Invent. math. 96, 103–176 (1989)

Inventiones mathematicae

The unitary dual for complex classical Lie groups

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1. Introduction

In this paper we classify the unitary spectrum of the complex classical groups.

Let G be any complex simply connected semisimple Lie group. There are two well known ways of constructing unitary representations from lower groups; unitary induction from a parabolic subgroup and complementary series. A reasonable way to describe the unitary dual is to find a distinguished set of representations for each group such that every other unitary representation is obtained from this set by the above constructions from strictly smaller parabolic subgroups. This is entirely similar to the classification of admissible modules where the distinguished representations are the discrete series.

For integral infinitesimal character, such a set is provided by the special unipotent representations introduced in [BV1]. Their existence and unitarity was conjectured earlier by [A]. They were shown to have many interesting properties such as the fact that their annihilator is a maximal primitive ideal. From this it was possible to obtain character formulas for them.

However it is clear that this set is not large enough to describe the entire unitary spectrum. For example in type C the metaplectic representation is not special unipotent or even one of the larger class of representations considered in [A].

The definition of special unipotent representations suggests that one should consider representations with maximal primitive ideal and infinitesimal character as small as possible. It is not clear which of them should be unitary. Imposing the condition that the annihilator be a completely prime ideal, which is necessary for unitarity, does not seem to be very helpful either. Thus one is left with the task of explicitly computing the $\mathfrak{g}$-invariant form and ruling out the cases when it is indefinite. This is an almost impossible calculation. A rough estimate says that one needs to compute signatures on the K-types in the principal series which have length $\leq \|2\rho\|$.

[1] Supported by NSF grant DMS8603172

For the classical groups the following elementary ideas simplify the calculations significantly.

Denote by $\mathfrak{g}(n)$ a classical algebra of type B, C, D of rank n and by $\mathscr{A}(m)$ the algebra $gl(m, \mathbb{C})$. Then we can view $\mathfrak{g}(n)\times\mathscr{A}(m)\subset\mathfrak{g}(n+m)$ as a Levi component. If $\bar{X}$ is a $(\mathfrak{g}(n), K(n))$ module and Y is a $(\mathscr{A}(m), U(m))$ module, then we can form

$$\bar{X} * Y = \mathrm{Ind}_{\mathfrak{g}(n)\times\mathscr{A}(m)}^{\mathfrak{g}(n+m)} \bar{X}\otimes Y]. \tag{1.1}$$

The main idea for obtaining necessary conditions for unitarity is the following.

If $\bar{X}$ and Y are unitary then so are all the irreducible composition factors of $\bar{X} * Y$.

This idea is made precise in the relation $\underset{u}{<}$ of Definition 7.1.

In order to be able to exploit this it is essential to have control over the composition factors of $\bar{X} * Y$. Character formulas are derived for the case when $\bar{X}$ and Y have maximal annihilator in the enveloping algebra in a similar manner as in [BV1].

This allows us to reduce the question of computing the signature on isolated representations (which are difficult), to the similar question for a simpler representation but on a higher rank group.

In order to treat the case when the annihilator is not maximal we combine this technique with the "bottom layer K-type" idea in [SV] or [V2]. Since it is not true (as in type A in [V3]) that the negativity of the form can be detected on the bottom layer K-types alone, we have to work much harder to get enough necessary conditions for unitarity. Here the character formulas combined with $\underset{u}{<}$ are crucial in simplifying the arguments.

This leaves a set of representations with half-integral infinitesimal character which we call unipotent. To show that they are unitary we introduce in Sect. 10 an equivalence relation $\underset{u}{\sim}$ among representations on different rank groups. This reduces the question of computing the signature of an isolated representation to the question of computing the signature on a finite set of K-types and the signatures of simpler representations on a higher rank group. In particular the finite set consists of K-types which are very close to the ones with fundamental extremal weight so it is much smaller than the bound $\|2\rho\|$.

This completes the classification of the unitary spectrum.

We summarize briefly the contents of each section.

Section 2 contains notation and some basic known results about $(\mathfrak{g}, K)$ modules and invariant hermitian forms. In particular we include the results we need about bottom layer K-types.

Section 3 introduces small K-types. These are the finite set mentioned earlier. They are used later to detect nonunitarity, to produce factors which are related via $\underset{u}{<}$ to the original $\bar{X}$ or to prove unitarity as in Sect. 10.

Sections 4 and 5 compute the character formulas for representations with maximal annihilator. The techniques are as in [BV1] so we omit most details.

Section 6 attaches a Levi component $\mathfrak{m}(\lambda)$ and parameters $\bar{X}(\lambda_0, \lambda_0)$, $\mathscr{F}^i$ to each integral λ and spherical $\bar{X}$.

Section 7 introduces the relation $\underset{u}{<}$ and shows (Propositions 7.3, 7.4) that any spherical integral $\bar{X}$ which is not quasiunipotent (Definition 6.5) or satisfies (1) of 7.2 can be bounded via $\underset{u}{<}$ by a $\bar{Z}$ of a simple form.

Section 8 shows that for spherical representations, a necessary condition for unitarity is that $\bar{X}$ be weakly preunitary (Definitions 7.5, 8.6 and Theorem 8.4). If not, the form is negative on a small K-type.

Section 9 extends these results to the case when the small K-types on $\bar{X}$ do not carry negative signature. The necessary condition for unitarity is called preunipotent and preunitary (Definitions 9.3, 9.4 and Theorem 9.5).

Section 10 determines the unitary dual at half-integral infinitesimal character (Definitions 9.6, 10.3 and Propositions 10.5, 10.6 and 10.7).

Section 11 gives a parametrization of the unipotent representations in terms of nilpotent orbits in the dual Lie algebra. The definitions apply to all Lie algebras, not just the classical ones. The obvious conjectures should hold.

Section 12 determines the unitary dual for arbitrary infinitesimal character (12.6, 12.7). The most noteworthy representations are the "edges of complementary series" (Sect. 12.5). These are subquotients at endpoints of complementary series which do not arise as lowest K-type factors of complementary series (or endpoints) from unitary representations on proper Levi components.

The treatment of half-integral infinitesimal character is the same as for the general case. I have separated them for the sake of the exposition.

Section 13 gives some low rank examples.

Section 14 gives an irreducibility result for unitary representations with half-integral infinitesimal character. The proof is independent of Sect. 6–13.

Previously, for complex groups, low rank examples were computed in [D2]. The case of regular integral infinitesimal character was done in [E]. For type A the unitary spectrum was derived in [V4].

The main bulk of the paper is devoted to finding necessary conditions for unitarity that are also sufficient. Throughout the paper I have systematically tried to use $\underset{u}{<}$ and information about composition factors instead of continuous deformations of parameters and K-type calculations. I believe this to be simpler for the case of classical groups.

In contrast, the proof that the unipotent representations are unitary is quite simple and elegant, only taking up the last part of section 10.

The technique of inducing and analyzing the composition factors was also used independently, in the case of $GL(n)$ by B. Speh in her thesis and by M. Tadic in the case of $GL(n)$ over the p-adics.

Throughout the paper, when inducing from a parabolic subgroup P only the Lie algebra of the Levi component is indicated. The classical Lie algebras are to be thought of as infinite series

$$\ldots\mathfrak{g}(n)\subset\mathfrak{g}(n+1)\ldots \tag{1.2}$$

ignoring the low rank isomorphisms.

The exposition and some of the proofs in this paper have benefitted from many valuable suggestions from S. Salamanca, M. Duflo, C. Moeglin and N. Wallach. I would like to thank them. The catchy title is due to J. Adams.

I would like to especially thank D. Vogan for patiently reading every line and pointing out its inadequacy. The fact that the paper is at all understandable is due to him; the remaining inaccuracies and mistakes are of course mine.

A preliminary version of this paper was written in 1987. I have substantially revised it while at MSRI in Berkeley. I would like to thank the institute for their hospitality.

2. General results and notation

2.1. Let G be a connected simply connected complex Lie group. We denote its Lie algebra viewed as a real algebra by $\mathfrak{g}_{\mathfrak{R}}$. Denote by K a fixed maximal compact subgroup and by B^+ a Borel subgroup. Let $\mathfrak{k}_{\mathfrak{R}}$ and $\mathfrak{b}^+_{\mathfrak{R}}$ be their respective Lie algebras. Then $T = K \cap B^+$ is a maximal torus in K. If $\mathfrak{t}_{\mathfrak{R}}$ is its Lie algebra, then $\mathfrak{a}_{\mathfrak{R}} = \sqrt{-1}\,\mathfrak{t}_{\mathfrak{R}}$ is a maximally split component. Let $A = \exp \mathfrak{a}_{\mathfrak{R}}$ and $H = T \cdot A$ be a Cartan subgroup. Its Lie algebra is denoted by $\mathfrak{h}_{\mathfrak{R}}$. Fix an automorphism σ that preserves $\mathfrak{k}_{\mathfrak{R}}$ and such that $\sigma|_{\mathfrak{h}_{\mathfrak{R}}} = -\mathrm{Id}$. Define

$$\check{X} = -X, \quad {}^tX = -\sigma X, \quad \text{for } X \in \mathfrak{g}_{\mathfrak{R}}. \tag{2.1.1}$$

Then $\check{}$ and ^{t} extend uniquely to antiautomorphisms of $\mathscr{U}(\mathfrak{g}_{\mathfrak{R}})$ by the formulas

$$\begin{aligned} (X_1 \ldots X_n)\check{} &= (-1)^n X_n \ldots X_1, \\ {}^t(X_1 \ldots X_n) &= {}^tX_n \ldots {}^tX_1, \end{aligned} \tag{2.1.2}$$

Let $\mathfrak{g} = (\mathfrak{g}_{\mathfrak{R}})_c$. Write j for the action of $\sqrt{-1}$ coming from the complexification and i for the action of $\sqrt{-1}$ coming from the original complex structure on $\mathfrak{g}_{\mathfrak{R}}$. Then $\mathfrak{g}$ can be written as the direct sum of two ideals each isomorphic to $\mathfrak{g}_{\mathfrak{R}}$,

$$\begin{aligned} \mathfrak{g}^L &= \{1/2(X + jiX) \mid X \in \mathfrak{g}_{\mathfrak{R}}\}, \\ \mathfrak{g}^R &\{1/2(X - jiX) \mid X \in \mathfrak{g}_{\mathfrak{R}}\}. \end{aligned} \tag{2.1.3}$$

They identify with $\mathfrak{g}_{\mathfrak{R}}$ via the maps

$$\begin{aligned} \phi^L(X) &= 1/2(X + jiX), \\ \phi^R(X) &= 1/2(-{}^t\bar{X} + ji\,{}^t\bar{X}), \end{aligned} \tag{2.1.4}$$

for $X \in \mathfrak{g}_{\mathfrak{R}}$. (The bar denotes complex conjugation with respect to the real form $\mathfrak{k}_{\mathfrak{R}}$ of $\mathfrak{g}_{\mathfrak{R}}$ viewed as a complex algebra this time). Then

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}^L) \otimes \mathfrak{U}(\mathfrak{g}^R). \tag{2.1.5}$$

The complexification of $\mathfrak{k}_{\mathfrak{R}}$ can be identified as the image of $\mathfrak{g}_{\mathfrak{R}}$ under the map

$$\begin{aligned} &\phi^D\colon \mathfrak{g}_{\mathfrak{R}} \to \mathfrak{g}, \\ &\phi^D(X) = \phi^L(-{}^tX) + \phi^R(X). \end{aligned} \tag{2.1.6}$$

Thus we can identify $\mathfrak{k} = (\mathfrak{k}_{\mathfrak{R}})_c$, $\mathfrak{t} = (\mathfrak{t}_{\mathfrak{R}})_c$, $\mathfrak{a} = (\mathfrak{a}_{\mathfrak{R}})_c$ with the following objects in $\mathfrak{g}_{\mathfrak{R}} \oplus \mathfrak{g}_{\mathfrak{R}}$.

$$\begin{aligned} \mathfrak{k} &= \{(-{}^tX, X) \mid X \in \mathfrak{g}_{\mathfrak{R}}\}, \\ \mathfrak{t} &= \{(H, -H) \mid H \in \mathfrak{h}_{\mathfrak{R}}\}, \\ \mathfrak{a} &= \{(H, H) \mid \mathfrak{h} \in \mathfrak{h}_{\mathfrak{R}}\}. \end{aligned} \tag{2.1.7}$$

Furthermore $j(X, Y) = (-{}^t\bar{Y}, X)$ so that $\mathfrak{g}_{\mathfrak{R}}$ identifies with

$$\mathfrak{g} \mid \mathfrak{R} = \{(-{}^t\bar{X}, X) \mid X \in \mathfrak{g}_{\mathfrak{R}}\}. \tag{2.1.8}$$

2.2. We now describe the classification of irreducible $(\mathfrak{g}, K)$ modules. Denote by Δ the roots and by Δ^+ the roots of the fixed Borel subgroup $B^+ = H \cdot N^+$.

Let $\lambda, \lambda' \in \mathfrak{h}^*_{\mathfrak{R}}$ be such that $\lambda - \lambda' = \mu$ is the weight of a finite dimensional holomorphic representation of G. Let $\nu = \lambda + \lambda'$. Define a character $\mathbb{C}_{(\lambda, \lambda')}$ of H via

$$\begin{aligned} \mathbb{C}_{(\lambda, \lambda')}|_T &= \mathbb{C}_\mu, \\ \mathbb{C}_{(\lambda, \lambda')}|_A &= \mathbb{C}_\nu. \end{aligned} \tag{2.2.1}$$

$\mathbb{C}_{(\lambda, \lambda')}$ can be extended to a character of B^+ by making it trivial on N^+. Let

$$\begin{aligned} X(\lambda, \lambda') &= [\mathrm{Ind}_B^G\, \mathbb{C}_{(\lambda, \lambda')}]_{\text{K-finite}}, \\ \bar{X}(\lambda, \lambda') &= \text{the unique irreducible subquotient of } X(\lambda, \lambda') \\ &\qquad \text{containing the K-type with extremal weight } \mu. \end{aligned} \tag{2.2.2}$$

Theorem (Parthasarathy-Rao-Varadarajan, Zhelobenko). *Fix (λ, λ') and (λ_1, λ'_1) as before. The following are equivalent.*

(1) *$X(\lambda, \lambda')$ and $X(\lambda_1, \lambda'_1)$ have the same composition factors with multiplicities.*
(2) *$\bar{X}(\lambda, \lambda') \simeq \bar{X}(\lambda_1, \lambda'_1)$.*
(3) *There is $w \in W$ such that $w\lambda = \lambda_1$ and $w\lambda' = \lambda'_1$.*

Any irreducible $(\mathfrak{g}, K)$ module is isomorhic to an $\bar{X}(\lambda, \lambda')$.

Proof. See [D1].

2.3. We will find it convenient to use the following part of the proof of Theorem 2.2.

Recall that $\lambda - \lambda' = \mu$ and $\lambda + \lambda' = \nu$. Then we can use (μ, ν) instead of (λ, λ'). For every $w \in W$, there is an intertwining operator ([D1], III 4)

$$B(w; \mu, \nu)\colon\ X(\mu, \nu) \to X(w\mu, w\nu). \tag{2.3.1}$$

This is the standard intertwinning operator normalized to be the identity on the lowest K-type. It is a rational function of ν defined almost everywhere.

Proposition.
(1) *If*

$$(\nu, \check{\alpha}) \neq -|(\mu, \check{\alpha})| - 2j \quad \textit{for any } \alpha \in \Delta^+, \quad j \in \mathbb{N}^+,$$

then $X(\mu, \nu)$ has a unique proper maximal submodule. It equals $\ker B(w_0;\, \mu, \nu)$, *where $w_0 \in W$ is the longest element.*

(2) *If*

$$(\nu, \check{\alpha}) \neq |(\mu, \check{\alpha})| + 2j \quad \text{for any } \alpha \in \Delta^+, \quad j \in \mathbb{N}^+,$$

then $X(\mu, \nu)$ has a unique irreducible submodule. It equals $\bar{X}(\mu, \nu)$. Any nonzero submodule of $X(\mu, \nu)$ contains $\bar{X}(\mu, \nu)$.

Proof. See [D1], Sect. 3.11.

2.4. A hermitian form on a $(\mathfrak{g}, K)$ module (π, V) is called $\mathfrak{g}_{\mathfrak{R}}$ invariant if

$$(\pi(X)\, v, w) = -(v, \pi(X^h)\, w)$$

where $X = X_1 + jX_2$ and $X^* = X_1 - jX_2$ for $X_1, X_2 \in \mathfrak{g}_{\mathfrak{R}}$. It is a well known result of Harish-Chandra that the problem of classifying the unitary irreducible G-modules is equivalent to the problem of classifying the irreducible $(\mathfrak{g}, K)$ modules admitting positive definite $\mathfrak{g}_{\mathfrak{R}}$ invariant hermitian forms.

Theorem. *$\bar{X}(\lambda, \lambda')$ admits a nondegenerate hermitian form if and only if there is $w \in W$ such that*

$$\begin{aligned} w\mu &= \mu, \\ w\nu &= -\bar{\nu} \end{aligned} \tag{2.4.1}$$

Proof. For the complex case see [D2]. For the more general real case see [KZ].

The form is given as follows.

For any (μ, ν) there is a natural nondegenerate hermitian pairing between $X(\mu, \nu)$ and $X(\mu, -\bar{\nu})$. That is,

$$X(\mu, \nu)^h \simeq X(-\bar{\mu}, \bar{\nu}) \simeq X(\mu, -\bar{\nu}). \tag{2.4.2}$$

Now suppose $\bar{X}(\mu, \nu)$ is hermitian. There is w_h satisfying (2.4.1). w_h defines an intertwining operator

$$B(w_h;\, \mu, \nu)\colon\ X(\mu, \nu) \to X(\mu, -\bar{\nu}). \tag{2.4.3}$$

If $\operatorname{Re} \nu$ is dominant, the form is given by

$$\langle v, w \rangle = (v, B(w_h;\, \mu, \nu)\, w). \tag{2.4.4}$$

Its radical is $\ker B(w_h;\, \mu, \nu)$. Indeed, since $\operatorname{Re} \nu$ is dominant,

$$(\nu, \check{\alpha}) \neq -|(\mu, \check{\alpha})| - 2j \quad \text{for any } \alpha \in \Delta^+, \quad j \in \mathbb{N}^+, \tag{2.4.5}$$

$$(-\bar{\nu}, \check{\alpha}) \neq |(\mu, \check{\alpha})| + 2j \quad \text{for any } \alpha \in \Delta^+, \quad j \in \mathbb{N}^+. \tag{2.4.6}$$

By 2.3(1), $X(\mu, \nu)$ has a unique proper maximal submodule. Then the image of $B(w_h;\, \mu, \nu)$ has $\bar{X}(\mu, \nu)$ as quotient. On the other hand, by Proposition 2.3(2), any proper submodule of $X(\mu, -\bar{\nu})$ must meet $\bar{X}(\mu, \nu)$. Thus the image of $B(w_h;\, \mu, \nu)$ is $\bar{X}(\mu, \nu)$ and the kernel is the maximal proper submodule.

The form on $\bar{X}(\mu, \nu)$ is obtained from (2.4.4) by passage to the quotient. It is clearly nondegenerate.

For simplicity we will refer to $\mathfrak{g}_{\mathfrak{R}}$-invariant nondegenerate hermitian forms as just non-degenerate forms. If $\bar{X}$ admits a nondegenerate form, we will say $\bar{X}$ is hermitian. The form will be normalized to be positive on the lowest K-type.

2.5. Let $\nu = \operatorname{Re} \nu + j\phi$. Assume ϕ is dominant for Δ^+ and $(\operatorname{Re} \nu, \alpha) \geqq 0$ for all $\alpha \in \Delta^+$ such that $(\alpha, \phi) = 0$. Let $P_\phi = M_\phi \cdot N_\phi$ be the parabolic subgroup defined by ϕ. Thus

$$\Delta(\mathfrak{m}_\phi, \mathfrak{a}) = \{\alpha \in \Delta \mid (\alpha, \phi) = 0\}, \tag{2.5.1}$$

$$\Delta(\mathfrak{n}_\phi, \mathfrak{a}) = \{\alpha \in \Delta \mid (\alpha, \phi) > 0\}. \tag{2.5.2}$$

(λ, λ') also defines a parameter for a standard $(\mathfrak{m}_\phi, M_\phi \cap K)$-module $X_\phi(\lambda, \lambda')$ containing the irreducible $\bar{X}_\phi(\lambda, \lambda')$. By induction in stages,

$$X(\lambda, \lambda') = \operatorname{Ind}_{P_\phi}^G [X_\phi \otimes \mathbf{1}]. \tag{2.5.3}$$

Proposition. (*Vogan*)

$$\bar{X}(\lambda, \lambda') = \operatorname{Ind}_{P_\phi}^G [\bar{X}_\phi \otimes \mathbf{1}].$$

Proof. Let w_ϕ be the longest element in $W(\mathfrak{m}_\phi)$. If $\alpha \in \Delta^+(\mathfrak{m}_\phi, \mathfrak{a})$, then

$$(\nu, \check{\alpha}) = (\operatorname{Re} \nu, \check{\alpha}) \geqq 0. \tag{2.5.4}$$

If $\alpha \notin \Delta(\mathfrak{m}_\phi, \mathfrak{a})$, then

$$(\nu, \check{\alpha}) \notin \mathbb{Z}. \tag{2.5.5}$$

Thus

$$(\nu, \check{\alpha}) \neq -|(\mu, \check{\alpha})| - 2j \quad \text{for any } \alpha \in \Delta^+, \quad j \in \mathbb{N}^+. \tag{2.5.6}$$

By 2.3(1), $X(\mu, \nu)$ has a unique proper maximal submodule. Similarly,

$$(w_\phi \nu, \check{\alpha}) = (\operatorname{Re} \nu, \check{\alpha}) \leqq 0 \quad \text{for } \alpha \in \Delta^+(\mathfrak{m}_\phi, \mathfrak{a}), \tag{2.5.7}$$

$$(w_\phi \nu, \check{\alpha}) \notin \mathbb{Z} \quad \text{for } \alpha \notin \Delta(\mathfrak{m}_\phi, \mathfrak{a}). \tag{2.5.8}$$

By Proposition 2.3(2), $X(w_\phi \mu, w_\phi \nu)$ has a unique irreducible submodule, namely $\bar{X}(\lambda, \lambda')$; and any nonzero submodule meets $\bar{X}(\lambda, \lambda')$. Thus the image of $B(w_\phi, \mu, \nu)$ must contain $\bar{X}(\lambda, \lambda')$ as a quotient and as a submodule. This can only happen if the image is $\bar{X}(\lambda, \lambda')$. On the other hand, $B(w_\phi, \mu, \nu)$ can be viewed as induced from the long intertwining operator on M_ϕ. Then the image is $\operatorname{Ind}_{P_\phi}^G [\bar{X} \otimes \mathbf{1}]$. This completes the proof.

Corollary. *Suppose $\bar{X}(\lambda, \lambda')$ is unitary. Then there is a parabolic subgroup P_ϕ, a unitary representation $\bar{X}_{\phi, \mathfrak{R}}$ with real infinitesimal character and a unitary character χ_ϕ such that*

$$\bar{X}(\lambda, \lambda') = \operatorname{Ind}_{P_\phi}^G [\bar{X}_{\phi, \mathfrak{R}} \otimes \chi_\phi \otimes \mathbf{1}].$$

Proof. This follows from the proposition since

$$\bar{X}_\phi = \bar{X}_{\phi, \mathfrak{R}} \otimes \chi_\phi.$$

Assumption. $\bar{X}(\lambda, \lambda')$ has real infinitesimal character.

This is justified in view of the corollary and will be assumed throughout the rest of the paper.

2.6. Assume μ is dominant for Δ^+. Let $P_\mu = M_\mu \cdot N_\mu$ be the real parabolic subgroup defined by

$$\begin{aligned} \Delta(\mathfrak{m}_\mu, \mathfrak{a}) &= \{(\alpha, i\mu) = 0\}, \\ \Delta(\mathfrak{n}_\mu, \mathfrak{a}) &= \{(\alpha, i\mu) > 0\}. \end{aligned} \tag{2.6.1}$$

Let $P = M \cdot N$ be a parabolic containing P_μ. Given a parameter (λ, λ'), we can define, as for P_ϕ, a standard module $X_M(\lambda, \lambda')$ for $(\mathfrak{m}, M \cap K)$ with lowest $M \cap K$-type subquotient $\bar{X}_M(\lambda, \lambda')$. Then

$$X(\lambda, \lambda') = \mathrm{Ind}_P^G[X_M(\lambda, \lambda') \otimes \mathbf{1}] \tag{2.6.2}$$

and $\bar{X}(\lambda, \lambda')$ is the lowest K-type subquotient of

$$\mathrm{Ind}_P^G[\bar{X}_M(\lambda, \lambda') \otimes \mathbf{1}].$$

Definition (Speh-Vogan). *A K-type γ is called* $\mathfrak{p}$*-bottom layer for $X(\lambda, \lambda')$ if*

$$\gamma = \mu + \sum_{\alpha \in \Delta(\mathfrak{m})} m_\alpha \cdot \alpha \quad \text{for } m_\alpha \in \mathbb{Z}. \tag{2.6.3}$$

2.7. Suppose (π, V) is a $(\mathfrak{g}, K)$-module admitting an invariant hermitian form. Let $\gamma \in K$. Define $[\gamma : V]_\pm$ by

$$[\gamma : V]_\pm = \text{the dimension of the } \pm\text{-signature of the } \gamma\text{-isotypic component of } V. \tag{2.7.1}$$

With the notation as in 2.6 assume that $\bar{X}_M(\lambda, \lambda')$ is hermitian. Then so is $\bar{X}(\lambda, \lambda')$. Assume the forms are normalized to be positive definite on the lowest $K \cap M$-type, K-type respectively.

Proposition (Vogan). *Let γ be* $\mathfrak{p}$*-bottom layer. Then*

$$[\gamma : \bar{X}_M(\lambda, \lambda')]_\pm = [\gamma : \bar{X}_G(\lambda, \lambda')]_\pm .$$

(By abuse of notation γ denotes both the K-type with highest weight γ as well as the corresponding $K \cap M$-type).

In particular the multiplicities also coincide.

We will prove this in 2.8. Let $\xi \in \mathfrak{h}_{\mathbb{R}}$ be dominant such that

$$\Delta(\mathfrak{p}) = \{\alpha \in \Delta^+ \mid (\alpha, \xi) \geqq 0\}. \tag{2.7.2}$$

Let $\alpha \in \Delta^+$. Then α determines a real group $M(\alpha) \supset H$ such that

$$\Delta(\mathfrak{m}(\alpha)) = \{\pm \alpha\}. \tag{2.7.3}$$

If $\alpha \in \Delta^+$ is simple, we denote by $P(\alpha) = M(\alpha)\, N(\alpha)$ the parabolic subgroup with roots in the nilradical given by

$$\Delta(\mathfrak{n}(\alpha)) = \{\beta \in \Delta^+ \mid \beta \neq \alpha\}. \tag{2.7.4}$$

Lemma. *Let γ be* $\mathfrak{p}$*-bottom layer. Then*

(1) $[\gamma : X(\mu, \nu)] = [\gamma : X_M(\mu, \nu)]$,

(2) *Let $\alpha \in \Delta(\mathfrak{n})$. The only $M(\alpha) \cap K$-type that occurs in both $X_{M(\alpha)}(\mu, \nu)$ and $\gamma|_{M(\alpha) \cap K}$ has extremal weight μ.*

Proof. The multiplicity of γ in $X(\mu, \nu)$ is obtained from Kostant's formula as follows.

Let $\mathfrak{l} \cap \mathfrak{k}$ be a reductive subalgebra containing $\mathfrak{t}$. Let $\mathfrak{s} \cap \mathfrak{k}$ be an $\mathfrak{l} \cap \mathfrak{k}$-invariant subspace complementary to $\mathfrak{l} \cap \mathfrak{k}$. Then $\Delta^+(\mathfrak{k}) = \Delta^+(\mathfrak{l} \cap \mathfrak{k}) \cup \Delta^+(\mathfrak{s} \cap \mathfrak{k})$. Let γ be the highest weight of a K-type, $\gamma_{L \cap K}$ the highest weight of an $L \cap K$-type. Define

$$W^+_{L \cap K} = \{w \in W \mid w(\gamma + \rho(\mathfrak{k})) \quad \text{is dominant for } \Delta(\mathfrak{l})^+\}, \tag{2.7.5}$$

$$\begin{aligned} P(\gamma, \gamma_{L \cap K}, w) = \operatorname{card}\{&(p_\beta)_{\beta \in \Delta^+(\mathfrak{s} \cap \mathfrak{k})} \mid p_\beta \in \mathbb{N} \\ &\text{and } w(\gamma + \rho(\mathfrak{k})) - \rho(\mathfrak{k}) = \gamma_{L \cap K} + \textstyle\sum p_\beta \beta\}, \end{aligned} \tag{2.7.6}$$

where $\rho(\mathfrak{k})$ is half the sum of positive roots in $\mathfrak{k}$. Then

$$[\gamma_{L \cap K} : \gamma|_{L \cap K}] = \sum_{w \in W^+_{L \cap K}} \operatorname{sgn}(w)\, P(\gamma, \gamma_{L \cap K}, w). \tag{2.7.7}$$

This can be proved by expanding the appropriate part of the denominator in Weyl's character formula.

By Frobenius reciprocity,

$$[\gamma : X(\mu, \nu)] = \sum [\gamma_{M \cap K} : \gamma|_{M \cap K}][\gamma_{M \cap K} : X_M(\mu, \nu)]. \tag{2.7.8}$$

We can write

$$\gamma = \mu + \sum_{\delta \in \Delta(\mathfrak{m} \cap \mathfrak{k})} m_\delta \delta. \tag{2.7.9}$$

Then on the one hand, in (2.7.6),

$$w(\gamma + \rho(\mathfrak{k})) - \rho(\mathfrak{k}) = \gamma_{M \cap K} + \sum_{\beta \in \Delta(\mathfrak{n} \cap \mathfrak{k})} p_\beta \beta \tag{2.7.10}$$

because in this case, $\mathfrak{s} \cap \mathfrak{k} = \mathfrak{n} \cap \mathfrak{k} + \bar{\mathfrak{n}} \cap \mathfrak{k}$. On the other hand,

$$w(\gamma + \rho(\mathfrak{k})) - \rho(\mathfrak{k}) = \gamma - \sum_{\beta \in \Delta^+(\mathfrak{k})} q_\beta \beta, \quad q_\beta \in \mathbb{N}. \tag{2.7.11}$$

Taking scalar product with ξ and taking into account that

$$\gamma_{M \cap K} = \mu + \sum_{\delta \in \Delta(\mathfrak{m} \cap \mathfrak{k})} n_\delta \delta, \tag{2.7.12}$$

we see that we must have $p_\beta, q_\beta = 0$ for all $\beta \in \Delta(\mathfrak{n} \cap \mathfrak{k})$. The proof of (1) follows.

For (2) we replace (2.7.9) by

$$\gamma=\mu+n\alpha+\sum_{\beta\in\Delta^+(\beta)} r_\beta\,\beta. \tag{2.7.13}$$

By taking scalar product with ξ we derive that $n=0$.

2.8. *Proof of Proposition 2.7.*

Let (μ,ν) be such that μ is dominant for $\Delta(\mathfrak{n})$ and ν is dominant for $\Delta(\mathfrak{m})^+$ (In particular, μ can be made dominant for Δ^+ using an element in $W(\mathfrak{m})$). Then the intertwining operator $B(w_h;\mu,\nu)$ is induced from the corresponding $B_M(w_h;\mu,\nu)$. Then $\mathrm{Ind}_P^G[\bar X_M(\mu,\nu)\otimes\mathbf{1}]$ inherits a hermitian form from $\bar X_M(\mu,\nu)$. We get

$$[\gamma:\bar X(\mu,\nu)]_\pm \leqq \sum_{\gamma_M\in\widehat{M\cap K}} [\gamma|_{M\cap K}:\gamma_M]\cdot[\gamma_M:\bar X_M(\mu,\nu)]_\pm. \tag{2.8.1}$$

In view of 2.7, if γ is bottom layer, this takes the form

$$[\gamma:\bar X(\mu,\nu)]_\pm \leqq [\gamma:\bar X_M(\mu,\nu)]_\pm. \tag{2.8.2}$$

To prove the assertion, it is enough to show

$$[\gamma:\bar X(\mu,\nu)]=[\gamma:\bar X_M(\mu,\nu)]. \tag{2.8.3}$$

If ν is dominant for Δ^+, then the image of

$$B(w_h;\mu,\nu):\ X(\mu,\nu)\to X(\mu,-\nu) \tag{2.8.4}$$

is $\bar X(\mu,\nu)$ in view of 2.4. Since $B(w_h;\mu,\nu)$ is induced from $B_M(w_h;\mu,\nu)$ we get

$$\bar X(\mu,\nu)=\mathrm{Ind}_P^G[\bar X_M(\mu,\nu)\otimes\mathbf{1}]. \tag{2.8.5}$$

Assume ν is not dominant for Δ^+. Let

$$\Delta^-(\nu)=\{\beta\in\Delta^+\,|\,(\beta,\nu)<0\}. \tag{2.8.6}$$

Then there is a simple $\alpha\in\Delta(n)$ such that $(\alpha,\nu)<0$. Thus

$$\Delta^-(s_\alpha\nu)=s_\alpha\Delta^-(\nu)-\{\alpha\}. \tag{2.8.7}$$

Continuing in this way we can find a sequence of reflections $s_1=s_\alpha, s_2,\dots,s_k$ so that the following holds.

Let $w_i=s_i\cdot\ldots\cdot s_1$. Then

$$\Delta^-(w_i\nu)=s_i[\Delta^-(w_{i-1}\nu)]-\{\alpha_i\}, \tag{2.8.8}$$

where $\alpha_i\in\Delta(\mathfrak{n})$ is simple and $w_{i-1}^{-1}\alpha_i\in\Delta(\mathfrak{n})$. In particular, k is determined by the fact that $w_k\nu$ is dominant. Let

$$B_i:X(w_i\mu,w_i\nu)\to X(w_{i-1}\mu,w_{i-1}\nu). \tag{2.8.9}$$

Similarly we can find reflections $t_1,\dots,t_l$ and $v_j=t_j\cdot\ldots\cdot t_1$ so that $v_l\nu$ is antidominant for Δ^+. Each t_j is a reflection about a simple root $\alpha_j\in\Delta(\mathfrak{n})$ such that $v_{j-1}^{-1}\alpha_j\in\Delta(\mathfrak{n})$. Let $w=s_k\cdot\ldots\cdot s_1$ and $v=t_l\cdot\ldots\cdot t_1$ and

$$B_j^*:X(w_{j-1}^{-1}\mu,-w_{j-1}^{-1}\nu)\to X(w_j^{-1}\mu,-w_j^{-1}\nu). \tag{2.8.10}$$

Then

$$B(vw^{-1};w\mu,w\nu):X(w^{-1}\mu,w^{-1}\nu)\to X(v\mu,v\nu) \tag{2.8.11}$$

factors as

$$B(vw^{-1};w\mu,w\nu)=\prod B_j^*\cdot B(w_h;\mu,\nu)\cdot\prod B_i. \tag{2.8.12}$$

Since the image of $B(vw^{-1};w\mu,w\nu)$ is $\bar X(\mu,\nu)$, (2.8.3) is equivalent to the fact that B_j^*, B_i are isomorphisms on the γ-isotypic component.

We show this for B_i. The argument for B_j^* is the same.

Note that B_i is induced from the $SL(2,\mathbb{C})$ intertwining operator

$$B_{\alpha_i}:X_{M(\alpha_i)}(w_i\mu,w_i\nu)\to X_{M(\alpha_i)}(w_{i-1}\mu,w_{i-1}\nu). \tag{2.8.13}$$

If γ is in the kernel of B_i it must occur in $X_{M(\alpha_i)}(w_{i-1}\mu,w_{i-1}\nu)$. By shifting to $\mathrm{Ind}^K_{M(w_{i-1}^{-1}\alpha_i)\cap K}[\mathbb{C}_\mu]$ we get

$$\gamma=\mu+n\,w_{i-1}^{-1}\alpha_i+\sum_{\alpha\in\Delta^+} p_\alpha\alpha. \tag{2.8.14}$$

Since $w_{i-1}^{-1}\alpha_i\in\Delta(\mathfrak{n})$ the claim follows from Lemma 2.7(2).

2.9. Corollary. *Let $P=MN$ and $\bar X(\mu,\nu)$ be such that $(\mu+\nu,\check\alpha)\notin 2\mathbb{Z}$ for any $\alpha\in\Delta(\mathfrak{n})$. Then*

$$\bar X(\mu,\nu)=\mathrm{Ind}_P^G[\bar X_M(\mu,\nu)\otimes\mathbf{1}].$$

Proof. Assume μ is dominant for Δ^+, ν dominant for $\Delta^+(\mathfrak{m})$. Write

$$X(\mu,\nu)=\mathrm{Ind}_P^G[X_M(\mu,\nu)\otimes\mathbf{1}]. \tag{2.9.1}$$

Let w_m be the long element in $W(\mathfrak{m})$. Then the image of $B_M(w_m;\mu,\nu)$ is $\bar X_M(\mu,\nu)$. The image of $B(w_m;\mu,\nu)$ is $\mathrm{Ind}_P^G[\bar X(\mu,\nu)\otimes\mathbf{1}]$. As in 2.8 we can define B_i and B_i^* and we must show that they are isomorphisms. This follows from proposition 2.3 and the fact that in this case $(w_{i-1}\mu+w_{i-1}\nu,\check\alpha_i)\notin 2\mathbb{Z}$.

2.10. Let Δ^+ be the positive system defined in [Bou] for the classical groups. A finite dimensional holomorphic representation F can be described by its infinitesimal character. We will write it as

$$F:(a_1,\dots,a_r). \tag{2.10.1}$$

The a_j will be called the coordinates and satisfy

TYPE A.	$a_j+a_{j+1}\in\mathbb{N}^+$,	
TYPE B.	$a_j-a_{j+1}\in\mathbb{N}^+$,	$2a_j\in\mathbb{N}^+$,
TYPE C.	$a_j-a_{j+1}\in\mathbb{N}^+$,	$a_j\in\mathbb{N}^+$,
TYPE D.	$a_j-a_{j+1}\in\mathbb{N}^+$,	$2a_j\in\mathbb{N}^+, 2\|a_r\|\in\mathbb{N}$.

A finite dimensional $(\mathfrak{g}, K)$ module $\mathscr{F}=F_L \otimes F_R$ will be written

$$\mathscr{F}:(a_{L,1}, \ldots, a_{L,r}) \times (a_{R,1}, \ldots, a_{R,r}). \tag{2.10.2}$$

In particular, for $2m \in \mathbb{N}$,

$$F(m):(m, m-1, \ldots, -m+1, -m) \tag{2.10.3}$$

denotes the trivial representation for type A_{2m}.

3. A special set of K-types

3.1. Let $\mathfrak{g}$ be of classical type. Then $\mathfrak{g}$ has a distinguished representation $\varepsilon_{\mathfrak{g}}$ called the standard representation. It has the following properties.

(1) There is a simple root α_1 such that $(\varepsilon_{\mathfrak{g}}, \beta)=(\varpi_{\alpha_1}, \beta)$ for all $\beta \in \Delta$. Here ϖ_{α_1} is the fundamental weight corresponding to α_1.

(2) Let $\mathfrak{g}(n-1) \subset \mathfrak{g}(n)$. Then $\varepsilon_{\mathfrak{g}(n-1)}=\varepsilon_{\mathfrak{g}(n)}-\alpha_1$ and the restriction of α_1 to $\mathfrak{g}(n-1)$ is $-\varepsilon_{\mathfrak{g}(n-1)}$ or $-\varepsilon^*_{\mathfrak{g}(n-1)}$.

(3) $\varepsilon_{\mathfrak{g}} \otimes \varepsilon^*_{\mathfrak{g}}$ contains $\operatorname{Ad}\mathfrak{g}$ and the irreducible finite dimensional representation with extremal weight $\breve{\beta}$ where β is the highest short root.

Definition. Assume K is of classical type. A K-type ε will be called small if it is in the root lattice and satisfies

$$(\varepsilon_{\mathfrak{k}}, \varepsilon), (\varepsilon^*_{\mathfrak{k}}, \varepsilon) \leqq 1,$$

where (,) is the usual scalar product.

In coordinates, the small K-types are as follows.

TYPE A. $\mu(r)=(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{r})$.

TYPE B. $\mu(r)=(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0)$.

TYPE C. $\mu(r)=(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0)$ with r even.

TYPE D. $\mu(r)=(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0)$ with r even,

$\mu^{\pm}(n)=(1, \ldots, \pm 1)$ with n even.

3.2. Let B be the Cartan-Killing form. For any $\beta \in \Delta$ define $h_\beta \in \mathfrak{t}$ to satisfy $B(h_\beta, H)=\breve{\beta}(H)$ for all $H \in \mathfrak{t}$.

Proposition. *Suppose* $\varepsilon \neq 0$ *is small. Then*

$$\varepsilon=\beta_1+\ldots+\beta_k,$$

where β_1 *is the highest short root in* $\mathfrak{k}$ *and* β_{i+1} *is a highest short root in* $C(\mathfrak{k}; h_{\beta_1}, \ldots, h_{\beta_i})$ *such that* $\varepsilon_{i+1}=\beta_1+\ldots+\beta_{i+1}$ *is small.*

Proof. This is clear if we express ε in coordinates. We give a proof using (1), (2), (3) of 3.1. By (3)

$$(\varepsilon, \breve{\beta}) \leqq 2 \tag{3.2.1}$$

for all $\beta \in \Delta^+$.

$(\varepsilon, \breve{\beta}_1)=2$. Suppose not. Then $(\varepsilon, \breve{\beta}_1)=1$. Write $\beta_1=\sum m_\delta \delta$ with δ simple. Then $1=(\varepsilon, \breve{\beta}_1)=\sum m_\delta(\varepsilon, \breve{\delta})$. Thus there is exactly one simple root δ such that $(\varepsilon, \breve{\delta})>0$; and for this δ, $m_\delta=1$ and $(\varepsilon, \breve{\delta})=1$. Then $\varepsilon=\varpi_\delta$, the fundamental weight corresponding to δ. By inspecting the coefficients m_δ for each classical simple algebra we see that if $m_\delta=1$ then ϖ_δ is not in the root lattice. The claim follows.

The argument also shows that either

$$(\varepsilon, \breve{\alpha}) \leqq 1 \quad \text{for all simple roots } \alpha, \tag{3.2.2}$$

or there is only one simple root such that $(\varepsilon, \alpha)>0$. In this case if α_i is simple,

$$(\varepsilon, \breve{\alpha}_i)=\begin{cases}0 & \text{if } \alpha_i \neq \alpha \\ 2 & \text{if } \alpha_i=\alpha.\end{cases} \tag{3.2.3}$$

We show that if (3.2.2) or (3.2.3) hold, then the assertion of the proposition follows. We do an induction on *rank*$(\mathfrak{k})$.

$\varepsilon^1=\varepsilon-\beta_1$ *is small for* $\mathfrak{k}(\beta_1)=C(\mathfrak{k}, h_{\beta_1})$. From the previous claim we know that $(\varepsilon^1, \breve{\beta}_1)=0$. Because β_1 is also a weight and ε^1 is an integral combination of roots, ε^1 is also in the root lattice of $\mathfrak{k}(\beta_1)$.

Note that

$$(\varepsilon, \breve{\beta})=(\varepsilon^1, \breve{\beta}) \quad \text{for } \beta \in \Delta^+(\mathfrak{k}(\beta_1)). \tag{3.2.4}$$

Thus if ε satisfies (3.2.2) then ε^1 satisfies (3.2.2) for $\Delta^+(\mathfrak{k}(\beta_1))$. The similar statement holds if ε satisfies (3.2.3). By induction, $\varepsilon^1=\beta_2+\ldots+\beta_k$ so $\varepsilon=\beta_1+\ldots+\beta_k$ as claimed.

We must also show that ε_i is small.

Note that

$$(\varepsilon_{\mathfrak{k}}, \varepsilon)=(\varepsilon_{\mathfrak{k}}, \beta_1)+\ldots+(\varepsilon_{\mathfrak{k}}, \beta_k) \tag{3.2.5}$$

and each $(\varepsilon_{\mathfrak{k}}, \beta_i) \in \mathbb{N}$. Thus

$$(\varepsilon_{\mathfrak{k}}, \beta_i)=\begin{cases}0 & \text{for } i \neq 1, \\ 1 & \text{for } i=1.\end{cases} \tag{3.2.6}$$

The claim follows.

3.3. With the notation as in 2.2, 2.6, write

$$\mathfrak{m}_\mu=\mathfrak{m}_{\mu,0} \times \mathscr{A}_{\mu,1} \times \ldots \times \mathscr{A}_{\mu,s}, \tag{3.3.1}$$

where $\mathscr{A}_{\mu,i}$ are $gl(n)$'s and $\mathfrak{m}_{\mu,0}$ is classical not type A.

Proposition. *Let ε be small for one of the factors $\mathfrak{m}^1$ in* (3.3.1) *with simple roots $\{\alpha_1, \ldots, \alpha_n\}$. Then $\mu+\varepsilon$ is dominant except in the following case:*

$\mathfrak{g}$ is type B, D, $\mathfrak{m}^1$ is type A and the restriction of μ to coordinates in $\mathfrak{m}^1$ is $(1/2, \ldots, 1/2, \pm 1/2)$. (These are the Spin representations.)

Proof. Identify μ with its highest weight. We must show that $\mu+\varepsilon$ is dominant except in the indicated cases. This is equivalent to

$$(\varepsilon, \check{\alpha}) \geqq -(\mu, \check{\alpha}) \quad \text{for all simple } \alpha \in \Delta^+. \tag{3.3.2}$$

Fix a simple root α. (3.3.2) is implied by

$$(\varepsilon, \check{\alpha}) > -1. \tag{3.3.3}$$

If α is one of the α_i then (3.3.3) follows from the dominance of μ and ε. If α is not adjacent to any α_i, it follows from the fact that ε is in the span of the $\{\alpha_i\}$. So assume that α is adjacent to to at least one of the α_i. Let $\mathfrak{m}^2$ be the subalgebra with roots $\{\alpha, \alpha_1, \ldots, \alpha_l\}$. Except when $\mathfrak{m}^1$ is type A and $\mathfrak{m}^2$ is type B or D, the restriction of $\check{\alpha}$ to $\mathfrak{m}^1$ is $-\varepsilon_{\mathfrak{m}^1}$ or $-\varepsilon^*_{\mathfrak{m}^1}$. In this case (3.3.3) is a consequence of the definition of small.

Finally suppose that $\mathfrak{m}^1$ is type A and $\mathfrak{m}^2$ is type B or D.

In type B the restriction of $\check{\alpha}$ to $\mathfrak{m}^1$ is $-2\varepsilon_{\mathfrak{m}^1}$ or $-2\varepsilon^*_{\mathfrak{m}^1}$.

In type D, let α_{n-1} be unique simple root adjacent to α. Then α_{n-1} occurs once in the expression of β_1 in terms of simple roots. The same is true for β_2 while the other β_i are orthogonal to α. Thus in both case B and D,

$$(\varepsilon, \check{\alpha}) = -2, \tag{3.3.4}$$

Since $\mu = (a, \ldots, a, \pm a)$ in coordinates, (3.3.2) is satisfied only if $a = 1/2$.

The proof is complete.

3.4. Let notation be as in 3.3.

Definition. (1) A K-type μ' is called level 0 for μ if it is equal to the lowest K-type μ.

(2) μ' is called level 1 for μ if it is of the form $\mu' = \mu + \varepsilon$ where $\varepsilon \neq 0$ is the highest weight of an $M_\mu \cap$ K-type of the form $\varepsilon = \mathbb{C} \otimes \ldots \otimes \mathbb{C} \otimes \varepsilon_i \otimes \mathbb{C} \otimes \ldots \otimes \mathbb{C}$ with ε_i small.

(3) μ' is called level 2 if it is not level 1 and is of the form $\mu' = \mu + \varepsilon_1 + \varepsilon_2$ with $\varepsilon_i \neq 0$ and $\mu + \varepsilon_1$ level 1 for μ and $\mu + \varepsilon_1 + \varepsilon_2$ level 1 for $\mu + \varepsilon_1$.

3.5. Theorem. *An irreducible* $(\mathfrak{g}, K)$ *module* $\bar{X}(\lambda, \lambda')$ *is unitary if and only if the nondegenerate form is positive on the level* $\leqq 2$ *isotypic components.*

Theorem (Vogan). *If $\mathfrak{g}$ is type A, $\bar{X}(\lambda, \lambda')$ is unitary if and only if the nondegenerate form is positive on the level $\leqq 1$ isotypic components.*

The proof of 3.5 will be complete only after chapter 12.

3.6. The following properties will be used without proof. Assume $\mu = 0$ and K is classical.

(1) A *level* 1 K-type is a small K-type. A *level* 2 K-type has highest weight ε satisfying

$$(\varepsilon_t^*, \varepsilon), \quad (\varepsilon_t, \varepsilon) \leqq 2$$

and equality holds for at least one of them.

(2) The restriction of a *level* 2 K-type to any Levi component $M \cap K$ is a tensor product of *level* $\leqq 2$ $M \cap$ K-types.

(3) The restriction of a *level* 1 K-type to a Levi component is a tensor product of *level* $\leqq 1$ $M \cap K$-types.

(4) In types B, C, D most small K-types are fundamental. They can be realized in terms of ε_t as follows.

TYPE B. $\mu(r) = \bigwedge^r \varepsilon_t$

TYPE D. $\mu(r) = \bigwedge^{2r}$ for $r < n/2$, $\quad \mu^+(n/2) \oplus \mu^-(n/2) = \bigwedge^n \varepsilon_t$

TYPE C. $\bigwedge^{2r} \varepsilon_t = \mu(r) \oplus \mu(r-2) \oplus \ldots$

This is useful for computing restrictions to Levi components.

4. Character theory of representations with maximal annihilator

4.1. Let $\lambda \in \mathfrak{h}^*_{\mathfrak{R}}$ and

$$\Delta(\lambda) = \{\alpha \in \Delta \mid (\check{\alpha}, \lambda) \in \mathbb{Z}\},$$
$$\Delta^+(\lambda) = \Delta(\lambda) \cap \Delta^+. \tag{4.1.1}$$

Then $\Delta(\lambda)$ is not necessarily a subroot system in Δ but ${}^L\Delta(\lambda) = \{\check{\alpha} \mid \alpha \in \Delta(\lambda)\}$ is the subroot system of roots vanishing on $\xi_\lambda = e^{2\pi i\lambda} \in {}^LH$. Let W^λ be the centralizer of λ in W. Let $W(\lambda)$ be the Weyl group corresponding to $\Delta(\lambda)$. (Recall that ${}^LG$ is adjoint.)

Given $w \in W$ and λ negative for Δ^+, let

$$M(w\lambda) = \mathfrak{U}(\mathfrak{g}_{\mathfrak{R}}) \underset{\mathcal{U}(\mathfrak{b}_{\mathfrak{R}})}{\otimes} C_{w\lambda-\rho}, \tag{4.1.2}$$

$$L(w\lambda) = \text{its unique irreducible quotient}, \tag{4.1.3}$$

$$I(w\lambda) = \text{the annihilator of } L(w\lambda) \text{ in } \mathcal{U}(\mathfrak{g}_{\mathfrak{R}}) \tag{4.1.4}$$

$$\tau(w) = \{\alpha \in \Delta^+(\lambda) \text{ simple}, w\alpha \notin \Delta^+(\lambda)\}, \tag{4.1.5}$$

$$\mathscr{P}[\mathfrak{U}(\mathfrak{g}_{\mathfrak{R}}), \lambda] = \text{the set of primitive ideals with infinitesimal character } \lambda. \tag{4.1.6}$$

We will put the algebra as a subscript whenever there is more than one involved and will suppress it when there is no ambiguity.

Suppose λ, λ' are such that $\lambda - \lambda'$ is the weight of a holomorphic finite dimensional representation of G. Let $\lambda'_1, \ldots, \lambda'_r$ be the set of elements in $W\lambda'$ which are integral for $\Delta(\lambda)$, antidominant for $\Delta^+(\lambda)$ and such that $\lambda - \lambda'_i$ is a weight. The set of equivalence classes of irreducible $(\mathfrak{g}, K)$ modules with infinitesimal character (λ, λ') decomposes into a disjoint union of

$$\mathscr{H}(\mathfrak{g}, \lambda, \lambda'_i) = \{\bar{X}(\lambda, w\lambda'_i) \mid w \in W^\lambda \backslash W(\lambda)/W^{\lambda_i}\}. \tag{4.1.7}$$

Then the Grothendieck group of virtual characters decomposes as

$$\mathscr{G}(\mathfrak{g}, \chi_\lambda, \chi_{\lambda'}) = \bigoplus_i \mathscr{G}(\lambda, \lambda_i'). \tag{4.1.8}$$

$W(\lambda)$ acts by the coherent continuation representation and preserves the direct sum decomposition. The notions of left, right and double cells carry over for λ, λ' regular. Let $w \in W(\lambda)$ be minimal in its double coset $W^\lambda w W^{\lambda'}$. Then w satisfies

$$\begin{aligned} &\text{if } \alpha \in \Delta^+, (\alpha, \lambda) = 0 \quad \text{then} \quad w^{-1}\alpha \in \Delta^+, \\ &\text{if } \alpha \in \Delta^+, (\alpha, \lambda') = 0 \quad \text{then} \quad w\alpha \in \Delta^+. \end{aligned} \tag{4.1.9}$$

Let

$$\begin{aligned} {}^L\mathfrak{g}(\lambda) &= C({}^L\mathfrak{g}, \lambda), \\ {}^L\mathfrak{s}(\lambda) &= {}^L\mathfrak{s}(\lambda') = C({}^L\mathfrak{g}; \xi_\lambda). \end{aligned} \tag{4.1.10}$$

The dual algebra is denoted by $\mathfrak{s}(\lambda)$. If $S(\lambda)$ is the connected simply connected Lie group corresponding to $\mathfrak{s}(\lambda)$, we can define maps

$$\begin{aligned} &T_i\colon \mathscr{H}_i(\mathfrak{g}, \lambda, \lambda') \to \mathscr{H}(\mathfrak{s}(\lambda); \lambda, \lambda_i') \\ &\bar{T}_i\colon \mathscr{P}[\mathfrak{U}(\mathfrak{g}_{\mathbb{R}}), \chi_\lambda'] \to \bigcup_i \mathscr{P}[\mathfrak{U}(\mathfrak{s}(\lambda)), \lambda_i'] \end{aligned} \tag{4.1.11}$$

by simply transferring the parameters. $\bar{T}_i$ is well defined for classical groups in view of the next theorem.

Theorem. *Let $\mathfrak{g}$ be classical. Then $I(w_1\lambda') = I(w_2\lambda')$ in $\mathscr{P}[\mathfrak{U}(\mathfrak{g}_{\mathbb{R}}), \lambda']$ if and only if $I(w_1\lambda')_{\mathfrak{s}(\lambda)} = I(w_2\lambda')_{\mathfrak{s}(\lambda)}$. Moreover, $\bar{T}_i$ induces an isomorphism on the level of cells as Weyl group representations.*

Proof. This is Theorem 30 in [BV2].

4.2. The Kazhdan-Lusztig conjectures for nonintegral infinitesimal character provide a sharper version.

Theorem. *Let $\mathfrak{g}$ be arbitrary and*

$$T_i\colon \mathscr{G}(\mathfrak{g}, \lambda, \lambda') \to \bigoplus_i \mathscr{G}(\mathfrak{s}(\lambda), \lambda, \lambda_i')$$

be the linear map determined by

$$T_i\colon X(\lambda, w\lambda_i') \mapsto X_{\mathfrak{s}(\lambda)}(\lambda, w\lambda_i').$$

Then

$$T_i(\bar{X}(\lambda, \lambda_i')) = \bar{X}_{\mathfrak{s}(\lambda)}(\lambda, \lambda_i').$$

We will be studying the character theory of the $\bar{X}(\lambda, \lambda_i')$ which have maximal left and right annihilators. For $\mathfrak{g}$ classical it is possible to reduce this study to the case of integral infinitesimal character via Theorem 4.1. If $\mathfrak{g}$ is not classical, one has to use Theorem 4.2. Its proof reduces via equivalence of categories

([BG], [ES], [J]), to the similar statement for Verma modules. In this case it is due to Beilinson-Bernstein but no proof is in print. See also [L] for the relevant calculations of intersection homology.

4.3. Assume λ is dominant integral. Let $\mathfrak{I}(\lambda)$ be the maximal primitive ideal with infinitesimal character λ.

Proposition.

$$L\mathrm{Ann}\,\bar{X}(\lambda, \lambda) = R\,\mathrm{Ann}\,\bar{X}(\lambda, \lambda) = \mathfrak{I}(\lambda).$$

Proof. It is enough to show that $L\mathrm{Ann}\,\bar{X}$, $R\,\mathrm{Ann}\,\bar{X}$ are maximal primitive ideals. Let $\mathfrak{I}$ be any bi-ideal with infinitesimal character $-\lambda$. Then $\mathfrak{U}(\mathfrak{g}_{\mathbb{R}})/\mathfrak{I}$ is a $(\mathfrak{g}, K)$ module via

$$(a \times b)\cdot v = {}^t\check{a}\cdot v\cdot \check{b}. \tag{4.3.1}$$

It contains $\bar{X}(\lambda, \lambda)$ as subquotient. Thus

$${}^t\mathfrak{I} \subset L\mathrm{Ann}\,\bar{X}(\lambda, \lambda), \qquad \mathfrak{I} \subset R\,\mathrm{Ann}\,\bar{X}(\lambda, \lambda). \tag{4.3.2}$$

The claim follows.

4.4. Let $\lambda_r \in \mathfrak{h}_{\mathbb{R}}^*$, ${}^L\lambda_r \in {}^L\mathfrak{h}_{\mathbb{R}}^*$ be dominant integral regular. There is an inclusion reversing map

$$\iota\colon \mathscr{P}[\mathfrak{U}(\mathfrak{g}), \lambda_r] \to \mathscr{P}[\mathfrak{U}({}^L\mathfrak{g}), {}^L\lambda_r], \tag{4.4.1}$$

$$\mathfrak{I}(w\lambda_r) \mapsto \mathfrak{I}(ww_0\,{}^L\lambda_r). \tag{4.4.2}$$

It induces a bijection

$$\mathcal{O} \mapsto {}^L\mathcal{O} \tag{4.4.3}$$

from special nilpotent orbits in $\mathfrak{g}$ to special nilpotent orbits in ${}^L\mathfrak{g}$ via WF-sets of $(\mathfrak{g}, K)$ modules.

Proposition. *Let $\mathcal{O} = WF(\bar{X}(\lambda, \lambda))$. Then*

$${}^L\mathcal{O} = \mathrm{ind}_{{}^L\mathfrak{g}(\lambda)}^{{}^L\mathfrak{g}}(0).$$

Proof. Recall from [LS] that, if $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is a parabolic subalgebra and $\mathcal{O}_{\mathfrak{m}} \subseteq \mathfrak{m}$ is nilpotent, then $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{m}})$ is the unique nilpotent orbit which meets $\mathcal{O}_{\mathfrak{m}} + \mathfrak{n}$ in a dense set. It depends only on the conjugacy class of $(\mathfrak{m}, \mathcal{O}_{\mathfrak{m}})$. By [BV1] $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is compatible with induction of representations via WF-sets.

Let $w(\lambda)$ be the longest element in W^λ. Then $w = w(\lambda)w_0$ is minimal in $W^\lambda w_0 W^{w_0\lambda}$ so that $\bar{X}(\lambda_r, w(\lambda)w_0[w_0\lambda_r])$ corresponds to $\bar{X}(\lambda, \lambda)$ via the appropriate translation functors. Then by 3.10 in [BV1]

$$R\,\mathrm{Ann}\,\bar{X}({}^L\lambda_r, w(\lambda)[w_0\,{}^L\lambda_r]) = \check{I}(-w(\lambda)\,{}^L\lambda_r) = I(w(\lambda)w_0\,{}^L\lambda_r). \tag{4.4.4}$$

Then there is a finite dimensional $F(\lambda)$ on ${}^L\mathfrak{g}(\lambda)$ such that

$$L(w(\lambda)[w_0\,{}^L\lambda_r]) = U(\mathfrak{g}) \bigotimes_{U({}^L\mathfrak{p})} F_\lambda \tag{4.4.5}$$

The proposition follows.

4.5. Definition. ${}^L\mathcal{O} = \operatorname{ind}_{{}^L\mathfrak{g}(\lambda)}^{{}^L\mathfrak{g}}[\{0\}]$ is called Richardson induced. $\mathcal{O}$ is called coinduced. Suppose λ is dominant integral. By 4.4 there is a coinduced orbit $\mathcal{O}(\lambda)$ attached to it. Let

$$\mathscr{X}(\lambda) = \{\bar{X}(\lambda, w\lambda) \mid WF(\bar{X}(\lambda, w\lambda)) \subset \overline{\mathcal{O}(\lambda)}\},$$

$\mathscr{G}(\lambda)$ = Grothendieck group generated by the characters of elements in $\mathscr{X}(\lambda)$. (4.5.1)

Proposition. *The following are equivalent.*

(1) $\bar{X}(\lambda, w\lambda) \in \mathscr{X}(\lambda)$.

(2) $WF[\bar{X}(\lambda, w\lambda)] = \overline{\mathcal{O}(\lambda)}$.

(3) $L\operatorname{Ann}\bar{X}(\lambda, w\lambda) = R\operatorname{Ann}\bar{X}(\lambda, w\lambda) = \bar{I}(-w_0\lambda) = I(\lambda)$.

Proof. This is the same as 5.20 in [BV 1]. We omit the details.

4.6. The cardinality of $\mathscr{X}(\lambda)$ is computed in the same way as for the special nilpotent representations in section 5 of [BV 1].

Proposition.

$$|\mathscr{X}(\lambda)| = \dim \operatorname{Hom}_W[V^L(w(\lambda)w_0) : V^R(w(\lambda)w_0)]$$

where $V^L(w(\lambda)w_0) \simeq V^R(w(\lambda)w_0)$ are the left and right cell representations respectively.

4.7. We make the following assumption.

Assumption. *$V^L(w(\lambda)w_0)$ decomposes with multiplicity 1 as a representation of W.*

This is true in all cases except for some λ's in the exceptional cases, the most interesting being ${}^L\mathfrak{g}(\lambda) \simeq C_2$ in F_4 and ${}^L\mathfrak{g}(\lambda) \simeq A_5 \times A_1$ in E_8.

For these cases the R_σ in 4.8 do not form a basis. However it is still possible to exhibit each element of $\mathscr{X}(\lambda)$ as an induced representation from a finite dimensional representation on a Levi component. We will give the details of this calculation elsewhere.

4.8. Proposition. *Let $\sigma \in V^L(w(\lambda)w_0)$. Then define*

$$R_\sigma(\lambda) = \frac{1}{|W^\lambda|} \sum_{w \in W} tr\,\sigma(w)\, X(\lambda, w\lambda).$$

Then

$$\{R_\sigma(\lambda)\}_{\sigma \in V^L(w(\lambda)w_0)}$$

forms a basis of $\mathscr{G}(\lambda)$.

Proof. This is the same as 5.31 in [BV 1].

4.9. Theorem. *There is a finite group $\bar{A}(\mathcal{O})$ and correspondences*

$$[x] \in [\bar{A}(\mathcal{O})] \mapsto \sigma_x \in V^L(w(\lambda)w_0)$$
$$\chi \in \hat{\bar{A}}(\mathcal{O}) \mapsto X_\chi \in \mathscr{X}(\lambda) \qquad (4.9.1)$$

such that at the level of characters,

$$X_\chi = \frac{1}{|\bar{A}(\mathcal{O})|} \sum_{[x] \in [\bar{A}(\mathcal{O})]} \operatorname{tr}\chi(x)\, R_{\sigma_x}(\lambda)$$
$$R_{\sigma_x}(\lambda) = \sum_{\chi \in \hat{\bar{A}}(\mathcal{O})} \operatorname{tr}\chi(x)\, X_\chi. \qquad (4.9.2)$$

The proof follows III of [BV 1] closely and will be sketched in the next sections. In particular, $\bar{A}(\mathcal{O})$ is the same as in [L] but the correspondence (4.9.1) is different.

4.10. Definition. Let λ, λ^1 be two dominant (singular) integral weights. We say that λ and λ^1 are linked if ${}^L\mathfrak{g}(\lambda)$ and ${}^L\mathfrak{g}(\lambda^1)$ are linked in the sense of §7 in [BV 1].

The main result about linked parameters is the following.

Proposition. *There is a canonical bijection*

$$P_{\lambda,\lambda^1} : \mathscr{X}(\lambda) \to \mathscr{X}(\lambda^1)$$

which when extended linearly to $\mathscr{G}(\lambda)$ has the property that

$$P_{\lambda,\lambda^1}(R_\sigma(\lambda)) = R_\sigma(\lambda^1).$$

The proof takes up most of §7 in [BV 1].

4.11. Definition. We say λ is final if

$$(\lambda, \check{\alpha}) \leqq 1 \quad \text{for all simple } \alpha \in \Delta^+,$$
$$\|\lambda\| \text{ is minimal among all } \lambda^1 \text{ linked to } \lambda.$$

Given any integral λ there is a final λ_f linked to λ. In view of 4.10 and the Jantzen-Zuckerman translation principle [Z], we may assume λ is final when proving character formulas.

Definition. We say λ is special unipotent if ${}^L\mathcal{O}$ is even and $\|\lambda\|$ is minimal among all λ's attached to $\mathcal{O}$. By [BV 1], $\lambda = \lambda(\mathcal{O}) = 1/2\, {}^Lh({}^L\mathcal{O})$.

4.12. Assume the following.

(1) There is ${}^L\mathfrak{p} = {}^L\mathfrak{m} + {}^L\mathfrak{n}$ such that ${}^L\mathcal{O}$ meets ${}^L\mathfrak{m}$. Let ${}^L\mathcal{O}_{\mathfrak{m}}$ be an orbit in the intersection.

(2) $\mathcal{O}$ is smoothly induced from $\mathcal{O}_{\mathfrak{m}}$.

(3) There is a conjugate $\lambda_{\mathfrak{m}}$ of λ such that

$${}^L\mathcal{O}_{\mathfrak{m}} = \operatorname{ind}_{{}^L\mathfrak{m}(\lambda_{\mathfrak{m}})}^{{}^L\mathfrak{m}}[\{0\}].$$

The unexplained notation is as in §8 of [BV 1].

Under these conditions, $\lambda_{\mathfrak{m}}$, λ give rise to $\mathscr{X}_{\mathfrak{m}}(\lambda_{\mathfrak{m}})$, $\mathscr{X}(\lambda)$ satisfying the following. If $\bar{X}_{\mathfrak{m}} \in \mathscr{X}_{\mathfrak{m}}$, then

$\operatorname{Ind}_{\mathfrak{m}}^{\mathfrak{g}}[\bar{X}_{\mathfrak{m}}]$ has WF-set $\mathcal{O}$ and all its composition factors are in $\mathscr{X}(\lambda)$. (4.12.1)

Proposition. *Let $X_m^1, \ldots, X_m^k$ be the elements of $\mathscr{X}_m(\lambda_m)$. Then*

(1) *$X^i = \mathrm{Ind}_m^g[X_m^i]$ is irreducible.*

(2) *Suppose $\sigma_m^i, \ldots, \sigma_m^k$ are the representations in $V^L(w(\lambda_m)\, w_0(m))$. Because $\mathcal{O}$ is smoothly induced,*

$$\sigma^i = J^W_{W(m)}[\sigma_m^i]$$

are the irreducible representations in $V^L(w(\lambda))$. If

$$X_m^i = \sum_{1 \leq j \leq k} a_{ij}\, R_{\sigma_m^j}(\lambda_m),$$

then

$$X^i = \sum_{1 \leq j \leq k} a_{ij}\, R_{\sigma^j}(\lambda).$$

Proof. The proof in §8 in [BV1] applies to this case. We omit the details.

4.13. Assume now that λ and λ_m are linked and $\mathcal{O}$ and $\mathcal{O}_m$ are the corresponding orbits. Assume, in addition, the following.

(1) There is ${}^L\mathfrak{p} = {}^L\mathfrak{m} + {}^L\mathfrak{n}$ such that ${}^L\mathcal{O}$ is smoothly induced from a special nilpotent ${}^L\mathcal{O}_m$.

(2) ${}^L\mathcal{O}_m = \mathrm{ind}^{L\mathfrak{m}}_{L\mathfrak{m}(\lambda_m)}(0)$.

Proposition.

(1) *If $\sigma_m^1 \ldots \sigma_m^k$ are the representations in $V^L(w(\lambda_m))$ then*

$$\sigma^i = [J^W_{W(m)}(\sigma_m^i \otimes \mathrm{sgn})] \otimes \mathrm{sgn}$$

are the representations in $V^L(w(\lambda))$.

(2) *If $X_m^1 \ldots X_m^k$ are the elements of $\mathscr{X}_m(\lambda_m)$ and*

$$X_m^i = \sum_j a_{ij}\, R_{\sigma_m^j}(\lambda_m)$$

then there is a canonical bijection

$$P_{\lambda_m, \lambda}\colon \mathscr{X}_m(\lambda_m) \to \mathscr{X}(\lambda)$$

such that

$$P_{\lambda_m, \lambda}(X_m^i) = \sum a_{ij}\, R_{\sigma^j}(\lambda).$$

This is proved for special unipotent representations in §7 in [BV1]. The proof applies to this case as well.

Definition. A pair $(\mathcal{O}, \lambda)$ is called *triangular* if it does not satisfy 4.12 or 4.13 for any proper parabolic subalgebra.

For $\lambda = \lambda(\mathcal{O})$ corresponding to special unipotent representations the orbits are described in [L] and the representations in [BV1]. The more general case is treated in detail for $\mathfrak{g}$ classical in Sect. 5.

5. The triangular case

In this section we describe the character theory of $\mathscr{X}(\lambda)$ for $\mathcal{O}$, λ triangular (Definition 4.13).

5.1. Type B, C. $\mathcal{O}$ and ${}^L\mathcal{O}$ have symbol

$$\begin{pmatrix} 0 & & 2 & & \ldots & & 2m \\ & 1 & & 3 & \ldots & 2m-1 & \end{pmatrix}. \tag{5.1.1}$$

The Levi components from which ${}^L\mathcal{O}$ is induced are as follows. For each $k = 0, \ldots, m$ there is a $\lambda(k)$ such that

$${}^L\mathfrak{g}(\lambda(k)) \simeq \mathscr{A}(1) \times \mathscr{A}(3) \times \ldots \times \mathscr{A}(2k-1) \times {}^L\mathfrak{g}(k) \times \mathscr{A}(2k+2) \times \ldots \times \mathscr{A}(2m). \tag{5.1.2}$$

According to proposition 4.6 and [L], $|\mathscr{X}(\lambda(k))| = 2^m$. Define

$$\begin{aligned} \mathfrak{m}(\lambda(k)) \simeq \mathscr{A}(1) \times \ldots \times \mathscr{A}(2m-2k-1) \\ \times \mathfrak{m}(m-k) \times \mathscr{A}(2m-2k+2) \times \ldots \times \mathscr{A}(2m). \end{aligned} \tag{5.1.3}$$

Relabel the factors so that

$$\mathfrak{m}(\lambda(k)) \simeq \mathscr{A}^1 \times \ldots \mathscr{A}^k \times \mathfrak{m}^0 \times \mathscr{A}^{k+2} \times \ldots \times \mathscr{A}^{m+1}, \tag{5.1.4}$$

where $\mathscr{A}^l = \mathscr{A}(2m-2l+2)$ for $1 \leqq l \leqq k$, $\mathscr{A}^l = \mathscr{A}(2m-2l+3)$ for $k+2 \leqq l \leqq m+1$ and $\mathfrak{m}^0 = \mathfrak{m}(m-k)$. Then let $\hat{\varepsilon}_l = \pm 1$ for all $l = 1, \ldots, k, k+2, \ldots, m+1$. For each such $\{\hat{\varepsilon}_l\}$ we define a finite dimensional representation

$$\mathscr{F}_{\hat{\varepsilon}_l}(\lambda(k)) = \bigotimes_{1 \leqq l \leqq k} \mathscr{F}^l_{\hat{\varepsilon}_l}(\lambda(k)) \otimes \bar{X}(\lambda^0(k), \lambda^0(k)) \otimes \bigotimes_{k+2 \leqq l \leqq m+1} \mathscr{F}^l_{\hat{\varepsilon}_l}(\lambda(k)) \tag{5.1.5}$$

of $\mathfrak{m}(\lambda(k))$ as follows. Let

$$\begin{aligned} &F^l(\lambda(k))\colon (m-l+1, \ldots, 1, 0, -1, \ldots, -m+l) \quad \text{for } 1 \leqq l \leqq k, \\ &F^0(\lambda(k))\colon (m-k, \ldots, 1), \\ &F^l(\lambda(k))\colon (m-l+2, \ldots, 1, -1, \ldots, -m+l-1) \quad \text{for } k+2 \leqq l \leqq m+1. \end{aligned} \tag{5.1.6}$$

For $1 \leqq l \leqq k$ or $k+2 \leqq l \leqq m+1$, let

$$\mathscr{F}^l_{+1}(\lambda(k)) = F^l(\lambda(k)) \otimes F^l(\lambda(k)), \qquad \mathscr{F}^l_{-1}(\lambda(k)) = F^l(\lambda(k)) \otimes (F^l(\lambda(k)))^*, \tag{5.1.7}$$

where $*$ denotes the dual. For $l = 0$ write

$$\bar{X}(\lambda^0(k), \lambda^0(k)) = F^0(\lambda(k)) \otimes F^0(\lambda(k)). \tag{5.1.8}$$

Let

$$X_{\{\varepsilon_l\}}(\lambda(k)) = \mathrm{Ind}_{\mathfrak{m}(\lambda)}^{\mathfrak{g}}\Big[\bigotimes_{1\leq l\leq k} \mathscr{F}^l_{\varepsilon_l}(\lambda(k)) \otimes \bar{X}(\lambda^0(k), \lambda^0(k)) \otimes \bigotimes_{k+2\leq l\leq m+1} \mathscr{F}^l_{\varepsilon_l}(\lambda(k))\Big]. \tag{5.1.9}$$

Because of the translation principle, the character theory of any $\mathscr{X}(\lambda)$ attached to $\mathcal{O}$ is equivalent to the character theory of some $\mathscr{X}(\lambda(k))$.

Proposition. *$X_{\{\varepsilon_l\}}(\lambda(k))$ is irreducible. In particular*

$$\mathscr{X}(\lambda(k)) = \{X_{\{\varepsilon_l\}}(\lambda(k))\}.$$

Proof. This is the same as the proof for special unipotents in [BV1] Sect. 5–9.

We will drop the "$(\lambda(k))$" from the notation whenever we deal with just one k.

Remark. All $\lambda(k)$ are final except for $k=0$ in type **B**. In this case the final λ should have half-integers as coordinates.

From [L] we recall that $V^L(w(\lambda)w_0)$ is formed of the following representations.

Consider the m pairs

$$(01)\dots(2k-2, 2k-1)(2k+1, 2k+2)\dots(2m-1, 2m). \tag{5.1.10}$$

For $\varepsilon_l = \pm 1$ with $1\leq l\leq k$ or $k+2\leq l\leq m+1$, let $\sigma_{\{\varepsilon_l\}}$ be the representation with symbol obtained from 5.1.1 by interchanging the elements in the pairs for which $\varepsilon_l = -1$. This gives a map

$$(\mathbb{Z}_2)^m \to V^L(w(\lambda)w_0). \tag{5.1.11}$$

The representations of $V^L(w(\lambda))$ are obtained in the same way from the pairs obtained by replacing each entry in (5.1.10) by its difference to $2m$.

Example. Let $m=3$. Then we have

(1) $k=3$, λ: $(3, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0)$,

$$\begin{aligned} &{}^L\mathfrak{g}(\lambda) = \mathscr{A}(1)\times\mathscr{A}(3)\times\mathscr{A}(5)\times{}^L\mathfrak{g}(3),\\ &\mathfrak{m}(\lambda) = \mathscr{A}^1\times\mathscr{A}^2\times\mathscr{A}^3 \simeq \mathscr{A}(6)\times\mathscr{A}(4)\times\mathscr{A}(2),\\ &V^L(w(\lambda)w_0)\colon (01)(23)(45), \quad V^L(w(\lambda))\colon (56)(34)(12). \end{aligned}$$

(2) $k=2$, λ: $(3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0)$,

$$\begin{aligned} &{}^L\mathfrak{g}(\lambda) = \mathscr{A}(1)\times\mathscr{A}(3)\times{}^L\mathfrak{g}(2)\times\mathscr{A}(6),\\ &\mathfrak{m}(\lambda) = \mathscr{A}^1\times\mathscr{A}^2\times\mathfrak{m}^0\times\mathscr{A}^4 \simeq \mathscr{A}(6)\times\mathscr{A}(4)\times\mathfrak{m}(1)\times\mathscr{A}(1),\\ &V^L(w(\lambda)w_0)\colon (01)(23)(56), \quad V^L(w(\lambda))\colon (56)(34)(01). \end{aligned}$$

(3) $k=1$, λ: $(3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0)$,

$$\begin{aligned} &{}^L\mathfrak{g}(\lambda) = \mathscr{A}(1)\times{}^L\mathfrak{g}(1)\times\mathscr{A}(4)\times\mathscr{A}(6),\\ &\mathfrak{m}(\lambda) = \mathscr{A}^1\times\mathfrak{m}^0\times\mathscr{A}^3\times\mathscr{A}^4 \simeq \mathscr{A}(6)\times\mathfrak{m}(2)\times\mathscr{A}(3)\times\mathscr{A}(1),\\ &V^L(w(\lambda)w_0)\colon (01)(34)(56), \quad V^L(w(\lambda))\colon (56)(23)(01). \end{aligned}$$

(4) $k=0$, λ: $(3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1)$,

$$\begin{aligned} &{}^L\mathfrak{g}(\lambda) = \mathscr{A}(2)\times\mathscr{A}(4)\times\mathscr{A}(6),\\ &\mathfrak{m}(\lambda) = \mathfrak{m}^0\times\mathscr{A}^2\times\mathscr{A}^3\times\mathscr{A}^4 \simeq \mathfrak{m}(3)\times\mathscr{A}(5)\times\mathscr{A}(3)\times\mathscr{A}(1),\\ &V^L(w(\lambda)w_0)\colon (12)(34)(56), \quad V^L(w(\lambda))\colon (45)(23)(01). \end{aligned}$$

5.2. Type D. $\mathcal{O}$ and ${}^L\mathcal{O}$ have symbol

$$\begin{pmatrix} 1 & 3 & 5 & \dots & 2m+1 \\ 0 & 2 & 4 & \dots & 2m \end{pmatrix}. \tag{5.2.1}$$

The Levi components from which ${}^L\mathcal{O}$ is Richardson induced are as follows.

For each $k=1, \dots, m$ there is a $\lambda(k)$ such that

$${}^L\mathfrak{g}(\lambda(k)) = \mathscr{A}(2)\times\dots\times\mathscr{A}(2k)\times{}^L\mathfrak{g}(k+1)\times\mathscr{A}(2k+3)\times\dots\times\mathscr{A}(2m+1). \tag{5.2.2}$$

According to 4.6 and [L], $|\mathscr{X}(\lambda)| = 2^m$. Define

$$\begin{aligned} \mathfrak{m}(\lambda(k)) = \mathscr{A}(1)\times\dots\times\mathscr{A}(2m-2k-1)\times\mathfrak{m}(2m-k+1)\\ \times A(2m-2k+2)\times\dots\times\mathscr{A}(2m). \end{aligned} \tag{5.2.3}$$

Relabel the factors so that

$$\mathfrak{m}(\lambda(k)) = \mathscr{A}^1\times\dots\times\mathscr{A}^k\times\mathfrak{m}^0\times\mathscr{A}^{k+2}\times\dots\times\mathscr{A}^{m+1}, \tag{5.2.4}$$

where $\mathscr{A}^l = \mathscr{A}(2m-2l+2)$ for $1\leq l\leq k$, $\mathscr{A}^l = \mathscr{A}(2m+2l+3)$ for $k+2\leq l\leq m+1$, $\mathfrak{m}^0 = \mathfrak{m}(2m-k+1)$. Let $F^l(\lambda(k))$ and $\bar{X}(\lambda^0(k), \lambda^0(k))$ be such that

$$\begin{aligned} &F^l(\lambda(k))\colon (m-l+2, \dots, 1, -1, \dots, -m+l+1), \quad k+2\leq l\leq m+1,\\ &F^l(\lambda(k))\colon (m-l+1, \dots, 1, 0, -1, \dots, -m+l), \quad 1\leq l\leq k,\\ &\lambda^0(k)\colon (m, \dots, 1, 0, -1, \dots, -m+k). \end{aligned} \tag{5.2.5}$$

Let

$$\mathscr{F}^l_1(\lambda(k)) = F^l(\lambda(k))\otimes F^l(\lambda(k)), \quad \mathscr{F}^l_{-1}(\lambda(k)) = F^l(\lambda(k))\otimes(F^l(\lambda(k)))^*, \tag{5.2.6}$$

where $*$ denotes the dual. For $\bar{\varepsilon}_l = \pm 1$ where $l = 1, \dots, k, k+2, \dots, m+1$, let

$$X_{\{\varepsilon_l\}}(\lambda(k)) = \mathrm{Ind}_{\mathfrak{m}(\lambda(k))}^{\mathfrak{g}}\Big[\bigotimes_{1\leq l\leq k} \mathscr{F}^l_{\varepsilon_l}(\lambda(k))\otimes\bar{X}(\lambda^0(k), \lambda^0(k))\otimes\bigotimes_{k+2\leq l\leq m+1}\mathscr{F}^l_{\varepsilon_l}(\lambda(k))\Big]. \tag{5.2.7}$$

For $k=0$ there are several possibilities. There are $\lambda^\pm(0)$ such that

$${}^L\mathfrak{g}(\lambda^\pm(0)) = \mathscr{A}(1)\times\dots\times\mathscr{A}(2m+1). \tag{5.2.8}$$

In addition there is $\lambda'(0)$ such that

$$^{L}\mathfrak{g}(\lambda'(0)) = {}^{L}\mathfrak{g}(1) \times \mathscr{A}(3) \times \ldots \times \mathscr{A}(2m+1). \tag{5.2.9}$$

Set

$$\mathfrak{m}(\lambda(0)) = \mathscr{A}^1 \times \mathscr{A}^2 \times \ldots \times \mathscr{A}^{m+1}, \tag{5.2.10}$$

where $\mathscr{A}^l = \mathscr{A}(2m+3-2l)$. For $\lambda^{\pm}(0)$ let

$$\begin{aligned} &F^l(\lambda^{\pm}(0))\colon\ ((2m+1-2l)/2, \ldots, 1/2, -1/2, \ldots, (-2m+1+2l)/2), \quad l \neq m+1,\\ &F^{m+1}(\lambda^{+0}(0))\colon\ (1/2)\\ &F^{m+1}(\lambda^{-0}(0))\colon\ (-1/2). \end{aligned} \tag{5.2.11}$$

Form

$$\mathscr{F}^l_1(\lambda^{\pm}(0)) = F^l(\lambda^{\pm}(0)) \otimes F^l(\lambda^{\pm}(0)), \quad \mathscr{F}^l_{-1}(\lambda^{\pm}(0)) = F^l(\lambda^{\pm}(0)) \otimes (F^l(\lambda^{\pm}(0)))^*. \tag{5.2.12}$$

Let $\mathscr{E}_l = \pm 1$, for $l = 2, \ldots, m+1$. Then let $\mathscr{E}_1 = \prod_{2 \leq l \leq m+1} \mathscr{E}_l$. Then form

$$X_{\{\mathscr{E}_l\}}(\lambda^{\pm}(0)) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{m}}\Big[\bigotimes_{1 \leq l \leq m+1} \mathscr{F}^l_{\mathscr{E}_l}(\lambda^{\pm}(0))\Big], \tag{5.2.13}$$

where $2 \leq l \leq m+1$ in the left hand side. For $\lambda'(0)$, let

$$\begin{aligned} &F^l(\lambda'(0))\colon\ (m-l+2, \ldots, 1, -1, \ldots, m+l+1) \quad \text{for } 2 \leq l \leq m+1,\\ &\lambda'(0)\colon\ (m, \ldots, 1, 0, -1, \ldots, -m). \end{aligned} \tag{5.2.14}$$

Form

$$\mathscr{F}^l_1(\lambda'(0)) = F^l(\lambda'(0)) \otimes F^l(\lambda'(0)), \quad \mathscr{F}^l_{-1}(\lambda'(0)) = F^l(\lambda'(0)) \otimes (F^l(\lambda'(0)))^*. \tag{5.2.15}$$

If $\mathscr{E}_l = \pm 1$, for $l = 2, \ldots, m+1$, let

$$X_{\{\mathscr{E}_l\}}(\lambda'(0)) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{m}}\Big[\bar{X}(\lambda'^0, \lambda'^0) \otimes \bigotimes_l \mathscr{F}^l_{\mathscr{E}_l}(\lambda'(0))\Big]. \tag{5.2.16}$$

Because of the translation principle the character theory of any $\mathscr{X}(\lambda)$ is equivalent to the character theory of one of the $\mathscr{X}(\lambda(k))$. Thus we may assume $\lambda = \lambda^{\varepsilon}(k)$ where ε is present only when $k=0$; and in that case equals $\pm$ or $'$.

We will drop the "$\lambda^{\varepsilon}(k)$" whenever we are dealing with a fixed k and ε.

Proposition. *In all cases $X_{\{\mathscr{E}_l\}}(\lambda^{\varepsilon})$ is irreducible. Furthermore,*

$$\mathscr{X}(\lambda(k)) = \{X_{\{\mathscr{E}_l\}}(\lambda^{\varepsilon})\}.$$

Proof. This is again as in 5–9 [BV1].

To get character formulas we also need to know the characters of the $\bar{X}(\lambda^{\varepsilon 0}(k), \lambda^{\varepsilon 0}(k))$.

Let

$$\mathscr{O}_0(k) = WF(\bar{X}(\lambda^{\varepsilon 0}(k)). \tag{5.2.17}$$

Then the pairs $(\mathscr{O}_0(k), \lambda^{\varepsilon 0}(k))$ are of the type dealt with in Sect. 4. In particular, except for the case $k=0$, $\varepsilon = \pm$, they are special unipotent as in [BV1]. In this case $\bar{X}(\lambda^{\pm 0}(0))$ is induced irreducible from a 1-dimensional on a maximal type A Levi component.

In all cases

$$|\mathscr{X}(\lambda^{\varepsilon 0}(k))| = 1. \tag{5.2.18}$$

Remark. $\lambda(k)$ is final for $k \neq 0$. For $k=0$, $\lambda^{\pm}(0)$ are final related by the outer automorphism of D_n. $\lambda'(0)$ is linked to them. In addition, in type D_4 even $\lambda'(0)$ is final.

Consider the m pairs

$$(0,1)\ldots(1,2)\ldots(2k-1,2k)(2k+2,2k+3)\ldots(2m-1,2m). \tag{5.2.19}$$

For $\{\varepsilon_l = \pm 1\}$ with $1 \leq l \leq k$, $k+2 \leq l \leq m+1$ let $\sigma_{\{\varepsilon_l\}}$ be the representation with symbol obtained from 5.2.1 by permuting the pairs for which $\varepsilon_l = -1$. This gives a map

$$(\mathbb{Z}_2)^m \to V^L(w(\lambda)\, w_0). \tag{5.2.20}$$

The representations in $V^L(w(\lambda))$ are obtained in the same way from the pairs obtained from (5.2.18) by replacing each entry by its difference to $2m+1$.

Example. Let $m=3$. Then we have

(1) $k=3$, λ: $(3,3,2,2,2,2,1,1,1,1,1,1,0,0,0,0)$,

$$\begin{aligned} &^{L}\mathfrak{g}(\lambda) = \mathscr{A}(2) \times \mathscr{A}(4) \times \mathscr{A}(6) \times {}^{L}\mathfrak{g}(4),\\ &\mathfrak{m}(\lambda) = \mathscr{A}^1 \times \mathscr{A}^2 \times \mathscr{A}^3 \times \mathfrak{m}^0 \simeq \mathscr{A}(6) \times \mathscr{A}(4) \times \mathscr{A}(2) \times \mathfrak{m}(4),\\ &V^L(w(\lambda)\, w_0)\colon\ (12)(34)(56), \qquad V^L(w(\lambda))\colon\ (56)(34)(12). \end{aligned}$$

(2) $k=2$, λ: $(3,3,2,2,2,2,1,1,1,1,1,1,1,0,0,0)$,

$$\begin{aligned} &^{L}\mathfrak{g}(\lambda) = \mathscr{A}(2) \times \mathscr{A}(4) \times {}^{L}\mathfrak{g}(3) \times \mathscr{A}(7),\\ &\mathfrak{m}(\lambda) = \mathscr{A}^1 \times \mathscr{A}^2 \times \mathfrak{m}^0 \times \mathscr{A}^4 \simeq \mathscr{A}(6) \times \mathscr{A}(4) \times \mathfrak{m}(5) \times \mathscr{A}(1),\\ &V^L(w(\lambda)\, w_0)\colon\ (12)(34)(67), \qquad V^L(w(\lambda))\colon\ (56)(34)(01). \end{aligned}$$

(3) $k=1$, λ: $(3,3,2,2,2,2,2,1,1,1,1,1,1,1,0,0)$,

$$\begin{aligned} &^{L}\mathfrak{g}(\lambda) = \mathscr{A}(2) \times {}^{L}\mathfrak{g}(2) \times \mathscr{A}(5) \times \mathscr{A}(7),\\ &\mathfrak{m}(\lambda) = \mathscr{A}^1 \times \mathfrak{m}^0 \times \mathscr{A}^3 \times \mathscr{A}^4 \simeq \mathscr{A}(6) \times \mathfrak{m}(6) \times \mathscr{A}(3) \times \mathscr{A}(1).\\ &V^L(w(\lambda)\, w_0)\colon (12)(45)(67), \qquad V^L(w(\lambda))\colon (56)(23)(01). \end{aligned}$$

(4) $k=0$, $\lambda^{\pm}$: $(7/2, \underbrace{5/2 \ldots 5/2}_{3}, \underbrace{3/2, \ldots, 3/2}_{5}, \underbrace{1/2, \ldots, \pm 1/2}_{7})$

$$\lambda'\colon (3,3,3,2,2,2,2,2,1,1,1,1,1,1,1,0).$$

$$\begin{aligned}{}^L\mathfrak{g}(\lambda) &= \mathscr{A}(1)\times\mathscr{A}(3)\times\mathscr{A}(5)\times\mathscr{A}(7) \quad \text{or}\\ &= \mathfrak{m}(1)\times\mathscr{A}(3)\times\mathscr{A}(5)\times\mathscr{A}(7),\\ \mathfrak{m}(\lambda) &= \mathfrak{m}^0\times\mathscr{A}^2\times\mathscr{A}^3\times\mathscr{A}^4 \simeq \mathfrak{m}(7)\times\mathscr{A}(5)\times\mathscr{A}(3)\times\mathscr{A}(1).\\ V^L(w(\lambda)\,w_0)&\colon (23)(45)(67), \qquad V^L(w(\lambda))\colon (45)(23)(01).\end{aligned}$$

Definition. For type D denote by $\mathscr{A}^0\subset\mathfrak{m}^0$, $\mathscr{F}^{\varepsilon 0}(\lambda(k))=F^{\varepsilon 0}(\lambda(k))\otimes F^{\varepsilon 0}(\lambda(k))$ a maximal type A Levi component and finite dimensional representation such that $\bar{X}(\lambda^{\varepsilon 0}(k),\lambda^{\varepsilon 0}(k))$ is the spherical subquotient in

$$\mathrm{Ind}_{\mathscr{A}^0}^{\mathfrak{m}^0}[\mathscr{F}^{\varepsilon 0}(\lambda(k))].$$

For type B, C we will use the similar notation.

5.3. We summarize the results obtained so far.

Proposition. *Let $\mathcal{O}$ be triangular and $\lambda(k)$ be as in 5.1, 5.2. If we identify $\hat{\varepsilon}_i$ in 5.1.9, 5.2.9 with the character of $(\mathbb{Z}_2)^m$ satisfying*

$$(\hat{\varepsilon}_i,\varepsilon_j)=\begin{cases}1 & i\neq j,\\ -1 & i=j,\end{cases}$$

then

$$X_\pi=\frac{1}{2^m}\sum_{x\in(\mathbb{Z}_2)^m}\operatorname{tr}\pi(x)\,R_{\sigma_x},$$

$$R_{\sigma_x}=\sum_{\pi\in\widehat{(\mathbb{Z}_2)^m}}\operatorname{tr}\pi(x)\,X_\pi.$$

The correspondences $x\mapsto\sigma_x$ and $\pi\mapsto X_\pi$ are defined explicitly in 5.1–5.2.

The proof of Theorem 4.9 follows if these correspondences are extended to the other cases to be compatible with the reduction techniques in Sect. 4.

6. A parabolic subalgebra

Assume the infinitesimal character is integral.

In analogy with $\mathfrak{m}(\lambda)$ defined in (5.1.3), (5.2.3), we define an $\mathfrak{m}(\lambda)$ and representations $\bar{X}(\lambda^0,\lambda^0)$ and $\mathscr{F}^l_{\bar{X}}$ for any λ. The main property will be that if the corresponding orbit $\mathcal{O}$ is induced from $\mathfrak{m}(\lambda)$ then $\bar{X}(\lambda,\lambda)$ is induced irreducible.

We will omit the subscript "$\bar{X}$" whenever there is no ambiguity.

6.1. Type A. The Levi component ${}^L\mathfrak{g}(\lambda)$ corresponds to a partition $n_1\geqq\ldots\geqq n_k$ of n. Let $\mathfrak{m}(\lambda)$ be the Levi component corresponding to the transpose partition. If we write

$$\mathfrak{m}(\lambda)=\mathscr{A}^1\times\ldots\times\mathscr{A}^k, \tag{6.1.1}$$

then there are finite dimensional $\mathscr{F}^l$ such that $\bar{X}(\lambda,\lambda)$ is the spherical subquotient of

$$\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{m}(\lambda)}\Big(\bigotimes_l \mathscr{F}^l\Big). \tag{6.1.2}$$

The $\mathscr{F}^l$ are unique. By Sects. 4–5, $\bar{X}(\lambda,\lambda)$ equals (6.1.2). (This particular result was conjectured by Enright and proved by D. Vogan and myself some time ago.)

The $\mathscr{F}^l$ are obtained as follows. Write λ in coordinates,

$$\lambda\colon (\lambda_1,\ldots,\lambda_1,\lambda_2,\ldots,\lambda_2,\ldots,\lambda_s,\ldots\lambda_s), \qquad \lambda_i>\lambda_{i+1}. \tag{6.1.3}$$

We call the λ_i entries.

Denote by $r(\lambda_i)$ the number of coordinates of λ equal to λ_i in (6.1.3). Then $\{n_i\}=\{r(\lambda_i)\}$. Let

$$F^1\colon (\lambda_1,\lambda_2,\ldots,\lambda_s). \tag{6.1.4}$$

The remainder has $r(\lambda_i)=\max\{r(\lambda_i)-1,0\}$.

The coordinates of F^2 are obtained in the same way from the remainder. Similarly for the other F^l. If $r(x+1)\leqq r(x)$ for all coordinates, then the F^l are 1-dimensional.

The ranks of the $\mathscr{A}^i$ are given by the transpose partition to $\{n_i\}$.

6.2. Type B, C. The representations in the left cell $V^L(w(\lambda))$ corresponding to ${}^L\mathcal{O}$ are parametrized by a symbol

$$(x_0,x_1)\ldots(x_{2m-2k-2},x_{2m-2k-1})(x_{2m-2k})(x_{2m-2k+1},x_{2m-2k+2})\ldots(x_{2m-1},x_{2m}) \tag{6.2.1}$$

satisfying $x_{2i-1}<x_{2i}$ for $i<m-k$, and $x_{2j}<x_{2j+1}$ for $j>m-k$. When $x_{2i-2}=x_{2i-1}$ for $1\leqq i\leqq m-k$ or $x_{2j+1}=x_{2j+2}$ for $m-k\leqq j<m$, the nilpotent $\mathcal{O}$ is smoothly induced from a Levi component of the form

$$\begin{aligned}\mathfrak{m}=\mathfrak{g}(*)\times\mathscr{A}(x_s+x_{s+1}-s), \quad &\text{for } 0\leqq s=2i-2\leqq 2m-2k-2\\ &\text{or } \ 2m-2k+1\leqq s=2j+1\leqq 2m-1,\end{aligned} \tag{6.2.2}$$

and a nilpotent orbit which is trivial on the $\mathscr{A}$ factor.

When $x_{2i-2}+1=x_{2i-1}$ for $1\leqq i\leqq m-k$ or $x_{2j+1}+1=x_{2j+2}$ for $m-k\leqq j<m$, the nilpotent $\mathcal{O}$ is induced but not smoothly from a Levi component of the same form as in (6.2.2).

The reduction techniques in 4 and the triangular case in 5 show that in either of these cases $\bar{X}(\lambda,\lambda)$ is induced irreducible from a representation on $\mathfrak{m}$ which is a finite dimensional representation of $\mathscr{A}(x_s+x_{s+1}-s)$.

The symbol determines a Levi component

$$\mathfrak{m}=\mathscr{A}^1\times\ldots\times\mathscr{A}^k\times\mathfrak{m}^0\times\mathscr{A}^{k+2}\times\ldots\times\mathscr{A}^{m+1}, \tag{6.2.3}$$

where

$$\mathfrak{m}^0 = \mathfrak{g}(x_{2m-2k} - m + k),$$
$$\mathscr{A}^i = \mathfrak{gl}(x_{2m-2i+2} + x_{2m-2i+1} - 2m + 2i - 1) \quad \text{for } 1 \leqq i \leqq k, \tag{6.2.4}$$
$$\mathscr{A}^j = \mathfrak{gl}(x_{2m-2j+3} + x_{2m-2j+2} - 2(m-j+1)) \quad \text{for } k+2 \leqq j \leqq m+1.$$

We will define finite dimensional $\mathscr{F}^l$ for $1 \leqq l \leqq k$, $k+2 \leqq l \leqq m+1$ and $\bar{X}(\lambda^0, \lambda^0)$, such that $\bar{X}(\lambda, \lambda)$ is a subquotient of

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{m}(\lambda)}[\bar{X}(\lambda^0, \lambda^0) \otimes \bigotimes_l \mathscr{F}^l]. \tag{6.2.5}$$

Let $\mathscr{A}^0 \subset \mathfrak{m}^0$ be a maximal type A Levi component. Then we also define a finite dimensional representation $\mathscr{F}^0$ of $\mathscr{A}^0$ such that $\bar{X}(\lambda^0, \lambda^0)$ is the spherical subquotient in $\operatorname{Ind}^{\mathfrak{m}^0}_{\mathscr{A}^0}[\mathscr{F}^0]$.

The $\mathscr{F}^l$ are as follows. Write λ in coordinates,

$$\lambda\colon (\lambda_1, \ldots, \lambda_1, \ldots, \lambda_s, \ldots, \lambda_s), \quad \text{with } \lambda_i - \lambda_{i+1} > 0, \lambda_s \geqq 0. \tag{6.2.6}$$

Assume $r(0) > 0$. Then we form F^1 as follows.

Extract one coordinate for each entry. This gives a decreasing sequence

$$(\lambda_1, \ldots, \lambda_s) \tag{6.2.7}$$

of length $x_{2m} - m + 1$. Then adjoin to it negatives of coordinates, one for each nonzero entry in the remainder. This adds $x_{2m-1} - m$ coordinates. These form the coordinates of F^1. Similarly, form $F^2, \ldots, F^k$ from the remainder. At each step extract a sequence of decreasing coordinates, one for each nonnegative entry. This gives $x_{2m-2i+2} - (m-i)$ coordinates. Then adjoin a decreasing sequence of negatives of coordinates, one for each nonzero entry in the remainder. This adds $x_{2m-2i+1} - (m-i+1)$ coordinates. k is determined by the fact that there are no zeroes left in the remainder. In particular, $k = r(0)$.

λ^0 is obtained by extracting one coordinate for each entry in the remainder. This gives a decreasing sequence of length $x_{2m-2k} - m + k$. It also determines the coordinates of F^0 by restriction to $\mathscr{A}^0$.

$F^{k+2}, \ldots, F^{m+1}$ are obtained in the same way as $F^1, \ldots, F^k$. At each step one extracts a sequence of decreasing coordinates, one for each entry. This gives $x_{2m-2j+3} - (m-j+1)$ coordinates. Then one adjoins a decreasing sequence of negatives of coordinates, one for each entry in the remainder. This adds $x_{2m-2j+2} - (m-j+1)$ coordinates. $F^{k+2}, \ldots, F^{m+1}$ occur precisely when λ has a coordinate x such that $r(x) > 2r(0) + 1$.

In case $r(0) = 0$, we start with the algorithm for λ^0 and continue in the same way to get $F^2, \ldots, F^{m+1}$. The same formulas as above hold with $k = 0$.

$\bar{X} \in \mathscr{X}(\lambda)$. The other elements of $\mathscr{X}(\lambda)$ are obtained in a way similar to section 5. For each (x_s, x_{s+1}) with $x_s \neq x_{s+1}$, with s as in (6.2.2), let $\{\varepsilon_l = \pm 1\}$ for $l = m - i$ or $m + 2 - j$. Let $\mathscr{F}^l_1 = F^l \otimes F^l$ and $\mathscr{F}^l_{-1} = F^l \otimes \tilde{F}^l$ where $\tilde{F}^l$ is obtained by changing the sign of smallest positive coordinate so that its negative does not occur in F^l and reordering in decreasing order. Then let $\bar{X}_{(\varepsilon_l)}$ be the lowest K-type subquotient of

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{m}(\lambda)}[\bar{X}(\lambda^0, \lambda^0) \otimes \bigotimes_l \mathscr{F}^l_{\varepsilon_l}]. \tag{6.2.8}$$

6.3. Type D. The left cell $V^L(w(\lambda) w_0)$ corresponding to ${}^L\mathcal{O}$ has symbol

$$(x_0, x_1) \ldots (x_{2m-2k-2}\, x_{2m-2k-1})(x_{2m-2k} x_{2m+1})$$
$$\cdot (x_{2m-2k+1}\, x_{2m-2k+2}) \ldots (x_{2m-1}\, x_{2m}) \tag{6.3.1}$$

satisfying $x_{2i-1} < x_{2i}$ for $i < m-k$, and $x_{2j} < x_{2j+1}$ for $j > m-k$. When $x_{2i-2} = x_{2i-1}$ for $1 \leqq i \leqq m-k$ or $x_{2j+1} = x_{2j+2}$ for $m-k \leqq j \leqq m-1$, the nilpotent $\mathcal{O}$ is smoothly induced from a Levi component of the form

$$\mathfrak{m} = \mathfrak{g}(*) \times \mathscr{A}(x_s + x_{s+1} - s), \quad \text{for } 0 \leqq s = 2i - 2 \leqq 2m - 2k - 2$$
$$\text{or} \quad 2m - 2k + 1 \leqq s = 2j + 1 \leqq 2m - 1, \tag{6.3.2}$$

and a nilpotent which is trivial on the $\mathscr{A}$ factor.

When $x_{2i-2} + 1 = x_{2i-1}$ for $1 \leqq i \leqq m-k$ or $x_{2j+1} + 1 = x_{2j+2}$ for $m-k \leqq j < m$, the nilpotent $\mathcal{O}$ is induced but not smoothly from a Levi component of the same form as in (6.2.3). The analogous results hold when $k = 0$ and $x_{2m} = x_{2m+1}$ or $x_{2m} + 1 = x_{2m+1}$.

Let

$$\mathfrak{m}(\lambda) = \mathscr{A}^1 \times \ldots \times \mathscr{A}^k \times \mathfrak{m}^0 \times \mathscr{A}^{k+2} \times \ldots \times \mathscr{A}^{m+1} \tag{6.3.3}$$

be given by

$$\mathfrak{m}^0 = \mathfrak{g}(x_{2m-2k} + x_{2m+1} + k - 2m),$$
$$\mathscr{A}^i = \mathfrak{gl}(x_{2m-2i+2} + x_{2m-2i+1} - (2m - 2i + 1)) \quad \text{for } 1 \leqq i \leqq k, \tag{6.3.4}$$
$$\mathscr{A}^j = \mathfrak{gl}(x_{2m-2j+3} + x_{2m-2j+2} - 2(m-j+1)) \quad \text{for } k+2 \leqq j \leqq m.$$

Then we will define a finite dimensional representation $\mathscr{F}^l$ of $\mathscr{A}^l$ and a representation $\bar{X}(\lambda^0, \lambda^0)$ of $\mathfrak{m}^0$ such that $\bar{X}(\lambda, \lambda)$ is the spherical subquotient of

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{m}}[\bar{X}(\lambda^0, \lambda^0) \otimes \bigotimes_l \mathscr{F}^l]. \tag{6.3.5}$$

In this case $\bar{X}(\lambda^0, \lambda^0)$ is not necessarily finite dimensional. Let $\mathscr{A}^0 \subset \mathfrak{m}^0$ be a maximal type A Levi component. Then we define a finite dimensional $\mathscr{F}^0$ such that $\bar{X}(\lambda^0, \lambda^0)$ is the spherical subquotient in

$$\operatorname{Ind}^{\mathfrak{m}^0}_{\mathscr{A}^0}(\mathscr{F}^0). \tag{6.3.6}$$

The $\mathscr{F}^l$ are obtained as follows. Write λ in coordinates,

$$\lambda\colon (\lambda_1, \ldots, \lambda_1, \ldots, \lambda_s, \ldots, \pm\lambda_s), \quad \text{with } \lambda_i - \lambda_{i+1} > 0, \quad \lambda_s \geqq 0. \tag{6.3.7}$$

Assume $r(0) > 0$. Then extract

$$\tilde{\lambda}^0\colon (\lambda_1, \ldots, \lambda_s) \tag{6.3.8}$$

of lengths $s = x_{2m+1} - m$ from λ.

Then $F^1, \ldots, F^k$ are obtained as in type B, C. For each F^i, extract one coordinate for each entry. This gives a decreasing sequence of length $x_{2m-2i+2} - (m-i)$. Adjoin to it negatives of coordinates, one for each nonzero entry in the remainder. This adds $x_{2m-2i+1} - (m-i+1)$ coordinates. The $\mathscr{A}^i$ are as in 6.3.4. k is determined by the fact that there are no zeroes in the remainder. In particular, $k = r(0) - 1$.

λ^0 is obtained by adjoining negatives of coordinates to $\tilde{\lambda}^0$, one from each entry in the remainder. This adds a decreasing sequence of length $x_{2m-2k} - m + k$. It also determines the coordinates of F^0 by restriction to $\mathscr{A}^0$.

$F^{k+2}, \ldots, F^{m+1}$ are obtained in the same way as $F^1, \ldots, F^k$. At each step one extracts a sequence of decreasing coordinates, one from each entry. This gives $x_{2m-2j+3} - (m-j+1)$ coordinates. Then one adjoins a decreasing sequence of negatives of coordinates, one one for each entry in the remainder. This adds $x_{2m-2j+2} - (m-j+1)$ coordinates.

$F^{k+2}, \ldots, F^{m+1}$ occur precisely when there is λ_i such that $r(\lambda_i) > 2r(0)$.

In case $r(0) = 0$ start with the algorithm for F^1 and continue in the same way to get $F^2, \ldots, F^{m+1}$. The same formulas as above hold with $k = 0$. $\bar{X}(\lambda, \lambda) \in \mathscr{X}(\lambda)$. The other elements of $\mathscr{X}(\lambda)$ are obtained as follows.

For each (x_s, x_{s+1}) with s as in (6.3.2) such that $x_s \neq x_{s+1}$, let $\{\varepsilon_l = \pm 1\}$ for $l = m+2-i$ or $m-j$. Let $\mathscr{F}^l_1 = F^l \otimes F^l$ and $\mathscr{F}^l_{-1} = F^l \otimes \bar{F}^l$, where $\bar{F}^l$ is obtained by changing the smallest positive coordinate of F^l whose negative does not occur, to its negative and reordering in decreasing order. Then let $\bar{X}_{\{\varepsilon_l\}}$ be the lowest K-type subquotient of

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{m}(\lambda)}[\bar{X}(\lambda^0, \lambda^0) \otimes \bigotimes \mathscr{F}^l_{\varepsilon_l}]. \tag{6.3.9}$$

In case there are no zeroes in the coordinates of λ form $\{\varepsilon_l = \pm 1\}$, for $l = m+1-i$ such that the corresponding $x_s \neq x_{s+1}$. Then let $\varepsilon_0 = \prod_{2 \leq l \leq m+1} \varepsilon_l$. Then $\bar{X}_{\{\varepsilon_l\}}(\lambda)$ is the lowest K-type subquotient of

$$X_{\{\varepsilon_l\}}(\lambda) = \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{m}}[\bigotimes \mathscr{F}^l_{\varepsilon_l}(\lambda)]. \tag{6.3.10}$$

6.4. If $\mathfrak{m}^l \times \mathscr{A}^l$ is a maximal Levi component with $\mathscr{A}^l$ one of the $\mathscr{A}^i$ or $\mathscr{A}^j$ then we will denote by $X^l = \bar{X}(\lambda^l, \lambda^l)$ the irreducible such that $\bar{X}(\lambda, \lambda)$ is the spherical subquotient of

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{m}^l \times \mathscr{A}^l}[X^l \otimes \mathscr{F}^l]. \tag{6.4.1}$$

(6.4.1) equals $\bar{X}(\lambda, \lambda)$ precisely when

$$\mathcal{O} = \operatorname{ind}^{\mathfrak{g}}_{\mathfrak{m}^l \times \mathscr{A}^l}[WF(\bar{X}(\lambda^l, \lambda^l)) \times WF(\mathscr{F}^l)], \tag{6.4.2}$$

which is the same condition as $x_{s+1} - x_s = 0$ or 1 for the appropriate s (see also 6.6.)

6.5. Definition. The λ's and $\bar{X} \in \mathscr{X}(\lambda)$ corresponding to the following cells will be called quasiunipotent.

Type A. any λ.

Type B, C. Either the coordinates are half-integers, or $k = m$. When $k = m$, $r(x) \leq 2r(0) + 1$ for all coordinates x of λ.

Type D. Either the coordinates are half-integers or $k = m$. When $k = m$, $r(x) \leq 2r(0)$ for all coordinates x of λ.

The following parameters are special unipotent.

Type B. The coordinates are half-integers and $r(x+1) \leq r(x)$.

Type C. The coordinates are integers, $k = m$ and $r(x+1) \leq r(x)$.

Type D. The coordinates are integers, $k = m$ and $r(x+1) \leq r(x)$.

6.6. Let λ be a parameter attached to the orbit $\mathcal{O}$. Assume that there is a maximal Levi component $\mathfrak{m} \times \mathscr{A}$ and a nilpotent orbit $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m}$ such that

$$\mathcal{O} = \operatorname{ind}^{\mathfrak{g}}_{\mathfrak{m} \times \mathscr{A}}[\mathcal{O}_{\mathfrak{m}} \times \operatorname{triv}]. \tag{6.6.1}$$

Furthermore, assume there is $\lambda_{\mathfrak{m}}$ attached to $\mathcal{O}_{\mathfrak{m}}$ and $\mathscr{F}$ a finite dimensional representation on $\mathscr{A}$ such that

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{m} \times \mathscr{A}}[\bar{X}(\lambda_{\mathfrak{m}}, \lambda_{\mathfrak{m}}) \otimes \mathscr{F}] \tag{6.6.2}$$

has infinitesimal character λ.

Proposition. *There is a subgroup $\bar{A}_{\mathfrak{m}}(\mathcal{O}) \subset \bar{A}(\mathcal{O})$ which has $\bar{A}(\mathcal{O}_{\mathfrak{m}})$ as quotient such that*

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{m} \times \mathscr{A}}[X_{\chi_{\mathfrak{m}}} \otimes \mathscr{F}] = \sum_{\chi \in \hat{A}(\mathcal{O})} [\chi|_{\bar{A}_{\mathfrak{m}}(\mathcal{O})} : \chi_{\mathfrak{m}}] \, X_\chi .$$

Proof. The induced representation on the left side of the equality has composition series formed of elements in $\mathscr{X}(\lambda, \mathcal{O})$. Since the character theory of $\mathscr{X}(\lambda, \mathcal{O})$ and $\mathscr{X}(\lambda_{\mathfrak{m}}, \mathcal{O}_{\mathfrak{m}})$ is known, the formula follows in the same way as 12.5 in [BV1].

In the classical groups case, $[\bar{A}(\mathcal{O})/\bar{A}_{\mathfrak{m}}(\mathcal{O})] = 1$ or 2 so there are at most two irreducible composition factors.

7. A relation among Hermitian modules

7.1. Let (π, V) *be a* $(\mathfrak{g}(n), K(n))$ module and Y be an $(\mathscr{A}(m), K_{\mathscr{A}(m)})$ module. Then let

$$V * Y = \operatorname{Ind}^{\mathfrak{g}(n+m)}_{\mathfrak{g}(n) \times \mathscr{A}(m)}[V \otimes Y]. \tag{7.1.1}$$

If V and Y are irreducible then $V * Y$ has a lowest K-type which occurs with multiplicity 1. We denote by $\overline{V * Y}$ the lowest K-type subquotient of $V * Y$. If V and Y are unitary then so is every factor of $V * Y$. In view of this we make the following definition.

Definition. Let $\bar{X}_{(n)}$ be an irreducible $(\mathfrak{g}(n), K(n))$ module and $\bar{X}_{(n+m)}$ be an irreducible $(\mathfrak{g}(n+m), K(n+m))$ module. We say that $\bar{X}_{(n)} \underset{u}{<} \bar{X}_{(n+m)}$ if there is a unitary $(\mathscr{A}(m), K_{\mathscr{A}(m)})$ module Y such that $\bar{X}_{(n+m)}$ is a factor of $\bar{X}_{(n)} * Y$.

Assume in 7.2–7.5 that the infinitesimal character is integral and $\bar{X}(\lambda, \lambda)$ is hermitian.

7.2. Let the coordinates of F^l defined in section 6 be given by

$$F^l\colon\ (m_1^l, \ldots, m_{r_l}^l). \tag{7.2.1}$$

We will distinguish two possibilities whenever $\dim \mathscr{F}^l > 1$.

(1) There are r, r' such that

$$\begin{aligned} &m_r^l = -m_{r'}^l > 0,\\ &m_{r'-1}^l - m_{r'}^l > 1. \end{aligned} \tag{7.2.2}$$

(2) (1) is not satisfied for any l. Then there is r such that

$$\begin{aligned} &m_r^l > 0,\\ &m_r^l - m_{r+1}^l > 1. \end{aligned} \tag{7.2.3}$$

In this case, if F^l has negative coordinates, there is a positive coordinate larger than or equal to $|m_{r_l}^l|$ so that all coordinates between it and $m_{r_l}^l$ occur.

7.3. *Assume the coordinates of λ are formed of half-integers.*

Proposition. *Assume there is an $l \neq 0$ such that $\mathscr{F}^l$ is as in (1) of 7.2 or $\dim F^0 > 1$. Then there is a spherical $\bar{Z}$ such that $\bar{X} \underset{u}{<} \bar{Z}$, satisfying the following condition.*

There is $j \neq 0$ such that
(1) $\bar{Z} = \bar{Z}^j * \mathscr{F}_{\bar{Z}}^j$ (notation 6.4),
(2) There is $M > 0$ such that

$$F_{\bar{Z}}^j\colon\ (M+1, M-1, \ldots, -M+1, -M-1), \quad \textit{or} \tag{a}$$
$$(M+1, M, \ldots, -M+1, -M-1). \tag{b}$$

Proof. (1) follows from (2) by 6.2, 6.3; for the description of how the symbol is calculated from the coordinates shows that the terms (x_s, x_{s+1}) corresponding to $F_{\bar{Z}}^j$ differ by at most one.

Type A. Note that, if $\mathscr{F}^l$ with $\dim F^l > 1$ admits a hermitian form, then (1) of 7.2 is satisfied and $\mathscr{F}^j$ is as in (a). The conclusion and proof are valid even when the coordinates are integers.

Because the parameter is hermitian, $r(x) = r(-x)$ for all entries x of λ (see (6.1.1)). 7.2 is equivalent to the the fact that

$$r(M+1) > r(M) \quad \text{for some } M \geqq 0. \tag{7.3.1}$$

Replacing $\bar{X}$ by $\bar{Z} = \overline{X * \mathscr{F}(m)}$ has the effect that

$$r_{\bar{Z}}(x) = \begin{cases} r_{\bar{X}}(x) + 1 & \text{for } -m \leqq x \leqq m,\\ r_{\bar{X}}(x) & \text{for } x < -m \text{ or } x > m. \end{cases} \tag{7.3.2}$$

Thus 7.3.1 is preserved if we use $m \neq M$ or $m = M$ fewer than $r(M+1) - r(M)$ times.

Thus it is easy to see that we can find $\bar{Z}$ such that $\bar{X} \underset{u}{<} \bar{Z}$ satisfying

$$\begin{aligned} r_{\bar{Z}}(x+1) &\leqq r_{\bar{Z}}(x) \quad \text{for } x \geqq 0, \quad x \neq M,\\ r_{\bar{Z}}(M+1) &= r_{\bar{Z}}(M) + 1,\\ r_{\bar{Z}}(-M-1) &= r_{\bar{Z}}(-M) + 1. \end{aligned} \tag{7.3.3}$$

Then $\bar{Z}$ satisfies the conclusion of the proposition.

Type B. In coordinates,

$$\dim F^0 > 1 \Leftrightarrow \text{there is } M \geqq 0 \text{ such that } r(M) = 0,\ r(M+1) > 0. \tag{7.3.4a}$$

$$\begin{aligned} (1) \text{ of } 7.2 \Leftrightarrow &\text{ there is } M \text{ such that } r(M+1) \geqq 3\\ &\text{and} \quad r(M+1) - \varepsilon(M+1) > r(M), \end{aligned} \tag{7.3.4b}$$

where

$$\varepsilon(M+1) = \begin{cases} 0 & \text{if } r(M+1) \text{ is odd},\\ 1 & \text{if } r(M+1) \text{ is even}. \end{cases} \tag{7.3.5}$$

Replacing $\bar{X}$ by $\bar{Z} = \overline{X * \mathscr{F}(m)}$ has the effect that

$$r_{\bar{Z}}(x) = \begin{cases} r_{\bar{X}}(x) + 2 & \text{for } x \leqq m,\\ r_{\bar{X}}(x) & \text{for } x > m. \end{cases} \tag{7.3.6}$$

Let M be such that (7.3.4a) is satisfied. Then use the largest coordinate of λ for m. If $r_{\bar{X}}(M+1)$ is even, then

$$r_{\bar{Z}}(M+1) = r_{\bar{X}}(M+1) + 2 \geqq 4 = r_{\bar{Z}}(M) + 2. \tag{7.3.7}$$

This is as in (7.3.4b). Similarly if $r(M+1)$ is odd. Thus (7.3.4a) reduces to (7.3.4b).

Assume $\dim F^0 = 1$ *and* 7.3.4b *is satisfied for some* F^l. This condition is not affected if we use $m \neq M$ in the definition of $\bar{Z}$. Thus we can find a $\bar{Z}$ such that $\bar{X} \underset{u}{<} \bar{Z}$ and

$$\begin{aligned} r_{\bar{Z}}(x+1) &< r_{\bar{Z}}(x) \quad \text{for } x \geqq 0, \quad x \neq M,\\ r_{\bar{Z}}(M+1) - \varepsilon(M+1) &\geqq r_{\bar{Z}}(M) + 1,\\ r_{\bar{Z}}(M+2) &\leqq r_{\bar{Z}}(M+1) - \varepsilon(M+1) - 2,\\ r_{\bar{Z}}(M+1) &< r_{\bar{Z}}(M-1), \quad \text{for } M \geqq 2. \end{aligned} \tag{7.3.8}$$

This satisfies the conclusion of the proposition.

Type D. In this case, $\mathfrak{m} = \mathscr{A}^1 \times \ldots \times \mathscr{A}^{m+1}$. In coordinates,

$$(1) \text{ of } 7.2 \Leftrightarrow \text{there is } M \text{ such that } r(M+1) - \varepsilon(M+1) > r(M), \tag{7.3.9}$$

where

$$\varepsilon(M+1) = \begin{cases} 0 & \text{if } r(M+1) \text{ is even},\\ 1 & \text{if } r(M+1) \text{ is odd}. \end{cases} \tag{7.3.10}$$

Replacing $\bar{X}$ by $\bar{Z}=\overline{\bar{X}*\mathscr{F}(m)}$ has the effect that

$$r_{\bar{Z}}(x)=\begin{cases} r_{\bar{X}}(x)+2 & \text{for } x\leqq m,\\ r_{\bar{X}}(x) & \text{for } x>m.\end{cases} \tag{7.3.11}$$

Choose an M satisfying (7.3.9). This condition is not affected if we use $m\neq M$ in the definition of $\bar{Z}$. We are reduced to the case when (7.3.8) holds. Then $\bar{Z}$ satisfies the conclusion of the proposition.

7.4. *Assume the coordinates of λ are integers.*

Proposition. *Let $\bar{X}$ be such that one of the following holds.*

$$\bar{X} \textit{ is not quasiunipotent,} \tag{I}$$
$$\textit{There is } 1\leqq l\leqq k \textit{ such that } \mathscr{F}^l \textit{ is as in (1) of 7.2.} \tag{II}$$

Then there is a spherical $\bar{Z}$ such that $\bar{X}\underset{u}{<}\bar{Z}$, satisfying the following condition.

There is $j\neq 0$ such that
(1) $\bar{Z}=\bar{Z}^j*\mathscr{F}^j_{\bar{Z}}$ (notation 6.4),
(2) There is $M>0$ such that

$$F^j_{\bar{Z}}:\ (M+1, M-1, \ldots, -M+1, -M-1), \tag{a}$$
$$(M+1, M, \ldots, -M+1, -M-1), \tag{b}$$

or

$$F^j_{\bar{Z}}:\ (1,-1) \ \textit{or}\ (1),\quad F^0_{\bar{Z}}:\ (1) \textit{ in types } B, C, \tag{c}$$
$$F^j_{\bar{Z}}:\ (1,-1) \ \textit{or}\ (1),\quad F^0_{\bar{Z}}:\ (M^0, \ldots, 1, 0, -1) \textit{ in type } D. \tag{d}$$

Proof. (1) follows as in 7.3.

Type A. This is the same as 7.3.

Type B, C. Conditions (I) and (II) are equivalent to the following.

$$(I)\Leftrightarrow \text{there is } M\geqq 0 \text{ such that } r(M+1)>2r(0)+1, \tag{7.4.1a}$$
$$(II)\Leftrightarrow \text{there is } M>0 \text{ such that } r(M)<2r(0)+1,$$
$$r(M+1)-\varepsilon(M+1)\geqq r(M), \tag{7.4.1b}$$

where

$$\varepsilon(M+1)=\begin{cases}0 & \text{if } r(M+1) \text{ is even},\\ 1 & \text{if } r(M+1) \text{ is odd}.\end{cases} \tag{7.4.2}$$

Replacing $\bar{X}$ by $\bar{Z}=\overline{\bar{X}*\mathscr{F}(m)}$ has the effect that

$$r_{\bar{Z}}(x)=\begin{cases} r_{\bar{X}}(x)+2 & \text{for } 0<x\leqq m,\\ r_{\bar{X}}(x) & \text{for } x>m,\\ r_{\bar{X}}(0)+1 & \text{for } x=0.\end{cases} \tag{7.4.3}$$

Assume (II) holds. (7.4.1b) is not affected if we use $m\neq M$ in the definition of $\bar{Z}$.

Then we can find $\bar{Z}$ such that $\bar{X}<\bar{Z}$ satisfying (7.3.8) and

$$r_{\bar{Z}}(x)\leqq 2r_{\bar{X}}(0)+1 \quad \text{for all } x. \tag{7.4.4}$$

Then $\bar{Z}$ satisfies the conclusion (a) or (b) of the proposition.

Assume (I) holds. Let $M\geqq 0$ be smallest such that $r(M+1)>2r(0)+1$. By using $m\geqq M+1$ the appropriate number of times, we reduce to the case when

$$\begin{aligned} r_{\bar{X}}(x+1)&<r_{\bar{X}}(x) && \text{for } x\geqq M+1,\\ r_{\bar{X}}(x)&\leqq 2r(0)+1, && \text{for } x>M+1.\end{aligned} \tag{7.4.5}$$

By using $m=M$ the appropriate number of times, we reduce to the case

$$2r(0)+2\leqq r(M+1)\leqq 2r(0)+3. \tag{7.4.6}$$

If $M=0$, $\bar{Z}$ satisfies the conclusion (c) of the proposition because of (7.4.5). $r_{\bar{X}}(1)$ even, gives the first case in (c), $r_{\bar{X}}(1)$ odd, gives the second.

If $M>0$ replace $\bar{X}$ by $\bar{Z}=\overline{\bar{X}*\mathscr{F}(0)}$. Because M was chosen smallest so that (7.4.1a) is satisfied, $r_{\bar{X}}(M)\leqq 2r(0)+1$. If $r_{\bar{X}}(M+1)$ is odd,

$$r_{\bar{Z}}(M+1)-\varepsilon(M+1)=r_{\bar{X}}(M+1)-\varepsilon(M+1)=2r(0)+2\geqq r_{\bar{X}}(M)+1=r_{\bar{Z}}(M)+1. \tag{7.4.7}$$

Similarly if $r_{\bar{X}}(M+1)$ is even. Thus $\bar{Z}$ is as in (II). The proof follows in this case.

Type D. Conditions (I) and (II) are equivalent to the following.

$$(I)\Leftrightarrow \text{there is } M\geqq 0 \text{ such that } r(M+1)>2r(0), \tag{7.4.8a}$$
$$(II)\Leftrightarrow \text{there is } M>0 \text{ such that } r(M)<2r(0),$$
$$r(M+1)-\varepsilon(M+1)\geqq r(M)+1, \tag{7.4.8b}$$

where

$$\varepsilon(M+1)=\begin{cases}1 & \text{if } r(M+1) \text{ is even},\\ 0 & \text{if } r(M+1) \text{ is odd}.\end{cases} \tag{7.4.9}$$

Replacing $\bar{X}$ by $\bar{Z}=\overline{\bar{X}*\mathscr{F}(m)}$ has the effect that

$$r_{\bar{Z}}(x)=\begin{cases} r_{\bar{X}}(x)+2 & \text{for } 0<x\leqq m,\\ r_{\bar{X}}(x) & \text{for } x>m,\\ r_{\bar{X}}(0)+1 & \text{for } x=0.\end{cases} \tag{7.4.10}$$

Assume (II) holds. 7.4.8b is not affected if we use $m\neq M$.

Then we can find $\bar{Z}$ such that $\bar{X}\underset{u}{<}\bar{Z}$ satisfying 7.3.8 and

$$r_{\bar{Z}}(x)\leqq 2r(0)+1 \quad \text{for all } x. \tag{7.4.11}$$

Then $\bar{Z}$ satisfies the conclusion of the proposition.

Assume (1) holds. Let $M \geqq 0$ be smallest such that $r(M+1) > 2r(0)$. By using $m \geqq M+1$ the appropriate number of times, we reduce to the case when

$$\begin{aligned} r_{\bar{X}}(x-1) &< r_{\bar{X}}(x) \quad \text{for } x \geqq M, \\ r_{\bar{X}}(x) &\leqq 2r(0)+1 \quad \text{for } x > M+1. \end{aligned} \tag{7.4.12}$$

By using $m = M$ the appropriate number of times, we reduce to the case

$$2r(0)+1 \leqq r(M+1) \leqq 2r(0)+2. \tag{7.4.13}$$

If $M=0$, this satisfies the conclusion (d) of the proposition because of (7.4.12). $r(1)$ even, gives the first case in (d), $r(1)$ odd, gives the second.

If $M>0$ replace $\bar{X}$ by $\bar{Z} = \overline{\bar{X} * F(0)}$. Because M was chosen smallest so that (7.4.8a) is satisfied, $r_{\bar{X}}(M) \leqq 2r(0)$. If $r_{\bar{X}}(M+1)$ is even,

$$r_{\bar{Z}}(M+1) - \varepsilon(M+1) = r_{\bar{X}}(M-1) - \varepsilon(M+1) = 2r(0)+1 \geqq r_{\bar{X}}(M)+1 = r_{\bar{Z}}(M)+1. \tag{7.4.14}$$

Similarly if $r_{\bar{X}}(M+1)$ is odd. Thus $\bar{Z}$ is as in (II).

The proof follows in this case.

7.5. Definition. We say λ or $\bar{X} \in \mathscr{X}(\lambda)$ is weakly preunipotent if λ is quasiunipotent and no $\mathscr{F}^1$ satisfies (1) of 7.2.

This is exactly the opposite of the conditions in 7.3, 7.4.

8. Reduction using level 1 K-types

Assume $\bar{X}(\lambda, \lambda)$ is hermitian with integral infinitesimal character.

The main result in this section is the following.

8.1. Proposition. *If $\bar{X}(\lambda, \lambda)$ is not weakly preunipotent then the form is negative on a small K-type.*

The proof will be by reduction via $\underset{u}{<}$ to some simple cases involving type A only.

8.2. We collect some results about composition factors of induced representations which will be used to derive necessary conditions for unitarity. Similar results in the Verma module case can be obtained from [BC] or [ES]. Assume $\mathfrak{g}$ is type A.

Lemma. *Let M be such that $2M \in \mathbb{N}$ and $\mathscr{F} \simeq F \otimes F$ be a spherical finite dimensional representation.*

(1) *Assume F is one of*

a) $F\colon (M+1, M-1, \ldots, -M+1)$,

b) $F\colon (M+1, M-1, \ldots, -M, -M-1)$, $M \neq 0$,

c) $F\colon (M+1, M-1, \ldots, -M+1, -M-1)$, $M \neq 0$.

Then $\mathscr{F} * \mathscr{F}(M)$ contains an irreducible subquotient $\bar{X}_u$ which is the lowest K-type subquotient of $\bar{X}[(M+1, M), (M, M+1)] * \mathscr{F}^1 * \mathscr{F}^2$ where $\mathscr{F}^1$ and $\mathscr{F}^2$ are the unique one dimensional spherical representations that match the infinitesimal characters.

(2) *Assume $F\colon (M+1, M, \ldots, -M+1)$. Then $\mathscr{F} * \mathscr{F}(M)$ contains in irreducible subquotient $\bar{X}_u$ which is the lowest K-type subquotient of $\mathscr{F}_1 * (\mathscr{F}^1)^*$ with*

$$\mathscr{F}^1\colon (M+1, M, \ldots, -M+1) \times (M, \ldots, -M).$$

(3) *Let $F\colon (M+1, M, \ldots, -M+1)$, $M \neq 0$. Then $\mathscr{F} * (\mathscr{F})^*$ contains an irreducible subquotient $\bar{X}_u$ which is the lowest K-type subquotient in $\mathscr{F}' * (\mathscr{F}^1) * (\mathscr{F}^1)^*$ where $\mathscr{F}'\colon (M+1, -M-1) \times (M+1, -M-1)$ and $\mathscr{F}^1\colon (M, \ldots, -M+1) \times (M-1, \ldots, -M)$.*

Proof. Recall the notation in 2 and 4.1. Let χ be an integral infinitesimal character. It is determined by a $\lambda(\chi) \in \mathfrak{h}^*$, dominant for $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Let α be a simple root. It determines a new infinitesimal character χ_α by

$$\lambda(\chi_\alpha) = \lambda(\chi) - (\lambda(\chi), \check{\alpha})\, \varpi_\alpha, \tag{8.2.1}$$

where ϖ_α is the fundamental weight corresponding to α. Then define a functor

$$\begin{aligned} &T_\alpha\colon \mathscr{H}(\chi) \to \mathscr{H}(\chi_\alpha), \\ &T_\alpha(X) = P_{\chi_\alpha}[X \otimes \mathscr{F}(-(\lambda(\chi), \check{\alpha})\, \varpi_\alpha)], \end{aligned} \tag{8.2.2}$$

where $\mathscr{F}(\eta)$ is the irreducible finite dimensional representation with extremal weight η. If X is an induced module from a representation $X_{\mathfrak{m}}$ of a Levi component $\mathfrak{m}$ then $T_\alpha(X)$ is computed by tensoring $X_{\mathfrak{m}}$ with the restriction of $\mathscr{F}$ to $\mathfrak{m}$ and projecting onto the appropriate infinitesimal character. T_α is exact and takes irreducible representations into irreducible representations or 0.

Proof of (1). Let (λ, λ) be the infinitesimal character of $\mathscr{F} * \mathscr{F}(M)$. For every simple root $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$ there are two simple roots $\alpha^L = (\alpha, 0)$, $\alpha^R = (0, \alpha)$ in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Then let $T_\alpha = T_{\alpha^L} \cdot T_{\alpha^R}$. Let $\alpha = \varepsilon_1 - \varepsilon_2$ where $\varepsilon_i(\lambda)$ equals the i-th coordinate of λ. Then

$$T_\alpha[\mathscr{F} * \mathscr{F}(M)] \neq 0. \tag{8.2.3}$$

On the other hand,

$$\overline{\mathscr{F} * \mathscr{F}(M)} = \mathscr{F}' * \mathscr{F}'' \tag{8.2.4}$$

by the results in 4–6. Then

$$T_\alpha[\mathscr{F}' * \mathscr{F}''] = 0. \tag{8.2.5}$$

Then there must be a composition factor $\bar{X}_u$ in $\mathscr{F} * \mathscr{F}(M)$ such that $T_\alpha[\bar{X}_u]$ equals the lowest K-type subquotient of $T_\alpha[\mathscr{F} * \mathscr{F}(M)]$. This is the $\bar{X}_u$ in the statement of (1).

(2) Let $\mu(r) = \beta_1 + \ldots + \beta_r$ be as in 3.1. Recall the known multiplicity formula

$$\operatorname{Ind}_{U(p) \times U(q)}^{U(p+q)}[\mathrm{Triv}] = \sum_{x_1 \geqq \ldots \geqq x_k \geqq 0} \mu(x_1 \beta_1 + \ldots + x_k \beta_k), \qquad k = \min\{p, q\}. \tag{8.2.6}$$

By Sects. 4–6,

$$\overline{\mathscr{F} * \mathscr{F}(M)} = \mathscr{F}' * \mathscr{F}'', \tag{8.2.7}$$

where

$$\begin{aligned}\mathscr{F}'&: (M+1, \ldots, -M) \times (M+1, \ldots, -M),\\ \mathscr{F}''&: (M, \ldots, -M+1) \times (M, \ldots, -M+1).\end{aligned} \tag{8.2.8}$$

By (8.2.6), $\mu(2M+1)$ is the lowest K-type in $\mathscr{F} * \mathscr{F}(M)$ not in the spherical subquotient.

Thus $\mathscr{F} * \mathscr{F}(M)$ has a unique irreducible composition factor with lowest K-type $\mu(2M+1)$. The proof now follows by observing that there is only one irreducible composition factor with infinitesimal character (λ, λ) and lowest K-type $\mu(2M+1)$, namely $\bar{X}_u$.

(3) Similarly, we find that the lowest K-type outside of $\mathscr{F} * \mathscr{F}(M)$ is $\mu(2M)$.

Thus $\mathscr{F} * \mathscr{F}^h$ has a unique irreducible composition factor with lowest K-type $\mu(2M)$. Since $\mathscr{F} * \mathscr{F}^h$ is hermitian, this composition factor has to be hermitian. Unfortunately, there are two possibilities for such a composition factor with infinitesimal character (λ, λ). There are two translation functors, T_{α_1}, $T_{\alpha_{4M+1}}$ such that $T_{\alpha_i}(\mathscr{F} * \mathscr{F}^h) = 0$. The same has to be true for any irreducible composition factor. This is equivalent to saying that the corresponding simple roots are in the left and right τ-invariant described in 4.1. The only irreducible composition factor satisfying this is $\bar{X}_u$ in the statement of the proposition. The claim follows.

Example. Let F: (2, 1, 0). The only possible hermitian parameters with lowest K-type $\mu = (1, 1, 0, 0, -1, -1)$ are

$$\begin{aligned}&[(1, 0, 2, -2, 0, -1) \times (0, -1, 2, -2, 1, 0)],\\ &[(2, -1, 0, 0, 1, -2) \times (1, -2, 0, -0, 2, -1)].\end{aligned} \tag{8.2.9}$$

The roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_5 = \varepsilon_5 - \varepsilon_6$. The second parameter does not have these roots in its τ-invariants.

Corollary. *In case* (2) *of Lemma* 8.2,

$$\mathscr{F} * \mathscr{F}(M) = \overline{\mathscr{F} * \mathscr{F}(M)} + \bar{X}_u.$$

Proof. By 14.2, $\mathscr{F}^1 * (\mathscr{F}^1)^h$ is irreducible. Formula (8.2.6) generalizes to

$$\bar{X}_u|_K = \sum_{x_1 \geqq \ldots \geqq x_{2M} > 0} \mu(x_1 \beta_1 + \ldots + x_{2M} \beta_{2M}) \tag{8.2.10}$$

Combining this with

$$\overline{\mathscr{F} * \mathscr{F}(M)}|_K = \sum_{x_1 \geqq \ldots \geqq x_{2M-1} \geqq 0} \mu(x_1 \beta_1 + \ldots + x_{2M-1} \beta_{2M-1}), \tag{8.2.11}$$

we get the desired result.

8.3. The proofs of the results in this section are essentially as in [V3]. They are included for completeness.

Assume $\mathfrak{g}$ *is type A and* $\bar{X}$ *is spherical with arbitrary infinitesimal character.*

Write λ as in (6.1.3). Then $\bar{X}$ is hermitian if and only if $r(x) = r(-x)$ for every coordinate x of λ. By Sects. 4–6 we can write

$$\bar{X} = Y^1 * \ldots * Y^r \tag{8.3.1}$$

such that

$$\begin{aligned}Y^l = \mathscr{F}^l &\approx (\mathscr{F}^l)^h \quad \text{or} &(8.3.2\text{a})\\ = \mathscr{F}^l &* (\mathscr{F}^l)^h, \quad \mathscr{F}^l \not\approx (\mathscr{F}^l)^h. &(8.3.2\text{b})\end{aligned}$$

In (8.3.2a) the coordinates are integers or half-integers, in (8.3.2b) they are neither.

Since each Y^l is hermitian, $\bar{X}$ is unitary if and only if all Y^l are unitary.

If $\mathscr{F}^l$ is hermitian, it is unitary if and only if it is a unitary character. Otherwise the form is negative on the small K-type equal to the highest root.

Thus we may assume (8.3.2b) holds and write $Y = \mathscr{F} * \mathscr{F}^h$ for Y^l. Let the coordinates of $\mathscr{F}$ be given by

$$(M_1 + t, \ldots, M_r + t), \quad \text{for } 2M_i \in \mathbb{Z}, \quad 0 < t < 1/2, \quad M_1 \geqq |M_r|. \tag{8.3.3}$$

This can always be arranged by interchanging $\mathscr{F}$ and $\mathscr{F}^h$ if necessary.

For $M \in \mathbb{N}$, $t \in \mathbb{R}$ let $\mathscr{F}(M; t) = F(M; t) \otimes F(M; t)$, $Y(M; t) = \mathscr{F}(M; t) * (\mathscr{F}(M; t))^h$, where

$$F(M; t)\colon (M + t, \ldots, -m + t). \tag{8.3.4}$$

The $Y(M; t)$ and their analogues with one dimensional lowest K-type will be called Stein complementary series. We will derive necessary and sufficient conditions for the unitarity of Y using $\underset{u}{<}$ and $Y(M; t)$.

Lemma 1. *$Y(M; t)$ is unitary for $0 \leqq t < 1/2$.*

Proof. $Y(M; t)$ is irreducible for $0 \leqq t < 1/2$. Because $Y(M; 0)$ is unitarily induced irreducible from the trivial representation on a Levi component, this is a complementary series.

Lemma 2. *$Y(M; 1)$ contains a composition factor $\bar{X}_u$ such that its form is negative on one of the small K-types $\mu(2M)$ and $\mu(2M+1)$.*

Proof. Since $Y(M; 1) = \mathscr{F} * \mathscr{F}^h$ as in (3) of Lemma 8.2, it contains a composition factor $\bar{X}_u$ with lowest K-type $\mu(2M)$. By bottom layers, $\mu(2M+1)$ has opposite sign.

Proposition. *$Y = \mathscr{F} * \mathscr{F}^h$ is unitary if and only if there is an m and $0 \leqq t < 1/2$ such that $Y = Y(m; t)$. If not, the form is negative on a small K-type.*

Proof. We reduce the number of cases by using $\underset{u}{<}$. In terms of the coordinates in 8.3.3,

$$Y = Y(m; t) \Leftrightarrow r(x + t) = \begin{cases} 1 & \text{for } -m \leqq x \leqq m,\\ 0 & \text{otherwise.}\end{cases} \tag{8.3.5}$$

If $M_r \neq -M_1$, replace Y by $Z = \overline{Y * Y(M_1 - 1; t)}$. This has an Y^l as in (8.3.2b) with $\mathscr{F}^l$ as in (2) of Lemma 8.2. Lemma 2 completes the proof in this case. If $M_r = -M_1$, let M be the largest coordinate such that one of $(\pm M + t)$ does not occur. Replace Y by $\overline{Y * Y(M+1; t) * Y(M-1; t)}$. This has a Y^l as in (8.3.2b) with $\mathscr{F}^l$ satisfying (1) of Lemma 8.2. Apply (1) of Lemma 8.2 to $Y^1 * Y(M_1 - 1; t)$, twice, once with $\mathscr{F} = \mathscr{F}^l$, and once with $\mathscr{F} = (\mathscr{F}^l)^h$. Then $Y^l * Y(M_1 - 1; t)$ contains two composition factors, $\bar{X}_u$ and $\bar{X}_u^h$, such that $\bar{X}_u^h \not\leq \bar{X}_u$ with small lowest K-type equal to the highest root. The proof follows.

8.4. *Assume* $\mathfrak{g}$ *is type A.*

Theorem ([V3]). *$\bar{X}(\mu, \nu)$ is unitary if and only if it is unitarily induced from a unitary representation on $\mathfrak{m}_\mu$. Otherwise $\bar{X}(\mu, \nu)$ is not unitary and the form is negative on a K-type $\mu + \varepsilon$ of level 1 for μ.*

Proof. Recall from 3.3,

$$\mathfrak{m}_\mu = \mathfrak{m}_{\mu,1} \times \ldots \times \mathfrak{m}_{\mu,s}. \tag{8.4.1}$$

There are representations with 1-dimensional lowest K-type $\bar{X}_{\mu,1}, \ldots, \bar{X}_{\mu,s}$ such that $\bar{X}$ is the lowest K-type in $\bar{X}_{\mu,1} * \ldots * \bar{X}_{\mu,s}$. By Proposition 8.3, 2.7 and 3.3, if one of the $\bar{X}_{\mu,i}$ is not unitary, then the form is negative on a K-type $\mu + \varepsilon_i$ where ε_i is small for $\mathfrak{m}_{\mu,i}$.

The stronger result that

$$\bar{X} = \bar{X}_{\mu,1} * \ldots * \bar{X}_{\mu,s} \tag{8.4.2}$$

whenever $\bar{X}_{\mu,i}$ are unitary is proved in [V3], Chap. 14.

Assume $\mathfrak{g}$ *is classical not type A.* Then there are $\bar{X}_{\mu,i}$ with 1-dimensional lowest K-type such that $\bar{X}(\lambda, \lambda')$ is the lowest K-type subquotient in $\bar{X}_{\mu,0} * \bar{X}_{\mu,1} * \ldots * \bar{X}_{\mu,s}$.

Corollary. *$\bar{X}$ is unitary only if $\bar{X}_{\mu,1}, \ldots, \bar{X}_{\mu,s}$ are unitary. Otherwise the hermitian form is negative on a level 1 K-type.*

Proof. The proof follows from Theorem 8.4, 2.7 and 3.3.

8.5. *Proof of Proposition* 8.1. It is enough to prove the proposition for a $\bar{Z}$ such that $\bar{X} \underset{u}{<} \bar{Z}$ that satisfies the conclusions of Proposition 7.3 or 7.4.

Assume $\bar{X} = \bar{Z}$ in order to simplify notation.

If F^j is as in (2a) of 7.3, 7.4, the assertion is obvious.

Assume we are in case (2c) *or* (2d) *of Proposition 7.4.*

If F^j: $(1, -1)$ then the assertion is trivial; so assume F^j: (1).

We find a Levi component $\mathfrak{m} \times \mathscr{A}$ and an orbit $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m}$ that satisfy the assumptions in 6.6. Recall the notation in 6.1.

Type B, C. The symbol of ${}^L\mathcal{O}$, with the notation as in (6.2.1), must satisfy $k = m - 1$,

$$x_2 - 1 = 1, \quad x_1 - (m - (m+1) + 1) = 1, \quad x_0 - (m - (m+1) + 1) = 0, \tag{8.5.1}$$

Therefore, $x_2 - x_1 = 1$ and the nilpotent is induced from an

$$\mathcal{O}_{\mathfrak{m}} \times \mathrm{Triv} \subset \mathfrak{m} \times \mathscr{A}(2). \tag{8.5.2}$$

Let $\mathscr{F} = F \otimes F$ be such that F: $(1, -1)$ and $\bar{X}_{\mathfrak{m}}$ be such that $\bar{X}$ is the lowest K-type subquotient in $\bar{X}_{\mathfrak{m}} * \mathscr{F}$. By 6.6,

$$X_{\mathfrak{m}} * \mathscr{F} = \bar{X}_{(1, \ldots, 1)} + \bar{X}_{(1, \ldots, -1)}. \tag{8.5.3}$$

The lowest K-type of $\bar{X}_{(1, \ldots, -1)}$ has highest weight $(2, 0, \ldots, 0)$. Thus if $\mu(1) = \beta_1$ is small,

$$[\mu(1) : \bar{X}] = [\mu(1) : X_{\mathfrak{m}} * \mathscr{F}]. \tag{8.5.4}$$

The signatures are computed by the formula in the right hand side of (2.8.1). The claim follows by observing that $\mathscr{F}$ is hermitian and its form is negative on the small K-type with highest weight equal to the highest root.

The proof follows in this case.

Type D. The symbol of ${}^L\mathcal{O}$, with the notation as in 6.3.1, satisfies $k = m - 1$. Then

$$x_{2m+1} - m = M^0 + 1, \quad x_2 - 1 = 1, \quad x_1 - 0 = 1, \quad x_0 - 0 = 0. \tag{8.5.5}$$

Thus $x_2 - x_1 = 1$ so $\mathcal{O}$ is induced from an

$$\mathcal{O}_{\mathfrak{m}} \times \mathrm{Triv} \subset \mathfrak{m} \times \mathscr{A}(2). \tag{8.5.6}$$

Let $\mathscr{F} = F \otimes F$ be such that F: $(1, -1)$ and $\bar{X}_{\mathfrak{m}}$ be such that $\bar{X}$ is the lowest K-type subquotient in $\bar{X}_{\mathfrak{m}} * \mathscr{F}$. By 6.6,

$$X_{\mathfrak{m}} * \mathscr{F} = \bar{X}_{(1, \ldots, 1)} + \bar{X}_{(1, \ldots, -1)}. \tag{8.5.7}$$

The proof follows as in types B, C.

Assume we are in case (2b) of proposition 7.3 or 7.4.

Let $\mathscr{F}_t = F_t \otimes F_t$ be such that

$$F_t\colon\ (M + 1 + t, M + t, \ldots, -M + 1 + t, -M - 1 + t), \qquad M \neq 0, \tag{8.5.8}$$

for $0 < t \ll 1/2$ very small. By (1) of Propositions 7.3, 7.4, $\bar{X} = \bar{X}^l * \mathscr{F}_0$. Then $\bar{X}(t) = \bar{X}^l * F_t$ is irreducible and hermitian except possibly in type D when the coordinates are half-integers. In that case replace $\bar{X}$ by $\bar{X} * \mathscr{F}(0)$ which is irreducible in view of 4.2. Then,

$$[\bar{X}]_\pm = [\bar{X}(t)]_\pm. \tag{8.5.9}$$

It is enough to prove the claim for $\bar{X}(t)$. Consider

$$\bar{X}(t) * Y(M; t) = \bar{X}^l * [\mathscr{F}_t * Y(M; t)]. \tag{8.5.10}$$

Apply (1) of Lemma 8.2 to $\mathscr{F}_t * Y(M; t)$. Then (8.5.10) has a composition factor $\bar{X}_u(t)$ which is the lowest K-type subquotient of $\bar{X}^l * \bar{X}_u$. This composition factor has small lowest K-type equal to a highest root, and $\bar{X}(t) \underset{u}{<} \bar{X}_u(t)$.

On the other hand, $\bar{X}_u(t)$ is the lowest K-type subquotient in

$$\bar{X}[(M + 1 - t, M - t), (M - t, M + 1 - t)] * Y(t), \tag{8.5.11}$$

where

$$Y(t)=[\bar{X}^{l} * \mathscr{F}(M+1/2; t+1/2)] * Y(M-1/2; t+1/2) \tag{8.5.12}$$

is irreducible. Because of bottom layers and the facts about small K-types in Sect. 3, the proof comes down to showing that the form on $Y(M-1/2;\ t+1/2)$ is indefinite on a set of small K-types. This follows from Lemma 2 in 8.3.

Remarks.

(1) Let $\mathfrak{g}=\mathrm{sp}(4)$ and $\bar{X}=\mathrm{Ind}_{\mathfrak{gl}(4)}^{\mathfrak{g}}[\mathscr{F}]$ where $\mathscr{F}=F\otimes F$ is of the form

$$F:(3,1,0,-1). \tag{8.5.13}$$

This is weakly preunipotent. The form is positive on (1, 1, 0, 0) and negative on (2, 0, 0, 0).

(2) The argument in the proof of Proposition 8.1 is designed to apply in the case of section 10. This is why it is important that it work for t very small. Otherwise there are simpler proofs.

Corollary. *Assume $\mathfrak{g}$ is type B, $\bar{X}$ is spherical and $\dim \mathscr{F}^0>1$.*
Then $\bar{X}$ is not unitary and the form is negative on a level $\leqq 1$ K-type.

Proof. In case λ has half-integer coordinates, Proposition 7.3 type B reduces to the case when the assumptions of Proposition 8.1 are satisfied.

When λ has integer coordinates, we can reduce to the case when

$$r(x+1)<r(x), \quad r(1)=2r(0)+1. \tag{8.5.14}$$

By 4.2 and the character formulas in 4–6, this is a complementary series from a unitarily induced from a hermitian finite dimensional representation. Since this representation is not the trivial representation and the highest root is a small K-type in this case, the proof follows.

8.6. Recall the notation in 8.3, 8.4.

Definition. $\bar{X}(\mu, \nu)$ will be called weakly preunitary if $\bar{X}_0$ is weakly preunipotent and $\bar{X}_1, \ldots, \bar{X}_s$ are unitary.

In particular, $\bar{X}_1, \ldots, \bar{X}_s$ are unitarily induced from unitary characters.

The results in this section can be summarized as follows.

Theorem. *A hermitian $\bar{X}(\mu, \nu)$ with integral infinitesimal character is unitary only if it is weakly preunitary.*

9. Reduction using level $\leqq 2$ K-types

Assume throughout that (λ, λ') is integral.
In 9.1–9.4 assume that μ is small.

9.1. We can write

$$\mathfrak{m}_\mu=\mathfrak{m}_{\mu,0}+\mathscr{A}_{\mu,1}. \tag{9.1.1}$$

With notation as in 2.6, let

$$X_\mu=X(\lambda_0,\lambda_0)\otimes X(\lambda_1,\lambda_1'). \tag{9.1.2}$$

Write

$$\mathfrak{m}(\lambda_0)=\mathfrak{m}_0^0(\lambda_0)\times\mathscr{A}_0^1\times\ldots\times\mathscr{A}_0^{k_0}\times\mathscr{A}_0^{k_0+2}\times\ldots\times\mathscr{A}_0^{m_0+1}\subset\mathfrak{m}_{\mu,0}. \tag{9.1.3}$$

Write the $\mathfrak{a}$ parameter of $X(\lambda_1,\lambda_1')$ as (ν_1,ν_1). Let

$$\mathscr{A}(\nu_1)=\mathscr{A}_1^1\times\ldots\times\mathscr{A}_1^{k_1}\subset\mathscr{A}_{\mu,1}. \tag{9.1.4}$$

Let λ_0^0, $\mathscr{F}_0^1, \ldots \mathscr{F}_0^{k_0}$, $\mathscr{F}_0^{k_0+2}, \ldots, \mathscr{F}_0^{m_0+1}$ and $\mathscr{F}_1^1, \ldots, \mathscr{F}_1^{k_1}$ be the corresponding objects as in Sect. 6.

Write λ_0 and ν_1 in coordinates as in Sect. 6. Let

$$\begin{aligned} r_0(x)&=\text{number of coordinates of } \lambda_0 \text{ equal to } x,\\ r_1(x)&=\text{number of coordinates of } \nu_1 \text{ equal to } x. \end{aligned} \tag{9.1.5}$$

Recall $\varepsilon(M)$ from Sect. 7 and write

$$\varepsilon(0)=\begin{cases} 1 & \text{type B, C, integer coordinates,}\\ \infty & \text{type B, D, half-integer coordinates,}\\ 0 & \text{type D, integer coordinates.}\end{cases} \tag{9.1.6}$$

Then the condition of weakly preunitary can be written as follows.

$$\begin{aligned} &r_0(x)\leqq 2r_0(0)+\varepsilon(0) \quad \text{for all } x,\\ &r_0(x+1)-\varepsilon(x+1)\leqq r_0(x) \quad \text{for all } x,\\ &r_1(x+1/2)\leqq r_1(x-1/2), \quad r_1(x+1/2)=r_1(-x-1/2) \quad \text{for all } x. \end{aligned} \tag{9.1.7}$$

9.2. Theorem. *A weakly preunipotent $\bar{X}(\lambda, \lambda')$ is unitary only if the following is satisfied.*

If $r_0(M+1)>r_0(M)$, $\quad\varepsilon_0(M+1)=1$, $\quad$ then $r_1(M+1/2)<r_1(M-1/2)$.

Proof. Denote by M_1+1, $M_2+1, \ldots, M_s+1$ the coordinates such that $\varepsilon(M_i+1)=1$. Fix an $M=M_i$. By replacing $\bar{X}$ with $\overline{\bar{X}*\mathscr{F}(m)}$ with $m\neq M$ the appropriate number of times as in 7.3, 7.4, we reduce to the case when

$$\begin{aligned} r_0(x+1)&\leqq r_0(x) \quad \text{for } x\neq M,\\ r_0(M+1)&=r_0(M)+1. \end{aligned} \tag{9.2.1.}$$

Then $\bar{X}$ is as in 7.3b or 7.4b. Let

$$F_0^j:(M+1, M-1, \ldots, -M+1). \tag{9.2.2}$$

This is as in (1) of Lemma 8.2. Then

$$\bar{X}*\mathscr{F}(M)=\bar{X}^j*[\mathscr{F}_0^j*\mathscr{F}(M)], \tag{9.2.3}$$

so it has a factor $\bar{Z}$ which is the lowest K-type subquotient of $\bar{X}^j * \bar{X}_u$. Its lowest K-type is $\mu+\mu_0(1)$ where $\mu_0(1)$ is small for $\mathfrak{m}_{\mu,0}$. Because $\bar{Z}$ has small lowest K-type, the notation in 8.1 applies. The relations

$$\begin{aligned} r_{\bar{Z},1}(M+1/2) &= r_{\bar{X},1}(M+1/2)+1, \\ r_{\bar{Z},1}(M-1/2) &= r_{\bar{X},1}(M-1/2), \end{aligned} \tag{9.2.4}$$

hold. If $r_{\bar{X},1}(M+1/2)=r_{\bar{X},1}(M-1/2)$ then $\bar{Z}$ cannot be unitary because it is not weakly preunitary. Moreover, the form on $\bar{Z}$ is negative on a K-type of the form $\mu+\mu_0(1)+\mu_1(1)$ where $\mu(1)$ is small for $\mathscr{A}_{\mu+\mu_0(1),1}$. Thus the form is negative on a *level* 2 K-type for $\bar{X}$. Of course, $\mu_0(1)$, $\mu_1(1)$ are equal to highest roots.

9.3. Definition. An integral λ satisfying

$$\begin{aligned} r(x+1) &\leqq r(x), \\ r(x) &\leqq 2r(0)+\varepsilon(0) \end{aligned}$$

for all its coordinates will be called preunipotent.

The irreducible representations in $\mathscr{X}(\lambda)$ will also be called preunipotent. If the group is a product, the representation is called preunipotent if all factors in the tensor product are preunipotent.

9.4. Definition. An $\bar{X}(\lambda, \lambda')$ will be called preunitary if it is the lowest K-type subquotient of a unitarily induced from a preunipotent representation on a Levi component.

Proposition. *Any weakly preunipotent representation $\bar{X}(\lambda, \lambda')$ with small lowest K-type satisfying the necessary condition of Proposition 9.2 is preunitary.*

Proof. It is enough to show the claim for a representation such that, in the notation of 9.1, the only $\mathscr{F}_1^j$ are precisely the

$$\mathscr{F}_1^j\colon (M_j, \ldots, -M_j+1)\times(M_j-1, \ldots, -M_j), \tag{9.4.1}$$

one for each M_j such that $\varepsilon(M_j+1)=1$.

We show that λ must be preunipotent. Assume for simplicity that there is only one M_j such that $\varepsilon(M_j+1)=1$. Denote it by M. Then

$$r_\lambda(x)=\begin{cases} r_{\lambda_0}(x) & \text{for } x\geqq M+1, \\ r_{\lambda_0}(x)+1 & \text{for } x=M, \\ r_{\lambda_0}(x)+2 & \text{for } 0<x<M, \\ r_{\lambda_0}(0)+1 & \text{for } x=0. \end{cases} \tag{9.4.2}$$

The claim follows in view of (9.1.7).

We show that $\bar{X}(\lambda, \lambda')\in\mathscr{X}(\lambda)$. Again assume for simplicity that there is only one M_j. Compare $\{\mathfrak{m}(\lambda), \mathscr{F}^1\}$ with $\{\mathfrak{m}_0^0, \mathscr{A}_0^i, \mathscr{A}_1^i, \mathscr{F}_0^i, \mathscr{F}_1^1\}$. There is an $i\geqq 0$ such that

$$F_0^i\colon (M^i, M^i-1, \ldots M+1, M-1, M-2, \ldots, -m^i) \tag{9.4.3}$$

M^i is determined by the condition $r(M^i+1)<r(M^i)=r(M+1)$.

Remove $M^i, \ldots, M+1$ from $\mathscr{F}_0^i$ and add them to $\mathscr{F}_1^1$. This forms two new finite dimensional representations

$$(M-1, M-2, \ldots, -m^i)\times(M-1, M-2, \ldots, -m^i), \quad \text{and} \tag{9.4.4a}$$

$$(M^i, \ldots, M+1, M, \ldots, -M+1)\times(M^i, \ldots, M+1, M-1, \ldots, -M). \tag{9.4.4b}$$

In the notation of 6.2, 6.3, (9.4.4a) is an $\mathscr{F}_1^i$ and (9.4.4b) is an $\mathscr{F}_{-1}^{i'}$ for $\mathscr{X}(\lambda)$.

The proof follows.

9.5. Theorem. *An $\bar{X}(\lambda, \lambda')$ with integral infinitesimal character and arbitrary lowest K-type is unitary only if it is preunitary.*

Otherwise, the form is negative on a level $\leqq 2$ K-type of the form $\mu+\mu_0(1)$ or $\mu+\mu_0(1)+\mu_1(1)$ where $\mu_0(1)$ is small for $\mathfrak{m}_{\mu,0}$ or $\mathfrak{m}_{\mu,i}$. The second possibility $\mu+\mu_0(1)+\mu_1(1)$, occurs only if $i=0$ and then $\mu_1(1)$ is small for $\mathscr{A}_{\mu+\mu_0(1),1}$.

Proof. This is an easy consequence of 9.1–9.4 and the results about bottom layer K-types in 2.9 and Sect. 3.

9.6. *Assume $\bar{X}$ is preunipotent with integral infinitesimal character.*

Definition. We say that λ is rigid if the corresponding nilpotent orbit in $\mathfrak{g}$ is not smoothly induced from any proper Levi component.

In coordinates these conditions are as follows.

Type B, C

Integers. The symbol in (6.2.1) is given by

$$(x_0)\ldots(x_{2i-1}\, x_{2i})\ldots(x_{2p-1}\, x_{2p}). \tag{9.6.1}$$

The rigidity condition is $x_{2i-1}<x_{2i}$ for all i. The coordinates of the F^i are given by

$$\begin{aligned} &F^0\colon (x_0, \ldots, 1), \\ &F^{p-i+1}\colon (x_{2i}-i, \ldots, 1, 0, -1, \ldots, -(x_{2i-1}-i)), \quad \text{for } 1\leqq i\leqq p. \end{aligned} \tag{9.6.2}$$

The representation in $\mathscr{X}(\lambda)$ are obtained by the procedure outlined in 6.2.

Replace $\bar{X}$ by $\overline{\bar{X}*\mathscr{F}(M)}$. This has the following effect.

Suppose there is i such that $x_{2i-1}-i<M<x_{2i}-i$. Then the symbol changes to

$$\begin{aligned} &(x_0)\ldots(x_{2i-1}, M+i)(M+i+1, x_{2i}+1)(x_{2i+1}+1, x_{2i+2}+1) \\ &\quad\ldots(x_{2p-1}+1, x_{2p}+1). \end{aligned} \tag{9.6.3}$$

If $M<x_0$ then the symbol changes to

$$(M)(M+1, x_0+1)\ldots(x_{2p-1}+1, x_{2p}+1). \tag{9.6.4}$$

If there is an i such that $x_{2i}-i \leqq M \leqq x_{2i+1}-2i-1$ then the new symbol is obtained by adding a pair $(M+i+1,\ M+i+1)$ and adding a 1 to all the pairs after (x_{2i-1}, x_{2i}). In case (9.6.3),

$$\bar{X} * \mathscr{F}(M) = \bar{X}_{\{1,\dots,1\}} + \bar{X}_{\{1,\dots,-1,-1,1,\dots,1\}} \tag{9.6.5}$$

where the -1 correspond to $(x_{2i-1}, M+i)$, $(M+i+1, x_{2i-1}+1)$.
In case (9.6.4),

$$\bar{X} * \mathscr{F}(M) = \bar{X}_{\{1,\dots,1\}} + \bar{X}_{\{1,\dots,1,-1\}}. \tag{9.6.6}$$

Half-integers. The symbol in 6.2 is given by

$$(y_0\, y_1)\dots(y_{2j}\, y_{2j+1})\dots(y_{2q-2}\, y_{2q-1})(y_{2q}). \tag{9.6.7}$$

The coordinates of the F^j are

$$\begin{aligned} F^0&: (y_{2q}-q-1/2, \dots, 1/2), \\ F^{q-j+1}&: (y_{2j+1}-j-1/2, \dots, 1/2, -1/2, \dots, -(y_{2j}-j-1/2)), \quad \text{for } 1 \leqq j \leqq q. \end{aligned} \tag{9.6.8}$$

If $y_{2j}-j < M < y_{2j+1}-j$, then $\overline{\bar{X} * \mathscr{F}(M)}$ has symbol

$$\begin{aligned} &(y_0\, y_1)\dots(y_{2j}, M+j)(M+j+1, y_{2j+1}+1) \\ &\quad \dots(y_{2q-2}+1, y_{2q-1}+1)(y_{2q}+1). \end{aligned} \tag{9.6.9}$$

If $y_{2q}-q < M$, then $\overline{\bar{X} * \mathscr{F}(M)}$ has symbol

$$(y_0\, y_1)\dots(y_{2j}\, y_{2j+1})\dots(y_{2q-2}, y_{2q-1})(y_{2q}, M+q)(M+q+1). \tag{9.6.10}$$

Similar to the integers case, in 9.6.9,

$$\bar{X} * \mathscr{F}(M) = \bar{X}_{\{1,\dots,1\}} + \bar{X}_{\{1,\dots,-1,-1,1,\dots,1\}}, \tag{9.6.11}$$

where the -1 correspond to $(y_{2j}, M+j)$, $(M+j+1, y_{2j+1}+1)$. In case 9.6.10,

$$\bar{X} * \mathscr{F}(M) = \bar{X}_{\{1,\dots,1\}} + \bar{X}_{\{-1,1,\dots,1\}}, \tag{9.6.12}$$

where the -1 corresponds to $(y_{2q}, M+q)$.

Type D

Integers. The symbol in (6.3.1) is

$$(x_0\, x_{2p+1})\dots(x_{2i-1}\, x_{2i})\dots(x_{2p-1}\, x_{2p}). \tag{9.6.13}$$

The rigidity condition is $x_{2i-1} < x_{2i}$ for all i. The F^i have coordinates

$$\begin{aligned} F^0&: (x_{2p+1}-p-1, \dots, 1, 0, -1, \dots, -x_0), \\ F^{p-i+1}&: (x_{2i}-i, \dots, 1, 0, -1, \dots, -(x_{2i-1}-i)), \quad \text{for } 1 \leqq i \leqq p. \end{aligned} \tag{9.6.14}$$

If $x_{2i-1}-i < M < x_{2i}-i$ then $\overline{\bar{X} * \mathscr{F}(M)}$ satisfies

$$\begin{aligned} &(x_0, x_{2p+1}+1)\dots(x_{2i-1}, M+i)(M+i+1, x_{2i}+1) \\ &\quad \dots(x_{2p-1}+1, x_{2p}+1), \end{aligned} \tag{9.6.15}$$

$$\bar{X} * \mathscr{F}(M) = \bar{X}_{\{1,\dots,1\}} + \bar{X}_{\{1,\dots,-1,-1,\dots,1\}},$$

where the -1 correspond to $(x_{2i-1}, M+i)$, $(M+i+1, x_{2i}+1)$. If $M > x_{2p+1}-p$, then $\overline{\bar{X} * \mathscr{F}(M)}$ satisfies

$$\begin{aligned} &(x_0, M+p+1)(x_1, x_2)\dots(x_{2p-1}, x_{2p})(x_{2p+1}, M+p), \\ &\bar{X} * \mathscr{F}(M) = \bar{X}_{\{1,\dots,1\}} + \bar{X}_{\{-1,1,\dots,1\}}, \end{aligned} \tag{9.6.16}$$

where the -1 corresponds to the pair $(x_{2p+1}+1, M+p)$.
If $M < x_0$, then $\overline{\bar{X} * \mathscr{F}(M)}$ satisfies

$$\begin{aligned} &(M, x_{2p+1})(M+1, x_0+1)(x_1+1, x_2+1)\dots(x_{2p-1}+1, x_{2p}+1), \\ &\overline{\bar{X} * \mathscr{F}(M)} = \bar{X}_{\{1,\dots,1\}} + \bar{X}_{\{1,\dots,1,-1\}}, \end{aligned} \tag{9.6.17}$$

where the -1 corresponds to the pair (M, x_0).

Half-integers. The symbol in (6.3.1) is

$$(y_0\, y_1)\dots(y_{2j}\, y_{2j+1})\dots(y_{2q}\, y_{2q+1}). \tag{9.6.18}$$

The rigidity condition is $y_{2j} < y_{2j+1}$ for all j. The F^j satisfy

$$\begin{aligned} F^{q-j+1}&: (y_{2j+1}-j-1/2, \dots, 1/2, -1/2, \dots \\ &\quad \dots, -(y_{2j}-j-1/2)) \quad \text{for } 1 \leqq j \leqq q. \end{aligned} \tag{9.6.19}$$

If $y_{2j}-j < M < y_{2j+1}-j$, then $\overline{\bar{X} * \mathscr{F}(M)}$ satisfies

$$\begin{aligned} &(y_0\, y_1)\dots(y_{2j}, M+j)(M+j+1, y_{2j+1}+1)\dots(y_{2q}+1, y_{2q+1}+1), \\ &\bar{X} * \mathscr{F}(M) = \bar{X}_{\{1,\dots,1\}} + \bar{X}_{\{1,\dots,-1,-1,\dots,1\}}, \end{aligned} \tag{9.6.20}$$

where the -1 correspond to $(y_{2j}, M+j)$, $(M+j+1, y_{2j}+1)$.
In the other cases, $\bar{X} * \mathscr{F}(M)$ is irreducible.

10. The unitary spectrum for half-integral infinitesimal character

10.1. Definition. Let V_1, V_2 be hermitian irreducible $(\mathfrak{g}_1, K_1)$ and $(\mathfrak{g}_2, K_2)$ modules respectively with the forms normalized to be positive on the lowest K-type. We say $V_1 \underset{u}{\sim} V_2$ if there exist unitary irreducible $(\mathscr{A}_i, K_i)$ modules Y_i, $Y_{3-i,j}$ and a reductive algebra $\mathfrak{g}$ such that

$$[V_i * Y_i]_\pm = \sum_j c_{i,j} [V_{3-i} * Y_{3-i,j}]_\pm, \tag{10.1.1}$$

for $c_{i,j}>0$ and $i=1$ or 2.

Clearly if $V_1 \underset{u}{\sim} V_2$, then V_1 is unitary if and only if V_2 is unitary.

Assume $\mathfrak{g}$ *is not type A in Sects. 10.2 to 10.6.*

10.2. Assume λ is such that ${}^L\mathfrak{s}(\lambda)$ is a proper maximal subalgebra. Write $\mathfrak{s}(\lambda)=\mathfrak{s}_e\times\mathfrak{s}_o$. If

$$T(\bar{X}(\lambda,\lambda))=\bar{X}_e*\bar{X}_o, \tag{10.2.1}$$

then $\mathfrak{s}_e$ is distinguished by the fact that the coordinates of the parameter of $\bar{X}_e$ are of the same parity as the trivial representation of $\mathfrak{g}$. In coordinates, the infinitesimal character is as follows.

Type B. Half-integers on $\mathfrak{s}_e$, integers on $\mathfrak{s}_o$.

Type C. Integers on $\mathfrak{s}_e$, half-integers on $\mathfrak{s}_o$.

Type D. Integers on $\mathfrak{s}_e$, half-integers on $\mathfrak{s}_o$.

Let Y be an $(\mathscr{A}, K_{\mathscr{A}})$ module whose parameter has integer or half-integer coordinates. Then

$$\begin{aligned} T(\bar{X}*Y)&=[\bar{X}_e*Y]\otimes\bar{X}_o \quad \text{or}\\ &=\bar{X}_e\otimes[\bar{X}_o*Y]. \end{aligned}\tag{10.2.2}$$

Because we did not use continuous deformations in the arguments in 8 and 9, the necessary conditions for unitarity hold for $\bar{X}_e, \bar{X}_o$ separately. Thus we obtain the following theorem.

Theorem. *An irreducible representation with half integral character λ is unitary only if both $\bar{X}_e, \bar{X}_o$ are preunitary. Otherwise the form is negative on a level $\leqq 2$ K-type of the form $\mu+\mu_0(1)$ or $\mu+\mu_0(1)+\mu_1(1)$ where $\mu_0(1)$ is a small K-type for $\mathfrak{m}_{\mu,0}$ and $\mu_1(1)$ is a small K-type for $\mathscr{A}_{\mu+\mu_0(1),1}$.*

We also say that λ is preunitary in this case.

Similarly, if $\mathfrak{g}$ is a product of reductive algebras, $\bar{X}$ is called preunitary if each factors in the tensor product is preunitary.

Note that if $\bar{X}_e$ is preunipotent, it is also special unipotent.

10.3. Assume that λ is half-integral preunipotent rigid. Define

$$\mathfrak{m}(\lambda)=\mathfrak{m}_e(\lambda)\times\mathscr{A}_e^1\times\ldots\times\mathscr{A}_e^{k_e}\times\mathscr{A}_o^0\times\ldots\times\mathscr{A}_o^{k_o}. \tag{10.3.1}$$

Definition. Let $\mathfrak{m}\times\mathscr{A}$ be a maximal Levi component so that $\mathscr{A}$ is one of the $\mathscr{A}_o^j$ and let ξ be a character of $\mathfrak{m}\times\mathscr{A}$. We say that λ is extremal, if for all such $\mathfrak{m}\times\mathscr{A}$,

$$\mathfrak{m}(\lambda)=\mathfrak{m}(\lambda+t\xi),$$

for all $t\in\mathbb{R}$ such that $\lambda+t\xi$ is half-integral and $\|\lambda+t\xi\|<\|\lambda\|$. In this case we refer to the irreducible representations in $\mathscr{X}(\lambda)$ as extremal as well.

We will say that $\bar{X}$ is strongly rigid if in addition to λ being preunipotent rigid, $\bar{X}$ is not of the form $\bar{Z}*Y_{\mathscr{A}}$ where $Y_{\mathscr{A}}$ is a unitary module.

Typically we will use this notion in the special case when all the $\mathscr{F}_e^i, \mathscr{F}_o^j$ satisfying

$$F_L:(M,\ldots,-M+1) \tag{10.3.2}$$

are spherical.

10.4. Lemma. *Assume λ is preunipotent rigid, not extremal. If $\bar{X}$ is not strongly rigid then there is $\bar{Z}$ such that $\bar{X}\underset{u}{<}\bar{Z}$ satisfying the following conditions.*

There are $I>0$, $J>0$ such that $\mathscr{F}_*^I$ is spherical satisfying

$$F_*^I:(M, M-1,\ldots,-M+1), \qquad F_{**,L}^J:(M+1/2, M-1/2,\ldots,-M+3/2).$$

*Here * and ** denote e or o.*

Proof.

Type B. The parameter fails to be extremal whenever there are i,j such that

$$y_{2j}-j-1/2<x_{2i}-i-1/2\leqq y_{2j+1}-j-1/2, \quad \text{or} \tag{10.4.1a}$$

$$y_{2j}-j-1/2\leqq x_{2i-1}-i+1-1/2<y_{2j+1}-j-1/2. \tag{10.4.1b}$$

Whenever $x_{2i-1}-i+1-1/2=x_{2i}-i-1/2$, strict inequalities must hold everywhere in (10.4.1). If $i=0$ only (10.4.1a) makes sense. If $j=2q$, in (10.4.1a) only the first inequality makes sense.

Assume $x_{2i-1}-i+1-1/2=x_{2i}-i-1/2$. Replace $\bar{X}$ by $\bar{Z}=\overline{\bar{X}*\mathscr{F}(m)}$ such that $y_{2j}-j-1/2\leqq m<x_{2i-1}-i+1-1/2$ and $x_{2i}-i+1/2<m\leqq y_{2j+1}-j-1/2$ with $2m+1\in\mathbb{N}$. This does not affect (10.4.1). In finitely many steps we get a $\bar{Z}$ such that $\bar{X}\underset{u}{<}\bar{Z}$ satisfying

$$y_{2J}-J-1/2+1=x_{2i}-i-1/2=y_{2J+1}-J-1/2-1 \tag{10.4.2}$$

for some J. Set $M=x_{2i}-i$. Then there are I, J such that

$$\begin{aligned} &F^I:(M, M-1,\ldots,-M+1),\\ &F^J:(M+1/2, M-1/2,\ldots,-M+3/2). \end{aligned}\tag{10.4.3}$$

Assume $x_{2i-1}-i+1-1/2<x_{2i}-i-1/2$ *and* (10.4.1a) *holds*. Replace $\bar{X}$ by $\bar{Z}=\overline{\bar{X}*\mathscr{F}(m)}$ with $m\in\mathbb{N}$ for $x_{2i-1}-i<m<x_{2i}-i$. This does not affect (10.4.1a). In finitely many steps we get a $\bar{Z}$ such that $\bar{X}\underset{u}{<}\bar{Z}$ satisfying

$$x_{2I-1}-I+2=x_{2i}-i=x_{2I}-I \tag{10.4.4}$$

for some I.

(10.4.1a) is also not affected if we replace $\bar{X}$ by $\bar{Z}=\overline{\bar{X}*\mathscr{F}(m)}$ with $y_{2j}-j-1/2<m\leqq x_{2i}-i-1/2$ or $x_{2i}-i-1/2\leqq y_{2j+1}-j-1/2$, $2m\in 2\mathbb{N}+1$. In finitely many steps we get a $\bar{Z}$ such that $\bar{X}\underset{u}{<}\bar{Z}$ satisfying

$$y_{2J}-J-1/2+1=x_{2i}-i-1/2=y_{2J+1}-J-1/2 \tag{10.4.5}$$

for some J. Set $M = y_{2J+1} - J - 1/2$. Then

$$F^I: (M, M-1, \ldots, -M+1),$$
$$F^J: (M+1/2, M-1/2, \ldots, -M+3/2). \tag{10.4.6}$$

Note also that in all cases $\mathscr{F}^I$ is spherical because of the condition that $\bar{X}$ is not strongly rigid.

Similarly for (10.4.1 b).

Type C. The parameter fails to be extremal whenever there is i, j such that

$$x_{2i-1} - i < y_{2j+1} - j - 1 \leqq x_{2i} - i, \tag{10.4.7a}$$

or

$$x_{2i-1} - i \leqq y_{2j} - j < x_{2i} - i. \tag{10.4.7b}$$

If $i=0$, only the second inequality in (10.4.7) makes sense. If $j=2q$, only (10.4.7b) makes sense. If $y_{2j+1} - j - 1 = y_{2j} - j$, strict inequality must hold in (10.4.7).

Assume $y_{2j} - j = y_{2j+1} - j - 1$. Then (10.4.7a) and (10.4.7b) are equivalent. Replace $\bar{X}$ by $\overline{X * \mathscr{F}(m)}$ with $x_{2i-1} - i < m < x_{2i} - i$ repeatedly until there is I such that

$$x_{2I} - I = x_{2I-1} - I + 2 = y_{2j+1} - j - 1. \tag{10.4.8}$$

Let $M = x_{2I} - I - 1/2$. Then (10.4.6) holds.

Assume $y_{2j} - j < y_{2j+1} - j - 1$. Replace $\bar{X}$ by $Z = \overline{X * \mathscr{F}(m)}$ with $2m \in 2\mathbb{N}+1$ for $y_{2j} - j - 1/2 < m \leqq y_{2j+1} - j$ repeatedly until there is I such that

$$y_{2I} - I = y_{2I+1} - I - 2 = y_{2j+1} - j - 2. \tag{10.4.9}$$

Then replace $\bar{X}$ by $Z = \overline{X * \mathscr{F}(m)}$ with $x_{2i-1} - i < m \leqq y_{2j+1} - j$ and $y_{2j+1} - j < m < x_{2i} - i$, $m \in \mathbb{N}$ until there is a J such that

$$x_{2J-1} - J = x_{2J} - J - 1 = y_{2I+1} - I. \tag{10.4.10}$$

Let $M = x_{2J} - J$. Then (10.4.6) holds.

Similarly for (10.4.7b).

Type D. The parameter fails to be extremal whenever there is i, j such that

$$x_{2i-1} - i < y_{2j+1} - j - 1 \leqq x_{2i} - i, \tag{10.4.11a}$$

or

$$x_{2i-1} - i \leqq y_{2j} - j < x_{2i} - i. \tag{10.4.11b}$$

If $i=0$, only the second inequality in (10.4.11) makes sense. If $y_{2j+1} - j - 1 = y_{2j} - j$, strict inequality must hold in (10.4.11).

This is as in type C, even though the nature of the symbol is different, because of the similarity of the formulas in 9.6.

The proof follows.

10.5. Proposition. *Assume λ is preunipotent rigid but not extremal. Then the form is negative on a K-type $\mu + \mu_0$ where μ_0 is small for $\mathfrak{m}_{\mu,0}$.*

Proof. It is enough to prove the statement for the $\bar{Z}$ in Lemma 10.4; for the relevant restriction of small K-types to the Levi component is formed only of small K-types by Sect. 3.

Thus assume $\bar{X}$ satisfies the conclusion of Lemma 10.4. At this point we have to use continuous deformations.

Recall $\bar{X}(\lambda_0, \lambda_0)$, $\bar{X}(\lambda_1, \lambda_1')$ from (9.1.2). Then we can write

$$\bar{X}(\lambda_0, \lambda_0) = \bar{X}^I * \mathscr{F}^I. \tag{10.5.1}$$

Thus,

$$X(t) = \bar{X}^I * \mathscr{F}(M-1/2; 1/2+t) \tag{10.5.2}$$

is irreducible for $0 \leqq t < 1/2$. By replacing $\bar{X}$ with $\bar{X} * \mathscr{F}(0)$ in type D if necessary, we may also assume $X(t)$ hermitian. Let $\bar{X}(1/2)$ be the lowest K-type subquotient at $t = 1/2$. Then because of the properties of small and bottom layer K-types in Sects. 2–3,

$$[\bar{X}^\gamma]_\pm \geqq [\bar{X}(1/2)^{\mu_0}]_\pm, \tag{10.5.3}$$

for $\gamma = \mu + \mu_0$ where μ_0 is any small $M \cap K$-type for $\mathfrak{m}_{\mu,0}$.

Assume $\mathscr{F}^J$ is not spherical. Then if $\bar{X}(1/2) = \bar{X}(\lambda_0(1/2), \lambda_0(1/2))$,

$$r_{\lambda_0(1/2)}(M+1/2) = r_{\lambda_0(1/2)}(M-1/2) + 1, \qquad \varepsilon(M+1/2) = 0. \tag{10.5.4}$$

Thus $\bar{X}(1/2)$ is not weakly unipotent. By 9.2, the form is negative on a small K-type $\mu + \mu_0(r)$.

Assume $\mathscr{F}^J$ is spherical. We show that the form on $\bar{X}(1/2)$ is negative on a *level* $\leqq 1$ K-type of the form $\mu + \mu_0$ with μ_0 small for $\mathfrak{m}_{\mu,0}$.

Note that $\bar{X}(1/2)$ is the lowest K-type subquotient in an $\bar{X}^{IJ} * \mathscr{F}^I * \mathscr{F}^J * \mathscr{F}(M-1/2; 1)$, for the appropriate $\bar{X}^{IJ}$. The representation $\mathscr{F}^J * \mathscr{F}(M-1/2; 1)$ is as in (3) of Lemma 8.1. It is also hermitian. The factor $\bar{X}_u$ from (3), Lemma 8.1 has lowest K-type $\mu(2M)$ and its hermitian form is negative on $\mu(2M+1)$. Thus the form on $\bar{X}^{IJ} * \mathscr{F}^I * \mathscr{F}^J * \mathscr{F}(M-1/2; 1) * \bar{X}(\lambda_1, \lambda_1')$ is indefinite on $\mu + \mu_0(2M)$, $\mu + \mu_0(2M+1)$ where $\mu_0(2M)$, $\mu_0(2M+1)$ are small for $\mathfrak{m}_{\mu,0}$. By the irreducibility of $X(t)$, the same must hold for $\bar{X}^{IJ} * \mathscr{F}^J * \mathscr{F}^I$.

To complete the proof it is enough to show that the multiplicity of $\mu + \mu_0(2M)$, $\mu + \mu_0(2M+1)$ is the same in $\bar{X}$ and $X(I, J) = \bar{X}^{IJ} * \mathscr{F}^J$.

By Corollary 8.2,

$$\bar{X}^{IJ} * \mathscr{F}^J * \mathscr{F}(M-1/2) = \bar{X}^{IJ} * \mathscr{F}' * \mathscr{F}'' + \bar{X}^{IJ} * \mathscr{F}^{IJ} * \mathscr{F}^{IJ}, \tag{10.5.5}$$

where

$$\mathscr{F}^{IJ}: (M+1/2, \ldots, -M+1/2) \times (M-1/2, \ldots, -M-1/2),$$
$$\mathscr{F}': (M+1/2, \ldots, -M+1/2) \times (M+1/2, \ldots, -M+1/2), \tag{10.5.6}$$
$$\mathscr{F}'': (M-3/2, \ldots, -M+3/2) \times (M-3/2, \ldots, -M+3/2).$$

$\bar{X}^{IJ} * \mathscr{F}' * \mathscr{F}''$ is irreducible in view of Sect. 6. The lowest K-type of $\bar{X}^{IJ} * \mathscr{F}^{IJ} * \mathscr{F}^{IJ}$ is $\mu(2M)$; and any small K-type of the form $\mu(2M) + \mu'$ with μ' small for the factor in the Levi component corresponding to $\bar{X}^{IJ}$ occurs with full multiplicity in its lowest K-type subquotient. On the other hand, by 6.6,

$$\overline{\bar{X}^{IJ} * \mathscr{F}^J} * \mathscr{F}(M-1/2) = \overline{\bar{X}^{IJ} * \mathscr{F}' * \mathscr{F}''} + \overline{\bar{X}^{IJ} * \mathscr{F}^{IJ} * \mathscr{F}^{IJ}}. \tag{10.5.7}$$

By restriction to the relevant Levi component, all small K-types occur with the same multiplicity in $\bar{X}^{IJ} * \mathscr{F}^J$ as in $\overline{\bar{X}^{IJ} * \mathscr{F}^J}$.

10.6. Proposition. *Assume λ is preunipotent rigid and extremal. Then all $\bar{X} \in \mathscr{X}(\lambda)$ are unitary.*

Proof. Let

$$l(\lambda) = \sum (x_{s+1} - x_s - 1) + \sum (y_{r+1} - y_r - 1). \tag{10.6.1}$$

We do an induction on $l(\lambda)$.

If $l(\lambda)=0$ then $\bar{X}$ is unitarily induced from a complementary series.

If $l(\lambda)>0$ let $\mathscr{F}^i$ be such that the corresponding pair (x_s, x_{s+1}) or (y_r, y_{r+1}) contributes a number larger than 0 in (10.6.1). Write

$$F_L^i: (M^i, \ldots, m^i+2, m^i+1, m^i, \ldots, -m^i) \tag{10.6.2}$$

where $M^i \geqq m^i+2$.

Assume $\mathscr{F}^i$ is spherical. If $M^i = m^i+2$ then $\bar{X}$ is at the end of a complementary series; for if we write $\bar{X}^i * \mathscr{F}(m^i+1; t)$, the condition of extremal shows that this is irreducible for $0 \leqq t < 1$. At $t=1$, $\overline{\bar{X}^i * \mathscr{F}(m^i+1; t)} = \bar{X}$. Thus assume $M^i > m^i+2$. Write

$$X(M^i-1) = \bar{X} * \mathscr{F}(M^i-1) = \bar{X}_1(M^i-1) + \bar{X}_{-1}(M^i-1) \tag{10.6.3}$$

where $\bar{X}_{\pm 1}$ are the X_χ in 6.6. Similarly define $X(m^i+1)$, $\bar{X}_1(m^i+1)$, $\bar{X}_{-1}(m^i+1)$.

Write $\mathscr{F}_1(m; t) = F(m; t) \otimes F(m; t)$, $\mathscr{F}_{-1}(m; t) = F(m; t) \otimes F(m; t)^h$. Then

$$\begin{aligned} \bar{X}_{\pm 1}(M^i-1) &= \overline{\bar{X}^i * \mathscr{F}_{\pm 1}((M^i+m^i-1)/2; (M^i-m^i-1)/2)} \\ &\quad * \mathscr{F}_{\pm 1}(M^i-1/2; 1/2), \end{aligned} \tag{10.6.4a}$$

$$\begin{aligned} \bar{X}_{(\pm 1)}(m^i+1) &= \overline{\bar{X}^i * \mathscr{F}_{\pm 1}((M^i+m^i+1)/2; (M^i-m^i-1)/2)} \\ &\quad * \mathscr{F}_{\pm 1}(m^i+1/2; 1/2), \end{aligned} \tag{10.6.4b}$$

where

$$\begin{aligned} &F((M^i+m^i-1)/2; (M^i-m^i-1)/2): (M^i-1, \ldots, -m^i), \\ &F(M^i-1/2; 1/2): (M^i, \ldots, -M^i+1), \\ &F((M^i+m^i+1)/2; (M^i-m^i-1)/2): (M^i, \ldots, -m^i-1), \\ &F(m^i+1/2; 1/2): (m^i+1, \ldots, -m^i). \end{aligned} \tag{10.6.5}$$

It is enough show that

$$\bar{X} \underset{u}{\sim} \overline{\bar{X}^i * \mathscr{F}_1((M^i+m^i+1)/2; (M^i-m^i-1)/2)}.$$

By induction on rank, this will complete the proof.

By induction $\bar{X}_{\pm 1}(M^i-1)$, $\bar{X}_{\pm 1}(m^i+1)$ are unitary. It is therefore enough to show that the induced form in (10.6.4b) is positive on the lowest K-type $\mu_{-1}(m^i+1)$ of $\bar{X}_{-1}(m^i+1)$ in $X(m^i+1)$. For this we note that, in $X(M^i-1)$ all K-types of the form

$$\mu(s) = \mu + \beta_1 + \ldots + \beta_s \tag{10.6.6}$$

with $s < 2M^i + 2m^i$ are contained in $\bar{X}_1(M^i-1)$ with full multiplicity since its lowest K-type corresponds to $s = M^i + 2m^i$. Thus the induced form on $X(M^i-1)$ is positive on them. Therefore the analogous statement is true for $\bar{X}$. The restriction of $\mu_{-1}(m^i+1) = \mu + \beta_1 + \ldots + \beta_{4m^i+4}$ to $\mathfrak{g}(n)$ only contains K-types of the form (10.6.6) with $s \leqq 4m^i+4$. Since

$$4m^i + 4 < 2m^i + 2(M^i-2) + 4 = 2m^i + 2M^i, \tag{10.6.7}$$

the claim follows.

Assume $\mathscr{F}^i$ is not spherical. Then a similar argument applies. The lowest K-types of $\bar{X}_{-1}(M^i-1)$ and $\bar{X}_{-1}(m^i+1)$ are

$$\begin{aligned} \mu_{-1}(M^i-1) &= \mu + \beta_1 + \ldots + \beta_{2M^i-1}, \\ \mu_{-1}(m^i+1) &= \mu + \beta_1 + \ldots + \beta_{2m^i+3}. \end{aligned} \tag{10.6.8}$$

The inequality (10.6.7) is replaced by

$$2m^i + 3 \leqq 2(M^i-2) + 3 = 2M^i - 1. \tag{10.6.9}$$

The proof is complete.

I wish to thank David Vogan for simplifying my original argument for the unitarity of the representations with extremal parameter.

10.7. Definition. The preunipotent representations with half-integral infinitesimal character which are unitary are called unipotent. These are the smoothly induced from the preunipotent rigid extremal representations.

We can summarize the results in this section as follows.

Theorem. *A representation $\bar{X}(\lambda, \lambda')$ with half-integral infinitesimal character is unitary if and only if it is the lowest K-type subquotient of a unitarily induced from a unipotent representation (tensored with a unitary character).*

Otherwise the form is negative on a level $\leqq 2$ K-type for μ.

11. A distinguished set of unipotent representations

In this section we define a subset of the unipotent representations which contains the set of cuspidal representations in the sense that the associated nilpotent orbit (in $\mathfrak{g}$) is not induced. Some of the proofs which can be read off from the explicit formulas for rigid extremal parameters will be left to the reader.

11.1. Let

$$^L\phi: sl(2, \mathbb{C}) \to {}^L\mathfrak{g} \tag{11.1.1}$$

correspond to an even rigid nilpotent orbit. Then there is an exact sequence

$$1 \to \kappa \to A(^L\phi) \to \bar{A}(^L\phi) \to 1. \tag{11.1.2}$$

For each semisimple $s \in A(^L\phi)$ such that $^LG(s)$ is maximal, let $\mathcal{O}_s \subset \mathfrak{g}(s)$ be the dual orbit to $^L\mathcal{O} \subset {}^L\mathfrak{g}(s)$ viewed as a nilpotent orbit in $^L\mathfrak{g}(s)$. Then there is an $\bar{X}$ with infinitesimal character λ_s satisfying

$$e^{2\pi i \lambda_s} = s, \qquad WF(T_{\mathfrak{g}(s)}(\bar{X})) = \bar{\mathcal{O}}_s. \tag{11.1.3}$$

In analogy with the special unipotent case, it is reasonable to impose the condition that $\|\lambda_s\|$ be minimal subject to (11.1.3). We are going to describe conditions on s under which the $\bar{X}$ are unitary.

Type B. In this case ^LG is type C. Then $^L\mathcal{O}$ is even with Jordan blocks of even size,

$$0 \leqq 2n_0 < 2n_1 \leqq 2n_2 < \ldots < 2n_{2i-1} \leqq 2n_{2i} < \ldots \leqq 2n_{2p}. \tag{11.1.4}$$

Then $G(^L\phi)$ is a product of $O(2)$'s, one for each pair $n_{2i-1} = n_{2i}$ and $O(1)$'s, one for each remaining $n_j > 0$.

An element in $G(^L\phi)$ which gives rise to half integral infinitesimal character can be written as

$$s = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2p}) \tag{11.1.5}$$

where $\varepsilon_0 = 1$ if $n_0 = 0$ and $\varepsilon_i = \pm 1$. Then for such elements,

$$s \in \kappa \Leftrightarrow \varepsilon_{2i-1} \cdot \varepsilon_{2i} = 1. \tag{11.1.6}$$

Given s, the element λ_s of minimal length satisfying (11.1.3) is obtained as follows.

We describe the infinitesimal characters of $\bar{X}_o$, $\bar{X}_e$ corresponding to $\bar{X}$ as in 10.2.

(1) For each $\varepsilon_i = 1$ add

$$(n_i - 1/2, \ldots, 1/2)$$

to the infinitesimal character of $\bar{X}_e$.

(2) For the $\varepsilon_i = -1$, reorder the corresponding n_i in increasing order and add a row of size 0 if necessary, so that there are an odd number, $m_0, m_1, \ldots, m_{2q}$. Then add

$$\begin{aligned} &(m_j, \ldots, 1) && \text{for } j \text{ odd}, \\ &(m_j - 1, \ldots, 1, 0) && \text{for } j \text{ even}, \end{aligned}$$

to the infinitesimal character of $\bar{X}_o$.

Every coset modulo κ has a representative of the form

$$\begin{aligned} &(\eta_0, \eta_0, \ldots, \eta_{p-1}, \eta_{p-1}, 1) && \text{if } n_0 \neq 0, \\ &(1, \eta_0, \ldots, \eta_{p-1}, \eta_{p-1}, 1) && \text{if } n_0 = 0. \end{aligned} \tag{11.1.7}$$

The results of chapter 10 show that the unitary representations satisfying (11.1.3) have infinitesimal character λ_s associated to a representative as in (11.1.7). These λ_s can be characterized as follows.

Fix an $\bar{s} \in \bar{A}(^L\phi)$. Then let λ_s be the smallest length semisimple element in the Lie algebra such that $e^{2\pi i \lambda_s}$ maps to $\bar{s}$ under the map in (11.1.2).

We show that the representative in (11.1.7) gives the minimum among the λ_s with s as in (11.1.5).

Let

$$s_j = (1, 1, \ldots, \underset{2j-1}{-1}, \underset{2j}{-1}, 1, \ldots, 1). \tag{11.1.8}$$

If $\varepsilon_{2p} = -1$ replace s by $s \cdot s_p$. Then λ_s changes as follows.

$$\begin{aligned} &(n_{2p}, \ldots, 1) && \to (n_{2p} - 1/2, \ldots, 1/2) \quad \text{and} \\ &(n_{2p-1} - 1, \ldots, 1, 0) && \to (n_{2p-1} - 1/2, \ldots, 1/2) \quad \text{or} \\ &(n_{2p-1} - 1/2, \ldots, 1/2) && \to (n_{2p-1}, \ldots, 1). \end{aligned} \tag{11.1.9}$$

Then $\|\lambda_{s \cdot s_p}\| \leqq \|\lambda_s\|$ with equality precisely when $n_{2p-1} = n_{2p}$. Continue in this way by replacing s with $s \cdot s_{j+1}$ every time $\varepsilon_{2j} \neq \varepsilon_{2j+1}$. The claim follows.

Type C. In this case ^LG is type B. Then $^L\mathcal{O}$ is even with Jordan blocks of odd size,

$$2n_0 + 1 \leqq 2n_1 + 1 < \ldots < 2n_{2i} + 1 \leqq 2n_{2i+1} + 1 < \ldots < 2n_{2p} + 1. \tag{11.1.10}$$

Then $G(^L\phi)$ is a product of $O(2)$'s, one for each pair $n_{2i} = n_{2i+1}$ and $O(1)$'s, one for each remaining n_j. An element in $G(^L\phi)$ which gives rise to half integral infinitesimal character can be written as

$$s = (\varepsilon_0, \ldots, \varepsilon_{2p}) \tag{11.1.11}$$

where $\varepsilon_0 \cdot \varepsilon_1 \cdot \ldots \cdot \varepsilon_{2p} = 1$. Then for such elements,

$$s \in \kappa \Leftrightarrow \varepsilon_{2i} \cdot \varepsilon_{2i+1} = 1. \tag{11.1.12}$$

Then every coset modulo κ has a representative of the form

$$(1, \eta_1, \eta_1, \ldots, \eta_p, \eta_p). \tag{11.1.13}$$

Given s, the element λ_s of minimal length satisfying (11.1.3) is obtained as follows.

(1) For the $\varepsilon_i = 1$ reorder the corresponding n_i in increasing order $m_0, m_1, \ldots, m_{2p}$ and add

$$\begin{aligned} &(m_i, \ldots, 1) && \text{if } i \text{ is even}, \\ &(m_i - 1, \ldots, 1, 0) && \text{if } i \text{ is odd}, \end{aligned}$$

to the infinitesimal character of $\bar{X}_e$.

(2) For the $\varepsilon_j = -1$, reorder the corresponding n_i in increasing order $m_0, \ldots, m_{2q+1}$ and add

$$\begin{array}{ll}(m_j+1/2, \ldots, 1/2) & \text{if } j \text{ is odd,}\\ (m_j-1/2, \ldots, 1/2) & \text{if } j \text{ is even,}\end{array}$$

to the infinitesimal character of $\bar{X}_o$.

Similar to type B, Sect. 10 implies that the unitary $\bar{X}$ satisfying (11.1.3), have infinitesimal character λ_s with s is as in (11.1.13). This λ_s can be characterized by the same property as for type B:

Fix an $\bar{s} \in \bar{A}(^L\phi)$. Then let λ_s be the smallest length semisimple element in the Lie algebra such that $e^{2\pi i \lambda_s}$ maps to $\bar{s}$ under the map in (11.1.2).

We show that the representative in (11.1.13) gives the minimum among the λ_s with s as in (11.1.11). Let

$$s_j = (1, 1, \ldots, \underset{2j-2}{-1}, \underset{2j-1}{-1}, 1, \ldots, 1). \tag{11.1.14}$$

If $\varepsilon_{2p} \neq \varepsilon_{2p-1}$ replace s by $s \cdot s_p$. Then $\lambda_s \leqq \|\lambda_{s \cdot s_p}\|$. Continue in this way by replacing s with s_j every time $\varepsilon_{2j-1} \neq \varepsilon_{2j}$.

Remark. In the case when the nilpotent orbit corresponds to the partition

$$2n_0+1 = 1 \leqq 2n_1+1 = 1 < 2n_2+1$$

we get the metaplectic representation attached to the element $s = (1, -1, -1)$.

Type D. In this case ^LG is type D. Then $^L\mathcal{O}$ is even with Jordan blocks of odd size,

$$2n_0+1 \leqq 2n_1+1 < \ldots 2n_{2i} \leqq 2n_{2i+1} < \ldots \leqq 2n_{2p+1}+1. \tag{11.1.15}$$

Similarly to types B, C, an element in $G(^L\phi)$ giving rise to half integral infinitesimal character can be written as

$$s = (\varepsilon_0, \ldots, \varepsilon_{2p+1}). \tag{11.1.16}$$

Then

$$s \in \kappa \Leftrightarrow \varepsilon_{2i} \cdot \varepsilon_{2i+1} = 1 \quad \text{for } 0 \leqq i \leqq p. \tag{11.1.17}$$

Then every coset modulo κ has a representative of the form

$$(1, \eta_1, \eta_1, \ldots, \eta_p, \eta_p, 1). \tag{11.1.18}$$

Given s, the element λ_s of minimal length satisfying (11.1.3) is obtained as follows.

(1) For the $\varepsilon_i = 1$ reorder the corresponding n_i in increasing order $m_0, m_1, \ldots, m_{2p+1}$ and add

$$\begin{array}{ll}(m_i, \ldots, 1, 0) & \text{if } i \text{ is odd,}\\ (m_i, \ldots, 1) & \text{if } i \text{ is even,}\end{array}$$

to the infinitesimal character of $\bar{X}_e$.

(2) For the $\varepsilon_j = -1$, reorder the corresponding n_i in increasing order $m_0, \ldots, m_{2q+1}$ and add

$$\begin{array}{ll}(m_j+1/2, \ldots, 1/2) & \text{if } j \text{ is odd,}\\ (m_j-1/2, \ldots, 1/2) & \text{if } j \text{ is even,}\end{array}$$

to the infinitesimal character of $\bar{X}_o$.

The same characterization of the unitary representations satisfying (11.1.3) as in types B, C holds. We omit the details.

11.2. In view of 11.1, we make the following general definition for an even rigid nilpotent orbit. If $s \in G(^L\phi)$, denote by $\mathcal{O}_s$ the dual nilpotent orbit to $^L\mathcal{O}$ viewed as an orbit in $^L\mathfrak{g}(s)$.

Definition. For each $\bar{s} \in \bar{A}$ let λ_s be the minimal length semisimple element such that $e^{2\pi i \lambda_s} \in A(^L\phi)$ maps to $\bar{s}$ and such that there is an $\bar{X}_{\mathfrak{g}(s)}$ with infinitesimal character λ_s satisfying

$$WF(\bar{X}_{\mathfrak{g}(s)}) = \bar{\mathcal{O}}_s.$$

Define $\mathcal{U}(^L\mathcal{O}, \bar{s}) = \mathcal{X}(\lambda_s)$ to be the set of preunipotent representations attached to λ_s via the maps T_i in 4.2.

11.3. Conjecture. *$\mathcal{U}(^L\mathcal{O}, \bar{s})$ is formed of unitary representations and each cuspidal unitary representation is an element of a $\mathcal{U}(^L\mathcal{O}, \bar{s})$.*

$\bar{X}$ with half-integral infinitesimal character is unitary if and only if it is unitarily induced or a complementary series from an element of a $\mathcal{U}(^L\mathcal{O}, \bar{s})$.

This conjecture is true in the classical groups and was motivated by my (and D. Vogan's) attempt to understand the unitary dual for the exceptional groups. It holds at least for G_2 using [D2], and for F_4.

Remarks. (1) The parameters in [A] corresponding to nilpotent orbits in $^L\mathfrak{g}$ that are not even do not arise in $\mathcal{U}(^L\mathcal{O}, \bar{s})$ because they are not rigid.

(2) The $\mathcal{U}(^L\mathcal{O}, \bar{s})$ do not parametrize all the rigid extremal unipotent representations. For example in type B, the spherical irreducible representation with infinitesimal character $\lambda = (1/2, 1/2, 1, 0)$ is rigid extremal but is not part of any $\mathcal{U}(^L\mathcal{O}, \bar{s})$ in type B. It is a complementary series so that conjecture 11.2 is still satisfied.

12. Complementary series

Assume (λ, λ') is arbitrary and $\bar{X}(\lambda, \lambda')$ is hermitian with small lowest K-type.

12.1. Write

$$^L\mathfrak{s}(\lambda) = {}^L\mathfrak{s}(\lambda)_e \times {}^L\mathfrak{s}(\lambda)_o \times {}^L\mathcal{A}_1 \times \ldots \times {}^L\mathcal{A}_k. \tag{12.1.1}$$

There are $\bar{X}_e$, $\bar{X}_o$, as in 10.2 and $\bar{X}_1, \ldots, \bar{X}_k$ such that

$$T(\bar{X}) = \bar{X}_e \otimes \bar{X}_o \otimes \bar{X}_1 \otimes \ldots \otimes \bar{X}_k. \tag{12.1.2}$$

The argument in 10.2 applies so Theorem 10.2 holds for $\bar{X}_e, \bar{X}_o$.

Assume that the necessary conditions for unitarity in Theorem 10.2 *hold for* $\bar{X}_e, \bar{X}_o$. *In type D, assume that* $\bar{X}_e$ *occurs.*

For each $\bar{X}_*$ as above, let $\mathscr{A}^i_{*0}$, $\mathscr{F}^i_{*0}$ and $\mathscr{A}^i_{*1}$, $\mathscr{F}^i_{*1}$ be the data defined in Sect. 9.1.

12.2. For each $\bar{X}_l$ write the parameter $\lambda_{l,0}$ as

$$(x_l^1+t_l, \dots, x_l^1+t_l, \dots, x_l^s+t_l, \dots, x_l^s+t_l) \tag{12.2.1}$$

where $x_l^i - x_l^{i+1} > 0$ and x_l^i are congruent mod $\mathbb{Z}$ to the coordinates of $\bar{X}_e$ and $0 < t_l < 1/2$. We will generally drop the subscript l whenever we deal with just a fixed one.

Recall r_0, r_1 from (9.1.5). Assume $\bar{X}$ is spherical for the moment.

Replace $\bar{X}$ by $\bar{Z} = \overline{\bar{X} * Y(m; t)}$. Then

$$r_Z(x+t) = \begin{cases} r_{\bar{X}}(x+t)+2 & -m \leqq x \leqq m, \\ r_{\bar{X}}(x+t) & x > m, \text{ or } x < -m. \end{cases} \tag{12.2.2}$$

Replace $\bar{X}$ by $\bar{Z} = \overline{\bar{X} * Y(m-1/2; 1/2-t)}$. Then

$$r_Z(x+t) = \begin{cases} r_{\bar{X}}(x+t)+2 & -m \leqq x < m, \\ r_{\bar{X}}(x+t) & x \geqq m, \text{ or } x < -m. \end{cases} \tag{12.2.3}$$

If $\bar{X}$ is not spherical, the same formulas hold for r_0.

Fix an $M \geqq 0$. By replacing $\bar{X}$ with $\bar{Z} = \overline{\bar{X} * Y(m; t)}$ with $m \leqq M$ sufficiently many times, we can find a $\bar{Z}$ such that $\bar{X} \underset{u}{<} \bar{Z}$, satisfying

$$\begin{aligned} &r_{Z,0}(x+t) > r_{Z,0}(y+t) \quad \text{for } |x| \leqq M < |y|,\ x, y \equiv M \bmod (\mathbb{Z}) \\ &r_{Z,0}(\pm(x+1)+t) < r_{Z,0}(\pm(x)+t), \quad \text{for } 0 \leqq x \leqq M. \end{aligned} \tag{12.2.4}$$

In particular, using the results in 4–6, we may conclude that for $M \geqq 1$,

$$\begin{aligned} &\bar{Z} * \mathscr{F}(M-1; t), \quad \bar{Z} * \mathscr{F}(M-1; 1-t), \\ &\bar{Z} * \mathscr{F}(M-1/2; 1/2 \pm t) \quad \text{are irreducible.} \end{aligned} \tag{12.2.5}$$

Similarly, 6.5 and Sect. 10 shows that if we replace $\bar{X}$ by

$$\overline{\bar{X} * \mathscr{F}(M; 0) * \mathscr{F}(M-1/2; 0)},$$

we may assume that $\bar{X} * \mathscr{F}(M; 0)$ and $\bar{X} * \mathscr{F}(M-1/2; 0)$ are irreducible.

In particular, we can single out $\bar{X}_l, M$ and perform these operations to the other $\bar{X}_i$. We obtain a $\bar{Z}$ with $\bar{X}_l$ among its data defined by (12.1.2), such that $\bar{X} \underset{u}{<} \bar{Z}$ and

$$\bar{Z} * \mathscr{F}(M-1; \tau) \quad \text{is irreducible for } 0 \leqq \tau < t, \tag{12.2.6a}$$

$$\bar{Z} * \mathscr{F}(M-1/2; \tau) \quad \text{is irreducible for } 0 \leqq \tau < 1/2 - t. \tag{12.2.6b}$$

Thus when we try to find necessary conditions for unitarity that involve one $\bar{X}_l$ only, we may replace it by any irreducible subquotient of one of the representations in (12.2.6) with $\tau = t$. Similar formulas to (12.2.2) and (12.2.3) hold for the coordinates of these representations.

Let

$${}^L\mathfrak{s}(\lambda)_{1/2} \times {}^L\mathscr{A}_1 \times \dots \times {}^L\mathscr{A}_k \supset {}^L\mathfrak{s}(\lambda)_e \times {}^L\mathfrak{s}(\lambda)_o \times {}^L\mathscr{A}_1 \times \dots \times {}^L\mathscr{A}_k \tag{12.2.7}$$

be such that ${}^L\mathfrak{s}(\lambda)_e \times {}^L\mathfrak{s}(\lambda)_o \subset {}^L\mathfrak{s}(\lambda)_{1/2}$ is proper maximal. Then the LHS of (12.2.7) is a Levi component and there is $\bar{X}_s$ such that

$$\bar{X} = \bar{X}_s * \bar{X}_1 * \dots * \bar{X}_k. \tag{12.2.8}$$

By applying the argument above to $\bar{X}_1, \dots, \bar{X}_k$ for the appropriate M's we can find a $\bar{Z}$ such that $\bar{X} \underset{u}{<} \bar{Z}$, satisfying $\bar{Z}_s = \bar{X}_s$ and such that all the deformation arguments in Sect. 10 apply. Thus we get the following corollary.

Corollary. *$\bar{X}$ is unitary only if $\bar{X}_s$ is unitary. In particular, $\bar{X}_s$ must satisfy* 10.7.

Definition. If (12.2.6) holds with $\bar{X}$ in place of $\bar{Z}$, we say that every composition factor Y of (12.2.6a) for $0 \leqq \tau \leqq t$ and every composition factor Y of (12.2.6b) for $0 \leqq \tau < 1/2 - t$ satisfy $\bar{X} \underset{u}{<} Y$. This extends $\underset{u}{<}$ and has the same properties. For $0 \leqq \tau < t$, $0 \leqq \tau < 1/2 - t$ respectively, we also say $\bar{Z} \underset{u}{\sim} Y$.

12.3. *Assume $\bar{X}$ is spherical hermitian and the necessary conditions for unitarity in Corollary* 12.2 *hold.*

Since all necessary conditions for unitarity to be established in this section and 12.4 are in terms of each $\bar{X}_l$ separately, we may assume that (12.2.6) holds with $\bar{X}$ instead of $\bar{Z}$. Fix an l and omit the subscript in this section.

Lemma. *If $\bar{X}$ is unitary, then the following must hold. For each $M \geqq 0$, $2M \in \mathbb{N}$, and $0 < t < 1/2$,*

$$r(M+1+t) \leqq r(M+t)+1, \tag{1a}$$

$$r(-(M+1)+t) \leqq r(-M+t)+1, \tag{1b}$$

$$r(M+1+t)+r(-M-1+t) \leqq r(M+t)+r(-M+t)+1, \tag{2}$$

$$\textit{not both } r(M+1+t),\, r(-M-1+t) > r(M+t), \tag{3a}$$

$$\textit{not both } r(M+t),\, r(-M-1+t) > r(-M+t). \tag{3b}$$

Otherwise, the form is negative on a small K-type.

Proof. These conditions are not affected if we replace $\bar{X}$ by $\bar{Z} = \overline{\bar{X} * \mathscr{F}(m; t)}$ with $m < M$ or $m \geqq M+1$. Then we can reduce to the case when the following hold.

$$\begin{aligned} &r(\pm x+t) > r(M+t), \\ &r(\pm x+t) > r(-M+t), \quad \text{for } 0 \leqq x < M, \end{aligned} \tag{12.3.1a}$$

$$\begin{aligned} &r(\pm y+t) < r(M+1+t), \\ &r(\pm y+t) < r(-M-1+t), \quad \text{for } y > M+1, \end{aligned} \tag{12.3.1b}$$

$$r(\pm x+t) < r(\pm y+t), \quad \text{for } 0 \leqq x < M < y. \tag{12.3.1c}$$

Assume (1) *does not hold for* $\bar{X}$. By replacing $\bar{X}$ with

$$Z=\overline{\bar{X}*\mathscr{F}(M-1/2;\,1/2-t)}$$

in (1 a), we reduce to the case when

$$r(-M+t)>r(\pm(M+1)+t). \tag{12.3.2}$$

By replacing $\bar{X}$ with $Z=\overline{\bar{X}*\mathscr{F}(M+1/2;\,1/2-t)}$ in (1 b), we can reduce to the case

$$r(M+1+t)<r(M+t). \tag{12.3.3}$$

If $r(-M-1+t)>r(M+t)$ in case (1 a), $\bar{X}$ has an $\mathscr{F}^i$ satisfying

$$F^i\colon (M+1+t,\,M-1+t,\,\ldots,\,-M+t,\,-M-1+t). \tag{12.3.4}$$

If $r(-M-1+t)\leqq r(M+t)$ in (1 a), $\bar{X}$ has a pair $\mathscr{F}^i\approx\mathscr{F}^{i+1}$ satisfying

$$F^i\colon (M+1+t,\,M-1+t,\,\ldots,\,-M+t). \tag{12.3.5}$$

Similarly in case (1 b), either $\bar{X}$ has an $\mathscr{F}^i$ satisfying

$$F^i\colon (M+t,\,M-1+t,\,\ldots,\,-M+1+t,\,-M-1+t), \tag{12.3.6}$$

or a pair $\mathscr{F}^i\approx\mathscr{F}^{i+1}$ satisfying

$$F^i\colon (M-1+t,\,\ldots,\,-M+1+t,\,-M-1+t). \tag{12.3.7}$$

Assume (2) *does not hold for* $\bar{X}$. Then we can reduce to the case when $\bar{X}$ has an $\mathscr{F}^i$ as in (12.3.6).

Assume (3) *does not hold for* $\bar{X}$. Then we can reduce to the case when $\bar{X}$ has an $\mathscr{F}^i$ as in (12.3.4) or (12.3.6).

In cases when (12.3.5) or (12.3.7) hold, $\bar{X}=\bar{X}^i*[\mathscr{F}^i*\mathscr{F}^{i+1}]$. Since $\mathscr{F}^i*\mathscr{F}^{i+1}$ is hermitian, the proof follows from Sect. 8.

(12.3.6) reduces to (12.3.4) by substituting $M=M'+1/2$, $t=1/2-t'$. Then the proof of Proposition 8.1, case (2 b) applies.

Remark. The necessary conditions for unitarity in this section are the analogues of the ones in Sects. 7 and 8. The representations satisfying these conditions are analogous to the weakly unipotent ones.

12.4. Proposition. *$\bar{X}$ is unitary only if the necessary conditions in* 12.3 *are satisfied for $\bar{X}_{\mu,0}$. In addition, $\bar{X}_{\mu,1}$ must satisfy*

$$r_1((M+1)+t)\leqq r_1(\pm M+t),$$
$$r_1(-M-1+t)\leqq r_1(-M+t),$$
$$r_1(M+1+t)\leqq r_1(-M-1+t),$$
$$r_1(M+t)\geqq r_1(-M-1+t),$$

for all $M\geqq 0$, $2M\in\mathbb{N}$, and $0<t<1/2$.

Proof. Because of the bottom layer arguments, if the form is negative on a small K-type on $\bar{X}_{\mu,0}$, $\bar{X}_{\mu,1}$, then $\bar{X}$ is not unitary. Thus $\bar{X}_{\mu,0}$ must satisfy 12.3 and $\bar{X}_{\mu,1}$ must satisfy 8.4. These are the conditions described in the statement of the proposition.

Assume that $\bar{X}$ satisfies the necessary conditions for unitarity of Proposition 12.4 *in the remainder of the section.*

Lemma. *Assume one of the following holds.*

$$r_0(M+1+t)=r_0(M+t)+1, \tag{1 a}$$

$$r_0(-M-1+t)=r_0(-M+t)+1, \tag{1 b}$$

$$r_0(M+1+t)>r_0(-M+t), \tag{2 a}$$

$$r_0(-M-1+t)>r_0(M-1+t). \tag{2 b}$$

Then $\bar{X}$ is not unitary unless

$$r_1(M+1/2+t)<r_1(M-1/2+t),\quad r_1(-M-1/2+t)\qquad \textit{in case } (1),$$
$$r_1(-M-1/2+t)<r_1(-M+1/2+t),\, r_1(M-1/2+t)\qquad \textit{in case } (2).$$

Otherwise, the form is negative on a level $\leqq 2$ K-type.

Proof. As in 12.3, we can reduce to the case when (12.3.1) holds.

If (1 a) or (1 b) holds, we can reduce to the case when there is $\mathscr{F}^i_{\mu,0}$ satisfying

$$F^i_{\mu,0}\colon (M+1+t,\, M-1+t,\,\ldots,\,-M+1+t),\quad \text{or} \tag{12.4.1 a}$$

$$(M-1+t,\,\ldots,\,-M+1+t,\,-M-1+t). \tag{12.4.1 b}$$

Case (12.4.1 b) reduces to (12.4.1 a) by substituting $M=M'+1/2$, $t=1/2-t'$. The proof follows as in Theorem 9.2 using $\mathscr{F}(M;t)$ instead of $\mathscr{F}(M)$.

If (2 a) or (2 b) holds, we can reduce to the case when there is $\mathscr{F}^i_{\mu,0}$ satisfying

$$F^i_{\mu,0}\colon (M+1+t,\,\ldots,\,-M+1+t), \tag{12.4.2 a}$$

$$(M-2+t,\,\ldots,\,-M_1+t). \tag{12.4.2 b}$$

Case (12.4.2 b) reduces to (12.4.2 a) by the substitution $M=M'+1/2$, $t=1/2-t'$.

Then $\bar{X}_{\mu,0}=\bar{X}^i*\mathscr{F}^i_{\mu,0}$. Consider $Z=\bar{X}*\mathscr{F}(M;t)$. Since $\mathscr{F}^i_{\mu,0}$ is as in (2) of Lemma 8.2, Z has a composition factor $\bar{Y}=\overline{\bar{X}*\bar{X}_u}$ with small lowest K-type. Then

$$r_{\bar{Y},1}(x+t)=\begin{cases} r_{\bar{X},1}(x+t)+1 & -M+1/2\leqq x\leqq M+1/2,\\ r_{\bar{X},1}(x+t) & \text{otherwise.}\end{cases} \tag{12.4.3}$$

Unless $r_{\bar{X},1}(M+1/2+t)<r_{\bar{X},1}(M-1/2+t)$, $r_{\bar{X},1}(-M-1/2+t)$, Proposition 12.4 is not satisfied.

12.5. In this section we define a class of representations called basic which satisfy the necessary conditions for unitarity in 12.1–12.4. To each such representation we attach an induced spherical $X(\underline{t}', \underline{t}'')$ and show in 12.7 that $\bar{X}$ is unitary if and only if the corresponding $X(\underline{t}', \underline{t}'')$ is a complementary series.

The necessary conditions in 12.1–12.4 can then be stated in words as follows.

Fix a $t = t_l$. Then either $\mathscr{F}^i_{\mu,0}$ is of the form

$$\begin{aligned} &(M^i + t, \ldots, -M^i + t),\\ &(M^i + t, \ldots, -M^i + 1 + t),\\ &(M^i - 1 + t, \ldots, -M^i + t),\\ &(M^i - 1 + t, \ldots, -M^i - 1 + t), \end{aligned} \tag{12.5.1}$$

or else one of the assumptions of Lemma 12.4 is satisfied. Thus one can pair up each $M+1+t$, $-M-1+t$ satisfying (1) or (2) in 12.4 with a $\mathscr{F}^i_{\mu,1}$, $\mathscr{F}^{i+1}_{\mu,1} \approx (\mathscr{F}^i_{\mu,1})^h$ of the form (12.5.2), (12.5.4). The remaining coordinates will form $\mathscr{F}^i_{\mu,0}$ as in (12.5.1) or $\mathscr{F}^j_{\mu,1}$ as in (12.5.2), (12.5.4) which can be grouped into Stein complementary series.

Definition. We say that $\bar{X}$, satisfying the necessary conditions for unitarity in 12.3, 12.4, is basic if the following hold.

(1) $\bar{X}_s$ is rigid extremal.

(2) No $(\bar{X}_l)_{\mu,0}$ can be written as $(\bar{X}_l)_{\mu,0} = (\bar{X}^\prime_l)_{\mu,0} * Y(m; t_l)$ (notation 12.1 and 8.3).

(3) For each M satisfying (1) or (2) as in Lemma 12.4, there is exactly one pair $\mathscr{F}^j_{\mu,1}$, $\mathscr{F}^{j+1}_{\mu,1} \approx (\mathscr{F}^j_{\mu,1})^h$. In other words, there are no Stein complementary series after pairing up the coordinates as described above.

If $\bar{X}$ is basic, we define new $\tilde{\mathscr{A}}^i$, $\tilde{\mathscr{F}}^i$ as follows.

Assume that $M+1+t_l$ satisfies (1) of Lemma 12.4. Then the conclusion of 12.4 and the definition of basic assures that there is exactly one pair $\mathscr{F}^j_{\mu,1}$, $\mathscr{F}^{j+1}_{\mu,1} \approx (\mathscr{F}^j_{\mu,1})^h$ of the form

$$\mathscr{F}^j_{\mu,1}: (M + t_l, \ldots, -M + t_l) \times (M - 1 + t_l, \ldots, -M - 1 + t_l). \tag{12.5.2}$$

Adjoin the corresponding $M+1+t_l$ and replace them by the spherical

$$\begin{aligned} &(M+1+t'_l, M+t'_l, \ldots, -M+t'_l) \times (M+1+t'_l, M+t'_l, \ldots, -M+t'_l),\\ &(M-1+t''_l, \ldots, -M-1+t''_l) \times (M-1+t''_l, \ldots, -M-1+t''_l), \end{aligned} \tag{12.5.3}$$

where $t'_l < t_l < t''_l$ and $t''_l - t'_l$ is very small.

Assume that $-M-1+t_l$ satisfies (2) of Lemma 12.4. Then the conclusion of 12.4 and the definition of basic assures that there is exactly one pair $\mathscr{F}^j_{\mu,1}$, $\mathscr{F}^{j+1}_{\mu,1} \approx (\mathscr{F}^j_{\mu,1})^h$ of the form

$$\mathscr{F}^j_{\mu,1}: (M - 1 + t_l, \ldots, -M + t_l) \times (M + t_l, \ldots, -M + 1 + t_l). \tag{12.5.4}$$

Adjoin the corresponding $-M-1+t_l$ and replace them by

$$\begin{aligned} &(M - t + t'_l, \ldots, -M - 1 + t'_l) \times (M - 1 + t'_l, \ldots, -M - 1 + t'_l),\\ &(M + t''_l, \ldots, -M + 1 + t''_l) \times (M + t''_l, \ldots, -M + 1 + t''_l), \end{aligned} \tag{12.5.5}$$

where $t'_l < t_l < t''_l$ and $t''_l - t'_l$ is very small.

The other spherical $\mathscr{F}^i_{\mu,0}$ are obtained from the remainder as in 9.1. Doing this to all M_l, t_l satisfying the assumptions of Lemma 12.4, we get a spherical induced irreducible module

$$X(\underline{t}', \underline{t}'') = \bar{X}_s * \tilde{\mathscr{F}}^1 * \ldots * \tilde{\mathscr{F}}^r \tag{12.5.6}$$

where $(\underline{t}', \underline{t}'')$ are vector parameters. At $t'_l = t''_l = t_l$, $X(\underline{t}', \underline{t}'')$ has $\bar{X}$ as subquotient due to (3) of Lemma 8.2.

To each spherical $\tilde{\mathscr{F}}^i$ we attach a pair $[-M^i + t, M^i + t]$, $[-M^i + 1 + t, M^i + t]$ or $[-M^i - 1 + t, M^i - 1 + t]$, $[-M^i - 1 + t, M^i + t]$ where the first entry is the smallest coordinate of $\tilde{\mathscr{F}}^i$, the second is the largest coordinate. We do the same to $\mathscr{F}^i_e$, $\mathscr{F}^j_o$ using the smallest and largest coordinates of the corresponding F_L and call the resulting pairs $[z_s, z_{s+1}]$. The trivial representations corresponding to types B, C, D will be singled out by the corresponding subscript. To case (12.5.3) we attach $[(-M+t'_l, M+1+t'_l)\,(-M-1+t''_l, M-1+t''_l)]$. Similarly to (12.5.5). These cases will be called *edges of complementary series.*

12.6. Given a parameter for a basic representation $\bar{X}$, Sect. 12.5 attaches a Levi component $\mathfrak{m}(\bar{X})$ and a spherical induced representation $X(\underline{t}', \underline{t}'')$ to it. We describe the conditions under which $X(\underline{t}', \underline{t}'')$ is a complementary series. This is very similar to the condition of extremal for half-integral parameter.

Let $[-m^i + t_l, M^i + t_l]$ be a pair such that $|M^i - m^i + 2t_l|$ is smallest. Deform t_l to τ in the factor $\tilde{\mathscr{F}}^i_l(t_l)$ so that the corresponding quantity $|M^i - m^i + 2\tau|$ becomes smaller. We get a representation $X(\underline{t}', \underline{t}'', \tau) = X^i(\underline{t}', \underline{t}'') * \tilde{\mathscr{F}}^i(\tau)$ which equals $X(\underline{t}', \underline{t}'')$ at $\tau = t_l$. The condition for irreducibility is as follows.

If $M^i + \tau - M^{j_k} - t_k \in \mathbb{Z}$, then the pairs for this new parameter should be obtained by just adjoining $[-m^i + \tau, M^i + \tau]$ to the pairs coming from $X^i(\underline{t}', \underline{t}'')$.

When $M^i - m^i + 2\tau = 0$, the condition insures that the corresponding $X(\underline{t}', \underline{t}'', \tau)$ is unitarily induced irreducible from $X^i(\underline{t}', \underline{t}'') *$ Triv. Then we require that $X^i(t', t'')$ satisfy the same condition.

12.7. Proposition. *Assume $\bar{X}$ is basic. Then $\bar{X}$ is unitary if and only if $X(\underline{t}', \underline{t}'')$ is a complementary series from a representation which is unitarily induced from a unipotent representation.*

Otherwise, the form is negative on a level 1 K-type.

Proof. We do an induction on the number of edges of complementary series. Each corresponds to an M such that

$$r_0(M+1+t_l)>r_0(M+t_l), \quad \text{or} \quad r_0(-M+t_l), \tag{12.7.1 a}$$

$$r_0(-M-1+t_l)>r_0(-M+t_l), \quad \text{or} \quad r_0(M-1+t_l). \tag{12.7.1 b}$$

Fix a $t_l=t$ and omit the subscript.

Assume there are no edges of complementary series. Fix a pair $[-m+t, m+t]$. Then we can write $\bar{X}=\bar{X}^i * \mathscr{F}(m;t)$. This is a complementary series unless there is a pair $[z_s, z_{s+1}]$ at half-integral infinitesimal character with the same parity as m such that $-z_s<m<z_{s+1}$. Then the proof of 10.5 applies. We have used the fact that in this case, any $\bar{\mathscr{F}}$ is of the form (12.5.1).

Now fix a pair $[-m+t, m+t+1]$ so that the corresponding $\bar{\mathscr{F}}$ is as in the second of (12.5.1). Then $\bar{X}=\bar{X}^i * \mathscr{F}(m+1/2;\, 1/2+t)$. This is a complementary series from an induced irreducible unless one of the following holds.

(1) There is a $[z_s, z_{s+1}]$ such that

$$z_s<m+1/2<z_{s+1}, \qquad z_s-m-1/2\in\mathbb{N}.$$

(2) There is a $[z_s, z_{s+1}]$ such that

$$z_s\leqq m<m+1\leqq z_{s+1}, \qquad z_s-m\in\mathbb{N}.$$

(3) There is a pair $[-m-1+\theta, m+\theta]$ with $\theta<t$ among the $\bar{\mathscr{F}}$.

In case (1) the proof of Proposition 10.5 applies. In cases (2) and (3), we can deform $\bar{X}$ to a representation

$$\tilde{X} * Y(m-1/2;\, 1/2+\tau) \tag{12.7.2}$$

where $\tilde{X}$ is irreducible. Lemma 8.3 completes the proof.

Similarly for $[-m+t, m-1+t]$, $[-m-1+t, m-1+t]$.

Assume there is at least one edge of complementary series. It is enough to prove that, if $X(\underline{t}', \underline{t}'')$ is not a complementary series, then the form on $\bar{X}_{\mu,0}$ is negative on a small K-type. Let M be the largest such that (12.7.1) holds. *Assume* (12.7.1 a) *holds for this* M. Let

$$X(M;t)=\bar{X} * Y(M;t). \tag{12.7.3}$$

Then the lowest K-type subquotient $\bar{X}(M;t)$ occurs in an

$$\tilde{X} * \mathscr{F}(M+1/2;\, 1/2+t) * [\mathscr{F} * \mathscr{F}^h] \tag{12.7.4}$$

as the lowest K-type subquotient, where

$$\mathscr{F}: (M+t, \ldots, -M+t)\times(M-1+t, \ldots, -M-1+t), \tag{12.7.5}$$

and $\tilde{X}$ is irreducible. Then $\tilde{X} * \mathscr{F}(M+1/2;\, 1/2+t)$ is irreducible, basic. The induction hypothesis applies. If it is not unitary then the form is negative on a small K-type. The same holds for $\bar{X}(M;t)$ and $\bar{X}$ by the properties of small and bottom layer K-types in Sect. 3. Thus a necessary condition for $\bar{X}$ to be unitary is that $\tilde{X} * \mathscr{F}(M+1/2;\, 1/2+t)$ is a complementary series.

Since (notation (12.5.6))

$$X(\underline{t}', \underline{t}'')=\tilde{X} * \mathscr{F}(M+1/2;\, 1/2+t') * \mathscr{F}(M;\, 1-t''), \tag{12.7.6}$$

$X(\underline{t}', \underline{t}'')$ stays irreducible when we deform $1/2+t'$ to 0. Thus the following are satisfied.

(1) There is no pair $[z_s, z_{s+1}]$ such that

$$z_s<M+1/2<z_{s+1}, \qquad z_s-M-1/2\in\mathbb{N}.$$

(2) There is no pair $[z_s, z_{s+1}]$ such that

$$z_s\leqq M<M+1\leqq z_{s+1}, \qquad z_s-M\in\mathbb{N}.$$

(3) There is no pair $[-M-1+\theta, M+\theta]$ with $\theta<t$.

(4) There is no $0<\theta<t$ such that

$$r_0(M+1+\theta)>r_0(M+\theta) \quad \text{or} \quad r_0(-M+\theta). \tag{12.7.7}$$

(5) There is no $0<\theta<t$ such that

$$r_0(-M-1+\theta)>r_0(M+\theta) \quad \text{or} \quad r_0(-M+\theta). \tag{12.7.8}$$

(4) and (5) follow in view of (12.5.2) and (12.5.4). In particular, by choosing $t=t_l$ largest possible, we see that for a given M there is at most one edge of complementary series as in (12.5.3) and (12.5.5).

The proof will be complete if we show that if $\bar{X}$ is unitary, $\tilde{X} * \mathscr{F}(M;\, 1-t'')$ is a complementary series when we deform $1-t''$ to 0.

Suppose not. Then one of the following must hold.

(1) There is a pair $[z_s, z_{s+1}]$ such that

$$z_s\leqq M-1/2<M+1/2\leqq z_{s+1}, \qquad z_s-M-1/2\in\mathbb{N}.$$

We can reduce this to the case $z_s=M-1/2$, $z_{s+1}=M+1/2$.

(2) There is a pair $[z_s, z_{s+1}]$ such that $z_s < M < z_{s+1}$, $z_s - M \in \mathbb{N}$. We can reduce this to the case $z_s = -M+1$, $z_{s+1} = M+1$.

$$\begin{aligned} r_0(-M-1+\theta) &> r_0(M+\theta) \quad \text{or} \quad r_0(-M+\theta), \\ r_0(-M-2+\theta) &> r_0(M+1+\theta) \quad \text{or} \quad r_0(-M-1+\theta). \end{aligned} \tag{12.7.9}$$

(4) There is $\theta > t$ such that

$$\begin{aligned} r_0(M+1+\theta) &> r_0(M+\theta) \quad \text{or} \quad r_0(-M+\theta), \\ r_0(M+\theta) &> r_0(M-1+\theta) \quad \text{or} \quad r_0(-M+1+\theta). \end{aligned} \tag{12.7.10}$$

(5) There is a pair $[-M+1+\theta, M+\theta]$ with $t < \theta \leqq 1/2$.

(6) There is a pair $[-M+\theta, M+\theta]$ with $t < \theta \leqq 1/2$.

Cases (3) and (4) cannot occur because of (12.7.7), (12.7.8), and the maximality of M.

In the remaining cases, we can reduce as in 12.3 to the case when there are $\mathscr{F}_t$, $\mathscr{F}_\theta$ and $\bar{X}_{\mu,0}$ satisfying one of the following.

$$F_t\colon (M+1+t, M-1+t, \ldots, -M+1+t, -M+t), \tag{12.7.11 a}$$

$$(M+1+t, M-1+t, \ldots, -M+1+t), \tag{12.7.11 b}$$

$$(M+1+t, M+t, M-1+t, \ldots, -M+1+t). \tag{12.7.11 c}$$

$$F_\theta\colon (M+\theta, \ldots, -M+\theta), \tag{12.7.11 d}$$

$$(M+\theta, \ldots, -M+1+\theta). \tag{12.7.11 e}$$

In all cases,

$$X_{\mu,0}(\tau) = \bar{X}_{\mu,0} * \mathscr{F}_t * \mathscr{F}_\tau \tag{12.7.12}$$

is irreducible for $\theta \leqq \tau < 1 - t''$ and equal to $\bar{X}_{\mu,0}$ at $\tau = \theta$.

Deform τ to $1-t''$. Then the assumptions of Lemma 12.3 or Lemma 2 of 8.3 are satisfied. Then the form is negative on a small K-type.

The argument is similar when (12.7.1 b) holds. We omit the details.

The proof follows.

12.8. Theorem. *A representation $\bar{X}$ with arbitrary infinitesimal character and lowest K-type is unitary if and only if it is the lowest K-type subquotient of a representation which is unitarily induced from a unitary basic representation tensored with unitary characters or Stein complementary series.*

Otherwise the form is negative on a level $\leqq 2$ K-type.

Proof. Assume $\bar{X}$ has small lowest K-type. Then 12.3, 12.4 show that it must be the lowest K-type subquotient of a unitarily induced from a basic representation. If the basic representation is not unitary, the proof in 12.7 combined with bottom layer K-types shows that there is an $\bar{X}_u$ with small lowest K-type not satisfying the necessary conditions for unitarity in Proposition 8.3, such that $\bar{X} \underset{u}{<} \bar{X}_u$.

Thus the theorem holds in this case.

If $\bar{X}$ does not have small lowest K-type, Corollary 8.4 completes the proof.

13. Unitary spectrum for small rank groups

13.1 Type D4

Nilpotent orbits

Jordan blocks	*Dual*	$\lvert\bar{A}(\phi)\rvert$	*special/nonspecial*	*Symbol*
71	1^8	1	special	$\begin{bmatrix}4\\0\end{bmatrix}$
53	$2^2 1^4$	1	special	$\begin{bmatrix}3\\1\end{bmatrix}$
51^3	$3^2 1^5$	1	special	$\begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}$
$4^2_{I,II}$	$2^4_{I,II}$	1	special	$\begin{bmatrix}2\\2\end{bmatrix}_{I,II}$
$3^2 1^2$	$3^2 1^2$	2	special	$\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$
$32^2 1$	–	–	nonspecial	$\begin{bmatrix}0 & 3\\1 & 2\end{bmatrix}$
31^5	51^3	1	special	$\begin{bmatrix}1 & 2 & 4\\0 & 1 & 2\end{bmatrix}$
$2^4_{I,II}$	$4^2_{I,II}$	1	special	$\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}_{I,II}$
$2^2 1^4$	53	1	special	$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 3\end{bmatrix}$
1^8	71	1	special	$\begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 3\end{bmatrix}$

Unipotent representations

$\mathcal{O}$	$\bar{X}$	*Type*
1^8	$(3, 2, 1, 0)\times(3, 2, 1, 0)$	Trivial
2^21^4	$(2, 1, 1, 0)\times(2, 1, 1, 0)$	endpoint of complementary series
$2^4_{I,II}$	$(3/2, 3/2, 1/2, 1/2)$	
	$\times(3/2, 3/2, 1/2, 1/2)$	
	$(3/2, 3/2, 1/2, -1/2)$	induced from A_3
	$\times(3/2, 3/2, 1/2, -1/2)$	induced from A_3
31^5	$(2, 1, 0, 0)\times(2, 1, 0, 0)$	unitarily induced from D_3.
32^21	$(1, 1, 0, 0)\times(1, 1, 0, 0)$	complementary series
	$(1, 0, 1, 0)\times(0, -1, 1, 0)$	unitarily induced irreducible
	$(3/2, 1/2, 1, 0)\times(3/2, 1/2, 1, 0)$	endpoint of complementary series
4^2	$(1/2, 1/2, 1/2, 1/2)$	induced from $A_1\times A_1$.
	$\times(1/2, 1/2, 1/2, 1/2)$	
	$(1/2, 1/2, 1/2, -1/2)$	induced from $A_1\times A_1$.
	$\times(1/2, 1/2, 1/2, -1/2)$	
51^3	$(0, 0, 1, 0)\times(0, 0, 1, 0)$	unitarily induced from D_2.
53	$(0, 0, 1/2, -1/2)$	unitarily induced from A_1.
	$\times(0, 0, 1/2, -1/2)$	
71	$(0, 0, 0, 0)\times(0, 0, 0, 0)$	tempered principal series.

Complementary series

The representation is $\bar{X}(\lambda, \lambda)$ where $\lambda=(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ satisfying $\lambda_1\geqq\lambda_2\geqq\lambda_3\geqq|\lambda_4|\geqq 0$.

$\lambda_4\neq 0$. $\bar{X}$ admits a hermitian form if and only if $\lambda_1=\lambda_2$ and $\lambda_3=\pm\lambda_4$. It is enough to discuss the case $\lambda_3=\lambda_4$.

$\mathfrak{m}(\lambda)$	λ	*Complementary series*
$\mathscr{A}(4)$	$(1/2+t, 1/2+t, 1/2-t, 1/2-t)$	$0\leqq t<1/2$.
$\mathscr{A}(1)^4$	$(t_1, t_2, -t_1, -t_2)$	$0\leqq t_2\leqq t_1<1, t_1\pm t_2\notin\mathbb{Z}$

$\lambda_4=0$.

$\mathfrak{m}(\lambda)$	λ	*Complementary series*
$D(3)\times\mathscr{A}(1)$	$(t, 2, 1, 0)$	$0\leqq t<1$.
$D(2)-\mathscr{A}(2)$	$(1/2+t, -1/2+t, 1, 0)$	$0\leqq t<1$.
$D(2)\times\mathscr{A}(1)\times\mathscr{A}(1)$	$(t_1, t_2, 1, 0)$	$0\leqq t_2<t_1, t_1+t_2<1$.
$D(1)\times\mathscr{A}(1)\times\mathscr{A}(2)$	$(1/2+t_1, -1/2+t_1, t_2, 0)$	$0\leqq t_1<1/2, 0\leqq t_2<1$.

All other representations are obtained by unitary induction from unitary representations on $\mathfrak{m}_\mu\subsetneqq\mathfrak{g}$.

13.2 Type C3

Nilpotent orbits

Jordan blocks	*Dual*	$\lvert\bar{A}(\phi)\rvert$	*special/nonspecial*	*Symbol*
6	1^7	1	special	$[3]$
42	31^4	2	special	$\begin{bmatrix}0 & & 3\\ & 1 & \end{bmatrix}$
41^2	–	–	nonspecial	$\begin{bmatrix}1 & & 3\\ & 0 & \end{bmatrix}$
3^2	32^2	1	special	$\begin{bmatrix}0 & & 2\\ & 2 & \end{bmatrix}$
2^3	3^21	1	special	$\begin{bmatrix}1 & & 2\\ & 1 & \end{bmatrix}$
2^21^2	51^2	2	special	$\begin{bmatrix}0 & & 1 & & 3\\ & 1 & & 2 & \end{bmatrix}$
21^4	–	–	nonspecial	$\begin{bmatrix}1 & & 2 & & 3\\ & 0 & & 1 & \end{bmatrix}$
1^6	7	1	special	$\begin{bmatrix}0 & & 1 & & 2 & & 3\\ & 1 & & 2 & & 3 & \end{bmatrix}$

Unipotent representations

$\mathcal{O}$	$\bar{X}$	*Type*
1^6	$(3, 2, 1)\times(3, 2, 1)$	Trivial
2^21^2	$(2, 1, 0)\times(2, 1, 0)$	endpoint of complementary series
	$(2, 1, 0)\times(2, 0, -1)$	
	$(5/2, 3/2, 1/2)\times(5/2, 3/2, 1/2)$	metaplectic representation
	$(5/2, 3/2, -1/2)$	metaplectic representation
	$\times(5/2, 3/2, 1/2)$	
2^3	$(1, 0, -1)\times(1, 0, -1)$	unitarily induced from $\mathscr{A}(3)$
3^2	$(1/2, -1/2, 1)\times(1/2, -1/2, 1)$	unitarily induced from $C(1)\times\mathscr{A}(2)$
42	$(1, 0, 0)\times(1, 0, 0)$	complementary series
	$(1, 0, 0)\times(0, -1, 0)$	unitarily induced from $\mathscr{A}(2)\times\mathscr{A}(1)$.
	$(3/2, 1/2, 1/2)\times(3/2, 1/2, 1/2)$	complementary series
	$(3/2, 1/2, -1/2)$	unitarily induced from metaplectic
	$\times(3/2, 1/2, 1/2)$	
6	$(0, 0, 0)\times(0, 0, 0)$	tempered principal series.

Complementary series

The representation is $\bar{X}(\lambda, \lambda)$ where $\lambda=(\lambda_1, \lambda_2, \lambda_3)$ satisfies $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq 0$.

$\mathfrak{m}(\lambda)$	λ	*Complementary series*
$C(2)\times\mathscr{A}(1)$	$(t, 2, 1)$	none
$C(1)\times\mathscr{A}(1)$	$(1/2+t, t_2, -1/2+t, 1)$	$0\leqq t<1/2$
$C(1)\times\mathscr{A}(1)^2$	(t_1, t_2)	none
$\mathscr{A}(3)$	$(1+t, t, -1+t)$	$0\leqq t<1/2.$
$\mathscr{A}(2)\times\mathscr{A}(1)$	$(1/2+t_1, -1/2+t_1, t_2)$	$0\leqq t_1, t_2<1, t_1+t_2<3/2.$ At $t_1+t_2=3/2$ get edge of complementary series
$\mathscr{A}(1)^3$	(t_1, t_2, t_3)	$0\leqq t_3\leqq t_2\leqq t_1, t_i+t_j<1/2.$

All other representations are obtained by unitary induction from unitary representations on $\mathfrak{m}_\mu \subsetneqq \mathfrak{g}$.

14. An irreducibility result

14.1. Let $\bar{X}=\bar{X}(\lambda, \lambda')$ be preunitary with half-integral infinitesimal character. Then there is a Levi component

$$\mathfrak{m}_u=\mathfrak{m}_{u,0}\times\mathscr{A}_{u,1}\times\ldots\times\mathscr{A}_{u,r}, \tag{14.1.1}$$

a preunipotent $\bar{X}_{u,0}\in\mathscr{X}(\lambda_{u,0})$ and unitary nonspherical characters $\bar{X}_{u,i}$ such that $\bar{X}$ is the lowest K-type subquotient in

$$X=\bar{X}_{u,0}*\bar{X}_{u,1}*\ldots*\bar{X}_{\mu,r}. \tag{14.1.2}$$

Theorem. $\bar{X}=\bar{X}_{u,0}*\bar{X}_{u,1}*\ldots*\bar{X}_{u,r}$.

The proof reduces to a statement about the relation of X to the ring of functions on a nilpotent orbit. This is not known and quite hard in this generality. In the case of classical groups, we will reduce the proof, by elementary means, to a simpler version of Theorem 14.1, stated in 14.2.

14.2. Theorem (Vogan). *Let $Q=LN$ be the parabolic subgroup from which $\bar{X}_{u,0}*\bar{X}_{u,1}*\ldots*\bar{X}_{u,r}$ is induced. Assume that all $\bar{X}_{u,i}$ are unitary characters. If $\mathcal{O}$ is normal and the moment map*

$$\mathscr{M}: T^*(G/Q)\to\mathcal{O}$$

is birational then $\bar{X}=X$. ($\mathcal{O}$ is the unique orbit whose intersection with $\mathfrak{n}$ is dense in $\mathfrak{n}$.)

Proof. The proof is in [V3] Sect. 14.

14.3. Proposition. *Sufficient conditions for $\mathcal{O}$ to be normal are as follows.*

Sp(*n*, C): *The row sizes in the Jordan block description of $\mathcal{O}$ are even only. Equivalently, denote by $c[k]$ the number of times the column of size k occurs. Then $c[k]$ are all even.*

Equivalently, the nilpotent orbit is induced Richardson from a Levi component satisfying

$$L=GL(1)^{c[1]/2}\times\ldots\times GL(k)^{c[k]/2}.$$

O(2*n*+1): *The row sizes in the Jordan block description of $\mathcal{O}$ are odd only. Equivalently, the $c[k]$ are all even except for the largest one which is of odd size say $2k+1$ for which $c[2k+1]$ is odd. Equivalently, the nilpotent orbit is induced Richardson from a Levi component satisfying*

$$L=O(2k+1)\times GL(1)^{c[1]/2}\times\ldots\times GL(2k+1)^{(c[2k+1]-1)/2}.$$

O(2*n*): *The row sizes in the Jordan block description of $\mathcal{O}$ are odd only. Equivalently, the $c[k]$ are all even except for the largest one which is of even size say $2k$ for which $c[2k]$ is odd. Equivalently, the nilpotent is induced Richardson from a Levi component satisfying*

$$L=O(2k)\times GL(1)^{c[1]/2}\times\ldots\times GL(2k)^{(c[2k])/2}.$$

Proof. This follows from [KP] and the description induction in terms of the Jordan block description of nilpotent orbits (see also 14.4).

14.4. Denote by $\operatorname{triv}(m)$ the trivial orbit in $\mathscr{A}=\mathscr{A}(2m+1)$ where m is an integer or half-integer. Then if $\mathscr{Z}$ is an $(\mathfrak{m}, K\cap M)$ module and $\mathfrak{m}\times\mathscr{A}\subset\mathfrak{g}$ is a Levi component,

$$WF(Z*\mathscr{F}(m))=\operatorname{ind}_{\mathfrak{m}\times\mathscr{A}}^{\mathfrak{g}}(WF(Z)\times\operatorname{triv}(m)). \tag{14.4.1}$$

If $\mathcal{O}=WF(Z)$ then denote the right hand side of (14.4.1) by $\mathcal{O}*\operatorname{triv}(m)$. The symbol of $\mathcal{O}*\operatorname{triv}(m)$ is obtained by adding a 1 to the last $2m+1$ entries of the symbol of $\mathcal{O}$.

Proposition. *Let $\mathcal{O}(n)$ be a special nilpotent orbit in $\mathfrak{g}(n)$. Then*

(1) *There are $n_1, \ldots, n_k$ such that*

$$\mathcal{O}(n, n_1, \ldots, n_k)=\mathcal{O}(n)*\operatorname{triv}(n_1)*\ldots*\operatorname{triv}(n_k)$$

satisfies the hypotheses in Proposition 14.3 (with possibly a different L than $G(n)\times GL(n_1)\times\ldots GL(n_k)$).

(2) *If $\{\mathcal{O}, L\}$ already satisfies the hypotheses of Proposition 14.3 then so will any $\{\mathcal{O}*\operatorname{triv}(m), G(n)\times GL(m)\}$ provided that in types B, D, m is less than or equal to the length of the largest column of $\mathcal{O}$.*

Proof. **Type B, D.** Label the rows of $\mathcal{O}$ in decreasing order of size. Let m_1, m_1+1 be the first rows of even size. Then $\mathcal{O} * \mathrm{triv}(m_1)$ adds a 2 to the size of each row earlier than m_1 and adds a 1 to rows m_1, m_1+1. In finitely many steps we get the desired result.

Type C. The same argument works but m_1, m_1+1 denote the first rows of odd size. Statement (2) follows by the description of induction given above.

Definition. Let $d(\mathcal{O})=k$ be the smallest number of factors necessary so that $\mathcal{O}(n, n_1, \ldots, n_k)$ satisfies the hypotheses of Proposition 14.3 for some L.

14.5. Proposition. *Let $\mathcal{O}$ be even. Let Q be the parabolic subgroup corresponding to the semisimple element $h(\mathcal{O})$ of the Lie triple attached to $\mathcal{O}$. Then the corresponding moment map is birational.*

Proof. This follows from Proposition 2.4 in [BV1].

Corollary. *The pairs $\{\mathcal{O}, L\}$ in* 14.3 *satisfy the hypotheses of Proposition* 14.2.

Proof. It is enough to show that the L's in 14.4 are isomorphic to the ones given by the corresponding $h(\mathcal{O})$. This is clear.

14.6. Proof of Theorem 14.2. Let

$$\bar{\mathcal{O}} = WF(\bar{X}_{u,0} * \bar{X}_{u,1} * \ldots * \bar{X}_{u,r}). \tag{14.6.1}$$

We first reduce to the case when $\bar{X}_{u,0}$ is induced from 1-dimensional characters (not necessarily unitary) and $\mathcal{O}$ and the corresponding L are as in 14.3–14.4.

Recall $T(\bar{X}_{u,0}) = \bar{X}_e \otimes \bar{X}_o$ and the symbols in 9.6. If (z_s, z_{s+1}) is a pair from the symbol of $\bar{X}_e$, $\bar{X}_o$ such that $z_{s+1} - z_s > 1$ we can write

$$\bar{X}_{u,0} * \mathcal{F}(m) = \bar{X}_{(1,\ldots,1)} + \bar{X}_{(1,\ldots,-1,-1,\ldots,1)}, \quad \text{for } z_s < m+s < z_{s+1}. \tag{14.6.2}$$

Then $X * \mathcal{F}(m)$ decomposes into a similar sum of induced representations each of the same type as X but with the corresponding $l(\lambda_{u,0})$ strictly less than for X. Since both $\bar{X}_{(1,\ldots,1)}, \bar{X}_{(1,\ldots,-1,-1,\ldots,1)}$ have small lowest K-type, the bottom layer arguments in Sect. 2 insure that the lowest K-type composition factors of the induced representations in $X * \mathcal{F}(m)$ occur already in $\bar{X} * \mathcal{F}(m)$. Then it is enough to show that these are irreducible. In finitely many steps, we reduce to the case when all $z_{s+1} - z_s = 0$ or 1.

We now show how to achieve 14.3–14.4 for each type.

Type B. Assume that $x_0 > 0$. Replace $\bar{X}_{u,0}$ by

$$\bar{X}_{u,0} * \mathcal{F}(x_0 - 1) * \ldots * \mathcal{F}(0). \tag{14.6.3}$$

By the argument above, it is enough to prove the theorem for the composition factors of (14.6.3). This reduces to the case when $\bar{X}_{u,0}$ has $x_0 = 0$. In this case, let

$$\mathfrak{m}(a) \times \mathcal{A}(a_1) \times \ldots \times \mathcal{A}(a_k), \quad a_i < a_{i+1} \tag{14.6.4}$$

be the Levi component from which X is induced from 1-dimensional representations. Then the corresponding $\mathcal{O}$, L do not satisfy 14.3–14.4 if $2a+1 < a_k$. Then replace $\bar{X}_{u,0}$ by

$$\bar{X}_{u,0} * \mathcal{F}(a+1/2) * \ldots * \mathcal{F}(a_k + 1/2). \tag{14.6.5}$$

Then it is enough to prove the theorem for the composition factors of (14.6.5). The corresponding $\mathcal{O}$, L satisfy 14.3–14.4.

Type C. Let (14.6.4) be the Levi component from which X is induced. 14.3–14.4 are satisfied precisely when $a=0$. Replace $\bar{X}_{u,0}$ by

$$\bar{X}_{u,0} * \mathcal{F}(x_0 - 1) * \ldots * \mathcal{F}(0). \tag{14.6.6}$$

This reduces by the same argument to the case when $a=0$. Then 14.3–14.4 is satisfied.

Type D. Let (x_0, x_{2p+1}) be the pair in the symbol for $\bar{X}_e$ as in (9.6.13). It is enough to prove the assertion for the composition factors of (14.6.6). Thus we may reduce to the case when $x_0 = 0$. In this case, $\bar{X}_{u,0}$ is induced from a 1-dimensional character on a Levi component as in (14.6.4). 14.3–14.4 are satisfied if $2a \geqq a_k$. If this is not the case, replace $\bar{X}_{u,0}$ by

$$\bar{X}_{u,0} * \mathcal{F}(a) * \ldots * \mathcal{F}(a_k). \tag{14.6.7}$$

Then it is enough to prove the theorem in the case when $\bar{X}_{u,0}$ is one of the composition factors of (14.6.7), which satisfy 14.3–14.4.

In order to be able to apply 14.2 we must reduce to the case when X is induced from a unitary character. We will do this in the same way as above, using $* \mathcal{F}(m)$ with m satisfying (2) of Proposition 14.4 so that the birationality and normality conditions are not affected.

Let $d(\bar{X})$ be the number of pairs (z_s, z_{s+1}) in the symbol of $\bar{X}_{u,0}$ such that the corresponding $\mathcal{F}^s = F^s * F^s$ satisfies

$$F^s: (M+1, M, \ldots, -M). \tag{14.6.8}$$

We will do an inductrion on $d(\bar{X})$. The case $d(\bar{X}) = 0$ is Theorem 14.2. Assume $d(\bar{X}) > 0$. Then in the notation 6.5,

$$\bar{X}_{u,0} * \mathcal{F}(M+1/2; 1/2) = \bar{X}^s_{u,0} * [\mathcal{F}^s * \mathcal{F}(M+1/2; 1/2)]. \tag{14.6.9}$$

Then $[\mathcal{F}^s * \mathcal{F}(M+1/2; 1/2]$ is as in (2) of Lemma 8.2. Thus it contains a spherical composition factor $X' = \mathcal{F}(M+1) * \mathcal{F}(M-1)$ and a nonspherical composition factor $X'' = \mathcal{F}'' * \mathcal{F}''$ where

$$\mathcal{F}'': (M+1, M, \ldots, -M) \times (M, \ldots, -M, -M-1). \tag{14.6.10}$$

By Sects. 4–6 and Theorem 14.2 for type A, X', X'' are induced irreducible. By Corollary 8.2,

$$[\mathcal{F}^s * \mathcal{F}(M+1/2; 1/2] = X' + X''. \tag{14.6.11}$$

Since in addition, X', X'' have small lowest K-types, the argument from before shows that it is enough to prove the theorem with $\bar{X}_{u,0}$ replaced by X', X''. Since they have strictly smaller $d(\,)$, the proof follows.

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Inventiones mathematicae

The shape of a figure-eight under the curve shortening flow

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When a closed curve immersed in the plane evolves by its curvature vector, it stays smooth until its curvature blows up. An interesting problem is to describe the exact circumstances surrounding the formation of such a singularity. Gage [4, 5], Gage and Hamilton [6], and the author [7] show that an embedded closed curve cannot develop a singularity until is shrinks to a point. The limiting shape of the curve is round, with convergence in the C^∞ topology. In marked contrast to this behaviour, we show that an immersed curve can evolve by the curvature flow until its unsigned area vanishes, but its isoperimetric ratio can converge to ∞. In some cases, the limiting shape can converge to a line segment. Abresch and Länger in [1], and Epstein and Weinstein in [3], study immersed curves which preserve their shapes, and thus their isoperimetric ratios, as they shrink to points.

In this note, we will deal with the simplest non-embedded curves: the figure-eights. A smooth curve C immersed in the plane is a figure-eight if

i. it has exactly one double point, and
ii. it has total rotation number $\int_C k\,ds = 0$.

Such a curve divides the plane into three regions, two of which have finite area.

Any immersed curve $C(0)$: $S^1 \to \mathbb{R}^2$ extends to a flow $C(t)$: $S^1 \to \mathbb{R}^2$ for $t \in [0, T)$ satisfying $C_t = kN$, where kN is the curvature vector of C. The curvature is unbounded as $t \to T$. The main result of this paper is

Theorem 1. *The isoperimetric ratio L^2/A converges to ∞ as $t \to T$ if and only if the loops of $C(0)$ bound regions of equal area.*

One nice feature of the curvature evolution for curves immersed in a surface is that the number of double points is a non-increasing function of time (see [8] or [2]). Therefore a figure eight stays a figure eight until one of its loops collapses or the flow encounters a singularity.

Choose cartesian coordinates in the plane and let θ denote the angle which the tangent to the curve makes with the $x-$axis. Since the total rotation number

* Research supported in part by an NSF Postdoctoral Research Fellowship