JOURNAL OF ALGEBRA **209**, 288–304 (1998)
ARTICLE NO. JA987530

Duality between $\mathfrak{sl}_n(\mathbb{C})$ and the Degenerate Affine Hecke Algebra

Tomoyuki Arakawa

Graduate School of Mathematics, Nagoya University, Japan
E-mail: tarakawa@math.nagoya-n.ac.jp

and

Takeshi Suzuki*

Research Institute for Mathematical Sciences, Kyoto University, Japan
E-mail: takeshi@kurims.kyoto-u.ac.jp

Communicated by Gordon James

Received February 5, 1998

We construct a family of exact functors from the Bernstein–Gelfand–Gelfand category $\mathscr{O}$ of $\mathfrak{sl}_n$-modules to the category of finite-dimensional representations of the degenerate affine Hecke algebra H_l of GL_l. These functors transform Verma modules to standard modules or zero, and simple modules to simple modules or zero. Any simple H_l-module can be thus obtained. © 1998 Academic Press

INTRODUCTION

The classical Frobenius–Schur–Weil duality gives a remarkable correspondence between the category of finite-dimensional representations of the symmetric group $\mathfrak{S}_l$ and the category of finite-dimensional representations of the special (or general) linear group SL_n. Its generalizations have been studied in, e.g., [5, 6, 12, 14, 20], where $\mathfrak{S}_l$ is replaced by other algebras, e.g., the Hecke algebras, the (degenerate) affine Hecke algebras, or the double affine Hecke algebras, and SL_n is replaced by the corresponding quantum groups.

In this paper, we present a new direction in generalizing the classical duality. Let $\mathscr{O}(\mathfrak{sl}_n)$ denote the BGG category of representations of the

* Supported by JSPS, the Research Fellowships for Young Scientists.

0021-8693/98 $25.00

complex Lie algebra $\mathfrak{sl}_n$, and let $\mathscr{R}(H_l)$ denote the category of finite-dimensional representations of the degenerate (or graded) affine Hecke algebra H_l of GL_l. With each weight λ of $\mathfrak{sl}_n$ such that $\lambda + \rho$ is dominant integral (where ρ is the half sum of the positive roots), we associate a functor F_λ from $\mathscr{O}(\mathfrak{sl}_n)$ to $\mathscr{R}(H_l)$. When we take $\lambda = 0$ and restrict the functor F_0 to the category of finite-dimensional representations of $\mathfrak{sl}_n$, we obtain the classical duality.

To be more precise, let $V_n = \mathbb{C}^n$ be the vector representation of $\mathfrak{sl}_n$, and let $M(\lambda)$ be the highest weight Verma module with highest weight λ. For $X \in \operatorname{obj} \mathscr{O}(\mathfrak{sl}_n)$, we construct in Section 2.2 an action of H_l on $X \otimes V_n^{\otimes l}$ commuting with the $\mathfrak{sl}_n$-action. This induces an H_l-action on a certain subquotient $F_\lambda(X)$ (see Section 2.1.1)) of $X \otimes V_n^{\otimes l}$. When $\lambda + \rho$ is dominant integral, $F_\lambda(X)$ is identified with $\operatorname{Hom}_{\mathfrak{sl}_n}(M(\lambda), X \otimes V_n^{\otimes l})$. (This space is an analogue of the space of the conformal blocks in the conformal field theory. See [1].)

Under the assumption that $\lambda + \rho$ is dominant integral, we prove that

(1) F_λ is exact.

(2) F_λ sends a Verma module to a "standard module" unless it is zero.

Here the standard module is an induced module from a certain one-dimensional representation of a parabolic subalgebra of H_l, and it has a unique simple quotient.

Moreover, in the case $n = l$, we prove that

(3) F_λ sends a simple module to a simple module unless it is zero.

We should remark that our proof of (3) relies on the formula (A.3.2) in the Appendix. This formula is a consequence of the fact that the irreducible decompositions of the standard modules of H_l [13] and those of the Verma modules [2, 4] are both described by the Kazhdan–Lusztig polynomials.

We also determine when the image of the functor is nonzero (Theorem 3.3.1, Theorem 3.4.1). Furthermore, it follows from Zelevinsky's results in [24] that any simple H_l-module with "integral weights" is of the form $F_\lambda(L)$ for some weight λ such that $\lambda + \rho$ is dominant integral and some simple $\mathfrak{sl}_l$-module L (see Corollary 3.5.1 for the precise statement).

1. BASIC DEFINITIONS

1.1. *Root data*

Let $\mathfrak{t}_n$ be an n-dimensional complex vector space with the basis $\{\epsilon_i^\vee \mid i = 1, \dots, n\}$ and the inner product defined by $(\epsilon_i^\vee \mid \epsilon_j^\vee) = \delta_{ij}$. Let $\mathfrak{t}_n^*$ be its dual and $\{\epsilon_i\}$ be the dual basis of $\{\epsilon_i^\vee\}$, which are orthonormal basis with respect to the induced inner product: $(\epsilon_i \mid \epsilon_j) = \delta_{ij}$. The natural pairing between $\mathfrak{t}_n$ and $\mathfrak{t}_n^*$ will be denoted by $\langle,\rangle$: $\mathfrak{t}_n^* \times \mathfrak{t}_n \to \mathbb{C}$. Put $\mathfrak{h}_n^* = \{\sum_{i=1}^n \lambda_i \epsilon_i \in \mathfrak{t}_n^* \mid \sum_{i=1}^n \lambda_i = 0\}$ and $\mathfrak{h}_n = \mathfrak{t}_n / \mathbb{C}\epsilon^\vee$, where $\epsilon^\vee = \sum_{i=1}^n \epsilon_i^\vee$, so that $\mathfrak{h}_n$ and $\mathfrak{h}_n^*$ are dual to each other. Define the roots and the simple roots by $\alpha_{ij} = \epsilon_i - \epsilon_j (i \neq j)$, $\alpha_i = \alpha_{ii+1}$, respectively, and put

$$R_n = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}, \tag{1.1.1}$$

$$R_n^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}, \tag{1.1.2}$$

$$\Pi_n = \{\alpha_i \mid i = 1, \ldots, n-1\}; \tag{1.1.3}$$

then $R_n \subseteq \mathfrak{h}_n^*$ is a root system of type A_{n-1}. Define the coroots and the simple coroots by $\alpha_{ij}^\vee = \epsilon_i^\vee - \epsilon_j^\vee (i \neq j)$ and $h_i = \alpha_{ii+1}^\vee$, respectively.

Let $W_n \subset GL(\mathfrak{t}_n^*)$ be the Weyl group associated with the above data, which is by definition generated by the reflections $s_\alpha (\alpha \in R_n)$ defined by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \qquad (\lambda \in \mathfrak{t}_n^*). \tag{1.1.4}$$

We often write $s_{\alpha_{ij}} = s_{ij}$. Observe that W_n preserves $\mathfrak{h}_n^* \subseteq \mathfrak{t}_n^*$ and W_n is isomorphic to the symmetric group $\mathfrak{S}_n$. We often use another action of W_n on $\mathfrak{h}_n^*$, which is given by

$$w \circ \lambda = w(\lambda + \rho) - \rho \qquad (w \in W_n, \lambda \in \mathfrak{h}_n^*), \tag{1.1.5}$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R_n^+} \alpha$. Put

$$Q_n = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i \subseteq \mathfrak{h}_n^*, \qquad Q_n^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\alpha_i, \tag{1.1.6}$$

$$P_n = \{\lambda \in \mathfrak{h}_n^* \mid \langle \lambda, h_i \rangle \in \mathbb{Z} \text{ for all } i = 1, \ldots, n-1\}, \tag{1.1.7}$$

$$P_n^+ = \{\lambda \in \mathfrak{h}_n^* \mid \langle \lambda, h_i \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } i = 1, \ldots, n-1\}. \tag{1.1.8}$$

An element of P_n (resp. P_n^+) is called an integral (resp. dominant integral) weight.

1.2. *Lie algebras of type A*

Let $\mathfrak{sl}_n$ be the complex Lie algebra of type A_{n-1}. We introduce an invariant inner product on $\mathfrak{sl}_n$ by $(x \mid y) = \mathrm{tr}_{\mathbb{C}^n}(xy)$, where $\mathbb{C}^n$ denotes the vector representation of $\mathfrak{sl}_n$. We identify $\mathfrak{h}_n$ (introduced in the previous subsection) and the Cartan subalgebra of $\mathfrak{sl}_n$ as inner product spaces. Let $\mathfrak{sl}_n = \mathfrak{n}_+ \oplus \mathfrak{h}_n \oplus \mathfrak{n}_-$ be the triangular decomposition with $\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_n^+} (\mathfrak{sl}_n)_{\pm\alpha}$, where $(\mathfrak{sl}_n)_\alpha$ denotes the root space corresponding to $\alpha \in R_n$. We choose a set of root vectors $\{e_\alpha \in (\mathfrak{sl}_n)_\alpha \mid \alpha \in R_n\}$ such that $(e_\alpha \mid e_{-\alpha}) = 1$ holds for all $\alpha \in R_n^+$. Let $\{h^i\}_{i=1,\ldots,n-1}$ be the dual basis of the coroots $\{h_i\}_{i=1,\ldots,n-1}$ in $\mathfrak{h}_n$, so that $(h^i \mid h_j) = \delta_{ij}$. We define the special elements of $\mathfrak{sl}_n \otimes \mathfrak{sl}_n$ by

$$r = \frac{1}{2}\sum_{i=1}^{n-1} h^i \otimes h_i + \sum_{\alpha \in R_n^+} e_\alpha \otimes e_{-\alpha}, \tag{1.2.1}$$

$$\Omega = \sum_{i=1}^{n-1} h^i \otimes h_i + \sum_{\alpha \in R_n^+} (e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha), \tag{1.2.2}$$

which will be used later. For an $\mathfrak{h}_n$-module X and $\lambda \in \mathfrak{h}_n^*$, put

$$X_\lambda = \{v \in X \mid hv = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}_n^*\} \tag{1.2.3}$$

$$P(X) = \{\lambda \in \mathfrak{h}_n^* \mid X_\lambda \neq 0\}. \tag{1.2.4}$$

The space X_λ is called the weight space of weight λ, and an element of $P(X)$ is called a weight of X.

1.3. *The BGG category $\mathscr{O}$*

Let $U(\mathfrak{sl}_n)$ denote the universal enveloping algebra of $\mathfrak{sl}_n$. Let $\mathscr{O}(\mathfrak{sl}_n)$ denote the category whose objects are those $\mathfrak{sl}_n$-modules X such that

(i) X is finitely generated over $U(\mathfrak{sl}_n)$.

(ii) X is $\mathfrak{n}_+$-locally finite, i.e., $\dim_{\mathbb{C}} U(\mathfrak{n}_+)v < \infty$ for each $v \in X$.

(iii) $X = \bigoplus_{\lambda \in \mathfrak{h}_n^*} X_\lambda$.

The morphisms of $\mathscr{O}(\mathfrak{sl}_n)$ are, by definition, all of the $\mathfrak{sl}_n$-homomorphisms. The category $\mathscr{O}(\mathfrak{sl}_n)$ is closed under the operations such as taking submodules, forming quotient modules, finite direct sums, and tensor products with finite-dimensional modules. For $\lambda \in \mathfrak{h}_n^*$ let $M(\lambda)$ denote the highest weight Verma module with highest weight λ. The unique simple quotient of $M(\lambda)$ will be denoted by $L(\lambda)$. The modules $M(\lambda)$ and $L(\lambda)$ are objects of $\mathscr{O}(\mathfrak{sl}_n)$.

Let χ_λ: $Z(U(\mathfrak{sl}_n)) \to \mathbb{C}$ denote the infinitesimal character of $M(\lambda)$ (i.e. $zv = \chi_\lambda(z)v$ for all $z \in Z(U(\mathfrak{sl}_n))$ and $v \in M(\lambda)$). It is known that $\chi_\lambda = \chi_\mu$ if and only if $\lambda = w \circ \mu$ for some $w \in W_n$. Let $[\lambda]$ denote the orbit $W_n \circ \lambda$ and put $Z_\lambda = \operatorname{Ker} \chi_\lambda \subset Z(U(\mathfrak{sl}_n))$. Define the full subcategory $\mathscr{O}(\mathfrak{sl}_n)_{[\lambda]}$ of $\mathscr{O}(\mathfrak{sl}_n)$ by

$$\operatorname{obj} \mathscr{O}(\mathfrak{sl}_n)_{[\lambda]} = \left\{X \in \operatorname{obj} \mathscr{O}(\mathfrak{sl}_n) \mid (Z_\lambda)^k X = 0 \text{ for some } k\right\}.$$

Then any $X \in \operatorname{obj} \mathcal{O}(\mathfrak{sl}_n)$ admits a decomposition

$$X = \bigoplus_{[\lambda]} X^{[\lambda]}, \tag{1.3.1}$$

such that $X^{[\lambda]} \in \operatorname{obj} \mathcal{O}(\mathfrak{sl}_n)_{[\lambda]}$. The correspondence $X \mapsto X^{[\lambda]}$ gives an exact functor on $\mathcal{O}(\mathfrak{sl}_n)$.

1.4. $\mathfrak{n}$*-homology of* $\mathfrak{sl}_n$*-modules*

We prove some facts on the zeroth $\mathfrak{n}_-$-homology space,

$$H_0(\mathfrak{n}_-, X) = X/\mathfrak{n}_- X, \tag{1.4.1}$$

of a $\mathfrak{sl}_n$-module X. The space $H_0(\mathfrak{n}_-, X)$ has a natural $\mathfrak{h}_n$-module structure, and it is known that

$$P(H_0(\mathfrak{n}_-, X)) \subseteq W_n \circ \mu, \tag{1.4.2}$$

for $X \in \mathcal{O}(\mathfrak{sl}_n)_{[\mu]}$ ($\mu \in \mathfrak{h}_n^*$). Hence we have

LEMMA 1.4.1. *For any $X \in \operatorname{obj} \mathcal{O}(\mathfrak{sl}_n)$ and $\lambda \in \mathfrak{h}_n^*$, we have*

$$H_0(\mathfrak{n}_-, X)_\lambda = H_0(\mathfrak{n}_-, X^{[\lambda]})_\lambda,$$

where $X^{[\lambda]}$ is defined by the decomposition (1.3.1).

Therefore we have the natural surjection

$$(X^{[\lambda]})_\lambda \to H_0(\mathfrak{n}_-, X)_\lambda. \tag{1.4.3}$$

PROPOSITION 1.4.2. *Let $\lambda + \rho \in P_n^+$. Then the above map* (1.4.3) *is bijective.*

Proof. By Lemma 1.4.1, it is enough to show that $(\mathfrak{n}_- X)_\lambda = 0$, assuming $X \in \operatorname{obj} \mathcal{O}(\mathfrak{sl}_n)_{[\lambda]}$. To show this, note that

$$P(X) \subseteq \{\lambda - \beta \mid \beta \in Q_n^+\}, \tag{1.4.4}$$

where we used the assumption $\lambda + \rho \in P_n^+$. Hence we have $P(\mathfrak{n}_- X) \subseteq \{\lambda - \beta \mid \beta \in Q_n^+ \setminus \{0\}\}$, which implies $\lambda \notin P(\mathfrak{n}_- X)$, as required. ∎

Remark 1.4.3. (i) There also exists a canonical bijection

$$\operatorname{Hom}_{\mathfrak{sl}_n}(M(\lambda), X) \xrightarrow{\sim} (X^{[\lambda]})_\lambda$$

if $\lambda + \rho \in P_n^+$.

(ii) Similar arguments prove that Proposition 1.4.2 holds for any $\lambda \in \mathfrak{h}_n^*$ such that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in R_n^+$. But we do not need this.

1.5. *Degenerate affine Hecke algebras*

Let $l \in \mathbb{Z}_{\geq 1}$. Let $S(\mathfrak{t}_l)$ be the symmetric algebra of $\mathfrak{t}_l$, which is isomorphic to the polynomial ring over $\mathfrak{t}_l^*$.

DEFINITION 1.5.1. The *degenerate (graded) affine Hecke algebra H_l of GL_l* is the unital associative algebra over $\mathbb{C}$ defined by the following properties:

(i) As a vector space, $H_l \cong \mathbb{C}[W_l] \otimes S(\mathfrak{t}_l)$, where $\mathbb{C}[W_l]$ denotes the group algebra of W_l.

(ii) The subspaces $\mathbb{C}[W_l] \otimes \mathbb{C}$ and $\mathbb{C} \otimes S(\mathfrak{t}_l)$ are subalgebras of H_l (their images will be identified with $\mathbb{C}[W_l]$ and $S(\mathfrak{t}_l)$, respectively).

(iii) The following relations hold in H_l:

$$s_\alpha \xi - s_\alpha(\xi) s_\alpha = -\langle \alpha, \xi \rangle \qquad (\alpha \in \Pi_l, \xi \in \mathfrak{t}_l). \tag{1.5.1}$$

The following two lemmas are well known.

LEMMA 1.5.2. *The center $Z(H_l)$ of H_l is*

$$S(\mathfrak{t}_l)^{W_l} := \{p \in S(\mathfrak{t}_l) \subset H_l \mid w(p) = p \textit{ for all } w \in W_l\}.$$

LEMMA 1.5.3. *There exists a unique algebra homomorphism* ev: $H_l \to \mathbb{C}[W_l]$ *(called the evaluation homomorphism) such that*

$$\mathrm{ev}(w) = w \ (w \in W_l), \qquad \mathrm{ev}(\epsilon_i^\vee) = \sum_{1 \leq j < i} s_{ji} \ (i = 1, \ldots, l).$$

2. EXACT FUNCTOR F_λ

2.1. *Representation space*

Let $V_n := \mathbb{C}^n$ be the vector representation of $\mathfrak{sl}_n$ and let $u_i \in V_n$ ($i = 1, \ldots, n$) be the vector with only nonzero entry 1 in the ith component. Let l be another positive integer. For $X \in \operatorname{obj} \mathcal{O}(\mathfrak{sl}_n)$, we regard $X \otimes V_n^{\otimes l}$ as an $\mathfrak{sl}_n$-module. For $\lambda \in \mathfrak{h}_n^*$, we put

$$F_\lambda(X) = H_0(\mathfrak{n}_-, X \otimes V_n^{\otimes l})_\lambda. \tag{2.1.1}$$

PROPOSITION 2.1.1. *Let $\lambda + \rho \in P_n^+$. Then F_λ is an exact functor from $\mathscr{O}(\mathfrak{sl}_n)$ to the category of finite-dimensional vector spaces.*

Proof. The proof follows from Proposition 1.4.2, because $(\cdot) \otimes V_n^{\otimes l}$, $(\cdot)^{[\lambda]}$, and $(\cdot)_\lambda$ are all exact functors. ∎

Remark 2.1.2. The space $F_\lambda(X)$ for $\lambda \in P_n^+$ has been studied by Zelevinsky [23]. He proved that F_λ transforms the BGG resolution of a finite-dimensional simple $\mathfrak{sl}_n$-module to an exact sequence.

2.2. H_l-action

We shall define an action of H_l on $F_\lambda(X)$ as follows. For $i = 0, 1, \ldots, l$, define $\pi_i\colon \mathfrak{sl}_n \to \mathfrak{sl}_n^{\otimes l+1}$ by $\pi_i(g) = 1^{\otimes i} \otimes g \otimes 1^{\otimes l-i}$. We let $\mathbb{C}[W_l]$ act on $X \otimes V_n^{\otimes l}$ naturally via permutations of components of $V_n^{\otimes l}$. The image of $w \in W_n$ in $\mathrm{End}_{\mathbb{C}}(X \otimes V_n^{\otimes l})$ will be denoted by the same symbol. Note that as operators on $X \otimes V_n^{\otimes l}$, the following equality holds:

$$s_{ij} = \Omega_{ij} + \frac{1}{n} \qquad (1 \le i \ne j \le l), \tag{2.2.1}$$

where $\Omega_{ij} = (\pi_i \otimes \pi_j)(\Omega)$, and Ω is as in (1.2.2). Consider the following operators on $X \otimes V_n^{\otimes l}$:

$$y_i = \Omega_{0i} + \sum_{1 \le j < i} \left(\Omega_{ji} + \frac{1}{n} \right) + \frac{n-1}{2} \qquad (i = 1, \ldots, l). \tag{2.2.2}$$

LEMMA 2.2.1. *As operators on $X \otimes V_n^{\otimes l}$, the following equations hold:*

$$[y_i, y_j] = 0 \qquad (i, j = 1, \ldots, l), \tag{2.2.3}$$

$$s_i y_j - y_{s_i(j)} s_i = -\langle \alpha_i, \epsilon_j^\vee \rangle \qquad (i = 1, \ldots, l-1,\ j = 1, \ldots, l). \tag{2.2.4}$$

Proof. The first equality (2.2.3) follows easily from the following relations:

$$[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0, \qquad [\Omega_{ij}, \Omega_{km}] = 0,$$

where $i, j, k, m \in \{1, \ldots, l\}$ are all distinct. To show (2.2.4), recall the homomorphism ev in Lemma 1.5.3, with which the operator y_j can be written as

$$y_j = \Omega_{0j} + \mathrm{ev}(\epsilon_j^\vee) + \frac{n-1}{2}. \tag{2.2.5}$$

Now the equality (2.2.4) follows by using $s_i \Omega_{0j} = \Omega_{0 s_i(j)} s_i$ $(i = 1, \ldots, l-1)$.
∎

THEOREM 2.2.2. (i) (cf. [9]). *There exists a unique homomorphism*

$$H_l \to \mathrm{End}_{\mathfrak{sl}_n}(X \otimes V_n^{\otimes l})$$

such that

$$\epsilon_i^\vee \mapsto y_i \qquad (i = 1, \ldots, l),$$
$$w \mapsto w \qquad (w \in W_l).$$

(ii) *The above homomorphism induces an action of H_l on $F_\lambda(X)$.*

Evidently the correspondence $X \mapsto F_\lambda(X)$ defines a functor from the category $\mathscr{O}(\mathfrak{sl}_n)$ to the category $\mathscr{R}(H_l)$ of finite-dimensional H_l-modules. Some remarks about this functor are in the sequel.

Remark 2.2.3. (i) The above construction of the functor F_λ arose from [1], in which representations of the *degenerate double affine Hecke algebra* are constructed from representations of the affine Lie algebra $\mathfrak{sl}_n$, using the Knizhnik–Zamolodchikov connections in the conformal field theory.

(ii) Let us consider the case $\lambda = 0$. Then for any $X \in \mathrm{obj}\,\mathscr{O}(\mathfrak{sl}_n)$, the action of H_l on $F_0(X)$ factors the evaluation homomorphism in Lemma 1.5.3. Namely, F_0 is regarded as a functor from $\mathscr{O}(\mathfrak{sl}_n)$ to the category of finite-dimensional representations of W_l. Restricting the functor F_0 to the category of finite-dimensional representations of $\mathfrak{sl}_n$, we obtain the classical Frobenius–Schur–Weil duality (see [23]).

3. IMAGES OF VERMA MODULES AND SIMPLE MODULES

3.1. *Standard modules and their simple quotients*

We will determine explicitly how Verma modules and their simple quotients in $\mathscr{O}(\mathfrak{sl}_n)$ are transformed by the functor F_λ. We first introduce some H_l-modules. A pair $[a, b]$ of complex numbers such that $b - a + 1 \in \mathbb{Z}_{\ge 0}$ is called *a segment*. For a segment $[a, b]$ such that $b - a + 1 = l$, there exists a unique one-dimensional representation $\mathbb{C}_{[a,b]} = \mathbb{C}\mathbf{1}_{[a,b]}$ of H_l (we put $H_0 = \mathbb{C}$ for convenience) such that

$$w\mathbf{1}_{[a,b]} = \mathbf{1}_{[a,b]} \qquad (w \in W_l), \tag{3.1.1}$$

$$\epsilon_i^\vee \mathbf{1}_{[a,b]} = (a + i - 1)\mathbf{1}_{[a,b]} \qquad (i = 1, \ldots, l). \tag{3.1.2}$$

Let $\Delta := ([a_1, b_1], \ldots, [a_k, b_k])$ be an ordered sequence of segments such that $b_i - a_i + 1 = l_i$ and $l = \sum_{i=1}^k l_i$. Regard $H_{l_1} \otimes H_{l_2} \otimes \cdots \otimes H_{l_k}$ as a subalgebra of H_l. Define an H_l-module $\mathscr{M}(\Delta)$ by

$$\mathscr{M}(\Delta) = H_l \bigotimes_{H_{l_1} \otimes \cdots \otimes H_{l_k}} \left(\mathbb{C}_{[a_1, b_1]} \otimes \cdots \otimes \mathbb{C}_{[a_k, b_k]}\right). \tag{3.1.3}$$

Evidently $\mathscr{M}(\Delta)$ is a cyclic module with a cyclic weight vector,

$$\mathbf{1}_\Delta := \mathbf{1}_{[a_1, b_1]} \otimes \cdots \otimes \mathbf{1}_{[a_k, b_k]}, \tag{3.1.4}$$

whose weight ζ_Δ is given by

$$\langle \zeta_\Delta, \epsilon_j^\vee \rangle = a_i + j - \sum_{k=1}^{i-1} l_k - 1 \qquad \text{for } \sum_{k=1}^{i-1} l_k < j \leq \sum_{k=1}^{i} l_k. \tag{3.1.5}$$

It is also obvious that $\mathscr{M}(\Delta) \cong \mathbb{C}[W_l/(W_{l_1} \times \cdots \times W_{l_k})]$ as a $\mathbb{C}[W_l]$-module. In particular

$$\dim \mathscr{M}(\Delta) = \frac{l!}{l_1! \cdots l_k!}. \tag{3.1.6}$$

3.2. *Relation between parameters*

Take a pair of weights $\lambda, \mu \in \mathfrak{h}_n^* \subset \mathfrak{t}_n^*$ of $\mathfrak{sl}_n$ such that $\lambda - \mu \in P(V_n^{\otimes l})$. Then there exist integers $(l_1, \ldots, l_n) \in \mathbb{Z}_{\geq 0}^n$ such that $l = \sum_{i=1}^n l_i$ and

$$\lambda - \mu \equiv \sum_{i=1}^{n} l_i \epsilon_i \bmod \mathbb{C}\epsilon, \tag{3.2.1}$$

where $\epsilon = \epsilon_1 + \cdots + \epsilon_n$. For $\lambda, \mu \in \mathfrak{h}_n^*$, we associate an ordered sequence of segments

$$\Delta_{\lambda, \mu} := ([\mu_1', \mu_1' + l_1 - 1], \ldots, [\mu_n', \mu_n' + l_n - 1]), \tag{3.2.2}$$

where $\mu_i' = \langle \mu + \rho, \epsilon_i^\vee \rangle$. We put

$$\mathscr{M}(\lambda, \mu) = \mathscr{M}(\Delta_{\lambda, \mu}), \qquad \mathbf{1}_{\lambda, \mu} = \mathbf{1}_{\Delta_{\lambda, \mu}}, \tag{3.2.3}$$

where $\mathbf{1}_{\Delta_{\lambda,\mu}}$ is as in (3.1.4). We call $\mathscr{M}(\lambda, \mu)$ a *standard module* if $\lambda + \rho \in P_n^+$. It is known that the standard module $\mathscr{M}(\lambda, \mu)$ has a unique simple quotient, which is denoted by $\mathscr{L}(\lambda, \mu)$ (see Theorem A.2.1).

3.3. *Images of Verma modules*

Our goal in this subsection is the following.

THEOREM 3.3.1. *For $\lambda, \mu \in P_n$, there is an isomorphism of H_l-modules*

$$F_\lambda(M(\mu)) \cong \begin{cases} \mathscr{M}(\lambda, \mu) & \text{if } \lambda - \mu \in P(V_n^{\otimes l}), \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathscr{M}(\lambda, \mu)$ is given by (3.2.3) and (3.1.3). In particular, if $\lambda + \rho \in P_n^+$ and $\lambda - \mu \in P(V_n^{\otimes l})$, then $F_\lambda(M(\mu))$ has a unique simple quotient.

To prove Theorem 3.3.1, we prepare some lemmas. For $\mu \in \mathfrak{h}_n^*$, let v_μ denote the highest-weight vector of $M(\mu)$.

LEMMA 3.3.2. *For $\lambda, \mu \in P_n$, the natural inclusion $(V_n^{\otimes l})_{\lambda - \mu} \hookrightarrow (M(\mu) \otimes V_n^{\otimes l})_\lambda$ given by $u \mapsto v_\mu \otimes u$ induces an isomorphism as W_l-modules,*

$$(V_n^{\otimes l})_{\lambda - \mu} \xrightarrow{\sim} F_\lambda(M(\mu)). \tag{3.3.1}$$

In particular, $F_\lambda(M(\mu)) = 0$ unless $\lambda - \mu \in P(V_n^{\otimes l})$.

Proof. The lemma follows from the following fact (known as the tensor product formula): For any $\mu \in \mathfrak{h}_n^*$ and any $\mathfrak{sl}_n$-module Y there exists a unique $\mathfrak{sl}_n$-isomorphism

$$M(\mu) \otimes Y \xrightarrow{\sim} \operatorname{Ind}_{U(\mathfrak{n}_+ \oplus \mathfrak{h})}^{U(\mathfrak{sl}_n)}(\mathbb{C}v_\mu \otimes Y),$$

which sends $v_\mu \otimes u$ to $v_\mu \otimes u (u \in Y)$. ∎

Recall that $\{u_i\}_{i=1,\ldots,n}$ is the standard basis of V_n. Fix $\lambda, \mu \in P_n$ such that $\lambda - \mu \in P(V_n^{\otimes l})$ and let $u_{\lambda, \mu} \in F_\lambda(M(\mu))$ be the image of

$$\tilde{u}_{\lambda, \mu} := v_\mu \otimes u_1^{\otimes l_1} \otimes \cdots \otimes u_l^{\otimes l_n} \in \left(M(\mu) \otimes V_n^{\otimes l}\right)_\lambda,$$

where l_i are as in (3.2.1). Let $\zeta_{\lambda, \mu} \in \mathfrak{t}_l^*$ be the weight of $\mathbf{1}_{\lambda, \mu}$ with respect to the action of $\mathfrak{t}_l$:

$$\xi \mathbf{1}_{\lambda, \mu} = \langle \zeta_{\lambda, \mu}, \xi \rangle \mathbf{1}_{\lambda, \mu} \qquad (\xi \in \mathfrak{t}_l) \qquad (\text{see } (3.1.4)\ (3.1.5)).$$

LEMMA 3.3.3. *Let $\bar{y}_i$ denote the image of the operator y_i (see (2.2.2)) in $\operatorname{End}_{\mathbb{C}}(F_\lambda(M(\mu)))$. Then we have*

$$\bar{y}_i u_{\lambda, \mu} = \langle \zeta_{\lambda, \mu}, \epsilon_i^\vee \rangle u_{\lambda, \mu} \qquad (i = 1, \ldots, l).$$

Proof. Let $p: (M(\mu) \otimes V_n^{\otimes l})_\lambda \to F_\lambda(M(\mu))$ be the natural surjection. Then it can be checked that $\bar{y}_i u_{\lambda,\mu} = p(v)$, where

$$v = \Bigg[\sum_{1 \le j < i} \left(r_{ji} + \frac{1}{2n} \right) - \sum_{i < j \le l} \left(r_{ij} + \frac{1}{2n} \right) - \pi_i \left(\frac{1}{2} \sum_{k=1}^{n-1} h_k h^k + \sum_{\alpha \in R_n^+} e_{-\alpha} e_\alpha + \frac{1}{2n} \right) + \frac{1}{2} \sum_{k=1}^{n-1} \langle \lambda + \mu, h^k \rangle \pi_i(h_k) - \frac{1}{2} + \frac{l}{2n} \Bigg] \tilde{u}_{\lambda,\mu}.$$

Here $r_{ij} = (\pi_i \otimes \pi_j)(r)$ (r is as in (1.2.1)), and we used $r_{ij} + r_{ji} = \Omega_{ij}$. Now the statement follows from the following formulas:

$$\left(r + \frac{1}{2n} \right)(u_j \otimes u_k) = \frac{1}{2} \delta_{jk} (u_j \otimes u_k) \qquad \text{for } j \le k,$$

$$\left(\frac{1}{2} \sum_{k=1}^{n-1} h_k h^k + \sum_{\alpha \in R_n^+} e_{-\alpha} e_\alpha + \frac{1}{2n} \right) u_j = -\frac{1}{2}(n - 2j + 1) u_j,$$

which are proved by direct calculations. ∎

Proof of Theorem 3.3.1. By Lemma 3.3.2, we have that

(i) $u_{\lambda,\mu}$ is a cyclic vector of $F_\lambda(M(\mu))$.

And obviously,

(ii) $w u_{\lambda,\mu} = u_{\lambda,\mu}$ for all $w \in W_{l_1} \times \cdots \times W_{l_n}$.

By Lemma 3.3.3 and (i), (ii) above, we have a surjective H_l-homomorphism $\mathscr{M}(\lambda, \mu) \to F_\lambda(M(\mu))$, which sends $\mathbf{1}_{\lambda,\mu}$ to $u_{\lambda,\mu}$, and it is a bijection by Lemma 3.3.2.

3.4. *Images of simple modules*

Next let us suppose that $n = l$ and determine the images of simple modules.

THEOREM 3.4.1. *Let $\lambda + \rho \in P_l^+$ and $w \in W_l$ be such that $\lambda - w \circ \lambda \in P(V_l^{\otimes l})$. Then we have the following:*

(i) *If w satisfies*

$$\langle w \circ \lambda + \rho, h_i \rangle \le 0 \qquad \textit{for any } i \in \{1, \dots, l\} \textit{ such that } \langle \lambda + \rho, h_i \rangle = 0, \tag{3.4.1}$$

then

$$F_\lambda(L(w \circ \lambda)) \cong \mathscr{L}(\lambda, w \circ \lambda), \tag{3.4.2}$$

where $\mathscr{L}(\lambda, \mu)$ is a unique simple quotient of $\mathscr{M}(\lambda, \mu)$ (see Theorem A.2.1).

(ii) *If w does not satisfy condition* (3.4.1), *then*

$$F_\lambda(L(w \circ \lambda)) = 0. \tag{3.4.3}$$

Remark 3.4.2. For $\lambda \in P_l^+ - \rho$, let $W_{\lambda+\rho}$ denote the stabilizer

$$W_{\lambda+\rho} = \{ w \in W_l \mid w(\lambda + \rho) = \lambda + \rho \}, \tag{3.4.4}$$

which is a parabolic subgroup of W_l. Let w_L and w_{LR} denote the unique longest element in the coset $W_{\lambda+\rho} w$ and $W_{\lambda+\rho} w W_{\lambda+\rho}$, respectively. Then condition (3.4.1) is equivalent to

$$w \circ \lambda = w_L \circ \lambda, \quad \text{or equivalently,} \quad w \circ \lambda = w_{LR} \circ \lambda. \tag{3.4.5}$$

Note that $w_L \circ \lambda = w_{LR} \circ \lambda$.

3.5. *Simple H_l-modules*

For $\zeta \in \mathfrak{t}_l^*$, let

$$\gamma_\zeta : Z(H_l) = S(\mathfrak{t}_l)^{W_l} \to \mathbb{C} \tag{3.5.1}$$

be the homomorphism given by the evaluation at ζ. The following is a consequence of Theorem 3.4.1 and the classification theorem of simple affine Hecke algebra modules (see Corollary A.2.4).

COROLLARY 3.5.1. *Let $\lambda + \rho \in P_l^+$. Then any finite-dimensional simple H_l-module with the action of $Z(H_l)$ via $\gamma_{\lambda+\rho}$ is isomorphic to $F_\lambda(L(w_L \circ \lambda))$ for some $w \in W_l$.*

Remark 3.5.2. For $c \in \mathbb{C}$, let t_c be the automorphism of H_l given by

$$t_c(s_i) = s_i (i = 1, \dots, l-1), \qquad t_c(\epsilon_i^\vee) = \epsilon_i^\vee + c (i = 1, \dots, l).$$

For an H_l-module Y, let Y^c denote the H_l-module given by the composition

$$H_l \xrightarrow{t_c} H_l \to \mathrm{End}_{\mathbb{C}}(Y).$$

It is known that any simple H_l-module is isomorphic to

$$H_l \bigotimes_{H_{l_1} \otimes \cdots \otimes H_{l_k}} \left(\mathscr{L}(\lambda^{(1)}, \mu^{(1)})^{c_1} \otimes \cdots \otimes \mathscr{L}(\lambda^{(k)}, \mu^{(k)})^{c_k} \right),$$

for some $\{(l_i, \lambda^{(i)}, \mu^{(i)}, c_i)\}_{i=1,\dots,k}$. Here $(l_1, \dots, l_k) \in \mathbb{Z}_{>0}$ is a partition of l, c_i is a complex number, and $(\lambda^{(i)}, \mu^{(i)}) \in P_{l_i} \times P_{l_i}$ satisfying $\lambda^{(i)} + \rho \in P_{l_i}^+$ and $\lambda^{(i)} - \mu^{(i)} \in P(V_i^{\otimes l_i})$ (see [10]).

3.6. *Proof of Theorem* 3.4.1

For $w, y \in W_n$ such that $w \le y$ (where $\le$ denotes the Bruhat order in W_n), let $P_{w,y}(q) \in \mathbb{Z}[q]$ denote the Kazhdan–Lusztig polynomial of the Hecke algebra associated with W_n (see [16, 17]). We put, for convenience, $P_{w,y}(q) = 0$ for $w \not\le y$.

The key formula in the following proof of Theorem 3.4.1 is the following formula (see Theorem A.1.1, Theorem A.3.1, and Corollary A.3.2):

$$[\mathscr{M}(\lambda, w \circ \lambda) : \mathscr{L}(\lambda, y \circ \lambda)] = P_{w_{LR}, y_{LR}}(1) = [M(w_L \circ \lambda) : L(y_L \circ \lambda)]. \tag{3.6.1}$$

Here $\lambda \in \mathfrak{h}_l^*$ and $w, y \in W_l$ are assumed to satisfy $\lambda + \rho \in P_l^+$ and $\lambda - w \circ \lambda$, $\lambda - y \circ \lambda \in P(V_l^{\otimes l})$; $[M : N]$ denotes the multiplicity of N in the composition series of M; and y_{LR} (resp. y_L) denotes the longest element in the coset $W_{\lambda+\rho} y W_{\lambda+\rho}$ (resp. $y W_{\lambda+\rho}$).

First we show the following lemma.

LEMMA 3.6.1. *Let $\lambda + \rho \in P_l^+$ and $w \in W_l$ be such that $\lambda - w \circ \lambda \in P(V_l^{\otimes l})$.*

(i) *If $w \in W_l$ satisfies condition* (3.4.1), *then $F_\lambda(L(w \circ \lambda)) \neq 0$.*

(ii) *If $w \in W_l$ does not satisfy condition* (3.4.1), *then $F_\lambda(L(w \circ \lambda)) = 0$.*

Proof. We first prove (ii). Suppose that w does not satisfy condition (3.4.1). Then we can find $s_i \in W_{\lambda+\rho}$ such that $\langle w \circ \lambda + \rho, h_i \rangle > 0$. This inequality means that $M(w \circ \lambda)$ contains $M(s_i w \circ \lambda)$ as a (proper) submodule, and hence $F_\lambda(L(w \circ \lambda))$ is a quotient of $F_\lambda(M(w \circ \lambda)/M(s_i w \circ \lambda)) \cong \mathscr{M}(\lambda, w \circ \lambda)/\mathscr{M}(\lambda, s_i w \circ \lambda)$. Since $s_i \in W_{\lambda+\rho}$, we have $\lambda - s_i w \circ \lambda \in P(V_l^{\otimes l})$ and $\dim \mathscr{M}(\lambda, w \circ \lambda) = \dim \mathscr{M}(\lambda, s_i w \circ \lambda)$ by (3.1.6), and thus we get $F_\lambda(L(w \circ \lambda)) = 0$.

Let us prove (i). Assume that w satisfies condition (3.4.1). Then by Remark 3.4.2, we can assume that w is the longest element in $W_{\lambda+\rho} w W_{\lambda+\rho}$. We can write in the Grothendieck group of $\mathscr{O}(\mathfrak{sl}_n)$ as

$$M(w \circ \lambda) = L(w \circ \lambda) + \sum_{y > w} P_{w,y}(1) L(y \circ \lambda) \tag{3.6.2}$$

(see Theorem A.1.1), and the sum runs over those elements $y \in W_l$ such that y is longest in $y W_{\lambda+\rho}$ and $y > w$. Applying F_λ to (3.6.2), we have

$$\mathscr{M}(\lambda, w \circ \lambda) = F_\lambda(L(w \circ \lambda)) + \sum_{y > w} P_{w,y}(1) F_\lambda(L(y \circ \lambda)) \tag{3.6.3}$$

in the Grothendieck group of $\mathscr{R}(H_l)$.

Now, let us assume that $F_\lambda(L(w \circ \lambda)) = 0$. Since $[\mathscr{M}(\lambda, w \circ \lambda) : \mathscr{L}(\lambda, w \circ \lambda)] > 0$, there must be a summand $F_\lambda(L(y \circ \lambda))$ in the right-hand side of (3.6.3) such that

$$[F_\lambda(L(y \circ \lambda)) : \mathscr{L}(\lambda, w \circ \lambda)] > 0. \tag{3.6.4}$$

Since $F_\lambda(L(y \circ \lambda))$ is a nonzero quotient of $F_\lambda(M(y \circ \lambda))$, we have

$$[\mathscr{M}(\lambda, y \circ \lambda) : \mathscr{L}(\lambda, w \circ \lambda)] \ge [F_\lambda(L(y \circ \mu)) : \mathscr{L}(\lambda, w \circ \lambda)] > 0. \tag{3.6.5}$$

On the other hand, (3.6.1) implies

$$[\mathscr{M}(\lambda, y \circ \lambda) : \mathscr{L}(\lambda, w \circ \lambda)] = P_{y_{LR}, w_{LR}}(1) = 0,$$

since $l(y_{LR}) \ge l(y) > l(w) = l(w_{LR})$. This contradicts (3.6.5) and shows that $F_\lambda(L(w \circ \lambda)) \neq 0$. ∎

Let us complete the proof of Theorem 3.4.1. Assume that $w \in W_l$ satisfies $\lambda - w \circ \lambda \in P(V_l^{\otimes l})$ and w is the longest element in $W_{\lambda+\rho} w W_{\lambda+\rho}$ (i.e., $w = w_{LR}$). We suppose that $F_\lambda(L(w \circ \lambda))$ has a constituent (a simple subquotient) other than $\mathscr{L}(\lambda, w \circ \lambda)$, and will deduce a contradiction. Such a constituent is isomorphic to $\mathscr{L}(\lambda, y \circ \lambda)$ for some $y = y_{LR} \in W_l$ such that $\lambda - y \circ \lambda \in P(V_l^{\otimes l})$ (see Corollary A.2.4):

$$[F_\lambda(L(w \circ \lambda)) : L(\lambda, y \circ \lambda))] \ge 1. \tag{3.6.6}$$

Since $F_\lambda(L(y \circ \lambda))$ is a nonzero quotient of $\mathscr{M}(\lambda, y \circ \lambda)$ by Lemma 3.6.1, we have

$$[F_\lambda(L(y \circ \lambda)) : L(\lambda, y \circ \lambda)] \ge 1. \tag{3.6.7}$$

Combining (3.6.3) and inequalities (3.6.6) and (3.6.7), we have

$$[\mathscr{M}(\lambda, w \circ \lambda) : \mathscr{L}(\lambda, y \circ \lambda)] \ge [F_\lambda(L(w \circ \lambda) : \mathscr{L}(\lambda, y \circ \lambda)]$$
$$+ P_{w,y}(1)[F_\lambda(L(y \circ \lambda)) : L(\lambda, y \circ \lambda)] \ge 1 + P_{w,y}(1),$$

which contradicts (3.6.1), since $y = y_{LR}$ and $w = w_{LR}$. ∎

APPENDIX A: SOME FACTS FROM REPRESENTATION THEORY

We will review some facts used in this paper.

A.1. *Composition series of Verma modules of* $\mathfrak{sl}_n$

For $w, y \in W_n$ such that $w \le y$, let $P_{w,y}(q) \in \mathbb{Z}[q]$ denote the Kazhdan–Lusztig polynomial of the Hecke algebra associated with W_n. We put, for convenience, $P_{w,y}(q) = 0$ for $w \not\leq y$. Let $W_{\lambda+\rho} := \{w \in W_n \mid w(\lambda+\rho) = \lambda+\rho\}$ be the stabilizer.

THEOREM A.1.1 [2, 4]. *Let $\lambda + \rho \in P_n^+$ and $w \in W_n$. Then any composition factor of $M(w \circ \lambda)$ is isomorphic to $L(y \circ \lambda)$ for some $y \in W_n$, and its multiplicity is given by*

$$[M(w \circ \lambda) : L(y \circ \lambda)] = P_{w, y_R}(1), \tag{A.1.1}$$

where y_R is the longest element in the right coset $yW_{\lambda+\rho}$.

A.2. *Finite-dimensional representations of H_l*

We review a classification of finite-dimensional simple H_l-modules following [24] in terms of our parameterizations. (Note that the representation theory of H_l is related to that of the corresponding affine Hecke algebra by Lusztig [18].)

THEOREM A.2.1 ([24, Theorem 6.1(a)]; see also [19, Theorem 5.2]). *Suppose that $\lambda, \mu \in \mathfrak{h}_n^*$ satisfy $\lambda + \rho \in P_n^+$ and $\lambda - \mu \in P(V_n^{\otimes l})$. Then $\mathscr{M}(\lambda, \mu)$ has a unique simple quotient, which is denoted by $\mathscr{L}(\lambda, \mu)$.*

LEMMA A.2.2 ([24, Theorem 6.1(b)]). *Suppose that $\lambda + \rho \in P_n^+$ and $\mu, \eta \in P_n$ satisfy $\lambda - \mu \in P(V_n^{\otimes l})$, $\lambda - \eta \in P(V_n^{\otimes l})$. Then the following conditions are equivalent:*

(i) $\mathscr{M}(\lambda, \mu) \cong \mathscr{M}(\lambda, \eta)$.

(ii) $\mathscr{L}(\lambda, \mu) \cong \mathscr{L}(\lambda, \eta)$.

(iii) *There exists $w \in W_{\lambda+\rho}$ such that $\eta = w \circ \mu$.*

Let us restrict ourselves to the case $n = l$. In this case, the module

$$\mathscr{M}(\lambda, \lambda) = H_l \underset{S(\mathfrak{t}_l)}{\otimes} \mathbb{C}_{\lambda+\rho},$$

is isomorphic to the regular representation as a $\mathbb{C}[W_l]$-module, where $\mathbb{C}_{\lambda+\rho}$ is a one-dimensional $S(\mathfrak{t}_l)$-module determined by the weight $\lambda + \rho \in \mathfrak{h}_l^* \subset \mathfrak{t}_l^*$. This module is called the principal series representation and is studied, e.g., in [8, 15, 19].

THEOREM A.2.3. *Let $\lambda \in \mathfrak{h}_l^*$ be such that $\lambda + \rho \in P_l^+ \subset \mathfrak{t}_l^*$.*

(i) [8, 19] *Any finite-dimensional simple module with action of $Z(H_l)$ via $\gamma_{\lambda+\rho}$ (where $\gamma_{\lambda+\rho}$ is as in (3.5.1)) is a constituent of $\mathscr{M}(\lambda, \lambda)$.*

(ii) ([24, *Theorem* 6.1(c), *Theorem* 7.1]) *For $w \in W_l$ such that $\lambda - w \circ \lambda \in P(V_l^{\otimes l})$, the module $\mathscr{L}(\lambda, w \circ \lambda)$ is a constituent of $\mathscr{M}(\lambda, \lambda)$, and any constituent is isomorphic to $\mathscr{L}(\lambda, w \circ \lambda)$ for some $w \in W_l$ such that $\lambda - w \circ \lambda \in P(V_l^{\otimes l})$.*

Put

$$\mathscr{S}(\lambda) := \{w \in W_l \mid \lambda - w \circ \lambda \in P(V_l^{\otimes l})\} \subseteq W_l,$$

and let $\overline{\mathscr{S}(\lambda)}$ denote the image of $\mathscr{S}(\lambda)$ in the double coset $W_{\lambda+\rho} \backslash W_l / W_{\lambda+\rho}$.

COROLLARY A.2.4. *Let $\lambda + \rho \in P_l^+$. Then there exists a one-to-one correspondence between $\overline{\mathscr{S}(\lambda)}$ and the set of equivalent classes of finite-dimensional simple H_l-modules with the action of $Z(H_l)$ via $\gamma_{\lambda+\rho}$, which is given by $W_{\lambda+\rho} w W_{\lambda+\rho} \mapsto \mathscr{L}(\lambda, w \circ \lambda)$.*

A.3. *Multiplicity Formulas*

Let us recall the multiplicity formula for (degenerate) affine Hecke algebras of GL_l. Zelevinsky conjectured in [21] that the multiplicity of simple modules in the composition series of standard modules is given in terms of the intersection cohomologies concerning the quiver variety. He proved in [22] that these intersection cohomologies are expressed by the Kazhdan-Lusztig polynomials of the symmetric group. Zelevinsky's conjecture was proved by Ginzburg [13] (see also [11]) in more general situations. The result is rephrased as follows:

THEOREM A.3.1 [13]. *Suppose that $\lambda + \rho \in P_l^+$ and $y, w \in W_l$ satisfy $\lambda - w \circ \lambda \in P(V_l^{\otimes l})$ and $\lambda - y \circ \lambda \in P(V_l^{\otimes l})$. Then*

$$[\mathscr{M}(\lambda, w \circ \lambda) : \mathscr{L}(\lambda, y \circ \lambda)] = P_{w_{LR}, y_{LR}}(1), \tag{A.3.1}$$

where w_{LR} and y_{LR} denote the longest elements in the double coset $W_{\lambda+\rho} w W_{\lambda+\rho}$ and $W_{\lambda+\rho} y W_{\lambda+\rho}$, respectively.

Combining Theorem A.1.1 and Theorem A.3.1, we have the following identity.

COROLLARY A.3.2. *Assume that $\lambda + \rho \in P_l^+$ and $w, y \in W_l$ satisfy $\lambda - w \circ \lambda \in P(V_l^{\otimes l})$ and $\lambda - y \circ \lambda \in P(V_l^{\otimes l})$. Then we have*

$$[\mathscr{M}(\lambda, w \circ \lambda) : \mathscr{L}(\lambda, y \circ \lambda)] = [M(w_L \circ \lambda) : L(y_L \circ \lambda)], \tag{A.3.2}$$

where w_L and y_L denote the longest elements in the left coset $W_{\lambda+\rho} w$ and $W_{\lambda+\rho} y$, respectively.

ACKNOWLEDGMENTS

Our work is motivated by the joint research with A. Tsuchiya presented in [1]. We are grateful to him for discussions and his encouragement. We thank T. Tanisaki for drawing our attention to [21, 22]. Thanks are also due to I. Cherednik and M. Kashiwara for valuable comments.

REFERENCES

1. T. Arakawa, T. Suzuki, and A. Tsuchiya, Degenerate double affine Hecke algebras and conformal field theory, *in* "Topological Field Theory, Primitive Forms and Related Topics, Proceedings of the 38th Taniguchi symposium, Kyoto," Birkhäuser, Basel.
2. A. Beilinson and J. Bernstein, Localisation de $\mathfrak{g}$-modules, *C. R. Acad. Sci. Paris* **292** (1981), 15–18.
3. I. N. Bernstein, I. M. Gel'fand, and S. I. Gel'fand, Structure of representations generated by vectors of highest weight, *Funct. Anal. Appl.* **5** (1971), 1–8.
4. J. L. Brylinski and M. Kashiwara, Kazhdan–Lusztig conjecture and holonomic systems, *Invent. Math.* **64** (1981), 387–410.
5. V. Chari and A. Pressley, Quantum affine algebras and affine Hecke algebras, *Pacific J. Math.* **174** (2) (1996), 295–326.
6. I. V. Cherednik, A new interpretation of Gelfand–Tzetlin basis, *Duke Math. J.* **54** (1987), 563–578.
7. I. V. Cherednik, An analogue of the character formulas for Hecke algebras, *Funct. Anal. Appl.* **21** (2) (1987), 94–95.
8. I. V. Cherednik, A unification of Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras, *Invent. Math.* **106** (1991), 411–431.
9. I. V. Cherednik, Monodromy representations for generalized Knizhnik–Zamolodchikov equations and Hecke algebras, *Publ. Res. Inst. Math. Sci.* **27** (1991), 711–726.
10. I. V. Cherednik, Special bases of irreducible representations of a degenerate affine Hecke algebra, *Funct. Anal. Appl.* **20** (1) (1986), 76–78.
11. N. Chriss and V. Ginzburg, "Representation Theory and Complex Geometry," Birkhäuser, Basel, 1997.
12. V. G. Drinfeld, Degenerate affine Hecke algebras and Yangians, *Funct. Anal. Appl.* **20** (1) (1986), 58–60.
13. V. A. Ginzburg, Proof of the Deligne–Langlands conjecture, *Soviet. Math. Dokl.* **35** (2) (1987), 304–308.
14. M. Jimbo, A q-analogue of $U(gl(N+1))$, Hecke algebra and the Yang–Baxter equation, *Lett. Math. Phys.* **11** (1986), 247–252.
15. S. Kato, Irreducibility of principal series representations for Hecke algebras of affine type, *J. Fac. Sci. Univ. Tokyo Sect. 1A Math.* **28** (3) (1982), 929–943.
16. D. Kazhdan and G. Lusztig, Representation of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.
17. D. Kazhdan and G. Lusztig, Schubert varieties and Poincare duality, *Proc. Symp. Pure Math.* **36** (1980), 185–203.
18. G. Lusztig, Affine Hecke algebras and their graded version, *J. Am. Math. Soc.* **2** (3) (1989), 599–635.
19. J. D. Rogawski, On modules over the Hecke algebra of a p-adic group, *Invent. Math.* **79** (1985), 443–465.
20. M. Varagnoro and E. Vasserot, Schur duality in the troidal setting, *Comm. Math. Phys.* **182** (2) (1996), 469–483.
21. A. Z. Zelevinsky, p-adic analogue of the Kazhdan–Lusztig hypothesis, *Funct. Anal. Appl.* **15** (2) (1981), 83–92.
22. A. Z. Zelevinsky, Two remarks on graded nilpotent classes, *Russian Math. Surveys* **40** (1) (1985), 249–250.
23. A. Z. Zelevinsky, Resolvents, dual pairs and character formulas, *Funct. Anal. Appl.* **21** (1987), 152–154.
24. A. Z. Zelevinsky, Induced representations of reductive p-adic groups II, *Ann. Sci. École Norm. Sup. (4)* **13** (1980), 165–210.

JOURNAL OF ALGEBRA **209**, 305–344 (1998)
ARTICLE NO. JA987511

Down–Up Algebras

Georgia Benkart*

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706-1388

and

Tom Roby†

Department of Mathematics and Computer Science, California State University, Hayward, California 94542-3092

Communicated by Walter Feit

Received March 9, 1998

The algebra generated by the down and up operators on a differential or uniform partially ordered set (poset) encodes essential enumerative and structural properties of the poset. Motivated by the algebras generated by the down and up operators on posets, we introduce here a family of infinite-dimensional associative algebras called down–up algebras. We show that down–up algebras exhibit many of the important features of the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$ including a Poincaré–Birkhoff—Witt type basis and a well-behaved representation theory. We investigate the structure and representations of down–up algebras and focus especially on Verma modules, highest weight representations, and category $\mathcal{O}$ modules for them. We calculate the exact expressions for all the weights, since that information has proven to be particularly useful in determining structural results about posets.

1. INTRODUCTION

Combinatorial Source of Down–Up Algebras

Assume P is a partially ordered set (poset), and let $\mathbf{C}P$ denote the complex vector space whose basis is the set P. For many posets there are

* The author gratefully acknowledges support from National Science Foundation Grant DMS-9622447 and from the Mathematical Sciences Research Institute, Berkeley. E-mail: benkart@math.wisc.edu.

† The author gratefully acknowledges support from National Science Foundation Grant DMS-9353149 and the University of Wisconsin, Madison. E-mail: troby@mcs.csuhayward.edu.

0021-8693/98 $25.00