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E_8 FOR UNDERGRADUATES

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1. Introduction

1.1. **Lie algebras and their representations.** We learned from Reyer's talk that a Lie algebra is an abstract structure formed of a vector space $\mathfrak{g}$ with a bracket operation

$$[\ ,\ ]\ :\ \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}, \tag{1.1.1}$$

satisfying

(1) $[x, y]$ is bilinear and skew symmetric, *i.e.*

$$[ax_1+bx_2,y]=a\,[x_1,y]+b\,[x_2,y]\,,\qquad [x,y]=-\,[y,x]\,,$$

(2) (Jacobi identity)

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

The prototype of a Lie algebra is $gl(V)$, the vector space of linear transformations (matricies) $L : V \longrightarrow V$, with bracket $[A, B] := AB - BA$. So the bracket measures how far two linear transformations are from commuting.

1.2. **Classification of Lie algebras.** In view of the formulation in (1.2.10), it is tempting to ask whether we can classify Lie algebras in the sense of listing all $\{c^k_{ij}\}$ which give Lie algebras up to change of basis. This is not feasible, but there is a kind of answer. The building blocks are the **simple** Lie algebras. These are analogues of the blocks for matrices. Complex simple Lie algebras were classified in the late 19th century by Elie Cartan and Killing. Nowadays we use the Dynkin diagrams to describe them. They are

(1.2.1)	A_n	$\circ-\circ-\cdots-\circ-\circ$
(1.2.2)	B_n	$\circ-\circ-\cdots-\circ=>\circ$
(1.2.3)	C_n	$\circ-\circ-\cdots-\circ<=\circ$
(1.2.4)		$/^{\circ}$
(1.2.5)	D_n	$\circ-\circ-\cdots-\circ$
(1.2.6)		$\backslash_{\circ}$
(1.2.7)	G_2	$\circ\equiv>\circ$
(1.2.8)	F_4	$\circ-\circ=>\circ-\circ$

E_6

$$\begin{array}{ccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & | & & & & \\ & & & & \circ & & & & \end{array}$$

E_7

$$\begin{array}{ccccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & | & & & & & & \\ & & & & \circ & & & & & & \end{array} \tag{1.2.9}$$

E_8

$$\begin{array}{ccccccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & | & & & & & & & & \\ & & & & \circ & & & & & & & & \end{array}$$

The first set, A_n B_n and C_n are the classical Lie algebras realized as $sl(n+1,\mathbb{C})$, $so(2n+1,\mathbb{C})$, $sp(2n,\mathbb{C})$ and $so(2n,\mathbb{C})$. Standard realizations for them are well known in terms of matrices of trace 0, skew symmetric and symplectic matrices. The remainder are called exceptional. While there are realizations, mathematicians usually work with them in terms of (abstract) data derived from the diagrams. For example E_8 is an algebra of dimension 248, and it has four real forms. The program **atlas** gives detailed information about the real Lie algebras, the corresponding groups, and representations computed from the diagram for any of the simple (and composite) groups and algebras.

Exercise. $\mathbb{R}^3$, the space of 3-dimensional vectors is a Lie algebra with bracket the cross product of vectors. $\square$

A Lie algebra can be described in terms of structure constants. Let $\{e_i\}$ be a basis of $\mathfrak{g}$. Then there are constants c_{ij}^k such that

$$[e_i, e_j] = \sum c_{ij}^k e_k. \tag{1.2.10}$$

For the example in the exercise, let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the usual coordinate vectors. Then these constants come from the relations

$$\vec{i} \times \vec{j} = \vec{k}, \qquad \vec{j} \times \vec{k} = \vec{i}, \qquad \vec{k} \times \vec{i} = \vec{j}. \tag{1.2.11}$$

Example 1. Let $su(2) := \{X \in gl(\mathbb{C}^2) \ : \ X + X^* = 0,\ \mathrm{trace}(X) = 0\}$. This is a (real) Lie subalgebra of $gl(\mathbb{C}^2)$. A basis is given by

$$e_1 := \begin{bmatrix} i/2 & 0 \\ 0 & -i/2 \end{bmatrix}, \qquad e_2 := \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}, \qquad e_3 := \begin{bmatrix} 0 & i/2 \\ i/2 & 0 \end{bmatrix}. \tag{1.2.12}$$

Example 2. Let $so(3) := \{X \in gl(\mathbb{R}^3) \ : \ X + X^t = 0,\ \mathrm{trace}(X) = 0\}$. This is a (real) Lie subalgebra of $gl(\mathbb{R}^3)$. A basis is given by

$$e_1 := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad e_3 := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{1.2.13}$$

Note that in fact these three algebras are **the same**. This leads us to the notion of a *representation* of a Lie algebra.

Definition (1)**.** *A representation of a Lie algebra* $\mathfrak{g}$ *is a linear map*

$$\rho : \mathfrak{g} \longrightarrow gl(V)$$

which commutes with the Lie bracket, i.e.

$$\rho([x,y]) = [\rho(x), \rho(y)] := \rho(x)\rho(y) - \rho(y)\rho(x).$$

Here V *is a complex vector space.*

The examples above are all representations of the (abstract) Lie algebra with basis $\{e_1, e_2, e_3\}$ and bracket defined by the constants $[e_i, e_j] = e_k$ (in the appropriate sense, i, j, k are $1, 2, 3$ cyclically permuted). In the third example you should think of the matrices as acting on vectors with complex entries.

Example 3. Let $\mathfrak{g} = \mathbb{R}$. The Lie bracket is trivial, $[a, b] := ab - ba = 0$. We want to describe all finite dimensional representations of $\mathbb{R}$. This is very easy. If $\rho(\mathbb{1}) = M$, a linear transformation of a vector space V, then $\rho(t) = \rho(t \cdot \mathbb{1}) = t\rho(\mathbb{1}) = tM$. $\square$

But when we talk about representations, we want to be able to detect if we are talking about **the same** representation in different disguises. For example, if we pick a basis for V, then M is given by an $n \times n$ matrix, where $n = \dim V$. If we change basis, the matrix will change from M to gMg^{-1} where g is an invertible $n \times n$ matrix. So (equivalence classes of) representations correspond to similarity classes of matrices.

Theorem (Jordan canonical form)**.** *Assume that* $M \in gl(V)$, *where* V *is a finite dimensional vector space. Then there is a basis such that* M *takes on the following block diagonal form:*

$$\begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \lambda_i & 1 \\ 0 & \dots & & 0 & \lambda_i \end{bmatrix}$$

The subspace generated by the basis vectors corresponding to a given block is the **generalized eigenspace** of the matrix,

$$V_{\lambda_i} := \{v \in V \ : \ (M - \lambda_i Id)^n v = 0\}. \tag{1.2.14}$$

This subspace has the property that $MV_{\lambda_i} \subset V_{\lambda_i}$. Then (ρ, V_{λ_i}) is called a **subrepresentation** of (ρ, V).

Definition (2)**.** *A representation* (ρ, V) *is called* ***irreducible*** *if the only invariant subspaces are* (0) *and* V.

A representation is called ***completely reducible*** *if any invariant space* $W \subset V$ *has an invariant complement, i.e. there is* W' *such that*

$$W \cap W' = (0), \qquad W + W' = V, \qquad \rho(x)W \subset W \text{ for every } x \in \mathfrak{g}.$$

A representation of $\mathbb{R}$ (matrix) is irreducible if and only if it is 1-dimensional. It is completely reducible if and only if it is diagonalizable. In general, the representation (ρ, V) corresponding to a matrix M decomposes into a direct sum

$$V = \bigoplus V_\lambda, \qquad V_\lambda := \big\{v \in V \ : \ (M - \lambda I)^n v = 0 \text{ for some } n > 0\big\}. \tag{1.2.15}$$

Each V_λ has a filtration of subspaces

$$\dots \subset V_\lambda^k \subset V_\lambda^{k+1} \subset \dots, \qquad V_\lambda^k := \big\{v \ : \ (M - \lambda I)^k = 0\big\}. \tag{1.2.16}$$

Then $MV_\lambda^k \subset V_\lambda^k$, so it is a subrepresentation of V_λ). Then M (so also $\mathfrak{g}$) will act on $V_\lambda^{k+1}/V_\lambda^k$ as well. This representation is completely reducible, because M acts by multiplication by λ. Its dimension is the number of blocks of size larger than or equal to $k+1$. The filtration (1.2.16) is called a **Jordan-Hölder series**.

1.3. **Abelian Lie algebras.** The next case to study, would be $\mathfrak{g} = \mathbb{R}^n$. Representations are in 1-1 correspondence with $n-$tuples of commuting matrices $(M_1, \dots, M_n)$, and their representation theory corresponds to Jordan canonical forms of commuting matrices.

1.4. **Lie groups and their representations.** We saw in Reyer's talk that there is a strong connection between Lie groups and Lie algebras. The case we are going to focus on is closedgroups $G \subset GL(V)$, where $GL(V)$ is the group of invertible linear transformations of a finite dimensional vector space. Given such a group G, define its Lie algebra as

$$\mathcal{L}(G) := \{X \in gl(V) \ : \ e^{tX} \in G \text{ for all } t \in \mathbb{R}\}. \tag{1.4.1}$$

A theorem of von Neumann states that $\mathcal{L}(G)$ is a Lie algebra. The general theory of Lie groups shows how one can recover all (most) of the properties of G from $\mathcal{L}(G)$.

Conversely, given a Lie subalgebra of $gl(V)$, there is a Lie group attached to it, namely the group generated by all e^X with $X \in \mathfrak{g}$. The exponential map plays a crucial role in the interplay between $\mathfrak{g}$ and G. For example, a **representation** of G is a continuous group homomorphism

$$\rho : G \longrightarrow GL(V). \tag{1.4.2}$$

When V is finite dimensional, there is a Lie algebra representation associated to ρ, namely

$$d\rho(X) := \frac{d}{dt}\bigg|_{t=0} \rho(e^{tX}) \tag{1.4.3}$$

Examples.

(1) Let $\mathbb{R}^+, \cdot)$ be the positive real numbers with the usual multiplication. Then $\mathcal{L}(G) = (\mathbb{R}, +)$.
(2) Let $\mathbb{R}^\times, \cdot)$ be the nonzero real numbers with the usual multiplication. Then $\mathcal{L}(G) = (\mathbb{R}, +)$.
(3) Let $G = SO(2) := \big\{g \in GL(\mathbb{R}^2) \ : \ g^* = g^{-1}, \ \det(g) = 1\big\}$. We write these matrices as $r(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta. \end{bmatrix}$. Then $\mathcal{L}(G) = so(2) = \left\{ \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \right\}$. If we drop the condition that $\det(g) = 1$, we get a larger group called $O(2)$ which is disconnected, but has the same Lie algebra.
(4) Let $G = SU(2) := \{g \in GL(\mathbb{C}^2) \ : \ g^* = g^{-1}, \ \det(g) = 1\}$. Then $\mathcal{L}(G) = su(2)$.
(5) Let $G = SO(3) := \{g \in GL(\mathbb{R}^3) \ : \ g^t = g^{-1}, \ \det(g) = 1\}$. Then $\mathcal{L}(G) = so(3)$.

While the Lie algebras are the same, the groups are not. They are only **locally** the same.

The way we recover the representation of the group from the Lie algebra is via formula (1.4.3);

$$\rho(g) = e^{d\rho(\log g)}. \tag{1.4.4}$$

In this formula g has to be very close to Id for the series to make sense. Furthermore, in example (1), log is 1-1, so any representation of G is of this form. In example (2) however, we need to insure that $M = d\rho(\mathbb{1})$ is such that $e^{2\pi M} = Id$. This forces M to be semisimple and have integer eigenvalues. The irreduible characters of $SO(2)$ are the $\chi_n(e^{i\theta}) = e^{in\theta}$ that appear in Fourier analysis.

Not all representations of the Lie algebra exponentiate to the corresponding Lie group, even if the Lie group is connected. For example, in quantum physics the groups $SU(2)$ and $SO(2)$ play an important role. They have the same Lie algebra. Every representation of the Lie algebra $su(2)$ exponentiates to a representation of $SU(2)$ essentially because

$$SU(2) = \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \ : \ |\alpha|^2 + |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{C} \right\} \tag{1.4.5}$$

is simply connected. On the other hand $SO(3)$ is a quotient of $SU(2)$ by $\pm Id$, and so only the representations of $SU(2)$ which are trivial on $-Id$ drop down to $SO(3)$. This has to do with half spin in physics.

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU(2) \longrightarrow SO(3) \longrightarrow 1. \tag{1.4.6}$$

1.5. **Adjoint Representation.** Note that

$$ge^X g^{-1} = e^{gXg^{-1}}. \tag{1.5.1}$$

It follows that there is a representation called the **adjoint representation**

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g}), \qquad \mathrm{Ad}(g) \cdot X := gXg^{-1}. \tag{1.5.2}$$

As in the case of Jordan canonical forms, you can ask for conditions of when two elements X, Y are conjugate by G. For the simple algebras that we are considering, (the technical definition is that they do not have any nontrivial proper ideals),

$$d\,\mathrm{Ad} = \mathrm{ad} : \mathfrak{g} \longrightarrow gl(\mathfrak{g}) \tag{1.5.3}$$

is an inclusion. Then there is a notion of **semisimple** elements, generalizing the notion of diagonalizable, namely $x \in \mathfrak{g}$ is semisimple if and only if $\mathrm{ad}\, x$ is diagonalizable. The map from $SU(2)$ to $SO(3)$ in (1.4.6) is really the adjoint representation of $SU(2)$. The Lie algebra $su(2)$ is 3-dimensional vector space, and has an positive definite inner product,

$$\langle x, y\rangle = \mathrm{trace}(xy^*). \tag{1.5.4}$$

Then $SU(2)$ acts by $\mathrm{Ad}\, g(x) := gxg^{-1}$, and it is easy to check that $\langle \mathrm{Ad}\, gx, \mathrm{Ad}\, gy\rangle = \langle x, y\rangle$. Thus for each $g \in SU(2)$, $\mathrm{Ad}\, g$ is an orthogonal 3×3 matrix.

1.6. **Cartan subalgebras and subgroups.**

Definition. *A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a Cartan subalgebra, if*

(1) it is abelian, i.e. ($[x, y] = 0$ for all $x, y \in \mathfrak{h}$,
(2) for every element $x \in \mathfrak{h}$, $\mathrm{ad}\, x$ is semisimple.

A group $H \subset G$ is called a Cartan subgroup, if it is the centralizer of a Cartan subalgebra.

A Lie algebra has finitely many conjugacy classes of Cartan subalgebras.

Example 4. Suppose $G = SL(\mathbb{R}^n)$. Then representatives of the Cartan subgroups are of the form

$$H_r = \left\{ \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \\ \dots & \cos\theta_i & \sin\theta_i & 0 & \dots & 0 \\ \dots & -\sin\theta_i & \cos\theta_i & 0 & \dots & 0 \\ & \dots & \dots & \dots & \dots & \\ & \dots & \dots & t_j & \dots & \\ \dots & \dots & \dots & \dots & \dots & \end{bmatrix} \right\} \tag{1.6.1}$$

where there are r boxes of the $SO(2)$ kind, and $n - r$ of the $\mathbb{R}^\times$ kind, with the product of the t_j equal to 1.

One of the main uses of the cartan subgroups is that their irreducible representations parametrize the irreducible infinite dimensional representations of the group G. For each $\chi \in \widehat{H}$ there is a **standard module** $X(\chi)$, which has a complicated composition series. But it has a unique subquotient $\overline{X}(\chi)$, so that every irreducible module of G is an $\overline{X}(\chi)$, and there is a precise easy rule how to decide when $\overline{X}(\chi)$ and $\overline{X}(\chi')$ are equivalent.

1.7. **Representations of $SL(2, \mathbb{R})$.** This is the subgroup of $GL(\mathbb{R}^2)$ of invertible matrices of determinant 1. its Lie algebra is the subalgebra of $gl(\mathbb{R}^2)$ of matrices of trace 0. Its complexification is $sl(2, \mathbb{C})$ called A_1 in the list of simple algebras.

Reminder. $\det(e^X) = e^{\mathrm{trace}(X)}$. □

We are looking to classify the infinite dimensional irreducible representations of $SL(2, \mathbb{R})$. The Lie algebra $sl(2, \mathbb{R})$ should act on such a representation (in fact it acts on a dense subspace). Then the complexification

$$sl(2, \mathbb{C}) := \big\{ X \in gl(\mathbb{C}^2) \ : \ \mathrm{trace}(X) = 0 \big\}. \tag{1.7.1}$$

should act as well (because our representations are complex vector spaces). We use the basis

$$h = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \qquad e = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}, \qquad f = \begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix}. \tag{1.7.2}$$

A representation of $sl(2, \mathbb{R})$ which is the derivative of a representation should have h act semisimply and with integer eigenvalues. If v_k is an eigenvector for h with eigenvalue k, then $\rho(e)v_k$ should be a multiple of of v_{k+2}, $\rho(f)v_k$ should be a multiple of v_{k-2}. Indeed, if $\rho(h)v = \lambda v$, then

$$\rho(h)\rho(e)v = \rho(e)\rho(h)v + \rho([h, e])v = \lambda\rho(v) + 2\rho(e)v = (\lambda + 2)\rho(e)v. \tag{1.7.3}$$

There are two Cartan subgroups in $SL(2, \mathbb{R})$,

(1.7.4)

$$T = \big\{ r(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \big\}, \qquad A = \big\{ a(t, \epsilon) = \begin{bmatrix} \epsilon e^t & 0 \\ 0 & \epsilon e^{-t} \end{bmatrix}, \ \epsilon = \pm 1 \big\}.$$

The standard modules which are *easy* to construct, are associated to irreducible representations of the Cartan subgroups. The compact Cartan subgroup T has irreducible representations $\chi_n(r(\theta) = e^{in\theta}$ with $n \in \mathbb{Z}$ as described earlier. Then

(1.7.5)
$$\begin{aligned}
&X(T,n) := \{v_{n+2k}\}_{k\geq 0},\\
&\rho(h)v_{n+2k} = (n+2k)v_{n+2k},\\
&\rho(e)v_{n+2k} = i(n+1+k)v_{n+2k+2},\ \rho(f)v_{n+2k} = i(-1+k)v_{n+2k-2}, \qquad n>0\\
&X(T,-n) := \{v_{-n-2k}\}_{k\geq 0},\\
&\rho(h)v_{-n-2k} = (-n-2k)v_{n+2k},\\
&\rho(e)v_{-n-2k} = i(1-k)v_{-n-2k+2},\ \rho(f)v_{n+2k} = (-n-1-k)v_{n+2k-2}, \quad n>0
\end{aligned}$$

These representations are irreducible. The Cartan subgroup A is disconnected, the irreducible representations are

$$\chi_{triv,\nu}(a(\epsilon,t) = |\epsilon|e^{t(\nu-1)}, \quad \chi_{sgn,\nu}(a(\epsilon,t) = \epsilon e^{t(\nu-1)}. \tag{1.7.6}$$

The standard modules are

$$\begin{aligned}
&X(A,triv,\nu) := \{v_{2k}\}_{k\in\mathbb{Z}},\\
&\rho(h)v_{2k} = (2k)v_{2k},\\
&\rho(e)v_{2k} = i(\nu+1/2+k)v_{2k+2},\ \rho(f)v_{2k} = i(-\nu-1/2+k)v_{2k-2},\\
&X(A,sgn,\nu) := \{v_{2k+1}\}_{k\in\mathbb{Z}},\\
&\rho(h)v_{2k+1} = (2k+1)v_{2k+1},\\
&\rho(e)v_{2k+1} = i(\nu+1+k)v_{2k+3},\ \rho(f)v_{2k+1} = i(-\nu+k)v_{2k-1}.
\end{aligned} \tag{1.7.7}$$

These representations are irreducible, except when $\nu = 1/2+n$ for $triv$, and $\nu = n$ with $n \in \mathbb{Z}$ for sgn. In these cases the module has a Jordan-Hölder series analogous to the case of the Jordan blocks. These Jordan-Hölder series can be drawn pictorially

$$\nu = n+1/2 \quad \dots \quad \underset{-2n-2}{\circ}\ ] \quad \underset{-2n}{\circ} \quad \dots \quad \underset{2n}{\circ} \quad [\ \underset{2n+2}{\circ} \quad \dots \tag{1.7.8}$$

$$\nu = -n-1/2 \quad \dots \quad \underset{-2n-2}{\circ}\ [\quad \underset{-2n}{\circ} \quad \dots \quad \underset{2n}{\circ} \quad]\ \underset{2n+2}{\circ} \quad \dots \tag{1.7.9}$$

The multiplicities and levels of the irreducible representations are encoded in polynomials in q called Kazhdan-Lusztig-Vogan polynomials

$$P_{a,b}(q) = \sum a_i q^i \tag{1.7.10}$$

where a_i is the multiplicity of $\overline{X}(b)$ at level i in the composition series of $X(a)$. These polynomials are very easy to compute for $SL(2,\mathbb{R})$ but quite difficult as the groups become larger.

Remark. The above account is of course an oversimplification. There is a partial order on the set of parameters so that $P_{a,b} = 0$ unless $a \leq b$. The algorithm to compute the $P_{a,b}$ is such that one needs to know the $P_{a',b'}$ for all the earlier (a',b'). Of course the initial standard modules are irreducible. The difficulty with the algorithm is that the number of parameters goes up very fast with the size of the group, but also that at each step, to compute $P_{a,b}$ one needs to look up all the earlier $P_{a',b'}$.

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