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The Unitary Spectrum for Real Rank one Groups

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0. Introduction

Let G be a connected real semi-simple Lie group with finite center. One of the main problems in harmonic analysis is to determine the unitary spectrum of G. In this paper we treat this question in the case when real rank of G is 1.

Although the answer was known for G of classical type previously ([2], [5], [7], [15]), we have redone this work sometimes giving simpler arguments. With the arbitrary rank case in mind we have tried to deal with the problem of classifying unitary representations in as systematic a way as possible proceeding by induction on the dimension of G. We have concentrated on the case when G is linear and has a compact Cartan subgroup, the other cases being known already.

As an application we also give a list of the unitary representations that contribute to the L^2-index formula for the Dirac operator with coefficients. As is apparent from the calculations in [19], this is intimately connected with one of our main techniques for determining the unitary spectrum, the Dirac inequality. We will deal with some related problems in a future paper.

The paper is organized as follows:

In Sect. 1, we introduce notation and summarize the results necessary for the reduction to lower dimension groups.

Section 2 deals with the Dirac inequality and reducibility of principal series in terms of parabolic induction via the derived functor reviewed in section one. We show for the representations in question how to reduce the question of unitarity to lower (not necessarily strictly) dimensional groups. The results are summarized in Theorem 2.3.

In Sect. 3 we deal with the cases not considered in the previous section and we prove in particular that certain representations that are isolated are unitary.

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Theorem 3.2 provides a classification of the unitary irreducible representations in real rank one, modulo certain cases in $\mathrm{Sp}(n,1)$, [2], as well as the spherical principal series case, [8].

Corollary 3.5 gives an affirmative answer to a conjecture of Zuckerman that a certain construction of irreducible representations via derived functors always leads to unitary representations.

In Sect. 4 we apply these results to determine the representations with $(\mathfrak{g}, K)$ cohomology and the representations contributing to the L^2-index formula alluded to earlier.

For a motivation for considering this problem we refer to the brief discussion at the beginning of Sect. 4 and to [19] and [13].

Our approach to classifying unitary representations owes a great deal to many conversations with D. Vogan. We would like to thank him for generously sharing his insight in the problems of reducibility of principal series and unitarity of representations.

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1. Preliminary Results

Let G be a semisimple real connected Lie group with finite center and Lie algebra $\mathfrak{g}_0$. We assume that $G \subseteq G_c$, where G_c is a linear complexification of G. We will denote by a subscript 0 a real vector space and we will drop the subscript for the complexification. For example, the Lie algebra of G_c will be denoted by $\mathfrak{g}$.

Extend θ, the Cartan involution for $\mathfrak{g}_0$, linearly to $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the corresponding Cartan decomposition. We fix $\mathfrak{t}_0$ a θ stable fundamental Cartan of $\mathfrak{g}$ and we denote by K and T the analytic subgroups corresponding to $\mathfrak{k} \cap \mathfrak{g}$ and $\mathfrak{t}_0$.

Let $\mathfrak{p}_0 = \mathfrak{m}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ be a minimal parabolic subalgebra of $\mathfrak{g}$ and $P_0 = M_0 A_0 N_0$ the corresponding subgroup.

We assume that $\dim \mathfrak{a}_0 = 1$ and $\mathfrak{t}_0 \subseteq \mathfrak{t}_0$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ be the roots of $\mathfrak{g}$ relative to $\mathfrak{t}$ and $\Delta(\mathfrak{k}) = \Delta(\mathfrak{k}, \mathfrak{t})$ the corresponding compact roots. If ψ is a positive system of roots for Δ, we denote by $\bar{\psi}$ the corresponding closed Weyl chamber in $i\mathfrak{t}^*$ and we set $\rho(\psi) = \frac{1}{2}\sum_{\alpha \in \psi} \alpha$, $\rho_c(\psi) = \frac{1}{2}\sum_{\alpha \in \psi \cap \Delta(\mathfrak{k})} \alpha$ and $\rho_n(\psi) = \rho(\Psi) - \rho_c(\psi)$. In general for any $\mathfrak{t}$ stable vector space V we will write $\rho(V) = \frac{1}{2}\sum_{\alpha \in \Delta(V)} \alpha$, $\Delta(V)$ the roots of $\mathfrak{t}$ in V.

We denote by W the Weyl group of Δ and W_K the Weyl group of $\Delta(\mathfrak{k})$. If α is a root then $\mathfrak{s}_\alpha$ is the reflection by α.

We now fix a system of positive compact roots $\Delta^+(\mathfrak{k})$ and list all the positive systems for Δ containing $\Delta^+(\mathfrak{k})$ as follows:

a) $SO_e(2n,1)$.

$\Delta^+(\mathfrak{k})$: $\varepsilon_i \pm \varepsilon_j$, $\quad 1 \leqq i < j \leqq n$

$$\tilde{B}_1 = \left\{ \sum_{k=1}^{n} a_i \varepsilon_i,\ a_1 \geqq \ldots \geqq a_n \geqq 0 \right\}$$

$$\tilde{B}_2 = \left\{ \sum_{k=1}^{n} a_i \varepsilon_i,\ a_1 \geqq \ldots \geqq a_{n-1} \geqq -a_n \geqq 0 \right\}.$$

Then B_1 and B_2 contain $\Delta^+(\mathfrak{k})$ and

$$\rho_c = \rho(\Delta^+(\mathfrak{k})) = \sum_{k=1}^{n} (n-k)\,\varepsilon_k$$

$$\rho_n^1 = \rho_n(B_1) = \sum_{k=1}^{n} \varepsilon_k, \quad \rho_n^2 = \rho_n(B_2) = \mathfrak{s}_{\varepsilon_n} \rho_n^1.$$

b) $SU(n,1)$.

$\Delta^+(\mathfrak{k})$: $\varepsilon_i - \varepsilon_j$, $\quad 1 \leqq i < j \leqq n$

$$\tilde{A}_i = \left\{ \sum_{k=1}^{n+1} a_k \varepsilon_k,\ a_1 \geqq \ldots \geqq a_i \geqq a_{n+1} \geqq a_{i+1} \geqq \ldots \geqq a_n \right\}, \qquad 0 \leqq i \leqq n$$

$$\rho_c = \tfrac{1}{2} \sum_{k=1}^{n} (n+1-2k)\,\varepsilon_k$$

$$\rho_n^i = \rho_n(A^i) = \tfrac{1}{2} \left\{ \sum_{k=1}^{i} \varepsilon_k - \sum_{k=i+1}^{n} \varepsilon_k + (n-2i)\,\varepsilon_{n+1} \right\}.$$

c) $Sp(n,1)$.

$\Delta^+(\mathfrak{k})$: $\varepsilon_i \pm \varepsilon_j$, $\quad 1 \leqq i < j \leqq n$; $\quad 2\varepsilon_i$, $\quad 1 \leqq i \leqq n+1$.

$$\tilde{C}_i = \left\{ \sum_{k=1}^{n+1} a_k \varepsilon_k:\ a_1 \geqq \ldots \geqq a_i \geqq a_{n+1} \geqq a_{i+1} \geqq \ldots \geqq a_n \geqq 0 \right\} \qquad 0 \leqq i \leqq n.$$

$$\rho_c = \sum_{k=1}^{n} (n-k+1)\,\varepsilon_k + \varepsilon_{n+1}$$

$$\rho_n^i = \rho_n(C_i) = \sum_{k=1}^{i} \varepsilon_k + (n-i)\,\varepsilon_{n+1}.$$

d) $F_{4,1}$.

$\Delta^+(\mathfrak{k})$: $\varepsilon_i \pm \varepsilon_j$, $\quad 1 \leqq i < j \leqq 4$; $\quad \varepsilon_i$, $\quad 1 \leqq i \leqq 4$.

0——0 $\Longrightarrow$ 0——0

F_3: $\varepsilon_2 - \varepsilon_3 \ \varepsilon_3 - \varepsilon_4 \ \varepsilon_4 \ \ 1/2(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$

$F_2 = \mathfrak{s}_{1/2(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)} F_3$

$F_1 = \mathfrak{s}_{1/2(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)} F_2$

$\rho_c = 7/2\,\varepsilon_1 + 5/2\,\varepsilon_2 + 3/2\,\varepsilon_3 + 1/2\,\varepsilon_4$

$\rho_n^1 = \rho_n(F_1) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

$\rho_n^2 = \rho_n(F_2) = 3/2\,\varepsilon_1 + 1/2(\varepsilon_2 + \varepsilon_3 + \varepsilon_4)$

$\rho_n^3 = \rho_n(F_3) = 2\,\varepsilon_1$.

If B is the Cartan Killing form on $\mathfrak{k}$ we let $\langle\,,\,\rangle$ denote the bilinear form defined by B on $i\mathfrak{t}^*$.

By a θ-stable parabolic subalgebra of $\mathfrak{g}$ we will mean a subalgebra constructed from some element $\gamma \in i\mathfrak{t}^*$ in the following way. We set $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ where $\mathfrak{t} \subseteq \mathfrak{l}$, $\Delta(\mathfrak{l}) = \{\alpha \in \Delta : \langle \alpha, \gamma \rangle = 0\}$ and $\Delta(\mathfrak{u}) = \{\alpha \in \Delta : \langle \alpha, \gamma \rangle > 0\}$. We will say that such a $\mathfrak{q}$ is determined by γ. Set $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$, then $\mathfrak{l}_0$ is reductive θ-stable subalgebra whose complexification is $\mathfrak{l}$. We fix, given such $\mathfrak{l}$, a positive system $\Delta^+(\mathfrak{l} \cap \mathfrak{k})$ of $\mathfrak{l} \cap \mathfrak{k}$ with respect to $\mathfrak{t}$ such that $\Delta^+(\mathfrak{l} \cap \mathfrak{k}) \subseteq \Delta^+(\mathfrak{k})$. $W(\mathfrak{l})$, $W_{L \cap K}$ will denote the corresponding Weyl groups, and L the subgroup corresponding to $\mathfrak{l}_0$. Irreducible K-types (respectively $L \cap K$ types) are always parametrized by the highest weight with respect to $\Delta^+(\mathfrak{k})$ (respectively $\Delta^+(\mathfrak{l} \cap \mathfrak{k})$). Let now $P = MAN$ be a parabolic subgroup of G. If δ is a discrete series representation of M and ν is a complex valued linear functional on the Lie algebra of A, then we define

$$\pi_P(\delta, \nu) = \mathrm{Ind}_P^G(\delta \otimes e^\nu \otimes 1),$$

where the induced representation is defined in such a way that G acts on the left and the representation is unitary of ν is imaginary. In [16], Sect. 7, it is shown how to obtain the Langlands data for an irreducible $(\mathfrak{g}, K)$ module π from its lowest (in the sense of [16]) K-type μ. This is useful for calculations involving the Dirac operator. We review this procedure in the special case of our setting.

We fix μ a lowest K-type. Then we choose a positive system ψ in $\Delta(\mathfrak{g}, \mathfrak{t})$ such that $\mu + 2\rho_c$ is dominant with respect to $i\mathfrak{t}$. Then $\mu + 2\rho_c - \rho(\psi)$, according to Proposition 4.1 in [16], is either dominant with respect to ψ or else there is a simple noncompact root $\beta \varepsilon \psi$ such that

$$2\langle \mu + 2\rho_c - \rho(\psi), \beta \rangle \langle \beta, \beta \rangle^{-1} = -1.$$

We define

$$\bar{\lambda}(\mu) = \begin{cases} \mu + 2\rho_c - \rho(\psi) & \text{if } \mu + 2\rho_c - \rho(\psi) \text{ is dominant} \\ \mu + 2\rho_c - \rho(\psi) + \tfrac{1}{2}\beta & \text{if it is not.} \end{cases}$$

In the real rank one case $\bar{\lambda}(\mu)$ is dominant with respect to ψ and is precisely the parameter associated to μ by Proposition 4.1 of [16].

We denote by $\mathfrak{b}^0 = \mathfrak{l}^0 + \mathfrak{u}^0$ the θ-stable parabolic subalgebra determined by $\bar{\lambda}(\mu)$.

Then $\mathfrak{b}^0$ determines a cuspidal parabolic subgroup of G, $P = MAN$ and a discrete series representation δ of M. This is either G and a discrete series or a minimal parabolic subgroup $P_0 = M_0 A_0 N_0$ and a finite dimensional unitary irreducible representation. Let $\nu \in \mathfrak{a}_0^*$ then $\pi_P(\delta, \nu)$ contains the K-type μ with multiplicity one. Moreover if $\pi_P(\delta, \nu, \mu)$ denotes the unique subquotient of $\pi_P(\delta, \nu)$ containing the K-type μ, then π is infinitesimally equivalent to it. If M is connected then $\delta = \bar{\lambda}(\mu)$. By abuse of notation we will write simply $\bar{\lambda}(\mu)$ for δ. (We also remind the reader that M_0 is disconnected only if $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$.)

In particular discrete series representations are parametrized by $\bar{\lambda}(\mu)$'s which are regular ($P = G$); and limit of discrete series by $(\bar{\lambda}(\mu), 0)$ and a positive system ψ such that $\bar{\lambda}(\mu)$ is dominant with respect to ψ and $\langle \bar{\lambda}(\mu), \alpha \rangle = 0$ for α simple implies α noncompact.

We also remark that if $P = P_0$ and $\mathrm{Re}\,\nu > 0$ then $\pi_{P_0}(\bar{\lambda}(\mu), \nu, \mu) \cong \bar{\pi}_{P_0}(\bar{\lambda}(\mu), \nu)$ where $\bar{\pi}_P(\bar{\lambda}(\mu), \nu)$ is the unique Langlands quotient of the corresponding induced representation. For a complete account of the quoted results compare [11, 10, 16].

It is well known that the problem of classifying the irreducible unitary representations of G amounts to describing precisely which representations $\bar{\pi}_{P_0}(\bar{\lambda}(\mu), \nu)$, $\nu > 0$, are infinitesimally unitary. We will consider only such representations and so from now on we will drop the subscript P_0.

Let $\mathfrak{u}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Fix a K-type $\mu \in \Delta^+(\mathfrak{k})$ and let $\bar{\lambda}(\mu)$ be defined as above and $\mathfrak{b}^0$ the corresponding parabolic. Let $\mathfrak{q} \supseteq \mathfrak{b}^0$ be a θ-stable parabolic subalgebra and set $\mu_L = \mu - 2\rho(\mathfrak{u} \cap \mathfrak{s})$. Set $\lambda^0 = \bar{\lambda}(\mu)$ and $\lambda_L^0 = \bar{\lambda}(\mu_L) = \bar{\lambda}(\mu - 2\rho(\mathfrak{u} \cap \mathfrak{s}))$.

The following theorem is an immediate consequence of 4.17–4.21 in [14].

Theorem 1.1. *In the setting described, assume also that*

$$(*) \qquad \mathrm{Re}\langle (\lambda^0, \nu), \alpha \rangle \geqq 0 \quad \textit{for each } \alpha \in \Delta(\mathfrak{u}).$$

Then

a) $\pi(\lambda^0, \nu)$ *has a number of composition factors which is less than or equal to the number of composition factors of* $\pi(\lambda_L^0, \nu)$. *In particular*

b) $\pi(\lambda^0, \nu)$ *is irreducible if* $\pi(\lambda_L^0, \nu)$ *is irreducible.*

We now need to recall the basic properties of the Harish-Chandra modules constructed by Vogan-Zuckerman in order to be able to compute various types of cohomology. We state the results we need without proof. A basic reference is [17].

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic and L as before. Let W be an $(\mathfrak{l}, L \cap K)$ module. Then in [17] Sect. 6, a family of $(\mathfrak{g}, K)$ modules $\{\mathscr{R}^i W\}$ is associated to W with the following properties. Set $S = \dim(\mathfrak{u} \cap \mathfrak{k})$.

1. [17, cor. 6.3.21] $\mathscr{R}^i W = 0$ for $i < 0$ and $i > S$.

2. [17, th. 6.3.12] Let $\delta \in \Delta^+(\mathfrak{k})$. Set $W_K^1 = \{w \in W_k : w^{-1}\alpha < 0,\ \alpha \in \Delta^+(\mathfrak{k})$ then $\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k})\}$. Suppose δ occurs in $\mathscr{R}^{S-i}(W)$. Then there is $w \in W_k^1$ of length i, non-negative integers n_β, $\beta \in \Delta(\mathfrak{u} \cap \mathfrak{s})$ and an $L \cap K$ type η occurring in W such that

$$\eta = w(\delta + \rho_c) - \rho_c - 2\rho(\mathfrak{u} \cap \mathfrak{s}) - \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{s})} n_\beta \beta.$$

The multiplicity of δ is at most the number of such expressions (counted with the multiplicity of η in W). Furthermore, if W has finite length, then

$$\sum (-1)^i m(\delta, \mathscr{R}^{S-i} W) = \sum_\gamma m(\eta, W) \sum_{w \in W_K} (-1)^{l(w)} P(w(\delta + \rho_c) - \rho_c - 2\rho(\mathfrak{u} \cap \mathfrak{s}) - \eta)$$

where $l(w)$ is the length of w, m is the multiplicity and P is the Kostant partition function with respect to the roots of $\mathfrak{u} \cap \mathfrak{s}$.

3. Let $\gamma_G=(\bar{\lambda}(\mu), \nu)$ be the Langlands data for a $(\mathfrak{g}, K)$ module. Let $\pi(\gamma_G)$ be the corresponding induced representation with lowest K-type μ. By abuse of notation if γ_G is the parameter for a limit of discrete series we still denote by $\pi(\gamma_G)$ the corresponding limit of discrete series. Assume $\mathfrak{g}\supseteq\mathfrak{b}^0$ and set $\gamma_{\mathfrak{q}}^L=\gamma_G-\rho(\mathfrak{u})$, $\mu_L=\mu-2\rho(\mathfrak{u}\cap\mathfrak{s})$. Then

$$\mathscr{R}^i(\pi(\gamma_{\mathfrak{q}}^L))=\begin{cases}0 & i\neq S\\ \pi(\gamma_G) & i=S\end{cases}$$

whenever $\mathrm{Re}\langle(\bar{\lambda}(\mu), \nu), \alpha\rangle\geqq 0$ for all $\alpha\in\Delta(\mathfrak{u})$ (cf. [17] a special case of 8.2.15).

4. If $0\to W_1\to W_2\to W_3\to 0$ is an exact sequence then there is a long exact sequence

$$\ldots\to\mathscr{R}^i W_1\to\mathscr{R}^i W_2\to\mathscr{R}^i W_3\to\mathscr{R}^{i+1} W_1\to\ldots.$$

Corollary 1.2. Let $\mathrm{Re}\,\nu>0$ and $\bar{\pi}_L=\bar{\pi}((\bar{\lambda}(\mu_L), \nu)-\rho(\mathfrak{u}))$ be the Langlands quotient of $\pi(\gamma_{\mathfrak{q}}^L)$. Then under the assumption in 3

$$\mathscr{R}^i(\bar{\pi}_L)=\begin{cases}0 & i\neq S\\ \bar{\pi}(\gamma_G) & i=S.\end{cases}$$

Proof. The proof is by descending induction on the length of $\bar{\lambda}(\mu)$. We give a sketch of it. By 3 it is true for $\langle(\bar{\lambda}(\mu), \bar{\lambda}(\mu))\rangle$ maximal because of 6.6.6 of [17]. Otherwise we have an exact sequence

$$0\to\bar{\pi}(\gamma_{\mathfrak{q}}^L)\to\pi(\gamma_{\mathfrak{q}}^L)\to X_L\to 0$$

and the composition factors are higher in the induction than $\pi(\gamma_{\mathfrak{q}}^L)$ (6.6.6 of [17]). Therefore

$$\mathscr{R}^i(X_L)=0 \quad\text{and}\quad \mathscr{R}^i\pi(\gamma_{\mathfrak{q}}^L)=0, \qquad i<S.$$

Using the long exact sequence in 4 we get $\mathscr{R}^i(\bar{\pi}(\gamma_{\mathfrak{q}}^L))=0$ if $i<S$, and also $\mathscr{R}^S\bar{\pi}(\gamma_{\mathfrak{q}}^L)\supset\bar{\pi}(\gamma_G)$ because of the induction hypothesis. Because of Theorem 1.1a the conclusion follows.

2. Cases With Only Complementary Series

We recall the definition of complementary series.

Definition 2.1 [9]. For a fixed M-parameter $\bar{\lambda}(\mu)$, the complementary series associated to it is defined as the set of $\pi(\bar{\lambda}(\mu), \nu)$, with $\nu>0$ such that $\pi(\bar{\lambda}(\mu), \nu)$ is unitary and irreducible. The boundary of this set, for $\nu>0$, is called the endpoints of the complementary series.

For convenience we will refer to *complementary series* as the *union of the two sets in Definition 2.1*.

The main result in this section is Theorem 2.3. In essence it reduces the problem of calculating complementary series of an arbitrary $\bar{\lambda}(\mu)$ to the corresponding question for the parameter of a spherical principal series. The main tools are the Dirac inequality and the reducibility results in Sect. 1. The technical statements of the reduction are Definition 2.2 and Theorem 2.3. It is enough to consider G and μ satisfying the following restrictions (the other cases are well known or can be reduced to this situation).

A. G is connected, simple, linear and has compact Cartan subgroups. Also $\mathfrak{g}\neq\mathfrak{sl}(2, \mathbf{R})$.

B. μ is not the lowest K-type of a discrete series or of a nondegenerate limit of discrete series.

We can exclude the case in which μ is the lowest K-type of a nondegenerate limit of discrete series because there is no complementary series attached to $\bar{\lambda}(\mu)$, in other words the nondegenerate limit of discrete series is the only unitary representation with that lowest K-type. This can be checked directly by using the Dirac inequality or by an argument as in [2].

Definition 2.2. Let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ be a θ-stable parabolic subalgebra. We say that μ is L-trivial if $\mathfrak{q}\supseteq\mathfrak{b}^0$ and the restriction of μ_L to $[\mathfrak{l}, \mathfrak{l}]\cap\mathfrak{k}$ is trivial.

Theorem 2.3. *Let μ be a K-type which is not L-trivial for any L that contains a factor isomorphic to $Sp(m, 1)$, $m\geqq 2$. Then there exists an (explicitly constructed) θ-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ with the following properties:*

(1) *For $\nu>0$, $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ and $\bar{\pi}(\bar{\lambda}(\mu_L), \nu)$ are either both unitary or both nonunitary. $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ and $\bar{\pi}(\bar{\lambda}(\mu_L), \nu)$ are unitary if and only if they are complementary series.*

(2) *On $[\mathfrak{l}, \mathfrak{l}]$, $(\bar{\lambda}(\mu_L), \nu)$ is either the parameter of a spherical principal series or else $[\mathfrak{l}, \mathfrak{l}]=\mathfrak{sp}(m, 1)$ with $m\geqq 2$ and $\mu_L=a\,\varepsilon_{m+1}$ with $a>0$.*

(3) *For $\nu>0$, if $\bar{\pi}=\bar{\pi}(\bar{\lambda}(\mu), \nu)$ and $\bar{\pi}_L=\bar{\pi}(\bar{\lambda}(\mu_L), \nu)$ are both unitary, then*

$$\mathscr{R}^i\bar{\pi}_L=\begin{cases}0 & \text{if } i<S\\ \bar{\pi} & \text{if } i=S.\end{cases}$$

Remarks. (a) The definition of the L-trivial K-type is made to avoid the case of the trivial representation which is isolated in $\mathrm{Sp}(n, 1)$ and $F_{4,1}$. The μ's that do not satisfy the hypotheses of the theorem will be classified in the course of the proof and a list of them appears at the start of Sect. 3. They will be handled separately in that section.

(b) Under the hypotheses on μ, the theorem reduces the question of whether $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ is unitary for the cases in (2) of the theorem. Then $L=G$ and the classification of the unitary spectrum is well known by [2] and [8, §7]. We will recall the necessary results as they are needed for the proof.

Proof. The proof will occupy the remainder of this section. Our objective will be to construct a θ-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ with the properties (2.1), (2.2), (2.3), (2.4) listed below. Before doing the actual construction, we show how the theorem follows once we have such a $\mathfrak{q}$. Then we establish a sufficient condition for (2.2) to be satisfied. Finally we give a case by case construction of $\mathfrak{q}$.

The subalgebra $\mathfrak{q}$ is to satisfy the following four properties:

(2.1) $\mathfrak{q} \supseteq \mathfrak{b}^0$.

(2.2) The Dirac inequality for $\bar{\pi}(\bar{\lambda}(\mu), \nu)$, stated in (2.5) below, gives an upper bound for unitary points $\nu > 0$ that is $\leqq$ the corresponding upper bound given by the Dirac inequality for $\bar{\pi}(\bar{\lambda}(\mu_L), \nu)$.

(2.3) $(\bar{\lambda}(\mu_L), \nu)$ is as in (2) of Theorem 2.3.

(2.4) If $(\bar{\lambda}(\mu_L), \nu)$ satisfies the Dirac inequality on $\mathfrak{l}$, then

$$\langle(\bar{\lambda}(\mu), \nu), \alpha\rangle \geqq 0 \quad \text{for } \alpha \in \Delta(\mathfrak{u}).$$

For the representations given by (2) of Theorem 2.3 it can be checked using [8] and [2] that the upper bound for unitary points ν coincides with the endpoint of the complementary series and essentially with the one given by the Dirac inequality (see the detailed calculations in the proof). Call it ν_L. Due to conditions (2.1), (2.4) we can apply Theorem 1.1 to deduce that $\pi(\bar{\lambda}(\mu), \nu)$ and $\pi(\bar{\lambda}(\mu_L), \nu)$ are both irreducible for $\nu < \nu_L$. By (2.2) if $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ is unitary then $\nu \leqq \nu_L$. Thus $\nu = \nu_L$ is the endpoint of the complementary series, so in particular $\pi(\bar{\lambda}(\mu), \nu)$ is also reducible. The theorem now follows from Corollary 1.2.

Thus Theorem 2.3 is proved once we construct $\mathfrak{q}$ satisfying (2.1)–(2.4).

We now obtain a sufficient condition on $\mathfrak{q}$ such that $\mathfrak{q}$ satisfies (2.2) whenever (2.3) holds. This condition, (2.6) below, is easier to check in practice.

Recall that by hypothesis, μ is not the lowest K-type of a discrete series or a nondegenerate limit of discrete series. Then $\bar{\lambda}(\mu) = \mu + 2\rho_c - \rho(\psi)$ if $\mu + 2\rho_c - \rho(\psi)$ is already dominant (but singular with respect to ψ) or $\bar{\lambda}(\mu) = \mu + 2\rho_c - \rho(\psi) + \frac{1}{2}\beta$, where β is a simple noncompact root.

It is well known (for example [2]) that any unitary representation $\bar{\pi}$, with lowest K-type μ, has to satisfy the Dirac inequality

$$\|\gamma + \rho_c\|^2 \geqq \|(\bar{\lambda}(\mu), \nu)\|^2$$

where γ is a K-type of $\bar{\pi} \otimes s$ (s being the spin representation).

Because of (2.3) and the discussion right after (2.4) we will only need it in the following form

(2.5) $\quad \|\sigma(\mu - \rho_n) + \rho_c\|^2 \geqq \|(\bar{\lambda}(\mu), \nu)\|^2$

where $\rho_n = \rho(\Phi) - \rho_c$ for some positive root system $\Phi \supset \Delta^+(\mathfrak{k})$ and $\sigma \in W_K$ is such that $\sigma(\mu - \rho_n)$ is dominant with respect to $\Delta^+(\mathfrak{k})$.

Consider all the systems Φ of the form $\Phi = \Phi(\mathfrak{l}) \cup \Delta(\mathfrak{u})$ where $\Phi(\mathfrak{l})$ is a positive system for $\mathfrak{l}$ compatible with $\Delta^+(\mathfrak{k})$, i.e. $\rho_c = \rho(\Phi(\mathfrak{l}) \cap \Delta(\mathfrak{k})) + \rho(\mathfrak{u} \cap \mathfrak{k})$. Set $\rho(\mathfrak{l}) = \rho(\Phi(\mathfrak{l}))$, $\rho(\mathfrak{l} \cap \mathfrak{k}) = \rho(\Phi(\mathfrak{l}) \cap \Delta(\mathfrak{k}))$ and $\rho_n(\mathfrak{l}) = \rho(\mathfrak{l}) - \rho(\mathfrak{l} \cap \mathfrak{k})$. Then $\rho_n = \rho_n(\Phi) = \rho_n(\mathfrak{l}) + \rho(\mathfrak{u} \cap \mathfrak{s})$.

(2.6) Our sufficient condition on $\mathfrak{q}$ for condition (2.2) to hold is that the σ used to make $\mu - \rho_n$ dominant is in $W_{L \cap K}$.

Then

(2.7) $\quad \mu - \rho_n = \mu_L - \rho_n(\mathfrak{l}) + \rho(\mathfrak{u} \cap \mathfrak{s})$.

Since $\sigma \in W_{L \cap K}$ then $\sigma(\rho(\mathfrak{u} \cap \mathfrak{s})) = \rho(\mathfrak{u} \cap \mathfrak{s})$ and so we have

(2.8)
$$\begin{aligned}\sigma(\mu - \rho_n) + \rho_c &= \sigma(\mu_L - \rho_n(\mathfrak{l})) + \rho(\mathfrak{u} \cap \mathfrak{s}) + \rho(\mathfrak{l} \cap \mathfrak{k}) + \rho(\mathfrak{u} \cap \mathfrak{k}) \\ &= \sigma(\mu_L - \rho_n(\mathfrak{l})) + \rho(\mathfrak{l} \cap \mathfrak{k}) + \rho(\mathfrak{u}).\end{aligned}$$

On the other hand

(2.9) $\quad \mu + 2\rho_c - \rho = \mu_L + 2\rho(\mathfrak{l} \cap \mathfrak{k}) - \rho(\mathfrak{l}) + \rho(\mathfrak{u})$

so $\bar{\lambda}(\mu) = \bar{\lambda}_L(\mu) + \rho(\mathfrak{u})$ because $\mathfrak{q} \supseteq \mathfrak{b}^0$.

Since $\rho(\mathfrak{u})$ is orthogonal to the roots of $\mathfrak{l}$ and $\beta \in \Delta(\mathfrak{l})$ we can write

(2.10) $\quad \langle\sigma(\mu_L - \rho_n(\mathfrak{l})) + \rho(\mathfrak{l} \cap \mathfrak{k}), \rho(\mathfrak{u})\rangle = \langle\mu_L, \rho(\mathfrak{u})\rangle$

and

(2.11) $\quad \langle(\bar{\lambda}(\mu), \nu), \rho(\mathfrak{u})\rangle = \langle\mu_L + \rho(\mathfrak{u}), \rho(\mathfrak{u})\rangle$.

Suppose that

(2.12) $\quad \|\sigma(\mu_L - \rho_n(\mathfrak{l})) + \rho(\mathfrak{l} \cap \mathfrak{k})\|^2 \geqq \|(\bar{\lambda}(\mu_L), \nu)\|^2$

then (2.8)–(2.12) give

$$\|\sigma(\mu - \rho_n) + \rho_c\|^2 \geqq \|(\bar{\lambda}(\mu), \nu)\|^2.$$

Thus $\mathfrak{q}$ satisfies condition (2.2) whenever (2.3) is satisfied. We now list the parabolic subalgebras $\mathfrak{q}$ for each μ.

We also remark that for calculating end points of complementary series it is enough to consider the simple noncompact factor of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$ and μ_L restricted to it.

Construction of $\mathfrak{q}$ *for* $\mathfrak{g} = \mathfrak{so}(2n, 1)$. We recall that a K-type is parametrized by

$$\mu = \sum_{k=1}^{n} \mu_k \varepsilon_k, \qquad \mu_1 \geqq \ldots \geqq \mu_{n-1} \geqq |\mu_n| \text{ and } \mu_k - \mu_j \in \mathbf{Z},\ 2\mu_k \in \mathbf{Z},\ 1 \leqq k,\ j \leqq n.$$

Because μ is not the lowest K-type of a discrete series or a limit of discrete series then $\mu_j \in \mathbf{Z}$ and $\mu_n = 0$. Set $\psi = B_1$ and let $1 \leqq t \leqq n$ be the smallest integer such that

$$\mu_t = \ldots = \mu_n = 0.$$

Then $\mu + 2\rho_c - \rho_1 = \sum_{k=1}^{n} (\mu_k + n - k - 1/2)\varepsilon_k$ and

$$(\bar{\lambda}(\mu), \nu) = \sum_{k=1}^{n-1} (\mu_k + n - k - 1/2)\varepsilon_k + \nu\varepsilon_n.$$

Define $\mathfrak{q}$ by means of the element $\gamma = \sum_{k=1}^{t-1} (n-k)\varepsilon_k$, then

$$[\mathfrak{l}_0, \mathfrak{l}_0] = \mathfrak{so}(2(n-t+1), 1)$$

and μ_L restricted to $[\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{k}$ is trivial.

The complementary series for $\pi(\bar{\lambda}(\mu_L), \nu)$ extends to the point $n-t+1/2$ ([8]). On the other hand condition (2.4) is

$$\mu_{t-1}+n-t+1/2 \geqq \nu \quad \text{if } 2 \leqq t \leqq n.$$

Because $\mu_{t-1}>0$ (if $t \geqq 2$) and the coefficient of ε_k in the expression of ρ_n^i is 1/2 if $1 \leqq k \leqq n-1$, and 1/2 or $-1/2$ if $k=n$, then $\mu-\rho_n^i$ can be made dominant by using only $\sigma \in W_{L\cap K}$. Thus (2.1)–(2.4) hold and the theorem follows.

Construction of $\mathfrak{q}$ *for* $\mathfrak{g}=\mathfrak{su}(n,1)$. In this case a K-type μ is parametrized by $\mu=\sum_{k=1}^{n+1}\mu_k\varepsilon_k$, where $\mu_k \in \mathbf{Z}$, $1 \leqq k \leqq n+1$, and $\mu_1 \geqq \ldots \geqq \mu_n$. Let $1 \leqq j \leqq n+1$ be the unique integer defined by:

$$\mu_{j-1}+n-2j+3>\mu_{n+1} \geqq \mu_j+n-2j+1.$$

Then $\mu+2\rho_c$ is dominant for A_{j-1} so we choose $\psi=A_{j-1}$.

$$\mu+2\rho_c-\rho_{j-1}=\sum_{k=1}^{j-1}\left(\mu_k+\frac{n}{2}-k\right)\varepsilon_k+\sum_{k=j}^{n}\left(\mu_k+\frac{n}{2}-k+1\right)\varepsilon_k + \left(\mu_{n+1}-\frac{n}{2}+j-1\right)\varepsilon_{n+1}.$$

There are four cases.

If $\mu+2\rho_c-\rho_{j-1}$ is regular and dominant for A_{j-1}, then $(\bar{\lambda}(\mu),\nu)=\bar{\lambda}(\mu)=\mu+2\rho_c-\rho_{j-1}$ and μ is the lowest K-type of a discrete series representation.

If $\mu+2\rho_c-\rho_{j-1}$ is dominant for A_{j-1}, i.e.

$$\mu_{j-1}+n-2j+2 \geqq \mu_{n+1} \geqq \mu_j+n-2j+2 \quad \text{and} \quad \mu_{j-1}>\mu_j,$$

then the only unitary irreducible representation with the lowest K-type μ is a limit of discrete series as can be seen by [16] or using the Dirac operator which forces the real Langlands parameter to be zero. Excluding these two cases we are left with

a$_1$) $\mu_{n+1}=\mu_j+n-2j+2$ and $\mu_{j-1}=\mu_j$, for some j, $2 \leqq j \leqq n$.

a$_2$) $\mu_{n+1}=\mu_j+n-2j+1$, for some j, $1 \leqq j \leqq n$.

We do the reduction in two steps. In the first step (2.6) holds and in the second step we have to check (2.3) directly. Therefore (2.3) will hold combining the two steps.

Step 1. Let $0 \leqq t_0 \leqq j-1$ and $0 \leqq t_1 \leqq n-j$ be the largest integers such that

$$\mu_{j-t_0}=\ldots=\mu_j=\ldots=\mu_{j+t_1}=y.$$

Let now

$$\gamma=\sum_{k=1}^{j-t_0-1}(n-k+1)\varepsilon_k+\sum_{k=j-t_0}^{j+t_1}(n-j+t_0+1)\varepsilon_k+\sum_{k=j+t_1+1}^{n}(n-k+1)\varepsilon_k + (n-j+t_0+1)\varepsilon_{n+1}.$$

Then the corresponding parabolic subalgebra $\mathfrak{q} \supseteq \mathfrak{b}^0$ and $[\mathfrak{l}_0,\mathfrak{l}_0]=\mathfrak{su}(t_0+t_1+1,1)$

$$\mu_L|_{\mathfrak{u}(t_0+t_1+1)\cap\mathfrak{t}}=\begin{cases}\sum_{k=j-t_0}^{j+t_1} y\varepsilon_k+(y+1+t_1-t_0)\varepsilon_{n+1} & \text{in case a}_1)\\ \sum_{k=j-t_0}^{j+t_1} y\varepsilon_k+(y+t_1-t_0)\varepsilon_{n+1} & \text{in case a}_2).\end{cases}$$

Assume now that a$_1$) holds. Then

$$(\bar{\lambda}(\mu),\nu)=\sum_{k=1}^{j-1}\left(\mu_k+\frac{n}{2}-k\right)\varepsilon_k+\sum_{k=j}^{n}\left(\mu_k+\frac{n}{2}-k+1\right)\varepsilon_k + \left(\mu_j+\frac{n}{2}-j+1\right)\varepsilon_{n+1}+\nu(\varepsilon_{n+1}-\varepsilon_j)$$

and the inequality in (2.4) is equivalent to

$$\nu \leqq \min(\mu_{j-t_0-1}-\mu_j+t_0,\ \mu_j-\mu_{j+t_1+1}+t_1+1),$$

where the minimum has to be taken only on the terms which are defined. The first term is defined only if $t_0 \leqq j-2$ and the second only if $t_1 \leqq n-j-1$. If none of the terms are defined there is no condition on ν. In the course of the proof we will often encounter an expression of this form, with one more term in the case of $\mathfrak{g}=\mathfrak{sp}(n,1)$. The meaning of such an expression will be always subjected to the above restrictions on t_0 and t_1.

Similarly if a$_2$) holds then

$$(\bar{\lambda}(\mu),\nu)=\sum_{k=1}^{j-1}\left(\mu_k+\frac{n}{2}-k\right)\varepsilon_k+\left(\mu_j+\frac{n+1}{2}-j\right)\varepsilon_j + \sum_{k=j+1}^{n}\left(\mu_k+\frac{n}{2}-k+1\right)\varepsilon_k+\left(\mu_j+\frac{n+1}{2}-j\right)\varepsilon_{n+1}+\nu(\varepsilon_{n+1}-\varepsilon_j)$$

and the inequality in (2.4) amounts to

$$\nu \leqq \min(\mu_{j-t_0-1}-\mu_j+t_0+1/2,\ \mu_j-\mu_{j+t_1+1}+t_1+1/2).$$

One checks easily that condition (2.6) is satisfied both in case a$_1$) and a$_2$). Thus condition (2.1) and (2.6) hold. On the other hand if μ satisfies a$_1$) (respectively a$_2$)), then $\mu_L|_{[\mathfrak{l}_0,\mathfrak{l}_0]\cap\mathfrak{t}}$ satisfies a$_1$) (respectively a$_2$)) for $\mathfrak{su}(t_1+t_0+1,1)$ with $j=t_0+1$. Therefore we are left to consider the following situation ($m=t_0+t_1+1$ in the applications).

Step 2. Let $\mathfrak{g}=\mathfrak{su}(m,1)$ and $\mu=\mu_{m+1}\varepsilon_{m+1}$ and either

a$'_1$) $\mu_{m+1}=m-2j+2 \quad 2 \leqq j \leqq m$ or

a$'_2$) $\mu_{m+1}=m-2j+1 \quad 1 \leqq j \leqq m$.

Let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ be the θ-stable parabolic determined by

$$\gamma=\sum_{k=1}^{j-t-1}(m-k+1)\varepsilon_k+\sum_{k=j-t}^{j+\tilde{t}}(m-j+t+1)\varepsilon_k$$
$$+\sum_{k=j+\tilde{t}+1}^{n}(m-k+1)\varepsilon_k+(m-j+t+1)\varepsilon_{m+1}$$

where $t=\tilde{t}=\min(j-1, m-j)$ if a$'_2$) holds and $t=\min(j-1, m-j+1)$, $\tilde{t}=t-1$ if a$'_1$) holds. As before we easily obtain that $[\mathfrak{l}_0, \mathfrak{l}_0]=\mathfrak{su}(t+\tilde{t}+1, 1)$ and μ_L is the trivial representation. For this case condition (2.6) is not satisfied but (2.2) can be checked directly by exhibiting a specific ρ_n giving an upper bound that is smaller than the corresponding upper bound on 1. Finally the inequality in (2.4) gives

$$v\leqq\begin{cases}t & \text{in case a}'_1)\\ t+1/2 & \text{in case a}'_2).\end{cases}$$

By [8, §7] the complementary series for L extends to the point t if a$'_1$) holds and $t+1/2$ if a$'_2$) holds.

Combining the two steps we get a parabolic subalgebra with $[\mathfrak{l}_0, \mathfrak{l}_0]=\mathfrak{su}(t+\tilde{t}+1, 1)$ satisfying (2.1)-(2.4). This completes the proof of the theorem in this case.

For the convenience of the reader we point out that if μ satisfies a$_1$) then μ_L satisfies a$'_1$) and then in particular the complementary series for $\bar{\lambda}(\mu)$ extends up to (inclusive)

$$t=\min(j-1, m-j+1)$$

with $m=t_0+t_1+1$ and $j=t_0+1$, in other words up to $t=\min(t_0, t_{1+1})$. Similarly if μ satisfies a$_2$) then μ_L satisfies a$'_2$) and the complementary series for $\bar{\lambda}(\mu)$ extends up to (inclusive)

$$t=\min(t_0, t_1)+1/2.$$

We now continue with the proof of the theorem.

Construction of $\mathfrak{q}$ *for* $\mathfrak{g}=\mathfrak{sp}(n, 1)$. A K-type μ is parametrized by $\mu=\sum_{k=1}^{n+1}\mu_k\varepsilon_k$, where μ_k are integers and $\mu_1\geqq\ldots\geqq\mu_n\geqq 0$, $\mu_{n+1}\geqq 0$. The fact that μ is not L trivial for any L with a factor isomorphic to $sp(m, 1)$ means that μ_{n-1}, μ_n and μ_{n+1} are not simultaneously zero. (Otherwise we could choose the parabolic determined by $\gamma=\varepsilon_{n-1}+\varepsilon_n+\varepsilon_{n+1}$ and get a contradiction).

Let $1\leqq j\leqq n+1$ be defined by

$$\mu_{j-1}+2n-2j+4>\mu_{n+1}+2\geqq\mu_j+2n-2j+2.$$

Then $\mu+2\rho_c$ is dominant for C_{j-1} and we choose $\psi=C_{j-1}$.

$$\mu+2\rho_c-\rho_{j-1}=\sum_{k=1}^{j-1}(\mu_k+n-k)\varepsilon_k+\sum_{k=j}^{n}(\mu_k+n-k+1)\varepsilon_k$$
$$+(\mu_{n+1}-n+j)\varepsilon_{n+1}.$$

Exactly as in the case of $\mathfrak{su}(n, 1)$, because of the hypothesis on μ we are left with the following possibilities:

c$_1$) $\mu_{n+1}=\mu_j+2n-2j$, for some j, $1\leqq j\leqq n$

c$_2$) $\mu_{n+1}=\mu_j+2n-2j+1$ and $\mu_{j-1}=\mu_j$, for some j, $2\leqq j\leqq n$.

Let t_0, t_1, y and γ be defined as in the case $\mathfrak{g}=\mathfrak{su}(n, 1)$ and first assume $y>0$. In this case we will show that Theorem (2.3) holds with $(\bar{\lambda}(\mu_L), v)$ a parameter for the spherical principal series. The reason for this assumption is that the σ needed to make the $\mu-\rho_n$'s dominant can be chosen in $W_{L\cap K}$, since the coefficients of ε_k in one such ρ_n are 0 or 1. Therefore in this case (2.6) is automatically satisfied. Let $\mathfrak{q}$ be the parabolic determined by γ (γ as in step 1 for $\mathfrak{su}(n, 1)$. Then

$$\mu_L|_{\mathfrak{u}(t_0+t_1+1, 1)\cap\mathfrak{t}}=\begin{cases}\displaystyle\sum_{k=j-t_0}^{j+t_1}(y-1)\varepsilon_k+(y-1+t_1-t_0)\varepsilon_{n+1} & \text{in case c}_1)\\ \displaystyle\sum_{k=j-t_0}^{j+t_1}(y-1)\varepsilon_k+(y+t_1-t_0)\varepsilon_{n+1} & \text{in case c}_2).\end{cases}$$

Assume now that c$_1$) holds. Then

$$(\bar{\lambda}(\mu), v)=\sum_{k=1}^{j-1}(\mu_k+n-k)\varepsilon_k+(\mu_j+n-j+1/2)\varepsilon_j$$
$$+\sum_{k=j+1}^{n}(\mu_k+n-k+1)\varepsilon_k+(\mu_j+n-j+1/2)\varepsilon_{n+1}+v(\varepsilon_{n+1}-\varepsilon_j)$$

and the inequality in (2.4) amounts to

$$v\leqq\min(\mu_j-\mu_{j+t_1+1}+t_1+1/2,\ \mu_{j-t_0-1}-\mu_j+t_0+1/2,\ \mu_j+n-j+1/2).$$

Similarly if c$_2$) holds then

$$(\bar{\lambda}(\mu), v)=\sum_{k=1}^{j-1}(\mu_k+n-k)\varepsilon_k+(\mu_j+n-j+1)\varepsilon_j+\sum_{k=j+1}^{n}(\mu_k+n-k+1)\varepsilon_k$$
$$+(\mu_j+n-j+1)\varepsilon_{n+1}+v(\varepsilon_{n+1}-\varepsilon_j)$$

and the inequality in (2.4) is

$$v\leqq\min(\mu_j-\mu_{j+t_1+1}+t_1+1,\ \mu_{j-t_0-1}-\mu_j+t_0,\ \mu_j+n-j+1).$$

Therefore in both cases (2.1) and (2.6) are satisfied.

We now observe that if μ satisfies c$_1$) (respectively c$_2$)) then μ_L is as in step 2 for $su(n, 1)$ (with $m=t_0+t_1+1$) and satisfies a$'_2$) (respectively a$'_1$)) with $j=t_0+1$. Therefore in both cases we can apply step 2 to get the theorem, under the assumption $y>0$. ($(\bar{\lambda}(\mu_L), v)$ is the parameter of a spherical principal series.) As a consequence if μ satisfies c$_1$) (respectively c$_2$)) then the complementary series for $\bar{\lambda}(\mu)$ extends to the point (inclusive) $\min(t_0, t_1)+1/2$ (respectively $\min(t_0, t_1+1)$).

We now assume that $y=0$. In this case we will shown that the theorem is true with $[\mathfrak{l}_0, \mathfrak{l}_0] = \mathfrak{sp}(m,1)$, $m \geqq 2$ and $\mu_L = a\varepsilon_{m+1}$. In the course of the argument we will point out the cases for which $a=0$.

Because $y=0$ then $t_1 = n-j$. Let $\gamma = \sum_{k=1}^{j-t_0-1} (n-k)\varepsilon_k$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the corresponding parabolic subalgebra. Then $[\mathfrak{l}_0, \mathfrak{l}_0] \simeq \mathfrak{sp}(n-j+t_0+1, 1)$ and

$$\mu_L|_{[\mathfrak{l}_0,\mathfrak{l}_0]\cap\mathfrak{t}} = \begin{cases} 2(n-j)\varepsilon_{n+1} & \text{if } c_1) \text{ holds} \\ 2(n-j+1/2)\varepsilon_{n+1} & \text{if } c_2) \text{ holds.} \end{cases}$$

If $j=n$ and $c_1)$ holds then by assumption $t_0=0$ (otherwise $\mu_{n-1} = \mu_n = \mu_{n+1} = 0$) and hence $[\mathfrak{l}_0, \mathfrak{l}_0] \simeq \mathfrak{so}(4,1)$ and μ_L is the trivial representation. In this case the complementary series for L extends to the point 3/2. Otherwise following [2] it is not too difficult to see that the complementary series for π_L extends to the point $t_0 + 1/2$ if $c_1)$ holds with $j \leqq n-1$, and to t_0 if $c_2)$ holds.

On the other hand the inequality in (2.4) is

$$\begin{aligned} &\nu \leqq \mu_{j-t_0-1} + t_0 + 1/2 && \text{if } t_0 \leqq j-2 \text{ and } c_1) \text{ holds} \\ &\nu \leqq \mu_{j-t_0-1} + t_0 && \text{if } t_0 \leqq j-2 \text{ and } c_2) \text{ holds} \end{aligned}$$

(no condition if $t_0 = j-1$).

With this choice of $\mathfrak{q}$ it is obvious that $\mu - \rho_n$ can be made dominant by an element of $W_{L\cap K}$, unless $\mu_{j-t_0-1} = 1$ and $t_0 \leqq j-2$. On the other hand if $\mu_{j-t_0-1} = 1$ then by [2] we may still prove that if $\pi(\bar{\lambda}(\mu), \nu)$ is unitary then $\nu \leqq \bar{t}$ where $t_0 + 1/2 \leqq \bar{t} < t_0 + 3/2$ if $c_1)$ holds and $t_0 \leqq \bar{t} < t_0 + 1$ if $c_2)$ holds. Therefore (2.2) is always satisfied. The theorem now follows also in this case.

Remark 2.4. In the situation just described, i.e. $y=0$, we could have actually considered two cases:

i) $\mu_{j-1} = 0$ and $t_0 \leqq n-j = t_1$, or $j=1$ or $\mu_{j-1} > 0$ (resp. $\mu_{j-2} = 0$ and $t_0 \leqq n-j+1 = t_1+1$ or $j=2$ or $\mu_{j-2} > 0$)
if $c_1)$ holds (resp. if $c_2)$ holds).

ii) $\mu_{j-1} = 0$ and $t_0 > n-j$ (resp. $t_0 > n-j+1$) if $c_2)$ holds).

The situation described in i) behaves exactly as the corresponding cases for $y \neq 0$. The only problem is that it is necessary to supply a direct proof of (2.2), since "σ" cannot be chosen in $W_{L\cap K}$. On the other hand for ii) we are forced to choose a parabolic for which the Levi factor contains a factor of type $\mathfrak{sp}(m,1)$. Therefore we prefer to treat the two cases together, in spite of the fact that they really behave differently.

Construction of $\mathfrak{q}$ *for* $\mathfrak{g} = \mathfrak{f}_{4,1}$. We recall that a K-type μ is parametrized by

$$\mu = \sum_{k=1}^{4} \mu_k \varepsilon_k = (\mu_1, \mu_2, \mu_3, \mu_4), \quad \text{where} \quad \mu_1 \geqq \ldots \geqq \mu_4 \geqq 0, \quad \mu_i - \mu_j \in \mathbf{Z}$$

and $2\mu_k \in \mathbf{Z}$. The assumption on μ implies that if $\mu_3 = \mu_4 = 0$ then $\mu_1 > \mu_2$. (Otherwise we could choose $\gamma = \varepsilon_1 + \varepsilon_2$ and get a contradiction). Then,

$$\mu + 2\rho_c = (\mu_1 + 7, \mu_2 + 5, \mu_3 + 3, \mu_4 + 1).$$

Assume first that $\mu + 2\rho_c$ is dominant for F_3. Because of the assumption on ν,

$$\mu_1 = \mu_2 + \mu_3 + \mu_4 - 2$$

and

$$(\bar{\lambda}(\mu), \nu) = (\mu_1 + 7/4 + \nu/2, \mu_2 + 9/4 - \nu/2, \mu_3 + 5/4 - \nu/2, \mu_4 + 1/4 - \nu/2).$$

Let $\gamma = \varepsilon_1 + \varepsilon_2$, then $[\mathfrak{l}_0, \mathfrak{l}_0] = \mathfrak{sp}(2,1)$ and the positive roots for

$$\Delta(\mathfrak{l} \cap \mathfrak{t}) \quad \text{are} \quad \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 \pm \varepsilon_4, \varepsilon_3, \varepsilon_4\}.$$

The inequality in (2.4) gives

$$\nu \leqq \mu_2 + \mu_4 + 5/2.$$

Let $\Delta^+(\mathfrak{l}) = F_3 \cap \Delta(\mathfrak{l})$. Then

$$\begin{aligned} \bar{\lambda}(\mu_L) &= \bar{\lambda}(\mathfrak{u}) + \rho(\mathfrak{w} = (\mu_1 - 9/4, \mu_2 - 7/4, \mu_3 + 5/4, \mu_4 + 1/4) \\ &= \tfrac{1}{2}(\mu_1 + \mu_2 - 4)(1,1,0,0) + \tfrac{1}{2}(\mu_1 - \mu_2 - 1/2)(1,-1,1,1) \\ &\quad + \tfrac{1}{2}(\mu_3 - \mu_4 + 1)(0,0,1,-1) \end{aligned}$$

and

$$\begin{aligned} \mu_L &= \tfrac{1}{2}(\mu_1 + \mu_2 - 4)(1,1,0,0) + \tfrac{1}{2}(\mu_1 - \mu_2)(1,-1,0,0) \\ &\quad + (\mu_3 + \mu_4)(0,0,0,1) + \mu_3(0,0,1,-1) \end{aligned}$$

where $(1,1,0,0)$ is orthogonal to the roots of $\mathfrak{l}$. In this case the Dirac inequality for μ_L is sharp (it gives 1/2 as upper bound) and is obtained by taking $\mu_L - \rho(\Delta^+(\mathfrak{l} \cap \mathfrak{s}))$ and $\sigma = \mathrm{id}$ in (2.5) (see [2], Proposition 4.6). On the other hand

$$\begin{aligned} &\|\mu_L - \rho(\Delta^+(\mathfrak{l} \cap \mathfrak{s})) + \rho(\mathfrak{l} \cap \mathfrak{t})\|^2 - \|(\bar{\lambda}(\mu_L), \nu)\|^2 \\ &= \|\mu - \rho_n^3 + \rho_c\|^2 - \|(\bar{\lambda}(\mu), \nu)\|^2 \end{aligned}$$

where $\mu - \rho_n^3$ is a K-type. Therefore $\mathfrak{q}$ satisfies (2.2). We can now apply to $\mathfrak{sp}(2,1)$ and μ_L the reduction argument from $\mathfrak{sp}(n,1)$ to find a parabolic satisfying (2.1)-(2.4) with $(\bar{\lambda}(\mu_L), \nu)$ a parameter for a spherical principal series. (In the usual parametrization for $\mathfrak{sp}(2,1)$,

$$\mu_L = (\mu_3 + \mu_4)\varepsilon_1 + (\mu_3 - \mu_4)\varepsilon_2 + (\mu_3 + \mu_4 + 2)\varepsilon_3$$

and μ_L satisfies $c_1)$ with $j=1$. If $\mu_3 + \mu_4 > 0$ then we are in the situation $y > 0$ and so $(\bar{\lambda}(\mu_L), \nu)$ is indeed the parameter for a spherical principal series. If $y=0$ then because $j=1$, Remark 2.4i) will apply to yield the same conclusion.)

Assume now that $\mu + 2\rho_c$ is dominant for F_1. Then because of the assumption on μ,

$$\mu_2 + \mu_3 = \mu_4 + \mu_1.$$

In this case

$$(\bar{\lambda}(\mu), \nu) = (\mu_1 + 9/4 - \nu/2, \mu_2 + 7/4 + \nu/2, \mu_3 + 3/4 + \nu/2, \mu_4 + 1/4 - \nu/2).$$

We have two possibilities

f_1) $\mu_1=\mu_2=\mu_3=\mu_4$ and $2\mu_1\geqq 1$

f_2) μ not as in f_1.

Suppose f_2) holds. Let $\gamma=\varepsilon_1+\varepsilon_2$ and $\mathfrak{q}$ be the corresponding parabolic. Then $[\mathfrak{l}_0,\mathfrak{l}_0]=\mathfrak{sp}(2,1)$. Let $\Delta^+(\mathfrak{l})=F_1\cap\Delta(\mathfrak{l})$. Then

$$\bar{\lambda}(\mu_L)=\tfrac{1}{2}(\mu_1+\mu_2-4)(1,1,0,0)+\tfrac{1}{2}(\mu_1-\mu_2+1/2)(1,-1,1,-1)$$
$$+\tfrac{1}{2}(\mu_3+\mu_4+1)(0,0,1,1)$$

and

$$\mu_L=\tfrac{1}{2}(\mu_1+\mu_2-4)(1,1,0,0)+\tfrac{1}{2}(\mu_1-\mu_2)(1,-1,0,0)$$
$$+(\mu_3+\mu_4)(0,0,0,1)+\mu_3(0,0,1,-1).$$

The Dirac inequality for μ_L is sharp: it gives the point 1/2 if $\mu_1>\mu_2$ (cf. [2], 2.5 and 4.6) and the point 3/2 if $\mu_1=\mu_2$ (cf. [2] 5.1).

Moreover the same values are obtained by applying the Dirac inequality (2.5) to the $L\cap K$-type $\mu_L-\rho(\Delta^+(\mathfrak{l})\cap\mathfrak{s})$ in the first case and $s_{\varepsilon_1-\varepsilon_2}(\mu_L-\rho(\Delta(\mathfrak{l})\cap F_3\cap\mathfrak{s}))$ in the second case.

Because $\rho(\Delta^+(\mathfrak{l})\cap\mathfrak{s})=(0,0,1,0)$, $\rho(\Delta(\mathfrak{l})\cap F_3\cap\mathfrak{s})=(1/2,-1/2,1/2,1/2)$ one easily obtains

$$\|\mu_L-\rho(\Delta^+(\mathfrak{l}\cap\mathfrak{s})+\rho(\mathfrak{l}\cap\mathfrak{k})\|^2-\|(\bar{\lambda}(\mu_L,\nu)\|^2=\|\mu-\rho_n^1+\rho_c\|^2-\|(\bar{\lambda}(\mu),\nu)\|^2$$

and

$$\|s_{\varepsilon_1-\varepsilon_2}(\mu_L-\rho(\Delta(\mathfrak{l})\cap F_3\cap\mathfrak{s}))+\rho(\mathfrak{l}\cap\mathfrak{k})\|^2-\|(\bar{\lambda}(\mu_L),\nu)\|^2$$
$$=\|s_{\varepsilon_1-\varepsilon_2}(\mu-\rho_n^2)+\rho_c\|^2-\|(\bar{\lambda}(\mu),\nu)\|^2$$

where $\mu-\rho_n^1$ and $s_{\varepsilon_1-\varepsilon_2}(\mu-\rho_n^2)$ are K-types. Therefore $\mathfrak{q}$ satisfies (2.2). The inequality in (2.4) gives

$$\nu\leqq\mu_1-\mu_3+3/2.$$

We now repeat the argument for the chamber F_3 to complete the proof in case f_2). To be precise in the usual parametrization for $\mathfrak{sp}(2,1)$,

$$\mu_L=(\mu_3+\mu_4)\varepsilon_1+(\mu_3-\mu_4)\varepsilon_2+(\mu_3-\mu_4)\varepsilon_3,$$

$j=2$ and μ_L satisfies c_1). If $\mu_3>\mu_4$ then we are in the situation $y>0$, otherwise $y=0$ and the parabolic to which we apply a further reduction is of the form $\mathfrak{so}(4,1)$ (compare the beginning of the discussion after the assumption $y=0$ in the case of $\mathfrak{sp}(n,1)$. In both cases $(\bar{\lambda}(\mu_L),\nu)$ is in the final reduction the parameter of a spherical principal series.

Suppose now that f_1) holds. Let $\gamma=\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4$ then $[\mathfrak{l}_0,\mathfrak{l}_0]=\mathfrak{so}(6,1)$. Let

$$\Delta^+(\mathfrak{l})=F_1\cap\Delta(\mathfrak{l})\quad\text{and}\quad\Delta^+(\mathfrak{l}\cap\mathfrak{k})=F_1\cap\Delta(\mathfrak{l}\cap\mathfrak{k}).$$

$\mu_{L|_{\mathfrak{so}(6,1)\cap\mathfrak{k}}}$ is the trivial representation for $L\cap K$. Therefore the complementary series for π_L extends to the point 5/2 [8] and the inequality in (2.4) is equivalent to

$$\nu\leqq 2\mu_1+1/2.$$

If $\mu_1\geqq 1$ then $s_{\varepsilon_1-\varepsilon_2}(\mu-\rho_n^1)$ gives the same estimate on $\mathfrak{g}$. If $\mu_1=1/2$, then the lower bound for the Dirac inequality (2.5) on G is $\bar{t}$, $7/2<\bar{t}<4$. This lower bound can be further improved by applying the Dirac operator to the K-type (3/2, 1/2, 1/2, 1/2) which is in the Langlands quotient at $\nu=7/2$. (The fact that (3/2, 1/2, 1/2, 1/2) is in the Langlands quotient can be checked by a lengthy but straight forward calculation of composition factors and multiplicity of K-types. We omit the details).

We now show that in fact the conclusion of Corollary 1.2 extends to $\nu=5/2$, even though $\mathrm{Re}\langle(\bar{\lambda}(\mu),\nu),\alpha)\rangle\geqq 0$ is not satisfied.

We first note that by [6] the composition series for the spherical principal series of $\mathfrak{so}(6,1)$ at $\nu=5/2$ gives an exact sequence

$$0\to W_1\to W\to W_2\to 0$$

where W_2 is a one dimensional representation and W_1 is the Langlands quotient with parameter $(\bar{\lambda}(1/2,1/2,-1/2,-1/2),3/2)$. Then, by Corollary 1.2 for W_1 and Theorem 8.2.15 for W, we get

$$\mathscr{R}^iW=\begin{cases}0 & i<S\\ \pi(\bar{\lambda}(1/2,1/2,1/2,1/2),5/2) & i=S\end{cases}$$

$$\mathscr{R}^SW_1=\begin{cases}0 & i<S\\ \bar{\pi}(\bar{\lambda}(2,2,1,1),\tfrac{3}{2}) & i=S.\end{cases}$$

Then the exact sequence 4 in Sect. 1 collapses to

$$0\to\mathscr{R}^{S-1}W_2\to\mathscr{R}^SW_1\to\mathscr{R}^SW\to\mathscr{R}^SW_2\to 0$$

(in particular, $\mathscr{R}^iW_2=0$ for $i<S$).

We first show that $\mathscr{R}^{S-1}W_2=0$. For this it is enough to see that Eq. 2 in Sect. 1 has no solution such that $w\in W_k^1$ has length 1. The only such w is $s_{\varepsilon_3+\varepsilon_4}$. But then if $\delta=(x_1,x_2,x_3,x_4)$ with $x_1\geqq x_2\geqq x_3\geqq x_4\geqq 0$, we get the following system of equations

$$\frac{x_1-x_2}{2}=n_4-n_5$$

$$\frac{x_1+x_4}{2}+1=n_3-n_5$$

$$\frac{x_1+x_3}{2}+1=n_2-n_5$$

$$\frac{x_2-x_4-x_3-x_1}{2}=5+n_1+2n_5$$

with n_1, n_2, n_3, n_4, n_5 non negative integers. This is a contradiction so

$$\mathscr{R}^{S-1}W_2=0.$$

It remains to show that $\mathscr{R}^SW_2=\bar{\pi}(\bar{\lambda}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),\tfrac{5}{2})$.

This can be seen either by a direct calculation of K-types or else by a computation of the length of the composition series of $\pi(\bar{\lambda}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \frac{5}{2})$. We omit the details. The proof for this case is now complete.

Assume now that $\mu+2\rho_c$ is dominant for F_2. Then unless $\mu_1=\mu_2+\mu_3+1$ and $\mu_4=0$ we either have that ν is not strictly positive or else μ is also dominant for F_1 or F_3, cases we have already considered. So assume $\mu_1=\mu_2+\mu_3+1$, $\mu_4=0$. Then

$$(\bar{\lambda}(\mu), \nu)=(\mu_1+2+\nu/2, \mu_2+2-\nu/2, \mu_3+1-\nu/2, \nu/2).$$

Suppose first that $\mu_2\geqq 1$. Then let $\gamma=\varepsilon_1+\varepsilon_2$ and $\mathfrak{q}$ be the corresponding parabolic. Then the Dirac inequality for μ_L is sharp and gives the value 1. By direct computation one can prove that the Dirac inequality (2.5) on $\mathfrak{g}$ has as lower bound $1\leqq E<2$ if $\mu_2\geqq 1$ and $3<E<4$ if $\mu_2=0$.

On the other hand because the inequality in (2.4) is equivalent to $\mu_2+2-\nu\geqq 0$ the result follows if $\mu_2\geqq 1$. If $\mu_2=0$ one argues as follows. The representation in question is reducible at the point $\nu=2$ and at the point $\nu=3$ is reducible.

Then the Dirac inequality applied to the K-type (1, 1, 1, 0) which is in $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ for $1<\nu\leqq 3$, says that such representation is not unitary. (Again all these facts can be checked by a straight forward calculation of composition factors and multiplicity of K-types. We omit the details).

We now repeat the argument as for the previous chamber to prove Theorem 2.3 with $(\bar{\lambda}(\mu_L), \nu)$ a parameter for the spherical principal series. Theorem 2.3 is now completely proved.

3. Cases with Isolated Representations

In this section we investigate the representations not covered by Theorem 2.3. Excluding the case of the spherical principal series which is done in [8], we are to treat

I. $\mathfrak{g}=\mathfrak{sp}(n,1)$, $\mu=\sum_{j=1}^{k}\mu_j\varepsilon_j$, $\mu_k>0$, $0<k<n-1$

II. $\mathfrak{g}=\mathfrak{f}_{4,1}$ $\quad \mu=\lambda(\varepsilon_1+\varepsilon_2)$, $\lambda>0$.

Proposition 3.1. *For μ as in case I or II, there is an (explicitly constructed) θ-stable parabolic subalgebra $\mathfrak{q}$ such that conclusion (3) of Theorem 2.3 holds. Moreover:*

(1′) For $\nu>0$, $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ and $\bar{\pi}(\bar{\lambda}(\mu_L), \nu)$ are either both unitary or both non-unitary.

(2′) On $[\mathfrak{l}, \mathfrak{l}]$, $(\bar{\lambda}(\mu), \nu)$ is the parameter of a spherical principal series.

Proof. Case 1. Let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ be the parabolic subalgebra determined by $\gamma=\sum_{i=1}^{k}(n-i+1)\varepsilon_i$. Then the inequality in (2.4) is satisfied for $\nu\leqq\mu_k+n-k-\frac{1}{2}$. On the other hand the Dirac inequality implies that $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ cannot be unitary unless

$$\nu\leqq n-k+\tfrac{1}{2} \quad\text{and}\quad n-k+\tfrac{1}{2}\leqq\mu_k+n-k-\tfrac{1}{2}$$

since $\mu_k>0$ is an integer. Thus by Corollary 1.2, if $\bar{\pi}$ is unitary then $\bar{\pi}=\mathscr{R}^S(\bar{\pi}(\bar{\lambda}(\pi_L), \nu))$ and $\bar{\pi}_L$ is a quotient of a spherical principal series.

For $k<n-1$, $[\mathfrak{l}_0, \mathfrak{l}_0]\simeq\mathfrak{sp}(n-k, 1)$. In this case the complementary series extends to $\nu=n-k-\frac{1}{2}$. The trivial representation is the Langlands quotient at $\nu=n-k+\frac{1}{2}$ and is isolated. The same happens with $\bar{\pi}(\bar{\lambda}(\mu), \nu)$.

We now show that $\bar{\pi}(\bar{\lambda}(\mu), n-k+\frac{1}{2})$ is unitary. This was first done in [2]. It depended on knowing the multiplicities of K-types of $\bar{\pi}$ and the intertwining operator explicitly. We give a somewhat simpler proof based on the multiplicity formula in 1.4. (The idea of the proof is due to D. Vogan.)

Let δ be a K-type of $\bar{\pi}(\lambda(\mu), n-k+\frac{1}{2})$. Then

$$\delta=2\rho(\mathfrak{u}\cap\mathfrak{s})+\sum n_i\beta_i \qquad \beta_i\in\Delta(\mathfrak{u}\cap\mathfrak{s}),\ n_i\in\mathbf{N}. \tag{1}$$

On the other hand, any K-type of $\pi(\bar{\lambda}(\mu), \nu)$ is of the form

$$\delta=\gamma+2\rho(\mathfrak{u}\cap\mathfrak{s})+\sum m_i\beta_i \qquad \beta_i\in\Delta(\mathfrak{u}\cap\mathfrak{s}),\ m_i\in\mathbf{N} \tag{2}$$

and γ a K-type of the spherical principal series of $\mathrm{sp}(n-k, 1)$. But according to [6], γ must be of the form $\gamma=\frac{p+q}{2}\varepsilon_{k+1}+\frac{p-q}{2}\varepsilon_{k+2}+q\varepsilon_{n+1}$ where $p-q\in 2\mathbf{N}$, $q\in\mathbf{N}$ and the roots in $\Delta(\mathfrak{u}\cap\mathfrak{s})$ are of the form $\beta=\varepsilon_S\pm\varepsilon_{n+1}$ with $s\leqq k$. Then if δ is both of the form (1) and (2) then γ must be a combination of roots in $\Delta(\mathfrak{u}\cap\mathfrak{s})$ which implies $p-q=0$ and $p+q=0$, in other words $\gamma=0$.

Thus, $m(\delta, \bar{\pi}(\bar{\lambda}(\mu), n-k+\frac{1}{2}))=0$ or it is the same as $m(\delta, \pi(\bar{\lambda}(\mu), \nu))$. In addition $\bar{\pi}(\bar{\lambda}(\mu), n-k-\frac{1}{2})$ as a K-representation contains $\bar{\pi}(\bar{\lambda}(\mu), n-k+\frac{1}{2})$ since this is true on L.

The proof that $\bar{\pi}(\bar{\lambda}(\mu), n-k+\frac{1}{2})$ is unitary is now completed as in [2].

Case 2. Let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ be the parabolic subalgebra determined by $\gamma=\varepsilon_1+\varepsilon_2$. Then $[\mathfrak{l}_0, \mathfrak{l}_0]\simeq\mathfrak{sp}(2,1)$ and the inequality in (2.4) is satisfied up to $\nu\leqq\lambda+\frac{3}{2}$. The Dirac inequality implies that $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ is not unitary for $\nu>\frac{5}{2}$. Thus, the complementary series for $\bar{\pi}(\bar{\lambda}(\mu), \nu)$ extends to $\nu=\frac{3}{2}$. Remains to see that $\bar{\pi}(\bar{\lambda}(\mu), \frac{5}{2})$ is unitary. The argument in Case I cannot be used. Instead, we use formula 1.4 for the multiplicity of a K-type in $\mathscr{R}^S\bar{\pi}_L$ to see that the K-types of $\bar{\pi}(\bar{\lambda}(\mu), \frac{5}{2})$ have a particularly simple form.

We get, for δ a K-type,

$$m(\delta, \bar{\pi})=\sum_{w\in W_K}(-1)^{l(w)}P(w(\delta+\rho_c)-\rho_c-\lambda\varepsilon_1-\lambda\varepsilon_2).$$

A combination of roots in $\Delta(\mathfrak{u}\cap\mathfrak{s})$ with non-negative integer coefficients is of the form $\alpha=\alpha_1\varepsilon_1+\alpha_1\varepsilon_2+\alpha_3\varepsilon_3+\alpha_4\varepsilon_4$ where

$$\alpha_1-\alpha_3=n_1+n_4$$
$$\alpha_1+\alpha_2=n_2+n_1$$
$$\alpha_3-\alpha_2=n_3-n_1.$$

Then $\delta=\mu_1\varepsilon_1+\mu_1\varepsilon_2+\mu_2\varepsilon_3+\mu_3\varepsilon_4$, $\mu_1\geqq\mu_2\geqq\mu_3\geqq 0$ and the μ_i are either all integers or half integers.

Since W_K is the group of permutations and sign changes in $\varepsilon_1,\dots,\varepsilon_4$, $P(w(\delta+\rho_c)-\rho_c-\lambda\varepsilon_1-\lambda\varepsilon_2)\neq 0$ only if w leaves the first two coordinates fixed. Then $m(\delta,\bar{\pi})$ is the sum with the listed sign changes of the number of solutions (n_1, n_2, n_3, n_4) of the following equation

$(+)$ $(\mu_1-\lambda,\mu_1-\lambda,\mu_2,\mu_3)=\alpha$	$\alpha_1-\mu_2+1$ solutions if $\alpha_1\geqq\mu_2$, 0 otherwise
$(-)$ $(\mu_1-\lambda,\mu_1-\lambda,\mu_2,-\mu_3-1)=\alpha$	$\alpha_1-\mu_2+1$ if $\mu_2>\mu_3$, $\alpha_1\geqq\mu_2$; $\alpha_1-\mu_2$ if $\mu_2=\mu_3$, $\alpha_1\geqq\mu_2+1$ 0 otherwise
$(-)$ $(\mu_1-\lambda,\mu_1-\lambda,-\mu_2-3,\mu_3)=\alpha$	$\alpha_1-\mu_2-2$ if $\alpha_1\geqq\mu_2+3$; 0 otherwise
$(+)$ $(\mu_1-\lambda,\mu_1-\lambda,-\mu_2-3,-\mu_3-1)=\alpha$	$\alpha_1-\mu_2-2$ if $\alpha_1\geqq\mu_2+3$; 0 otherwise
$(-)$ $(\mu_1-\lambda,\mu_1-\lambda,\mu_3-1,\mu_2+1)=\alpha$	$\alpha_1-\mu_2$ if $\alpha_1\geqq\mu_2+1$; 0 otherwise
$(+)$ $(\mu_1-\lambda,\mu_1-\lambda,\mu_3-1,-\mu_2-2)=\alpha$	$\alpha_1-\mu_2-1$ if $\alpha_1\geqq\mu_2+2$; 0 otherwise
$(+)$ $(\mu_1-\lambda,\mu_1-\lambda,-\mu_3-2,\mu_2+1)=\alpha$	$\min(\alpha_1-\mu_2-1,\alpha_1-\mu_3-2)+1$ if $\alpha_1\geqq\mu_2+1$, $\alpha_1\geqq\mu_3+2$; 0 otherwise
$(-)$ $(\mu_1-\lambda,\mu_1-\lambda,-\mu_3-2,-\mu_2-2)=\alpha$	$\alpha_2-\mu_2-1$ if $\alpha_1\geqq\mu_2+2$; 0 otherwise.

We can now easily verify that $m(\delta,\bar{\pi})=0$ unless $\mu_1-\lambda=\mu_2=\mu_3$ in which case the multiplicity is 1.

Then the argument in [18] applies and we can conclude that $\bar{\pi}(\bar{\lambda}(\mu),\frac{5}{2})$ is unitary. The proof of the proposition is now complete.

Remark. It is possible to obtain the result on the multiplicity of the K-types in the same way as in [18] be computing the τ-invariant and then using the equations for the extremal K-types as in [18]. The calculations are about as lengthy as in our approach. Since this requires more notation just for this particular calculation, we have preferred to do a direct multiplicity calculation.

For the convenience of the reader we now summarize Theorem 2.3 and Proposition 3.1 in the following.

Theorem 3.2. *Let μ be a K-type and assume that μ is not the lowest K-type of a discrete series representation or a nondegenerate limit of discrete series. Then there is an (explicitly constructed) θ-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ depending on μ such that:*

(1) *For $\nu>0$, $\bar{\pi}(\bar{\lambda}(\mu),\nu)$ is unitary if and only if $\bar{\pi}(\bar{\lambda}(\mu_L),\nu)$ is unitary.*

(2) *On $[\mathfrak{l},\mathfrak{l}]$, $(\bar{\lambda}(\mu_L),\nu)$ is the parameter of the spherical principal series or else $[\mathfrak{l},\mathfrak{l}]=\mathfrak{sp}(m,1)$ and $\mu_L=a\varepsilon_{n+1}$ with $a>0$.*

(3) *For $\nu>0$, if $\bar{\pi}(\bar{\lambda}(\mu),\nu)$ and $\bar{\pi}(\bar{\lambda}(\mu_L),\nu)$ are both unitary, then*

$$\mathscr{R}^i(\bar{\pi}(\bar{\lambda}(\mu_L),\nu))=\begin{cases}0 & \text{if } i<S\\ \bar{\pi}(\bar{\lambda}(\mu),\nu) & \text{if } i=S.\end{cases}$$

We now give a list of the unitary spectrum for each group. We have omitted the case of lowest K-types of discrete series or of a nondegenerate limit of discrete series (see 2, §2).

Table 1. Unitary parameters ν

	Complementary series	Isolated representations
$\mathfrak{g}=\mathfrak{so}(n,1)$		
$\mu=\sum_{i=1}^{n}\mu_i\varepsilon_i$ such that $\mu_n=0$ $t=\min\{1\leqq j\leqq n;\mu_j=0\}$	$\nu\leqq n-t+\frac{1}{2}$	
$\mathfrak{g}=\mathfrak{su}(n,1)$		
$\mu=\sum_{i=1}^{n+1}\mu_i\varepsilon_i$ and $j\in\{1,\dots,n\}$ such that $\mu+2\rho_c$ dominant for A_{j-1} but not for A_{j-2} a$_1$) $j\geqq 2$, $\mu_{n+1}=\mu_j+n-2j+2$ and $\mu_{j-1}=\mu_j$ a$_2$) $j\geqq 1$ and $\mu_{n+1}=\mu_j+n-2j+1$ $t_0=\max\{0\leqq i\leqq j-1;\mu_{j-i}=\mu_j\}$ $t_1=\max\{0\leqq i\leqq n-j;\mu_j=\mu_{j+i}\}$	$\nu\leqq\begin{cases}\min(t_0,t_1+1) \text{ if a}_1\text{) holds}\\ \min(t_0,t_1)+\frac{1}{2} \text{ if a}_2\text{) holds}\end{cases}$	
$\mathfrak{g}=\mathfrak{sp}(n,1)$		
$\mu=\sum_{i=1}^{n+1}\mu_i\varepsilon_i$ such that $\mu+2\rho_c$ dominant for C_{j-1} but not for C_{j-2} c$_1$) $j\geqq 1$ and $\mu_{n+1}=\mu_j+2n-2j$ c$_2$) $j\geqq 2$, $\mu_{n+1}=\mu_j+2n-2j+1$ and $\mu_{j-1}=\mu_j$ $t_0=\max\{0\leqq i\leqq j-1;\mu_{j-i}=\mu_j\}$ $t_1=\max\{0\leqq i\leqq n-j;\mu_j=\mu_{j+i}\}$	$\nu\leqq\begin{cases}\min(t_0,t_1)+\frac{1}{2} \text{ if c}_1\text{) holds}\\ \text{and } \mu_j>0\ t_0+\frac{1}{2} \text{ if c}_1\text{) holds and}\\ \mu_j=0, j\leqq n-1\ \frac{3}{2} \text{ if c}_1\text{) holds,}\\ j=n,\ \mu_{n-1}>\mu_n=0\ n-k-\frac{1}{2}\\ \text{if c}_1\text{) holds, } j=n,\\ \mu_k>\mu_{k+1}=\dots=\mu_{n-1}=\mu_n=0,\\ 0\leqq k\leqq n-2\end{cases}$	$\nu=n-k+\frac{1}{2}$ if c$_1$) holds, $j=n$, $\mu_k>\mu_{k+1}=\dots=\mu_{n-1}=\mu_n=0$, $0\leqq k\leqq n-2$
$\mathfrak{g}=\mathfrak{f}_{4,1}$		
$\mu=\sum_{k=1}^{4}\mu_k\varepsilon_k$ such that f$_1$) $\mu_1=\mu_2+\mu_3+\mu_4+2$ f$_2$) $\mu_1=\mu_2+\mu_3+1$, $\mu_4=0$ f$_3$) $\mu_2+\mu_3=\mu_4+\mu_1$ and $\mu_1>\mu_2$ or $\mu_1=\mu_2>\mu_3$ f$_4$) $\mu_1=\mu_2=\mu_3=\mu_4$	$\nu\leqq\frac{1}{2}$ $\nu\leqq 1$ $\nu\leqq\frac{1}{2}$ if $\mu_1<\mu_2$ $\nu\leqq\frac{3}{2}$ if $\mu_1=\mu_2>\mu_3$ $\nu\leqq\frac{5}{2}$	$\nu=\frac{5}{2}$ if f$_3$) holds and $\mu_1=\mu_2\geqq 0$, $\mu_3=\mu_4=0$ $\nu=\frac{11}{2}$ if $\mu_1=0$

As a consequence of the results proved up to now and of the following lemma we are able to show in Corollary 3.5 that Zuckerman's conjecture is true for real rank one groups.

Lemma 3.4. *Let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ be a θ-stable parabolic subalgebra and π_L an irreducible, finite dimensional representation of L with lowest $(L\cap K)$ type μ_L such that $(\mu_L,\alpha)=0$ for $\alpha\in\Delta(\mathfrak{l})$ and $(\mu_L,\alpha)\geqq 0$ $\alpha\in\Delta(\mathfrak{u})$. Assume that the Langlands parameter $(\bar\lambda(\mu_L),\nu)$ is such that*

$$\langle(\bar\lambda(\mu_L),\nu)+\rho(\mathfrak{u}),\alpha\rangle>0 \quad \textit{for } \alpha\in\Delta(\mathfrak{u}).$$

Then $\mu=\mu_L+2\rho(\mathfrak{u}\cap\mathfrak{s})$ is a K-type and

$$\mathscr{R}^i\pi_L=\begin{cases}0 & i<S\\ \bar\pi(\bar\lambda(\mu),\nu) & i=S.\end{cases}$$

Proof. This is Corollary 1.2 once we prove that μ is a K-type, $\bar\lambda(\mu)=\bar\lambda(\mu_L)+\rho(\mathfrak{u})$ and $\mathfrak{q}\supseteq\mathfrak{b}^0$.

The fact that μ is a K-type, $\bar\lambda(\mu)=\bar\lambda(\mu_L)+\rho(\mathfrak{u})$ and $\mathfrak{q}\supseteq\mathfrak{b}^0$ are proved in [17] page 395 under the condition that $(\bar\lambda(\mu_L)+\rho(\mathfrak{u}),\alpha)>0$ for $\alpha\in\Delta(\mathfrak{u})$. To check this we argue as follows. Consider the Cayley transform that maps the simple noncompact root β into the real root. This maps $(\bar\lambda(\mu_L),\nu)$ onto a parameter on the maximally split Cartan subgroup $\mathfrak{g}=\mathfrak{t}^-+\mathfrak{a}$. By abuse of notation we call it $(\bar\lambda(\mu_L),\nu)$ again. The Cayley transform also leaves $\Delta(\mathfrak{u})$ invariant since $\beta\in\Delta(\mathfrak{l})$. Then

$$((\bar\lambda(\mu_L),\nu)+\rho(\mathfrak{u}),\alpha)>0$$

and

$$((\bar\lambda(\mu_L),\nu)+\rho(\mathfrak{u}),\theta\alpha)>0 \quad \text{for } \alpha\in\Delta(\mathfrak{u}).$$

But $\theta((\bar\lambda(\mu),\nu))=(\bar\lambda(\mu),-\nu)$ and $\theta\rho(\mathfrak{u})=\rho(\mathfrak{u})$ so

$$((\bar\lambda(\mu_L),0)+\rho(\mathfrak{u}),\alpha)=\tfrac{1}{2}((\bar\lambda(\mu_L),\nu)+\rho(\mathfrak{u})+\theta((\bar\lambda(\mu_L),\nu)+\rho(\mathfrak{u})),\alpha)>0$$

and the proof is complete.

Corollary 3.5. *Let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ be a θ-stable parabolic subalgebra. Let $\psi(\mathfrak{l})$ be a positive chamber for $\mathfrak{l}$ such that $2\rho(\mathfrak{l}\cap\mathfrak{k})$ is dominant with respect to it and $\psi=\psi(\mathfrak{l})\cup\Delta(\mathfrak{u})$. Let λ: $\mathfrak{q}\to\mathbf{C}$ be a character such that*

$$\langle\lambda+\rho(\psi),\alpha\rangle>0 \quad \textit{for } \alpha\in\psi.$$

Let π^L be the representation of L corresponding to λ. Then

$$\mathscr{R}^i\pi^L=\begin{cases}0 & i\neq S\\ \pi & i=S\end{cases}$$

and π is an irreducible unitary representation.

Conversely, for any irreducible unitary representation π with non-singular integral infinitesimal character, there is $\mathfrak{q}$ and λ as above such that $\pi=\mathscr{R}^S\pi^L$.

Proof. Assume first that $\mathfrak{l}_0$ does not contain any factor of type $\mathfrak{sp}(m,1)$. If $\mathfrak{l}_0$ also does not contain any compact factor, then the result follows from the proof of Theorem 2.3. If $\mathfrak{l}_0$ does contain compact factors, one has to use induction by stages ([17] 6.3.6). We omit the details.

If $\mathfrak{l}_0$ contains a factor of type $\mathfrak{sp}(m,1)$, the result is contained in the proof of Theorem 3.1.

4. The L^2 Index and Unitarity

In this section we determine the representations that contribute to the index of the Dirac operator. We briefly review what is involved. Let $\Gamma\subseteq G$ be a discrete torsion free subgroup such that vol $(\Gamma\backslash G)<\infty$. Let η be a K-type with representation space V_η. Let s^+ and s^- be the two spin representations. We fix a positive chamber Δ^+ ($\tilde B_1$, in case $\mathfrak{so}(2n,1)$, $\tilde A_n$ in case $\mathfrak{su}(n,1)$, $\tilde C_n$ in case $\mathfrak{sp}(n,1)$, F_3F_4, in case $\mathfrak{f}_{4,1}$) such that $2\rho_c$ is dominant and normalize s^+ and s^- such that on T', the set of regular elements in T, the characters of $s^\pm$ satisfy

$$(\operatorname{ch}s^+-\operatorname{ch}s^-)|_{T'}=\prod_{\alpha\in\Delta^+-\Delta(\mathfrak{l})}(e^{\alpha/2}-e^{-\alpha/2}).$$

Then there are two bundles $\varepsilon^\pm$ over $X=\Gamma\backslash G/K$ with fibers $E^\pm=V_\eta\otimes s^\pm$ and an elliptic operator D: $\varepsilon^+\to\varepsilon^-$ called the Dirac operator. Then (cf. [12])

$$\operatorname{index}D=\sum_{\pi\in L^2_d(\Gamma\backslash G)} m_\Gamma(\pi)\{\dim\operatorname{Hom}_K(\pi,E^+)-\dim\operatorname{Hom}_K(\pi,E^-)\}$$

where $L^2_d(\Gamma\backslash G)$ is the discrete spectrum of $L^2(\Gamma\backslash G)$. In this section we calculate

$$m(\pi,\eta)=\dim\operatorname{Hom}_K(\pi,E^+)-\dim\operatorname{Hom}_K(\pi,E^-)$$

for all η and all π unitary. We need the following facts from [1].

1. $m(\pi,\eta)\neq 0$ only if the infinitesimal character of π, denoted by χ_π, is $\eta+\rho_c$.

2. Let $\operatorname{ch}\pi$ be the distribution character of π and for η a K-type, we denote by $\operatorname{ch}\eta$ its character. Then if $\chi_\pi=\chi_{\eta+\rho_c}$

$$\operatorname{ch}\pi|_{T'}\cdot\Delta=\sum_{w\in W_c}a_w e^{w(\eta+\rho_c)}$$

where $\Delta=\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})$.

Then

$$\operatorname{ch}\pi(\operatorname{ch}s^+-\operatorname{ch}s^-)|_{T'}=\sum a_{\eta_i}\operatorname{ch}\eta_i|_{T'}$$

and

$$m(\pi,\eta)=(-1)^p a_\eta,\qquad p=\tfrac{1}{2}(\dim\mathfrak{g}-\dim\mathfrak{k}).$$

We will consider two cases. Let $\psi(\eta+\rho_c)$ be a positive system such that $\eta+\rho_c$ is dominant.

Case I. $\lambda=\eta+\rho_c-\rho(\psi)$ is dominant with respect to ψ.

Case 2. $\lambda = \eta + \rho_c - \rho(\psi)$ is not dominant with respect to ψ.

We treat Case 1 first. It can be reduced to the computations in [19]. Let F_λ be the finite dimensional representation of $\mathfrak{g}$ with highest weight λ. Let F_λ^* be its dual and $H^*(\mathfrak{g}, K, \pi \otimes F_\lambda^*)$ be Lie algebra cohomology as defined in [4].

Lemma 4.1. *If $m(\pi, \eta) \neq 0$ and π is unitary irreducible then $H^*(\mathfrak{g}, K, \pi \otimes F_\lambda^*) \neq 0$.*

Proof. Since π is unitary, a standard argument is [4] shows that $H^*(\mathfrak{g}, K, \pi \otimes F_\lambda^*) \neq 0$ is equivalent to

a) $\chi_\pi = \chi_{F_\lambda}$

b) $\mathrm{Hom}_K(\pi \otimes F_\lambda^*, \Lambda^* \mathfrak{s}) \neq 0$.

We check that these two properties are satisfied.

a) is clear from property 1 listed earlier. For b) we write

$$\Lambda^* \mathfrak{s} = (s^+ \oplus s^-) \otimes (s^+ \oplus s^-) = s \otimes s.$$

Then

$$\mathrm{Hom}_K(\pi \otimes F_\lambda^*, \Lambda^* \mathfrak{s}) \cong \mathrm{Hom}_K(\pi, F_\lambda \otimes s \otimes s) = \mathrm{Hom}_K(\pi \otimes s^*, F_\lambda \otimes s).$$

Then $\mathrm{Hom}_K(\pi \otimes s^*, F_\lambda \otimes s) \neq 0$ by the property that $\eta \in \pi \otimes s^*|_K$.

We now recall the following proposition. For a character λ: $\mathfrak{l} \to C$ we denote by π_λ^L the corresponding representation.

Proposition 4.2 ([19] Sects. 5–7). Let π be unitary and F an irreducible finite dimensional representation of $\mathfrak{g}$. Suppose $H^*(\mathfrak{g}, K, \pi \otimes F) \neq 0$. Then there is a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and a one dimensional representation π_λ^L of L such that $\pi \cong \mathscr{R}^S \pi_\lambda^L$. We fix Δ_L^+ a positive system for which $2\rho(\mathfrak{l} \cap \mathfrak{k})$ is dominant. Then if $\psi = \Delta_L^+ \cup \Delta(\mathfrak{u})$, λ must be the highest weight for F with respect to ψ.

We now compute $m(\pi, \eta)$ for such a representation. For any irreducible representation π^L of L we denote by $e(\pi^L)$ the virtual character

$$e(\pi^L) = \sum_i (-1)^i c(\mathscr{R}^i \pi^L)$$

and refer to it as the Euler characteristic. It is a well defined map from the ring of virtual characters of L to the ring of virtual characters of G. Furthermore it commutes with coherent continuation.

Suppose

$$\mathrm{ch}\, \pi_\lambda^L = \sum C_j \,\mathrm{ch}\, \pi^L(\bar{\lambda}(\mu_j), \nu_j) = \sum C_j \,\mathrm{ch}\, \pi^L(\gamma_j)$$

with C_j integers and $\pi^L(\gamma_j)$, discrete series or principal series. Then

$$e(\pi_\lambda^L) = \sum C_j (\pi^L(\gamma_j)).$$

We note that $\gamma_j + \rho(\mathfrak{u}) = w(\lambda + \rho)$ with $w \in W(\mathfrak{l})$. Since $\langle \lambda + \rho, \alpha \rangle > 0$ for $\alpha \in \Delta(\mathfrak{u})$, $\langle \gamma_j + \rho(\mathfrak{u}), \alpha \rangle > 0$. It follows that

$$e(\pi_\lambda^L) = (-1)^S \,\mathrm{ch}\, \mathscr{R}^S \pi_\lambda^L$$

and by using 8.2.15 in [17],

$$e(\pi^L(\gamma_j)) = (-1)^S \,\mathrm{ch}\, \mathscr{R}^S \pi^L(\gamma_j) = (-1)^S \,\mathrm{ch}\, \pi(\gamma_j + \rho(\mathfrak{u})).$$

Thus

$$\mathrm{ch}\, \pi = \sum C_j \,\mathrm{ch}\, \pi(\gamma_j + \rho(\mathfrak{u})).$$

Since we only need $\mathrm{ch}\, \pi|_{T'}$ we are only interested in the C_j's for which $\pi^L(\gamma_j)$ is a discrete series.

By comparing the expressions of such characters we can write for $\rho = \rho(\psi)$,

$$\mathrm{ch}\, \pi^L_{\lambda|T'} = (-1)^{P_L} \sum_{\psi_L \supseteq \Delta^+(\mathfrak{l} \cap \mathfrak{k})} \mathrm{ch}\, \pi(\psi_L, \lambda + \rho(\mathfrak{l}))|_{T'}.$$

Then

$$\mathrm{ch}\, \pi|_{T'} = (-1)^{P_L} \sum_{\substack{\psi \supseteq \Delta(\mathfrak{u}) \\ \psi \supseteq \Delta^+(\mathfrak{k})}} \mathrm{ch}\, \pi(\psi, \lambda + \rho)|_{T'},$$

so writing out the expression for $\mathrm{ch}\, \pi(\psi, \lambda + \rho)$ we get

$$m(\pi, \eta) = (-1)^{P_L} \varepsilon(\psi)$$

where ψ is the chamber for which $\eta + \rho_c$ is dominant and $\varepsilon(\psi) = (-1)^{|\psi \cap \Delta^+|}$. Furthermore $\eta + \rho_c = w(\lambda + \rho_c)$ for a unique $w \in W(\mathfrak{l})$.

Case 2. In this case, $\eta + \rho_c$ must be singular. Since $\eta + \rho_c$ is regular with respect to $\Delta(\mathfrak{k})$, there cannot be two adjacent noncompact roots in the Dynkin diagram for the simple roots of ψ such that $\eta + \rho_c$ is singular with respect to both of them. Thus in real rank 1, $\eta + \rho_c$ is singular with respect to exactly one simple noncompact root in ψ say β. Then χ_π coincides with the infinitesimal character of a limit of discrete series.

We also note that if $w(\eta + \rho_c)$ is dominant with respect to $\Delta^+(\mathfrak{k})$ then $w(\eta + \rho_c)$ must be singular with respect to a compact root unless $w(\eta + \rho_c) = \eta + \rho_c$ and $w = \mathrm{id}$ or the simple reflection about β. Thus, if $\bar{\lambda}(\mu, \nu) = \gamma$ is the parameter of π, $\eta + \rho_c$ is the unique element in the W-orbit of γ which is dominant, regular with respect to $\Delta^+(\mathfrak{k})$ and

$$\mathrm{ch}\, \pi|_{T'} = (-1)^P m(\eta + \rho_c, \pi) \Delta^{-1} \Big(\sum_{w \in W_K} \varepsilon(w)\, e^{w(\eta + \rho_c)} \Big).$$

To compute $m(\eta + \rho_c, \pi)$ we argue as follows. If $\pi = \mathscr{R}^S \pi_\lambda^L$ for $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ then λ must satisfy $(\lambda, \alpha) = 0$ for $\alpha \in \Delta(\mathfrak{l})$, $(\lambda + \rho, \alpha) \geqq 0$ for $\alpha \in \Delta(\mathfrak{u})$ and equality for at least one $\alpha \in \Delta(\mathfrak{u})$. In this case $m(\pi, \eta)$ is computed in the same way as in Case 1. We note that there is a unique $\tilde{w} \in W(\mathfrak{l})$ such that $\tilde{w}(\lambda + \rho) = \eta + \rho_c$. The existence of $\tilde{w}$ is clear by the calculation in Case 1. If there were $w_1, w_2 \in W(\mathfrak{l})$ then $w_1(\lambda + \rho) = w_2(\lambda + \rho)$ implies $\lambda + \rho(\mathfrak{u}) + w_1 \rho(\mathfrak{l}) = \lambda + \rho(\mathfrak{u}) + w_2 \rho(\mathfrak{l})$ so $w_1 = w_2$. Choose a chamber $\psi(\lambda + \rho)$ for which $\lambda + \rho$ is dominant. Then let $\psi = \tilde{w} \psi(\lambda + \rho)$. We have

$$m(\pi, \eta) = (-1)^{P_L} \varepsilon(\psi).$$

However, there are cases when $\pi \neq \mathscr{R}^S \pi_\lambda^L$. These are the cases $\mathfrak{g}=\mathfrak{sp}(n,1)$ and the representations with lowest K-type

$$\mu=\sum_{k=1}^{j-1}\mu_k\varepsilon_k+\mu_{n+1}\varepsilon_{n+1} \text{ where either } c_1:\ \mu_{n+1}=2n-2j \text{ and}$$

$$(\bar\lambda(\mu),\nu)=\sum_{k=1}^{j-1}(\mu_k+n-k)\varepsilon_K+(n-j+\tfrac{1}{2}-\nu)\varepsilon_j$$

$$+\sum_{k=j+1}^{n}(n-k+1)\varepsilon_k+(n-j+\tfrac{1}{2}+\nu)\varepsilon_{n+1},\ \nu=t_0+\tfrac{1}{2},\ 1\leqq j\leqq n-1$$

or c_2: $\mu_{n+1}=2n-2j+1$ and

$$(\bar\lambda(\mu),\nu)=\sum_{k=1}^{j-1}(\mu_k+n-k)\varepsilon_k+(n-j+1\ \nu)_j$$

$$+\sum_{k=j+1}^{n}(n-k+1)\varepsilon_k+(n-j+1+\nu)_{n+1}$$

$\nu=t_0$, $\mu_{j-1}=0$ and $2\leqq j\leqq n$.

In this case we have to calculate the character of π using the results in [3]. We sketch how to do the calculation for case c_1. In [3] the Langlands parameter is given by a $\gamma=\sum_{i=1}^{n+1}a_i e_i$ (cf. page 448 in [3], e_i is a certain basis of the split Cartan subalgebra) where $a_1>a_2$, $a_1\geqq -a_2$, $a_3>a_4>\ldots>a_{n+1}>0$ $a_3,\ldots,a_n\in\mathbf{Z}$. The relation to $(\bar\lambda(\mu),\nu)$ is given by $a_1=\nu+(n-j+\frac{1}{2})$, $a_2=\nu-(n-j+\frac{1}{2})$ $a_K=\mu_{k-2}+n-k+2$ for $3\leqq k\leqq j+1$, $a_K=n-k+2$ for $j+1<k\leqq n+1$. We set $\nu=t_0+\frac{1}{2}$, the first reducibility point. Write $\pi(\gamma)$ and $\bar\pi(\gamma)$ for the principal series determined by γ and the corresponding Langlands quotient.

Lemma 4.3. *Suppose that, for* $\gamma=\sum_{i=1}^{n+1}a_i e_i$, *there are* $\tilde i$, $\tilde j$ *such that*

$$a_2=a_{\tilde i},\quad a_{\tilde j+1}>a_1>a_{\tilde j+2},\quad 3\leqq\tilde i,\quad \tilde j\leqq n+1.$$

Then

$$\operatorname{ch}\bar\pi(\gamma)|_{T'}=(-1)^{\tilde i+\tilde j+1}\operatorname{ch}\pi(C_{\tilde i-1},\tilde\gamma)|_{T'}$$

($C_{\tilde i-1}$ *the chamber defined in Sect. 1 and* $\bar\pi(C_{\tilde i-1},\tilde\gamma)$ *the corresponding limit of discrete series) and*

$$\tilde\gamma=\sum_{k=1}^{\tilde j-1}a_{k+2}\varepsilon_k+a_1\varepsilon_{\tilde j}+\sum_{k=\tilde j+1}^{n}a_{k+1}\varepsilon_k+a_{\tilde i}\varepsilon_{n+1}.$$

Proof. The proof is just a straightforward application of Theorem 4.2 in [3] and we refer to it for more details. What is involved is the following. $\gamma_0=\gamma$ is of type V according to the quoted result. This just means that γ_0 is dominant with respect to a certain chamber with singularity determined by the coefficient

of e_2. In any case due to the assumption we have a character identity of the form

$$\pi(\gamma_0)=\bar\pi(\gamma_0)+\bar\pi(\gamma_1)\quad\text{where } \gamma_1=s_{e_1-e_{j+2}}(\gamma_0).$$

Now we can write a relation of the same sort for the character of $\bar\pi(\gamma_1)$. Proceeding in this way we will be able to define parameters $\{\gamma_i\}_{i=0}^{k-1}$ such that

$$\pi(\gamma_i)=\bar\pi(\gamma_i)+\bar\pi(\gamma_{i+1})$$

and γ_k has a singularity determined (most of the time) by the coefficient of e_1. Because γ_k is of a particular sort,

$$\pi(\gamma_k)=\bar\pi(\gamma_k)+(\text{limit of discrete series}).$$

Solving all the equations for $\bar\pi(\gamma_0)$ and restricting the character to T' will give the required identity. For the reader willing to go through the details we define the parameters involved.

For $0\leqq k\leqq 2n-\tilde i-\tilde j+1$, let γ_k be defined by

$$\gamma_0=\gamma,$$
$$\gamma_k=s_{e_1-e_{\tilde j+k+1}}\gamma_{k-1},\ 1\leqq k\leqq\tilde i-\tilde j-2$$
$$\gamma_{\tilde i-\tilde j-1}=s_{e_1-e_{\tilde i}}s_{e_2-e_{\tilde i+1}}\gamma_{\tilde i-\tilde j-2}$$
$$\gamma_k=s_{e_2-e_{k+\tilde j+2}}\gamma_{k-1},\ \tilde i-\tilde j\leqq k\leqq n-\tilde j-1$$
$$\gamma_{n-\tilde j}=s_{2e_2}\gamma_{n-\tilde j-1},\ k=n-\tilde j\leqq 2n-\tilde i-\tilde j$$
$$\gamma_k=s_{e_2+e_{2n-k-\tilde j+2}}\gamma_{k-1},\ n-\tilde j+1\leqq k\leqq 2n-\tilde i-\tilde j$$
$$\tilde\gamma=\gamma_{2n-\tilde i-\tilde j+1},\ k=2n-\tilde i-\tilde j+1$$

then the required identity is

$$\bar\pi(\gamma)=\sum_{k=0}^{2n-\tilde i-\tilde j}(-1)^k\pi(\gamma_k)+(-1)^{\tilde i+\tilde j}\pi^+(C_{\tilde i-2},\tilde\gamma).$$

Lemma 4.4. *Suppose that for* $\gamma=\sum_{i=1}^{n+1}a_i e_i$ *there are* $\tilde i$, $\tilde j$ *such that* $-a_2=a_{\tilde i}$, $a_{\tilde j+1}>a_1>a_{\tilde j+2}$, $3\leqq\tilde i$, $\tilde j\leqq n+1$.

Then

$$\operatorname{ch}\bar\pi(\gamma)|_{T'}=(-1)^{\tilde i+\tilde j+1}\operatorname{ch}\bar\pi(C_{\tilde i-2},\tilde\gamma)|_{T'}$$

where

$$\tilde\gamma=\sum_{k=1}^{\tilde j-1}a_{k+2}\varepsilon_k+a_1\varepsilon_{\tilde j}+\sum_{k=\tilde j+1}^{n}a_{k+1}\varepsilon_k+a_{\tilde i}\varepsilon_{n+1}.$$

Proof. This is similar to Lemma 4.3. We omit the proof.

Similarly we get

Lemma 4.5. *Suppose* $\gamma=\sum_{k=1}^{n+1}a_k\varepsilon_k$ *satisfies* $a_2=0$ *and there is* $3\leqq\tilde j\leqq n+1$ *then*

$$\operatorname{ch}\bar\pi(\gamma)|_{T'}=0.$$

We now summarize these results in

Proposition 4.6. *As character identity we have: in case c_1*

$$\bar\pi(\bar\lambda(\mu), t_0+\tfrac{1}{2})|_{\Gamma'} = \begin{cases} -\bar\pi(C_{2n-2j-t_0+1}, \bar\gamma)|_{\Gamma'}, m(\bar\pi,\eta)=(-1)^{t_0} & t_0>n-j \\ -\bar\pi(C_{j+t_0}, \bar\gamma)|_{\Gamma'}, m(\bar\pi,\eta)=(-1)^{j+t_0+1} & t_0<n-j \\ 0 & t_0=n-j. \end{cases}$$

In case c_2,

$$\bar\pi(\bar\lambda(\mu), t_1)|_{\Gamma'} = \begin{cases} \bar\pi(C_{2n-j-t_0+2}, \bar\gamma)|_{\Gamma'}, m(\bar\pi,\eta)=(-1)^{t_0} & t_0>n-j+1 \\ \bar\pi(C_{j+t_0-1}, \bar\gamma)|_{\Gamma'}, m(\bar\pi,\eta)=(-1)^{j+t_0+1} & t_0<n-j+1 \\ 0 & t_0=n-j+1. \end{cases}$$

$\bar\gamma$ is obtained from $(\bar\lambda(\mu), \nu)$ by conjugating by W so that it becomes dominant with respect to the corresponding chamber and $\eta=\bar\gamma-\rho_c$.

Proof. Let's consider $(\bar\lambda(\mu), t_0+1/2)$.

If $t_0>n-j$ then $(\bar\lambda(\mu), t_0+1/2)$ satisfies the hypothesis of Lemma 4.3 with $\tilde i=2n-j-t_0+2$ and $\tilde j=j-t_0$. Therefore

$$\operatorname{ch}\bar\pi(\bar\lambda(\mu), t_0+1/2)|_{\Gamma'} = -(-1)^{2n-j-t_0+2+j-t_0}\operatorname{ch}\bar\pi(C_{2n-j-t_0+1}, \bar\gamma)|_{\Gamma'}.$$

If $t_0<n-j$ then we are in the situation described by Lemma 4.4 with $\tilde i=j+t_0+2$, $\tilde j=j-t_0$. Therefore

$$\bar\pi(\bar\lambda(\mu), t_0+1/2)|_{\Gamma'} = -\bar\pi(C_{j+t_0}, \bar\gamma)|_{\Gamma'}.$$

Similarly for $(\bar\lambda(\mu), t_0)$.

We summarize the results in the following theorems.

Theorem 4.7. *Suppose $\bar\pi=\bar\pi(\bar\lambda(\mu), \nu)$ is unitary such that $\gamma=(\bar\lambda(\mu), \nu)$ is regular. Then the η's such that $m(\bar\pi,\eta)\neq 0$ are given by the following procedure.*

Choose $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ as in Theorem 2.1 so that $\bar\pi=\mathscr{R}^S\pi_\lambda^L$ with $\lambda=\gamma-\rho(\mathfrak{u})$. Then $\eta+\rho_c=w\cdot\gamma$ with $w\in W(\mathfrak{l})$ and let ψ be the chamber for which $\eta+\rho_c$ is dominant. Then $m(\pi,\eta)=(-1)^{P_L}\varepsilon(\psi)$.

Conversely given η with $\eta+\rho_c$ regular, the unitary representations for which $m(\pi,\eta)\neq 0$ are given by the following.

Let $\lambda=\eta+\rho_c-\rho(\psi)$ and take $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ a θ-stable parabolic subalgebra such that $(\lambda,\alpha)=0$ for $\alpha\in\Delta(\mathfrak{u})$ and $(\lambda,\alpha)\geqq 0$ for $\alpha\in\Delta(\mathfrak{l})$. Then $\pi=\mathscr{R}^S\pi_\lambda^L$ (by 2.10 all such representations are unitary) and $m(\pi,\eta)=(-1)^{P_L}\varepsilon(\psi)$.

Theorem 4.8. *Suppose $\bar\pi=\bar\pi(\bar\lambda(\mu), \nu)$ is unitary and $\gamma=(\bar\lambda(\mu), \nu)$ is singular. If $m(\bar\pi,\eta)\neq 0$ then there is a unique root such that γ is singular with respect to it, and a unique K-type η such that $w\gamma=\eta+\rho_c$. $m(\bar\pi,\eta)$ is obtained as follows.*

A. $\mathfrak{g}=\mathfrak{sp}(n,1)$. The answer is given by Proposition 4.6.

B. In the other cases $\bar\pi=\mathscr{R}^S\pi_\lambda^L$ for $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ as in Theorem 3. Let $\psi(\gamma)$ be a chamber for which γ is dominant. Then $m(\bar\pi,\eta)=(-1)^{P_L}\varepsilon(\psi)$

where ψ is the unique chamber such that $\eta+\rho_c$ is dominant for ψ and there is $w\in W(\mathfrak{l})$ such that $\psi=w\psi(\gamma)$.

Conversely, given η, the unitary representations such that $m(\pi,\eta)\neq 0$ are given by

A. $\mathfrak{g}=\mathfrak{sp}(n,1)$, the cases in Proposition 4.6.

B. $\pi=\mathscr{R}^S\pi_\lambda^L$ such that $(\lambda+\rho,\alpha)\geqq 0$ for $\alpha\in\Delta(\mathfrak{u})$, $(\lambda,\alpha)=0$ for $\alpha\in\Delta(\mathfrak{l})$ and such that $\lambda+\rho$ is singular with respect to exactly one root.

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