SOME SMALL UNIPOTENT REPRESENTATIONS OF INDEFINITE ORTHOGONAL GROUPS AND THE THETA CORRESPONDENCE

ANNEGRET PAUL AND PETER TRAPA

Abstract.

1. Introduction

2. Preliminaries and notation

Recall that each K-type μ which occurs in $\mathcal{F}$ has associated to it a degree (the minimum degree of polymomials in the μ-isotypic subspace), and that if π and π' are representations of $\widetilde{G}$ and $\widetilde{G}'$, respectively, which correspond to each other, then each K-type for $\widetilde{G}$ which is of minimal degree in π will occur in $\mathcal{H}$ and correspond to a K-type for $\widetilde{G}'$ of minimal degree in π'. Since for given choice of n, $Sp(2n,\mathbb{R})$ is a member of many dual pairs, we refer to the degree of a K-type σ for $\widetilde{Sp}(2n,\mathbb{R})$ for the dual pair $\big(Sp(2n,\mathbb{R}), O(p,q)\big)$ as the (p,q)-*degree* of σ. As we will see below in Proposition 2.1, The degree of a K-type for $O(p,q)$ depends on the difference $p-q$ only, and the degree of a K-type for $O(p,q)$ is independent of n.

Proposition 2.1. *Let p, q, and n be non-negative integers, $p_0=[\frac{p}{2}]$, and $q_0=[\frac{q}{2}]$. The correspondence of K-types in the space of joint harmonics $\mathcal{H}$ for the dual pair $\big(Sp(2n,\mathbb{R}), O(p,q)\big)$ is given as follows.*

(1) *Let*

$$\mu=(a_1,a_2,\dots,a_x,0,\dots,0;\epsilon)\otimes(b_1,b_2,\dots,b_y,0,\dots,0;\eta) \tag{2.2}$$

be a K-type for $O(p,q)$, with $a_x>0$ and $b_y>0$. Then μ occurs in $\mathcal{H}$ if and only if $n\geq x+\frac{1-\epsilon}{2}(p-2x)+\frac{1-\eta}{2}(q-2y)+y$. In that case, μ corresponds to

$$\big(\tfrac{p-q}{2},\tfrac{p-q}{2},\dots,\tfrac{p-q}{2}\big)+(a_1,\dots,a_x,\underbrace{1,\dots,1}_{\frac{1-\epsilon}{2}(p-2x)},0,\dots,0,\underbrace{-1,\dots,-1}_{\frac{1-\eta}{2}(q-2y)},-b_y,\dots,-b_1). \tag{2.3}$$

(2) *If a K-type μ for $O(p,q)$ as in (2.2) occurs in the Fock space, then the degree of μ is*

$$\sum_{i=1}^{x}a_i+\sum_{i=1}^{y}b_i+\frac{1-\epsilon}{2}(p-2x)+\frac{1-\eta}{2}(q-2y). \tag{2.4}$$

For a K-type for $\widetilde{Sp}(2n,\mathbb{R})$ ξ which occurs in $\mathcal{F}$, write

$$\xi=\big(\tfrac{p-q}{2},\tfrac{p-q}{2},\dots,\tfrac{p-q}{2}\big)+(a_1,a_2,\dots,a_n). \tag{2.5}$$

Then the (p,q)-degree of ξ is $\sum_{i=1}^{n}|a_i|$.

Proposition 2.6. *Let p, q, and n be non-negative integers, and suppose π and π' are genuine irreducible admissible representations of $\widetilde{Sp}(2n,\mathbb{R})$ and $O(p,q)$ which correspond to each other in the correspondence for the dual pair $\big(Sp(2n,\mathbb{R}),O(p,q)\big)$. If $p+q \leq 2n+1$ then every LKT of π is of minimal (p,q)-degree in π, and if $2n \leq p+q$ then every LKT of π' is of minimal degree in π'.*

Proof.

$\square$

Proposition 2.7. *Let p, q, m, and n be non-negative integers such that $p+q \leq m \leq \frac{n}{2}$. Let $\pi = \theta_m(\mathbb{1}_{p,q})$, and $\pi' = \theta_{2m,n}(\pi)$.*

(1) *The K-types of π are precisely those of the form*

$$\left(\frac{p-q}{2},\dots,\frac{p-q}{2}\right) + (a_1,a_2,\dots,a_p,0,\dots,0,-b_q,\dots,-b_1), \tag{2.8}$$

where a_i and b_i are non-negative even integers for all i, and each K-type occurs with multiplicity one.

(2) *Let $a = \frac{p-q+n}{2} - m$. Then π' has a unique LKT Λ, given by*

$$\Lambda = \begin{cases} (a,a,\dots,a,\underbrace{0,\dots,0}_{q};0,\dots,0) & \text{if } a \geq 0 \text{ and even;} \\ (a,a,\dots,a,\underbrace{1,\dots,1}_{q};0,\dots,0) & \text{if } a > 0 \text{ and odd;} \\ (0,\dots,0;\underbrace{-a,\dots,-a}_{m-p},0,\dots,0) & \text{if } a \leq 0 \text{ and even;} \\ (0,\dots,0;\underbrace{-a,\dots,-a}_{m-p},\underbrace{1,\dots,1}_{p},0,\dots,0) & \text{if } a < 0 \text{ and odd.} \end{cases} \tag{2.9}$$

Proof. The first part of the proposition is Proposition 2.1, together with Corollary 2.7(c), of [22]. Let

$$\sigma = \left(\tfrac{p-q}{2},\dots,\tfrac{p-q}{2}\right) + (a_1,a_2,\dots,a_p,0,\dots,0,-b_q,\dots,-b_1), \tag{2.10}$$

be a K-type of π. Then

$$\sigma = \left(\tfrac{2m-n}{2},\dots,\tfrac{2m-n}{2}\right) + (a+a_1,\dots,a+a_p,a,\dots,a,a-b_q,\dots,a-b_1), \tag{2.11}$$

and the $(2m,n)$-degree of σ is

$$d = \sum_{i=1}^{p} |a+a_i| + \sum_{i=1}^{q} |a-b_q| + (m-p-q)|a|. \tag{2.12}$$

Suppose a is even. Then the degree of σ is minimized if $a_i = 0$ and $b_i = a$ for all i if $a \geq 0$, and by choosing $a_i = -a$ and $b_i = 0$ for all i if a is negative. So π has a unique K-type σ_0 of minimal degree. The K-type Λ corresponding to σ_0 in the space of joint harmonics (see Proposition 2.1) must then be the unique LKT of π' by Proposition 2.6.

If a is odd then π has more than one K-type of minimal (p,q)-degree: for $a < 0$ choose $b_i = 0$ for all i, and $a_i = -a+1$ or $-a-1$ in such a way that the resulting weight is dominant. The corresponding K-types for $O(p,q)$ are then those of the form

$$\Lambda_{k,l} = (\underbrace{1,\dots,1}_{k},0,\dots,0;\underbrace{-a,\dots,-a}_{m-p},\underbrace{1,\dots,1}_{l},0,\dots,0) \tag{2.13}$$

with $k+l=p$. By Proposition 2.6, the LKT's of π' are precisely those $\Lambda_{k,l}$ for which the Vogan-norm ([46] Definition 5.4.18) of $\Lambda_{k,l}$,

$$||\Lambda_{k,l}|| =< \Lambda_{k,l}+2\rho_c, \Lambda_{k,l}+2\rho_c > \tag{2.14}$$

is minimal. Since the quantity $< \Lambda_{k,l}, \Lambda_{k,l} >$ only depends on $k+l=p$, we can minimize the quantity $n_{k,l} = \frac{1}{2}\big(||\Lambda_{k,l}|| - < 2\rho_c, 2\rho_c > - < \Lambda_{k,l}, \Lambda_{k,l} >\big)$ instead. If $r = [\frac{n}{2}]$ then $2\rho_c = (2m-2, 2m-4, \dots, 2, 0; n-2, n-4, \dots, n-2r)$. So

$$n_{k,l} = \sum_{i=1}^{k}(2m-2i) + \sum_{i=1}^{m-p}(-a)(n-2i) + \sum_{i=1}^{l}(n-2m+2p-2i). \tag{2.15}$$

The second sum is independent of k and l, and it is clear that $n_{k,l}$ will be minimized by $k=p$, $l=0$ if $2m \le n-2m+2p$, and by $k=0$, $l=p$ if $2m \ge n-2m+2p$. It remains to show that we must have $4m > n+2p$. We know that $a = \frac{p-q+n}{2} - m < 0$, so $p-q+n < 2m$, Moreover, we have assumed that $p+q \le m$, so we get $2p+n < 3m \le 4m$, and we are done.

The case $a > 0$ is much easier and left to the reader.

□

References

[1] J. Adams, *Unitary Highest Weight Modules*, Adv. Math. **63** (1987), 113–137.
[2] J. Adams and D. Barbasch, *Reductive dual pair correspondence for complex groups*, J. Funct. Anal. **132** No. 1 (1995), 1–42.
[3] ———, *Genuine representations of the metaplectic group*, Comp. Math. **113** (1998), 23–66.
[4] J. Adams, D. Barbasch, and D. Vogan, *The Langlands Classification and Characters for Real Reductive Groups*, Progress in Mathematics vol. 104, Birkhäuser, Boston-Basel-Berlin, 1992.
[5] J. Arthur, *On Some Problems Suggested by the Trace Formula*, Lecture Notes in Mathematics vol. 1024, Springer Verlag, Berlin-Heidelberg-New York, 1983.
[6] D. Barbasch and D. Vogan, *Unipotent representations of complex semisimple groups*, Ann. Math. **121** (1985), pp. 41–110.
[7] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Chapman and Hall (London), 1994.
[8] T. Enright, R. Howe, and N. Wallach, *Classification of unitary highest weight modules*, *in* "Representation Theory of Reductive Groups" (P.C. Trombi, Ed.), pp. 97–144, Birkhäuser, Boston, 1983.
[9] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.
[10] R. Howe, *Elements of the oscillator representation*, Manuscript based on a course on dual pairs given at Yale University (19?), unpublished.
[11] R. Howe, *L^2-duality for stable dual pairs*, preprint.
[12] ———, *θ-series and invariant theory*, *in* Proc. Sympos. Pure Math. Vol. 33, Amer. Math. Soc., Providence, RI (1979), 275–285.
[13] ———, *On a notion of rank for unitary representations of the classical groups*, Harmonic Analysis and Group Representations, C.I.M.E. II Ciclo 1980, Cortona-Arezzo, A. Figa-Talamanca coord., Liquori Editore, Naples (1982), 223–331.
[14] ———, *Small unitary representations of classical groups*, in Group representations, ergodic theory, operator algebras and math. physics, C.C. Moore, ed. Springer 1986, pp. 121-150.
[15] ———, *Transcending classical invariant theory*, J. Am. Math. Soc. **2** (1989), 535–552.
[16] R. Howe and C. Zhu, *Eigendistributions for orthogonal groups and representations of symplectic groups*, J. reine angew. Math., to appear.
[17] J. Huang and C. Zhu, *On certain small representations of indefinite orthogonal groups*, Representation Theory **1** (1997), 190–206.

[18] A. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton, NJ, 1986.
[19] A. Knapp and D. Vogan, *Cohomological Induction and Unitary Representations*, Princeton University Press, Princeton, NJ, 1995.
[20] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Inven. Math. 44 (1978), 1–47.
[21] S. Kudla, *On the local theta correspondence*, Invent. Math. **83** (1986), 229–255.
[22] S. Kudla and S. Rallis, *Degenerate Principal Series and Invariant Distributions*, Isr. J. Math. **69** (1990), 25-45.
[23] S. Lee and H. Loke, *Degenerate principal series representations of $U(p,q)$ and $Spin_0(p,q)$*, preprint.
[24] S. Lee and C. Zhu, *Degenerate principal series and local theta correspondence*, Trans. Amer. Math. Soc. **350** (1998), no. 12, 5017–5046.
[25] ———, *Degenerate principal series and local theta correspondence II*, Israel J. Math. **100** (1997), 29–59.
[26] J.-S. Li, *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), 237–255.
[27] ———, *On the classification of irreducible low rank unitary representations of classical groups*, Comp. Math. **71** (1989), 29–48.
[28] ———, *Local theta lifting for unitary representations with nonzero cohomology*, Duke Math. J. **61** (1990), 913–937.
[29] J.-S. Li, A. Paul, E.-C. Tan, C. Zhu, *The explicit duality correspondince of $(Sp(p,q), O^*(2n))$*, preprint.
[30] C. Moeglin, *Correspondance the Howe pour les paires reductives duales; quelques calculs dans le cas archimédian*, J. Funct. Anal. **85** (1989), 1–85.
[31] R. Parthasarathy, *Criteria for the unitarizability of some highest weight modules*, Proc. Indian Acad. Sci. **89** (1980), 1–24.
[32] A. Paul, *Howe correspondence for real unitary groups*, J. Funct. Anal. **159** (1998), 384–431.
[33] ———, *Howe correspondence for real unitary groups II*, Proc. Amer. Math. Soc. **128**(2000), 3129–3136.
[34] ———, *On the Howe correspondence for symplectic-orthogonal dual pairs*, preprint.
[35] A. Paul and E. C. Tan, *On the dual pairs $(O(p,q), SL(2,\mathbb{R}))$, $(U(p,q), U(1,1))$, and $(Sp(p,q), O^*(4))$*, Pac. J. Math. **187** no. 2 (1999), 349–378.
[36] A. Paul and P. Trapa, *One-dimensional representations of $U(p,q)$ and the Howe correspondence*, J. Funct. Anal. **195** no.1 (2002), 129–166.
[37] ———, *Some small unipotent representations of indefinite orthogonal groups and the theta correspondence*, preprint.
[38] T. Przebinda, *On Howe's duality theorem*, J. Funct. Anal. **81**(1988), 160–183.
[39] ———, *The duality correspondence of infinitesimal characters*, Colloq. Math. **70**(1996), no. 1, 93–102.
[40] ———, *Characters, dual pairs, and unitary representations.* Duke Math. J., **69**(1993), no. 3, 547–592.
[41] R. Rao, *On some explicit formulas in the theory of the Weil representation*, Pac. J. Math. **157** (1993), no. 2, 335–371.
[42] B. Roberts, *The Weil representation and dual pairs*, Notes from a workshop on the theta correspondence at the University of Maryland, May 1994.
[43] S. Salamanca-Riba, *On the unitary dual of real reductive Lie groups and the $A_{\mathfrak{q}}(\lambda)$-modules: the strongly regular case*, Duke Math. J. **96**(1999), no. 3, 521–546.
[44] B. Speh and D. Vogan, *Reducibility of Generalized Principal Series Representations*, Acta Math. **145**(1980), 227–299.
[45] P. Trapa, *Annihilators and associated varieties of $A_{\mathfrak{q}}(\lambda)$ modules for $U(p,q)$*, Compositio Math., to appear.
[46] D. Vogan, *Representations of Real Reductive Lie Groups*, Progess in Math. **15**(1981), Birkhäuser(Boston).
[47] D. Vogan, *Unitarizability of certain series of representations*, Ann. Math. **120**(1984), 141–187.
[48] D. Vogan, *Unitary representations of reductive Lie groups*, Annals of Mathematical Studies **118**(1987), Princeton University Press (Princeton).
[49] D. Vogan and G. Zuckerman, *Unitary representations with non-zero cohomology*, Compositio Math. **53**(1984), 51–90.
[50] C. Zhu, *Representations with scalar K-types and applications*, preprint.

Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008
E-mail address: annegret.paul@wmich.edu

Department of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: ptrapa@math.utah.edu