Gamma$ with multiplication $\tau \cdot (g_1, g_2) = (g_2, g_1) \cdot \tau$. $\square$

In **Example II**, we get ${}^L G = PSl(2, \mathbb{C}) \times \Gamma$, direct product. $\square$

The *Weil group* $W_{\mathbb{R}}$ is the group

$$W_{\mathbb{R}} = \mathbb{C}^\times \times \{1, j\}, \quad jzj^{-1} = \bar{z}, \quad j^2 = -1. \quad (3.3.1)$$

Definition. A Langlands parameter is a (real) Lie group homomorphism

$$\phi : W_{\mathbb{R}} \longrightarrow {}^L G \quad (3.3.2)$$

compatible with the maps into Γ and such that $\phi(\mathbb{C}^\times)$ is formed of semisimple elements.

Write this set as $\Pi({}^L G)$ and let ${}^\vee G$ act in the obvious way.

Definition. The pure Langlands component group is the component group

$$A_\phi^{loc} = {}^\vee G(\phi) / {}^\vee G(\phi)_0, \quad (3.3.3)$$

where ${}^\vee G(\phi)$ is the centralizer of the image of ϕ .

The pairs (ϕ, τ) , where $\tau \in \widehat{A_\phi^{loc}}$, are called *complete Langlands parameters* and the set of equivalence classes under the action of ${}^\vee G$ will be denoted $\Xi(G/\mathbb{R})$.

Here is the main theorem. The nature of the bijection will be described later.

Theorem. There is a natural bijection

$$\Pi(G/\mathbb{R}) \leftrightarrow \Xi(G/\mathbb{R}).$$

Remark. This is one of the main results in [ABV]. But Langlands and Shelstad already proved such statements much earlier. The difference is perhaps in the details. For example the set $\widehat{A}(\phi)$ always has a distinguished element, namely the trivial local system. To make the correspondence as canonical as possible, one has to choose a similar irreducible representation in each packet. For this one can fix a Whittaker model. Another difference is that representations of strong real forms may be attached to local systems on a cover (still algebraic) of ${}^\vee G$.

We describe the parametrization of $\Pi(G/\mathbb{R})$ in the case of regular infinitesimal character without recourse to the Weil group homomorphisms. Suppose that $G(\mathbb{R})$ is the set of fixed points of the conjugation σ . Let θ be a Cartan involution commuting with σ . Then a parameter is a triplet $\gamma = (H(\mathbb{R}), \Lambda, \lambda)$ as follows:

- $H = T \times A$ is a θ -stable σ -stable Cartan subgroup with Cartan subalgebra $\mathfrak{h}$,
- Λ is a character of $H(\mathbb{R})$, and λ is a character of $\mathfrak{h}(\mathbb{R})$; we are assuming that λ is regular.
- Let $R_{im}(\lambda) = R_{im,c} \cup R_{im,nc}$ be the positive system of imaginary roots determined by λ . Then

$$d\Lambda = \lambda + \rho(R_{im}) - 2\rho(R_{im,nc}).$$

By conjugating by the (real) Weyl group, we can also choose the parameter so that λ is dominant with respect to some fixed positive system of real roots. Then to such a parameter one can associate a *Standard module* $M(\gamma)$ which is induced from a Discrete series on a Levi component and has a unique irreducible quotient $L(\gamma)$.

The singular situation is worse, but of the same type.

3.4. Langlands Parameters. We now describe the Langlands parameters more explicitly. First an elementary lemma.

Lemma. *Let H be a complex group with Lie algebra $\mathcal{L}(H) = \mathfrak{h}$. Then there is a bijection between the set of real Lie homomorphisms $\phi : \mathbb{C}^\times \longrightarrow H$ and the set of pairs $(\lambda, \mu) \in \mathfrak{h} \times \mathfrak{h}$ such that $[\lambda, \mu] = 0$ and $\exp(2i\pi(\lambda - \mu)) = 1$.*

We recall the formula

$$\phi(e^t) = \exp(t\lambda + \bar{t}\mu).$$

Proposition. *The set of Langlands parameters can be put in 1-1 correspondence with the set of pairs*

$$\{ (y, \lambda) \in {}^L G \times {}^\vee \mathfrak{g} \mid \lambda \text{ semisimple} \}$$

subject to the conditions

- (1) $[\lambda, \text{Ad}(y)\lambda] = 0$,
- (2) $y^2 = \exp(2i\pi\lambda)$,
- (3) $y \in {}^\vee G^\Gamma - {}^\vee G$ is semisimple.

We write $e(\lambda) = \exp(2i\pi\lambda)$. We also recall that the μ in lemma 3.4 is $\mu = \text{Ad}(\phi(j))\lambda$ and $y = \exp(i\pi\lambda)\phi(j)$. In **Example I**, write $y = [(a, b), \tau]$ and $\lambda = (\lambda_L, \lambda_R)$. By conjugating everything by ${}^\vee G$, we may assume that $y = [(1, \beta), \tau]$. Then conditions (1)–(3) become $\beta = \exp(2i\pi\lambda_L) = \exp(2i\pi\lambda_R)$ and $[\lambda_L, \lambda_R] = 0$. The component group is trivial.

Suppose $\beta = 1$; then these data give the usual conditions for a parameter of an irreducible Harish-Chandra module. There is a single conjugacy class of Cartan subgroups with representative $H = TA$. Since $\mu = \lambda_L - \lambda_R$ is integral, it defines a character $\mathbb{C}_\mu$ of T and $\nu = \lambda_L + \lambda_R$ defines a character $\mathbb{C}_\nu$ of A . Let $B \supset H$ be a Borel subgroup. Then we can take

$$M(\gamma) = \text{Ind}_B^G(\mathbb{C}_\mu \otimes \mathbb{C}_\nu)_{K\text{-finite}}.$$

This has a unique irreducible subquotient $L(\gamma)$ determined by the fact that it contains the K-type $V(\mu)$ with extremal weight μ . This version is equivalent to the above. Furthermore, we note that the λ in Proposition 3.4 corresponds to the infinitesimal character of the representation being parametrized.

In **Example II**, let $y = (a, \tau)$. Then $a^2 = e(\lambda)$ and $[\lambda, \text{Ad}(a)\lambda] = 0$. Conjugate the data so that $\lambda, \text{Ad}(a)\lambda$ are in the diagonal Cartan subalgebra. Assume $e(\lambda) = 1$ and λ is regular for simplicity. Then there are three orbits, $a = 1$, $a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In the third case, the component group is a $\mathbb{Z}_2$.

On the other hand, in $Sl(2, \mathbb{R})$ there are two conjugacy classes of Cartan subgroups:

$$\begin{aligned} H_c &= \left\{ \begin{bmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{bmatrix} \right\} & \mathfrak{h}_c &= \left\{ \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix} \right\}, \\ H_s &= \left\{ \pm \begin{bmatrix} e^t & 0 \\ 0 & \pm e^{-t} \end{bmatrix} \right\} & \mathfrak{h}_s &= \left\{ \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} \right\}. \end{aligned}$$

Given a fixed integral infinitesimal character λ , there are two parameters attached to H_c and two attached to H_s . These are the two discrete series and the two principal series. We would like to say that they are simply lifted from the tori; but this gives trouble because typically it is Λ which is a representation of $T(\mathbb{R})$ not λ .

An *infinitesimal character* is an orbit $\mathcal{O}_{\mathbb{R}} = (\mathcal{Y}, \mathcal{O})$ where

$$\begin{aligned} \mathcal{O} &\text{ is a semisimple } {}^\vee G\text{-conjugacy class of an element } \lambda \in {}^\vee \mathfrak{g} \\ \mathcal{Y} &\text{ is a } {}^\vee G\text{-conjugacy class of an element } y \in {}^\vee G^\Gamma - {}^\vee G, \\ y^2 &\text{ is conjugate to } e(\lambda) = \exp(2i\pi\lambda) \text{ under } {}^\vee G. \end{aligned} \quad (3.4.1)$$

Denote by $\mathcal{C}(\mathcal{O}_{\mathbb{R}})$ the ${}^\vee G$ -conjugacy class $\mathcal{Y}^2 = e(\mathcal{O})$.

Definition. *The set of Langlands parameters with infinitesimal character $\mathcal{O}_{\mathbb{R}}$ is*

$$P(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma) = \{ \phi = (y, \lambda) \in P({}^L G) : y \in \mathcal{Y}, \lambda \in \mathcal{O} \}. \quad (3.4.2)$$

We make the simplifying assumption that $\mathcal{C}(\mathcal{O}_{\mathbb{R}})$ is trivial.

This is the same as requiring that the infinitesimal character be integral.

Then the $\mathcal{Y}$ in the orbit $\mathcal{O}_{\mathbb{R}}$ can be identified with just $\{y \mid y^2 = 1\}$. For $y \in \mathcal{Y}$, set

$$K(y) := \text{centralizer of } y \text{ in } {}^\vee G. \quad (3.4.3)$$

Then y defines an involutive automorphism $\theta(y)$ of ${}^\vee G$ with fixed point group $K(y)$. Then set

$$\mathcal{O}^{comm}(y) := \{ \lambda \in \mathcal{O} \mid [\lambda, \theta(y)\lambda] = 0 \}. \quad (3.4.4)$$

Proposition. *There is a ${}^\vee G$ -equivariant map between $P(\mathcal{O}_{\mathbb{R}}, {}^L G)$ and the fiber product*

$${}^\vee G \times_{K(y)} \mathcal{O}^{comm}(y).$$

In particular, the orbits of ${}^\vee G$ on $P(\mathcal{O}_{\mathbb{R}}, {}^L G)$ are in one-to-one correspondence with orbits of $K(y)$ on $\mathcal{O}^{comm}(y)$; and there are finitely many orbits. Furthermore, these orbits are closed.

This is the analog of the maps $\lambda : W_F \longrightarrow {}^L G$ in the p -adic case. But because the orbits are closed, this is not good for doing geometry.

3.5. Geometric Parameter Space. We now give the description of the *geometric parameters* which will have singular orbits, even though the group acting and the parametrization of orbits will be the same.

Given λ , let ${}^\vee \mathfrak{g}_i = \{ x : [\lambda, x] = ix \}$, and recall the map $e(\lambda) = \exp(2i\pi\lambda)$. The *canonical flat through λ* is defined to be

$$\mathcal{F}(\lambda) := \lambda + \sum_{i \in \mathbb{N}^+} {}^\vee \mathfrak{g}_i. \quad (3.5.1)$$

Write $\mathcal{F}({}^\vee \mathfrak{g})$ for the set of all canonical flats. For a fixed λ , we consider

$$\begin{aligned} {}^\vee \mathfrak{g}(\lambda) &:= \sum_{n \in \mathbb{Z}} {}^\vee \mathfrak{g}(\lambda)_n, & H(\lambda) &:= C_G(e(\lambda)) \\ {}^\vee \mathfrak{l}(\lambda) &:= {}^\vee \mathfrak{g}(\lambda)_0 = C_{{}^\vee \mathfrak{g}}(\lambda)_n, & L(\lambda) &:= C_H(\lambda) \\ {}^\vee \mathfrak{n}(\lambda) &:= \sum_{n \in \mathbb{N}} {}^\vee \mathfrak{g}(\lambda)_n, & N(\lambda) &\text{ the corresponding Lie group,} \\ {}^\vee \mathfrak{p}(\lambda) &:= {}^\vee \mathfrak{l}(\lambda) + {}^\vee \mathfrak{n}(\lambda), & P(\lambda) &:= L(\lambda)N(\lambda). \end{aligned} \quad (3.5.2)$$

Then $H(\lambda)$ is a reductive group, $P(\lambda)$ is a parabolic subgroup and most important,

$$N(\lambda) \cong \lambda + {}^\vee \mathfrak{n}(\lambda), \quad \text{via} \quad n \mapsto \text{Ad}(n)\lambda.$$

so that

$$\mathcal{F}(\lambda) = \text{Ad}(P(\lambda)) \cdot \lambda = \text{Ad}(N(\lambda)) \cdot \lambda = \lambda + {}^\vee \mathfrak{n}(\lambda). \quad (3.5.3)$$

Thus $\mathcal{F}(\lambda)$ is an affine space for $N(\lambda)$ and a homogeneous space for $P(\lambda)$.

We now specialize to the case when $\lambda \in \mathcal{O}$ for a fixed orbit $\mathcal{O}$. The corresponding space $\mathcal{F}(\mathcal{O})$ can be viewed as the analog of ${}^\vee \mathfrak{g}(q)$ in the obvious way. Fix $\Lambda \in \mathcal{F}(\mathcal{O})$. Write $P(\Lambda)$ for the stabilizer of Λ in ${}^\vee G$. Then $P(\Lambda) = L(\Lambda)N(\Lambda)$ is a parabolic subgroup, and if $\lambda \in \Lambda$, then $\mathcal{F}(\lambda) = \text{Ad}(N)\lambda$. Thus e is constant on Λ and $\mathcal{F}(\mathcal{O})$ can be identified with the variety $\mathcal{P}$ of parabolic subgroups of type $P(\Lambda)$.

Definition. *The set of Langlands parameters with infinitesimal character $\mathcal{O}_{\mathbb{R}}$ is*

$$P(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma) = \{ \phi = (y, \lambda) : y \in \mathcal{Y}, \lambda \in \mathcal{O} \}.$$

The geometric parameter space of infinitesimal character $\mathcal{O}_{\mathbb{R}}$ is the fiber product

$$X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma) = \mathcal{Y} \times_{\mathcal{C}(\mathcal{O}_{\mathbb{R}})} \mathcal{F}(\mathcal{O}),$$

obtained by using the map e and the squaring map from $\mathcal{Y}$ to $\mathcal{C}(\mathcal{O}_{\mathbb{R}})$.

Remark. In general (even when $e(\lambda) \neq 1$), the space $\mathcal{F}(\mathcal{O})$ inherits an algebraic structure and so does $\mathcal{C}(\mathcal{O}) \cong {}^\vee G / C_{{}^\vee G}(e(\lambda))$. The map $e : \mathcal{F}(\mathcal{O}) \rightarrow \mathcal{C}(\mathcal{O})$ is then algebraic.

Fix $y \in \mathcal{Y}$ such that $y^2 = 1$ (recall $\mathcal{C}(\mathcal{O}_{\mathbb{R}}) = 1$). Write $K(y)$ for the centralizer of y , a symmetric subgroup of ${}^\vee G$.

Proposition. *The group ${}^\vee G$ acts on $P(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma)$ and $X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma)$ in a natural fashion. There is a ${}^\vee G$ -equivariant isomorphism*

$$X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma) \cong {}^\vee G \times_{K(y)} \mathcal{P}.$$

The orbits of ${}^\vee G$ on X are in 1-1 correspondence with the orbits of $K(y)$ on $\mathcal{P}$; in particular there are only finitely many of them. The isotropy group ${}^\vee G^x$ at a point $x = (y, \Lambda) \in X$ is $K(y) \cap P(\Lambda)$.

The map

$$p : P(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma) \rightarrow X(\mathcal{O}_{\mathbb{R}}, {}^\vee G^\Gamma)$$

induces a bijection between ${}^\vee G$ -conjugacy classes on these two spaces. The component groups of the centralizers correspond.

3.6. Example. Let $G = \text{Sl}(2, \mathbb{R})$. Then ${}^\vee G^\Gamma = \text{PSl}(2) \times \{1, \tau\}$, direct product, where $\tau^2 = 1$. Take λ to be regular integral, in fact such that $e(\lambda) = 1$. Then $\mathcal{P} = \mathcal{B}$, the variety of Borel subgroups. There are two conjugacy classes of y 's,

$$\begin{aligned} y = \tau, \quad K(y) &= \text{PSl}(2, \mathbb{C}), \\ y = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tau, \quad K(y) &= \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \cup \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned} \quad (3.6.1)$$

Consider the second case. Identifying $\mathcal{B}$ with a sphere, there are two orbits, the union of the North and South pole, and the rest. The component group for the open orbit is $\mathbb{Z}_2$. The two characters correspond to $DS(\pm)$. The component group for the poles is trivial. The corresponding representation is the trivial representation occurring in the spherical principal $PS(\text{sph})$. The other (irreducible) principal series $PS(\text{sgn})$ of $\text{Sl}(2)$, matches the orbit of the *other* y .

4. Local Systems and Characteristic Cycles

4.1. We consider the setting of a (complex) algebraic variety Y with an action of an algebraic group H with finitely many orbits. There are several categories of interest:

$$\begin{aligned}\mathcal{C}(Y, H) &= \text{category of } H\text{-equivariant constructible sheaves on } Y, \\ \mathcal{P}(Y, H) &= \text{category of } H\text{-equivariant perverse sheaves on } Y, \\ \mathcal{D}(Y, H) &= \text{category of } H\text{-equivariant regular holonomic } D\text{-modules on } Y.\end{aligned}\tag{4.1.1}$$

Each of these categories is abelian and objects have finite length. In each of these categories, irreducible objects are parametrized by equivariant local systems $(S_\xi, \mathcal{V}_\xi)$ on orbits. We describe these irreducible objects. Let $S = H \cdot y$ be an orbit. There are inclusion maps

$$j = j_S : S \longrightarrow \overline{S}, \quad i = i_S : \overline{S} \longrightarrow Y.\tag{4.1.2}$$

We can form the *local* component group

$$A_y^{loc} := C_H(y)/C_H(y)^0.\tag{4.1.3}$$

A local system (equivariant with respect to the action of H) is defined to be an equivariant vector bundle with a flat connection. These are parametrized by $\xi = (S, \tau)$ where $\tau \in \widehat{A_y^{loc}}$. Explicitly, if (τ, V) is such a representation, then the bundle is

$$\mathcal{V}_\xi := H \times_{H_y} V.\tag{4.1.4}$$

In $\mathcal{C}(Y, H)$ the irreducible objects are given by

$$\mu(\xi) := i_* j_! \mathcal{V}, \quad \text{the extension by zero.}\tag{4.1.5}$$

In $\mathcal{P}(Y, H)$, the irreducible objects are given by the following construction. Let $d = \dim(S)$. Make a complex $\mathcal{V}[-d]$ out of $\mathcal{V}$ by putting zero in every degree except $-d$, where you put $\mathcal{V}$. Then the irreducible object attached to $\mathcal{V}$ is

$$P(\xi) = i_* j_{!*} \mathcal{V}[-d], \quad \text{where } j_{!*} \text{ is the intermediate extension functor.}\tag{4.1.6}$$

In the case of $\mathcal{D}(Y, H)$, consider $\mathcal{M}$, the sheaf of sections of $\mathcal{V}$. Because $\mathcal{V}$ has a flat connection, it naturally has a structure of H -equivariant $\mathcal{D}_S$ -module. Then

$$D(\xi) := \text{direct image of } \mathcal{M}.\tag{4.1.7}$$

In fact the *Riemann–Hilbert* correspondence provides an equivalence of categories between $\mathcal{P}$ and $\mathcal{D}$ via the deRham functor. Furthermore, the Grothendieck groups of these three categories are isomorphic. We analyze the isomorphism between $\mathcal{C}$ and $\mathcal{P}$ more closely. It is given by the map

$$\chi : \text{Ob } \mathcal{P}(Y, H) \longrightarrow K\mathcal{C}(Y, H), \quad P \mapsto \sum (-1)^i H^i P.\tag{4.1.8}$$

(The cohomology sheaves of perverse sheaves are constructible.) In particular we can write

$$P(\gamma) = \sum (-1)^{d(\xi)} c_g(\xi, \gamma) \mu(\xi), \quad \text{where } c_g(\xi, \gamma) = (-1)^{d(\xi)} \sum (-1)^i [\mathcal{V}_\xi : H^i P(\gamma)|_{S_\xi}].\tag{4.1.9}$$

Then the coefficients c_g satisfy

$$c_g(\gamma, \gamma) = 1, \quad c_g(\xi, \gamma) \neq 0 \text{ only if } S_\xi \subset \overline{S}_\gamma.\tag{4.1.10}$$

Typically, theorems are proved separately in each category; each version illuminates a different aspect of the theory.

4.2. We write $K(Y, H)$ for the Grothendieck group. In view of the expression for c_g , we define functionals χ_S^{loc} . Let $y \in S$, be an element in the orbit Y . Then for a constructible sheaf C , define

$$\chi_S^{loc} : K(Y, H) \longrightarrow \mathbb{N}, \quad \chi_S^{loc}(C) = \dim C_y \quad (4.2.1)$$

On a perverse sheaf P the formula is the Euler characteristic

$$\chi_S^{loc}(P) = \sum (-1)^i \dim(H^i P)_y. \quad (4.2.2)$$

Since A_y acts on the stalk of C at y , we can also define

$$\begin{aligned} Q_S^{loc} : \mathcal{C}(Y, H) &\longrightarrow (\text{local systems}) & C &\mapsto C|_S, \\ \tau_S^{loc} : \mathcal{C}(Y, H) &\longrightarrow (\text{representations of } A_y^{loc}) & C &\mapsto \text{representation of } A_y^{loc} \text{ on } C_y. \end{aligned} \quad (4.2.3)$$

On $\mathcal{P}$ one gets similar functors $(Q_S^{loc})^i$ and $(\tau_S^{loc})^i$.

Definition. A $\mathbb{Z}$ -linear functional $\eta : K(Y, H) \longrightarrow \mathbb{Z}$ is called geometrically stable if it is a $\mathbb{Z}$ -linear combination of χ_S^{loc} .

Remark. If we specialize to the case of (Y, H) the parameter space of representations of rational forms, the geometrically stable functionals are expected to be the stable combinations of characters defined by Langlands–Shelstad. In particular the χ^{loc} can be identified with the basis of stable combinations obtained from tempered parameters (or rather standard modules). In the real case, these are all theorems. In the case of p -adic groups, these are conjectures.

In the next section we define a different basis, which should correspond to the stable combinations conjectured by Arthur. We will analyze to what extent they have the required properties.

4.3. Given an orbit S , recall the conormal bundle T_S^*Y (with fiber at y given by the annihilator of $T_y S$) and the conormal bundle attached to the H -action,

$$T_H^*Y := \bigcup_S T_S^*Y. \quad (4.3.1)$$

A vector $(y, \nu) \in T_S^*Y$ is called degenerate if it is contained in the closure of some other conormal bundle $T_{S'}^*Y$. It is called regular if it is not degenerate. We denote the subset of non-degenerate vectors $T_H^*Y_{reg}$.

Attached to every H -equivariant regular holonomic $\mathcal{D}$ -module, there is a $\mathbb{Z}$ -linear combination of conormal bundles

$$Ch(P) = \sum_S \chi_S^{mic}(P) \overline{T_S^*(Y)}, \quad (4.3.2)$$

called the *Characteristic Cycle* of P . The χ_S^{mic} are integer valued and additive for short exact sequences, so they define functionals on $K(Y, H)$. In the $\mathcal{D}$ -module version this is due to Kashiwara. Another (independent) definition is due to MacPherson.

The $\mathcal{D}$ -module version seems more intuitive, so let us give a rough sketch. The ring of differential operators has a filtration by degrees

$$\mathcal{O}_Y \equiv \mathcal{D}_Y(0) \subset \mathcal{D}(1) \dots, \quad \mathcal{D}(p)\mathcal{D}(q) \subset \mathcal{D}(p+q) \quad (4.3.3)$$

such that

$$gr\mathcal{D} \cong p_*(\mathcal{O}_{T^*Y}), \quad (4.3.4)$$

Because $p : T^*Y \longrightarrow Y$ is affine, an equivalence of categories identifies $gr\mathcal{M}$ with a sheaf of modules for $\mathcal{O}_{T^*Y}$. A coherent $\mathcal{D}$ -module $\mathcal{M}$ has a “good” filtration so that the ensuing $\mathcal{O}_{T^*Y}$ -module $gr\mathcal{M}$ is coherent; view it as a sheaf.

On the other hand, given an algebraic variety and a coherent sheaf $\mathcal{N}$, one can define the characteristic variety and the characteristic cycle

$$\mathcal{V}(\mathcal{N}) = \{v \in V : \mathcal{N}_v \neq 0\}, \quad Ch(\mathcal{N}) = \sum m_W(\mathcal{N})W, \quad (4.3.5)$$

where the W ’s are the irreducible components of $\mathcal{V}(\mathcal{N})$. For a $\mathcal{D}$ -module $\mathcal{M}$, we set

$$Ch(\mathcal{M}) = Ch(gr\mathcal{M}). \quad (4.3.6)$$

In the case of equivariant $\mathcal{D}$ -modules, the W ’s are the conormal bundles T_S^*Y . The multiplicities are the χ_S^{mic} .

The main point of considering the notion of characteristic cycles is that they define another basis for the geometrically stable functionals. Here is the main result.

Theorem. (*Kashiwara–MacPherson*) *The functionals χ_S^{mic} are geometrically stable. More precisely, there are integers $c(S, S')$ such that*

$$\chi_S^{mic}(P) = \sum_{S \subset \overline{S'}} c(S, S') \chi_{S'}^{loc}(P).$$

Let us record some properties:

- $c(S, S) = (-1)^{\dim S}$,
- $c(S', S) = 0$ unless $S \subset \overline{S'}$,
- $c(S, S')$ is invertible, (a consequence of the above two facts),
- If $S \neq S'$ is such that $\overline{S}$ is contained in the smooth part of $\overline{S'}$, then $c(S, S') = 0$.

Example. Suppose the orbit S is open in Y . Then there is no $S' \neq S$ containing S in its closure, and therefore $\chi_S^{mic} = (-1)^{\dim S} \chi_S^{loc}$.

Remark. We emphasize again, that we have the space of *geometrically stable* functionals of $K(Y, H)$ with two bases, χ_S^{loc} , and χ_S^{mic} . The first one is easy to describe at least the values on one of the bases of $K(Y, H)$, namely

$$\chi_S^{loc}(\mu(S', \tau')) = \delta_{S, S'} \dim(\tau'). \quad (4.3.7)$$

The second one is not easy to describe explicitly in either case.

4.4. Let us consider the case of a group H acting algebraically on a vector space Y with finitely many orbits. In this case there is an extra structure that we can bring to bear upon the problem. In the case of $\mathcal{D}$ -modules we can use the usual Fourier transform. Equivalently, we can use the Fourier–Deligne transform in the case of sheaves. This transforms an irreducible perverse sheaf to an irreducible perverse sheaf, or a local system to another local system. The cotangent space to Y can be identified with $Y \times Y^*$, and similarly the cotangent space of Y^* can be identified with $Y^* \times Y$. A result of Piasetsky states that there is a bijection between orbits of H on Y and orbits on Y^* by the prescription

$$\mathcal{O} \longleftrightarrow \mathcal{O}^* \quad \text{iff} \quad \overline{T_{\mathcal{O}}^*Y} \cong \overline{T_{\mathcal{O}^*}^*Y^*}. \quad (4.4.1)$$

Precisely, the conormal bundle to $\mathcal{O}$ is $\{(v^*, y) : v^*(X \cdot y) = 0 \text{ for all } X \in \mathfrak{h}\}$. There are two projections $p : T^*Y \longrightarrow Y$ and $p^* : T^*Y \longrightarrow Y^*$. Then

$$\mathcal{O} \longleftrightarrow \mathcal{O}^* \quad \text{iff} \quad \overline{p^{-1}(\mathcal{O})} = \overline{p^{*-1}(\mathcal{O}^*)}. \quad (4.4.2)$$

Call this bijection on orbits FT as well. The name is justified by the next result.

Lemma. (*Hotta–Kashiwara*) $Ch(FT(P)) = Ch(P)$.

This is more or less the well known fact that the Fourier transform of the Dirac Delta function is the constant function.

This can be rewritten as

$$\chi_S^{mic} \circ FT = \chi_{FT(S)}^{mic}. \quad (4.4.3)$$

Example. Consider $H = Gl(2)$ acting on $Y = S(2, \mathbb{C})$, and recall the three orbits $\mathcal{O}_2$, $\mathcal{O}_1$ and $\mathcal{O}_0$. It is very easy to see that

$$FT(\mathcal{O}_2) = \mathcal{O}_0, \quad FT(\mathcal{O}_1) = \mathcal{O}_1. \quad (4.4.4)$$

Recall that the orbit $\mathcal{O}_2$ has two irreducible local systems attached to it. To compute χ^{mic} we could resort to the geometric definitions. The nature of the singularities of the closures of these orbits allow one to compute the numbers $c(S, S')$. But I don't know this well enough.

Instead, let me approach it as follows. We know that each local system is attached to a representation with $\mathcal{I}$ -fixed vectors for the split p -adic group $Sp(4)$. There is an involution on the set of such representations, called the Iwahori–Matsumoto involution. On the level of the Hecke algebra it tensors Weyl group representations with sgn and changes weights into their negatives. It is reasonable to conjecture that on the level of parameters this is FT . Evens–Mirkovic have in fact proved exactly such a result. Using this, one can then see that

$$FT(P(\mathcal{O}_2, 2)) = P(\mathcal{O}_0), \quad FT(P(\mathcal{O}_2, 1^2)) = P(\mathcal{O}_1, 1). \quad (4.4.5)$$

Then it is easy to see what χ_S^{mic} is:

$$\begin{aligned} \chi_{\mathcal{O}_2}^{mic} &= \delta(\mathcal{O}_2, 2) + \delta(\mathcal{O}_2, 1^2), \\ \chi_{\mathcal{O}_1}^{mic} &= \delta(\mathcal{O}_1) - \delta(\mathcal{O}_2, 1^2), \\ \chi_{\mathcal{O}_0}^{mic} &= \delta(\mathcal{O}_0) - \delta(\mathcal{O}_1). \end{aligned} \quad (4.4.6)$$

Here I am abusing the notation in the next section, $\delta(\xi)$ is the dual basis to $P(\xi)$. (One reason this works out nicely is that the infinitesimal character satisfies $\mathcal{O}(s) = \mathcal{O}(s^{-1})$.)

4.5. Consider still the case of H acting algebraically with finitely many orbits on a vector space Y . In the above example, we only used the following facts about χ^{mic} :

- (1) $\chi_S^{mic}(S, \tau) = (-1)^{\dim S} \dim(\tau)$,
- (2) $\chi_S^{mic}(S', \tau') = 0$, unless $S \subset \overline{S'}$,
- (3) $\chi_S^{mic} \circ FT = \chi_{FT(S)}^{mic}$,
- (4) $\chi_S^{mic}(S)$ is geometrically stable.

In view of this, it is natural to make the following conjectures, joint with Evens and Moy.

Conjecture 1. *The microlocal characters χ^{mic} are uniquely determined by the above four requirements.*

This conjecture can be sharpened in the case of the representations with $\mathcal{I}$ -fixed vectors. Recall that the infinitesimal character is a semisimple orbit given by an element s . Write $S = \mathcal{O}(u)$ for an orbit under ${}^vG(s)$ and $FT(S) = \mathcal{O}(u')$. This is a pair of nilpotent orbits such that $Ad(s)u = qu$ and $Ad(s)u' = -qu'$.

Conjecture 2. *Write*

$$\chi_S^{mic} = \sum a_{\xi'} \delta(\xi').$$

Let $\pi(\xi)$ be the irreducible module of the Hecke algebra corresponding to ξ . Then the identity of A -characters,

$$\sum (-1)^{(\dim \xi' - \dim \xi)} a_{\xi'} \pi(\xi') = \sum_i (-1)^i H^i(\mathcal{B}^s(u, u')),$$

holds. Here $\mathcal{B}^s(u, u')$ is the variety of Borel subgroups containing s, u, u' , and H^* may be equivariant K -homology as in Kazhdan–Lusztig or equivariant cohomology as in Lusztig in the case of the graded Hecke algebra. In particular, $H^*(\mathcal{B}^s(u, u'))$ should support a Hecke algebra representation.

Remark. The space $H^*(\mathcal{B}^s(u, u'))$ admits a representation of A , coming from tensoring with line bundles. So the formula above makes sense even *without* having to define an action of the full Iwahori–Hecke algebra.

We have computed numerous examples much more sophisticated than the one above. There are obvious generalizations to Hecke algebras with parameters, and the analogy to the constructions of Kazhdan–Lusztig or Ginzburg is also clear.

5. Multiplicity Matrices

5.1. In the previous section I kept writing χ^{mic} as a sum of elements in the Grothendieck group of irreducible characters of the original group. We now make the connection between the categories in section 4 and representation theory in the case of $Y = X(\mathcal{O}_{\mathbb{R}}, {}^L G)$ and $H = {}^\vee G$.

In general the orbits in X and the irreducible objects in Π are parametrized by the same set. Thus, given a parameter ξ , we write

$$\begin{aligned} \pi(\xi) &\text{ for the irreducible representation parametrized by } \xi, \\ M(\xi) &\text{ for the standard module parametrized by } \xi, \\ \mu(\xi) &\text{ for the irreducible constructible sheaf parametrized by } \xi, \\ P(\xi) &\text{ for the irreducible perverse sheaf parametrized by } \xi. \end{aligned} \tag{5.1.1}$$

The *geometric* multiplicity matrix $\{m_g(\gamma, \xi)\}$ captures the relation between the bases $\mu(\xi)$ and $P(\xi)$ in the Grothendieck group $KX({}^\vee G^\Gamma)$,

$$\mu(\xi) = (-1)^{\dim(\xi)} \sum m_g(\gamma, \xi) P(\gamma). \tag{5.1.2}$$

Here we have made a direct sum over all possible orbits $\mathcal{O}$. Note also the connection that if we know m_g , then we can compute χ^{loc} on irreducible perverse sheaves; the formula is simple only in the case of $\mu(\xi)$.

The *representation-theoretic* matrix $m_r(\gamma, \xi)$ captures the relation between $\pi(\xi)$ and $M(\xi)$,

$$M(\xi) = \sum m_r(\gamma, \xi) \pi(\gamma). \tag{5.1.3}$$

The main result about these matrices is that they are inverse transpose of each other. Precisely, the (general) result is as follows.

Theorem. (Vogan IC4) *There is a natural perfect pairing*

$$\langle \cdot, \cdot \rangle: K\Pi(G/\mathbb{R}) \times KX({}^\vee G^\Gamma) \longrightarrow \mathbb{Z}$$

defined by

$$\langle M(\xi), \mu(\xi') \rangle = \delta_{\xi, \xi'}.$$

This pairing then has the property that

$$\langle \pi(\xi), P(\xi') \rangle = (-1)^{\dim(\xi)} \delta_{\xi, \xi'}.$$

In other words, if the pairing is set up so that $\{M\}$ and $\{\mu\}$ are dual bases, then so are $\{\pi\}$ and $\{P\}$.

In the case of complex groups, there is a sharper version due to Beilinson–Ginzburg–Soergel, but it is not needed for this analysis.

Corollary. *Let $\overline{K\Pi}(G/\mathbb{R})$ be the set of formal linear combinations of irreducible representations of pure rational forms. Then $\overline{K\Pi}$ may be identified with the space $\mathbb{Z}$ -linear functionals of $KX({}^\vee G^\Gamma)$. In this identification,*

- (1) $M(\xi) : C \mapsto m(\mathcal{V}_\xi, C|_{S_\xi}),$
- (2) $\pi(\xi) : Q \mapsto (-1)^{d(\xi)} m(P(\xi), Q|_{S_\xi}).$

We can then ask what virtual character corresponds to χ_S^{loc} ; the answer is that it is a certain linear combination of standard modules $M(\xi)$ corresponding to the orbit S . In the general real case these are the L-packets defined by Langlands–Shelstad so they are interesting.

5.2. Given the relation in 5.1, it is interesting to identify the virtual combinations of characters corresponding to χ^{mic} . This is more delicate. But we can still use χ^{mic} to define packets.

Definition. (19.16 in [ABV]) *The micro-local packet attached to ϕ is the set of equivalence classes of irreducible representations*

$$\Pi(G/\mathbb{R})_\phi^{mic} := \{ (\pi(\xi), \delta(\xi)) : \chi_{S(\phi)}^{mic}(P(\xi)) \neq 0. \}$$

The corresponding virtual character is

$$\eta_\phi^{mic} := \sum_{\pi' \in \Pi(G/\mathbb{R})_\phi^{mic}} e(\pi') (-1)^{d(\pi') - d(\phi)} \chi_{S(\phi)}^{mic}(P') \pi'$$

Here,

- $d(\phi)$ and $d(\pi')$ are the dimensions of the orbits corresponding to ϕ , π' ,
- P' is the irreducible perverse sheaf corresponding to π' via 5.1.
- $e(\pi')$ is Kottwitz's constant attached to the real form for π' .

It is not clear what this packet is in an explicit sense. In the case of an Arthur parameter, it should be his conjectured packet. In the case of special unipotent parameters, it coincide with the definition of unipotent in [BV].

5.3. We begin the study of the micro-local packet, with a view towards its relation to Arthur's conjectured packet.

We saw in the previous section that in the case Y is a vector space, there is a dense subset in the conormal bundle T_S^*Y which plays a special role. The set of nondegenerate vectors plays a similar role in general. Fix $y \in Y$. Then the isotropy group H_y acts on the fiber $T_{y,S}^*Y$. Then given a $\nu \in T_{y,S}^*Y$, we can define the component group

$$H_{y,\nu} := C_H(y, \nu) / C_H(y, \nu)^0. \quad (5.3.1)$$

This is constant over a certain dense subset of nondegenerate vectors with y in the given orbit S . So we define

$$A_S^{mic} := C_H(y, \nu) / C_H(y, \nu)^0, \quad (y, \nu) \in T_S^*Y_{reg}. \quad (5.3.2)$$

There is a natural homomorphism

$$i_S : A_S^{loc} \longrightarrow A_S^{mic}. \quad (5.3.3)$$

The point of introducing this group is the following. Recall Q_S^{loc} and τ_S^{loc} from 4.2.

Theorem. (MacPherson) (24.8 in [ABV]) Suppose P is an H -equivariant perverse sheaf on Y . Then attached to P there is an H -equivariant local system $Q^{mic}(P)$ of complex vector spaces on $T_H^*(Y)_{reg}$ satisfying the following properties:

- (1) the rank of $Q^{mic}(P)$ at (y, ν) equals the multiplicity $\chi_S^{mic}(P)$,
- (2) if P is supported on $\bar{S}$, then the restriction of $Q^{mic}(P)$ to $T_S^*Y_{reg}$ is the pullback of $Q^{-dim S}(P)$ by the projection $T_S^*Y_{reg} \rightarrow S$,
- (3) Q^{mic} is exact from $\mathcal{P}(Y, H)$ to H -equivariant local systems on $T_H^*Y_{reg}$.

Remark. For a constructible sheaf C , one attaches a family $(Q^{mic})^i$; in the case of perverse sheaf they are zero for $i \neq 0$.

The difficulty in theorem 5.3 is proving that $\chi_S^{mic}(P)$ is the dimension of a local system, i.e. that it comes from a representation which is trivial on the connected component $H_{y, \nu}^0$.

This is part of what Arthur required, namely that one should attach a character to each representation.

As with τ^{loc} , we define $(\tau_S^{mic})^i(C)$ to be the representation of A_S^{mic} on $(Q_S^{mic})^i(C)$, and $\tau_S^{mic}(C)$ to be the corresponding Euler sum $\sum_i (-1)^i (\tau_S^{mic})^i(C)$. Its dimension is the multiplicity χ_S^{mic} .

5.4. Another requirement that Arthur imposes on his packets is that they should behave nicely under endoscopy. This has a counterpart in the geometry as well.

Suppose we have an algebraic action of a connected group G on a variety Y with finitely many orbits; in addition, suppose that there are compatible automorphisms of finite order

$$\sigma : G \rightarrow G, \quad \sigma : X \rightarrow X. \quad (5.4.1)$$

Fix a subgroup H of G such that $(G^\sigma)_0 \subset H \subset G^\sigma$ and a subvariety Y of X^σ such that $H \cdot Y = Y$ and Y is open and closed in X^σ . Write ϵ for the corresponding inclusions $H \subset G$ and $Y \subset X$. Suppose that C is a G -equivariant constructible complex endowed with a compatible automorphism σ of finite order. Suppose we have a point $(x, \nu) \in T_G^*X_{reg}$ which is fixed by σ . Then we get a map

$$(\tau_{x, \nu}^{mic})^i(C) : (Q^{mic})^i(C)_{x, \nu} \rightarrow (Q^{mic})^i(C)_{x, \nu}, \quad (5.4.2)$$

Then we can take the trace (and Euler characteristic)

$$\chi_{x, \nu}^{mic}(C)(\sigma) = \sum (-1)^i \text{tr}(\tau_{x, \nu}^{mic})^i(C)(\sigma). \quad (5.4.3)$$

The main result here is the following Lefschetz fixed point formula.

Theorem. (MacPherson, 25.8 in [ABV]) Write ϵ^*C for the restriction of C to Y . Then

$$\chi_{x, \nu}^{mic}(C)(\sigma) = (-1)^{\dim G \cdot x - \dim H \cdot x} \chi_{x, \nu}^{mic}(\epsilon^*C)(\sigma).$$

6. Arthur Parameters

6.1. Motivated by considerations coming from the trace formula, Arthur has introduced certain parameters which should describe the residual spectrum for automorphic forms. They should have profound consequences for the classification of the unitary dual of $G(F)$ as well as the general structure of admissible modules.

To describe these parameters, recall first that the Langlands classification (without the Weil group) is given in terms of standard modules which are induced from tempered irreducible representations. In terms of the geometric parametrization, these are open orbits. In terms of maps of the Weil group, i.e. in terms of (y, λ) , a parameter is *tempered* if and only if $\lambda + \theta_y(\lambda)$ is elliptic.

Example. Consider the inner forms $So(p, q)$ with $p+q = 2n+1$. Then the dual group is ${}^L G = Sp(2n, \mathbb{C}) \times \Gamma$. Suppose λ is regular integral so that we are parametrizing *Discrete Series* for $So(p, q)$. In this case,

$$y = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \lambda = \begin{bmatrix} 0 & z \\ z & 0 \end{bmatrix}, \quad (6.1.1)$$

where z is a nondegenerate diagonal matrix. Then $A_\phi^{loc} \cong \mathbb{Z}_2^n$. All these local systems cannot correspond to just a single $So(p, q)$, in fact they correspond to the *Discrete Series* over *all* the $p+q = 2n+1$ satisfying $q \leq n$.

On the other hand, recall that we defined a *geometrically stable* combination of characters as a linear combination of

$$\eta_S^{loc} = \sum e(\xi) M(\xi).$$

In the example, this combination is stable in the sense of Langlands–Shelstad; the only difference might be that it is an average over several real forms. Here are the definitions and results.

Definition. Suppose

$$\eta = \sum n(\xi) (\pi(\xi), \delta(\xi))$$

is a formal virtual character which is locally finite, in other words, for each real form δ there are only finitely many ξ with $n(\xi) \neq 0$. We say that η is *strongly stable* if the corresponding distribution character

$$\Theta(\eta, \delta) = \sum_{\delta(\xi)=\delta} \Theta(\pi(\xi), \delta(\xi))$$

satisfies the following condition:

$$\text{If } g \in G(\mathbb{R}, \delta) \cap G(\mathbb{R}, \delta'), \quad \text{then } \Theta(\eta, \delta)(g) = \Theta(\eta, \delta')(g).$$

Here is the main result, essentially a reformulation of results of Shelstad.

Theorem, 1.29 in [ABV]. *Using the identification of linear functionals of $K\Pi(G/\mathbb{R})$ with representations, the geometrically stable functionals correspond to strongly stable combinations of characters. Conversely, if Θ is a stable finite integer combination of characters on a real form $G(\mathbb{R}, \delta)$, then there is a strongly stable η with $\Theta(\eta, \delta) = \Theta$.*

6.2. Let us recall the situation. We have inclusions

$$\Pi(G/F)_{temp} \subset \Pi(G/F)_{unit} \subset \Pi(G/F). \quad (6.2.1)$$

In terms of the parameters, $\Phi(G/F)$, the L–packet of representations attached to a ϕ is the set

$$\Pi_\phi = \{\pi(y, \lambda, \tau) : \tau \in A_S^{loc}\}. \quad (6.2.2)$$

It is not easy to see how to attach a combination of characters of irreducible representations to Π_ϕ that will be stable in the sense of Langlands–Shelstad; the only reasonable one seems to be χ_S^{loc} from earlier, but this is $\sum e(\xi) M(\xi)$ from before. In terms of irreducible modules, this is quite large and unwieldy.

Arthur on the other hand, introduced the space $Q({}^\vee G^\Gamma)$ of maps (definition 22.4 in [ABV])

$$\psi : W_\mathbb{R} \times SL(2, \mathbb{C}) \longrightarrow {}^\vee G^\Gamma. \quad (6.2.3)$$

These maps should satisfy the additional properties

- (1) the restriction of ψ to $W_\mathbb{R}$ is tempered,
- (2) the restriction of ψ to $Sl(2)$ is holomorphic.

Two such parameters are called *equivalent* if they are conjugate by ${}^\vee G$. The set of equivalence classes is denoted $\Psi({}^\vee G^\Gamma)$. Define also

$${}^\vee G(\psi) = \text{centralizer of the image of } \psi \text{ in } {}^\vee G^\Gamma, \quad A_\psi = {}^\vee G_\psi / ({}^\vee G_\psi)_0. \quad (6.2.4)$$

The *associated* Langlands parameter is defined as

$$\phi_\psi(w) := \psi(w, \begin{bmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{bmatrix}). \quad (6.2.5)$$

This determines an L–packet of representation of the group G , or rather its quasisplit rational form. Arthur poses the following problems:

Problem 1. Define a finite set (called a *packet*) of representations Ψ_{ϕ_ψ} attached to each such map. It should contain the usual L–packet Φ_ϕ .

Problem 2. Attach a finite dimensional representation $\tau(\pi)$ of A_ψ to each $\pi \in \Psi_{\phi_\psi}$.

Problem 3. Show that

$$\sum \epsilon(\pi) \dim \tau(\pi) \Theta(\pi)$$

is stable in the sense of Langlands–Shelstad.

Problem 4. Prove analogues of the theorems on lifting tempered representations from endoscopic groups.

Problem 5. Show that the $\pi \in \Psi_{\phi_\psi}$ are unitary.

A good answer to Problem 1 should immediately give the resolution of the other ones. The microlocal packets defined earlier resolve these problems quite well. Here is the answer to Problem 1.

Definition. *The Arthur packet attached to ψ is the micropacket*

$$\Pi(G/\mathbb{R}) = \Pi(G/\mathbb{R})_{\phi_\psi}^{mic}.$$

6.3. We recall the description of such parameters (before Proposition 22.9 in [ABV]). Write ψ_0 for the restriction of ψ to $\mathcal{W}_\mathbb{R}$ and ψ_1 for its restriction to $SL(2)$. Then ψ_0 is determined by

$$(y_0, \lambda_0), \quad y_0 \in {}^\vee G^\Gamma - {}^\vee G, \quad \lambda_0 \in {}^\vee \mathfrak{g}. \quad (6.3.1)$$

For ψ_1 , define

$$y_1 = \psi_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \lambda_{\tilde{O}} = d\psi_1 \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \quad (6.3.2)$$

Then the parameter ϕ_ψ is associated to the pair

$$y = y_0 y_1, \quad \lambda = \lambda_0 + \lambda_{\tilde{O}}. \quad (6.3.3)$$

To answer problems 2–4, we just have to relate them to the geometry described earlier. Mainly we have to relate $A_{\phi_\psi}^{mic}$ to A_ψ .

Write θ_y for the conjugation by y on ${}^\vee G(\lambda)$; this gives a Cartan decomposition

$${}^\vee \mathfrak{g}(\lambda) = {}^\vee \mathfrak{k}(y) + {}^\vee \mathfrak{s}(y). \quad (6.3.4)$$

Let $x = (y, \Lambda) \in X(\mathcal{O}, {}^L G)$ be a point in the geometric parameter space and let $S = K(y) \cdot \Lambda \in \mathcal{P}(\Lambda)$ be the corresponding orbit. Then the conormal space to S at x can be identified as a module for $P(\Lambda) \cap K(y)$ with $[{}^\vee \mathfrak{g}(\Lambda) / ({}^\vee \mathfrak{p}(\Lambda) \cap {}^\vee \mathfrak{k})^*]$. Using the Cartan–Killing form, we can identify this latter space with ${}^\vee \mathfrak{n}(\Lambda) \cap {}^\vee \mathfrak{s}(y)$. Then let

$$E_\psi = d\psi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (6.3.5)$$

Then a simple calculation shows that $E_\psi \in {}^\vee \mathfrak{s}(y)$. Thus E_ψ is an element of this conormal space.

Theorem. *The orbit ${}^L G \cdot E_\psi$ is Zariski dense in T_S^* . In particular, A_ψ coincides with A_S^{mic} .*

For the remainder we just apply the machinery in section 5.

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