LIFTING OF CHARACTERS ON ORTHOGONAL AND METAPLECTIC GROUPS

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§1. Introduction.

An important principle in representation theory and automorphic forms is that of lifting or transfer of representations between reductive algebraic groups. Endoscopic transfer and base change are primary examples. Another type of example is provided by theta-lifting between members of a reductive dual pair. In this paper we study lifting, defined directly on characters, between special orthogonal groups $SO(2n+1)$ over $\mathbb{R}$ and the non-linear metaplectic group $\widetilde{Sp}(2n,\mathbb{R})$. This is closely related both to endoscopy and theta-lifting, and is an aspect of the duality between root systems of type B_n and C_n.

Let π be an irreducible representation of $SO(p,q)$, the special orthogonal group of a symmetric bilinear form in $p+q=2n+1$ real variables; π has a non-zero theta-lift to a representation π' of $\widetilde{Sp}(2n,\mathbb{R})$. A natural question is: what is the relationship, if any, between the global characters of π and π'? When $n=1$ this is closely related to the Shimura correspondence, which has been the subject of extensive study.

Evidence for such a relation is provided by the orbit correspondence, which provides a matching of semisimple conjugacy classes of $SO(p,q)$ and $Sp(2n,\mathbb{R})$. This is analogous to the matching of stable conjugacy classes in the theory of endoscopy. In fact there is a natural bijection between (strongly) regular, semisimple, stable conjugacy classes in the split groups $SO(n+1,n)$ and $Sp(2n,\mathbb{R})$. In elementary terms two such conjugacy classes correspond if they have the same non-trivial eigenvalues. Alternatively there is a bijection between conjugacy classes of Cartan subgroups in these two groups.

The main ideas are best illustrated by the example of the discrete series. Let $\pi_{SO}(\lambda)$ be a discrete series representation of $SO(n+1,n)$. We have fixed a compact Cartan subgroup T, and $\lambda\in\mathfrak{t}^*$ is a Harish-Chandra parameter. In the usual coordinates

$$\lambda=(a_1,\dots,a_k;b_1,\dots,b_\ell) \tag{1.1)(a}$$

with $a_i,b_j\in\mathbb{Z}+\frac12, a_1>\dots>a_k>0, b_1>\dots>b_\ell>0$.

Fix a compact Cartan subgroup T' of $Sp(2n,\mathbb{R})$, with inverse image $\tilde{T}'$ in $\widetilde{Sp}(2n,\mathbb{R})$. The theta-lift of $\pi_{SO}(\lambda)$ to $\widetilde{Sp}(2n,\mathbb{R})$ is the discrete series representation $\pi_{Sp}(\lambda')$ with Harish-Chandra parameter

$$\lambda'=(a_1,\dots,a_k,-b_\ell,\dots,-b_1)\in\mathfrak{t}'^*. \tag{b}$$

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The character $\Theta_{SO}(\lambda)$ of $\pi_{SO}(\lambda)$ has the following formula on T (all such formulas are on the regular elements):

$$\Theta_{SO}(\lambda)(t) = \frac{\sum_{w\in W_K}\operatorname{sgn}(w)e^{w\lambda}(t)}{D_{SO}(t)} \quad (t\in T). \tag{1.2)(a}$$

Here W_K is the Weyl group of T in $K\simeq S(O(n+1)\times O(n))$, and $D=\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})$ is a Weyl denominator, which depends on a choice of positive roots Δ^+. (The usual steps must be taken to interpret this formula since D_{SO} does not factor to T.)

The character formula for $\pi_{Sp}(\lambda')$ on $\tilde{T}'$ is similar:

$$\Theta_{Sp}(\lambda')(t') = \frac{\sum_{w\in W_{K'}}\operatorname{sgn}(w)e^{w\lambda'}(t')}{D_{Sp}(t')} \quad (t'\in\tilde{T}') \tag{b}$$

where $K'\simeq U(n)$ and the Weyl denominator is for $Sp(2n,\mathbb{R})$.

Now T and T' are isomorphic, and it makes sense to compare (a) and (b), at least on the Lie algebras so the issue of covering groups does not arise. We choose an isomorphism $\phi_T: T\to T'$ such that $d\phi_T^*:\lambda'\to\lambda$; then t corresponds to $\phi_T(t)$ under the correspondence of stable conjugacy classes. Because of the difference between W_K and $W_{K'}$ it is natural to replace $\pi_{SO}(\lambda)$ and $\pi_{Sp}(\lambda')$ by stable sums of discrete series. Therefore let $\overline{\pi}_{SO}(\lambda)=\sum_{w\in W_K\backslash W}\pi_{SO}(w\lambda)$, where $W=W(\mathfrak{so}(2n+1,\mathbb{C}),\mathfrak{t})\simeq W(\mathfrak{sp}(2n,\mathbb{C}),\mathfrak{t}')$ is the Weyl group of type B_n/C_n. The character of $\overline{\pi}_{SO}(\lambda)$ on T is

$$\overline{\Theta}_{SO}(\lambda)(t) = \frac{\sum_{w\in W}\operatorname{sgn}(w)e^{w\lambda}(t)}{D_{SO}(t)} \quad (t\in T) \tag{1.3)(a}$$

and similarly $\overline{\pi}_{Sp}(\lambda')=\sum_{w\in W_{K'}\backslash W}\pi_{Sp}(w\lambda')$ has formula

$$\overline{\Theta}_{Sp}(\lambda')(t') = \frac{\sum_{w\in W}\operatorname{sgn}(w)e^{w\lambda'}(t')}{D_{Sp}(t')} \quad (t'\in\tilde{T}'). \tag{b}$$

Roughly speaking (b) may be obtained from (a) (using $\phi_T: T\simeq T'$) by multiplying by

$$\Phi=\frac{D_{SO}}{D_{Sp}}. \tag{1.4)(a}$$

This a transfer factor of the type considered in endoscopy [23]. More precisely, suppose t' is a regular element of $\tilde{T}'$, with image $p(t')\in T'$ ($p:\widetilde{Sp}(2n,\mathbb{R})\to Sp(2n,\mathbb{R})$). Let $t=\phi_T^{-1}(t')\in T$ and define

$$\Phi(t')=\frac{D_{SO}(t)}{D_{Sp}(t')}. \tag{b}$$

Although D_{Sp} factors to T', D_{SO} is only well-defined on a cover of T. Nevertheless $\Phi(t')$ is a well-defined function on $\tilde{T}'$. In fact, let $\omega=\omega_{even}\oplus\omega_{odd}$ be the oscillator representation, with its decomposition into two irreducible components, and let $\Omega = \Omega_{even} + \Omega_{odd}$ be its character, considered as a function on the regular semisimple elements. The key observation (cf. [1]) is:

$$\pm\Phi(t') = \Omega_{even}(t') - \Omega_{odd}(t') \quad (t' \in \tilde{T}') \tag{1.5}$$

and we conclude

$$\overline{\Theta}_{Sp}(\lambda')(t') = \overline{\Theta}_{SO}(\lambda)(t)\Phi(t'). \tag{1.6}$$

The problem remains to extend (1.6) to other Cartan subgroups. For each Cartan subgroup H of $SO(n+1,n)$ we choose an isomorphism $\phi_H : H \to H'$ with a Cartan subgroup H' of $Sp(2n,\mathbb{R})$ (satisfying certain conditions, essentially that ϕ_H is conjugate over the complex groups to ϕ_T). We *define* Φ by (1.5): for g' a regular element of H' let

$$\Phi(g') = \Omega_{even}(g') - \Omega_{odd}(g'). \tag{1.7}$$

This is a reasonable definition: let D_{SO} and D_{Sp} be Weyl denominators on H and H' respectively. These depend on choices of positive roots, and in the case of D_{SO} is only defined on a covering group. However their absolute values are well-defined, and (Theorem 8.2)

$$|\Phi(g')| = \frac{|D_{SO}(g)|}{|D_{Sp}(g')|} \tag{1.8}$$

where $g = \phi_H^{-1}(p(g'))$. We emphasize that the character of the *difference* of the two halves of the oscillator representation satisfies this identity, not the sum. The analogue of (1.6) now holds on all Cartan subgroups. The main result (see below) is that the same statement holds for any stably invariant eigendistribution of $SO(n+1,n)$.

The analogue of (1.8) for endoscopic groups is one of the main requirements of a set of transfer factors ([29],§3). Definition (1.7) has the advantage that it defines transfer factors for all Cartan subgroups simultaneously.

We now state the main result (Theorem 4.6). Some of the terms here remain to be defined, for example stability for $\widetilde{Sp}(2n,\mathbb{R})$. See section 4 for details.

Let Θ be a stable invariant eigendistribution on $SO(n+1,n)$, considered as a function on the (strongly) regular semisimple elements. Define a function Θ' on the regular semisimple elements of $\widetilde{Sp}(2n,\mathbb{R})$ as follows. Fix a regular semisimple element $g' \in \widetilde{Sp}(2n,\mathbb{R})$. Choose $g \in SO(n+1,n)$ corresponding to $p(g') \in Sp(2n,\mathbb{R})$ via the correspondence of stable conjugacy classes. Equivalently let H' be the centralizer of $p(g')$ in $Sp(2n,\mathbb{R})$, choose an isomorphism $\phi_H : H \to H'$ as above, and let $g = \phi^{-1}(p(g'))$. Define

$$\Theta'(g') = \Theta(g)\Phi(g'). \tag{1.9}$$

This is independent of the choices since Θ is stable.

Theorem.

(1) The map $\Gamma : \Theta \to \Theta'$ is a bijection between stably invariant eigendistributions on $SO(n+1,n)$, and genuine stably invariant eigendistributions on $\widetilde{Sp}(2n,\mathbb{R})$,

(2) Γ *restricts to a bijection of stable virtual characters,*

(3) Γ *restricts to a bijection of stable tempered invariant eigendistributions,*

(4) Suppose Θ *is the character of a stable tempered virtual representation* π *of* $SO(n+1,n)$*. Consider* π *to be defined for all real forms of* $SO(2n+1,\mathbb{C})$ *simultaneously (cf. §2). Then* Θ' *is the character of the theta-lift of* π*.*

Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{R}$, with real points G. Write ${}^\vee G$ for the (connected, complex) dual group of G. Recall an endoscopic group H for G is (in part) a quasisplit group such that ${}^\vee H$ is the centralizer of a semisimple element s of ${}^\vee G$. In particular if s is central $H = G_{qs}$, the quasi-split inner form of G, and in this case endoscopic transfer takes stable distributions on H to stable distributions on G. The preceding theorem may be interpreted as realizing $SO(n+1,n)$ as an endoscopic group for $\widetilde{Sp}(2n,\mathbb{R})$—analogous to the quasi-split inner form, since Γ preserves stability.

The dual group of $Sp(2n,\mathbb{R})$ is $SO(2n+1,\mathbb{C})$, so $SO(n+1,n)$ is not an endoscopic group for $Sp(2n,\mathbb{R})$ (it is on the dual side). Interpreting $SO(n+1,n)$ as an endoscopic group for $\widetilde{Sp}(2n,\mathbb{R})$ suggests that for non-linear groups the roles of the group and dual group are combined in some way that is not understood. Heuristically it makes sense to consider $SO(n+1,n)$ to be an endoscopic group defined by a central element of $\widetilde{Sp}(2n,\mathbb{R})$ lying over $-1 \in Sp(2n,\mathbb{R})$ (4.12).

In the theory of endoscopy, transfer factors are used to define a map $f \to f_H$ between smooth compactly supported functions on G and H. This is compatible with a matching of orbital integrals. Dual to this is a transfer of characters, which is shown to satisfy character formulas on each Cartan subgroup. We have proceeded in reverse order, defining the lifting of characters directly. Our lifting is characterized by the condition that the characters of an invariant eigendistribution on $SO(n+1,n)$ and its lift to $\widetilde{Sp}(2n,\mathbb{R})$ have the same Weyl numerator on each Cartan subgroup (Proposition 9.3). It would be interesting to obtain a map $f \to f_H$ and matching of orbital integrals as a consequence. It should also be possible to define $f \to f_H$ directly.

The precise statement of the main theorem is Theorem 4.6. A sketch of the proof is found there — it is an application of Hirai's matching conditions and the induced character formula. The proof takes up sections 5-12. In Section 13 we discuss the case of complex groups. The map Γ commutes with coherent continuation; this is discussed, together with some consequences, in section 14.

The case of $n = 1$, in which most of the main features of the argument may be seen, is discussed in detail in section 15. This is the case of the Shimura correspondence, and related character formulas appear in [9].

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§2. Orbit Correspondence.

In this section we describe the matching of semisimple elements which underlies character lifting. This is analogous to the matching of conjugacy classes in a group G and an endoscopic group H [23].

Let V be a vector space of dimension $2n+1$, equipped with a non-degenerate symmetric bilinear form $(\,,)$, and let W be a vector space of dimension $2n$, with non-degenerate symplectic form $<,>$. Let $G = SO(V)$ and $G' = Sp(W)$ be the isometry groups of the forms, with Lie algebras $\mathfrak{g}_0 = \mathfrak{so}(V)$ and $\mathfrak{g}_0' = \mathfrak{sp}(W)$. The orbit map ([19], [25]) defines a correspondence between orbits (under the adjoint action) in $\mathfrak{g}_0$ and $\mathfrak{g}_0'$. We recall the definition.

Let $\mathbf{W} = V \otimes W$. Identifying $V^* = \mathrm{Hom}(V, \mathbb{R})$ with V via $(,)$ gives an isomorphism $\mathbf{W} \simeq \mathrm{Hom}(V, W)$. For $T \in \mathrm{Hom}(V, W)$ define $T^* \in \mathrm{Hom}(W, V)$ by $< Tv, w >= (v, T^*w)$. Then $\alpha : T \to T^*T$ defines a map from $\mathbf{W}$ to $\mathfrak{g}_0$, identified as the set of elements X in $\mathrm{Hom}(V, V)$ satisfying

$$(Xv, v') + (v, Xv') = 0. \tag{2.1)(a}$$

Similarly $\alpha'(T) = TT^* \in \mathfrak{g}_0'$, where $\mathfrak{g}_0'$ consists of the elements of $\mathrm{Hom}(W, W)$ satisfying

$$< Xv, w > + < v, Xw >= 0. \tag{b}$$

This defines the graph of a correspondence: we say $X \in \mathfrak{g}_0$ corresponds to $X' \in \mathfrak{g}_0'$ if there exists $T \in \mathbf{W}$ such that $\alpha(T) = X, \alpha'(T) = X'$. This is equivariant for the actions of G and G', and defines a correspondence of orbits. We write $X \leftrightarrow X'$, and if $\mathcal{O} = G \cdot X, \mathcal{O}' = G' \cdot X'$ write $\mathcal{O} \leftrightarrow \mathcal{O}'$.

If $X \leftrightarrow X'$ then X is of maximal rank if and only if X' is of maximal rank (i.e. 2n). If this is the case Witt's theorem implies

$$\alpha^{-1}(X) \text{ is a single } G'\text{-orbit}, \tag{2.2}$$

so $\mathcal{O} = G \cdot X$ corresponds to a single G'-orbit $\mathcal{O}'$, and vice versa. Therefore the orbit correspondence is a bijection when restricted to orbits of maximal rank.

If v is an eigenvector of T^*T with non-zero eigenvalue λ (over $\mathbb{C}$), then

$$(TT^*)(Tv) = T(T^*Tv) = \lambda(Tv) \neq 0, \tag{2.3}$$

so λ is an eigenvalue of TT^*. The semisimple elements of maximal rank are precisely the regular semisimple elements. It follows that the orbit correspondence also preserves regular semisimple elements. Furthermore if $X \leftrightarrow X'$ are regular and semisimple, then X and X' have the same non-zero eigenvalues (X in addition has the eigenvalue 0 with multiplicity one). This fact is useful for computing the orbit correspondence explicitly.

We now collect the real forms of $SO(2n+1, \mathbb{C})$ as in ([5],§1). We specialize to the present case, and thereby avoid some of the subtleties of [5]. In particular since $SO(2n+1, \mathbb{C})$ is adjoint, the notions of real form and strong real form coincide. The groups $SO(p, q)$ with $p + q = 2n + 1$ and the parity of q fixed form a set of representatives of the equivalence classes of (strong) real forms of $SO(2n+1, \mathbb{C})$.

Definition 2.4. *Fix $\delta = \pm 1$, and a set $\mathcal{V} = \{V\}$ of representatives of the equivalence classes of orthogonal spaces of dimension $2n+1$ and discriminant δ. In other words $\mathcal{V}$ is parameterized by signatures (p, q), $p+q = 2n+1$ and $(-1)^q = \delta$. Write $V = V_{p,q}$ and $SO(V) = SO(V_{p,q}) = SO(p, q)$. Then $SO(2n + 1)$ will stand for the set of groups $SO(V)$ ($V \in \mathcal{V}$), i.e. $\{SO(p, q) \,|\, p + q = 2n + 1, (-1)^q = \delta\}$. A representation π (resp. conjugacy class $\mathcal{C}$) is a pair (V, π_V) (resp. $(V, \mathcal{C}_V)$) with $V \in \mathcal{V}$ and π_V (resp. $\mathcal{C}_V$) a representation (resp. conjugacy class) of $SO(V)$.*

Proposition 2.5. *The orbit correspondence is a bijection between the regular semisimple adjoint orbits of $Sp(2n,\mathbb{R})$ and $SO(2n+1)$.*

Proof. This uses ideas from [19]. Fix $V\in\mathcal{V}$. For $X\in\mathfrak{so}(V)$, by (2.1)(a) $< v,v' >_X \overset{def}{=} (v,Xv')$ is a skew-symmetric form on V. The radical of $<,>_X$ is $Ker(X)$, so $<,>_X$ is a non-degenerate symplectic form on $V/Ker(X)$. Now suppose X has maximal rank, i.e. $2n$. Then $dim(V/Ker(X))=2n$, and there is an isomorphism of symplectic spaces $\phi: V/Ker(X)\rightarrow W$. Let $T=\phi\circ p: V\rightarrow W$ where p is projection of V on $V/Ker(X)$. Then for all $v,v'\in V$, $< Tv,Tv' >=< v,v' >_X$, and

$$\begin{aligned}(v,Xv') &=< v,v' >_X\\ &=< Tv,Tv' >\\ &=(v,T^*Tv')\end{aligned}\tag{2.6}$$

so $X=T^*T$. Therefore α maps onto the regular semisimple elements of $\mathfrak{so}(V)$, and the correspondence is a bijection between all regular semisimple orbits of $\mathfrak{so}(V)$ and a subset of the regular semisimple orbits of $\mathfrak{sp}(W)$.

Now suppose $Y\in\mathfrak{sp}(W)$. Then $(w,w')_Y\overset{def}{=}< Yw,w' >$ is a symmetric bilinear form on W, non-degenerate if Y is of maximal rank. Let V be the orthogonal space $(W,(,)_Y)\oplus V_0$ (orthogonal direct sum) where V_0 is a one-dimensional non-degenerate orthogonal space. Let $S:W\rightarrow V$ be inclusion in the first factor, so $(Sw,Sw')=(w,w')_Y$. It follows as in (2.6) that $Y=S^*S$, where S^* is defined by $< S^*v,w >=(v,Sw)$. Letting $T=S^*$ and noting $S^{**}=S$ we conclude $Y=TT^*$.

Therefore Y is in the image of the orbit map for two isomorphism classes of V of opposite discriminant, depending on the two choices of V_0 (positive or negative definite, up to isomorphism). In fact these are the only choices of V. For suppose $Y\leftrightarrow X'$ for some V', i.e. $Y=TT^*$ for some $T\in\mathrm{Hom}(V',W)$. Then $w\rightarrow T^*w$ is an isomorphism of W with $Im(T^*)\subset V'$, taking the form $(,)_Y$ to the restriction of the form on V'. Therefore Y determines the form on V up to isomorphism on a space of codimension one, and the claim follows. This completes the proof.

This information is enough to compute the orbit correspondence explicitly (at least on regular semisimple elements). We make this explicit in §7, once we have chosen coordinates on Cartan subgroups. We now stabilize the orbit correspondence.

Let $W_{\mathbb{C}}=W\otimes\mathbb{C}$, and let $Sp(2n,\mathbb{C})=Sp(W_{\mathbb{C}})$. Fix a non-degenerate orthogonal space $V^0_{\mathbb{C}}$ of dimension $2n+1$ and let $SO(2n+1,\mathbb{C})=SO(V^0_{\mathbb{C}})$. For $V\in\mathcal{V}$ let $V_{\mathbb{C}}=V\otimes\mathbb{C}$ and choose an isomorphism $V_{\mathbb{C}}\rightarrow V^0_{\mathbb{C}}$ of orthogonal spaces. This induces an embedding $\iota_V: SO(V)\rightarrow SO(2n+1,\mathbb{C})$. Any two such maps are conjugate by $SO(2n+1,\mathbb{C})$.

Definition 2.7.
(1) Let X,X' be regular semisimple elements of $\mathfrak{sp}(W)$. We say X,X' are in the same stable orbit if $X'=Ad(g)X$ for some $g\in Sp(2n,\mathbb{C})$.

(2) Let $V,V'\in\mathcal{V}$ (Definition 2.6). Let $X\in\mathfrak{so}(V),X'\in\mathfrak{so}(V')$ be regular semisimple elements. We say X,X' are in the same stable orbit of $SO(2n+1)$ if $d\iota_V(X')=Ad(g)d\iota_{V'}(X')$ for some $g\in SO(2n+1,\mathbb{C})$.

Condition (2) is independent of the choice of $\iota_V,\iota_{V'}$.

Lemma 2.8. *The stable orbit of a semisimple element of* $\mathfrak{so}(V)$ *or* $\mathfrak{sp}(W)$ *is determined by its (unordered) set of eigenvalues.*

Proof. This is just the fact that a semisimple orbit for $SO(2n+1,\mathbb{C})$ or $Sp(2n,\mathbb{C})$ is determined by its eigenvalues.

Let $\mathcal{O}_{st}$ be a stable orbit of $SO(2n+1)$. Let $\mathcal{O}'_{st} = \cup_{X\in\mathcal{O}_{st}}\{X' \mid X \leftrightarrow X'\}$ — this is a union of stable orbits. By the lemma $\mathcal{O}'_{st}$ is a single orbit. Therefore:

Proposition 2.9. *The orbit correspondence induces a bijection between the set of stable regular semisimple adjoint orbits of* $SO(2n+1)$ *and* $Sp(W)$*. Two such orbits correspond if and only if they have the same (non-zero) eigenvalues.*

We write $\mathcal{O}_{st} \overset{stable}{\longleftrightarrow} \mathcal{O}'_{st}$ for the stabilized orbit correspondence.

We now extend the orbit correspondence to the groups. One method is to use the exponential map. This has the disadvantage of not being surjective, even onto the regular semisimple elements. Instead we use the Cayley transform ([19], [25]).

Let $G = SO(V)$ or $Sp(W)$, and $\mathfrak{g}_0 = Lie(G)$. For $X \in G$ or $\mathfrak{g}_0$ such that $1+X$ is invertible, define

$$C(X) = (1-X)(1+X)^{-1}. \tag{2.10}$$

Then $C : \mathfrak{g}_0 \to G$, $C : G \to \mathfrak{g}_0$, and $C^2 = Id$, so C is a bijection between subsets of G and $\mathfrak{g}_0$. It is equivariant for the adjoint action of G on $\mathfrak{g}_0$ and conjugation on G. Finally X is semisimple if and only if $C(X)$ is semisimple, so C defines a bijection between semisimple adjoint orbits and conjugacy classes.

Note that if X is contained in a Cartan subalgebra $\mathfrak{h}_0$ then $C(X)$ is contained in the corresponding Cartan subgroup $H = Cent(\mathfrak{h}_0)$. Furthermore if X is semisimple, with eigenvalues λ_i then $C(X)$ has eigenvalues $(1-\lambda_i)(1+\lambda_i)^{-1}$.

Recall [30] a semisimple element of a connected reductive algebraic group $\mathbf{G}$ is strongly regular if its centralizer $\mathbf{H}$ is a Cartan subgroup, and regular if the identity component of $\mathbf{H}$ is a Cartan subgroup. Strongly regular implies regular, and the converse holds if $\mathbf{G}$ is simply connected.

A regular element g of $SO(2n+1,\mathbb{C})$ is strongly regular if and only if -1 is not an eigenvalue of g. For $Sp(2n,\mathbb{C})$ a regular element cannot have -1 as an eigenvalue. Therefore the Cayley transform maps onto the strongly regular semisimple elements of $SO(V)$ and $Sp(W)$.

Definition 2.11. We say strongly regular semisimple elements g of $SO(V)$ and g' of $Sp(W)$ correspond, written $g \leftrightarrow g'$, if $C(g) \leftrightarrow C(g')$. This defines a correspondence of conjugacy classes, written $\mathcal{C} \leftrightarrow \mathcal{C}'$.

The next result follows immediately from Proposition 2.5.

Proposition 2.12. *The correspondence of conjugacy classes is a bijection between the strongly regular semisimple conjugacy classes of* $SO(2n+1)$ *and* $Sp(W)$*.*

This result is a little misleading: this correspondence is somewhat unnatural, and does not lend itself to lifting of characters.

Example 2.13. Let $X_{3,0}(\theta)$ or $X_{1,2}(\theta)$ equal $\begin{pmatrix} 0 & \theta & \\ -\theta & 0 & \\ & & 0 \end{pmatrix}$, considered as an element of $\mathfrak{so}(3,0)$ or $\mathfrak{so}(1,2)$ respectively. (The orthogonal groups are defined with respect to the forms I and $diag(-1,-1,1)$ respectively.) Let $X_{Sp}(\theta)=\begin{pmatrix}0&\theta\\-\theta&0\end{pmatrix}\in\mathfrak{sl}(2,\mathbb{R})$. Then

$$X_{Sp}(\theta)\leftrightarrow\begin{cases}X_{3,0}(\theta)&\theta>0\\X_{1,2}(\theta)&\theta<0.\end{cases}\tag{2.14(a)}$$

For $(i,j)=(3,0)$ or $(1,2)$ let

$$\begin{aligned}g_{i,j}(\theta)&=exp(X_{i,j}(\theta))=\begin{pmatrix}\cos(\theta)&\sin(\theta)&\\-\sin(\theta)&\cos(\theta)&\\&&1\end{pmatrix},\\g_{Sp}(\theta)&=exp(X_{Sp}(\theta))=\begin{pmatrix}\cos(\theta)&\sin(\theta)\\-\sin(\theta)&\cos(\theta)\end{pmatrix}.\end{aligned}\tag{b}$$

Write the regular elements of the compact Cartan subgroup T of $SL(2,\mathbb{R})$ as $T_+\cup T_-$ where $T_\pm=\{g_{Sp}(\theta)\mid\pm\theta>0\}$. Then

$$g_{Sp}(\theta)\leftrightarrow\begin{cases}g_{3,0}(\theta)&g\in T_+\\g_{1,2}(\theta)&g\in T_-.\end{cases}\tag{c}$$

The numerator of the character of an invariant eigendistribution is analytic on T. In particular its values on T_- are determined by its values on T_+. This suggests identifying the compact Cartan subgroups of $SO(3,0)$ and $SO(1,2)$, which is precisely what stable conjugacy accomplishes. Therefore the matching of stable conjugacy classes of the next theorem is the crucial one.

Recall strongly regular elements g,g' of the real points $G=\mathbf{G}(\mathbb{R})$ of $\mathbf{G}$ are said to be stably conjugate if they are conjugate in $\mathbf{G}(\mathbb{C})$, and a stable conjugacy class is an equivalence class for this relation. Stable conjugacy classes for real forms of $SO(2n+1)$ considered simultaneously are defined analogously to stable adjoint orbits (Definition 2.7). That is strongly regular semisimple elements $g\in SO(V),g'\in SO(V')$ are defined to be stably conjugate if $\iota_V(g)$ is conjugate to $\iota_{V'}(g')$ via $SO(2n+1,\mathbb{C})$. As in Lemma 2.8 we have

$$g\text{ is stably conjugate to }g'\Leftrightarrow g,g'\text{ have the same eigenvalues.}\tag{2.15}$$

Theorem 2.16. *The correspondence of conjugacy classes induces a bijection of stable (strongly regular, semisimple) conjugacy classes. Two stable conjugacy classes correspond if and only if they have the same eigenvalues.*

Explicitly let $\mathcal{C}_{st}$ be stable conjugacy class for $SO(2n+1)$. Let $\mathcal{C}'_{st}=\cup_{g\in\mathcal{C}_{st}}\{g'\in Sp(W)\mid g\leftrightarrow g'\}$. Then $\mathcal{C}'_{st}$ is a single stable conjugacy class, and $\mathcal{C}_{st}\leftrightarrow\mathcal{C}'_{st}$ is a bijection. We write $g\overset{stable}{\longleftrightarrow}g'$ and $\mathcal{C}_{st}\overset{stable}{\longleftrightarrow}\mathcal{C}'_{st}$, and refer to this as the correspondence of stable conjugacy classes.

Proof. This follows immediately from Proposition 2.9.

Remark 2.17. Theorem 2.16 could be taken as the definition of the correspondence.

Remark 2.18. The elements $diag(1,-1,-1)$ of $SO(2,1)$ and $-I$ of $SL(2,\mathbb{R})$ correspond. The former element is regular but not strongly regular, whereas the latter is not regular. Hence the correspondence preserves strong regularity, but not regularity. This is another reason to restrict to strongly regular semisimple elements.

§3. Stable Conjugacy (Continued).

We have defined stable conjugacy for $SO(2n+1)$. This is a minor variation of the usual definition since we have grouped together different real forms. We now define stable conjugacy of regular semisimple elements of $\widetilde{Sp}(2n,\mathbb{R})$, which is non-standard since $\widetilde{Sp}(2n,\mathbb{R})$ is not a linear group.

We first recall results of ([28],§2). Let g, g' be strongly regular elements of a connected reductive algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. If g, g' are stably conjugate, after conjugating by $G = \mathbf{G}(\mathbb{R})$ if necessary, we may assume g, g' are contained in a single Cartan subgroup $H = \mathbf{H}(\mathbb{R})$. Let $\mathbf{A}$ be the maximal $\mathbb{R}$-split torus in $\mathbf{H}$, and let $\mathbf{M} = Cent_{\mathbf{G}}(\mathbf{A})$. Let $W(\mathbf{M},\mathbf{H}) \simeq W(\mathfrak{m},\mathfrak{h})$ be the Weyl group of $\mathbf{H}$ in $\mathbf{M}$, and $W(G,H)$ the Weyl group of H in G. Then $W(\mathbf{M},\mathbf{H})$ acts on $\mathfrak{h}_0$, and this action exponentiates to H. Furthermore g is stably conjugate to g' if and only if $g = wg'$ for some $w \in W_{st}(G,H)$ where

$$W_{st}(G,H) = W(\mathbf{M},\mathbf{H})W(G,H). \tag{3.1}$$

Therefore the "additional" conjugacy is that of $W(\mathbf{M},\mathbf{H})$. Note that $W(G,H)$ normalizes $W(\mathbf{M},\mathbf{H})$ and there is an exact sequence

$$1 \to W(M,H) \to W(G,H) \ltimes W(\mathbf{M},\mathbf{H}) \to W_{st}(G,H) \to 1. \tag{3.2}$$

Now let H be a Cartan subgroup of $Sp(2n,\mathbb{R})$, with $\mathbf{A}$ and $\mathbf{M}$ as above. Let $\tilde{H}$ be the inverse image of H in $\widetilde{Sp}(2n,\mathbb{R})$ under the projection map $p : \widetilde{Sp}(2n,\mathbb{R}) \to Sp(2n,\mathbb{R})$; this is a Cartan subgroup of $\widetilde{Sp}(2n,\mathbb{R})$ (cf. §6).

Lemma 3.3. *The action of $W(\mathbf{M},\mathbf{H})$ on $\mathfrak{h}$ exponentiates to an action on $\tilde{H}$.*

Proof. This follows from ([28], Proposition 2.2). It also follows from the explicit information of Lemma 7.4.

We say a semisimple element g of $\widetilde{Sp}(2n,\mathbb{R})$ is regular if its image in $Sp(2n,\mathbb{R})$ is regular. We define strongly regular to be equivalent to regular, since this holds for $Sp(2n,\mathbb{R})$.

Definition 3.4. *Stable conjugacy of (strongly) regular semisimple elements of $\widetilde{Sp}(2n,\mathbb{R})$ is the equivalence relation generated by relations:*

(a) *conjugacy by $\widetilde{Sp}(2n,\mathbb{R})$,*
(b) *the action of $W(\mathbf{M},\mathbf{H})$ on $\tilde{H}$ (Lemma 3.3).*

In other words letting

$$W_{st}(\widetilde{Sp}(2n,\mathbb{R}),\tilde{H}) = W(\widetilde{Sp}(2n,\mathbb{R}),\tilde{H})W(\mathbf{M},\mathbf{H}), \tag{3.5}$$

(a subgroup of the automorphism group of $\tilde{H}$), two strongly regular elements g, g' of $\tilde{H}$ of $\widetilde{Sp}(2n,\mathbb{R})$ are stably conjugate if and only if they are conjugate by $W_{st}(\widetilde{Sp}(2n,\mathbb{R}),\tilde{H})$.

For $G = SO(V)$ or $\widetilde{Sp}(2n,\mathbb{R})$ let G_0 be the strongly regular semisimple elements of G. This is an open and dense set. By [12] an invariant eigendistribution on G is represented by an invariant function on G_0. An invariant eigendistribution on $SO(2n+1)$ is represented by a function on $SO(2n+1)_0 = \cup_{V\in\mathcal{V}} SO(V)_0$.

Definition 3.6.
(1) An invariant function on $SO(2n+1)_0$ is a sum

$$\Theta = \sum_{V\in\mathcal{V}} \Theta_V \tag{3.7}$$

of invariant functions Θ_V on $SO(V)_0$ (cf. Definition 2.4).
(2) An invariant function on $SO(2n+1)_0$ or $\widetilde{Sp}(2n,\mathbb{R})_0$ is stably invariant if it is constant on stable conjugacy classes.
(3) An invariant eigendistribution on $G = SO(2n+1)$ or $\widetilde{Sp}(2n,\mathbb{R})$ is stably invariant if the function on G_0 representing it is stably invariant.

We identify a stably invariant eigendistribution with the function representing it.

Remark 3.8. Every (strongly regular, semisimple) stable conjugacy class for $SO(2n+1)$ contains a representative in the split group $SO(n+1,n)$ (or $SO(n,n+1)$, depending on δ). Equivalently every Cartan subgroup H of $SO(V)$ is stably conjugate to a Cartan subgroup of $SO(n+1,n)$. A stable invariant eigendistribution on $SO(2n+1)$ is therefore determined by its restriction to $SO(n+1,n)$. (The distributions on the other real forms are obtained by lifting from the split form as in [28]). Therefore we may identify a stably invariant eigendistribution on $SO(2n+1)$ with its restriction to $SO(n+1,n)$. Furthermore, isomorphic Cartan subgroups H_1, H_2 of $SO(V_1), SO(V_2)$ are identified by an isomorphism unique up to stable conjugation.

§4. Character Lifting.

Fix a stably invariant eigendistribution Θ on $SO(2n+1)$, identified as in §3 with a stably invariant function on $SO(2n+1)_0$. Define a stably invariant function $\tau(\Theta)$ on $Sp(2n,\mathbb{R})_0$ by the correspondence of stable conjugacy classes (Theorem 2.16): for $g' \in Sp(2n,\mathbb{R})_0$,

$$\tau(\Theta)(g') = \Theta(g) \qquad (g \overset{stable}{\longleftrightarrow} g'). \tag{4.1}$$

This is independent of the choice of g since Θ is stably invariant.

Definition 4.2. Fix a non-trivial unitary additive character ψ of $\mathbb{R}$. Let

$$\omega(\psi) = \omega(\psi)_{even} \oplus \omega(\psi)_{odd} \tag{4.3)(a}$$

be the corresponding oscillator representation of $\widetilde{Sp}(2n,\mathbb{R})$. Here $\omega(\psi)_{even}$ has a line invariant under a maximal compact subgroup; in the usual models $\omega(\psi)_{even}$ (resp. $\omega(\psi)_{odd}$) is realized on even (resp. odd) functions. Write

$$\Omega(\psi) = \Omega(\psi)_{even} + \Omega(\psi)_{odd} \tag{b}$$

for the character of $\omega(\psi)$. The transfer factor associated to ψ is the invariant eigendistribution

$$\Phi(\psi) = \Omega(\psi)_{even} - \Omega(\psi)_{odd} \tag{c}$$

considered as a function on $\widetilde{Sp}(2n,\mathbb{R})_0$. If ψ is understood we write $\omega = \omega(\psi), \Omega = \Omega(\psi)$ and $\Phi = \Phi(\psi)$.

By Corollary 8.10 Φ is stably invariant. By ([19], Appendix, cf. [1], 1.7), for $\tilde{g} \in \widetilde{Sp}(2n,\mathbb{R})_0$ and $g = p(\tilde{g})$,

$$\Phi(\tilde{g})^2 = \frac{\pm 1}{\det(1+g)}. \tag{4.4}$$

In particular Φ takes values in $\mathbb{R}^* \cup i\mathbb{R}^*$ and is bounded in a neighborhood of the identity. By standard results, $\Phi(a^2\psi) = \Phi(\psi)$ $(a \in \mathbb{R}^*)$ so without loss of generality we may take $\psi = \psi_\pm$ where $\psi_\pm(x) = e^{\pm 2\pi i x}$. Finally $\Phi(\overline{\psi}) = \overline{\Phi(\psi)}$, a consequence of the fact that $\omega(\psi)$ and $\omega(\overline{\psi})$ are dual.

It is worth noting that Ω, which might seem a more natural candidate for a transfer factor than Φ, is neither stable nor non-singular near the identity.

Definition 4.5. Let Θ be a stably invariant function on $SO(2n+1)_0$ (Definition 3.6). For $g' \in \widetilde{Sp}(2n,\mathbb{R})_0$ let

$$\begin{aligned}\Gamma(\psi)(\Theta)(g') &= \tau(\Theta)(p(g'))\Phi(\psi)(g') \\ &= \Theta(g)\Phi(\psi)(g') \qquad (g \overset{stable}{\longleftrightarrow} p(g')).\end{aligned}$$

If ψ is understood we write $\Gamma = \Gamma(\psi)$.

This is a stably invariant function on $\widetilde{Sp}(2n,\mathbb{R})_0$, and is genuine: $\Gamma(\Theta)(zg') = -\Gamma(\Theta)(g')$ for z the non-trivial element of $p^{-1}(1)$. We often write $\Theta' = \Gamma(\Theta)$.

By a virtual representation (resp. character) we mean a finite linear combination of irreducible representations (resp. characters) with integral coefficients. We identify a virtual representation π with its virtual character $\Theta(\pi)$. We say π is stable if $\Theta(\pi)$ is stable, and define $\Gamma(\pi) = \Gamma(\Theta(\pi))$.

Theorem 4.6.

1. *The map $\Gamma : \Theta \to \Theta' = \Gamma(\Theta)$ is a bijection between stably invariant eigendistributions on $SO(2n+1)$ and genuine stably invariant eigendistributions on $\widetilde{Sp}(2n,\mathbb{R})$.*
2. *Θ' is a virtual character if and only if Θ is a virtual character.*
3. *Θ' is tempered if and only if Θ is tempered.*
4. *Γ takes the (stable) discrete series of $SO(2n+1)$ to the (stable) genuine discrete series of $\widetilde{Sp}(2n,\mathbb{R})$.*
5. *Γ commutes with parabolic induction (cf. Theorem 12.16).*
6. *If Θ is a tempered virtual character then Θ' is the (normalized) theta-lift of Θ (Theorem 12.27).*

Remark 4.7. Using Remark 3.8, we could define Γ on stably invariant eigendistributions on the split group $SO(n+1,n)$ or $SO(n,n+1)$. Statements (1-5) of the theorem hold without change, but (6) would fail (cf. Remark 12.31).

Here is an outline of the proof. The map Γ is determined by the property that the numerators in the character formulas for Θ and Θ' are the same on all Cartan subgroups (Proposition 9.3). In section 10 we show this implies Θ satisfies Hirai's matching conditions [18] if and only if the same holds for Θ'. This proves

(1) (Theorem 10.1). Temperedness is determined by the character formulas on all Cartans, and this proves (3) (Corollary 10.13).

If Θ is a stable sum of discrete series representations, then by (3) Θ' is tempered. The character of Θ' on the compact Cartan subgroup is determined by Proposition 9.3, and by a result of Harish-Chandra this is enough to show Θ' is a stable sum of discrete series representations (Theorem 11.3), proving (4). That Γ commutes with parabolic induction is a straightforward application of the induced character formula (Theorem 12.16). The stable virtual characters are spanned by certain stable induced representations, and (2) follows from (5) (Corollary 12.22).

Finally theta-lifting from $SO(p,q)$ to $\widetilde{Sp}(2n,\mathbb{R})$ has been computed explicitly in [3], and (6) follows from this (Theorem 12.27). We note that the failure of Γ to agree with theta-lifting for some non-tempered representations is analogous to the failure of the sum of the representations in a non-tempered L-packet to be stable. A standard (full induced) module has a simple character formula, which is what is seen by Γ. On the other hand theta-lifting is defined on the irreducible quotient of a standard module, which may be proper in the non-tempered case.

We record a few simple properties of Γ. Recall [7] the *order* of an invariant eigendistribution Θ is a measure of the singularity of Θ at the identity. In particular the order of the character of an irreducible representation of Gelfand-Kirillov dimension d is $d/2$. By (4.4) Φ is bounded in a neighborhood of the identity. The next lemma is an immediate consequence of the definitions (cf. [7], Corollary 2.4).

Lemma 4.8. Θ *and* $\Gamma(\Theta)$ *have the same order.*

For example the trivial representation of $SO(2n+1)$ has Gelfand-Kirillov dimension 0. The lift to $\widetilde{Sp}(2n,\mathbb{R})$ is $\omega(\psi)_{even}-\omega(\psi)_{odd}$, the difference of the two halves of the oscillator representation. Each $\omega(\psi)_{even,odd}$ has Gelfand-Kirillov dimension n; these singularities cancel and the difference has order 0.

The dependence of Γ on ψ follows from $\Phi(\overline{\psi})=\overline{\Phi(\psi)}$ and the definitions:

$$\Gamma(\overline{\psi})(\Theta)=\overline{\Gamma(\psi)(\overline{\Theta})}. \tag{4.9}$$

Alternatively it is an entertaining exercise to compute $\Phi(\psi)/\Phi(\overline{\psi})$. This factors to $Sp(2n,\mathbb{R})$, and for g' a strongly regular element of $Sp(2n,\mathbb{R})$:

$$\frac{\Phi(\psi)}{\Phi(\overline{\psi})}(g')=\zeta(g)\quad(g\overset{stable}{\longleftrightarrow}g'). \tag{4.10}$$

Here ζ is the unique non-trivial one-dimensional representation of $SO(V)$ (trivial if $SO(V)$ is compact), considered as a stable virtual character of $SO(2n+1)$. Therefore

$$\Gamma(\overline{\psi})(\Theta)=\Gamma(\psi)(\Theta\zeta). \tag{4.11}$$

An alternative interpretation of Γ is obtained by considering invariant eigendistributions on $\widetilde{Sp}(2n,\mathbb{R})$ in a neighborhood of a central element x of $\widetilde{Sp}(2n,\mathbb{R})$ lying over $-1\in Sp(2n,\mathbb{R})$. Let $\chi(\psi)(x)$ be the scalar by which x acts in the even half $\omega(\psi)_{even}$ of $\omega(\psi)$. Then $\omega(\psi)_{odd}(x)=-\chi(\psi)(x)$ and

$$\Phi(\psi)(g')=\chi(\psi)(x)^{-1}\Omega(\psi)(xg') \tag{4.12(a)}$$

and (dropping ψ from the notation):

$$\Gamma(\Theta)(g')=\chi(x)^{-1}\Theta(g)\Omega(xg')\quad(g\overset{stable}{\longleftrightarrow}p(g')). \tag{b}$$

§5. Matching of Cartan Subgroups: Linear Groups.

We use the correspondence of stable conjugacy classes (Theorem 2.16) to identify each Cartan subgroup of $SO(p,q)$ with a Cartan subgroup of $Sp(2n,\mathbb{R})$. These identifications depends on some choices, but any two such choices are stably conjugate.

Fix $X \overset{stable}{\longleftrightarrow} X'$ where $X \in \mathfrak{so}(V), X' \in \mathfrak{sp}(W)$ are regular semisimple elements. Let $\mathfrak{h}_0, \mathfrak{h}_0'$ be the centralizers of X and X' respectively. By Proposition 2.9 X and X' have the same non-zero eigenvalues. Considering the Lie algebras as complex matrices, after conjugating by $GL(2n+1,\mathbb{C})$ and $GL(2n,\mathbb{C})$ respectively we may assume $\mathfrak{h}_0, \mathfrak{h}_0'$ are diagonal, $X = diag(\lambda_1,\dots,\lambda_{2n},0)$ and $X' = diag(\lambda_1,\dots,\lambda_{2n})$. It follows from Proposition 2.9 that a neighborhood of X in $\mathfrak{h}_0$ corresponds (via the stable orbit correspondence) to a neighborhood of X' in $\mathfrak{h}_0'$, and the correspondence is given by $diag(z_1,\dots,z_{2n},1) \to diag(z_1,\dots,z_{2n})$. This extends to a linear isomorphism $\gamma : \mathfrak{h}_0 \to \mathfrak{h}_0'$ with complexification $\gamma : \mathfrak{h} \to \mathfrak{h}'$.

Let H (resp. H') be the centralizer of $\mathfrak{h}_0$ (resp. $\mathfrak{h}_0'$). The kernel $X(H)$ of the exponential map $exp : \mathfrak{h}_0 \to H$ consists of those elements of $\mathfrak{h}_0$ all of whose eigenvalues are in $2\pi i\mathbb{Z}$. The corresponding statement holds for $X(H')$ so γ exponentiates to an isomorphism $\phi : \mathbf{H} \to \mathbf{H}'$ which restricts to $\phi : H \to H'$

Suppose $\phi_i : H \to H_i'$ $(i=1,2)$ are two isomorphisms constructed in this manner. Then for any (strongly regular) $g_1' \in H_1'$, $g_2' = \phi_2(\phi_1^{-1}(g_1'))$ has the same eigenvalues as g_1', and hence $g_2' = x g_1' x^{-1}$ for some $x \in Sp(2n,\mathbb{C})$. It follows that $xH_1'x^{-1} = H_2'$ and $\phi_2 = int(x) \circ \phi_1$.

Similarly a Cartan subgroup H' of $Sp(2n,\mathbb{R})$ may be isomorphic to two Cartan subgroups H_1, H_2, where now H_1 and H_2 may be subgroups of different orthogonal groups $SO(V_1), SO(V_2)$. We summarize this discussion.

Lemma 5.1. *Let H be a Cartan subgroup of $SO(V)$, and choose a strongly regular element $g \in H$. Suppose $g \overset{stable}{\longleftrightarrow} g'$, and let $H' = Cent(g')$. Then there is an isomorphism $\phi : H \to H'$ taking g to g'. Any two isomorphisms $\phi_i : H \to H_i'$ $(i=1,2)$ obtained this way are stably conjugate: there exists $x \in Sp(2n,\mathbb{C})$ such that $\phi_2 = int(x) \circ \phi_1$. In particular if $H_1' = H_2'$ then $\phi_2 = w \circ \phi_1$ for some $w \in W_{st}(Sp(2n,\mathbb{R}), H_1')$.*

Analogous statements hold with the roles of $SO(2n+1)$ and $Sp(2n,\mathbb{R})$ reversed. In this case a Cartan subgroup of $Sp(2n,\mathbb{R})$ may be isomorphic to Cartan subgroups H_V of different orthogonal groups $SO(V)$.

Definition 5.2. We refer to the isomorphisms ϕ constructed in Lemma 5.1 as *standard* isomorphisms.

The standard isomorphisms play a fundamental role in what follows. We record some elementary properties. Fix a standard isomorphism $\phi : H \to H'$. Write $\mathfrak{so} = \mathfrak{so}(2n+1,\mathbb{C}), \mathfrak{sp} = \mathfrak{sp}(2n,\mathbb{C})$. We normalize the invariant form on $\mathfrak{h}'$ so the long roots have length 2. For α a root of $\mathfrak{h}'$ in $\mathfrak{sp}$, let

$$\phi^\vee(\alpha) = 2d\phi^*(\alpha)/(\alpha,\alpha) \in \mathfrak{h}^*. \tag{5.3)(a}$$

Then $\phi^\vee(\alpha)$ is a root of $\mathfrak{h}$ in $\mathfrak{so}$. Of course this is just the duality between root systems of type B_n and C_n, and $\phi^\vee(\alpha)$ corresponds to the coroot of α. We have resisted the temptation to label this $\alpha^\vee$, to emphasize the role of ϕ and so as not to confuse roots and coroots. Then for all $g \in H$,

$$e^{\phi^\vee(\alpha)}(g) = e^{2\alpha/(\alpha,\alpha)}(\phi(g)) \tag{b}$$

which we also write

$$e^{\alpha}(\phi(g)) = \begin{cases} e^{\phi^\vee(\alpha)}(g) & \alpha \text{ long, } \phi^\vee(\alpha) \text{ short} \\ e^{\phi^\vee(\alpha)}(g)^2 & \alpha \text{ short, } \phi^\vee(\alpha) \text{ long.} \end{cases} \tag{c}$$

Suppose $\Delta^+(H')$ is a set of positive roots of $\mathfrak{h}'$ in $\mathfrak{sp}$. Then

$$\Delta^+(H) = \phi^\vee(\Delta^+(H')) \tag{d}$$

is a set of positive roots of $\mathfrak{h}$ in $\mathfrak{so}$. There is a unique isomorphism $\phi^W : W(\mathfrak{so},\mathfrak{h}) \to W(\mathfrak{sp},\mathfrak{h}')$ satisfying

$$\phi^W(\sigma)\circ d\phi = d\phi\circ\sigma \tag{e}$$

for all $\sigma \in W(\mathfrak{so},\mathfrak{h})$. Furthermore ϕ^W restricts to an isomorphism

$$\phi^W : W_{st}(\mathfrak{so},\mathfrak{h}) \to W_{st}(\mathfrak{sp},\mathfrak{h}'). \tag{f}$$

The last statement follows from Lemmas 7.6 and 7.8.

§6. Matching of Cartan Subgroups: Covering Groups.

We extend the preceding results to Cartan subgroups of $\widetilde{Sp}(2n,\mathbb{R})$. It is convenient to pass to a certain covering group $Spin(p,q)$ of $SO(p,q)$ as well. In this way Lemma 5.1 extends to an isomorphism of Cartan subgroups of $Spin(p,q)$ and $\widetilde{Sp}(2n,\mathbb{R})$. Also $Spin(p,q)$ is an acceptable group and plays a role in the matching conditions (§10).

Given a Cartan subgroup $\mathbf{H}$ of an algebraic group $\mathbf{G}$, let

$$Q^\vee \subset X_*(\mathbf{H}) \subset P^\vee \tag{6.1(a)}$$

be the co-root lattice, lattice of one-parameter subgroups, and co-weight lattice respectively. These play an important role in what follows, and we give a little detail in our case. Let $G = Sp(2n,\mathbb{R})$. In the usual coordinates (a) is isomorphic to

$$\mathbb{Z}^n \subseteqq \mathbb{Z}^n \subsetneqq \mathbb{Z}^n \oplus (\mathbb{Z}+\frac{1}{2})^n. \tag{b}$$

On the other hand for $G = SO(2n+1)$ the picture is

$$(\mathbb{Z}^n)_{even} \subsetneqq \mathbb{Z}^n \subseteqq \mathbb{Z}^n \tag{c}$$

where $(\mathbb{Z}^n)_{even}$ is the sublattice of sequences with even sum. The kernel $X(H)$ of the exponential map $exp : \mathfrak{h} \to \mathbf{H}$ is $2\pi i X_*(\mathbf{H})$.

Now let $G = SO(V)$, and embed G in $SO(V_{\mathbb{C}}) \simeq SO(2n+1,\mathbb{C})$ as in §2. Let $Spin(V)$ be the inverse image of $SO(V)$ in $Spin(2n+1,\mathbb{C})$. Note that if $SO(V)$ is non-compact, $Spin(V)$ is *not* the real points of a connected algebraic group: the real points of $Spin(2n+1,\mathbb{C})$ are connected and of index two in $Spin(V)$. For example $Spin(2,1)$ is isomorphic to the subgroup of $SL(2,\mathbb{C})$ generated by $SL(2,\mathbb{R})$ and $diag(i,-i)$.

Let $G = Sp(W)$ or $SO(V)$, $\tilde{G} = \widetilde{Sp}(W)$ or $Spin(V)$. A Cartan subgroup of $\tilde{G}$ is by definition the centralizer of a Cartan subalgebra of $\mathfrak{g}_0$, or equivalently the inverse image of a Cartan subgroup of G.

Lemma 6.2.
(a) The Cartan subgroups of $\tilde{G}$ are abelian.

(b) Let $\mathcal{C}$ be a regular semisimple conjugacy class in $G = Sp(W)$. Then $p^{-1}(\mathcal{C})$ is the disjoint union of two conjugacy classes in $\tilde{G}$.

Proof. Since $G = Spin(V)$ is a subgroup of $Spin(2n+1,\mathbb{C})$, (a) is immediate in this case, so suppose $G = Sp(W)$.

Write $\tilde{G} = \{(g,\epsilon)\,|\,g \in G, \epsilon = \pm 1\}$, with multiplication given by the standard cocycle $c(,)$ [26]: $(g,\epsilon)(g',\epsilon') = (gg', \epsilon\epsilon' c(g,g'))$. It is well known that the restriction of $c(,)$ to a Cartan subgroup is symmetric (cf. [1], §5, for example). Part (a) follows immediately. For (b) it is enough to show for all regular semisimple elements $x \in \tilde{G}$, x is not conjugate to zx, where $p^{-1}(1) = \{1, z\}$. Suppose $gxg^{-1} = zx$ for some $g \in \tilde{G}$. Then $p(g)p(x)p(g)^{-1} = p(x)$, so $p(g) \in H = Cent_G(p(x))$. But then $g \in \tilde{H}$ and $gxg^{-1}x^{-1} = z$, contradicting the fact that $\tilde{H}$ is abelian.

Remark 6.3. Lemma 6.2(b) is in contrast to the situation for $\widetilde{GL}(n)$, in which g is conjugate to zg for some regular semisimple elements g, which forces the character of a genuine representation to vanish on g ([9],§1.1).

We continue to let $G = Sp(W)$ or $SO(V)$. Fix a maximal compact subgroup K of G, and a Cartan subgroup T of K (and of G). Write T_{Sp} or T_{SO} accordingly, and $\tilde{T}_{Sp}, \tilde{T}_{SO}$ for their inverse images in $\tilde{G}$.

Now let H be a Cartan subgroup of G, with inverse image $\tilde{H}$. After conjugating by G if necessary we may assume $H \cap K \subset T$, and write $H = T_H A$ with $T_H = H \cap T$ as usual. Then $\tilde{H} = \tilde{T}_H A$ (identifying A with $\tilde{A}$).

Lemma 6.4. *Let $\phi : H \to H'$ be a standard isomorphism. Then ϕ lifts to an isomorphism $\tilde{\phi} : \tilde{H} \to \tilde{H}'$.*

Proof. We first assume $H = T_{Sp}$, $H' = T_{SO}$. In this case the results is immediate from the Lie algebra: $\tilde{T}_{Sp}$ and $\tilde{T}_{SO}$ are connected, and $d\phi$ takes the kernel of $exp : \mathfrak{t}_{Sp} \to \tilde{T}_{Sp}$ to the kernel of $exp : \mathfrak{t}_{SO} \to \tilde{T}_{SO}$. The general case follows from this: ϕ restricted to T_H lifts to an isomorphism $\tilde{T}_H \to \tilde{T}_{H'}$.

A subtle point is that if H is not connected the isomorphism $\tilde{\phi}$ of Lemma 6.4 is not unique. We specify the choice by the following Lemma. Let $\tilde{H} = \tilde{T}_H A$ be a Cartan subgroup of $\widetilde{Sp}(W))$. As in ([27], §2), suppose $c : \mathfrak{t} \to \mathfrak{h}$ is a Cayley transform, with adjoint $c^* : \mathfrak{h}^* \to \mathfrak{t}^*$ (also see ([1], §5) for some explicit choices). Suppose $\lambda \in \mathfrak{h}^*$ satisfies: $c^*\lambda \in \mathfrak{t}^*$ exponentiates to $\tilde{T}$. For $g = t \cdot exp(X) \in \tilde{T}_H A$ we define

$$e^{\lambda}(g) = e^{c^*\lambda}(t) e^{\lambda(X)}. \tag{6.5}$$

Lemma 6.6. *In the setting of Lemma 6.4, let $\Delta^+(\mathfrak{h}')$ be a set of positive roots of $\mathfrak{h}'$ in $\mathfrak{sp}(2n,\mathbb{C})$, and let $\Delta^+(\mathfrak{h}) = \phi^\vee(\Delta^+(\mathfrak{h}'))$ (cf. (5.3)(d)). Let $\rho = \rho(\Delta^+(\mathfrak{h}))$ and let $\lambda = d\phi^{*-1}(\rho)$. Then there is a unique choice of $\tilde{\phi} : \tilde{H} \to \tilde{H}'$, lifting ϕ, such that*

$$e^{\lambda}(\tilde{\phi}(g)) = e^{\rho}(g) \quad (g \in \tilde{H}). \tag{6.7}$$

Proof. The Cayley transform is trivial on $\mathfrak{t} \cap \mathfrak{h}$, so (6.7) is immediate on the Lie algebra, and on the identity component of $\tilde{H}$. The result then reduces to the split

Cartan subgroup $A \simeq \mathbb{R}^*$ of $SL(2,\mathbb{R})$. The inverse image $\tilde{A}$ of A in $\widetilde{SL}(2,\mathbb{R})$ is isomorphic to $\mathbb{R}^* \cup i\mathbb{R}^*$, with covering map $z \to z^2$. The choice of ϕ is unique up to automorphisms of $\tilde{A}$ lying over the identity map on A; there is a unique non-trivial such automorphism $\tau : z \to \overline{z}$. Then $e^\lambda(z) = \overline{e^\lambda(\tau z)}$ so this indeterminacy is eliminated by condition (6.7). See §15.

We use the term *standard* also for the isomorphisms of Lemma 6.6.

§7. Cartan Subgroups.

It is convenient to choose representatives for the conjugacy classes of Cartan subgroups of $Sp(2n,\mathbb{R})$ and $SO(V)$. For $Sp(2n,\mathbb{R})$ this was done in [1], and we briefly recall the definitions. For $2m+r+s=n$, $H^{m,r,s} \simeq \mathbb{C}^{*m} \times S^{1r} \times \mathbb{R}^{*s}$ is a Cartan subgroup of $Sp(2n,\mathbb{R})$, its inverse image $\tilde{H}^{m,r,s}$ in $\widetilde{Sp}(2n,\mathbb{R})$ is a Cartan subgroup of $\widetilde{Sp}(2n,\mathbb{R})$, and these form a set of representatives of conjugacy classes of Cartan subgroups in $Sp(2n,\mathbb{R})$ and $\widetilde{Sp}(2n,\mathbb{R})$ respectively. The Lie algebra of $H^{m,r,s}$ is denoted $\mathfrak{h}_0^{m,r,s}$, with complexification $\mathfrak{h}^{m,r,s}$. The compact Cartan subgroup is $T = H^{0,n,0}$, and each $H^{m,r,s}$ comes with a Cayley transform $c^{m,r,s} : \mathfrak{t} \to \mathfrak{h}^{m,r,s}$. We write $H^{m,r,s}_{Sp}, \tilde{H}^{m,r,s}_{Sp}$ etc. to indicate the larger group.

The elements of $\mathfrak{h}_0^{m,r,s}$ are written

$$X = \mathfrak{h}^{m,r,s}_{Sp}(w_1,\dots,w_m,\theta_1,\dots,\theta_r,c_1,\dots,c_s) \tag{7.1)(a}$$

and those of $H^{m,r,s}_{Sp}$ are written

$$g = H^{m,r,s}_{Sp}(z_1,\dots,z_m,u_1,\dots,u_r,x_1,\dots,x_s) \tag{7.1)(b}$$

$(w_i \in \mathbb{C}, \theta_i, c_i \in \mathbb{R}, z_i \in \mathbb{C}^*, u_i \in S^1, x_i \in \mathbb{R}^*)$.

We make similar choices in $SO(p,q)$. Let $v_1,\dots,v_p,v'_1,\dots,v'_q$ be a basis of V so that $(v_i,v_j) = -(v'_i,v'_j) = \delta_{i,j}$ and $(v_i,v'_j) = 0$ for all i,j. Suppose $2m+s \le min(p,q)$. Write $V = V_1 \oplus V_2 \oplus V_3$ where V_1 is spanned by $\{v_i, v'_j \mid 1 \le i,j \le 2m\}$, $V_2 =< \{v_i,v'_j \mid 2m{+}1 \le i,j \le 2m{+}s\} >$ and $V_3 =< \{v_i,v'_j \mid 2m{+}s < i \le p, 2m{+}s < j \le q\} >$. Then $SO(V_i)$ is embedded naturally in $SO(V)$ and we identify $SO(V_i)$ and $\mathfrak{so}(V_i)$ with their images in $SO(V)$ and $\mathfrak{so}(V)$. For $w_j = x_j + iy_j \in \mathbb{C}$ let

$$\mathfrak{h}^{m,0,0}(w_1,\dots,w_m) = \begin{pmatrix} & Y & & X \\ -Y & & X & \\ & X & & -Y \\ X & & Y & \end{pmatrix} \in \mathfrak{so}(V_1) \tag{7.2}$$

where $X = diag(x_1,\dots,x_m), Y = diag(y_1,\dots,y_m)$. For $c_j \in \mathbb{R}$ let

$$\mathfrak{h}^{0,0,s}(c_1,\dots,c_s) = \begin{pmatrix} & X \\ X & \end{pmatrix} \in \mathfrak{so}(V_2)$$

where $X = diag(c_1,\dots,c_s)$. Finally let $r_1 = [\frac{p-2m-s}{2}]$, $r_2 = [\frac{q-2m-s}{2}]$, and for $\theta_i, \phi_j \in \mathbb{R}$ let

$$\mathfrak{h}^{0,r_1+r_2,0}(\theta_1,\dots,\theta_{r_1},\phi_1,\dots,\phi_{r_2}) = diag(\hat{\theta}_1,\dots,\hat{\theta}_{r_1},\hat{\phi}_1,\dots,\hat{\phi}_{r_2}) \in \mathfrak{so}(V_3)$$

with $\hat\theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$.

Taking the sum of these elements gives us an element

$$X = \mathfrak{h}^{m,r,s}_{p,q}(w_1,\dots,w_m,\theta_1,\dots,\theta_{r_1},\phi_1,\dots,\phi_{r_2},c_1,\dots,c_s) \in \mathfrak{so}(p,q) \tag{7.3(a)}$$

and this defines a Cartan subalgebra $\mathfrak{h}^{m,r,s}_{p,q,0}$, with complexification $\mathfrak{h}^{m,r,s}_{p,q}$. Let $H^{m,r,s}_{p,q} \simeq \mathbb{C}^{*m} \times S^{1r} \times \mathbb{R}^{*s}$ be the Cartan subgroup of $SO(p,q)$ with Lie algebra $\mathfrak{h}^{m,r,s}_{p,q,0}$. This gives a set of representatives of the conjugacy classes of Cartan subgroups of $SO(p,q)$. The compact Cartan subgroup T is $H^{0,n,0}$.

We choose coordinates on $H^{m,r,s}_{p,q}$: for $z_i \in \mathbb{C}^*$, $u_i, v_i \in S^1$ and $x_i \in \mathbb{R}^*$ let

$$g = H^{m,r,s}_{p,q}(z_1,\dots,z_m,u_1,\dots,u_{r_1},v_1,\dots,v_{r_2},x_1,\dots,x_s) \tag{b}$$

be the exponential of $\mathfrak{h}^{m,r,s}_{p,q}(\log(z_1),\dots,\log(x_s))$.

We now make some of the construction of the preceding sections explicit.

Lemma 7.4. *Let $H = H^{m,r,s}_{p,q}$, $H' = H^{m,r,s}_{Sp}$, and use notation of (7.1) and (7.3).*

(1) Let

$$X = \mathfrak{h}^{m,r,s}_{p,q}(w_1,\dots,w_m,\theta_1,\dots,\theta_{r_1},\phi_1,\dots,\phi_{r_2},c_1,\dots,c_s)$$

be a regular element. After conjugating by $W(SO(p,q),H)$ we may assume $\theta_i,\phi_i > 0$ for all i. Then the orbit of X corresponds to the orbit of

$$X' = \mathfrak{h}^{m,r,s}_{Sp}(w_1,\dots,w_m,\theta_1,\dots,\theta_{r_1},-\phi_{r_2},\dots,-\phi_1,c_1,\dots,c_s).$$

(2) The conjugacy class of a strongly regular element

$$H^{m,r,s}_{p,q}(z_1,\dots,z_m,u_1,\dots,u_{r_1},v_1,\dots,v_{r_2},x_1,\dots,x_s)$$

corresponds to the conjugacy class of

$$H^{m,r,s}_{Sp}(z_1,\dots,z_m,u_1,\dots,u_{r_1},\overline{v}_{r_2},\dots,\overline{v}_1,x_1,\dots,x_s).$$

(3) The stable orbit of a regular element

$$\mathfrak{h}^{m,r,s}_{SO}(w_1,\dots,w_m,\theta_1,\dots,\theta_r,c_1,\dots,c_s)$$

corresponds to the stable orbit of

$$\mathfrak{h}^{m,r,s}_{Sp}(w_1,\dots,w_m,\theta_1,\dots,\theta_r,c_1,\dots,c_s).$$

(4) The stable conjugacy class of a strongly regular element

$$H^{m,r,s}_{SO}(z_1,\dots,z_m,u_1,\dots,u_r,x_1,\dots,x_s)$$

corresponds to the stable conjugacy class of

$$H^{m,r,s}_{Sp}(z_1,\dots,z_m,u_1,\dots,u_r,x_1,\dots,x_s).$$

Proof. The stabilized correspondences (3) and (4) follows immediately from the eigenvalues by Theorems 2.9 and 2.16. The extra work in proving (1) and (2) comes down to the compact Cartan subgroups (subalgebras). The result follows from the proof of Proposition 2.5. We omit the details. (See [2] for similar computations.)

Given an additive character ψ (cf. §4) let $\Delta^+(T,\psi)$ be a set of positive roots of $\mathfrak{t}$ in $\mathfrak{sp}(2n,\mathbb{C})$ satisfying: $\omega(\overline{\psi})$ has a vector annihilated by the root vectors corresponding to the positive non-compact roots. (The reason for using $\overline{\psi}$ here is so Theorem 8.2 will hold without it.) Let $H = H_{Sp}^{m,r,s}$ be a standard Cartan subgroup with Cayley transform $c = c^{m,r,s} : \mathfrak{t} \to \mathfrak{h}$. Let $\Delta^+(H,\psi) = c^{*-1}(\Delta^+(T,\psi))$.

Up to conjugation by the compact Weyl group, we may take $\Delta^+(T,\psi_-)$ to be the usual set of positive roots $2i\theta_j, i(\theta_j-\theta_k)\ (j<k)$ (notation (7.1)(a)); $\Delta^+(T,\psi_+)$ is obtained from this by replacing each $2\theta_i$ by $-2\theta_i$.

For each p,q,m,r,s let $\phi : H_{p,q}^{m,r,s} \to H_{Sp}^{m,r,s}$ be the isomorphism taking

$$H_{p,q}^{m,r,s}(z_1,\dots,z_m,u_1,\dots,u_r,x_1,\dots,x_s) \tag{7.5)(a}$$

to

$$H_{Sp}^{m,r,s}(z_1,\dots,z_m,u_1,\dots,u_r,x_1,\dots,x_s). \tag{b}$$

This is a standard isomorphism. Let

$$\Delta^+(H_{p,q}^{m,r,s},\psi) = \phi^\vee(\Delta^+(H_{Sp}^{m,r,s},\psi)) \tag{c}$$

(cf. (5.3)(d)). By Lemma 6.5 we obtain a standard isomorphism

$$\tilde{\phi} : \tilde{H}_{p,q}^{m,r,s} \to \tilde{H}_{Sp}^{m,r,s} \tag{d}$$

The long real roots of $\Delta^+(H_{Sp}^{m,r,s},\psi_\pm)$ are

$$\mp\gamma_1,\dots,\mp\gamma_s; \quad e^{\gamma_i}(g) = x_i^2 \tag{e}$$

(g as in 7.1(b)). The short real roots of $\Delta^+(H_{p,q}^{m,r,s},\psi_\pm)$ are

$$\mp\gamma_1,\dots,\mp\gamma_s; \quad e^{\gamma_i}(g) = x_i \tag{f}$$

(g as in (7.3)(b)).

Let $W(B_n) \simeq W(C_n)$ be the Weyl group of type B_n, acting in the usual way by permutations and sign changes, and let S^n be the symmetric group acting by permutations. We compute the Weyl groups of our Cartan subgroups by standard procedures, for example see ([21], cf. [32], Proposition 4.16). The results are as follows.

Lemma 7.6.
(1) $W(SO(V),H^{m,r,s})$ is isomorphic to

$$S^m \ltimes (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})^m \times W(B_{r_1}) \times W(B_{r_2}) \times W(B_s). \tag{7.7)(a}$$

In the notation of (7.3)(b), the first term acts by permutations of $z_1,\dots z_m$ *and* $z_i \to \overline{z}_i, z_i^{-1}, \overline{z}_i^{-1}$*. The factors of type* $W(B)$ *act as usual on coordinates* $u_1,\dots,u_{r_1}$*;* $v_1,\dots,v_{r_2}$ *and* $x_1,\dots,x_s$ *respectively.*

(2) In the setting of (1), $W(\mathbf{M},\mathbf{H})$ *is isomorphic to*

$$(\mathbb{Z}/2\mathbb{Z})^m \times W(B_r) \tag{b}$$

($r = r_1+r_2$*), with the first term acting by* $z_i \to \overline{z}_i$ *and the second on the coordinates* $u_1,\dots u_{r_1}, v_1,\dots,v_{r_2}$*.*

(3) $W_{st}(SO(V), H^{m,r,s})$ *is obtained from (a) by replacing* $W(B_{r_1})\times W(B_{r_2})$ *by* $W(B_r)$*.*

Lemma 7.8.
(1) $W(Sp(2n,\mathbb{R}), H^{m,r,s})$ *is isomorphic to*

$$S^m \ltimes (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})^m \times S^r \times W(B_s). \tag{7.9)(a}$$

In the notation of (7.1)(b), the action on $z_1,\dots,z_m$ *and* $x_1,\dots,x_s$ *is as in the previous lemma, and* S^r *acts by permutations on* $u_1,\dots,u_r$*.*

(2) In the setting of (1), $W(\mathbf{M},\mathbf{H})$ *is isomorphic to*

$$(\mathbb{Z}/2\mathbb{Z})^m \times W(B_r) \tag{b}$$

with the first term acting by $z_i \to \overline{z}_i$ *and the second on* $u_1,\dots,u_r$*.*

(3) $W_{st}(Sp(2n,\mathbb{R}), H^{m,r,s})$ *is obtained from (a) by replacing* S^r *by* $W(B_r)$*.*

§8. Transfer Factors.

Recall (Definition 4.2) the transfer factor $\Phi = \Phi(\psi) = \Omega(\psi)_{even} - \Omega(\psi)_{odd}$. We show this is the quotient of (normalized) Weyl denominators (Theorem 8.2), which is a defining property of transfer factors (cf. §1).

Let $G = \widetilde{Sp}(2n,\mathbb{R})$ or $Spin(p,q)$, and choose a Cartan subgroup H of G and a set of positive roots Δ^+ of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\rho = \rho(\Delta^+) = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Then e^ρ is defined and following [18] for $g \in H$ let

$$D(g) = e^\rho(g)\prod_{\alpha\in\Delta^+}(1-e^{-\alpha}(g)) \tag{8.1)(a}$$

and

$$\epsilon(g) = \prod_{\alpha\in\Delta^+(R)} \operatorname{sgn}(1-e^{-\alpha}(g)). \tag{b}$$

Here $\Delta^+(R)$ denotes the real roots in Δ^+. For $w \in W_{st}(G,H)$ (cf. 3.5) let

$$\epsilon(w,g) = (\epsilon D)(g)/(\epsilon D)(wg) = \pm 1. \tag{c}$$

This is independent of the the choice of Δ^+ (cf. [18], Lemma 2.2).

Consider the standard Cartan subgroups $H^{m,r,s}_{p,q}$ and $H^{m,r,s}_{Sp}$ of $Spin(p,q)$ and $\widetilde{Sp}(2n,\mathbb{R})$, with positive roots $\Delta^+(H^{m,r,s}_{p,q},\psi)$ and $\Delta^+(H^{m,r,s}_{Sp},\psi)$ as in §7 (since we will be working entirely in the covers, we drop ˜ from the notation.) Let $D^\psi_{p,q}, \epsilon^\psi_{p,q}, D^\psi_{Sp}, \epsilon^\psi_{Sp}$ be the corresponding functions of (8.1). Let $\phi : H^{m,r,s}_{p,q} \to H^{m,r,s}_{Sp}$ be the isomorphism of Lemma 6.5.

Theorem 8.2. *Let g' be a regular element of $H_{Sp}^{m,r,s}$ and set $g=\phi^{-1}(g')\in H_{p,q}^{m,r,s}$. Then*

$$\Phi(\psi)(g')=\frac{(D_{p,q}^{\psi}\epsilon_{p,q}^{\psi})(g)}{(D_{Sp}^{\psi}\epsilon_{Sp}^{\psi})(g')}. \tag{8.3}$$

Note that all terms on the right hand side factor to the linear groups with the exception of $D_{p,q}^{\psi}$.

This follows from [1]. Write $H_{Sp}=H_{Sp}^{m,r,s},\Delta_{Sp}^{+}=\Delta^{+}(H_{Sp}^{m,r,s},\psi),\rho=\rho(\Delta_{Sp}^{+})\in\mathfrak{h}^*$ and $W_{Sp}=W(\mathfrak{sp}(2n,\mathbb{C}),\mathfrak{h})$. Write $H_{p,q},\Delta_{p,q}^{+},\rho_{p,q}$ and W_{SO} similarly. Let

$$\begin{aligned}\lambda_0&=d\phi^{*-1}(\rho_{p,q})\\&=d\phi^{*-1}(\frac{1}{2}\sum_{\alpha\in\Delta_{p,q}^{+}}\alpha)\end{aligned} \tag{8.4)(a}$$

which by (5.3)(a,d) equals

$$\frac{1}{2}\sum_{\alpha\in\Delta_{Sp}^{+}}\frac{2\alpha}{(\alpha,\alpha)}. \tag{8.4)(b}$$

This is the infinitesimal character of the oscillator representation. Recall (cf. (7.5)(f)) the long real roots of $\Delta^{+}(H_{Sp}^{m,r,s},\psi_{\pm})$ are $\mp\gamma_1,\dots,\mp\gamma_s$.

Theorem 8.5 ([1], Theorem 3.11). *Let g' be a regular element of H_{Sp}. Then*

$$\Phi(\psi_{\pm})(g')=\prod_{i=1}^{s}sgn(1+e^{\pm\gamma_i/2}(g'))\frac{\sum_{w\in W_{SO}}sgn(w)e^{w\lambda_0}(g')}{\sum_{w\in W_{Sp}}sgn(w)e^{w\rho}(g')}. \tag{8.6}$$

Proof of Theorem 8.2. The denominator on the right hand side of (8.6) equals $D_{Sp}(g')$, and the numerator is $D_{p,q}(g)$. To see the latter, by (8.4)(b) the numerator is

$$e^{\lambda_0}(g')\prod_{\alpha\in\Delta_{Sp}^{+}}(1-e^{-2\alpha/(\alpha,\alpha)}(g'))$$

which by (5.3)(b) and the definition of ϕ (Lemma 6.5) equals

$$e^{\rho_{p,q}}(g)\prod_{\alpha\in\Delta_{p,q}^{+}}(1-e^{-\alpha}(g))=D_{p,q}(g).$$

Therefore

$$\Phi(\psi_{\pm})(g')=\prod_{i=1}^{s}\mathrm{sgn}(1+e^{\pm\gamma_i/2}(g'))\frac{D_{p,q}(g)}{D_{Sp}(g')}. \tag{8.7}$$

Furthermore (with $\psi = \psi_{\pm}$):

$$\begin{aligned}\frac{\epsilon_{p,q}(g)}{\epsilon_{Sp}(g')} &= \frac{\prod_{\alpha\in\Delta^+_{p,q}(R)} \mathrm{sgn}(1-e^{-\alpha}(g))}{\prod_{\alpha\in\Delta^+_{Sp}(R)} \mathrm{sgn}(1-e^{-\alpha}(g'))} \\ &= \frac{\prod_{\alpha\in\Delta^+_{Sp}(R)} \mathrm{sgn}(1-e^{-\phi^\vee(\alpha)}(g))}{\prod_{\alpha\in\Delta^+_{Sp}(R)} \mathrm{sgn}(1-e^{-\alpha}(g'))} \quad \text{by (5.3)(d)} \\ &= \prod_{i=1}^{s} \frac{\mathrm{sgn}(1-e^{\pm\gamma_i/2}(g'))}{\mathrm{sgn}(1-e^{\pm\gamma_i}(g'))} \quad \text{by (7.5)(e,f)} \\ &= \prod_{i=1}^{s} \mathrm{sgn}(1+e^{\pm\gamma_i/2}(g')).\end{aligned} \tag{8.8}$$

Inserting (8.8) in (8.7) gives (8.3), completing the proof of the Theorem.

From Theorem 8.5 we obtain the following formula for Φ ([1], Theorem 3.11):

$$\Phi(\psi_{\pm})(g') = \frac{\prod_{i=1}^{s} \mathrm{sgn}(1+e^{\pm\gamma_i/2}(g'))}{\prod_{\alpha\in\Delta^+_{Sp}(long)} (e^{\alpha/4}+e^{-\alpha/4})(g')}. \tag{8.9}$$

(The product in the denominator, although not each individual term $e^{\alpha/4}$, is well defined on $H^{m,r,s}_{Sp}$.)

Corollary 8.10. *Φ is stable.*

Proof. Since Φ is the character of a virtual representation it is invariant under $W(\widetilde{Sp}(2n,\mathbb{R}),H)$, so we only need to check

$$\Phi(wg) = \Phi(g) \quad (g \in H, w \in W(M,H)\backslash W(\mathbf{M},\mathbf{H})).$$

In the notation of (7.1)(a) let $\alpha_i(X) = 2\theta_i$. By Lemma 7.8 $W(M,H)\backslash W(\mathbf{M},\mathbf{H})$ is generated by $s_i = s_{\alpha_i}$ $(i=1,\dots,r)$. Then $e^{\alpha_i/4}(s_i g') = e^{-\alpha_i/4}(g')$, and the result follows from (8.6) and (8.9).

§9. Character Formulas.

We write an invariant function Θ on $SO(2n+1)_0$ or Θ' on $\widetilde{Sp}(2n,\mathbb{R})_0$ in terms of functions on our standard Cartan subgroups. When $\Theta' = \Gamma(\Theta)$ (Definition 4.5), we show the numerators of Θ and Θ' are the same (Proposition 9.3).

As in the preceding section we lift an invariant function Θ on $SO(p,q)$ to the acceptable group $Spin(p,q)$. So let Θ be an invariant function on $G = \widetilde{Sp}(2n,\mathbb{R})_0$ or $Spin(p,q)_0$ (the strongly regular semisimple elements, cf. §4).

We work with our fixed set of Cartan subgroups $H^{m,r,s}_{p,q}$ of $Spin(p,q)$ and $H^{m,r,s}_{Sp}$ of $\widetilde{Sp}(2n,\mathbb{R})$ (§7). As in §8 we drop ˜ from the notation. Recall we have defined positive systems $\Delta^+(H^{m,r,s}_{Sp},\psi)$ and $\Delta^+(H^{m,r,s}_{p,q},\psi)$, and D,ϵ (cf. (8.1)) are defined accordingly. Let $\phi: H^{m,r,s}_{p,q} \to H^{m,r,s}_{Sp}$ be as in Lemma 6.5.

For g a strongly regular element of $H = H^{m,r,s}_{Sp}$ or $H^{m,r,s}_{p,q}$ let

$$\kappa^{m,r,s}(\Theta,g) = \Theta(g)\epsilon(g)D(g) \tag{9.1}$$

(cf. [18], 1.13). We refer to this as the *normalized Weyl numerator of* Θ *on* H. We often write $\kappa = \kappa^{m,r,s}$. By (8.1)(c)

$$\kappa(\Theta, wg) = \epsilon(w,g)\kappa(\Theta, g) \text{ for all } w \in W(G,H). \tag{9.2)(a}$$

Conversely given the family $\{\kappa^{m,r,s}\}$, (9.1) extends to an invariant function Θ if and only if (9.2)(a) holds for all m, r, s, in which case the extension is unique.

The preceding discussion generalizes immediately to *stably* invariant functions of $\widetilde{Sp}(2n,\mathbb{R})$. If Θ is a stably invariant function then $\{\kappa^{m,r,s}\}$ as defined by (9.1) satisfy

$$\kappa(\Theta, wg) = \epsilon(w,g)\kappa(\Theta, g) \text{ for all } w \in W_{st}(G,H). \tag{b}$$

Conversely such a family $\{\kappa^{m,r,s}\}$ define a unique stably invariant function if and only if (b) holds.

For $G = Spin(2n+1)$ a similar statement holds once the different real forms are taken into account. A stably invariant function Θ on $Spin(2n+1)$ is a family of invariant analytic functions $\Theta_{p,q}$. Then (9.1) defines a family of functions $\kappa_{p,q}^{m,r,s}$ satisfying (9.2)(b) for each p, q. Furthermore

$$\kappa_{p,q}^{m,r,s}(g) = \kappa_{p',q'}^{m,r,s}(g') \tag{c}$$

whenever there is an element of $Spin(2n+1,\mathbb{C})$ conjugating $H_{p,q}^{m,r,s}$ to $H_{p',q'}^{m,r,s}$, $\Delta^+(H_{p,q}^{m,r,s},\psi)$ to $\Delta^+(H_{p',q'}^{m,r,s},\psi)$ and g to g'. Conversely a family $\{\kappa_{p,q}^{m,r,s}\}$ satisfying (c) defines a stably invariant function Θ on $Spin(2n+1)$ by (9.1). Note that $H_{p,q}^{m,r,s}$ is conjugate to $H_{p',q'}^{m,r,s}$ by a unique element of $Spin(2n+1,\mathbb{C})$ preserving the positive roots, so these Cartan subgroups may be canonically identified.

Hence it makes sense to associate to a stably invariant function Θ on $Spin(2n+1)$ a family $\kappa_{SO}^{m,r,s}$ of Weyl numerators parameterized by m, r, s, satisfying (9.2)(b): define $\kappa_{SO}^{m,r,s}(g) = \kappa_{p,q}^{m,r,s}(g)$ whenever $g \in H_{p,q}^{m,r,s}$. Conversely a family $\{\kappa_{SO}^{m,r,s}\}$ satisfying (9.2)(b) defines a stably invariant function Θ.

For Θ a stably invariant function on $SO(2n+1)_0$ we lift Θ to $Spin(2n+1)_0$, and define κ accordingly. The following theorem is an immediate consequence of Theorem 8.2.

Proposition 9.3. *Let* Θ *be a stably invariant function on* $SO(2n+1)_0$*, and set* $\Theta' = \Gamma(\psi)(\Theta)$ *(Definition 4.5). Then for all strongly regular elements* g *of* $H_{SO}^{m,r,s}$

$$\kappa(\Theta, g) = \kappa(\Theta', g')$$

where $g' = \phi(g) \in H_{Sp}^{m,r,s}$*.*

§10. Matching Conditions.

The main result of this section is

Theorem 10.1. *The map*

$$\Theta \to \Gamma(\Theta)$$

is a bijection between stably invariant eigendistributions on $SO(2n+1)$ *and genuine stably invariant eigendistributions on* $\widetilde{Sp}(2n,\mathbb{R})$*.*

The correspondence of infinitesimal characters is given by the orbit correspondence (Lemma 10.14).

The proof uses Hirai's matching conditions [18], which give a necessary and sufficient condition for a function to represent an invariant eigendistribution. We summarize the result in terms convenient to our situation.

Hirai's result applies to acceptable groups [12]. The primary requirement is that one-half the sum of the positive roots exponentiate to a Cartan subgroup. This holds for $\widetilde{Sp}(2n,\mathbb{R})$ but fails for $SO(p,q)$. We follow the standard procedure of lifting to a finite acceptable cover: $Spin(p,q)$ is acceptable. (Both $Spin(p,q)$ and $\widetilde{Sp}(2n,\mathbb{R})$ also satisfy Condition B of ([18], §7)).

Let H be a Cartan subgroup of $G=\widetilde{Sp}(2n,\mathbb{R})$ or $Spin(p,q)$, and choose a set of positive roots Δ^+. For α a real or non-compact imaginary root, let

$$\Sigma_\alpha(H)=\{h\in H\,|\,e^\alpha(h)=1\} \tag{10.2)(a}$$

and let $\Sigma^0_\alpha=\Sigma^0_\alpha(H)$ be the *semiregular elements of type* α:

$$\Sigma^0_\alpha(H)=\{h\in H\,|\,e^\alpha(h)=1, e^\beta(h)\neq 1\ \forall\beta\in\Delta^+,\beta\neq\alpha\}. \tag{b}$$

Let $\Delta^+(R)$ be the real roots of Δ^+, and define

$$\begin{aligned} H(R)&=\{h\in H\,|\,e^\alpha(h)\neq 1\ \forall\alpha\in\Delta^+(R)\}\\ &=H-\cup_{\alpha\in\Delta^+(R)}\Sigma_\alpha(H). \end{aligned} \tag{c}$$

For α a real root, let $\mathfrak{j}$ be a Cayley transform of $\mathfrak{h}$. Thus $\mathfrak{j}$ is a Cartan subalgebra of $\mathfrak{g}$, which comes with an isomorphism $c_\alpha:\mathfrak{h}\simeq\mathfrak{j}$, and let $\beta=c_\alpha^{*-1}(\alpha)\in\mathfrak{j}^*$. Let J be the corresponding Cartan subgroup of G. Then

$$\Sigma_\alpha(H)=\Sigma_\beta(J)=H\cap J. \tag{d}$$

Using c_α we transfer Δ^+ to $\Delta^+(J)$ for J. Given an invariant function Θ, together with its normalized Weyl numerators $\kappa^{m,r,s}$ (§9) we define $\kappa^J(g)=(D\epsilon)(g')\kappa^{m,r,s}(g')$ where there is an element of G conjugating J to $H^{m,r,s}$, $\Delta^+(J)$ to $\Delta^+(H^{m,r,s})$ and $g\in H$ to $g'\in H^{m,r,s}$.

Let $H_\alpha\in\mathfrak{h}$ be the coroot of α (H'_α of [18], 1.7), and similarly $J_\beta\in\mathfrak{j}$, considered as differential operators on H and J respectively. Then $c_\alpha(H_\alpha)=J_\beta$.

Let $I(\mathfrak{h})$ be the $W(\mathfrak{g},\mathfrak{h})$ invariant polynomials on $\mathfrak{h}$, which we identify with the left-invariant differential operators on H. We identify $I(\mathfrak{h})$ with the center $\mathfrak{z}(\mathfrak{g})$ of the universal enveloping algebra via the Harish-Chandra homomorphism, and identify each element $\lambda\in\mathfrak{h}^*$ with an infinitesimal character as usual.

Theorem 10.3 ([18], Theorem 3). *Let $G=\widetilde{Sp}(2n,\mathbb{R})$ or $Spin(p,q)$. For each Cartan subgroup $H^{m,r,s}$ of G let $\kappa^{m,r,s}$ be an analytic function on the regular elements of $H^{m,r,s}$. Let Θ be the invariant function defined by (9.1). Then Θ is an invariant distribution with infinitesimal character λ if and only if for all m,r,s (let $H=H^{m,r,s},\kappa=\kappa^{m,r,s}$):*

(C1) *$D\kappa=\lambda(D)\kappa$ for all $D\in I(\mathfrak{h})$,*
(C2) *κ extends to an analytic function on $H(R)$,*

(C3) *For any* $\alpha \in \Delta^+(R)$, *let* J *be a Cayley transform of* H. *Then for any* $g \in \Sigma^0_\alpha(H)$, κ *and* κ^J *in a neighborhood of* g *satisfy*

$$J_\beta(\kappa^J)(g) = \frac{1}{2}[H_\alpha(\kappa)^+(g) - H_\alpha(\kappa)^-(g)]. \tag{10.4)(a}$$

Here κ^J *is analytic at* g *and*

$$H_\alpha(\kappa)^\pm(g) = \frac{d}{dt}\kappa(g\,exp(tH_\alpha))|_{t\to 0^\pm}. \tag{b}$$

Remark 10.5. If (C1) holds then $\kappa(\Theta, h)$ may be written as follows. Identify λ with an element $\lambda \in \mathfrak{h}^*$. Then for g a regular element of H and X in a sufficiently small neighborhood of 0 in $\mathfrak{h}_0$,

$$\kappa(\Theta, g\,expX) = \sum_{w\in W} p(w,g)(X)e^{w\lambda(X)} \tag{10.6}$$

for some polynomials $p(w,g)$ on $\mathfrak{h}$ ([11], Theorem 4, cf. [18], Lemma 2.7).

Before turning to the proof of Theorem 10.1, we state a simple result relating the data of Theorem 10.3 on our two groups. Let H be a Cartan subgroup of $Spin(p,q)$, with standard isomorphism $\phi = \phi_H : H \to H' \subset \widetilde{Sp}(2n,\mathbb{R})$ as in Lemma 6.5. Let α be a real root of $\mathfrak{h}$ and let J be a Cayley transform of H. Let $\alpha' = {\phi^\vee}^{-1}(\alpha)$ be the corresponding root of $\mathfrak{h}'$, and define J' similarly. Let $\phi_J : J \to J'$ be a standard isomorphism. Without loss of generality we may assume $\phi_H|_{H\cap J} = \phi_J|_{H\cap J}$.

Lemma 10.7.

1. $\phi_H(\Sigma_\alpha(H)) \subset \Sigma_{\alpha'}(H')$, *with equality if and only if* α *is long (i.e.* α' *is short),*
2. $\phi_H(H(R)) \supset H'(R)$,
3. $\phi_H(H_\alpha) = \begin{cases} H'_{\alpha'} & \alpha \textit{ long}, \\ \frac{1}{2}H'_{\alpha'} & \alpha \textit{ short}, \end{cases}$
4. *Fix* $g \in \Sigma_\alpha(H), g' = \phi_H(g) \in \Sigma_{\alpha'}(H')$. *Then* $\phi_H \cup \phi_J$ *is a diffeomorphism from a neighborhood of* g *in* $H \cup J$ *to a neighborhood of* g' *in* $H' \cup J'$.

Proof. Since e^α factors to the Cartan subgroups of the linear groups, $\Sigma_\alpha(H) = p^{-1}(\Sigma_\alpha(p(H)))$, and it is enough to prove the corresponding statements in the linear groups. Then (1) follows immediately from (5.3)(b,c), and (1) implies (2). Statement (3) follows from (5.3) applied to the Lie algebra, and (4) also follows from the Lie algebra.

Remark 10.8. The main subtlety occurs in the case of $SO(2,1)$,$SL(2,\mathbb{R})$. The main point is that the split Cartan subgroups of $SO(2,1)$ and $SL(2,\mathbb{R})$ are isomorphic to $\mathbb{R}^*$, and $-1 \in \mathbb{R}^*$ is regular in $SO(2,1)$ but not in $SL(2,\mathbb{R})$.

More precisely, take $H = A$, the split Cartan subgroup of $SO(2,1)$. Then α is short (α' is long) and $\phi_H(\Sigma_\alpha)(H) \subsetneqq \Sigma_{\alpha'}(H')$. The Cayley transform of A is a compact Cartan subgroup T, and $T \cap A = 1$. Writing $\tilde{T}, \tilde{A}$ for the inverse images in $Spin(2,1)$, we have $\tilde{T} \simeq S^1, \tilde{A} \simeq \mathbb{R}^* \cup i\mathbb{R}^*$, and $\Sigma_\alpha(\tilde{A}) = \tilde{T} \cap \tilde{A} = \pm 1$. On the other hand in $SL(2,\mathbb{R})$ the compact and split Cartan subgroups T, A satisfy $T \cap A = \pm 1$, and $\Sigma_\alpha(\tilde{A}) = \tilde{T} \cap \tilde{A} = \{\pm 1, \pm i\}$.

Proof of Theorem 10.1. Fix a stably invariant function Θ on $SO(2n+1)_0$ and let $\Theta' = \Gamma(\Theta)$. It is enough to show: Θ satisfies conditions (C1-C3) of Theorem 10.3 if and only if Θ' does as well. By Proposition 9.3, Θ and Θ' have the same numerators κ on each Cartan subgroup (identified via ϕ).

The equivalence of (C1) for Θ and Θ' follows immediately. By Lemma 10.7(2) conditions (C2-3) for Θ and Θ' are equivalent except possibly in the neighborhood of a regular element $g \in SO(2n+1)$ satisfying: $e^\alpha(g) = -1$ for some short real root α. By (5.3)(c) $e^{\alpha'}(\phi(g)) = 1$, and $\phi(g)$ is semiregular. Of course the numerator of an invariant eigendistribution Θ must be analytic in a neighborhood of g. The same is not true of the numerator of an arbitrary invariant eigendistribution Θ' in a neighborhood of $\phi(g)$. However this *does* hold if Θ' is stable and genuine, as follows from the next Lemma.

Lemma 10.9. *Let $J = H^{m,r,s}_{Sp}$ be a standard Cartan subgroup of $\widetilde{Sp}(2n,\mathbb{R})$. Let $g \in J$ be a semiregular element of type α for some long imaginary root α, satisfying $e^{\alpha/2}(g) = -1$. Let Θ be a genuine, stably invariant function on $\widetilde{Sp}(2n,\mathbb{R})_0$, with normalized Weyl numerator of κ^J on J. Then (with notation as in Theorem 10.3):*

$$J_\alpha(\kappa^J)(g) = 0. \tag{10.10}$$

Proof. In the notation of (7.1)(a) $\alpha(X) = 2i\theta_i$ and

$$J_\alpha = \mathfrak{h}^{m,r,s}_{Sp}(0, \dots, 0, \theta_i = 1, 0, \dots, 0) \tag{10.11)(a}$$

and

$$p(g) = H^{m,r,s}_{Sp}(z_1, \dots, u_i = -1, \dots, x_s). \tag{b}$$

Write $g = g_0 \exp(\pi J_\alpha)$ for some $g_0 \in J$, and let $\tilde{\kappa}(\theta) = \kappa^J(g_0\, exp(\theta J_\alpha))$. We need to show

$$\frac{d}{d\theta}\tilde{\kappa}(\theta)|_\pi = 0. \tag{c}$$

Let

$$Z = 2\pi J_\alpha; \tag{d}$$

then $exp(Z) = z \in \widetilde{Sp}(2n,\mathbb{R})$, where z is the non-trivial element of $\widetilde{Sp}(2n,\mathbb{R})$ in the inverse image of the identity in $Sp(2n,\mathbb{R})$.

The condition that Θ is genuine is $\kappa^J(zg) = -\kappa^J(g)$, which implies $\tilde{\kappa}(\theta + 2\pi) = -\tilde{\kappa}(\theta)$. On the other hand $s_\alpha(g_0 exp(\theta J_\alpha)) = g_0 exp(-\theta J_\alpha)$ (note that by Lemma 6.2(b) $s_\alpha(g_0) = g_0$). Therefore stability implies $\tilde{\kappa}(-\theta) = -\tilde{\kappa}(\theta)$, and we conclude

$$\begin{aligned}\tilde{\kappa}(\pi - \theta) &= -\tilde{\kappa}(-\pi + \theta) \\ &= \tilde{\kappa}(\pi + \theta).\end{aligned} \tag{e}$$

By (C2) $\tilde{\kappa}$ is analytic in a neighborhood of π. The result follows from the fact that an even analytic function has derivative zero at the center of symmetry.

Now suppose $g \in H$ is a regular element of $SO(2n+1)$ satisfying $e^{\alpha}(g) = -1$ for exactly one short real root α of H. Let $g' = \phi(g) \in H'$. The Lemma implies the numerator $\kappa^{H'}$ of Θ' on H' is analytic in a neighborhood of g' (cf. [18], §10, proof of Theorem 6). This shows conditions (C2-C3) for Θ and Θ' in neighborhoods of g, g' respectively are equivalent. A similar argument holds in the case g is a semi-regular element of $SO(2n+1)$ and g' is not semi-regular. In this case $e^{\alpha}(g') = e^{\beta}(g') = 1$ for some roots α, β and $e^{\alpha/2}(g') = -1$; by the previous case first extend the function analytically in a neighborhood of g' for which $e^{\beta}(g') \neq 1$. This completes the proof of Theorem 10.1.

Remark 10.12. The assumptions of genuine and stable in Lemma 10.9 are necessary. For example let π be an irreducible discrete series representation of $\widetilde{SL}(2,\mathbb{R})$ (genuine or not), or a stable sum of discrete series representations of $SL(2,\mathbb{R})$ (lifted to $\widetilde{SL}(2,\mathbb{R})$). Then the numerator κ of π does not satisfy (10.10), so κ has a jump at the elements $\pm i$ of the split Cartan subgroup of $\widetilde{SL}(2,\mathbb{R})$ (cf. Remark 10.8). On the other hand since the corresponding elements of $SO(2,1)$ are regular, the numerator of a character of $SO(2,1)$ must be analytic at these elements. Therefore π cannot match a character of $SO(2,1)$. See §15.

Corollary 10.13. *In the setting of Theorem 10.1, Θ is tempered if and only if $\Gamma(\Theta)$ is tempered.*

Proof. By a basic result of Harish-Chandra ([13], Theorem 7) an invariant eigendistribution is tempered if and only if the numerators of its character on each Cartan subgroup have at most polynomial growth. The corollary follows from Proposition 9.3.

The correspondence of infinitesimal characters follows immediately from the proof of Theorem 10.1. The set of infinitesimal characters for $SO(2n+1)$ or $\widetilde{Sp}(2n,\mathbb{R})$ is canonically identified with the set of semisimple coadjoint orbits of $SO(2n+1,\mathbb{C})$ or $Sp(2n,\mathbb{C})$ on $\mathfrak{so}(2n+1,\mathbb{C})^*$ or $\mathfrak{sp}(2n,\mathbb{C})^*$. The (dualized) orbit correspondence for the groups $SO(2n+1,\mathbb{C})$ and $Sp(2n,C)$ restricts to a bijection $\mathcal{O} \leftrightarrow \mathcal{O}'$ of these orbits (cf. §13).

Lemma 10.14. *Suppose $\Theta' = \Gamma(\Theta)$ are corresponding invariant eigendistributions as in Theorem 10.1. Let $\mathcal{O}$ (resp. $\mathcal{O}'$) be the infinitesimal character of Θ (resp. Θ'). Then $\mathcal{O} \leftrightarrow \mathcal{O}'$.*

In the obvious coordinates if Θ has infinitesimal character (the orbit of)

$$\lambda = \mathfrak{h}^*_{SO}(z_1, \dots, z_n)$$

then the infinitesimal character of Θ' is (the orbit of)

$$\lambda' = \mathfrak{h}^*_{Sp}(z_1, \dots, z_n).$$

§11. Discrete Series.

We now show that Γ takes discrete series representations of $SO(2n+1)$ to discrete series representations of $\widetilde{Sp}(2n,\mathbb{R})$.

Let $T_{Sp} = H^{0,n,0}_{Sp}$ be our fixed compact Cartan subgroup of $Sp = Sp(2n, \mathbb{R})$. Write $X^*(T_{Sp}) \subset \mathfrak{t}^*_{Sp}$ for the differentials of characters of T_{Sp}, and let $\lambda_0 \in \mathfrak{t}^*_{Sp}$ be the infinitesimal character of the oscillator representation (cf. (8.4)(a)). The genuine discrete series representations of $\widetilde{Sp}(2n, \mathbb{R})$ are parametrized by regular elements λ in $\lambda_0 + X^*(T_{Sp})$, modulo $W(Sp, T_{Sp})$. We write $\pi_{Sp}(\lambda)$ for the discrete series representation corresponding to λ.

For G any group with maximal compact subgroup K of equal rank, let $q = q(G) = \frac{1}{2}dim(G/K)$. Let

$$\overline{\pi}_{Sp}(\lambda) = (-1)^{q(Sp)} \sum_{w \in W(Sp,T_{Sp})\backslash W} \pi_{Sp}(w\lambda), \tag{11.1}$$

the averaged discrete series of $\widetilde{Sp}(2n, \mathbb{R})$ with infinitesimal character λ. Here $W = W(\mathfrak{sp}(2n, \mathbb{C}), \mathfrak{t}_{Sp})$.

We make a similar definition for $SO(2n + 1)$. Here we identify the compact Cartan subgroups $H^{0,n,0}_{p,q}$ (resp. positive systems $\Delta^+(H^{0,n,0}_{p,q}, \psi)$) of each $SO(p, q)$, and call it T_{SO} (resp. $\Delta^+(T_{SO}, \psi)$). For λ a regular element of $\rho + X^*(T_{SO})$ we let

$$\overline{\pi}_{SO}(\lambda) = \sum_{V \in \mathcal{V}} (-1)^{q(SO(V))} \sum_{w \in W(SO(V),T_{SO})\backslash W} \pi(w\lambda). \tag{11.2}$$

(We have identified $W = W(\mathfrak{sp}(2n, \mathbb{C}), \mathfrak{t}_{Sp})$ with $W(\mathfrak{so}(2n + 1, \mathbb{C}), \mathfrak{t}_{SO})$.) By ([13], cf. [28],§5) $\overline{\pi}_{SO}$ and $\overline{\pi}_{Sp}$ are stable.

Proposition 11.3.

Suppose $\lambda \in \mathfrak{t}^*_{SO}$ corresponds to $\lambda' \in \mathfrak{t}^*_{Sp}$ via the stabilized orbit correspondence (Proposition 2.9). Then

$$\Gamma(\overline{\pi}_{SO}(\lambda)) = \overline{\pi}_{Sp}(\lambda').$$

Here $\Gamma = \Gamma(\psi)$ (Definition 4.5) for any ψ.

Proof. Without loss of generality we assume λ is dominant for $\Delta^+ = \Delta^+(T_{SO}, \psi)$. By Harish-Chandra's formula for discrete series characters [13] the character Θ of $\overline{\pi}_{SO}(\lambda)$ satisfies

$$\Theta(g) = \frac{\sum_{w\in W} \operatorname{sgn}(w)e^{w\lambda-\rho}(g)}{\prod_{\alpha\in\Delta^+}(1 - e^{-\alpha}(g))} \quad (g \in T_{SO}). \tag{11.4)(a}$$

Upon lifting to the inverse image $\tilde{T}_{SO}$ of T_{SO} in $Spin$ this may be written (cf. (8.1)(a))

$$\Theta(g) = \frac{\sum_{w\in W} \operatorname{sgn}(w)e^{w\lambda}}{D_{SO}}(g) \quad (g \in \tilde{T}_{SO}). \tag{b}$$

By Proposition 9.3 and our choice of positive systems (§7), taking λ' dominant for $\Delta^+(T_{Sp}, \psi)$ gives:

$$\Gamma(\Theta)(g') = \frac{\sum_{w\in W} \operatorname{sgn}(w)e^{w\lambda'}}{D_{Sp}}(g') \quad (g' \in \tilde{T}_{Sp}). \tag{c}$$

By Harish-Chandra's formula again this is the restriction of the character of $\overline{\pi}_{Sp}(\lambda')$ to $\tilde{T}_{Sp}$. Therefore $\Gamma(\overline{\pi}_{SO}(\lambda))$ and $\overline{\pi}_{Sp}(\lambda')$ have the same restriction to $\tilde{T}_{Sp}$.

The numerators, on each Cartan subgroup, of the character of a discrete series representation are bounded ([13], cf. [22], Theorem 12.1). By Theorem 10.1 $\Gamma(\pi_{SO}(\lambda))$ is an invariant eigendistribution, with bounded numerators by Proposition 9.3. The theorem now follows from ([13], Theorem 3): the character of a discrete series representation is characterized, among all invariant eigendistributions, by its infinitesimal character, formula on the compact Cartan subgroup, and having bounded numerators on each Cartan subgroup.

Remark 11.5. The restriction of the transfer factor $\Phi(\psi)$ to $\tilde{T}_{Sp}$ is independent of ψ, and therefore $\Gamma(\psi)$ restricted to the discrete series is independent of ψ. Equivalently (cf. (4.11)) the characters of the discrete series vanish off of the identity component of $SO(V)$.

§12. Induced Representations.

We now show Γ commutes with induction (Theorem 12.16). Furthermore a character induced from a stable sum of discrete series representations on a parabolic subgroup $P = MN$ is stable, and these span the stable virtual characters. As a consequence Γ is a bijection of stable virtual characters (Corollary 12.22). Finally we prove that Γ agrees with theta-lifting for tempered representations (Theorem 12.27).

We first discuss induction for $G = SO(2n+1)$. We work in the setting of Definition 2.4. We need to consider different discriminants so let $\delta = \pm 1$ and write $\mathcal{V}(\delta)$ for the set $\mathcal{V} = \{V_{p,q}\}$ of Definition 2.4. Fix m, r, s and a Cartan subgroup $H = H_{SO}^{m,r,s}$ of $SO(2n+1)$. Recall this is really a family $H_{p,q}^{m,r,s}$ of Cartan subgroups which have been identified (cf. §9). Here p, q run over

$$p + q = 2n + 1, (-1)^q = \delta \text{ and } p, q \geq 2m + s. \tag{12.1)(a}$$

Then the centralizer of the split component of $H_{p,q}^{m,r,s}$ in $SO(p,q)$ is isomorphic to

$$M_{p,q} = SO(p', q') \times GL(2)^m \times GL(1)^s \tag{b}$$

$(GL(j) = GL(j, \mathbb{R}))$ with $p' = p - 2m - s, q' = q - 2m - s$.

As p, q vary over the set (12.1)(a), the orthogonal group factors of $M_{p,q}$ run over the real forms of $SO(2n'+1)$ $(n' = n - 2m - s)$, with discriminant $\delta(-1)^s$. We write $\mathcal{V}_0$ for the set of orthogonal spaces so obtained; this depends on δ as well as m, r, s. For $V_0 \in \mathcal{V}_0$ we write $M(V_0)$ for the corresponding Levi factor. We let M denote the family of Levi factors $M(V_0)$ $(V_0 \in \mathcal{V}_0)$.

Using (7.5)(d) we identify the $GL(2)$ and $GL(1)$ factors of each Levi factor. A stable distribution on M is defined to be a sum

$$\sum_{V_0 \in \mathcal{V}_0(\delta)} \theta(V_0), \tag{12.2}$$

where each $\theta(V_0)$ is a distribution on $M(V_0)$ which is stable on the factors $SO(V_0)$ as in Definition 3.6, and such that each $\theta(V_0)$ agrees on the GL factors. Thus a stable distribution is given by a stable distribution on $SO(2n'+1)$ as in §3, together with families σ_i $(1 \le i \le m)$ of distributions on $GL(2)$ and τ_i $(1 \le i \le s)$ on $GL(1)$.

Let Θ_M be a stable distribution on M. We define

$$Ind_{MN}^{G}(\Theta_M) = \sum_{V_0 \in \mathcal{V}_0} Ind_{M(V_0)N(V_0)}^{SO(V)}(\Theta(V_0)). \tag{12.3}$$

Here $N(V_0)$ is the nilpotent radical of a parabolic subgroup $P(V_0) = M(V_0)N(V_0)$ and $\Theta(V_0)$ is extended trivially to $N(V_0)$ as usual. Since we are interested only in characters the choice of $N(V_0)$ is irrelevant. (The notation MN is purely symbolic.)

Letting $H' = H_{Sp}^{m,r,s}$, we obtain a Levi factor

$$M' \simeq Sp(2n') \times GL(2)^m \times GL(1)^s \tag{12.4}$$

of $Sp(2n, \mathbb{R})$. The isomorphism $\phi : H \simeq H'$ extends to an isomorphism of the GL factors of M and M', which we use to identify them without further comment. Let $M'N'$ be a parabolic subgroup of $Sp(2n, \mathbb{R})$.

The preceding discussion applies to $\widetilde{Sp}(2n, \mathbb{R})$ with only minor changes. We identify the inverse image of a parabolic subgroup MN with $\tilde{M}N$ where $\tilde{M}$ is the inverse image of M.

The matching of stable conjugacy classes for $SO(2n+1)$ and $Sp(2n, \mathbb{R})$ restricts to a bijection of stable conjugacy classes of M and M'. We now define a transfer factor for lifting of invariant eigendistributions from M to the inverse image $\tilde{M}'$ of M' in $\widetilde{Sp}(2n, \mathbb{R})$. This is a slight modification of the obvious one (cf. Remark 12.6).

For H a Cartan subgroup of M let $\Delta_M^+(H)$ be any choice of positive roots of $\mathfrak{h}$ in $\mathfrak{m}$. Choose a standard isomorphism ϕ of H with a Cartan subgroup H' of M', and let $\Delta_{M'}^+(H') = \phi^{\vee -1}\Delta_M^+(H)$. Define $D_M, \epsilon_M, D_{M'}$ and $\epsilon_{M'}$ accordingly. Let $\tilde{\phi}$ be as in Lemma 6.5.

Definition 12.5.
(1) For g' a semisimple element of $\tilde{M}'$, strongly regular for $\widetilde{Sp}(2n, \mathbb{R})$, let

$$\Phi_0(g') = \frac{(D_M \epsilon_M)(g)}{(D_{M'}\epsilon_{M'})(g')} \tag{a}$$

where $g = \tilde{\phi}^{-1}(g')$, and

$$\Phi_0^\dagger(\psi) = \Phi(\psi)\frac{|\Phi_0|}{|\Phi(\psi)|}. \tag{b}$$

(2) For θ_M a stable invariant eigendistribution on M let

$$\Gamma_0(\psi)(\Theta_M)(g') = \Theta_M(g)\Phi_0^\dagger(\psi)(g') \quad (g \overset{stable}{\longleftrightarrow} p(g')).$$

Although Φ_0 depends on the choice of $\Delta_M^+(H)$, its absolute value does not (and factors to $Sp(2n, \mathbb{R})$). Consequently $\Phi_0^\dagger(\psi)$ and $\Gamma_0(\psi)$ depend only on ψ.

Remark 12.6. We may write

$$\Phi_0^\dagger(\psi) = \Phi_0 \frac{\Phi(\psi)/|\Phi(\psi)|}{\Phi_0/|\Phi_0|}$$

and the quotient on the right hand side takes values in $\pm 1, \pm i$. Up to this factor $\Phi_0^\dagger$ is given by (a), which is formally similar to the formula for Φ of Theorem 8.2.

We have one more covering group issue to take care of. Write $\widetilde{Sp}(2n')$ for the inverse image of $Sp(2n')$ in $\widetilde{Sp}(2n,\mathbb{R})$, and $\widetilde{GL}(2)$ and $\widetilde{GL}(1)$ similarly. Then taking $(g_0, g_1, \dots, g_{m+s})$ to $g_0 g_1 \dots g_{m+s}$ (product in $\widetilde{Sp}(2n,\mathbb{R})$) defines a map from

$$\overline{M}' = \widetilde{Sp}(2n') \times \widetilde{GL}(2)^m \times \widetilde{GL}(1)^s$$

to $\tilde{M}' \subset \widetilde{Sp}(2n,\mathbb{R})$. We pull Φ_0 and $\Phi_0^\dagger$ back to functions on (a subset of) $\overline{M}'$. Then (cf. 12.8) Φ_0 and $\Phi_0^\dagger$ factor as products according to this decomposition, and it makes sense to speak of the restriction to each factor.

Lemma 12.7.
(1) The restriction of $\Phi_0^\dagger(\psi)$ to $\widetilde{Sp}(2n')$ is equal the transfer factor $\Phi(\psi)$ defined in §4, applied to $Sp(2n')$. In other words this equals the character of the difference of the two halves of the oscillator representation (defined by ψ) of $\widetilde{Sp}(2n',\mathbb{R})$.
(2) The restriction of $\Phi_0^\dagger(\psi)$ to each copy of $\widetilde{GL}(2)$ or $\widetilde{GL}(1)$ is equal to the character of a genuine one-dimensional representation.

Proof. Just as in (8.9), we have:

$$\Phi_0 = \frac{\prod_{i=1}^t \operatorname{sgn}(1+e^{\beta_i/2})}{\prod_{\alpha\in\Delta^+_{M'}(long)}(e^{\alpha/4}+e^{-\alpha/4})}. \tag{12.8}$$

Here $\beta_1, \dots, \beta_t$ are the long real roots of $\Delta^+_{M'}$.

Since the roots of $\mathfrak{h}'$ in the GL factors of M' are short, Φ_0 is trivial on these factors. Furthermore (12.8) agrees with (8.9) (at least in absolute value) when restricted to the Sp factor of M'. Therefore (dropping ψ from the notation)

$$\Phi_0^\dagger(g') = \begin{cases} \Phi(g') & g' \in \widetilde{Sp}(2n') \\ \Phi(g')/|\Phi(g')| & g' \in \widetilde{GL}(j). \end{cases} \tag{12.9}$$

The long roots of $\Delta^+(H')\backslash\Delta^+_{M'}(H')$ are trivial on $Sp(2n')$, so (1) follows by comparing (12.8) and (8.9). So suppose g' is contained in a copy of $\widetilde{GL}(j)$ $(j=1,2)$ and let $x = p(g')$, considered as an element of $GL(j)$. By (4.4)

$$\begin{aligned}\Phi_0^\dagger(g')^2 &= \operatorname{sgn}(\det(1+x)\det(1+x^{-1})) \\ &= \operatorname{sgn}(\det(x(x^{-1}+1))\det(1+x^{-1})) \\ &= \operatorname{sgn}(\det(x)).\end{aligned} \tag{12.10}$$

In fact $\widetilde{GL}(j)$ is isomorphic to the cover of $GL(j)$ defined by the square-root of the determinant ([26], [33]). This cover comes equipped with a genuine character χ satisfying $\chi^2(g') = \det(x)$, i.e. $(\chi/|\chi|)^2 = \operatorname{sgn}(\det)$, and $\Phi = \chi/|\chi|$. We leave the details to the reader.

The following lemma is now immediate. On the orthogonal/symplectic factors, Γ_0 agrees with the lifting defined previously. On each $GL(j)$ factor it is a bijection between invariant eigendistributions on $GL(j)$ and genuine invariant eigendistribution on $\widetilde{GL}(j)$, given by tensoring with a genuine one-dimensional representation.

Lemma 12.11. $\Theta'_{M'} = \Gamma_0(\Theta_M)$ *is a stable invariant eigendistribution on* $\tilde{M}'$.

We now state a version of the induced character formula (Lemma 12.13). Let $G = SO(2n+1), Sp(2n,\mathbb{R})$, or $\widetilde{Sp}(2n,\mathbb{R})$, with parabolic subgroup $P = MN$, and invariant eigendistribution Θ_M on M, pulled back to P as usual (for $SO(2n+1)$ we are following the conventions discussed at the beginning of this section). Let $\Theta = Ind_P^G(\Theta_M)$. Let g be a strongly regular semisimple element of G, with centralizer H (a Cartan subgroup).

If H is not G-conjugate to a Cartan subgroup of M, then $\Theta(g) = 0$, so assume $g \in M$. Let $H_1 = H, H_2, \dots, H_k$ be representatives of the M-conjugacy classes of Cartan subgroups of M which are conjugate in G. For each H_i choose $g_i \in H_i$ which is G-conjugate to g. For $x \in H_i$ let

$$\begin{aligned}\tau(x) &= |\det(Ad(x)-1)|_{\mathfrak{g}/\mathfrak{m}}|^{-\frac{1}{2}} \\ &= |D_M(x)|/|D_G(x)|. \end{aligned} \tag{12.12}$$

Here D_G and D_M are Weyl denominators for G and M respectively (cf. (8.1)(a)). These depend on choices of positive roots; however $|D_G|$ and $|D_M|$ are independent of the choices (and are well defined even for non-acceptable groups). The induced character formula ([14], [15], [35]) applied to a stable distribution on M gives:

Lemma 12.13. *Let* Θ_M *be a stably invariant eigendistribution on* M. *Then*

$$\Theta(g) = \sum_i \sum_{w \in W_{st}(M,H_i)\backslash W_{st}(G,H_i)} \Theta_M(wg_i)\tau(wg_i).$$

Remark 12.14. The usual induced character formula involves a sum over

$$W(M,H_i)\backslash W(G,H_i).$$

For Θ_M stable we have replaced this by

$$W(M,H_i) \ltimes W(\mathbf{M},\mathbf{H}_i)\backslash W(G,H_i) \ltimes W(\mathbf{M},\mathbf{H}_i) \simeq W_{st}(M,H_i)\backslash W_{st}(G,H_i)$$

(cf. 3.2).

As in ([28], §5) the next lemma follows immediately:

Lemma 12.15. *Let* Θ_M *(resp.* $\Theta_{M'}$*) be a stable invariant eigendistribution on* M *(resp.* $\tilde{M}'$*). Then* $Ind_{MN}^{SO(2n+1)}(\Theta_M)$ *and* $Ind_{\tilde{M}'N'}^{\widetilde{Sp}(2n,\mathbb{R})}(\Theta_{M'})$ *are stable.*

Theorem 12.16. *Let* Θ_M *be a stable invariant eigendistribution on* M. *Then*

$$\Gamma(Ind_{MN}^{SO(2n+1)}(\Theta_M)) = Ind_{\tilde{M}'N'}^{\widetilde{Sp}(2n,\mathbb{R})}(\Gamma_0(\Theta_M)). \tag{12.17}$$

Before proving this we state a result comparing the data of Lemma 12.13 on $SO(2n+1)$ and $\widetilde{Sp}(2n,\mathbb{R})$. Let g be a strongly regular semisimple element of $SO(V)$, and let $H = Cent(g)$. Choose a standard isomorphism $\phi : H \to H'$ with some Cartan subgroup $H' \in Sp(2n,\mathbb{R})$. Let $\{g_i, H_i\}$ $(1 \le i \le k)$ be as in Lemma 12.13, applied to g. Let $g' = \phi(g)$ and choose $\{g'_i, H'_i\}$ $(1 \le i \le k')$ by Lemma 12.13 applied to g'.

Lemma 12.18. *The Cartan subgroups H_i are pairwise isomorphic: $k = k'$, and (renumbering if necessary) we can choose standard isomorphisms*

$$\phi_i : H_i \to H_i', \quad \phi_i(g_i) = g_i' \quad (1 \le i \le k). \tag{12.19)(a}$$

Furthermore we can find isomorphisms

$$\phi_i^W : W_{st}(SO(V), H_i) \to W_{st}(Sp(2n, \mathbb{R}), H_i') \tag{b}$$

satisfying

$$\phi_i^W : W(\mathbf{M}, \mathbf{H}_i) \to W(\mathbf{M}', \mathbf{H}') \tag{c}$$

and

$$\phi_i(wg) = \phi_i^W(w)\phi_i(g) \quad (w \in W_{st}(SO(V), H_i), g \in H_i). \tag{d}$$

Proof. The fact that $k = k'$ and the Cartan subgroups are pairwise isomorphic follows from a simple enumeration of conjugacy classes of Cartan subgroups of M and M'. The remaining assertions follows from Lemmas 7.6 and 7.8.

Proof of Theorem 12.16. We calculate (12.17) at a strongly regular semisimple element of $\widetilde{Sp}(2n, \mathbb{R})$. Since we will be working both in $\widetilde{Sp}(2n, \mathbb{R})$ and $Sp(2n, \mathbb{R})$, let $\tilde{g}'$ be such an element and $g' = p(\tilde{g}')$. If g' is not $Sp(2n, \mathbb{R})$-conjugate to an element of M', then both sides of (12.17) are zero, so assume $g' \in M'$. We compute the value of the left hand side of (12.17) at $\tilde{g}'$.

Let $\tilde{H}' = Cent(\tilde{g}')$ and $H' = p(\tilde{H}') = Cent(g')$. Choose a standard isomorphism $\phi : H \to H'$, and let $g = \phi^{-1}(g')$. Choose $\{g_i, H_i\}, \{\tilde{g}_i', \tilde{H}_i'\}$ as in Lemma 12.13, applied to g and $\tilde{g}'$. Let $g_i' = p(\tilde{g}_i'), H_i = p(\tilde{H}_i')$; equivalently this is the set of Lemma 12.13, applied to g'. By (12.19)(d)

$$\phi_i(wg_i) = p(\phi_i^W(w)\tilde{g}_i') \quad (w \in W_{st}(SO(V), H_i)). \tag{12.20}$$

By the definition of Γ and Lemma 12.13,

$$\Theta'(\tilde{g}') = \Theta(g)\Phi(\tilde{g}') \tag{12.21)(a}$$

$$= \sum_i \sum_w \Theta_M(wg_i)\frac{|D_M(wg_i)|}{|D_G(g)|}\Phi(\tilde{g}') \tag{b}$$

with the inner sum over $W_{st}(M, H_i)\backslash W_{st}(G, H_i)$. By definition of lifting on M we have

$$\Theta_M(wg_i) = \Theta_{M'}(x)/\Phi_0^\dagger(x) \tag{c}$$

for any $x \in p^{-1}(\phi_i(wg_i))$. By (12.20) choose $x = \phi_i^W(w)\tilde{g}_i'$. Inserting this in (b) gives

$$\sum_i \sum_w \Theta_{M'}(\phi_i^W(w)\tilde{g}_i')\frac{|D_M(wg_i)|}{|D_G(g)|}\frac{\Phi(\tilde{g}')}{\Phi_0^\dagger(\phi_i^W(w)\tilde{g}_i')}. \tag{d}$$

By Lemma 12.18 we may write this

$$\sum_i \sum_{w'} \Theta_{M'}(w'\tilde{g}_i')\frac{|D_M(wg_i)|}{|D_G(g)|}\frac{\Phi(\tilde{g}')}{\Phi_0^\dagger(w'\tilde{g}_i')}. \tag{e}$$

with the inner sum over $W_{st}(M',H'_i)\backslash W_{st}(G',H'_i)$, and $w = (\phi_i^W)^{-1}(w')$. This equals

$$\sum_i \sum_{w'} \Theta_{M'}(w'\tilde{g}'_i)\frac{|D_{M'}(w'g'_i)|}{|D_{G'}(g')|}\frac{|D_M(wg_i)|}{|D_G(g)|}\frac{\Phi(\tilde{g}')}{\Phi_0^\dagger(w'\tilde{g}'_i)}\frac{|D_{G'}(g')|}{|D_{M'}(w'g'_i)|}. \tag{f}$$

We claim the product of the final three terms is equal to 1. By stability $\Phi(\tilde{g}') = \Phi(w'\tilde{g}'_i)$, and similarly $|D_G(g)|$ and $|D_{G'}(g')|$. Replacing wg_i by g and $w'\tilde{g}'_i$ by $\tilde{g}'$ it is enough to show

$$\Phi_0^\dagger(\tilde{g}') = \Phi(\tilde{g}')\frac{|D_M(g)|/|D_{M'}(g')|}{|D_G(g)|/|D_{G'}(g')|}, \tag{g}$$

which follows from Theorem 8.2 and Definition 12.5. Therefore

$$\Theta'(g') = \sum_i \sum_{w'} \Theta_{M'}(w'g'_i)\frac{|D_{M'}(w'g'_i)|}{|D_{G'}(g')|} \tag{h}$$

which by Lemma 12.13 is the character of $Ind_{M'N'}^{G'}(\Gamma_0(\Theta_M))$. This completes the proof.

Corollary 12.22. *Let Θ be an invariant eigendistribution on $SO(2n+1)$ and let $\Theta' = \Gamma(\Theta)$. Then Θ is a virtual character if and only if Θ' is a virtual character.*

Proof. The definition of the stable sum of discrete series representations with a given infinitesimal character for $SO(2n+1)$ or $\widetilde{Sp}(2n,\mathbb{R})$ extends in the obvious way to Levi factors M, M' (with relative discrete series representations on the $GL(2)$ factors). We refer to a (virtual) representation induced from such a sum as a *stable standard module*. By Theorem 12.17 and Proposition 11.3 Γ takes stable standard modules for $SO(2n+1)$ to stable standard modules for $\widetilde{Sp}(2n,\mathbb{R})$, and vice-versa. By ([28], Lemma 5.2, cf. [5], Lemma 18.11) the stable standard modules span the stable virtual characters of $SO(2n+1)$. Therefore if Θ is a virtual character then Θ' is also.

Conversely suppose Θ' is a stable virtual representation of $\widetilde{Sp}(2n,\mathbb{R})$. By [24] the complex-virtual characters are characterized among all invariant eigendistributions by the condition that all coefficients $p(w,g)$ of (10.6) are complex constants. (By complex-virtual character we mean a finite linear combination of irreducible characters with complex coefficients.) Therefore the coefficients appearing in (10.6) for Θ' are (complex) constants, and by Proposition 9.3 the same holds for Θ. Therefore Θ is a sum of stable standard modules with complex coefficients. By Theorem 12.17 the same holds for Θ', and these coefficients must therefore be integers. Therefore Θ is a virtual character.

Remark 12.23. Corollary 12.22 implies the stable virtual characters for $\widetilde{Sp}(2n,\mathbb{R})$ are spanned by the stable standard characters. This can also be proven directly, from which the Corollary follows. Alternatively, it is a folk-theorem that [24] holds with integers in place of complex numbers, i.e. the virtual characters are precisely the invariant eigendistributions for which all coefficients (10.6) are integers. The Corollary follows from this as well.

We now discuss the relation with theta-lifting. Associated to the dual pair $(O(p,q), Sp(2n,\mathbb{R}))$ in $Sp(2n(2n+1),\mathbb{R}))$ $(p+q=2n+1)$ is the dual pair correspondence, or theta-lifting [20]. This is a correspondence between irreducible genuine representations of $\widetilde{Sp}(2n,\mathbb{R})$ and a certain two-fold cover of $O(p,q)$. After tensoring with a genuine one-dimensional representation of this cover it may be written on $O(p,q)$. Furthermore for each irreducible representation π of $O(p,q)$ precisely one of π and $\pi\otimes sgn$ occur in the correspondence, and we may restrict to $SO(p,q)$.

Proposition 12.24 ([3], Corollary 5.3). *Fix $\delta=\pm1$ and ψ. Then the dual pair correspondence defines a bijection*

$$\bigcup_{\substack{p+q=2n+1\\(-1)^q=\delta}} SO(p,q)\widehat{\ } \leftrightarrow \widetilde{Sp}(2n,\mathbb{R})\widehat{\ }_{genuine}.$$

Suppose π is an irreducible representation of $SO(p,q)$, and $\pi\leftrightarrow\pi'$ in this correspondence. Define the *normalized theta–correspondence*

$$\Gamma_\theta(\psi,\delta)(\pi)=(-1)^{q(V)+q(Sp)}\pi' \tag{12.25}$$

and extend by linearity to virtual representations. By ([3], Lemma 1.3)

$$\Gamma_\theta(\psi,\delta)=\Gamma_\theta(-\psi,-\delta). \tag{12.26}$$

Theorem 12.27. *Let π be a stable, tempered virtual representation of $SO(2n+1)$. For $\delta=\pm1$,*

$$\Gamma(\psi)(\pi)=\Gamma_\theta(\delta(-1)^n\psi,\delta)(\pi). \tag{12.28}$$

Proof. For discrete series representations this follows from Proposition 11.3 and ([3], Theorem 3.3). Note that in this case the choice of ψ is irrelevant. In general it follows by comparing Theorem 12.16 and the main theorem (5.1) of [3]. The only issue is to compare the character, call it $\eta(\psi)$, on GL factors coming from (12.9) with the character $\chi(\psi,V_\delta)$ of [3] (here V is an orthogonal space of dimension $2n+1$ and discriminant δ). The result is

$$\eta(\psi)=\chi(\delta(-1)^n\psi,V_\delta) \tag{12.29}$$

and the theorem follows from this.

Remark 12.30. By (12.26) the right hand side of (12.28) is independent of δ, as it must be.

Remark 12.31. It is necessary to consider π to be defined on all real forms for this result to hold. For example the stable discrete series of $SO(2n+1)$ with a given infinitesimal character is the sum of 2^n irreducible representations on various $SO(p,q)$. These correspond bijectively to the 2^n discrete series representations of $\widetilde{Sp}(2n,\mathbb{R})$, both via Γ and Γ_θ. While Γ gives the same result when applied to $SO(n+1,n)$, Γ_θ does not.

Remark 12.32. Theorem 12.27 fails for some non-tempered representations. For example the trivial representation of $SO(2n+1)$, considered as the constant function, corresponds via Γ to $\omega(\psi)_{even} - \omega(\psi)_{odd}$. However as a virtual representation it is the sum of the trivial representations on the $n+1$ real forms of $SO(2n+1)$, which correspond via Γ_θ to the sum of $n+1$ irreducible representations of $\widetilde{Sp}(2n, \mathbb{R})$. These clearly cannot agree for $n > 1$.

§13. Complex Groups.

The analogues for complex groups of the preceding results are substantially easier and in some sense well known. We will be very brief.

Throughout this section let $G = SO(2n+1, \mathbb{C})$, $G' = Sp(2n, \mathbb{C})$. The orbit and conjugacy class correspondences are defined as in the real case. The conjugacy class correspondence is a bijection between strongly regular semisimple conjugacy classes of G and G'. Recall for a complex group stable conjugacy is the same as conjugacy, and every invariant distribution is stable.

Let $\Omega = \Omega_{even} + \Omega_{odd}$ be the character of the oscillator representation, considered as a function on the strongly regular semisimple elements of H'. Let

$$\Phi = \Omega_{even} - \Omega_{odd}. \tag{13.1}$$

If Θ is an invariant function on G, let

$$\Gamma(\Theta)(g') = \Theta(g)\Phi(g') \quad (g \leftrightarrow g'). \tag{13.2}$$

Let H be a Cartan subgroup of G. The virtual characters of G are spanned by the standard characters $I_G(\lambda_1, \lambda_2)$ [36]. Here $\lambda_i \in \mathfrak{h}^*$ and

$$\lambda_1 - \lambda_2 \in X(H) \tag{13.3}$$

where $X(H)$ is the kernel of the exponential map $exp : \mathfrak{h}_0 \to H$. Analogous statements hold for H' a Cartan subgroup of G'. Let $\phi : H \to H'$ be an isomorphism satisfying $g \leftrightarrow \phi(g)$ for all strongly regular elements $g \in H$ (cf. Lemma 5.1). Then $d\phi$ takes $X(H)$ to $X(H')$.

The proofs in section 10 and 12 carry over easily to this setting. Together with [4] this gives:

Theorem 13.4. *The map $\Gamma : \Theta \to \Theta' = \Gamma(\Theta)$ is a bijection between the invariant eigendistributions on G and G'. Furthermore*

(1) *Θ is tempered if and only if Θ' is tempered,*
(2) *Θ is a virtual character if and only if Θ' is a virtual character,*
(3) *Suppose $I_G(\lambda_1, \lambda_2)$ is a standard module, and let $\lambda_i' = d\phi^{*-1}\lambda_i$ $(i = 1, 2)$. Then*
$$\Gamma(I_G(\lambda_1, \lambda_2)) = I_{G'}(\lambda_1', \lambda_2').$$
(4) *If Θ is a tempered virtual character then $\Gamma(\Theta)$ is the theta-lift of Θ.*

Choose positive roots $\Delta^+(H)$ of $\mathfrak{h}$ in $\mathfrak{g}$, and define the Weyl denominator D and numerator $\kappa(\Theta, g)$ accordingly. Similarly define D' and $\kappa(\Theta', g')$ for G' where $d\phi^*(\Delta^+(H')) = \Delta^+(H)$. Then (cf. [1], §6)

$$\Phi(g') = \frac{D(g)}{D'(g')} \quad (g = \phi^{-1}(g')). \tag{13.5}$$

If $\Theta' = \Gamma(\Theta), g' = \phi(g)$, then as in Proposition 9.3

$$\kappa(\Theta', g') = \kappa(\Theta, g) \tag{13.6}$$

and this could be taken as the definition of $\Gamma : \Theta \to \Theta'$.

For $\lambda \in \mathfrak{h}^*$ let $\mathcal{F}(H, \lambda)$ be the space of analytic functions on H satisfying

$$\begin{aligned} \kappa(wg) &= \kappa(g) \qquad (w \in W(G, H)) \\ D(\kappa) &= \lambda(D)\kappa \quad (D \in I(\mathfrak{h})). \end{aligned} \tag{13.7}$$

Here $I(\mathfrak{h})$ is the left invariant differential operators on $\mathfrak{h}$ (cf. §10). Define $\mathcal{F}(H', \lambda')$ similarly.

By the matching conditions the map $\Theta \to \kappa(\Theta, *)$ is an isomorphism between invariant distributions with eigenvalue λ and $\mathcal{F}(H, \lambda)$, and similarly for G' [17]. Therefore Γ may be interpreted as an isomorphism $\Gamma : \mathcal{F}(H, \lambda) \to \mathcal{F}(H', \lambda')$ with $\lambda' = d\phi^{*-1}(\lambda)$. Then Γ is simply the isomorphism induced by ϕ, i.e.

$$\Gamma(f)(g') = f(\phi^{-1}(g')). \tag{13.8}$$

In these terms the isomorphism is discussed in detail in ([16], Appendix II).

For example Γ takes the trivial representation of $SO(2n+1, \mathbb{C})$ to the difference of the two halves of the oscillator representation of $Sp(2n, \mathbb{C})$. It is interesting to note that the trivial representation is a special unipotent representation in the sense of [8] while this is not the case for the oscillator representation.

§14. Coherent Continuation.

It follows directly from the definition that Γ takes a coherent family for $SO(2n+1)$ to one for $\widetilde{Sp}(2n, \mathbb{R})$. As a consequence it commutes with the coherent continuation action of the Weyl group, and certain translation functors.

As in §11 let T be a compact Cartan subgroup of $SO(2n+1)$, with $X^*(T)$ the lattice of differentials of characters of T. This is the lattice of weights of finite dimensional representations of $SO(2n+1)$. Fix an arbitrary element λ_0 of $\mathfrak{t}^*$

Following ([31], Definition 7.2.5) and ([14], 3.38) we define a coherent family for $SO(V)$ to be a collection of virtual characters $\{\Theta(\lambda)\}$ of $SO(V)$ parametrized by $\lambda \in \lambda_0 + X^*(T)$, satisfying the following two conditions:

(1) $\Theta(\lambda)$ has infinitesimal character λ,
(2) For any finite dimensional representation F of $SO(2n+1, \mathbb{C})$

$$\Theta(\lambda) \otimes F = \sum_{\mu \in \Delta(F)} \Theta(\lambda + \mu). \tag{14.2}$$

Here F is considered a representation of $SO(V)$ by restriction and $\Delta(F) \subset \mathfrak{t}^*$ is the set of weights of $\mathfrak{t}$ in F.

Since $SO(V)$ may be disconnected, this is slightly cruder than ([31], Definition 7.2.5), in that we impose (14.2) only for finite dimensional representations of $SO(V)$ which extend to $SO(2n+1, \mathbb{C})$. This excludes, for example, the non-trivial one-dimensional representation ζ of $SO(p, q)$ $(pq \neq 0)$. In particular a coherent family in the sense of [31] must satisfy $\Theta(\lambda)\otimes\zeta\simeq\Theta(\lambda)$; we do not impose this condition. With this definition it is equivalent to work with coherent families based on T rather than on the split Cartan subgroup. In any event it is this definition which matches up correctly with coherent continuation on $\widetilde{Sp}(2n,\mathbb{R})$.

The condition that $\Theta(\lambda)$ is stable is independent of λ (for λ regular) and we say the family is stable if each $\Theta(\lambda)$ is stable. It then makes sense to define stable coherent families for $SO(2n+1)$. Such a family may be identified with a stable coherent family for the split form (cf. Remark 3.8).

Now fix a compact Cartan subgroup T' of $Sp(2n,\mathbb{R})$ and consider the usual coherent families parametrized by $\lambda'_0+X^*(T')\subset\mathfrak{t}'^*$. Note that $X^*(T')$ is the lattice of differentials of weights of finite dimensional representations of $\widetilde{Sp}(2n,\mathbb{R})$, equivalently $Sp(2n,\mathbb{R})$, and since $Sp(2n,\mathbb{R})$ is connected there is no question of which finite dimensional representations to use.

Choose any standard isomorphism $\phi:T\to T'$; then $d\phi^{*-1}:X^*(T)\to X^*(T')$. In particular we may choose T,T' and ϕ as in §7.

Proposition 14.3. *Let $\{\Theta(\lambda)\,|\,\lambda\in\lambda_0+X^*(T)\}$ be a coherent family of stable virtual characters of $SO(2n+1)$. Let $\lambda'_0=d\phi^{*-1}(\lambda_0)$. For $\lambda'\in\lambda'_0+X^*(T')$ let $\lambda=d\phi^*(\lambda')\in\lambda_0+X^*(T)$ and set*

$$\Theta'(\lambda')=\Gamma(\Theta(\lambda)).$$

Then $\{\Theta'(\lambda')\,|\,\lambda'\in\lambda'_0+X^*(T')\}$ is a coherent family of genuine virtual characters.

Conversely Γ^{-1} takes a genuine coherent family for $\widetilde{Sp}(2n,\mathbb{R})$ to a coherent family for $SO(2n+1)$.

Proof. This follows immediately from Proposition 9.3 and ([14], Lemma 3.44), which characterizes coherent families in terms of character formulas on each Cartan subgroup.

For $\lambda\in\mathfrak{t}^*$, let

$$W(\lambda)=\{w\in W(\mathfrak{so}(2n+1,\mathbb{C}),\mathfrak{t})\,|\,w\lambda-\lambda\in X^*(T)\}. \qquad (14.4)(a)$$

Since $X^*(T)$ is the root lattice of $\mathfrak{t}$ in $\mathfrak{so}(2n+1,\mathbb{C})$, this agrees with ([31],7.2.16). The coherent continuation action of $W(\lambda)$ on virtual characters of $SO(V)$ with infinitesimal character λ is defined as in ([31], Definition 7.2.28). We write $\Theta\to w\cdot\Theta$. This extends to an action on stable virtual $SO(2n+1)$ modules (for example using Remark 3.8).

For $\lambda'\in\mathfrak{t}'^*$ let

$$W(\lambda')=\{w\in W(\mathfrak{sp}(2n,\mathbb{C}),\mathfrak{t}')\,|\,w\lambda'-\lambda'\in X^*(T')\}. \qquad (b)$$

Since the root lattice is strictly contained in $X^*(T')$ this condition is weaker than ([31], 7.2.16); nevertheless the definition in [31] extends to this case and defines an action of $W(\lambda')$ on virtual characters for $\widetilde{Sp}(2n,\mathbb{R})$ with infinitesimal character λ' (cf. [1]).

Corollary 14.5. *Let Θ be a stable virtual $SO(2n+1)$ module with infinitesimal character λ, and suppose $w \in W(\lambda)$. Then*

$$\Gamma(w \cdot \Theta) = w' \cdot \Gamma(\Theta).$$

where $w' = \phi^W(w)$.

Proof. This follows immediately from the definition of coherent continuation and Proposition 14.3.

Remark 14.6. Care must be taken when applying this result to w for which $w\lambda' - \lambda'$ is in $X^*(T')$ but not in the root lattice. (This is the case where w satisfies (14.4)(b) but not the definition of $W(\lambda')$ in ([31],7.2.16).) For example holds if w is reflection in a long root (cf. [1], Remark 2.10).

Now suppose $\lambda_1, \lambda_2 \in \mathfrak{t}^*$, and $\lambda_1 - \lambda_2 \in X^*(T)$. The translation functor $\psi_{\lambda_1}^{\lambda_2}$ ([31], Definition 4.5.7) takes virtual characters for $SO(V)$ with infinitesimal character λ_1 to virtual characters with infinitesimal character λ_2. This definition makes sense for stable virtual $SO(2n+1)$ modules, as well as for stable virtual $\widetilde{Sp}(2n, \mathbb{R})$ modules.

Corollary 14.7. *Suppose $\lambda_1, \lambda_2 \in \mathfrak{t}^*$ satisfy*

(1) $\lambda_2 - \lambda_1 \in X^*(T)$,
(2) λ_1 *is regular,*
(3) λ_2 *is dominant for the positive integral roots defined by λ_1 ([31], Definition 7.2.16).*

Let Θ be a stable virtual character on $SO(2n+1)$ with infinitesimal character λ_1. Then

$$\Gamma\psi_{\lambda_1}^{\lambda_2}(\Theta) = \psi_{\lambda_1'}^{\lambda_2'}(\Gamma(\Theta))$$

where $\lambda_i' = d\phi^{*-1}(\lambda_i) \quad (i = 1, 2)$.

Proof. This follows immediately from Proposition 14.3 and ([31], Lemma 7.2.15).

By a result of Zuckerman the trivial representation (of a connected group) may be written as an alternating sum of coherently continued discrete series representations. We stabilize this formula and apply it to $SO(2n+1)$. Let $\overline{\pi}_{SO}$ be the averaged discrete series of $SO(2n+1)$ with the same infinitesimal character as the trivial representation (cf. 11.2). Zuckerman's result (cf. [6], Lemma 7.3) implies:

$$\frac{2}{|W|}\sum_{w \in W} \mathrm{sgn}(w) w \cdot \overline{\pi}_{SO} = \mathrm{trivial} + \zeta. \tag{14.8}$$

(The factor of 2 comes from the fact that $SO(p, q)$ is disconnected if $pq \neq 0$.) Here $W = W(\mathfrak{so}(2n+1, \mathbb{C}), \mathfrak{t}) \simeq W(\mathfrak{sp}(2n, \mathbb{C}), \mathfrak{t}')$. Let $\overline{\pi}_{Sp}$ be the averaged discrete series for $\widetilde{Sp}(2n, \mathbb{R})$ with the same infinitesimal character as the oscillator representation. We apply Γ to both sides. Recall $\Gamma(trivial) = \omega(\psi_+)_{even} - \omega(\psi_+)_{odd}$, and by (4.9, 4.11) $\Gamma(\zeta)=\omega(\psi_-)_{even}-\omega(\psi_-)_{odd}$. By (8.4), Corollary 14.5 and Proposition 11.3 we conclude

$$\begin{aligned}\frac{2}{|W|}\sum_{w\in W}\mathrm{sgn}(w)w\cdot\overline{\pi}_{Sp}&=(\omega(\psi_+)_{even}-\omega(\psi_+)_{odd})+(\omega(\psi_-)_{even}-\omega(\psi_-)_{odd})\\&=\Phi(\psi_+)+\Phi(\psi_-).\end{aligned}\tag{14.9}$$

This is a slightly weaker version of ([1], Theorem 2.3) which was used to compute $\Phi(\psi)$.

§15. Example: n=1.

We discuss the case of $(SO(3),\widetilde{SL}(2,\mathbb{R}))$ in some detail. As is well known this is closely related to the Shimura correspondence (cf. [34] and the references there), via the isomorphism $PGL(2,\mathbb{R})\simeq SO(2,1)$. Theorem 5.2 of [9] gives similar character relations (in the p-adic case as well).

Let $SO(2,1)$ be the set of 3×3 real matrices g satisfying $gKg^t=K$ with $K=diag(1,1,-1)$. Then $\mathfrak{so}(2,1)=\{X\,|\,XK+KX^t=0\}$. As in Remark 3.8 we work only on the split group $Spin(2,1)$ which is the inverse image of $SO(2,1)$ in $Spin(3,\mathbb{C})$.

For $z\in\mathbb{C}$ let $X(z)=\begin{pmatrix}0&&\\&0&z\\&z&0\end{pmatrix}, Y(z)=\begin{pmatrix}0&z&\\-z&0&\\&&0\end{pmatrix}\in\mathfrak{so}(2,1)$. For $c\in\mathbb{R}\oplus\pi i\mathbb{Z}$ let $\tilde a(c)=exp(X(c))\in Spin(2,1)$. This gives the split Cartan subgroup A of $Spin(2,1)$, which is isomorphic to $\mathbb{R}^*\cup i\mathbb{R}^*$. For $\theta\in\mathbb{R}$ let $t(\theta)=exp(Y(\theta))\in Spin(2,1)$; this gives the compact Cartan subgroup T. Note that $\tilde a(c+4\pi i)=\tilde a(c)$, $\tilde t(\theta+4\pi)=\tilde t(\theta)$, $p(\tilde a(c+2\pi i))=p(\tilde a(c))$, and $p(\tilde t(\theta+2\pi))=p(\tilde t(\theta))$ ($p: Spin(2,1)\to SO(2,1)$).

The intersection of A and T consists of $1=\tilde a(0)=\tilde t(0)$ and $z=\tilde a(2\pi i)=\tilde t(2\pi)$; z is the non-trivial element in the center of $Spin(2,1)$.

Now consider $SL(2,\mathbb{R})$ and $\widetilde{SL}(2,\mathbb{R})$. Let

$$X'(z)=diag(z,-z), Y'(z)=\begin{pmatrix}0&z\\-z&0\end{pmatrix}\in\mathfrak{sl}(2,\mathbb{C}).$$

Let $\tilde t'(\theta)=exp(Y(\theta))$ ($\theta\in\mathbb{R}$), this is the compact Cartan subgroup $\tilde T'$ of $\widetilde{SL}(2,\mathbb{R})$. For $c=c_0+i\theta\in\mathbb{R}\oplus\pi i\mathbb{Z}$ let $\tilde a'(c)=\tilde t'(\theta)exp(X'(c_0))$; this is an element of a split Cartan subgroup A' of $\widetilde{SL}(2,\mathbb{R})$. The intersection of A' and T' is the center of $\widetilde{SL}(2,\mathbb{R})$, consisting of the four elements $\tilde a'(k\pi i)=\tilde t'(k\pi)$ ($0\le k\le 3$). Let $z'=\tilde a'(\pi i)=\tilde t'(\pi)$. The inverse image of 1 in $\widetilde{SL}(2,\mathbb{R})$ is $1,{z'}^2$, so a representation π is genuine if $\pi({z'}^2)=-1$, i.e. $\pi(z')=\pm i$.

Let $\psi=\psi_\pm$. The positive roots are $\alpha(X(z))=\mp z$, $\alpha'(X'(z))=\mp 2z$, $\beta(Y(z))=\mp iz$, $\beta'(Y'(z))=\mp 2iz$ (recall (§7) the corresponding root vectors annihilate a vector in $\omega(\overline{\psi})$). Choose the Cayley transform $c: Y'(z)=X'(iz)$. The isomorphisms ϕ of Lemma 6.5 are given by $\phi(\tilde a(c))=\tilde a'(c)$ and $\phi(\tilde t(\theta))=\tilde t'(\theta)$. Note that the map $\phi(\tilde a(c))=\tilde a'(\overline{c})$ is also an isomorphism of A with A' with the same differential. However this map does not satisfy the conditions of Lemma 6.5.

We next compute the Weyl denominators. On the compact Cartan subgroups we have

$$(D\epsilon)(\tilde{t}(\theta)) = e^{\mp i\theta/2} - e^{\pm i\theta/2} \tag{15.1)(a}$$

$$(D'\epsilon')(\tilde{t}'(\theta)) = e^{\mp i\theta} - e^{\pm i\theta} \tag{b}$$

The formulas on the split Cartan subgroups are as follows. We always write

$$c = c_0 + \pi i k \quad (c_0 \in \mathbb{R}, k \in \mathbb{Z}). \tag{15.2}$$

With this notation:

$$\begin{aligned}(D\epsilon)(\tilde{a}(c)) &= (e^{\mp c/2} - e^{\pm c/2})sgn(1 - e^{\pm c})\\ &= (\mp i)^k |e^{c/2} - e^{-c/2}|\end{aligned} \tag{15.3)(a}$$

$$\begin{aligned}(D'\epsilon')(\tilde{a}'(c)) &= (e^{\mp c} - e^{\pm c})sgn(1 - e^{\pm 2c})\\ &= (-1)^k |e^{c} - e^{-c}|.\end{aligned} \tag{b}$$

Then

$$\frac{(D\epsilon)(\tilde{t}(\theta))}{(D'\epsilon')(\tilde{t}'(\theta))} = \frac{1}{e^{i\theta/2} + e^{-i\theta/2}} \tag{15.4)(a}$$

and

$$\frac{(D\epsilon)(\tilde{a}(c))}{(D'\epsilon')(\tilde{a}'(c))} = \frac{(\pm i)^k}{|e^{c/2} + e^{-c/2}|} \tag{b}$$

which by a short calculation agree with $\Phi(\psi_\pm)$ (cf. (8.9)).

We describe the standard characters of $Spin(2,1)$; these form a basis of the virtual characters. In fact both for $Spin(2,1)$ and $\widetilde{SL}(2,\mathbb{R})$ all invariant eigendistributions are virtual characters ([18],5.3). For Θ an invariant eigendistribution write $\Theta_T(\theta) = \Theta(\tilde{t}(\theta))$ and $\Theta_A(c) = \Theta(\tilde{a}(c))$.

$Spin(2,1), \pi_{DS}(\lambda) \quad (\lambda = \frac{1}{2}, 1, \frac{3}{2}, \dots)$:

$$\Theta_T(\theta) = \frac{-e^{i\lambda\theta} + e^{-i\lambda\theta}}{e^{i\theta/2} - e^{-i\theta/2}} \tag{15.5)(a}$$

$$\Theta_A(c) = \begin{cases} 0 & k \notin 2\mathbb{Z} \\ \dfrac{2e^{\pi i k(\lambda + \frac{1}{2})} e^{-\lambda|c_0|}}{|e^{c/2} - e^{-c/2}|} & k \in 2\mathbb{Z}. \end{cases} \tag{b}$$

This is the discrete series representation with Harish-Chandra parameter λ. The central element z acts by $-(-1)^{2\lambda}$ so $\pi_{DS}(\lambda)$ factors to $SO(2,1)$ if and only if $\lambda \in \mathbb{Z} + \frac{1}{2}$.

Next consider the principal series character $\pi_{PS}(m,\lambda)$ induced from the character $\chi(\tilde{a}(c)) = e^{\pi i k m} e^{\lambda c_0}$ of A. By the induced character formula this character vanishes on T and on A is given by:

$Spin(2,1), \pi_{PS}(m,\lambda) \quad (m \in \frac{1}{2}\mathbb{Z}, \lambda \in \mathbb{C})$:

$$\Theta(c) = \frac{e^{\pi i k m}(e^{\lambda c_0} + (-1)^{2km} e^{-\lambda c_0})}{|e^{c/2} - e^{-c/2}|}. \tag{15.6}$$

Note that $\pi_{PS}(m,\lambda)=\pi_{PS}(m+2,\lambda)$ and $\pi_{PS}(m,\lambda)=\pi_{PS}(-m,-\lambda)$. In this case z acts by $(-1)^{2m}$, so $\pi_{PS}(m,\lambda)$ factors to $SO(2,1)$ if and only if $m\in\mathbb{Z}$.

The infinitesimal character of $\pi_{DS}(\lambda)$ and $\pi_{PS}(m,\lambda)$ is λ.

We turn next to $\widetilde{SL}(2,\mathbb{R})$.

$\widetilde{SL}(2,\mathbb{R}),\pi'_{DS}(\epsilon,\lambda)\quad(\epsilon=\pm,\lambda=\frac{1}{2},1,\frac{3}{2},\dots)$:

$$\Theta_{T'}(\theta)=\frac{-\epsilon e^{i\epsilon\lambda\theta}}{e^{i\theta}-e^{-i\theta}}\tag{15.7)(a}$$

$$\Theta_{A'}(c)=\frac{e^{\epsilon\pi ik(\lambda+1)}e^{-\lambda|c_0|}}{|e^c-e^{-c}|}.\tag{b}$$

This is the discrete series. In addition taking $\lambda=0$ gives the limit of discrete series. The element z' of the center of $\tilde{G}'$ acts by $i^{-2\epsilon\lambda}$, so ${z'}^2$ acts by $(-1)^{2\lambda}$. In particular π is genuine if and only if $\lambda\in\mathbb{Z}+\frac{1}{2}$. Note that $\pi'_{DS}(+,\lambda),\pi'_{DS}(-,\lambda)$ have distinct central characters in the genuine case.

We stabilize the discrete series: let $\overline{\pi}'_{DS}(\lambda)=\pi'_{DS}(+,\lambda)+\pi'_{DS}(-,\lambda)$ (we have omitted the sign $(-1)^q$ of (11.1)). The character formula for $\overline{\pi}'_{DS}(\lambda)$ is:

$\widetilde{SL}(2,\mathbb{R}),\overline{\pi}'_{DS}(\lambda)\quad\lambda=\frac{1}{2},1,\frac{3}{2},\dots$:

$$\Theta_{T'}(\theta)=\frac{-e^{i\lambda\theta}+e^{-i\lambda\theta}}{e^{i\theta}-e^{-i\theta}}\tag{15.8)(a}$$

$$\Theta_{A'}(c)=\begin{cases}0 & \lambda\notin\mathbb{Z},k\notin2\mathbb{Z}\\ \dfrac{2e^{\pi ik(\lambda+1)}e^{-\lambda|c_0|}}{|e^c-e^{-c}|} & \lambda\in\mathbb{Z}\text{ or }k\in2\mathbb{Z}.\end{cases}\tag{b}$$

Note that in the genuine case, i.e. $\lambda\in\mathbb{Z}+\frac{1}{2}$, the character of $\overline{\pi}'_{DS}(\lambda)$ vanishes on the two components $\pm i\mathbb{R}^*$ (given by $k=1,3$) of A. This is an example of Lemma 10.7, and corresponds to the fact that the character of a discrete series representation of $SO(2,1)$ vanishes on $-\mathbb{R}^+$, since this not in the identity component of $SO(2,1)$ (cf. Remark 10.12).

Finally we have the principal series attached to the character $\chi(\tilde{a}'(c))=e^{\pi ikm}e^{\lambda c_0}$.

$\widetilde{SL}(2,\mathbb{R}),\pi'_{PS}(m,\lambda)\quad(m\in\frac{1}{2}\mathbb{Z},\lambda\in\mathbb{C})$:

$$\Theta_{A'}(c)=\frac{e^{\pi ikm}(e^{\lambda c_0}+e^{-\lambda c_0})}{|e^c-e^{-c}|}.\tag{15.9}$$

Note that $\pi'_{PS}(m,\lambda)=\pi'_{PS}(m+2,\lambda)$ and $\pi'_{PS}(m,\lambda)=\pi'_{PS}(m,-\lambda)$. In this case z' acts by i^{2m}, so this is genuine if and only if $m\in\mathbb{Z}+\frac{1}{2}$.

These characters span the virtual characters. The stable virtual characters are spanned by the principal series and the averaged discrete series.

We see (15.5)(a,b)×(15.4)(a,b) equals (15.8)(a,b) (using ϕ) provided $\lambda\in\mathbb{Z}+\frac{1}{2}$. This shows $\Gamma(\pi_{DS}(\lambda))=\overline{\pi}'_{DS}(\lambda)$ (independent of ψ). Similarly for $m\in\mathbb{Z}$ (15.6)×(15.4)(b) equals (15.9), provided we replace m by $m\pm\frac{1}{2}$ in (15.9). This proves $\Gamma(\psi_\pm)\pi_{PS}(m,\lambda)=\pi'_{PS}(m\pm\frac{1}{2},\lambda)$. We summarize these results.

Proposition 15.10. *Let* $\Gamma = \Gamma(\psi_\pm)$.

(1) $\Gamma(\pi_{DS}(\lambda)) = \overline{\pi}'_{DS}(\lambda), \quad \lambda = \frac{1}{2}, \frac{3}{2}, \dots,$

(2) $\Gamma(\pi_{PS}(m,\lambda)) = \pi'_{PS}(m \pm \frac{1}{2}, \lambda) \quad (m \in \mathbb{Z}; \lambda \in \mathbb{C})$.

Since we are only interested in stable characters we don't need to compute all irreducible characters, i.e. reducibility of the principal series. For representations factoring to $SL(2,\mathbb{R})$ this is well-known, and for genuine representations see ([10], Lemma 4.1). We mention only that for $m \in \mathbb{Z} + \frac{1}{2}$, $\pi'_{PS}(m,\lambda)$ is reducible if and only if $\lambda \in \mathbb{Z} + \frac{1}{2}$. In this case it has two composition factors, a discrete series representation $\pi'_{DS}((-1)^{|\lambda|+m}, |\lambda|)$ and an irreducible non-tempered representation.

For example consider representations of $\widetilde{SL}(2,\mathbb{R})$ with the same infinitesimal character as the oscillator representation, i.e. $\lambda = \frac{1}{2}$. The discrete series representations $\pi'_{DS}(\pm, \frac{1}{2})$ are the odd halves of the oscillator representations $\omega(\pm)_{odd}$ (with lowest K-type $\pm\frac{3}{2}$). The (characters of the) two principal series representations $\pi'_{PS}(\pm\frac{1}{2}, \frac{1}{2})$ decompose as

$$\begin{aligned}\pi'_{PS}(\tfrac{1}{2}, \tfrac{1}{2}) &= \omega(\psi_+)_{even} + \pi'_{DS}(-, \tfrac{1}{2}) \\ &= \omega(\psi_+)_{even} + \omega(\psi_-)_{odd} \qquad (15.11)(a) \\ \pi'_{PS}(-\tfrac{1}{2}, \tfrac{1}{2}) &= \omega(\psi_-)_{even} + \pi'_{DS}(+, \tfrac{1}{2}) \\ &= \omega(\psi_-)_{even} + \omega(\psi_+)_{odd} \qquad (b)\end{aligned}$$

and $\omega(\psi_\pm)_{even}$ are non-tempered (with lowest K-type $\pm\frac{1}{2}$). There are four representations $\omega(\psi_\pm)_{odd,even}$ with this infinitesimal character. The subspace of stable virtual characters is three dimensional, spanned by $\overline{\pi}'_{DS}(\frac{1}{2})$ and $\pi'_{PS}(\pm\frac{1}{2}, \frac{1}{2})$. Another stable sum is

$$\begin{aligned}\pi'_{PS}(\tfrac{1}{2}, \tfrac{1}{2}) - \overline{\pi}'_{DS}(\tfrac{1}{2}) &= \omega(\psi_+)_{even} + \omega(\psi_-)_{odd} - (\omega(\psi_+)_{odd} + \omega(\psi_-)_{odd}) \\ &= \omega(\psi_+)_{even} - \omega(\psi_+)_{odd}\end{aligned} \qquad (15.12)$$

(cf. Corollary 8.10) and $\omega(\psi_-)_{even} - \omega(\psi_-)_{odd}$ similarly.

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