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Representation of the Category $\mathcal{O}$ in Whittaker Categories

Erik Backelin

1 Introduction

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a complex semisimple Lie algebra and let $\eta\colon U(\mathfrak{n}_+) \to \mathbf{C}$ be an algebra homomorphism. Kostant [8] studied the invariants $\mathrm{Wh}_\eta(M) := \{m \in M;\ \mathrm{Ker}\,\eta \cdot m = 0\}$ in a $\mathfrak{g}$-module M. We call these invariants Whittaker vectors.

If a $\mathfrak{g}$-module M is the direct sum of its weight spaces with respect to $\mathfrak{h}$ (and $\eta|\mathfrak{n}_+ \neq 0$), then M contains no nonzero Whittaker vectors. If M is an object of the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand, one should therefore consider the set $\overline{\mathrm{Wh}}_\eta(M)$ of Whittaker vectors in the completion (definition 3.1) $\overline{M}$ of M. This gives the functor $\overline{\mathrm{Wh}}_\eta$ from $\mathcal{O}$ to the category of $Z(\mathfrak{g})$-modules. Kostant's results implies that $\overline{\mathrm{Wh}}_\eta$ is exact when η is regular, i.e., nonvanishing on each simple root vector.

Although Kostant did not work with the category $\mathcal{O}$, he did prove that $\dim \overline{\mathrm{Wh}}_\eta(M_\chi) = 1$ for any simple Verma module M_χ when η is regular. This is generalized to arbitrary Verma modules in Proposition 4.5. As a consequence, we prove in Corollary 4.6 a result about primitive vectors.

Let W_λ be the integral Weyl group of a dominant weight λ, and let w_0 be its longest element. $\mathcal{O}_\lambda$ denotes the subcategory of $\mathcal{O}$ whose objects have all their composition factors isomorphic to $L_{w\lambda}$, the simple module of highest weight $w\lambda - \rho$, $w \in W_\lambda$. Let $P_{w\lambda}$ be the projective cover of $L_{w\lambda}$ and consider the functor

$$\mathbb{V}\colon\ M \rightsquigarrow \mathrm{Hom}_{\mathfrak{g}}(P_{w_0\lambda}, M)$$

from $\mathcal{O}_\lambda$ to $\mathrm{End}_{\mathfrak{g}}(P_{w_0\lambda})$-mod. Soergel proved the important result that the multiplication homomorphism $Z(\mathfrak{g}) \to \mathrm{End}_{\mathfrak{g}}(P_{w_0\lambda})$ is surjective [12]. Therefore, $\mathbb{V}(M)$ may just as well be

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considered as a $Z(\mathfrak{g})$-module. He also proved that $\mathbb{V}$ restricted to projective objects in $\mathcal{O}_\lambda$ is fully faithful (see Theorem 5.1).

The study of $\mathbb{V}$ as well as the other $\mathrm{Hom}_\mathfrak{g}(P_{w\lambda}, \ \cdot\ )$ is motivated by the following. Put $P := \oplus_{w \in W_\lambda} P_{w\lambda}$. Then for abstract reasons, $M \rightsquigarrow \mathrm{Hom}_\mathfrak{g}(P, M)$ gives an equivalence between the categories $\mathcal{O}_\lambda$ and the category of finitely generated right $\mathrm{End}_\mathfrak{g}(P)$-modules [1]. In [2] this connection is studied and in particular it is proved that $\mathrm{End}_\mathfrak{g}(P)$ is a Koszul algebra.

$\mathrm{End}_\mathfrak{g}(P_{w_0\lambda})$ is a graded algebra (see Section 5). The best known way to see this grading is that it comes from a grading on $Z(\mathfrak{g})$ via the surjection $Z(\mathfrak{g}) \twoheadrightarrow \mathrm{End}_\mathfrak{g}(P_{w_0\lambda})$. One would, however, like to see the grading on $\mathrm{End}_\mathfrak{g}(P_{w_0\lambda})$ more directly. Joseph Bernstein therefore suggested that one should try to find another realization of the functor $\mathbb{V}$ where the grading becomes more obvious.

The first main result of this paper, suggested by Bernstein, is the proof for the assertions that $\mathbb{V}$ is determined, by its exactness and the fact that $\mathbb{V}(M_{w\lambda})$ is a one-dimensional vector space for each Verma module $M_{w\lambda}$, $w \in W_\lambda$, and that $\mathbb{V}$ is isomorphic to $\overline{\mathrm{Wh}}_\eta \mid \mathcal{O}_\lambda$ when η is regular. (See Theorem 5.2 and Corollary 5.4.) Unfortunately, the original attempt to see the grading on $\mathrm{End}_\mathfrak{g}(P_{w_0\lambda})$ failed, since we do not have any natural grading on $\overline{\mathrm{Wh}}_\eta(P_{w_0\lambda})$.

In Section 6 we consider an arbitrary η. We prove some results about $\overline{\mathrm{Wh}}_\eta$ and consider yet another functor

$$\overline{\Gamma}_\eta\colon\ M \rightsquigarrow \{v \in \overline{M};\ \exists k\colon\ (\mathrm{Ker}\,\eta)^k \cdot v = 0\}$$

from $\mathcal{O}$ to $U(\mathfrak{g})$-mod. This functor is exact and it maps the category $\mathcal{O}$ into the category Ω_η whose objects are finitely generated $U(\mathfrak{g})$-modules, which are locally annihilated by some power of $\mathrm{Ker}\,\eta$ and locally finite over $Z(\mathfrak{g})$.

It is known that Ω_η has certain "standard"-modules (and simple modules) parameterized by the cosets in the Weyl group of a certain subgroup (McDowell [9]) (see also Miličić-Soergel [10], [11]). Soergel (unpublished) has proved that the Verma modules are mapped onto the set of standard modules in Ω_η by the functor $\overline{\Gamma}_\eta$; we prove in Proposition 6.9 that $\overline{\Gamma}_\eta$ maps the simple objects in $\mathcal{O}$ onto the set of simple objects in Ω_η.

This is used to establish our second main result, Theorem 6.2, stating that the multiplicities of simple modules in standard modules in Ω_η follows from multiplicities of corresponding modules in $\mathcal{O}$, which can be calculated by the Kazhdan-Lusztig algorithm.

2 Preliminaries

In the sequel, we fix a semisimple complex Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$.

We shall use the following notation (cf. [5], [6]). R denotes the set of roots, B is a basis of R, and R_+ is the set of positive roots. This gives the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let Q denote the root lattice, Q_{++} the semigroup generated by the simple roots, and P the integral weights, $P_{++} := \{\chi \in \mathfrak{h}^*; \chi(H_\alpha) \in \mathbf{N}$ for $\alpha \in B\}$. ρ denotes half the sum of the positive roots.

For each $\alpha \in R_+$, choose $X_\alpha \in \mathfrak{g}^\alpha$ and $X_{-\alpha} \in \mathfrak{g}^{-\alpha}$ such that $\alpha([X_\alpha, X_{-\alpha}]) = 2$, and put $H_\alpha = [X_\alpha, X_{-\alpha}]$. Let W denote the Weyl group. Let s_α be the reflection in W corresponding to the root α. Hence, $s_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha$ for $\lambda \in \mathfrak{h}^*$. Let $S := \{s_\alpha; \alpha \in B\}$ be the set of simple reflections.

Elements in $\mathfrak{h}^*$ are called weights. A weight χ is called integral if $\chi(H_\alpha) \in \mathbf{Z}$, dominant if $\chi(H_\alpha) \notin \{-1, -2, \ldots\}$, and anti-dominant if $\chi(H_\alpha) \notin \{1, 2, \ldots\}$ for each $\alpha \in R_+$. For any $\mathfrak{g}$-module M, put $M^\chi = \{m \in M; Hm = \chi(H)m, \forall H \in \mathfrak{h}\}$. M^χ is called the weight space of χ in M and elements in M^χ weight vectors. A primitive vector is a weight vector annihilated by $\mathfrak{n}_+$.

For any Lie algebra $\mathfrak{a}$, let $U(\mathfrak{a})$ denote its enveloping algebra and $Z(\mathfrak{a})$ be the center of $U(\mathfrak{a})$. $U(\mathfrak{a})$-mod denotes the category of left $U(\mathfrak{a})$-modules and $Z(\mathfrak{a})$-mod the category of $Z(\mathfrak{a})$-modules. If $M, N \in U(\mathfrak{a}) - \text{mod}$, then $M \otimes_{\mathbf{C}} N$ denotes the $U(\mathfrak{a})$-module where $x \cdot m \otimes n := (xm) \otimes n + m \otimes (xn)$ for $x \in \mathfrak{a}$, $m \in M$, and $n \in N$.

The Bernstein-Gelfand-Gelfand category $\mathcal{O}$, [5], is the set of $M \in U(\mathfrak{g})$-mod, such that

(1) M is finitely generated over $U(\mathfrak{g})$;

(2) $M = \oplus_{\chi \in \mathfrak{h}^*} M^\chi$;

(3) M is locally finite over $U(\mathfrak{n}_+)$.

For $\chi \in \mathfrak{h}^*$, denote by M_χ the Verma module with highest weight $\chi - \rho$, L_χ its simple quotient, and P_χ the projective cover of L_χ. These are objects in $\mathcal{O}$.

Put $\Theta_\chi := \{z \in Z(\mathfrak{g}); z \cdot M_\chi = 0\}$. Then Θ_χ is a maximal ideal in $Z(\mathfrak{g})$ and each maximal ideal is of the form Θ_χ, for some $\chi \in \mathfrak{h}^*$, and $\Theta_\chi = \Theta_\phi$ if and only if $\phi \in W\chi$. Let $\mathcal{O}_{\Theta_\chi}$ be the subcategory of $\mathcal{O}$, whose objects are annihilated by some power of Θ_χ; equivalently, $M \in \mathcal{O}_{\Theta_\chi}$ if its composition factors belong to $\{L_{w\lambda}\}_{w \in W}$. Then $\mathcal{O}$ splits into the orthogonal direct sum

$$\mathcal{O} = \oplus_{\chi \in \mathfrak{h}^*/W} \mathcal{O}_{\Theta_\chi}.$$

Orthogonal here means that $\operatorname{Ext}_{\mathcal{O}}(M, N) = 0$ for $M \in \mathcal{O}_{\Theta_\chi}$, $N \in \mathcal{O}_{\Theta_\phi}$, $i \geq 0$, when $\chi \neq \phi$ in $\mathfrak{h}^*/W$.

Given a maximal ideal $\Theta = \Theta_\chi$ in $Z(\mathfrak{g})$, let $\chi_1, \ldots, \chi_s$ be the dominant weights in $W\chi$. Let $W_{\chi_i} := \{w \in W; w\chi_i - \chi_i \in P\}$ be the integral Weyl group corresponding to χ_i. Define

$\mathcal{O}_{\chi_i} \subset \mathcal{O}_{\Theta_\chi}$ as the subcategory of all modules whose composition factors are isomorphic to $L_{w\chi}$, $w \in W_{\chi_i}$. Then $\mathcal{O}_{\Theta_\chi}$ decomposes into the orthogonal direct sum $\mathcal{O}_{\Theta_\chi} = \oplus \mathcal{O}_{\chi_i}$.

We denote by $K(\mathcal{O})$ the Grothendieck group of $\mathcal{O}$. The Verma (resp. simple) modules form a basis of the free abelian group $K(\mathcal{O})$. If $M \in \mathcal{O}$, $\chi \in \mathfrak{h}^*$, then $[M : M_\chi]$ (resp. $[M : L_\chi]$) denotes the coefficient of M_χ (resp. L_χ) in the expression of M in the Verma (resp. simple) module basis of $K(\mathcal{O})$.

In this paper a category Ω which generalizes $\mathcal{O}$ is studied. Ω is the subcategory of $U(\mathfrak{g})$-mod, whose objects M satisfy the following:

(1) M is finitely generated over $U(\mathfrak{g})$;
(2) M is locally finite over $Z(\mathfrak{g})$;
(3) M is locally finite over $U(\mathfrak{n}_+)$.

Each object in Ω has finite length ([9], [11]), and a geometric proof is found in [10]. In Section 6.2 we define certain standard modules in Ω.

Let $\eta\colon \mathfrak{n}_+ \to \mathbf{C}$ be a Lie algebra homomorphism ($\mathbf{C}$ is considered as a commutative Lie algebra). It is called a character on $\mathfrak{n}_+$. It is clear that $\eta([\mathfrak{n}_+, \mathfrak{n}_+]) = 0$. Denote also by η the induced algebra homomorphism $U(\mathfrak{n}_+) \to \mathbf{C}$. η is said to be regular if $\eta(X_\alpha) \neq 0$ for all $\alpha \in B$. If η vanishes on $\mathfrak{n}_+$, we write $\eta = 0$ for short. Denote by Ω_η the subcategory of Ω whose objects are locally annihilated by some power of $\operatorname{Ker}\eta$. We refer to objects in Ω as Whittaker modules.[1]

Then the category Ω splits into an orthogonal direct sum over characters on $\mathfrak{n}_+$

$$\Omega = \oplus_\eta \Omega_\eta.$$

We also have the orthogonal direct sum $\Omega_\eta = \oplus_{\Theta \in \max Z(\mathfrak{g})} \Omega_{\eta,\Theta}$, where $\Omega_{\eta,\Theta}$ denotes the set of objects with generalized central character Θ, i.e., objects that are annihilated by some power of Θ.

It is proved in Kostant [8] that for η regular, Wh_η gives an equivalence between Ω_η and the category of Artinian $Z(\mathfrak{g})$-modules. This shows that Ω_η for regular η gives the least complicated blocks in Ω. In the other extreme, Ω_0 is the smallest subcategory of $U(\mathfrak{g})$-mod containing the category $\mathcal{O}$ which is closed under extensions. Clearly, properties of Ω_η for general η should follow from these two cases. For instance, Miličić-Soergel [11] proved that in most cases (at least for integral central character Θ), $\Omega_{\eta,\Theta}$ is equivalent to $\Omega_{0,\xi}$ for certain singular central character ξ.

[1] Our terminology here differs from that of Kostant [8] who used the term Whittaker module only in the case of a regular η, and he used the word nondegenerate instead of regular.

3 The Whittaker functor

Definition 3.1. To a module $M = \oplus_{\chi\in\mathfrak{h}^*} M^\chi$ in $\mathcal{O}$ we associate the $\mathfrak{g}$-module $\overline{M} := \prod_{\chi\in\mathfrak{h}^*} M^\chi$, defining an exact functor from $\mathcal{O}$ to $U(\mathfrak{g})$-mod.

There exists a duality functor on $\mathcal{O}$ which gives us a useful description of $\overline{M}$.

For $M \in U(\mathfrak{g})$-mod, the full dual $M' := \mathrm{Hom}_{\mathbf{C}}(M, \mathbf{C})$ has the $\mathfrak{g}$-module structure defined by $x\phi(m) = \phi(x^t m)$ for $x \in \mathfrak{g}, \phi \in M', m \in M$. Here ^{t} is the anti-automorphism of $\mathfrak{g}$, defined by $^t|\mathfrak{h} = \mathrm{id}_{\mathfrak{h}}$, $X_\alpha^t = X_{-\alpha}$, $\alpha \in R$. Then $m \to \phi$: $\psi \to \psi(m)$ defines a canonical homomorphism $M \to M''$.

In M' we have the submodule $M^* := \{m \in M'; \dim U(\mathfrak{n}_+)m < \infty\}$. If $M \in \mathcal{O}$, then $M^* \in \mathcal{O}$ and the canonical homomorphism $M \to M^{**}$ is an isomorphism, extending to an isomorphism of $\mathfrak{g}$-modules

$$\overline{M} \to M^{*\prime}. \tag{3.1}$$

Let η be a character on $\mathfrak{n}_+$. For $M \in U(\mathfrak{g})$-mod we define

$$\Gamma_\eta(M) := \{m \in M; \exists k\colon\ (\mathrm{Ker}\,\eta)^k \cdot m = 0\} \in U(\mathfrak{g}) - \mathrm{mod};$$

and for $M \in \mathcal{O}$ we put

$$\overline{\Gamma}_\eta(M) := \{m \in \overline{M}; \exists k\colon\ (\mathrm{Ker}\,\eta)^k \cdot m = 0\} \in U(\mathfrak{g}) - \mathrm{mod}\,.$$

Here $\mathrm{Ker}\,\eta$ denotes the kernel of η in $U(\mathfrak{n}_+)$.

Lemma 3.2. $\overline{\Gamma}_\eta$ defines an exact functor from $\mathcal{O}$ to Ω_η. □

Proof. Let $\mathcal{C}$ be the subcategory of $U(\mathfrak{g})$-mod whose objects are finitely generated over $U(\mathfrak{n}_+)$. Let δ be the functor on $U(\mathfrak{g})$-mod, defined by $\delta(M) := \mathrm{Hom}_{\mathbf{C}}(M, \mathbf{C})$ with the $U(\mathfrak{g})$-module structure defined by $(x\phi)(m) := \phi(-xm)$ for $m \in M$, $\phi \in \delta(M)$ and $x \in \mathfrak{g}$. Kostant [8, Lemma 4.5] proved that $\Gamma_\eta \circ \delta$ defines an exact functor from $\mathcal{C}$ to Ω_η. Kostant's proof is based on the Artin-Rees lemma for nilpotent Lie algebras.

Define $\mathcal{O}^-$ as the category $\mathcal{O}$ with the roles of $\mathfrak{n}_+$ and $\mathfrak{n}_-$ interchanged. Then $\mathcal{O}^- \subset \mathcal{C}$. There is an equivalence of categories δ_{fin} from $\mathcal{O}$ to $\mathcal{O}^-$, defined by $\delta_{\mathrm{fin}}(M) := \{\phi \in \delta(M); \exists k\colon\ (\mathfrak{n}_+)^k \cdot \phi = 0\}$. Then $\delta \circ \delta_{\mathrm{fin}}$ is isomorphic to the completion functor; hence $\overline{\Gamma}_\eta \cong \Gamma_\eta \circ \delta \circ \delta_{\mathrm{fin}}$ and we conclude that $\overline{\Gamma}_\eta\colon\ \mathcal{O} \rightsquigarrow \Omega_\eta$ is exact. ■

Remark 3.3. The functor $\overline{\Gamma}_\eta\colon\ \mathcal{O} \rightsquigarrow \Omega_\eta$ is not surjective. If $\eta = 0$, then $\overline{\Gamma}_\eta$ is the inclusion of $\mathcal{O}$ into Ω_0. If η is regular, then Ω_η is equivalent to the category of Artinian $Z(\mathfrak{g})$-modules by Proposition 4.3. But if $M \in \mathcal{O}$ has generalized central character Θ_λ, then M is annihilated by $J := \cap_{w\in W}\{z \in Z(\mathfrak{g}); zP_{w\lambda} = 0\}$, and thus also $J \cdot \overline{\Gamma}_\eta(M) = 0$. Therefore, no object in $\mathcal{O}$

is mapped by $\overline{\Gamma}_\eta$ to the object in Ω_η corresponding to the finite-dimensional $Z(\mathfrak{g})$-module $Z(\mathfrak{g})/J^2$.

However, if we define $\overline{\Gamma}_\eta\colon\ \Omega_0 \rightsquigarrow \Omega_\eta$ by $\overline{\Gamma}_\eta(M) := \Gamma_\eta(M^{*\prime})$, then this functor is probably surjective.

Each $M \in U(\mathfrak{g})$-mod has the $Z(\mathfrak{g})$-submodule $Wh_\eta(M) := \{m \in M; \operatorname{Ker}\eta \cdot m = 0\}$. An element in this set is called a Whittaker vector. It is easy to verify that $M \rightsquigarrow Wh_\eta(M)$ defines a left exact functor on $U(\mathfrak{g})$-mod. The composition of Wh_η with the completion functor on $\mathcal{O}$ gives a left exact functor $\overline{Wh}_\eta$.

Definition 3.4. The Whittaker functor $\overline{Wh}_\eta\colon\ \mathcal{O} \to Z(\mathfrak{g})$-mod is defined as

$$M \rightsquigarrow \overline{Wh}_\eta(M) := \{m \in \overline{M}; \operatorname{Ker}\eta \cdot m = 0\}.$$

Thus $\overline{Wh}_\eta = Wh_\eta \circ \overline{\Gamma}_\eta$. Notice that if $\eta = 0$ and $M \in \mathcal{O}$, then

$$\overline{Wh}_\eta(M) := \{m \in \overline{M};\ \mathfrak{n}_+ \cdot m = 0\} = \{m \in M; \mathfrak{n}_+ \cdot m = 0\},$$

i.e., the set of primitive vectors in M.

Let $\tilde{\eta}$ be the character on $\mathfrak{n}_-$ defined by $\tilde{\eta}(X_{-\alpha}) = \eta(X_\alpha)$, $\alpha \in B$. Using the identification of $M^{*\prime}$ with $\overline{M}$ in (3.1) we get

$$\overline{Wh}_\eta(M) = \{\phi \in M^{*\prime};\ \operatorname{Ker}\eta \cdot \phi = 0\} = \{\phi \in M^{*\prime};\ \phi|\operatorname{Ker}\tilde{\eta} \cdot M^* = 0\}. \tag{3.2}$$

This shows that $\dim \overline{Wh}_\eta(M) = \operatorname{codim}_{M^*}(\operatorname{Ker}\tilde{\eta} \cdot M^*)$. Since M^* is finitely generated over $U(\mathfrak{n}_-)$, this dimension is finite. (Similarly, $\overline{\Gamma}_\eta(M) = \{\phi \in M^{*\prime};\ \exists k\colon\ \phi|(\operatorname{Ker}\tilde{\eta})^k \cdot M^* = 0\}$.)

Up to isomorphism of functors $\overline{Wh}_\eta$ is determined by those simple root vectors on which η vanishes.

Assume η and η' are two characters on $\mathfrak{n}_+$ which vanish on the same simple root vectors. We define an isomorphism of functors $\overline{Wh}_\eta \rightsquigarrow \overline{Wh}_{\eta'}$ as follows: Let $f\colon\ Q \to \mathbf{C}^*$ be a group homomorphism such that $\eta'(X_\alpha) = f(\alpha)\eta(X_\alpha)$ for each $\alpha \in B$. Choose a set $\Psi \subset \mathfrak{h}^*$ of representatives of the cosets $\mathfrak{h}^*/Q$. Let $M \in \mathcal{O}$ and define for each $m \in \overline{Wh}_\eta(M)$ an element $m' \in \overline{Wh}_{\eta'}(M)$ by

$$m'^{(\psi+\chi)} := f(\chi)m^{(\psi+\chi)}, \quad \text{for} \quad \psi \in \Psi \quad \text{and} \quad \chi \in Q.$$

Here $m^{(\psi+\chi)}$ denotes the component of m in $M^{(\psi+\chi)}$. This defines a (functorial) isomorphism $\overline{Wh}_\eta(M) \to \overline{Wh}_{\eta'}(M)$.

The converse is also true, i.e., if $\overline{Wh}_\eta \cong \overline{Wh}_{\eta'}$, then η and η' vanish on the same weight spaces of $\mathfrak{n}_+$. This can be deduced, e.g., from Proposition 6.4 below. Hence there are $2^{\operatorname{rank}\mathfrak{g}}$ isomorphism classes of Whittaker functors.

4 Properties of the Whittaker functor in the regular case

Lemma 4.1 below is due to Kostant [8, Theorem 4.3]. Let us mention that a stronger version of it—where Ω_η is replaced by the category of all objects in $U(\mathfrak{g})$-mod which are locally annihilated by some power of $\operatorname{Ker}\eta$—can be deduced from a vanishing theorem of Wallach [13, Theorem 2.2].

Lemma 4.1 (η regular). The functor Wh_η restricted to Ω_η is exact. □

Combining this with Lemma 3.2, we get the following.

Theorem 4.2 (η regular). $\overline{\mathrm{Wh}}_\eta$ is exact. □

The following proposition is proved in [8, Theorem 4.3], and in [10]. Lemma 4.1 is the main ingredient in the proof.

Proposition 4.3 (η regular). The functor Wh_η defines an equivalence between Ω_η and the category of Artinian $Z(\mathfrak{g})$-modules. The inverse functor is given by $V \rightsquigarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{n}_+)\otimes_{\mathbf{C}} Z(\mathfrak{g})} V$, where the $U(\mathfrak{n}_+)$ action on the $Z(\mathfrak{g})$-module V is given by η. In particular, each object in Ω_η is generated by its Whittaker vectors. □

Remark 4.4. The functor $\overline{\mathrm{Wh}}_\eta$ is not exact when η is not regular. For instance, let $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$ and let $\eta = 0$; we have the surjection $P_{-\rho} \to M_{-\rho}$, but the induced homomorphism $\mathbf{C}^2 \cong \overline{\mathrm{Wh}}_\eta(P_{-\rho}) \to \overline{\mathrm{Wh}}_\eta(M_{-\rho}) \cong \mathbf{C}$ is the zero map.

The next theorem was proved in [8] for simple Verma modules.

Proposition 4.5 (η regular). Let $\lambda \in \mathfrak{h}^*$. Then $\dim \overline{\mathrm{Wh}}_\eta(M_\lambda) = 1$. □

Proof. (a) Let $M \in \mathcal{O}$ and denote by (M) its image in the Grothendieck group $K(\mathcal{O})$. Since $L^* \cong L$, when L is simple, we see that $(M^*) = (M)$. Since the Whittaker functor is exact, this shows that $\dim \overline{\mathrm{Wh}}_\eta(M) = \dim \overline{\mathrm{Wh}}_\eta(M^*)$. In particular, $\dim \overline{\mathrm{Wh}}_\eta(M_\lambda) = \dim \overline{\mathrm{Wh}}_\eta(M_\lambda^*)$.

(b) By 3.2 we have $\overline{\mathrm{Wh}}_\eta(M_\lambda^*) = \{\phi \in M_\lambda'; \phi |\operatorname{Ker}\tilde{\eta} \cdot M_\lambda = 0\}$. Hence,

$$\dim \overline{\mathrm{Wh}}_\eta(M_\lambda^*) = \operatorname{codim}_{M_\lambda}(\operatorname{Ker}\tilde{\eta} \cdot M_\lambda) = \operatorname{codim}_{U(\mathfrak{n}_-)}(\operatorname{Ker}\tilde{\eta} \cdot U(\mathfrak{n}_-)) = 1.$$

The second equality follows from the fact that M_λ is a free $U(\mathfrak{n}_-)$-module of rank 1; the last equality holds since $U(\mathfrak{n}_-)/\operatorname{Ker}\tilde{\eta} \cong \mathbf{C}$. ■

Proposition 4.5 casts some light on the structure of primitive vectors in Verma modules.

Let $R_+ = \{\alpha_1, \ldots, \alpha_l, \ldots, \alpha_n\}$, where the first l roots are simple. By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{n}_-)$ has the vector space basis

$$D := \{X_{-\alpha_1}^{t_1} \cdots X_{-\alpha_n}^{t_n};\ t_1, \ldots, t_n \in \mathbf{N}\}.$$

Let $M_\lambda \subset M_\chi$ be two Verma modules and μ_λ, μ_χ be their canonical generators, respectively. Then $\mu_\lambda = P\mu_\chi$ for a uniquely determined $P \in U(\mathfrak{n}_-)$. There is a unique l-tuple of natural numbers, $(t_1^0, \ldots, t_l^0)$, such that $\sum_{i=1}^{l} t_i^0 \alpha_i = \chi - \lambda$. Denote by $\lambda_{t_1^0,\ldots,t_l^0}$ the coefficient of $X_{-\alpha_1}^{t_1^0} \cdots X_{-\alpha_l}^{t_l^0}$ in P with respect to the basis D.

Corollary 4.6. With the notations above, $\lambda_{t_1^0,\ldots,t_l^0} \neq 0$. □

Proof. Let η be a regular character on $\mathfrak{n}_+$. Note that $\tilde{\eta}(P) = \lambda_{t_1^0,\ldots,t_l^0} \cdot \eta(X_{\alpha_1})^{t_1^0} \cdots \eta(X_{\alpha_l})^{t_l^0}$, so it suffices to prove $\tilde{\eta}(P) \neq 0$.

Put $V = M_\chi/M_\lambda$ and identify $\overline{V^*}$ with V'. We have $\overline{\mathrm{Wh}}_\eta(V^*) = \{\phi \in V';\ \phi|\operatorname{Ker}\tilde{\eta} \cdot V = 0\}$. Since $\overline{\mathrm{Wh}}_\eta(V^*) = 0$ (Theorem 4.2 and Proposition 4.5), we conclude that

$$\operatorname{Ker}\tilde{\eta} \cdot V = V.$$

In particular, $\overline{\mu}_\chi$, the image of μ_χ in V, belongs to $\operatorname{Ker}\tilde{\eta} \cdot V$. So there is a $S \in U(\mathfrak{n}_-)$ such that $S\overline{\mu}_\chi = \overline{\mu}_\chi$ and $\tilde{\eta}(S) = 0$. It is clear that we can write $S = 1 - S'$, where $\tilde{\eta}(S') = 1$ and $S'\overline{\mu}_\chi = 0$. Hence, $S' = S'' \cdot P$ and we conclude that $\tilde{\eta}(P) \neq 0$. ■

5 Characterization of Soergel's functor

We shall work in the category $\mathcal{O}_\lambda$ for a dominant weight λ. Let $\omega_0 \in W$ be such that $\omega_0\lambda$ is anti-dominant. If $M_{w\lambda} \in \mathcal{O}_\lambda$ is a Verma module, then $M_{w\lambda} \supset L_{w_0\lambda} = M_{w_0\lambda}$. The projective cover $P_{w_0\lambda}$ of $L_{w_0\lambda}$ belongs to $\mathcal{O}_\lambda$.

Soergel [12] considered the algebra $C = \operatorname{End}_{\mathfrak{g}}(P_{w_0\lambda})$ and the exact functor $\mathbb{V} = \operatorname{Hom}_{\mathfrak{g}}(P_{w_0\lambda},\ \cdot)\colon \mathcal{O}_\lambda \rightsquigarrow C\text{-mod}$. Here C-mod denotes the category of *finitely generated* right C-modules. Thus $C = \mathbb{V}(P_{w_0\lambda})$. Let $M \in \mathcal{O}$. By Proposition 1 in [5], we have

$$\dim \mathbb{V}(M) = [M,\ L_{w_0\lambda}]. \tag{5.1}$$

It follows from Verma's theorem that $[M_{w\lambda},\ L_{w_0\lambda}] = 1$ for $w \in W_\lambda$. This fact and the above formula imply $\dim \mathbb{V}(M_{w\lambda}) = 1$.

Let us collect some results about the algebra C which are indispensable in the discussion below.

The multiplication map $Z(\mathfrak{g}) \to C$ is surjective (and so C is commutative). Let $J = \operatorname{Ann}_{Z(\mathfrak{g})}(P_{w_0\lambda}) := \{z \in Z(\mathfrak{g});\ zP_{w_0\lambda} = 0\}$ be the kernel of this map. Then $Z(\mathfrak{g})/J \cong C$ can be identified with the cohomology algebra of a partial flag manifold. This gives C the structure of a graded algebra on which *Poincaré duality* holds. Put $C_+ = \oplus_{i\geq 1} C_i = \Theta_\lambda \cdot C$. The Poincaré duality implies that the socle of C, $\{c \in C;\ C_+ \cdot c = 0\}$, is one-dimensional;

therefore C is Gorenstein. These results were proved in Soergel [12]; see also Bernstein [3].

Since the algebra C is a finite-dimensional vector space, its Krull dimension is zero. But for Gorenstein algebras, the Krull dimension and injective dimension coincide. We conclude that C is injective as a module over itself. This gives a duality

$$F \rightsquigarrow F^* = \mathrm{Hom}_C(F, C) \tag{5.2}$$

in the category C-mod; so $F \cong F^{**}$ canonically.

The following result is due to Soergel.

Theorem 5.1 ([12] Struktursatz 9). Assume $M, P \in \mathcal{O}_\lambda$, P projective, then

$$\mathrm{Hom}_{\mathfrak{g}}(M,\ P) \longrightarrow \mathrm{Hom}_{Z(\mathfrak{g})}(\mathbb{V}(M),\ \mathbb{V}(P))$$

is bijective. □

Since $\mathcal{O}$ has enough projectives [5], this result in a sense gives us a complete description of the category $\mathcal{O}$.

The next theorem shows that $\mathbb{V}$ is determined by formula (5.1) and its exactness.

Theorem 5.2. Let $T\colon\ \mathcal{O}_\lambda \rightsquigarrow Z(\mathfrak{g})$-mod be an (additive) functor. Assume that the following holds.

(1) $\dim T(M_{w\lambda}) = 1$, for each $w \in W_\lambda$;

(2) T is exact;

(3) If $\mathfrak{a}$ is an ideal in $Z(\mathfrak{g})$, $M \in \mathcal{O}_\lambda$, then $T(\mathfrak{a}M) = \mathfrak{a}T(M)$ and

$$T(\{m \in M;\ \mathfrak{a}m = 0\}) = \{v \in T(M);\ \mathfrak{a}v = 0\}.$$

Then T is isomorphic to $\mathbb{V}$. □

Remark 5.3. Note that $\mathbb{V}$ satisfies the hypothesis of Theorem 5.2. Also, the assumptions (1) and (2) of Theorem 5.2 imply

$$(1')\quad \dim T(M) = [M : L_{w_0\lambda}].$$

Corollary 5.4 (η regular). $\overline{\mathrm{Wh}}_\eta$ is isomorphic to $\mathbb{V}$. □

Proof of Corollary 5.4. This follows from Theorem 4.2 and Proposition 4.5, since $\overline{\mathrm{Wh}}_\eta$ clearly satisfies (3). ■

Corollary 5.5. Let T satisfy the hypothesis of Theorem 5.2. For $M, P \in \mathcal{O}_\lambda$, P projective, the map defined by T

$$\mathrm{Hom}_{\mathfrak{g}}(M,\ P) \longrightarrow \mathrm{Hom}_{Z(\mathfrak{g})}(TM,\ TP)$$

is bijective. □

Proof of Corollary 5.5. This follows from Theorems 5.2 and 5.1. ■

Proof of Theorem 5.2. (a) Let $M \in \mathcal{O}_\lambda$. Recall that we put $J = \mathrm{Ann}_{Z(\mathfrak{g})}(P_{w_0\lambda})$. We shall prove $J \cdot T(M) = 0$.

Since, clearly, $J \cdot \mathbb{V}(M) = 0$, it suffices to prove that $\mathrm{Ann}_{Z(\mathfrak{g})}(\mathbb{V}(M)) \subseteq \mathrm{Ann}_{Z(\mathfrak{g})}(T(M))$.

If M contains a submodule isomorphic to $L_{w\lambda}$, $w \neq w_0$, then $T(M) \cong T(M/L_{w\lambda})$ and $\mathbb{V}(M) \cong \mathbb{V}(M/L_{w\lambda})$. Therefore we may assume

$$\text{for each } w \neq w_0, \text{there exists no injection } L_{w\lambda} \hookrightarrow M. \tag{*}$$

Now, put $M' = \sum_{f \in \mathbb{V}(M)} f(P_{w_0\lambda})$; obviously, the inclusion $\mathbb{V}(M') \to \mathbb{V}(M)$ is onto, so we get $\mathbb{V}(M/M') = 0$. Hence, by (1′), $[M/M' : L_{w_0\lambda}] = 0$. Now, by the construction of M', $zM' = 0$ if $z \in \mathrm{Ann}_{Z(\mathfrak{g})}(\mathbb{V}(M))$. So, if $zM = z(M/M') \subset M$ is nonzero, it contains a simple submodule not isomorphic to $L_{w_0\lambda}$. This contradicts $(*)$. It follows that $z \in \mathrm{Ann}_{Z(\mathfrak{g})}(M)$; hence $\mathrm{Ann}_{Z(\mathfrak{g})}(\mathbb{V}(M)) \subseteq \mathrm{Ann}_{Z(\mathfrak{g})}(M)$.

Clearly, $\mathrm{Ann}_{Z(\mathfrak{g})}(M) \subseteq \mathrm{Ann}_{Z(\mathfrak{g})}(T(M))$ and we conclude that $\mathrm{Ann}_{Z(\mathfrak{g})}(\mathbb{V}(M)) \subseteq \mathrm{Ann}_{Z(\mathfrak{g})}(T(M))$. Hence, $T(M)$ is a $C \cong Z(\mathfrak{g})/J$-module.

(b) $C \cong T(P_{w_0\lambda})$ as C- (or $Z(\mathfrak{g})$-) modules.

We first prove that $T(P_{w_0\lambda})$ is cyclic over $Z(\mathfrak{g})$. The support of the $Z(\mathfrak{g})$-module $T(P_{w_0\lambda})$ is the single maximal ideal Θ_λ. By Nakayama's lemma, $T(P_{w_0\lambda})$ is cyclic if and only if $\dim T(P_{w_0\lambda})/\Theta_\lambda T(P_{w_0\lambda}) = 1$. We have

$$\begin{aligned}\dim T(P_{w_0\lambda})/\Theta_\lambda T(P_{w_0\lambda}) &= \dim T(P_{w_0\lambda}/\Theta_\lambda P_{w_0\lambda}) = \dim \mathbb{V}(P_{w_0\lambda}/\Theta_\lambda P_{w_0\lambda}) \\ &= \dim \mathbb{V}(P_{w_0\lambda})/\Theta_\lambda \mathbb{V}(P_{w_0\lambda}) = \dim C/\Theta_\lambda C = 1,\end{aligned}$$

where the first and third equalities follow from (3), the second from (1′), and the last equality follows because C is cyclic over $Z(\mathfrak{g})$.

Choosing a generator of $T(P_{w_0\lambda})$, we get a surjective $Z(\mathfrak{g})$-linear map $Z(\mathfrak{g}) \to T(P_{w_0\lambda})$. By (a), the kernel of this map contains J. Hence we get the surjection

$$C \to T(P_{w_0\lambda}),$$

which moreover is an isomorphism because $\dim T(P_{w_0\lambda}) = \dim C$, by (1′).

(c) Consider the diagram of morphisms of functors

$$\begin{array}{ccc} \mathrm{Hom}_C(T(\cdot), T(P_{w_0\lambda})) & \xrightarrow{\alpha} & \mathrm{Hom}_C(T(\cdot), \mathbb{V}(P_{w_0\lambda})) = T(\cdot)^* \\ {\scriptstyle \pi_T}\uparrow & & \\ \mathrm{Hom}_{\mathfrak{g}}(\cdot, P_{w_0\lambda}) & \xrightarrow{\pi_{\mathbb{V}}} & \mathrm{Hom}_C(\mathbb{V}(\cdot), \mathbb{V}(P_{w_0\lambda})) = \mathbb{V}(\cdot)^* \end{array}$$

where π_T and $\pi_{\mathbb{V}}$ are the morphisms defined by T and $\mathbb{V}$ respectively, and α is defined by the isomorphism $\mathbb{V}(P_{w_0\lambda}) \cong T(P_{w_0\lambda})$ in (b). We shall prove that all these morphisms are isomorphisms. Then it follows that $\mathbb{V}(\cdot)^* \cong T(\cdot)^*$ and so $\mathbb{V} \cong T$.

(d) It is clear that α is an isomorphism.

(e) $\pi_{\mathbb{V}}$ is an isomorphism. (Soergel [12], part of the proof of Struktursatz 9.)

(f) π_T is an isomorphism. (The proof we give here is identical with Soergel's proof for (e), with $\mathbb{V}$ replaced by T.)

Let $M \in \mathcal{O}_\lambda$. We must prove that $\pi_T(M)$ is an isomorphism. By definition, $\pi_T(M)(\phi) = T(\phi)$, for $\phi \in \mathrm{Hom}_{\mathfrak{g}}(M, P_{w_0\lambda})$. Assume $\phi \neq 0$. Then $\mathrm{Im}\,\phi$ is a nonzero submodule of $P_{w_0\lambda}$. Since $P_{w_0\lambda}$ admits a Verma flag, $[\mathrm{Im}\,\phi : L_{w_0\lambda}] \neq 0$. Hence, by (1′), $T(\mathrm{Im}\,\phi) \neq 0$. The exactness of T implies $\mathrm{Im}\,T(\phi) = T(\mathrm{Im}\,\phi)$ and we conclude that $T(\phi) \neq 0$. This proves $\pi_T(M)$ is injective.

To see that $\pi_T(M)$ is also surjective, we only have to show that both terms to the left in the diagram in (c) have the same dimension. It suffices to do this when M is simple, because $\mathrm{Hom}_{\mathfrak{g}}(\cdot\,, P_{w_0\lambda})$ and $\mathrm{Hom}_C(T(\cdot), T(P_{w_0\lambda}))$ are exact functors. The first functor is exact because $P_{w_0\lambda}$ is an injective object in $\mathcal{O}$, [7]; the second because $T(P_{w_0\lambda}) \cong C$ is an injective C-module and T is exact.

When $M = L_{w\lambda}$, $w \neq w_0$, $T(M) = 0$, and so the upper term is zero; hence the lower term is zero by the injectivity just established.

When $M = L_{w_0\lambda}$, the lower term is isomorphic to

$$\mathrm{Hom}_{\mathfrak{g}}(L_{w_0\lambda}, \{v \in P_{w_0\lambda}; \Theta_\lambda v = 0\}),$$

because $\Theta_\lambda L_{w_0\lambda} = 0$. However, $\{v \in P_{w_0\lambda}; \Theta_\lambda v = 0\} = M_\lambda$ ([12], Lemma 7) and $\dim \mathrm{Hom}_{\mathfrak{g}}(L_{w_0\lambda}, M_\lambda) = 1$ by Verma's theorem; the lower term is one-dimensional. From (3) it follows that the upper term is isomorphic to

$$\begin{aligned}\mathrm{Hom}_C(T(L_{w_0\lambda}), \{v \in T(P_{w_0\lambda}); \Theta_\lambda v = 0\}) &\cong \mathrm{Hom}_C(T(L_{w_0\lambda}), T(\{v \in P_{w_0\lambda}; \Theta_\lambda v = 0\})) \\ &\cong \mathrm{Hom}_C(T(L_{w_0\lambda}), T(M_\lambda)) \cong \mathrm{Hom}_C(\mathbf{C}, \mathbf{C}),\end{aligned}$$

and this space is also one-dimensional. (Here $\mathbf{C} = C/C_+$ is the trivial C-module.) ∎

6 Multiplicities of standard Whittaker modules

Let η be a character on $\mathfrak{n}_+$. Put $B^1 := \{\alpha \in B; \eta(X_\alpha) \neq 0\}$ and $B^2 := \{\alpha \in B; \eta(X_\alpha) = 0\}$. Let $\mathfrak{g}^i$, $i = 1, 2$, be the Lie algebra generated by $X_\alpha, X_{-\alpha}$, $\alpha \in B^i$. It is clear that $\mathfrak{g}^i$ is semisimple, and it has the triangular decomposition

$$\mathfrak{g}^i = \mathfrak{n}^i_+ \oplus \mathfrak{h}^i \oplus \mathfrak{n}^i_-,$$

where $\mathfrak{n}_+^i = \mathfrak{n}_+ \cap \mathfrak{g}^i$, $\mathfrak{h}^i = \mathfrak{h} \cap \mathfrak{g}^i$, and $\mathfrak{n}_-^i = \mathfrak{n}_- \cap \mathfrak{g}^i$. The Weyl group $W_i = W(\mathfrak{g}^i, \mathfrak{h}^i)$ is identified with the subgroup of W generated by the reflections $s_\alpha, \alpha \in B_i$.

The category Ω_η contains certain standard modules $M(\chi, \eta)$, $\chi \in \mathfrak{h}^*$, constructed as follows ([9], [11]): Denote by $\mathfrak{p}_\eta$ the parabolic subalgebra $\mathfrak{n}_-^1 \oplus \mathfrak{n}_+ \oplus \mathfrak{h}$ and put $\mathfrak{g}_\eta := \mathfrak{n}_-^1 \oplus \mathfrak{n}_+^1 \oplus \mathfrak{h}$. Thus $\mathfrak{g}_\eta$ is reductive. Denote by $\xi_\eta^\sharp\colon Z(\mathfrak{g}_\eta) \to S(\mathfrak{h})$ the Harish-Chandra homomorphism of $U(\mathfrak{g}_\eta)$ normalized by $\xi_\eta^\sharp(z) - z \in U(\mathfrak{g}_\eta)\mathfrak{n}_+^1$. It induces on the maximal ideals a map $\xi_\eta\colon \mathfrak{h}^* \to \operatorname{Max} Z(\mathfrak{g}_\eta)$. Let $\mathbf{C}_{\eta_1}$ be the one-dimensional representation of $\mathfrak{n}_+^1$ defined by η_1 and put

$$Y(\chi, \eta) := (U(\mathfrak{g}_\eta)/\xi_\eta(\chi - \rho)U(\mathfrak{g}_\eta)) \otimes_{U(\mathfrak{n}_+^1)} \mathbf{C}_{\eta_1}.$$

Then $Y(\chi, \eta) \in U(\mathfrak{g}_\eta) - \mathrm{mod}$, but the first projection $\mathfrak{p}_\eta = \mathfrak{g}_\eta \oplus (\mathfrak{n}_+^2 + [\mathfrak{n}_+^1, \mathfrak{n}_+^2]) \to \mathfrak{g}_\eta$ defines an $U(\mathfrak{p}_\eta)$-module structure on $Y(\chi, \eta)$. Now put

$$M(\chi, \eta) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} Y(\chi, \eta).$$

Note that we have $M(\chi, 0) = M_\chi$. $M(\chi, \eta)$ is irreducible when η is regular [10].

Proposition 6.1 ([9] and [11]). (1) $M(\chi, \eta) \cong M(\mu, \eta)$ if and only if $W^1\chi = W^1\mu$.

(2) $M(\chi, \eta)$ has a unique simple quotient $L(\chi, \eta)$. $L(\chi, \eta) \cong L(\mu, \eta)$ if and only if $W^1\chi = W^1\mu$.

(3) Each simple object in Ω_η is isomorphic to $L(\chi, \eta)$ for some χ. □

Denote by $[M(\chi, \eta) : L(\mu, \eta)]$ the multiplicity of $L(\mu, \eta)$ in $M(\chi, \eta)$.

Theorem 6.2. Let $M(\chi, \eta)$ and $L(\mu, \eta)$ be given. If $\mu \in W\chi$ and there exists $w \in W^1$ such that $(w\mu)^1$ is anti-dominant and $\chi > w\mu$, then

$$[M(\chi, \eta) : L(\mu, \eta)] = [M(\chi) : L(w\mu)].$$

Otherwise $[M(\chi, \eta) : L(\mu, \eta)] = 0$. □

This multiplicity problem was formulated and partially solved in [11]. Since our solution is given in terms of some multiplicities in the category $\mathcal{O}$, the multiplicity $[M(\chi, \eta) : L(\mu, \eta)]$ can be calculated using the Kazhdan-Lusztig algorithm. The proof of Theorem 6.2 is postponed until the end of this section.

Before we start to analyze the functor $\overline{\Gamma}_\eta$, we establish some properties of $\overline{\mathrm{Wh}}_\eta$.

Let us introduce some more notation. For $\chi \in \mathfrak{h}^*$, put $\chi^i := \chi|\mathfrak{h}^i \in {\mathfrak{h}^i}^*$. Put $\eta^i := \eta|\mathfrak{n}_+^i$. Thus, η^1 is regular and $\eta^2 = 0$. Let

$$\mathrm{Wh}_{\eta^i}(M) := \{m \in M;\ \operatorname{Ker}\eta^i \cdot m = 0\},$$

$$\overline{\mathrm{Wh}}_{\eta^i}(M) := \{m \in \overline{M};\ \operatorname{Ker}\eta^i \cdot m = 0\}$$

for $M \in U(\mathfrak{g}^i)$-mod and $M \in \mathcal{O}(\mathfrak{g}^i)$ respectively. The functors Γ_{η^i} and $\overline{\Gamma}_{\eta^i}$ are analogously defined. As usual, $\tilde{\eta}$ denotes the character on $\mathfrak{n}_-$ defined by $\tilde{\eta}(X_{-\alpha}) = \eta(X_\alpha)$; We similarly define $\tilde{\eta}^i$ on $\mathfrak{n}_-^i$.

Lemma 6.3. Let $L \in \mathcal{O}$ be simple. Then $\dim \overline{\mathrm{Wh}}_\eta(L) \leq 1$. $\square$

Proof. We have $\overline{\mathrm{Wh}}_\eta(L) = \{\phi \in L^{*\prime};\ \phi|\operatorname{Ker}\tilde{\eta}\cdot L^* = 0\} = \{\phi \in L';\ \phi|\operatorname{Ker}\tilde{\eta}\cdot L = 0\}$, since simple modules are self-dual. This gives $\dim \overline{\mathrm{Wh}}_\eta(L) = \operatorname{codim}_L(\operatorname{Ker}\tilde{\eta}\cdot L)$. Since L is isomorphic to a quotient of $U(\mathfrak{n}_-)$ and $\operatorname{Ker}\tilde{\eta} \subset U(\mathfrak{n}_-)$ is an ideal of codimension one, we get $\dim \overline{\mathrm{Wh}}_\eta(L) \in \{0, 1\}$. ■

Proposition 6.4. (1) χ^1 anti-dominant implies $\dim \overline{\mathrm{Wh}}_\eta(L_\chi) = 1$.
(2) χ^1 not anti-dominant implies $\overline{\mathrm{Wh}}_\eta(L_\chi) = 0$. $\square$

Proof. (1): Let μ be the canonical generator of the $U(\mathfrak{g})$ Verma module M_χ. Then $U(\mathfrak{g}^1)\cdot\mu$ is isomorphic to the $U(\mathfrak{g}^1)$ Verma module M_{χ^1}, which by assumption is simple. Let

$$0 \neq \nu = \prod \nu^\psi \in \overline{\mathrm{Wh}}_{\eta^1}(M_{\chi^1})$$

(such a ν exists by Proposition 4.5). Let ϕ' be a maximal element in the set $\{\psi \in \mathfrak{h}^{1*};\ \mathfrak{n}^1_+\cdot\nu^\psi \neq 0\}$, with respect to the ordering $>$ on $\mathfrak{h}^{1*}$. Then choose $\alpha \in B^1$ such that $X_\alpha\cdot\nu^{\phi'} \neq 0$, and let $\phi = \phi' + \alpha$. Then $\nu^\phi = X_\alpha\cdot\nu^{\phi'}$ is a primitive vector. Since each primitive vector in M_{χ^1} belongs to $M_{\chi^1}^{\chi^1-\rho^1}$, we conclude that $\phi = \chi^1 - \rho^1$; hence $\nu^{\chi^1-\rho^1} \neq 0$.

Since $M_{\chi^1} = U(\mathfrak{n}^1_-)\cdot\mu$ and $[X_\alpha, U(\mathfrak{n}^1_-)] = 0$, for each $\alpha \in B^2$, we see that $X_\alpha\cdot\nu = 0$ for such α, and so ν belongs to $\overline{\mathrm{Wh}}_\eta(M_x)$. It follows that $\nu^{\chi-\rho} \neq 0$, hence the image of ν in $\overline{\mathrm{Wh}}_\eta(L_\chi)$ is nonzero. This proves $\dim \overline{\mathrm{Wh}}_\eta(L_\chi) = 1$.

(2): We have $\overline{\mathrm{Wh}}_\eta(L_\chi) = \{\phi \in L'_\chi;\ \phi|\operatorname{Ker}\tilde{\eta}\cdot L_\chi = 0\}$. Therefore it suffices to show that $\operatorname{Ker}\tilde{\eta}\cdot L_\chi = L_\chi$. By assumption, the $U(\mathfrak{g}^1)$ Verma module M_{χ^1} (defined in the proof of (1)) is not simple. Hence we can find a proper Verma submodule M_{ψ^1} of M_{χ^1}. Let $P \in U(\mathfrak{n}^1_-)$ be such that $P\mu$ generates M_{ψ^1} over $U(\mathfrak{n}^1_-)$. Then Corollary 4.6 implies $\tilde{\eta}^1(P) \neq 0$.

Let $\overline{\mu}$ be the image of μ in L_χ. It is clear that $P\mu$ is a primitive vector in M_χ, not proportional to μ, and we conclude that $P\overline{\mu} = 0$. Hence, $\overline{\mu} = (1 - P/\tilde{\eta}^1(P))\overline{\mu} = (1 - P/\tilde{\eta}(P))\overline{\mu} \in \operatorname{Ker}\tilde{\eta}\cdot L_\chi$. Hence, $\operatorname{Ker}\tilde{\eta}\cdot L_\chi = L_\chi$. ■

Recall that $\overline{\Gamma}_\eta\colon\ M \rightsquigarrow \{m \in \overline{M};\ \exists k:\ (\operatorname{Ker}\eta)^k\cdot m = 0\}$ defined an exact functor from $\mathcal{O}$ to Ω_η. The following lemma is crucial.

Lemma 6.5. $\overline{\Gamma}_\eta(M_\phi) = U(\mathfrak{g})\cdot\overline{\mathrm{Wh}}_\eta(M_\phi)$ for each Verma module M_ϕ. $\square$

Proof. (a) Let Q^i_{++} be the semigroup generated by B^i, $i = 1, 2$. For $\omega \in Q^2_{++}$, put

$$M[\omega] := \sum_{\sigma\in Q^1_{++}} M^{\phi-\rho-\sigma-\omega}.$$

Put $\mathcal{R} := U(\mathfrak{n}^2_- \oplus [\mathfrak{n}^2_-, \mathfrak{n}^1_-])$. The Lie algebra $\mathfrak{h}$ acts on $\mathcal{R}$ by means of the adjoint representation, and so $\mathcal{R}$ decomposes into a direct sum $\sum_{\chi\in\mathfrak{h}^*}\mathcal{R}^\chi$ of $\mathfrak{h}$ weight spaces. Let

$\tilde{\mathfrak{h}} := \{H \in \mathfrak{h};\ \alpha(H) = 0, \forall \alpha \in B^1\}$. Hence $\mathcal{R}$ decomposes into a direct sum $\sum_{\chi \in \tilde{\mathfrak{h}}^*} \mathcal{R}^{\chi}$ of $\tilde{\mathfrak{h}}$ weight spaces. The reader can verify that for each $\chi \in \tilde{\mathfrak{h}}^*$, $\mathcal{R}^{\chi}$ is finite dimensional.

The Poincaré-Birkhoff-Witt theorem together with the fact that M is cyclic over $U(\mathfrak{n}_-)$ shows that for each $\omega \in Q^2_{++}$,

$$\mathcal{R}^{(\omega|\tilde{\mathfrak{h}})} \cdot M[0] = M[\omega]. \tag{6.1}$$

Since $M[0]$ is cyclic over $U(\mathfrak{g}^1)$, 6.1 shows that $M[\omega]$ is finitely generated over $U(\mathfrak{g}^1)$. It is clear that $M[\omega]$ is locally finite over $\mathfrak{n}^1_+$ and $\mathfrak{h}^1$ semisimple. Thus $M[\omega] \in \mathcal{O}(\mathfrak{g}^1)$.

(b) From Lemma 3.2 we get $\Gamma_{\eta_1}(\overline{M[\omega]}) \in \Omega_{\eta_1}$. Since η_1 is regular, Proposition 4.3 shows that

$$\Gamma_{\eta_1}(\overline{M[\omega]}) = U(\mathfrak{g}^1) \cdot \mathrm{Wh}_{\eta_1}(\overline{M[\omega]}). \tag{6.2}$$

The fact that $\mathcal{R}^{(\omega|\tilde{\mathfrak{h}})}$ is finite-dimensional and 6.1 shows

$$\mathcal{R}^{(\omega|\tilde{\mathfrak{h}})} \cdot \overline{M[0]} = \overline{M[\omega]}. \tag{6.3}$$

Also, the Poincaré-Birkhoff-Witt theorem together with the fact that M is $U(\mathfrak{n}_-)$-free implies that the multiplication map

$$\mathcal{R}^{(\omega|\tilde{\mathfrak{h}})} \otimes_{\mathbf{C}} \overline{M[0]} \to \mathcal{R}^{(\omega|\tilde{\mathfrak{h}})} \cdot \overline{M[0]} \tag{6.4}$$

is an isomorphism.

(c) We prove $\Gamma_{\eta_1}(\overline{M[\omega]}) \subseteq \mathcal{R}^{(\omega|\tilde{\mathfrak{h}})} \cdot \Gamma_{\eta_1}(\overline{M[0]})$. Let $\nu \in \Gamma_{\eta_1}(\overline{M[\omega]})$. By 6.3 we can write $\nu = \sum_{i=1}^{k} P_i \nu_i$, for some $\nu_i \in \overline{M[0]}$, and the P_i's are linearly independent elements of $\mathcal{R}^{(\omega|\tilde{\mathfrak{h}})}$. We must prove that each ν_i is in $\Gamma_{\eta_1}(\overline{M[0]})$, i.e., each ν_i is killed by some power of $X_\alpha - \eta(X_\alpha)$ for each $\alpha \in B^1$.

Fix $\alpha \in B^1$ and note that $\mathcal{R}^{(\omega|\tilde{\mathfrak{h}})}$ is ad_{X_α}-stable and that ad_{X_α} is a nilpotent operator on $\mathcal{R}^{(\omega|\tilde{\mathfrak{h}})}$. Thus we can choose $i_0 \in \{1, \ldots, k\}$ such that

$$V := \mathrm{span}\{\mathrm{ad}^t_{X_\alpha}(P_i);\ i = 1, \ldots, k, t \geq 1\} \subset \mathcal{R}^{(\omega|\tilde{\mathfrak{h}})}$$

does not contain P_{i_0}. We have $(X_\alpha - \eta(X_\alpha))^n \nu = 0$ for some n. But

$$(X_\alpha - \eta(X_\alpha))^n \nu \in P_{i_0}(X_\alpha - \eta(X_\alpha))^n \nu_{i_0} + V \cdot \overline{M[0]}.$$

Equation 6.4 now implies $(X_\alpha - \eta(X_\alpha))^n \nu_{i_0} = 0$.

It follows that $P_{i_0}\nu_{i_0}$ is killed by some power of $X_\alpha - \eta(X_\alpha)$ and we conclude that $\sum_{i=1, i \neq i_0}^{k} P_i \nu_i$ is annihilated by some power of $X_\alpha - \eta(X_\alpha)$. An induction over k now shows that each ν_i is killed by some power of $X_\alpha - \eta(X_\alpha)$.

(d) We have $\overline{M} = \prod_{\omega \in Q^2_{++}} \overline{M[\omega]}$, so $\overline{\Gamma}_\eta(M) = \Gamma_\eta\left(\prod_{\omega\in Q^2_{++}} \overline{M[\omega]}\right)$. It is not hard to check that in fact

$$\overline{\Gamma}_\eta(M) = \Gamma_\eta\left(\bigoplus_{\omega\in Q^2_{++}} \overline{M[\omega]}\right) = \Gamma_{\eta_1}\left(\bigoplus_{\omega\in Q^2_{++}} \overline{M[\omega]}\right).$$

Each $\overline{M[\omega]}$ is $\operatorname{Ker}\eta_1$-stable, so $\overline{\Gamma}_\eta(M) = \sum_{\omega\in Q^2_{++}} \Gamma_{\eta_1}(\overline{M[\omega]})$. By (c) and 6.2 we have

$$\Gamma_{\eta_1}(\overline{M[\omega]}) \subseteq U(\mathfrak{g}) \cdot \Gamma_{\eta_1}(\overline{M[0]}) = U(\mathfrak{g}) \cdot \mathrm{Wh}_{\eta_1}(\overline{M[0]}).$$

Noting that $\mathrm{Wh}_{\eta_1}(\overline{M[0]}) \subseteq \overline{\mathrm{Wh}}_\eta(M)$, we get $\overline{\Gamma}_\eta(M) \subseteq U(\mathfrak{g}) \cdot \overline{\mathrm{Wh}}_\eta(M)$. ■

Lemma 6.6. Let $L_\phi \in \mathcal{O}$ be simple and assume that ϕ^1 is anti-dominant. Then $\overline{\Gamma}_\eta(L_\phi)$ is simple. □

Proof. (a) $\overline{\Gamma}_\eta(L_\phi) = U(\mathfrak{g}) \cdot \overline{\mathrm{Wh}}_\eta(L_\phi)$: the right hand side is nonzero by Proposition 6.4, and $\overline{\Gamma}_\eta(M_\phi) = U(\mathfrak{g}) \cdot \overline{\mathrm{Wh}}_\eta(M_\phi)$ by Lemma 6.5. The surjection $M_\phi \twoheadrightarrow L_\phi$ together with the exactness of $\overline{\Gamma}_\eta$ now proves the assertion.

(b) Let V be a nonzero submodule of $\overline{\Gamma}_\eta(L_\phi)$. Each element in V is annihilated by a power of $\operatorname{Ker}\eta$, so it is clear that V contains a nonzero Whittaker vector v. But according to Proposition 6.4, $\overline{\mathrm{Wh}}_\eta(L_\phi)$ is one-dimensional, so $\overline{\mathrm{Wh}}_\eta(L_\phi) = \mathbf{C} \cdot v$. We conclude from (a) that $\overline{\Gamma}_\eta(L_\phi) = U(\mathfrak{g}) \cdot v$. Thus $V = \overline{\Gamma}_\eta(L_\phi)$ and $\overline{\Gamma}_\eta(L_\phi)$ is simple. ■

Lemma 6.7. Each simple object in Ω_η is isomorphic to $\overline{\Gamma}_\eta(L_\phi)$ for some ϕ such that ϕ^1 is anti-dominant. If ϕ^1 and ψ^1 are anti-dominant, then $\overline{\Gamma}_\eta(L_\phi) \cong \overline{\Gamma}_\eta(L_\psi)$ if and only if $W^1\phi = W^1\psi$. □

Proof. (a) By Dixmier's theorem, each simple $U(\mathfrak{g})$-module admits a central character and $\overline{\Gamma}_\eta$ preserves central character. So let us fix a central character $\Theta = \Theta_\chi$ and prove the assertions in the lemma with Ω_η replaced by $\Omega_{\eta,\Theta}$.

Denote by $n := n_{\eta,\Theta}$ the number of isomorphism classes of simple modules in $\Omega_{\eta,\Theta}$. By Proposition 6.1 we have $n = \operatorname{Card} W\chi/W^1$. Noting that each W^1-orbit in $W\chi$ contains an element $w\chi$ such that $(w\chi)^1$ is anti-dominant, we conclude that

$$n = \operatorname{Card}\{w\chi;\ w \in W, (w\chi)^1 \text{ is anti-dominant }\}/W^1.$$

Recalling that $\overline{\Gamma}_\eta(L_\phi)$ is simple when ϕ^1 is anti-dominant (Lemma 6.6), we see—counting elements—that in order to prove the lemma it suffices to show that $\overline{\Gamma}_\eta(L_\phi) \cong \overline{\Gamma}_\eta(L_\psi)$ implies $W^1\phi = W^1\psi$.

(b) Put $\tilde{\mathfrak{h}} = \{H \in \mathfrak{h};\ \alpha(H) = 0, \forall\, \alpha \in B^1\}$. Let μ_ϕ be the canonical generator of L_ϕ. From the proof of Proposition 6.4.1 we have $\overline{\mathrm{Wh}}_\eta(L_\phi) \subset \overline{U(\mathfrak{g}^1) \cdot \mu_\phi}$. Since $[\tilde{\mathfrak{h}}, U(\mathfrak{g}^1)] = 0$, we get $(H - (\phi - \rho)(H))\overline{\mathrm{Wh}}_\eta(L_\phi) = 0, \forall H \in \tilde{\mathfrak{h}}$.

(c) The $U(\mathfrak{g}^1)$-module $U(\mathfrak{g}^1)\cdot\mu_\phi$ is isomorphic to the Verma module M_{ϕ^1}. We know that $\overline{Wh}_\eta(L_\phi) \subset \overline{U(\mathfrak{g}^1)\cdot\mu_\phi}$. This clearly implies $\mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\phi^1}) \subseteq \mathrm{Ann}_{Z(\mathfrak{g}^1)}(\overline{Wh}_\eta(L_\phi))$. Since $\mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\phi^1})$ is a maximal ideal in $Z(\mathfrak{g}^1)$, we conclude that $\mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\phi^1}) = \mathrm{Ann}_{Z(\mathfrak{g}^1)}(\overline{Wh}_\eta(L_\phi))$.

(d) Assume that $\overline{\Gamma}_\eta(L_\phi) \cong \overline{\Gamma}_\eta(L_\psi)$. This isomorphism induces the bijection $\overline{Wh}_\eta(L_\phi) \cong \overline{Wh}_\eta(L_\psi)$. Hence

(1) $\mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\phi^1}) = \mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\psi^1})$ and

(2) $\phi|\tilde{\mathfrak{h}} = \psi|\tilde{\mathfrak{h}}$

by (b) and (c) respectively. The Harish-Chandra theorem and (1) give the existence of an element $w \in W^1$ such that $w\phi|\mathfrak{h}^1 = \psi|\mathfrak{h}^1$. Combined with (2), noting that W^1 acts trivially on $\tilde{\mathfrak{h}}$ and that $\mathfrak{h} = \tilde{\mathfrak{h}} \oplus \mathfrak{h}^1$, we conclude $w\phi = \psi$. ■

Lemma 6.8. Let E be a finite dimensional $\mathfrak{g}$-module and let $P(E)$ denote its multiset of weights. There exists a $U(\mathfrak{g})$-module filtration of $E \otimes_{\mathbf{C}} M(\phi,\eta)$ with subquotients isomorphic to $M(\phi+\nu_i,\eta)$ for $\nu_i \in P(E)$. □

Proof. We have

$$E \otimes_{\mathbf{C}} M(\phi,\eta) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} (E \otimes_{\mathbf{C}} Y(\phi,\eta)) \tag{6.5}$$

(where the $U(\mathfrak{p}_\eta)$-module structure on $E \otimes_{\mathbf{C}} Y(\chi,\eta)$ is given by the projection $\mathfrak{p}_\eta \to \mathfrak{g}_\eta$). To prove 6.5 we just have to note that for any $L \in U(\mathfrak{g}) - \mathrm{mod}$,

$$\begin{aligned}
\mathrm{Hom}_{U(\mathfrak{g})}(E \otimes_{\mathbf{C}} M(\phi,\eta),\, L) &= \mathrm{Hom}_{U(\mathfrak{g})}(M(\phi,\eta),\, \mathrm{Hom}_{\mathbf{C}}(E,\, L)) \\
&= \mathrm{Hom}_{U(\mathfrak{p}_\eta)}(Y(\phi,\eta),\, \mathrm{Hom}_{\mathbf{C}}(E,\, L)) \\
&= \mathrm{Hom}_{U(\mathfrak{p}_\eta)}(E \otimes_{\mathbf{C}} Y(\phi,\eta),\, L) \\
&= \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} E \otimes_{\mathbf{C}} Y(\phi,\eta),\, L).
\end{aligned}$$

Consider the Verma module $M_{\phi^1} \in \mathcal{O}(\mathfrak{g}^1)$. Choose $w \in W^1$ such that $w\phi^1$ is anti-dominant. Then $\overline{\Gamma}_{\eta^1}(M_{\phi^1}) = \overline{\Gamma}_{\eta^1}(M_{w\phi^1})$ by Proposition 6.1. Thus $\overline{\Gamma}_{\eta^1}(M_{\phi^1})$ is simple (over $U(\mathfrak{g}^1)$) by Lemma 6.6. Let $\xi_1 = \xi_\eta|\mathfrak{h}^{1*}$ be the Harish-Chandra homomorphism from $\mathfrak{h}^{1*} \to \mathrm{Max}\, Z(\mathfrak{g}^1)$. Then

$$Y(\phi,\eta)|\mathfrak{g}^1 = U(\mathfrak{g}^1)/(\xi_1(\phi^1-\rho^1)) \otimes_{U(\mathfrak{n}^1)} \mathbf{C}_{\eta^1}$$

is simple by Proposition 4.3. Since $\overline{\Gamma}_{\eta^1}(M_{\phi^1})$ and $Y(\phi,\eta)|\mathfrak{g}^1$ both have central character $\xi_1(\phi^1)$, we conclude again from Proposition 4.3 that $\overline{\Gamma}_{\eta^1}(M_{\phi^1}) \cong Y(\phi,\eta)|\mathfrak{g}^1$. Let M_ϕ be the $U(\mathfrak{g}_\eta)$ Verma module whose restriction to $\mathfrak{g}^1$ is M_{ϕ^1} where the action of $H \in \mathfrak{h}$ on the highest weight space $M_{\phi^1}^{\phi^1-\rho}$ is multiplication by the scalar $(\phi-\rho)(H)$. Considering $\overline{\Gamma}_{\eta^1}$ as a functor on $U(\mathfrak{g}_\eta) - \mathrm{mod}$, it is now clear that

$$\overline{\Gamma}_{\eta^1}(M_\phi) = Y(\phi,\eta).$$

It follows from [6, Lemma 7.6.14] that $E \otimes M_\phi$ has a filtration F_i of $U(\mathfrak{g}_\eta)$ modules such that $F_i/F_{i-1} \cong M(\phi + \nu_i)$, $\nu_i \in P(E)$. One checks easily that $\overline{\Gamma}_{\eta^1}(E \otimes_C M_\phi) - E \otimes_C \overline{\Gamma}_{\eta^1}(M_\phi)$. Thus $E \otimes_C Y(\phi, \eta)$ has a filtration $V_i := \overline{\Gamma}_\eta(F_i)$ of $U(\mathfrak{g}_\eta)$-modules such that

$$V_i/V_{i-1} \cong Y(\phi + \nu_i, \eta),\ \nu_i \in P(E).$$

Since $U(\mathfrak{g})$ is a flat (free) $U(\mathfrak{p}_\eta)$-module, the lemma now follows from 6.5. ■

Proposition 6.9. (1)[2] $\overline{\Gamma}_\eta(M_\phi) = M(\phi, \eta)$.

(2) $\overline{\Gamma}_\eta(L_\phi) = L(\phi, \eta)$ if ϕ^1 is anti-dominant.

(3) $\overline{\Gamma}_\eta(L_\phi) = 0$ if ϕ^1 is not anti-dominant. □

Proof. (1): (a) Assume that ϕ is anti-dominant. Then $M_\phi = L_\phi$ is simple and also $M(\phi, \eta)$ is simple by [9]. We conclude from Lemma 6.7 that $M(\phi, \eta) \cong \overline{\Gamma}_\eta(L_\psi)$ for some weight ψ such that ψ^1 is anti-dominant. This isomorphism induces the bijection

$$Wh_\eta(M(\phi, \eta)) \cong \overline{Wh}_\eta(L_\psi). \tag{6.6}$$

Let $\mu := 1 \otimes 1 \in M(\phi, \eta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} Y(\chi, \eta)$. Then $\mu \in Wh_\eta(M(\phi, \eta))$ and we see that

$$(H - (\phi - \rho)(H))\mu = 0,\ \forall H \in \tilde{\mathfrak{h}} \tag{6.7}$$

where $\tilde{\mathfrak{h}} := \{H \in \mathfrak{h};\ \alpha(H) = 0,\ \forall\, \alpha \in B^1\}$, and

$$\mathrm{Ann}_{Z(\mathfrak{g}^1)}(\mu) = \mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\phi^1}). \tag{6.8}$$

It was proved in (b) in the proof of Lemma 6.7 that $(H - (\psi - \rho)(H))\overline{Wh}_\eta(L_\psi) = 0, \forall H \in \tilde{\mathfrak{h}}$. The bijection 6.6 and 6.7 imply

$$\phi|\tilde{\mathfrak{h}} \cong \psi|\tilde{\mathfrak{h}}. \tag{6.9}$$

In (c) in the proof of Lemma 6.7 we proved $\mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\psi^1}) = \mathrm{Ann}_{Z(\mathfrak{g}^1)}(\overline{Wh}_\eta(L_\psi))$. Thus 6.6 and 6.8 imply

$$\mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\phi^1}) = \mathrm{Ann}_{Z(\mathfrak{g}^1)}(M_{\psi^1}). \tag{6.10}$$

The same argument as in (d) in the proof of Lemma 6.7 now shows that 6.9 and 6.10 imply $\psi \in W^1\phi$. Hence by Lemma 6.7 $\overline{\Gamma}_\eta(L_\psi) \cong \overline{\Gamma}_\eta(L_\phi)$, and since $\overline{\Gamma}_\eta(L_\phi) = \overline{\Gamma}_\eta(M_\phi)$, it follows that $\overline{\Gamma}_\eta(M_\phi) \cong M(\phi, \eta)$.

[2]Soergel has already proved (1) in an unpublished manuscript. My proof of (1a) is new, but the idea to use translation functors in (1b) is taken from Soergel.

(b) Let $\phi \in \mathfrak{h}^*$ be arbitrary. By an induction starting with (a) we may assume that $\overline{\Gamma}_\eta(M_\psi) \cong M(\psi, \eta)$ for each $\psi < \phi$. If ϕ is anti-dominant, we are done by (a); so let us assume there is a reflection $s := s_\alpha$, $\alpha \in R^+$, such that $s\phi < \phi$. Let E be a finite dimensional irreducible $\mathfrak{g}$-module with highest weight $\phi - s\phi$. Denote by $P(E)$ the multiset of weights of E. It is known ([6], Lemma 7.6.14) that there exists a filtration

$$E \otimes_{\mathbf{C}} M_{s\phi} = M_n \supset M_{n-1} \supset \cdots \supset M_0 = 0$$

such that $M_1 \cong M_\phi$, and $M_i/M_{i-1} \cong M_{s\phi+\nu_i}$ for $\nu_i \in P(E) \setminus \{\phi - s\phi\}$, $i \geq 2$.

Put $M' := (E \otimes_{\mathbf{C}} M_{s\phi})/M_1$ and $M'_i := M_i/M_1$. Then $\overline{\Gamma}_\eta(M'_i)/\overline{\Gamma}_\eta(M'_{i-1}) \cong \overline{\Gamma}_\eta(M'_i/M'_{i-1}) \cong \overline{\Gamma}_\eta(M_{s\phi+\nu_i}) \cong M(s\phi + \nu_i, \eta)$, where the last isomorphism is given by induction hypothesis, since $s\phi + \nu_i < \phi$. This shows that

$$(\overline{\Gamma}_\eta(M')) = \sum_{\nu_i \in P(E)\setminus\{\phi - s\phi\}} (M(s\phi + \nu_i)) \tag{6.11}$$

in the Grothendieck group $K(\Omega_\eta)$. On the other hand, note that $\overline{\Gamma}_\eta(E \otimes_{\mathbf{C}} M_{s\phi}) = E \otimes_{\mathbf{C}} \overline{\Gamma}_\eta(M_{s\phi})$. By induction hypothesis the latter module is isomorphic to $E \otimes_{\mathbf{C}} M(s\phi, \eta)$. Lemma 6.8 now gives

$$(\overline{\Gamma}_\eta(E \otimes_{\mathbf{C}} M_{s\phi})) = \sum_{\nu_i \in P(E)} (M(s\phi + \nu_i)) \text{ in } K(\Omega_\eta). \tag{6.12}$$

We have $(\overline{\Gamma}_\eta(M_\phi)) = (\overline{\Gamma}_\eta(E \otimes_{\mathbf{C}} M_{s\phi})) - (\overline{\Gamma}_\eta(M'))$ in $K(\Omega_\eta)$. Hence 6.11 and 6.12 imply

$$(M(\phi, \eta)) = (\overline{\Gamma}_\eta(M_\phi)) \text{ in } K(\Omega_\eta). \tag{6.13}$$

We define a map $\pi\colon M(\phi, \eta) \to \overline{\Gamma}_\eta(M_\phi)$ as follows. Let $1 \otimes 1$ be the canonical generator of $M(\phi, \eta)$. Let ν be any nonzero element of $\overline{\mathrm{Wh}}_{\eta_1}(M_\phi[0]) \subset \overline{\mathrm{Wh}}_\eta(M_\phi)$—the space $M_\phi[0]$ was defined in the proof of Lemma 6.5. We see that $\pi(1 \otimes 1) := \nu$ gives us a well-defined homomorphism. The proof of Lemma 6.5 shows that $\overline{\Gamma}_\eta(M_\phi) = U(\mathfrak{g}) \cdot \overline{\mathrm{Wh}}_{\eta_1}(M_\phi[0])$. Moreover, since $M_\phi[0]$ is a $U(\mathfrak{g}^1)$ Verma module, we have $\dim \overline{\mathrm{Wh}}_{\eta_1}(M_\phi[0]) = 1$, by Proposition 4.5; thus $\overline{\Gamma}_\eta(M_\phi) = U(\mathfrak{g}) \cdot \nu$, and we conclude that π is surjective. 6.13 now shows that π is an isomorphism.

(2): By Lemma 6.6 we know that $\overline{\Gamma}_\eta(L_\phi)$ is simple; by exactness, $\overline{\Gamma}_\eta(L_\phi)$ is a quotient of $\overline{\Gamma}_\eta(M_\phi) \cong M(\phi, \eta)$; hence $\overline{\Gamma}_\eta(L_\phi)$ is isomorphic to the unique simple quotient $L(\phi, \eta)$ of $M(\phi, \eta)$.

(3): We know from Lemma 6.4 that $\overline{\mathrm{Wh}}_\eta(L_\phi) = 0$ in this case. This implies $\overline{\Gamma}_\eta(L_\phi) = 0$. ∎

Proof of Theorem 6.2. Since $\overline{\Gamma}_\eta$ is exact, it induces a homomorphism $K(\mathcal{O}) \to K(\Omega_\eta)$ between Grothendieck groups. By Proposition 6.9 we have the following equalities in $K(\Omega_\eta)$.

$$\begin{aligned}(M(\chi,\eta)) = (\overline{\Gamma}_\eta(M_\chi)) &= \sum_{\lambda\in\mathfrak{h}^*}[M_\chi\colon L_\lambda]\cdot((\overline{\Gamma}_\eta(L_\lambda)) \\ &= \sum_{\lambda\in\mathfrak{h}^*,\lambda^1\,\mathrm{anti-dominant}}[M_\chi\colon L_\lambda]\cdot(L(\lambda,\eta)).\end{aligned}$$

Since $(L(\lambda,\eta)) = (L(\psi,\eta))$ if and only if $\lambda \in W^1\psi$, we get

$$[M(\chi,\eta)\colon L(\mu,\eta)] = \sum_{w\in W^1,(w\mu)^1\,\mathrm{anti-dominant}}[M_\chi\colon L_{w\mu}].$$

Note that $[M_\chi : L_{w\mu}] \neq 0$ implies that w belongs to the integral Weyl group W_χ of χ. Clearly, the $W^1 \cap W_\chi$-orbit of μ contains precisely one element $w\mu$ such that $(w\mu)^1$ is anti-dominant. Theorem 6.2 now follows easily. ■

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Department of Mathematics, Stockholm University, S-106 91 Stockholm, Sweden;
erikb@matematik.su.se