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A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors

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Contents

1. Introduction

One of the most important developments in the theory of quantum groups has been the discovery of canonical bases which have remarkable integrality and positivity properties ([Lu]). Using these bases one expects to formulate the representation theory of quantum groups entirely over natural numbers. In the case of the simplest quantum group $U_q(\mathfrak{sl}_2)$, such a formulation can be achieved via the Penrose-Kauffman graphical calculus (see [FK]). The positive integral structure of representation theory suggests that it is itself a Grothendieck ring of a certain tensor 2-category. A strong support of this anticipation comes from identifying coefficients of the transition matrix between the canonical and elementary bases in the n-th tensor power of the two-dimensional fundamental representation V_1 of $U_q(\mathfrak{sl}_2)$ with the Kazhdan-Lusztig polynomials associated to $\mathfrak{gl}_n$ for the maximal parabolic subalgebras ([FKK]).

In this paper we will take one more step towards constructing a tensor 2-category with the Grothendieck ring isomorphic to the representation category for $U_q(\mathfrak{sl}_2)$. The construction of tensor categories or 2-categories with given Grothendieck groups will be referred to as "categorification". We obtain a categorification of the $U(\mathfrak{sl}_2)$ action in $V_1^{\otimes n}$ and the action of its commutant, the Temperley-Lieb algebra, using projective and Zuckerman functors between certain representation categories of $\mathfrak{gl}_n$. We extend this categorification to the comultiplication of $U(\mathfrak{sl}_2)$. Our results are strongly motivated by the papers [BLM], [GrL] and [Gr], where the authors use the geometric rather than algebraic approach. In the geometric setting the categorification can be obtained via the categories of perverse sheaves. It is expected that the algebraic and geometric languages will be equivalent, however, at the present moment the dictionary is still incomplete and the majority of our results do not allow a direct translation into the geometric language. Such a translation would require a nice geometric realization, so far unknown, of singular blocks of the highest weight categories for $\mathfrak{gl}_n$ and projective functors between these blocks.

The main results of the paper are contained in Sections 3 and 4 and provide two categorifications of $U(\mathfrak{sl}_2)$ and Temperley-Lieb algebra actions. Preliminary facts and definitions are collected in Section 2. The basis constituents of our construction are singular and parabolic categories of highest weight modules together with projective and Zuckerman functors acting on these categories. Projective functors in categories of $\mathfrak{gl}_n$ modules, defined as direct summands of functors of tensoring with a finite-dimensional $\mathfrak{gl}_n$-modules [J], [Zu], are extensively used in representation theory, (see [BG] and [KV]). Being exact, projective functors induce linear maps in Grothendieck groups of categories of representations. Zuckerman functors are defined for any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{gl}_n$ by taking the maximal $U(\mathfrak{p})$-locally finite submodule [KV]. Derived functors of Zuckerman functors are exact and also descend to Grothendieck groups. An important property of Zuckerman functors, namely their commutativity with the projective functors, yields in both categorifications what we consider the Schur-Weyl duality for $U_q(\mathfrak{sl}_2)$ and the Temperley-Lieb algebra actions.

In Section 3 we construct a categorification via singular blocks of the category $\mathcal{O}(\mathfrak{gl}_n)$ of highest weight $\mathfrak{gl}_n$-modules. More specifically, we realize $V_1^{\otimes n}$ as a Grothendieck group of the category

$$\mathcal{O}_n = \overset{n}{\underset{k=0}{\oplus}} \mathcal{O}_{k,n-k}, \tag{1}$$

where $\mathcal{O}_{k,n-k}$ is a singular block of $\mathcal{O}(\mathfrak{gl}_n)$ corresponding to the subgroup $\mathbb{S}_k \times \mathbb{S}_{n-k}$ of $\mathbb{S}_n$. The simplest projective functors constructed by means of tensoring with the fundamental representation of $\mathfrak{gl}_n$ and its dual descend on the Grothendieck group level to the action of generators E and F of $\mathfrak{sl}_2$ (Section 3.1.1). Various equalities between products of E and F result from functor isomorphisms (Section 3.1.2). Moreover, we show that indecomposable projective functors in $\mathcal{O}_n$ correspond to elements of Lusztig canonical basis in the modified universal enveloping algebra $\dot{U}(\mathfrak{sl}_2)$ (Section 3.1.3). Construction of the comultiplication for $U(\mathfrak{sl}_2)$ requires studying the relation between categories $\mathcal{O}(\mathfrak{gl}_n) \times \mathcal{O}(\mathfrak{gl}_m)$ and $\mathcal{O}(\mathfrak{gl}_{n+m})$, given by the induction functor from the maximal parabolic subalgebra of $\mathfrak{gl}_{n+m}$ that contains $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$. In particular, a categorification of the comultiplication formulas $\Delta E = E \otimes 1 + 1 \otimes E$ and $\Delta F = F \otimes 1 + 1 \otimes F$ for generators E and F is expressed by short exact sequences that employ certain properties of the induction functor (Section 3.1.4).

To categorify the action of the Temperley-Lieb algebra on $V_1^{\otimes n}$, we use derived functors of Zuckerman functors. We verify that defining relations for the Temperley-Lieb algebra result from appropriate functor isomorphisms. Projective and Zuckerman functors commute and that can be considered a "functor" version of the commutativity between the action of $\dot{U}(\mathfrak{sl}_2)$ and the Temperley-Lieb algebra (Section 3.2).

In Section 4 we construct another categorification, this time using parabolic subcategories of $\mathfrak{gl}_n$ to realize $V_1^{\otimes n}$ as a Grothendieck group. There we consider the category

$$\mathcal{O}^n = \bigoplus_{k=0}^{n} \mathcal{O}^{k,n-k}, \tag{2}$$

where $\mathcal{O}^{k,n-k}$ is a parabolic subcategory of a regular block, corresponding to the parabolic subalgebra of $\mathfrak{gl}_n$ that contains $\mathfrak{gl}_k \oplus \mathfrak{gl}_{n-k}$. In this picture the role of projective and Zuckerman functors is reversed, namely, the categorification of the Temperley-Lieb algebra action is given by projective functors, while the action of $U(\mathfrak{sl}_2)$ is achieved via Zuckerman functors. We show that the composition of translation functors on and off the i-th wall at the Grothendieck group level yields the i-th generator of the Temperley-Lieb algebra by verifying that equivalences between these projective functors correspond to relations of the Temperley-Lieb algebra (Section 4.1). This realization of the Temperley-Lieb algebra by functors was inspired and can be derived from the work [ES] of Enright and Shelton.

In the second picture the action of $U(\mathfrak{sl}_2)$ is categorified by Zuckerman functors (Section 4.2). This result can be extracted from the geometric approach of [BLM] and [Gr], which uses correspondences between flag varieties. The latter correspondences define functors between derived categories of sheaves, which are equivalent to the derived category of $\mathcal{O}^n$.

To summarize, we have two categorifications of the Temperley-Lieb algebra action on the n-th tensor power of the fundamental representation $V_1^{\otimes n}$: one by Zuckerman functors acting in singular blocks and the other by projective functors acting in parabolic categories. We also have two categorifications of the $U(\mathfrak{sl}_2)$ algebra action on the same space: by projective functors between singular categories and by Zuckerman functors between parabolic categories. We conjecture that the Koszul duality functor of [BGS] exchanges these pairs of categorifications and that, more generally, the Koszul duality functor exchanges projective and Zuckerman functors.

The categorification of the representation theory of $U(\mathfrak{sl}_2)$ presented in our work explains the nature of integrality and positivity properties established in [FK] by a direct approach based on the Penrose-Kauffman calculus. However, in this paper we did not reconsider some of the positivity and integrality results of [FK], e.g., positivity and integrality of the $6j$-symbol factorization coefficients. We expect that an extension of our approach will allow us to interpret these coefficients as dimensions of vector spaces of equivalences between appropriate functors. The problem of passing from categorifying $U(\mathfrak{sl}_2)$-representations to $U_q(\mathfrak{sl}_2)$-representations can most likely be solved by working with mixed versions of projective and Zuckerman functors and the category $\mathcal{O}(\mathfrak{gl}_n)$.

Moreover, many of our constructions admit a straightforward generalization from $U(\mathfrak{sl}_2)$ to $U(\mathfrak{sl}_m)$. In this case one should consider singular and parabolic categories corresponding to the subgroups $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_m}, i_1 + \cdots + i_m = n$, of $\mathbb{S}_n$. The Temperley-Lieb algebra will be replaced by appropriate quotients of the Hecke algebra of $\mathbb{S}_n$. A more difficult problem is to categorify the representation theory of $U(\mathfrak{g})$ for an arbitrary simple Lie algebra $\mathfrak{g}$. Another interesting generalization of our results would be a categorification of the affine version of the Schur-Weyl duality. It is expected that in this case one should consider certain singular and parabolic categories of highest weight modules for affine Lie algebras $\widehat{\mathfrak{gl}}_n$. The functors of tensoring with a finite-dimensional $\mathfrak{gl}_n$-module should be replaced by the Kazhdan-Lusztig tensoring with a tilting $\widehat{\mathfrak{gl}}_n$-module (see [FM]).

Finally we would like to discuss applications of our categorification results to a construction of topological invariants, which was the initial motivation for this work (see [CF]). It is well-known that the graphical calculus for representation theory of $U_q(\mathfrak{sl}_2)$ and in particular for representations of the Temperley-Lieb algebra in tensor powers of V_1 is intimately related ([Ka]) to the Jones polynomial ([Jo]), which is a quantum invariant of links and can be extended to give an invariant of tangles. An arbitrary tangle in the three-dimensional space is a composition of elementary pieces such as braiding and local maximum and minimum tangles. To construct the Jones polynomial one attaches to these elementary tangles operators from $V_1^{\otimes m}$ to $V_1^{\otimes n}$ for suitable m and n and obtains an isotopy invariant.

Extending both categorifications of the Temperley-Lieb algebra at the end of Sections 3.1.4 and 4.1 we define functors from derived categories of $\mathcal{O}_m$ to $\mathcal{O}_n$ and $\mathcal{O}^m$ to $\mathcal{O}^n$ corresponding to elementary tangles. Given a plane partition of a tangle, we can associate to it a functor, which is a composition of these basic functors. We conjecture that different plane projections produce isomorphic functors and we would get functor invariants of links and tangles. For links these invariants will take the form of $\mathbb{Z}$-graded homology groups. Given a diagram of a cobordism between two tangles, we can associate to it a natural transformation of functors. We expect that these natural transformations are isotopy invariants of tangle cobordisms, and, in the special case of a cobordism between empty tangles, invariants of 2-knots. To prove this conjecture one needs to present an arbitrary cobordism as a composition of elementary ones and verify all the relations between them. A complete set of generators and relations has been found in [CS], [CRS] and was interpreted as a tensor 2-category in [Fi]. The match between tensor 2-categories arising from topology and representation theory will yield a graphical calculus for a categorification of the representation theory of $U_q(\mathfrak{sl}_2)$ based on two-dimensional surfaces and as a consequence new topological invariants.

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2. Lie algebra $\mathfrak{sl}_2$ and categories of highest weight modules

2.1. $\dot{U}(\mathfrak{sl}_2)$ and its representations

2.1.1. Algebra $\dot{U}(\mathfrak{sl}_2)$. The universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ is given by generators E, F, H and defining relations

$$EF - FE = H, \quad HE - EH = 2E, \quad HF - FH = -2F.$$

We will denote this algebra by $\mathbf{U}$. Throughout the paper we consider it as an algebra over the ring of integers $\mathbb{Z}$. We will also need two other versions of this algebra, $\mathbf{U}_{\mathbb{Z}}$ and $\dot{U}(\mathfrak{sl}_2)$.

Let $\mathbf{U}_{\mathbb{Z}}$ be the integral lattice in $\mathbf{U} \otimes \mathbb{Q}$ spanned by

$$E^{(a)} \binom{H}{b} F^{(c)}$$

for $a, b, c \geq 0$. Here

$$E^{(a)} = \frac{E^a}{a!}, \quad F^{(c)} = \frac{F^c}{c!}, \quad \binom{H}{b} = \frac{H(H-1)\dots(H-b+1)}{b!} \tag{3}$$

$E^{(a)}$, $F^{(a)}$ are known as *divided powers* of E and F. The lattice $\mathbf{U}_{\mathbb{Z}}$ is closed under multiplication, and therefore inherits the algebra structure from that of $\mathbf{U} \otimes \mathbb{Q}$. Thus, $\mathbf{U}_{\mathbb{Z}}$ is an algebra over $\mathbb{Z}$ with multiplicative generators

$$1, \ E^{(a)}, \ F^{(a)}, \ \binom{H}{a}, \qquad a > 0.$$

Some of the relations between the generators are written below

$$E^{(a)} E^{(b)} = \binom{a+b}{a} E^{(a+b)} \tag{4}$$

$$F^{(a)} F^{(b)} = \binom{a+b}{a} F^{(a+b)} \tag{5}$$

$$E^{(a)} F^{(b)} = \sum_{j=0}^{\min(a,b)} F^{(b-j)} \binom{H-a-b+2j}{j} E^{(a-j)}. \tag{6}$$

It is easy to see that the comultiplication in $\mathbf{U} \otimes \mathbb{Q}$ preserves the lattice $\mathbf{U}_{\mathbb{Z}}$, i.e. $\Delta\mathbf{U}_{\mathbb{Z}} \subset \mathbf{U}_{\mathbb{Z}} \otimes \mathbf{U}_{\mathbb{Z}}$, and the algebra $\mathbf{U}_{\mathbb{Z}}$ is actually a Hopf algebra. Note that on $E^{(a)}, F^{(a)}$ the comultiplication is given by

$$\Delta E^{(a)} = \sum_{b=0}^{a} E^{(b)} \otimes E^{(a-b)} \tag{7}$$

$$\Delta F^{(a)} = \sum_{b=0}^{a} F^{(b)} \otimes F^{(a-b)}. \tag{8}$$

Algebra $\dot{U}(\mathfrak{sl}_2)$ is obtained by adjoining a system of projectors, one for each element of the weight lattice, to the algebra $\mathbf{U}_{\mathbb{Z}}$. Start out with a $\mathbf{U}_{\mathbb{Z}}$-bimodule, freely generated by the set 1_n, $n \in \mathbb{Z}$. Quotient it out by relations

$$\binom{H}{a} 1_i = \binom{i}{a} 1_i \tag{9}$$

$$1_i \binom{H}{a} = \binom{i}{a} 1_i \tag{10}$$

$$E^{(a)} 1_i = 1_{i+2a} E^{(a)} \tag{11}$$

$$F^{(a)} 1_i = 1_{i-2a} F^{(a)}. \tag{12}$$

The quotient $\mathbf{U}_{\mathbb{Z}}$-bimodule has a unique algebra structure, compatible with the $\mathbf{U}_{\mathbb{Z}}$-bimodule structure and such that

$$1_n 1_m = \delta_{n,m} 1_n. \tag{13}$$

Denote the resulting $\mathbb{Z}$-algebra by $\dot{U}(\mathfrak{sl}_2)$. As a $\mathbb{Z}$-vector space, it is spanned by elements

$$E^{(a)} 1_n F^{(b)} \quad \text{for} \quad a, b \geq 0,\ n \in \mathbb{Z}.$$

As a left $\mathbf{U}_{\mathbb{Z}}$-module, $\dot{U}(\mathfrak{sl}_2)$ decomposes into a direct sum

$$\dot{U}(\mathfrak{sl}_2) = \bigoplus_{i \in \mathbb{Z}} \dot{U}(\mathfrak{sl}_2)_i$$

where

$$\dot{U}(\mathfrak{sl}_2)_i = \{x \in \dot{U}(\mathfrak{sl}_2) | x 1_i = x\}.$$

$\dot{U}(\mathfrak{sl}_2)_i$ is spanned by $E^{(a)} 1_{i-2b} F^{(b)}$, $a, b \geq 0$.

We will be using Lusztig's basis $\dot{\mathbb{B}}$ of $\dot{U}(\mathfrak{sl}_2)$, given by

$$E^{(a)} 1_{-i} F^{(b)} \quad \text{for} \quad a, b, i \in \mathbb{N},\ i \geq a+b \tag{14}$$

$$F^{(b)} 1_i E^{(a)} \quad \text{for} \quad a, b, i \in \mathbb{N},\ i > a+b. \tag{15}$$

Remark. $E^{(a)}1_{-a-b}F^{(b)} = F^{(b)}1_{a+b}E^{(a)}$.

Define $\dot{\mathbb{B}}_i = \dot{\mathbb{B}} \cap \dot{U}(\mathfrak{sl}_2)_i$, $i \in \mathbb{Z}$.

The important feature of Lusztig's basis is the positivity of the multiplication: for any $x, y \in \dot{\mathbb{B}}$

$$xy = \sum_{z \in \dot{\mathbb{B}}} m^z_{x,y} z$$

with all structure constants $m^z_{x,y}$ being nonnegative integers. Although this positivity property is very easy to verify, its generalization to quantum groups $U_q(\mathfrak{g})$, $\mathfrak{g}$ symmetrizable, conjectured by Lusztig, is, apparently, still unproved. The algebra $\dot{U}(\mathfrak{sl}_2)$ is a special case ($q = 1$, $\mathfrak{g} = \mathfrak{sl}_2$) of the Lusztig's algebra $\dot{U}_q(\mathfrak{g})$ (see [Lu]), obtained from the quantum group $U_q(\mathfrak{g})$ by adding a system of projectors, one for each element of the weight lattice. Lusztig defined a basis in $\dot{U}_q(\mathfrak{g})$ and conjectured that the multiplication and comultiplication constants in this basis lie in $\mathbb{N}[q, q^{-1}]$. A proof of this conjecture, most likely, will require interpreting Lusztig's basis in terms of perverse sheaves on suitable varieties.

Although in this paper we do not venture beyond $\mathfrak{sl}_2$, our results suggest a close link between the Lusztig's basis of $\dot{U}_q(\mathfrak{sl}_N)$ and indecomposable projective functors for $\mathfrak{sl}_n$, N and n being independent parameters.

2.1.2. Representations of $\dot{U}(\mathfrak{sl}_2)$. Let V_1 be the two-dimensional representation over $\mathbb{Z}$ of $\mathbf{U}$ spanned by v_1 and v_0 with the action of generators of $\mathbf{U}$ given by

$$Hv_1 = v_1 \qquad Ev_1 = 0 \qquad Fv_1 = v_0 \tag{16}$$

$$Hv_0 = -v_0 \qquad Ev_0 = v_1 \qquad Fv_0 = 0. \tag{17}$$

In the obvious way, V_1 is also a representation of both $\mathbf{U}_{\mathbb{Z}}$ and $\dot{U}(\mathfrak{sl}_2)$. Using comultiplication, the tensor powers of V_1 become representations of $\mathbf{U}, \mathbf{U}_{\mathbb{Z}}$ and $\dot{U}(\mathfrak{sl}_2)$.

Denote by V_0 the one-dimensional representation of $\mathbf{U}$ given by the augmentation homomorphism $\mathbf{U} \to \mathbb{Z}$ of the universal enveloping algebra. Again, V_0 is a $\mathbf{U}_{\mathbb{Z}}$ and $\dot{U}(\mathfrak{sl}_2)$ module in a natural way.

Let δ be the module homomorphism $V_0 \to V_1 \otimes V_1$ given by

$$\delta(1) = v_1 \otimes v_0 - v_0 \otimes v_1. \tag{18}$$

For a sequence $I = a_1 \dots a_n$ of ones and zeros, let I_+ be the number of ones in the sequence. We will denote the vector $v_{a_1} \otimes \dots \otimes v_{a_n} \in V_1^{\otimes n}$ by $v(I)$.

We define a $\mathbb{Q}$-linear map (the symmetrization map) $p_n : V_1^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q} \to V_1^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$p_n(v(I)) = \binom{n}{I_+}^{-1} \sum_{J, J_+ = I_+} v(J) \tag{19}$$

where the sum on the right hand side is over all sequences J of length n with $J_+ = I_+$. Then p_n is a $\mathbf{U} \otimes_{\mathbb{Z}} \mathbb{Q}$-module homomorphism; in fact, it is the projection onto the unique $(n+1)$-dimensional irreducible $\mathbf{U} \otimes_{\mathbb{Z}} \mathbb{Q}$ subrepresentation of $V_1^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.2. Temperley-Lieb algebra

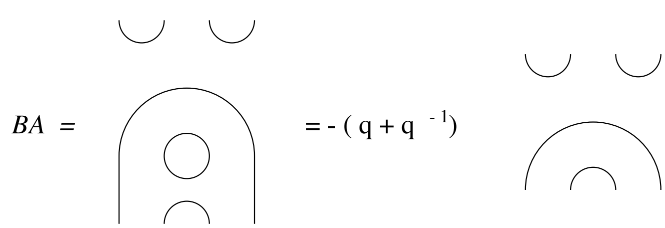
Definition 1. The Temperley-Lieb algebra $TL_{n,q}$ is an algebra over the ring $R = \mathbb{Z}[q, q^{-1}]$, where q is a formal variable, with generators $U_1, \dots, U_{n-1}$ and defining relations

$$U_i U_{i\pm 1} U_i = U_i \tag{20}$$
$$U_i U_j = U_j U_i \qquad |i-j| > 1 \tag{21}$$
$$U_i^2 = -(q+q^{-1}) U_i. \tag{22}$$

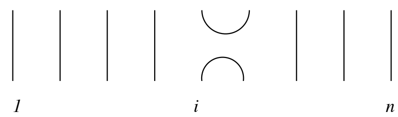
The Temperley-Lieb algebra admits a geometric interpretation via systems of arcs on the plane. Namely, as a free R-module, it has a basis enumerated by isotopy classes of systems of simple, pairwise disjoint arcs that connect n points on the bottom of a horizontal plane strip with n points on the top. We only consider systems without closed arcs. Two diagrams are multiplied by concatenating them. If simple closed loops appear as a result of concatenation, we remove them, each time multiplying the diagram by $-q-q^{-1}$.

Example. Let diagrams A and B be as depicted below.

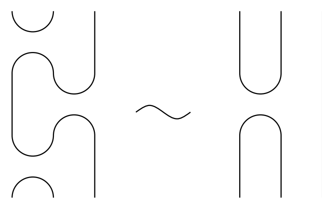
Then their composition BA can be depicted by



The generator U_i of $TL_{n,q}$ is given by the diagram



The defining relations have geometric interpretations. For instance, the first relation says that the diagrams below are isotopic



Definition 2. The Temperley-Lieb algebra $TL_{n,1}$, respectively $TL_{n,-1}$, is an algebra over the ring of integers, obtained from $TL_{n,q}$ by setting q to 1, respectively to -1, everywhere in the definition of the latter.

Thus, in $TL_{n,1}$ the value of a closed loop is -2, in $TL_{n,-1}$ the value of a closed loop is 2, while in $TL_{n,q}$ a closed loop evaluates to $-q-q^{-1}$.

Recall that we denoted by V_1 the fundamental representation of $\mathbf{U}_{\mathbb{Z}}$. Let u be an intertwiner $V_1^{\otimes 2} \to V_1^{\otimes 2}$ given by

$$\begin{aligned} u(v_1 \otimes v_0) = -u(v_0 \otimes v_1) &= v_0 \otimes v_1 - v_1 \otimes v_0 \\ u(v_1 \otimes v_1) = u(v_0 \otimes v_0) \quad &= 0. \end{aligned}$$

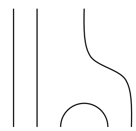
Then $V_1^{\otimes n}$ is a representation of $TL_{n,1}$ with U_i acting by $\mathrm{Id}^{\otimes(i-1)} \otimes u \otimes \mathrm{Id}^{\otimes(n-i-1)}$. This action commutes with the Lie algebra $\mathfrak{sl}_2$ action on the same space.

The Temperley-Lieb algebra allows a generalization into the so-called Temperley-Lieb category, as we now explain (for more details, see [KaL], [Tu]).

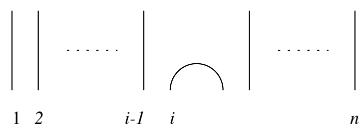
Definition 3. The Temperley-Lieb category TL has objects enumerated by non-negative integers: $\mathrm{Ob}(TL) = \{\overline{0}, \overline{1}, \overline{2}, \ldots\}$. The set of morphisms from $\overline{n}$ to $\overline{m}$ is a free R-module with a basis over R given by the isotopy classes of systems of $\frac{n+m}{2}$ simple, pairwise disjoint arcs inside a horizontal strip on the plane that connect in pairs n points on the bottom and m points on the top in some order.

Morphisms are composed by concatenating their diagrams. If closed loops appear after concatenation, we remove them, multiplying the diagram by $-q-q^{-1}$ to the power equal to the number of closed loops.

An example of a morphism from $\overline{5}$ to $\overline{3}$ is depicted below.

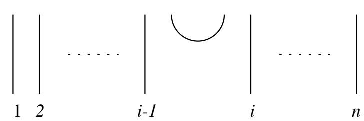


If $n+m$ is odd, there are no morphisms from $\overline{n}$ to $\overline{m}$. Denote by $\cap_{i,n}$ for $n \geq 2, 1 \leq i \leq n-1$ the morphism of TL from $\overline{n}$ to $\overline{n-2}$ given by the following diagram

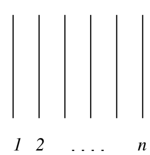


The diagram consists of $n-1$ arcs. One of the arcs connects the i-th bottom point (counting from the left) with the $(i+1)$-th bottom point. The remaining arcs connect the k-th bottom point for $1 \leq k < i$ with the k-th top point and the k-th bottom point for $i+2 \leq k \leq n$ with the $(k-2)$-th top point.

Denote by $\cup_{i,n}$, $n \geq 0$, $1 \leq i \leq n+1$ the morphism in TL from $\overline{n}$ to $\overline{n+2}$ given by the diagram



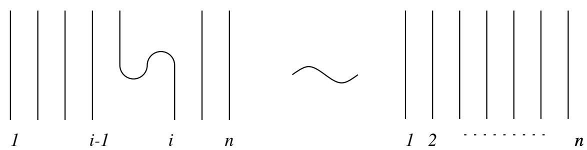
Denote by $\mathrm{Id}_{\overline{n}}$ the identity morphism from $\overline{n}$ to $\overline{n}$. This morphism can be depicted by a diagram that is made of n vertical lines:



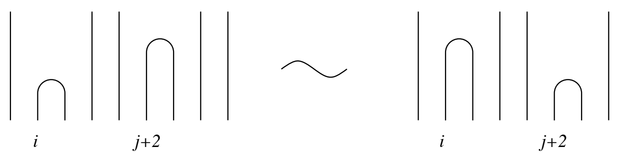
The morphisms $\cap_{i,n}$ and $\cup_{i,n}$ will serve as generators of the set of morphisms in the Temperley-Lieb category. The following is a set of defining relations for TL

$$\cap_{i+1,n+2} \circ \cup_{i,n} = \mathrm{Id}_{\overline{n}} \tag{23}$$
$$\cap_{i,n+2} \circ \cup_{i+1,n} = \mathrm{Id}_{\overline{n}} \tag{24}$$
$$\cap_{j,n} \circ \cap_{i,n+2} = \cap_{i,n} \circ \cap_{j+2,n+2} \qquad i \le j \tag{25}$$
$$\cup_{j,n-2} \circ \cap_{i,n} = \cap_{i,n+2} \circ \cup_{j+2,n} \qquad i \le j \tag{26}$$
$$\cup_{i,n-2} \circ \cap_{j,n} = \cap_{j+2,n+2} \circ \cup_{i,n} \qquad i \le j \tag{27}$$
$$\cup_{i,n+2} \circ \cup_{j,n} = \cup_{j+2,n+2} \circ \cup_{i,n} \qquad i \le j \tag{28}$$
$$\cap_{i,n+2} \circ \cup_{i,n} = -(q+q^{-1})\,\mathrm{Id}_{\overline{n}}\,. \tag{29}$$

The first 6 types of relations come from isotopies of certain pairs of diagrams. For example, relations (23) and (25) correspond to the isotopies



and



respectively. Algebra $TL_{n,q}$ is the algebra of endomorphisms of the object $\overline{n}$ of the Temperley-Lieb category.

2.3. The category of highest weight modules over a reductive Lie algebra

2.3.1. Definitions. In this section all Lie algebras and their representations are defined over the field $\mathbb{C}$ of complex numbers. Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra and $U(\mathfrak{g})$ its universal enveloping algebra. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Let R_+ be the set of positive roots and ρ the half-sum of positive roots. For $\lambda \in \mathfrak{h}^*$ denote by M_λ the Verma module with highest weight $\lambda - \rho$ and by L_λ the irreducible quotient of M_λ. The module L_λ is finite-dimensional if and only if $\lambda - \rho$ is an integral dominant weight.

Denote by $\mathcal{O}(\mathfrak{g})$ the category of finitely generated $U(\mathfrak{g})$-modules that are $\mathfrak{h}$-diagonalizable and locally $U(\mathfrak{n}_+)$-nilpotent. The category $\mathcal{O}(\mathfrak{g})$ is called *the* *category of highest weight* $\mathfrak{g}$*-modules*. Let P_λ denote the projective cover of L_λ (see [BGG] for the existence of projective covers).

If A is an additive category, denote by $K(A)$ the Grothendieck group of A. Denote by $[M]$ the image of an object $M \in \mathrm{Ob}(A)$ in the Grothendieck group of A. Denote by $D^b(A)$ the bounded derived category of an abelian category A.

2.3.2. Projective functors. This section is a brief introduction to projective functors. We refer the reader to Bernstein-Gelfand paper [BG] for a detailed treatment and further references.

Denote by Θ the set of maximal ideals of the center Z of $U(\mathfrak{g})$. We can naturally identify Θ with the quotient of the weight space $\mathfrak{h}^*$ by the action by reflections of the Weyl group W of $\mathfrak{gl}_n$. We will denote by η the quotient map $\mathfrak{h}^* \to \Theta$. For $\theta \in \Theta$ denote by J_θ the corresponding maximal ideal of Z. Thus, the Verma module M_λ with the highest weight $\lambda - \rho$ is annihilated by the maximal central ideal $J_{\eta(\lambda)}$.

For $\theta \in \Theta$ denote by $\mathcal{O}_\theta(\mathfrak{g})$ a full subcategory of $\mathcal{O}(\mathfrak{g})$ consisting of modules that are annihilated by some power of the central ideal J_θ :

$$M \in \mathcal{O}_\theta(\mathfrak{g}) \iff M \in \mathcal{O}(\mathfrak{g}) \text{ and } J_\theta^N M = 0 \text{ for sufficiently large } N. \tag{30}$$

A module $M \in \mathcal{O}(\mathfrak{g})$ belongs to $\mathcal{O}_\theta(\mathfrak{g})$ if and only if all of the simple subquotients of M are isomorphic to simple modules $L_\lambda, \lambda \in \eta^{-1}(\theta)$. We will call modules in $\mathcal{O}_\theta(\mathfrak{g})$ *highest weight modules with the generalized central character* θ. The category $\mathcal{O}(\mathfrak{g})$ splits as a direct sum of categories $\mathcal{O}_\theta(\mathfrak{g})$ over all $\theta \in \Theta$.

Denote by proj_θ the functor from $\mathcal{O}(\mathfrak{g})$ to $\mathcal{O}_\theta(\mathfrak{g})$ that, to a module M, associates the largest submodule of M with the generalized central character θ. Let F_V be the functor of tensoring with a finite-dimensional $\mathfrak{g}$-module V.

Definition 4. $F : \mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathfrak{g})$ is a projective functor if it is isomorphic to a direct summand of the functor F_V for some finite dimensional module V.

The functor proj_θ is an example of a projective functor, since it is a direct summand of the functor of tensoring with the one-dimensional representation. We have an isomorphism of functors

$$F_V = \underset{\theta_1,\theta_2\in\Theta}{\oplus} (\mathrm{proj}_{\theta_1} \circ F_V \circ \mathrm{proj}_{\theta_2}). \tag{31}$$

Any projective functor takes projective objects in $\mathcal{O}(\mathfrak{g})$ to projective objects. The composition of projective functors is again a projective functor. Each projective functor splits as a direct sum of indecomposable projective functors.

Projective functors are exact. Therefore, they induce endomorphisms of the Grothendieck group of the category $\mathcal{O}(\mathfrak{g})$. The following result is proved in [BG]:

Proposition 1. *Let λ be a dominant integral weight, $\theta = \eta(\lambda)$ and F, G projective functors from $\mathcal{O}_\theta(\mathfrak{g})$ to $\mathcal{O}(\mathfrak{g})$. Then*

1. *Functors F and G are isomorphic if and only if the endomorphisms of $K(\mathcal{O}(\mathfrak{g}))$ induced by F and G are equal.*

2. *Functors F and G are isomorphic if and only if modules FM_λ and GM_λ are isomorphic.*

We will be computing the action of projective functors on Grothendieck groups of certain subcategories of the category of highest weight modules. The simplest basis in the Grothendieck group of $\mathcal{O}(\mathfrak{g})$ is given by images of Verma modules. The following proposition shows that this basis is also handy for writing the action of projective functors on the Grothendieck group of $\mathcal{O}(\mathfrak{g})$.

Proposition 2. *Let V be a finite-dimensional $\mathfrak{g}$-module, $\mu_1, \dots, \mu_m$ a multiset of weights of V, M_χ the Verma module with the highest weight $\chi - \rho$, then*

1. *The module $V \otimes M_\chi$ admits a filtration with successive quotients isomorphic to Verma modules $M_{\chi+\mu_1}, \dots, M_{\chi+\mu_m}$ (in some order).*
2. *We have an equality in the Grothendieck group $K(\mathcal{O}(\mathfrak{g}))$:*

$$[V \otimes M_\chi] = \sum_{i=1}^{m} [M_{\chi+\mu_i}].$$

2.3.3. Parabolic categories. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ that contains $\mathfrak{n}_+ \oplus \mathfrak{h}$. Denote by $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$ the full subcategory of $\mathcal{O}(\mathfrak{g})$ that consists of $U(\mathfrak{p})$ locally finite modules. Notice that projective functors preserve subcategories $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$.

A generalized Verma module relative to a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ (see [Lp], [RC]) will be called a $\mathfrak{p}$-Verma module. The Grothendieck group of $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$ is generated by images $[M]$ of generalized Verma modules.

For a central character θ and a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, denote by $\mathcal{O}_\theta(\mathfrak{g}, \mathfrak{p})$ the full subcategory of $\mathcal{O}(\mathfrak{g})$ consisting of $U(\mathfrak{p})$-locally finite modules annihilated by some power of the central ideal J_θ. The category $\mathcal{O}_\theta(\mathfrak{g}, \mathfrak{p})$ is the intersection of subcategories $\mathcal{O}_\theta(\mathfrak{g})$ and $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$ of $\mathcal{O}(\mathfrak{g})$.

The following lemma is an obvious generalization of a special case of Lemma 3.5 in [ES].

Lemma 1. *Let T, S be covariant, exact functors from $\mathcal{O}_\theta(\mathfrak{g}, \mathfrak{p})$ to some abelian category $\mathcal{A}$ and let f be a natural transformation from T to S. If $f_M : T(M) \to S(M)$ is an isomorphism for each generalized Verma module $M \in \mathcal{O}_\theta(\mathfrak{g}, \mathfrak{p})$, then f is an isomorphism of functors.*

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be reductive Lie algebras, with fixed Cartan subalgebras $\mathfrak{h}_j \subset \mathfrak{g}_j, j = 1, 2$. Suppose we have two parabolic subalgebras $\mathfrak{p}_1, \mathfrak{p}_2$ such that $\mathfrak{h}_j \subset \mathfrak{p}_j \subset \mathfrak{g}_j$. Fix central characters θ_j of $\mathfrak{g}_j$.

Lemma 2. *Suppose that we have two exact functors*

$$f_{12} : \mathcal{O}_{\theta_2}(\mathfrak{g}_2, \mathfrak{p}_2) \to \mathcal{O}_{\theta_1}(\mathfrak{g}_1, \mathfrak{p}_1)$$
$$f_{21} : \mathcal{O}_{\theta_1}(\mathfrak{g}_1, \mathfrak{p}_1) \to \mathcal{O}_{\theta_2}(\mathfrak{g}_2, \mathfrak{p}_2)$$

such that f_{21} *is isomorphic to both left and right adjoint functors of* f_{12}*,* f_{21} *takes* $\mathfrak{p}_1$*-Verma modules to* $\mathfrak{p}_2$*-Verma modules,* f_{12} *takes* $\mathfrak{p}_2$*-Verma modules to* $\mathfrak{p}_1$*-Verma modules,* $f_{12}f_{21}(M)$ *is isomorphic to* M *for any* $\mathfrak{p}_1$*-Verma module* M *and* $f_{21}f_{12}(M)$ *is isomorphic to* M *for any* $\mathfrak{p}_2$*-Verma module* M*. Then* f_{12} *and* f_{21} *are equivalences of categories and the natural transformations*

$$a : f_{12}f_{21} \to \mathrm{Id}, \quad b : \mathrm{Id} \to f_{12}f_{21}$$

coming from the adjointness are isomorphisms of functors.

Proof. By Lemma 1 it suffices to prove that, for any $\mathfrak{p}_1$-Verma module M, the module morphisms

$$a_M : f_{12}f_{21}(M) \to M, \quad b_M : M \to f_{12}f_{21}(M)$$

are isomorphisms. Note that $f_{21}(M)$ is a $\mathfrak{p}_2$-Verma module and $f_{12}f_{21}(M)$ is isomorphic to M. The hom space $\mathrm{Hom}_{\mathfrak{g}_1}(M, M)$ is one-dimensional and all morphisms are just scalings of the identity morphism. We have a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{g}_1}(f_{12}f_{21}(M), M) = \mathrm{Hom}_{\mathfrak{g}_2}(f_{21}M, f_{21}M).$$

Under this isomorphism, the map $a_M : M \to M$ corresponds to the identity map $f_{21}M \to f_{21}M$. This identity map generates the space $\mathrm{Hom}_{\mathfrak{g}_2}(f_{21}M, f_{21}M)$, therefore a_M generates the hom space $\mathrm{Hom}_{\mathfrak{g}_1}(M, M)$, and thus, a_M is an isomorphism of M, being a non-zero multiple of the identity map. Therefore, a is an isomorphism of functors. Similarly, b is a functor isomorphism. □

This lemma is used in Section 3.2 in the proof of Theorem 5.

2.3.4. Zuckerman functors. Here we recall the basic properties of Zuckerman derived functors, following [ES] and [EW]. Knapp and Vogan's book [KV] contains a complete treatment of Zuckerman functors, but here we will only need some basic facts.

Throughout the paper we restrict Zuckerman functors to the category of highest weight modules.

Let $\mathfrak{g}, \mathfrak{p}$ be as in Section 2.3.3. The parabolic Lie algebra $\mathfrak{p}$ decomposes as a direct sum $\mathfrak{m} \oplus \mathfrak{u}$ where $\mathfrak{m}$ is the maximal reductive subalgebra of $\mathfrak{p}$ and $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{p}$. The reductive subalgebra $\mathfrak{m}$ contains the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $d = \dim(\mathfrak{m}) - \dim(\mathfrak{h})$. We denote by $*$ the contravariant duality functor in $\mathcal{O}$.

Let $\Gamma_{\mathfrak{p}}$ be the functor from $\mathcal{O}(\mathfrak{g})$ to $\mathcal{O}(\mathfrak{g}, \mathfrak{p})$ that to a module $M \in \mathcal{O}(\mathfrak{g})$ associates its maximal locally $U(\mathfrak{p})$-finite submodule. $\Gamma_{\mathfrak{p}}$ is called the Zuckerman functor. Functor $\Gamma_{\mathfrak{p}}$ is left exact and the category $\mathcal{O}(\mathfrak{g})$ has enough injectives, so we can define the derived functor

$$\mathcal{R}\Gamma_{\mathfrak{p}} : D^b(\mathcal{O}(\mathfrak{g})) \longrightarrow D^b(\mathcal{O}(\mathfrak{g}, \mathfrak{p}))$$

and its cohomology functors

$$\mathcal{R}^i\Gamma_{\mathfrak{p}} : \mathcal{O}(\mathfrak{g}) \longrightarrow \mathcal{O}(\mathfrak{g}, \mathfrak{p}).$$

Proposition 3.

1. *For* $i > d$, $\mathcal{R}^i\Gamma_{\mathfrak{p}} = 0$.
2. *Projective functors commute with Zuckerman functors. More precisely, if* F *is a projective functor, then there are natural isomorphisms of functors*

$$F \circ \Gamma_{\mathfrak{p}} \cong \Gamma_{\mathfrak{p}} \circ F$$
$$F \circ \mathcal{R}\Gamma_{\mathfrak{p}} \cong \mathcal{R}\Gamma_{\mathfrak{p}} \circ F.$$

3. *The functors* $M \longmapsto \mathcal{R}^i\Gamma_{\mathfrak{p}}(M)$ *and* $M \longmapsto \mathcal{R}^{d-i}\Gamma_{\mathfrak{p}}(M^*)^*$, $M \in \mathcal{O}(\mathfrak{g})$ *are naturally equivalent.*
4. $\mathcal{R}^d\Gamma_{\mathfrak{p}}$ *is isomorphic to the functor that to a module* $M \in \mathcal{O}(\mathfrak{g})$ *associates the maximal locally* $\mathfrak{p}$*-finite quotient of* M.

Proof. See [EW]. Zuckerman functors commute with functors of tensor product by a finite-dimensional module. A projective functor is a direct summand of a tensor product functor. Part 2 of the proposition follows. □

2.4. Singular blocks of the highest weight category for $\mathfrak{gl}_n$

2.4.1. Notations. We fix once and for all a triangular decomposition $\mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ of the Lie algebra $\mathfrak{gl}_n$. The Weyl group of $\mathfrak{gl}_n$ is isomorphic to the symmetric group S_n. Choose an orthonormal basis $e_1, \dots, e_n$ in the Euclidean space $\mathbb{R}^n$ and identify the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n$ with the dual $\mathfrak{h}^*$ of Cartan subalgebra so that $R_+ = \{e_i - e_j, i < j\}$ is the set of positive roots and $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1$ are simple roots. The generator s_i of the Weyl group $W = S_n$ acts on $\mathfrak{h}^*$ by permuting e_i and e_{i+1}. Denote by ρ the half-sum of positive roots

$$\rho = \frac{n-1}{2}e_1 + \frac{n-3}{2}e_2 + \cdots + \frac{1-n}{2}e_n.$$

Sometimes we will use the notation ρ_n instead of ρ.

For a sequence $a_1, \dots, a_n$ of zeros and ones, denote by $M(a_1 \dots a_n)$ the Verma module with the highest weight $a_1e_1 + \cdots + a_ne_n - \rho$. Similarly, $L(a_1 \dots a_n)$ will denote the simple quotient of $M(a_1 \dots a_n)$ and $P(a_1 \dots a_n)$ the minimal projective cover of $L(a_1 \dots a_n)$.

The sequence of n zeros, respectively ones, will be denoted by 0^n, respectively 1^n. If I_1, I_2 are two sequences of 0's and 1's, we denote their concatenation by I_1I_2.

Recall that $\lambda_i = e_1 + \cdots + e_i$ is a fundamental weight of $\mathfrak{gl}_n$. Denote by $\theta_i = \eta(\lambda_i)$ the corresponding central character of $\mathfrak{gl}_n$. We denote the category $\mathcal{O}_{\theta_i}(\mathfrak{gl}_n)$ by $\mathcal{O}_{i,n-i}$. A module $M \in \mathcal{O}(\mathfrak{gl}_n)$ lies in $\mathcal{O}_{i,n-i}$ if and only if all of its simple subquotients are isomorphic to $L(a_1 \dots a_n)$ for sequences $a_1 \dots a_n$ of zeros and ones with exactly i ones.

Denote by $\mathcal{O}_n$ the direct sum of categories $\mathcal{O}_{i,n-i}$ as i ranges over all integers from 0 to n:

$$\mathcal{O}_n = \bigoplus_{i=0}^{n} \mathcal{O}_{i,n-i}. \tag{32}$$

When $i < 0$ or $i > n$, denote by $\mathcal{O}_{i,n-i}$ the subcategory of $\mathcal{O}(\mathfrak{gl}_n)$ consisting of the zero module. We have an isomorphism of Grothendieck groups

$$K(\mathcal{O}_n) = \bigoplus_{i=0}^{n} K(\mathcal{O}_{i,n-i}). \tag{33}$$

Let Υ_n be the isomorphism of abelian groups $\Upsilon_n : K(\mathcal{O}_n) \to V_1^{\otimes n}$ given by

$$\Upsilon_n[M(a_1 \dots a_n)] = v_{a_1} \otimes \dots \otimes v_{a_n}. \tag{34}$$

2.4.2. Simple and projective module bases in the Grothendieck group $K(\mathcal{O}_n)$. Abelian group isomorphism Υ_n identifies the Grothendieck group of $\mathcal{O}_n$ and the $\dot{U}(\mathfrak{sl}_2)$-module $V_1^{\otimes n}$. In this section we describe the images under Υ_n of simple modules and indecomposable projectives in $\mathcal{O}_n$. The only result of this section that we use later is the formula (35) for the image of the indecomposable projective $P(0^j 1^k 0^l 1^m)$. This formula is used in the proof of Theorem 4.

Let us define 3 bases in $V_1^{\otimes n}$. We are working over $\mathbb{Z}$ and thus $V_1^{\otimes n}$ is a free abelian group of rank 2^n. We will parametrize basis elements by sequences of ones and zeros of length n.

First, the basis $\{v(a_1 \dots a_n), a_i \in \{0,1\}\}$ will be given by

$$v(a_1 \dots a_n) = v_{a_1} \otimes \dots \otimes v_{a_n}$$

so that, for a sequence I of length n of zeros and ones, we have $\Upsilon_n[M(I)] = v(I)$. We will call this basis *the product basis of* $V_1^{\otimes n}$.

Next we introduce the basis $\{l(a_1 \dots a_n), a_i \in \{0,1\}\}$ by induction of n as follows:

(i) $l(1) = v_1, l(0) = v_0$,
(ii) $l(0a_2 \dots a_n) = v_0 \otimes l(a_2 \dots a_n)$,
(iii) $l(a_1 \dots a_{n-1}1) = l(a_1, \dots, a_{n-1}) \otimes v_1$,
(iv)

$$l(a_1 \dots a_{i-1} 10 a_{i+2} \dots a_n) = \\ (\mathrm{Id}^{\otimes(i-1)} \otimes \delta \otimes \mathrm{Id}^{\otimes(n-i-1)}) l(a_1 \dots a_{i-1} a_{i+2} \dots a_n),$$

where δ is the intertwiner $V_0 \to V_1 \otimes V_1$ defined by the formula (18) and Id denotes the identity homomorphism of V_1. These rules are consistent and uniquely define $l(a_1 \dots a_n)$ for all $a_1 \dots a_n$.

Let us now define the basis $\{p(a_1 \dots a_n), a_i \in \{0,1\}\}$. Let I be a sequence of zeros and ones. Then define the basis inductively by the rules

(i) $p(1I) = v_1 \otimes p(I)$,
(ii) $p(I0) = p(I) \otimes v_0$,
(iii) If a sequence I_1 is either empty or ends with at least j zeros, and a sequence I_2 is either empty or starts with at least k ones, then

$$p(I_1 0^j 1^k I_2) = \binom{j+k}{j} (\mathrm{Id}^{\otimes |I_1|} \otimes p_{j+k} \otimes \mathrm{Id}^{\otimes |I_2|}) p(I_1 1^k 0^j I_2)$$

where $|I|$ stands for the length of I.

Proposition 4. *These rules are consistent, and for each sequence I define an element of $V_1^{\otimes |I|}$.*

Proposition follows from the results of [K], Section 3, setting q to 1. Note that it is not even obvious that $p(I)$ lies in $V_1^{\otimes n}$ because the projector p_i is defined (see Section 2.1.2) as an operator in $V_1^{\otimes i} \otimes_{\mathbb{Z}} \mathbb{Q}$, rather than in $V_1^{\otimes i}$.

Proposition 5. *The isomorphism $\Upsilon_n : K(\mathcal{O}_n) \to V_1^{\otimes n}$ takes the images of simple, resp. indecomposable projective modules to elements of the basis $\{l(I)\}_I$, resp. $\{p(I)\}_I$ of $V_1^{\otimes n}$:*

$$\Upsilon_n[L(I)] = l(I)$$
$$\Upsilon_n[P(I)] = p(I).$$

Proof. The rules (i)–(iii) for $p(I)$ can be used to write down the relations between the coefficients of the transformation matrix from the basis $\{v(I)\}_I$ to the basis $\{p(I)\}_I$ of $V_1^{\otimes n}$. These relations are equivalent to the $q = 1$ specialization of Lascoux-Schützenberger's recursive formulas (see [LS] and [Z]) for the Kazhdan-Lusztig polynomials in the Grassmannian case, as follows from the computation at the end of [FKK] (again, setting q to 1).

Kazhdan-Lusztig polynomials in the Grassmannian case for $q = 1$ are coefficients of the transformation matrix from the Verma module to the simple module basis of Grothendieck groups of certain parabolic subcategories of a regular block of $\mathcal{O}(\mathfrak{gl}_n)$. These parabolic subcategories are Koszul dual (see [BGS], Theorem 3.11.1 for the general statement) to the singular blocks $\mathcal{O}_{i,n-i}$, $0 \le i \le n$ of $\mathcal{O}(\mathfrak{gl}_n)$. Koszul duality functor descends to the isomorphism of Grothendieck groups that exchanges simple and projective modules in corresponding categories. Therefore, Lascoux-Schützenberger's formulas also describe coefficients of decomposition of projective modules in the Verma module basis of $K(\mathcal{O}_n)$. It now follows that $\Upsilon_n[P(I)] = p(I)$.

Introduce a bilinear form on $K(\mathcal{O}_n)$ by $< [M(I)], [M(J)] >= \delta_I^J$. The BGG reciprocity implies $\langle [P(I)], [L(J)] \rangle = \delta_I^J$, i.e., the basis $\{[L(I)]\}$ of $K(\mathcal{O}_n)$ is dual to the basis $\{[P(I)]\}$ of $K(\mathcal{O}_n)$. Abelian group isomorphism Υ_n transform this bilinear form to a bilinear form on $V_1^{\otimes n}$ such that $\langle v(I), v(J)\rangle = \delta_I^J$. From the main computation of Chapter 3 of [K], specializing q to 1, it follows that bases $\{l(I)\}$ and $\{p(I)\}$ are orthogonal relative to this form. We have $\Upsilon_n[M(I)] = v(I)$ by definition of Υ_n and we have already established that $\Upsilon_n[P(I)] = p(I)$. We conclude that $\Upsilon_n[L(I)] = l(I)$. $\square$

To prove Theorem 4 we will need an explicit formula for $p(0^j1^k0^l1^m)$:

$$\begin{aligned}
&p(0^j1^k0^l1^m) = \\
&= \binom{j+k}{k}\binom{j+l+m}{m}(p_{j+k}\otimes \mathrm{Id}^{\otimes(l+m)})(\mathrm{Id}^{\otimes k}\otimes p_{j+l+m}) \\
&(v_1^{\otimes(k+m)}\otimes v_0^{\otimes(j+l)}) \quad \text{if} \quad k\le l \\
&= \binom{l+m}{m}\binom{j+k+m}{j}(\mathrm{Id}^{\otimes(j+k)}\otimes p_{l+m})(p_{j+k+m}\otimes \mathrm{Id}^{\otimes l}) \\
&(v_1^{\otimes(k+m)}\otimes v_0^{\otimes(j+l)}) \quad \text{if} \quad k\ge l.
\end{aligned} \tag{35}$$

This formula follows from recurrent relations (i)–(iii) for $p(I)$ that we gave earlier in this section.

3. Singular categories

3.1. Projective functors and $\mathfrak{sl}_2$

3.1.1. Projective functors $\mathcal{E}$ and $\mathcal{F}$. Let L_n be the n-dimensional representation of $\mathfrak{gl}_n$ with vweights $e_1, e_2, \dots, e_n$. The dual representation L_n^* has weights $-e_1, -e_2, \dots, -e_n$.

Recall that $\mathcal{O}_{i,n-i}$, $i = 0, 1, \dots n$ is the singular block of $\mathcal{O}(\mathfrak{gl}_n)$ consisting of modules with generalized central character $\theta_i = \eta(\lambda_i)$. For $i < 0$ and $i > n$ we defined $\mathcal{O}_{i,n-i}$ to be the trivial subcategory of $\mathcal{O}(\mathfrak{gl}_n)$.

Denote by $\mathcal{E}_i$ the projective functor

$$(\mathrm{proj}_{\theta_{i+1}})\circ F_{L_n}: \quad \mathcal{O}_{i,n-i}\to \mathcal{O}_{i+1,n-i-1}$$

given by tensoring with the n-dimensional representation L_n and then taking the largest submodule of this tensor product that lies in $\mathcal{O}_{i+1,n-i-1}$.

Similarly, denote by $\mathcal{F}_i$ the projective functor from $\mathcal{O}_{i,n-i}$ to $\mathcal{O}_{i-1,n-i+1}$ given by tensoring with L_n^* and then taking the largest submodule that belongs to $\mathcal{O}_{i-1,n-i+1}$

Theorem 1. *For $i = 0, 1, \dots n$ there are isomorphisms of projective functors*

$$(\mathcal{E}_{i-1}\circ \mathcal{F}_i)\oplus \mathrm{Id}^{\oplus(n-i)} \cong (\mathcal{F}_{i+1}\circ \mathcal{E}_i)\oplus \mathrm{Id}^{\oplus i} \tag{36}$$

where Id *denotes the identity functor* $\mathrm{Id} : \mathcal{O}_{i,n-i} \to \mathcal{O}_{i,n-i}$.

Proof. By Proposition 1 it suffices to check the equality of endomorphisms of the Grothendieck group of $\mathcal{O}_{i,n-i}$ induced by these projective functors.

Denote by $[\mathcal{E}_i]$ and $[\mathcal{F}_i]$ the homomorphisms of the Grothendieck groups induced by functors $\mathcal{E}_i$ and $\mathcal{F}_i$:

$$[\mathcal{E}_i] : K(\mathcal{O}_{i,n-i}) \to K(\mathcal{O}_{i+1,n-i-1})$$
$$[\mathcal{F}_i] : K(\mathcal{O}_{i,n-i}) \to K(\mathcal{O}_{i-1,n-i+1}).$$

The Grothendieck group $K(\mathcal{O}_{i,n-i})$ is free abelian of rank $\binom{n}{i}$ and is spanned by images $[M(a_1 \dots a_n)]$ of Verma modules $M(a_1 \dots a_n)$ for all possible sequences $a_1 \dots a_n$ of zeros and ones with i ones.

By Proposition 2

$$[M(a_1 \dots a_n) \otimes L_n] = \sum_{j=1}^{n} [M(a_1 \dots a'_j \dots a_n)] \tag{37}$$

where $a'_j = a_j + 1$.

The functor $\mathcal{E}_i$ is a composition of tensoring with L_n and a projection onto $\mathcal{O}_{i+1,n-i-1}$, hence we get

Proposition 6. *Let $a_1 \dots a_n$ be a sequence of zeros and ones that contains i ones. Then*

$$[\mathcal{E}_i M(a_1 \dots a_n)] = \sum_{j=1, a_j=0}^{n} [M(a_1 \dots a_{j-1} 1 a_{j+1} \dots a_n)]. \tag{38}$$

In the same fashion, we obtain

Proposition 7. *Let $a_1 \dots a_n$ be a sequence of zeros and ones that contains i ones. Then*

$$[\mathcal{F}_i M(a_1 \dots a_n)] = \sum_{j=1, a_j=1}^{n} [M(a_1 \dots a_{j-1} 0 a_{j+1} \dots a_n]. \tag{39}$$

Therefore, after identifying the Grothendieck group $K(\mathcal{O}_n)$ with $V_1^{\otimes n}$ via the isomorphism Υ_n (formula (34), we see that maps $[\mathcal{E}_i], [\mathcal{F}_i]$ from $K(\mathcal{O}_{i,n-i})$ to $K(\mathcal{O}_{i+1,n-i-1})$ and $K(\mathcal{O}_{i-1,n-i+1})$ coincide with maps induced by the Lie algebra $\mathfrak{sl}_2$ generators E and F on the weight $2i-n$ subspace of the module $V_1^{\otimes n}$. This immediately gives

Proposition 8. *We have the following equality of endomorphisms of the abelian group $K(\mathcal{O}_{i,n-i})$:*

$$[\mathcal{E}_{i-1}][\mathcal{F}_i] + (n-i)\,\mathrm{Id} = [\mathcal{F}_{i+1}][\mathcal{E}_i] + i \cdot \mathrm{Id}\,.$$

Theorem 1 follows. $\square$

This proof also implies

Corollary 1. *Considered as an $\mathfrak{sl}_2$-module with E, resp. F acting as $\sum_i [\mathcal{E}_i]$, resp. $\sum_i [\mathcal{F}_i]$, the Grothendieck group $K(\mathcal{O}_n)$ is isomorphic to the n-th tensor power of the fundamental two-dimensional representation of $\mathfrak{sl}_2$.*

3.1.2. Realization of $\dot{U}(\mathfrak{sl}_2)$ by projective functors. Next we provide a realization of the divided powers of E and F by projective functors.

Let $\mathcal{E}_i^{(k)}$ be the functor from $\mathcal{O}_{i,n-i}$ to $\mathcal{O}_{i+k,n-i-k}$ given by tensoring with the k-th exterior power of L_n and then projecting onto the submodule with the generalized central character θ_{i+k} :

$$\mathcal{E}_i^{(k)}(M) = \mathrm{proj}_{\theta_{i+k}}(\Lambda^k L_n \otimes M). \tag{40}$$

Similarly,

$$\mathcal{F}_i^{(k)} : \mathcal{O}_{i,n-i} \to \mathcal{O}_{i-k,n-i+k} \tag{41}$$

is given by

$$\mathcal{F}_i^{(k)}(M) = \mathrm{proj}_{\theta_{i+k}}(\Lambda^k L_n^* \otimes M). \tag{42}$$

Denote by $[\mathcal{E}_i^{(k)}], [\mathcal{F}_i^{(k)}]$ the induced homomorphisms of the Grothendieck group $K(\mathcal{O}_n) = \oplus_{j\in\mathbb{Z}} K(\mathcal{O}_{j,n-j})$. Note that $[\mathcal{E}_i^{(k)}], [\mathcal{F}_i^{(k)}]$ map $K(\mathcal{O}_{j,n-j})$ to 0 unless $i = j$. The following theorem is proved in the same way as Theorem 1.

Theorem 2. *Under the abelian group isomorphism*

$$\Upsilon_n : K(\mathcal{O}_n) \to V_1^{\otimes n}$$

the endomorphism $[\mathcal{E}_i^{(k)}]$, resp. $[\mathcal{F}_i^{(k)}]$ of $K(\mathcal{O}_n)$ coincides with the endomorphism of $V_1^{\otimes n}$ given by the action of $E^{(k)}1_{2i-n} \in \dot{U}(\mathfrak{sl}_2)$, resp. $F^{(k)}1_{2i-n}$. In other words, the following diagrams are commutative

$$\begin{tikzcd} K(\mathcal{O}_n) \arrow[r, "\Upsilon_n"] \arrow[d, "{[\mathcal{E}_i^{(k)}]}"] & V_1^{\otimes n} \arrow[d, "E^{(k)}1_{2i-n}"] \\ K(\mathcal{O}_n) \arrow[r, "\Upsilon_n"] & V_1^{\otimes n} \end{tikzcd} .$$

$$\begin{tikzcd} K(\mathcal{O}_n) \arrow[r, "\Upsilon_n"] \arrow[d, "{[\mathcal{F}_i^{(k)}]}"] & V_1^{\otimes n} \arrow[d, "F^{(k)}1_{2i-n}"] \\ K(\mathcal{O}_n) \arrow[r, "\Upsilon_n"] & V_1^{\otimes n} \end{tikzcd}$$

Let S be the set $\{E^{(a)}1_j, F^{(a)}1_j | a, j \in \mathbb{Z}\}$. It is a subset of $\dot{U}(\mathfrak{sl}_2)$. To each element of S we can now associate a projective functor from the category $\mathcal{O}_n$ to itself as follows. The functor associated to an element $x \in S$ will be denoted $f_n(x)$.

$$f_n(E^{(a)}1_j) = \begin{cases} \mathcal{E}^{(a)}_{\frac{j+n}{2}} & \text{if } j+n = 0 (\mathrm{mod}\ 2) \\ 0 & \text{otherwise} \end{cases}$$
$$f_n(F^{(a)}1_j) = \begin{cases} \mathcal{F}^{(a)}_{\frac{j+n}{2}} & \text{if } j+n = 0 (\mathrm{mod}\ 2) \\ 0 & \text{otherwise.} \end{cases}$$

Let c be an arbitrary product $x_1 \dots x_m$ of elements of S. To c we associate a functor, denoted $f_n(c)$, from $\mathcal{O}_n$ to $\mathcal{O}_n$ by

$$f_n(c) = f_n(x_1) \circ \dots \circ f_n(x_m).$$

Proposition 1 implies

Theorem 3. *Let $c_1, \dots, c_s, d_1, \dots, d_t$ be arbitrary products of elements of S. The endomorphisms of $V_1^{\otimes n}$ induced by the elements $c_1 + \dots + c_s$ and $d_1 + \dots + d_t$ of $\dot{U}(\mathfrak{sl}_2)$ coincide if and only if the functors $\oplus_{i=1}^s f_n(c_i)$ and $\oplus_{j=1}^t f_n(d_i)$ are isomorphic.*

Corollary 2. *There exist isomorphisms of projective functors*

$$\mathcal{E}^{(b)}_{i+a} \circ \mathcal{E}^{(a)}_i \cong (\mathcal{E}^{(a+b)}_i)^{\oplus \binom{a+b}{a}}$$
$$\mathcal{F}^{(b)}_{i-a} \circ \mathcal{F}^{(a)}_i \cong (\mathcal{F}^{(a+b)}_i)^{\oplus \binom{a+b}{a}}$$
$$\bigoplus_{k=0}^{\min(a,b)} (\mathcal{E}^{(a-k)}_{i-b+k} \circ \mathcal{F}^{(b-k)}_i)^{\oplus \binom{n-i-a+b}{k}} \cong$$
$$\cong \bigoplus_{l=0}^{\min(a,b)} (\mathcal{F}^{(b-l)}_{i+a-l} \circ \mathcal{E}^{(a-l)}_i)^{\oplus \binom{i}{l}}.$$

Remark. We expect that functor isomorphisms in the above theorem can be made canonical so that they satisfy certain relations, including associativity relations.

3.1.3. Canonical basis of $\dot{U}(\mathfrak{sl}_2)$ and indecomposable projective functors.

Recall that the canonical basis $\dot{\mathbb{B}}$ of $\dot{U}(\mathfrak{sl}_2)$ is given by

$$E^{(a)}1_{-i}F^{(b)} \quad \text{for} \quad a, b, i \in \mathbb{N},\ i \geq a+b$$
$$F^{(b)}1_i E^{(a)} \quad \text{for} \quad a, b, i \in \mathbb{N},\ i > a+b.$$

Conveniently, each element of $\dot{\mathbb{B}}$ is a product of $E^{(a)}1_i, F^{(a)}1_i$ for various a and i, specifically,

$$E^{(a)}1_{-i}F^{(b)} = E^{(a)}1_{-i}F^{(b)}1_{-i+2b} \quad \text{for} \quad a, b, i \in \mathbb{N},\ i \geq a+b$$
$$F^{(b)}1_i E^{(a)} = F^{(b)}1_i E^{(a)}1_{i-2a} \quad \text{for} \quad a, b, i \in \mathbb{N},\ i > a+b.$$

In the previous section to each such product and each $n = 1, 2, \ldots$ we associated a projective functor. Therefore, we can associate a projective functor to each element of the canonical basis $\dot{\mathbb{B}}$ of $\dot{U}(\mathfrak{sl}_2)$. To $x \in \dot{\mathbb{B}}$ we associate a projective functor $f_n(x)$ from $\mathcal{O}_n$ to $\mathcal{O}_n$ by the following rule.

$$\begin{aligned}
f_n(E^{(a)}1_{-i}F^{(b)}) &= \mathcal{E}^{(a)}_{\frac{-i+n}{2}}\mathcal{F}^{(b)}_{\frac{-i+n}{2}+b} \\
&\text{for} \quad a, b, i \in \mathbb{N}, i \geq a+b, i+n = 0(\text{mod } 2), \\
f_n(F^{(b)}1_iE^{(a)}) &= \mathcal{F}^{(b)}_{\frac{i+n}{2}}\mathcal{E}^{(a)}_{\frac{i+n}{2}-a} \\
&\text{for} \quad a, b, i \in \mathbb{N}, i > a+b, i+n = 0(\text{mod } 2), \\
f_n(E^{(a)}1_{-i}F^{(b)}) &= f_n(F^{(b)}1_iE^{(a)}) = 0 \\
&\text{for} \quad i+n = 1(\text{mod } 2).
\end{aligned}$$

In this way to each canonical basis element $b \in \dot{\mathbb{B}}$ there is associated an exact functor

$$f_n(b) : \mathcal{O}_n \to \mathcal{O}_n$$

The multiplication in $\dot{U}(\mathfrak{sl}_2)$ correspond to composition of projective functors: for $x, y \in \dot{\mathbb{B}}$ the product xy is a linear combination of elements of $\dot{\mathbb{B}}$ with integral nonnegative coefficients $xy = \sum_{z \in \dot{U}(\mathfrak{sl}_2)} m^z_{x,y} z$. In turn, the functor $f_n(x) \circ f_n(y)$ decomposes as a direct sum of functors $f_n(z)$ with multiplicities $m^z_{x,y}$:

$$f_n(x)f_n(y) = \bigoplus_{z \in \dot{\mathbb{B}}} f_n(z)^{\oplus m^z_{x,y}}$$

Theorem 4. *Fix $n \in \mathbb{N}$. Let $x \in \dot{\mathbb{B}}$. Then the projective functor $f_n(x)$ is either* 0 *or isomorphic to an indecomposable projective functor. Moreover, for each indecomposable projective functor $A : \mathcal{O}_n \to \mathcal{O}_n$ there exists exactly one $x \in \dot{\mathbb{B}}$ such that $f_n(x)$ is isomorphic to A.*

Proof. Let $x \in \dot{\mathbb{B}}_{2j-n}$. Recall that in our notations the dominant Verma module in $\mathcal{O}_{j,n-j}$ is $M(1^j0^{n-j})$. From the properties of projective functors we know that $f_n(x)M(1^j0^{n-j})$ is a projective module in $\mathcal{O}_n$, $f_n(x)$ is the trivial functor if and only if $f_n(x)M(1^j0^{n-j})$ is the trivial module, and $f_n(x)$ is an indecomposable projective functor if and only if $f_n(x)M(1^j0^{n-j})$ is an indecomposable projective module. Isomorphism classes of projective modules in the category of highest weight modules are determined by their images in the Grothendieck group. Thus, all computations to check whether $f_n(x)M(1^j0^{n-j})$ is indecomposable, trivial, etc. can be done in the Grothendieck group of the category $\mathcal{O}_n$. We claim that $[f_n(x)M(1^j0^{n-j})] = 0$ or $[f_n(x)M(1^j0^{n-j})] = [P(0^k1^l0^m1^s)]$ for some k, l, m, s where, we recall, $P(0^k1^l0^m1^s)$ denotes the indecomposable projective cover of the simple module with highest weight $e_{l+1} + \cdots + e_{k+l} + e_{k+l+m+1} + \cdots + e_{k+l+m+s} - \rho$ (see Section 2.4.1).

Due to the isomorphism Υ_n between the Grothendieck group of $\mathcal{O}_n$ and the abelian group $V_1^{\otimes n}$ and Proposition 5 we can work with the latest group instead. The notations of Section 2.4.2 are used below.

Proposition 9.

$$E^{(a)}1_i v(1^b0^c) = \\ = \begin{cases} \binom{c}{a}(\mathrm{Id}^{\otimes b}\otimes p_c)v(1^{b+a}0^{c-a}) & \textit{if } i=b-c \textit{ and } c\geq a \\ 0 & \textit{otherwise} \end{cases}$$

$$F^{(a)}1_i v(1^b0^c) = \\ = \begin{cases} \binom{b}{a}(p_b\otimes \mathrm{Id}^{\otimes c})v(1^{b-a}0^{c+a}) & \textit{if } i=b-c \textit{ and } b\geq a \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Clearly, $1_i v(1^b0^c) = v(1^b0^c)$ if $i=b-c$ and $1^i v(1^b0^c)=0$ otherwise. $E^{(a)}$ acts on $v(1^b0^c)$ in the following way

$$E^{(a)}v(1^b0^c) = \sum_I v(1^b I)$$

where the sum is over all sequences I that contain a ones and $c-a$ zeros. Proposition follows. $\square$

Let a,b,i be non-negative integers with $i\geq a+b$. Then $E^{(a)}1_{-i}F^{(b)}$ is an element of the canonical basis $\dot{\mathbb{B}}$. We compute its action on the element $v(1^c0^d)$ of $V_1^{\otimes(c+d)}$. We restrict to the case $i=2b+d-c$ and $c\geq b$ as otherwise the result is 0. The condition $i\geq a+b$ can now be written as $c-b\leq d-a$.

$$\begin{aligned} & E^{(a)}1_{-i}F^{(b)}v(1^c0^d) = \\ & E^{(a)}\binom{c}{b}(p_c\otimes \mathrm{Id}^{\otimes d})v(1^{c-b}0^{d+b}) = \\ & \binom{c}{b}(p_c\otimes \mathrm{Id}^{\otimes d})E^{(a)}v(1^{c-b}0^{d+b}) = \\ & \binom{c}{b}(p_c\otimes \mathrm{Id}^{\otimes d})\binom{d+b}{a}(\mathrm{Id}^{\otimes(c-b)}\otimes p_{d+b})v(1^{c-b+a}0^{d+b-a}) = \\ & \binom{c}{b}\binom{d+b}{a}(p_c\otimes \mathrm{Id}^{\otimes d})(\mathrm{Id}^{\otimes(c-b)}\otimes p_{d+b})v(1^{c-b+a}0^{d+b-a}) = \\ & p(0^b1^{c-b}0^{d-a}1^a). \end{aligned}$$

The last equality follows from formula (35) (substituting $j=b$, $k=c-b$, $l=d-a$, $m=a$) and the condition $c-b\leq d-a$.

In the same fashion, for $a, b, c, d, i \in \mathbb{Z}_+$ such that $i = c - d + 2a, i \geq a + b$ (which implies $c - b \geq d - a$) we get

$$F^{(b)} 1_i E^{(a)} v(1^c 0^d) = p(0^b 1^{c-b} 0^{d-a} 1^a). \tag{43}$$

Thus, for any $x \in \dot{\mathbb{B}}$ and $c, d \in \mathbb{Z}_+$ the element $xv(1^c 0^d) \in V_1^{\otimes n}$ is either 0 or equal to the image under Υ_n of the Grothendieck class of the indecomposable projective module $P(0^k 1^l 0^m 1^s)$ for some quadruple (k, l, m, s).

Therefore, for any $x \in \dot{\mathbb{B}}_{2j-n}$ and a dominant Verma module $M(1^j 0^{n-j})$ the projective module $f_n(x) M(1^j 0^{n-j})$ is either the trivial or an indecomposable projective module, i.e. the projective functor $f_n(x)$ is either trivial or indecomposable projective. All other statements of Theorem 4 follow easily from the above analysis and the classification of projective functors (see [BG]). □

3.1.4. Comultiplication. In previous sections we studied projective functors in categories $\mathcal{O}_n$. We established that on the Grothendieck group level these functors descend to the action of generators of $\dot{U}(\mathfrak{sl}_2)$ on the n-th tensor power of the fundamental representation V_1 and that the composition of functors descends to the multiplication in $\dot{U}(\mathfrak{sl}_2)$. Yet, we do not have a functor realization of the whole algebra $\dot{U}(\mathfrak{sl}_2)$, only of its finite-dimensional quotients, also called Schur quotients, that are the homomorphic images or $\dot{U}(\mathfrak{sl}_2)$ in the endomorphisms of $V_1^{\otimes n}$. It is unconvenient to think about comultiplication in $\dot{U}(\mathfrak{sl}_2)$ if only some of its finite quotients are available. We notice, although, that comultiplication allows one to introduce a module structure in a tensor product and, with a categorification of $V_1^{\otimes n}$ at hand, we can try to construct functors between $\mathcal{O}_n \times \mathcal{O}_m$ and $\mathcal{O}_{n+m}$ corresponding to module isomorphism

$$V_1^{\otimes n} \otimes V_1^{\otimes m} \cong V_1^{\otimes (n+m)}. \tag{44}$$

Before we considered projective projective functors $\mathcal{E}_i^{(a)}, \mathcal{F}_i^{(a)} : \mathcal{O}_n \longrightarrow \mathcal{O}_n$ for a fixed n and in the notations for these functors we suppressed the dependence on n. In this section the rank of $\mathfrak{gl}$ will vary (we'll have functors between categories $\mathcal{O}(\mathfrak{gl}_n) \times \mathcal{O}(\mathfrak{gl}_m)$ and $\mathcal{O}(\mathfrak{gl}_{n+m})$ and we redenote the functors $\mathcal{E}_i^{(a)}, \mathcal{F}_i^{(a)} : \mathcal{O}_n \longrightarrow \mathcal{O}_n$ by $\mathcal{E}_{i,n}^{(a)}, \mathcal{F}_{i,n}^{(a)}$.

For the rest of the section we fix positive integers n and m. Let $\mathfrak{p}$ be the maximal parabolic subalgebra of $\mathfrak{gl}_{n+m}$ that contains $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ and $\mathfrak{n}_+$, i.e. $\mathfrak{p}$ is the standard subalgebra of block uppertriangular matrices. Denote by $\mathfrak{u}$ the nilpotent radical of $\mathfrak{p}$. Denote by $\mathcal{M}(\mathfrak{p})$ the category of finitely-generated $U(\mathfrak{p})$-modules that are $\mathfrak{h}$-diagonalizable and $U(\mathfrak{n}_+)$-locally nilpotent.

Let Ind be the induction functor Ind : $\mathcal{M}(\mathfrak{p}) \to \mathcal{O}(\mathfrak{gl}_{n+m})$. To a $U(\mathfrak{p})$-module $N \in \mathcal{M}(\mathfrak{p})$ it associates the $U(\mathfrak{gl}_{n+m})$-module $U(\mathfrak{gl}_{n+m}) \otimes_{U(\mathfrak{p})} N$. Recall that we denoted by F_V the functor of tensoring with a finite dimensional $\mathfrak{g}$-module V.

Lemma 3. *Let V be a finite dimensional $\mathfrak{gl}_{n+m}$-module. Denote by F'_V the functor from $\mathcal{M}(\mathfrak{p})$ to $\mathcal{M}(\mathfrak{p})$ given by tensoring with V, considered as a $U(\mathfrak{p})$-module. Then there is a canonical isomorphism of functors*

$$F_V \circ \mathrm{Ind} \cong \mathrm{Ind} \circ F'_V. \tag{45}$$

Proof. For $N \in \mathcal{M}(\mathfrak{p})$ we have natural isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{U(\mathfrak{gl}_{n+m})}(\mathrm{Ind} \circ F'_V(N), F_V \circ \mathrm{Ind}(N)) \\ = &\mathrm{Hom}_{U(\mathfrak{p})}(F'_V(N), F_V \circ \mathrm{Ind}(N)) = \\ = &\mathrm{Hom}_{U(\mathfrak{p})}(V \otimes N, V \otimes (U(\mathfrak{gl}_{n+m}) \otimes_{U(\mathfrak{p})} N)), \end{aligned}$$

the first isomorphism coming from the adjointness of induction and restriction functors. The third hom-space has a distinguished element coming from the $U(\mathfrak{p})$-module map $V \otimes N \longrightarrow V \otimes (U(\mathfrak{gl}_{n+m}) \otimes_{U(\mathfrak{p})} N)$ given by $v \otimes n \longmapsto v \otimes (1 \otimes n)$. It is easy to see that the corresponding map of $U(\mathfrak{gl}_{n+m})$ modules

$$\mathrm{Ind} \circ F'_V(N) \longrightarrow F_V \circ \mathrm{Ind}(N)$$

is an isomorphism. □

Let Y be the one-dimensional representation of $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ which has weight $\frac{m}{2}(e_1 + \cdots + e_n)$ as a representation of $\mathfrak{gl}_n$ and weight $\frac{-n}{2}(e_1 + \cdots + e_m)$ as a representation of $\mathfrak{gl}_m$.

Let $\mathcal{C}_0$ be the functor $\mathcal{O}_n \times \mathcal{O}_m \longrightarrow \mathcal{M}(\mathfrak{p})$ defined as follows. Tensor a product $M \times N \in \mathcal{O}_n \times \mathcal{O}_m$ with Y, and then make $Y \otimes M \otimes N$ into a $\mathfrak{p}$-module with the trivial action of the nilpotent radical $\mathfrak{u}$ of $\mathfrak{p}$.

Define $\mathcal{C} : \mathcal{O}_n \times \mathcal{O}_m \longrightarrow \mathcal{O}_{n+m}$ to be the composition of $\mathcal{C}_0$ and Ind :

$$\mathcal{C} = \mathrm{Ind} \circ \mathcal{C}_0.$$

Bifunctor $\mathcal{C}$ is exact and when applied to a product of Verma modules $M(a_1 \dots a_n) \times M(b_1 \dots b_m)$ produces the Verma module $M(a_1 \dots a_n b_1 \dots b_m)$. Therefore, we have a commutative diagram of isomorphisms of abelian groups:

$$\begin{tikzcd} K(\mathcal{O}_n) \times K(\mathcal{O}_m) \arrow[r, "\Upsilon_n \times \Upsilon_m"] \arrow[d, "{[\mathcal{C}]}"] & V_1^{\otimes n} \times V_1^{\otimes m} \arrow[d, equal] \\ K(\mathcal{O}_{n+m}) \arrow[r, "\Upsilon_{n+m}"] & V_1^{\otimes (n+m)}. \end{tikzcd}$$

Let us present more evidence that $\mathcal{C}$ is the bifunctor that categorifies the identity (44) by showing how to use $\mathcal{C}$ to categorify the comultiplication formula $\Delta E = E \otimes 1 + 1 \otimes E$.

Recall from Section 3.1.1 that L_n denotes the n-dimensional fundamental representation of $\mathfrak{gl}_n$. We can make L_n into a $U(\mathfrak{p})$-module by making $\mathfrak{u}\oplus\mathfrak{gl}_m \subset \mathfrak{p}$ act by 0. Similarly, L_m and L_{n+m} are $U(\mathfrak{p})$-modules in the natural way and we have a short exact sequence of $U(\mathfrak{p})$-modules

$$0 \to L_n \to L_{n+m} \to L_m \to 0, \tag{46}$$

which gives rise to a short exact sequence of functors from $\mathcal{M}(\mathfrak{p})$ to $\mathcal{M}(\mathfrak{p})$:

$$0 \to F'_{L_n} \to F'_{L_{n+m}} \to F'_{L_m} \to 0,$$

where F'_{L_n} is the functor of tensoring with L_n, considered as a $U(\mathfrak{p})$-module, and so on.

Functors Ind and $\mathcal{C}_0$ are exact; and composing with them we obtain an exact sequence

$$0 \to \mathrm{Ind}\circ F'_{L_n}\circ\mathcal{C}_0 \to \mathrm{Ind}\circ F'_{L_{n+m}}\circ\mathcal{C}_0 \to \mathrm{Ind}\circ F'_{L_m}\circ\mathcal{C}_0 \to 0$$

of functors. We have isomorphisms of functors from $\mathcal{O}_n\times\mathcal{O}_m$ to $\mathcal{M}(\mathfrak{p})$:

$$F'_{L_n}\circ\mathcal{C}_0 \cong \mathcal{C}_0\circ(F_{L_n}\times\mathrm{Id})$$
$$F'_{L_m}\circ\mathcal{C}_0 \cong \mathcal{C}_0\circ(\mathrm{Id}\times F_m)$$

and the isomorphism (see Lemma 3)

$$\mathrm{Ind}\circ F'_{L_{n+m}} \cong F_{L_{n+m}}\circ\mathrm{Ind}\,.$$

We thus get an exact sequence of functors

$$0 \to \mathrm{Ind}\circ\mathcal{C}_0\circ(F_{L_n}\times\mathrm{Id}) \to F_{L_{n+m}}\circ\mathrm{Ind}\circ\mathcal{C}_0 \to \mathrm{Ind}\circ\mathcal{C}_0\circ(\mathrm{Id}\times F_{L_m}) \to 0, \tag{47}$$

and recalling that $\mathcal{C}=\mathrm{Ind}\circ\mathcal{C}_0$ we obtain an exact sequence

$$0 \to \mathcal{C}\circ(F_{L_n}\times\mathrm{Id}) \to F_{L_{n+m}}\circ\mathcal{C} \to \mathcal{C}\circ(\mathrm{Id}\times F_{L_m}) \to 0 \tag{48}$$

$\mathcal{E}_{i,n}$ is a direct summand of the functor F_{L_n} and from this we derive

Proposition 10. *Exact sequence* (48) *contains the following exact sequence of functors as a direct summand*

$$0 \to \mathcal{C}\circ(\mathcal{E}_{i,n}\times\mathrm{Id}) \to \mathcal{E}_{i+j,n+m}\circ\mathcal{C} \to \mathcal{C}\circ(\mathrm{Id}\times\mathcal{E}_{j,m}) \to 0. \tag{49}$$

This exact sequence can be considered as a categorification of the comultiplication formula

$$\Delta(E) = E\otimes 1 + 1\otimes E. \tag{50}$$

In the same fashion, using the exact sequence dual to (46) we obtain an exact sequence of functors

$$0 \to \mathcal{C}\circ(\mathrm{Id}\times\mathcal{F}_{j,m}) \to \mathcal{F}_{i+j,n+m}\circ\mathcal{C} \to \mathcal{C}\circ(\mathcal{F}_{i,n}\times\mathrm{Id}) \to 0. \tag{51}$$

We proceed to "categorify" the comultiplication rules for the divided powers $\mathcal{E}^{(a)}_{i,n}, \mathcal{F}^{(a)}_{i,n}$.

The $\mathfrak{gl}_{n+m}$-module $\Lambda^a L_{n+m}$, considered as a $\mathfrak{p}$-module, admits a filtration $\Lambda^a L_{n+m} = G_{a+1}\supset G_a\supset\cdots\supset G_0 = 0$ such that the module $G_{k+1}/G_k, k = 0,\ldots,a$ is isomorphic to $\Lambda^{a-k}L_n\otimes\Lambda^k L_m$. Therefore, we obtain

Proposition 11. *The exact functor $\mathcal{E}^{(a)}_{i+j,n+m} \circ \mathcal{C}$ has a filtration by exact functors*

$$\mathcal{E}^{(a)}_{i+j,n+m} \circ \mathcal{C} = \mathcal{G}_{a+1} \supset \mathcal{G}_a \supset \cdots \supset \mathcal{G}_0 = 0$$

together with short exact sequences of functors

$$0 \to \mathcal{G}_k \to \mathcal{G}_{k+1} \to \mathcal{C} \circ (\mathcal{E}^{(a-k)}_{i,n} \times \mathcal{E}^{(k)}_{j,m}) \to 0.$$

The exact functor $\mathcal{F}^{(a)}_{i+j,n+m} \circ \mathcal{C}$ has a filtration by exact functors

$$\mathcal{F}^{(a)}_{i+j,n+m} \circ \mathcal{C} = \mathcal{G}_{a+1} \supset \mathcal{G}_a \supset \cdots \supset \mathcal{G}_0 = 0$$

such that the sequences below are exact

$$0 \to \mathcal{G}_k \to \mathcal{G}_{k+1} \to \mathcal{C} \circ (\mathcal{F}^{(k)}_{i,n} \times \mathcal{F}^{(a-k)}_{j,m}) \to 0.$$

This can be considered as a categorification of the comultiplication formulas (7) and (8).

Remark. In [Gr] Grojnowsky uses perverse sheaves to categorify the comultiplication rules for $U_q(\mathfrak{sl}_k)$. His approach appears to be Koszul dual to ours.

3.2. Zuckerman functors and Temperley-Lieb algebra

3.2.1. Computations with Zuckerman functors. Let $\mathfrak{g}_i$, $1 \le i \le n-1$ be a subalgebra of $\mathfrak{gl}_n$ consisting of matrices that can have non-zero entries only on intersections of i-th or $(i+1)$-th rows with i-th or $(i+1)$-th columns. $\mathfrak{g}_i$ is isomorphic to $\mathfrak{gl}_2$. Denote by $\mathcal{O}^i_{k,n-k}$ the subcategory of $\mathcal{O}_{k,n-k}$ consisting of locally $U(\mathfrak{g}_i)$-finite modules. Denote by $\Gamma_i : \mathcal{O}_{k,n-k} \longrightarrow \mathcal{O}^i_{k,n-k}$ the Zuckerman functor of taking the maximal locally $U(\mathfrak{g}_i)$-finite submodule, and by $\mathcal{R}\Gamma_i$ the derived functor of Γ_i. The derived functor goes from the bounded derived category $D^b(\mathcal{O}_{k,n-k})$ to $D^b(\mathcal{O}^i_{k,n-k})$. Denote by Γ^j_i the cohomology functor $\mathcal{R}^j\Gamma_i : \mathcal{O}_{k,n-k} \to \mathcal{O}^j_{k,n-k}$. This functor is zero if $j > 2$ by Proposition 3.

Recall that for a sequence $a_1 \dots a_n$ of zeros and ones with exactly k ones, the Verma module $M(a_1 \dots a_n)$ belongs to $\mathcal{O}_{k,n-k}$.

Proposition 12. *If $a_i = 1$, $a_{i+1} = 0$,*

$$\Gamma^2_i M(a_1 \dots a_n) = M(a_1 \dots a_n)/M(a_1 \dots a_{i-1} 0 1 a_{i+2} \dots a_n).$$

If $a_i = 0, a_{i+1} = 1$,

$$\Gamma^1_i M(a_1 \dots a_n) = M(a_1 \dots a_{i-1} 1 0 a_{i+2} \dots a_n)/M(a_1 \dots a_n).$$

For all other values of $(i, j, a_1, \dots, a_n)$ *with* $i \in 1, \dots, n-1, j \in \mathbb{Z}, a_1, \dots, a_n \in \{0,1\}$ *we have*

$$\Gamma_i^j M(a_1 \dots a_n) = 0.$$

Proof. Functors Γ_i^0 and Γ_i^2 have a simple description as functors of taking the maximal $U(\mathfrak{g}_i)$-locally finite submodule/quotient. So the proposition is easy to check for $j \neq 1$. For $j = 1$ it is a special case of Proposition 5.5 of [ES]. □

The simple module $L(a_1 \dots a_n)$ with $a_1, \dots, a_n \in \{0,1\}$ and $a_1 + \cdots + a_n = k$ belongs to $\mathcal{O}_{k,n-k}^i$ if and only if $a_i = 1, a_{i+1} = 0$. Therefore, a simple object of $\mathcal{O}_{k,n-k}$ cannot belong simultaneously to $\mathcal{O}_{k,n-k}^i$ and $\mathcal{O}_{k,n-k}^{i+1}$.

Corollary 3. *If a module* $M \in \mathcal{O}_{k,n-k}$ *lies in both* $\mathcal{O}_{k,n-k}^i$ *and* $\mathcal{O}_{k,n-k}^{i+1}$, *it is trivial.*

Proposition 13. *For any* $M \in \mathcal{O}_{k,n-k}^i$,

$$\Gamma_{i\pm 1}^j M = 0 \textit{ if } j \neq 1.$$

Proof. We know that $\Gamma_{i\pm1}^j M$ can be nontrivial only for $0 \leq j \leq 2$. But $\Gamma_{i\pm1}^0 M$ is the maximal locally $\mathfrak{g}_{i\pm1}$-finite submodule of M, i.e. the zero module (by the last corollary). Similarly, $\Gamma_{i\pm1}^2 M = 0$. □

Corollary 14.

1. *Functors* $\mathcal{R}\Gamma_{i\pm1}[1]$ *and* $\Gamma_{i\pm1}^1$, *restricted to the subcategory* $\mathcal{O}_{k,n-k}^i$, *are naturally equivalent (more precisely, the composition of* $\Gamma_{i\pm1}^1$, *restricted to* $\mathcal{O}_{k,n-k}^i$, *with the embedding functor* $\mathcal{O}_{k,n-k}^{i\pm1} \longrightarrow D^b(\mathcal{O}_{k,n-k}^{i\pm1})$ *is equivalent to the restriction of the functor* $\mathcal{R}\Gamma_{i\pm1}[1]$ *to the subcategory* $\mathcal{O}_{k,n-k}^i$ *of* $D^b(\mathcal{O}_{k,n-k}^i)$.*)*
2. *The functor* $\Gamma_{i\pm1}^1 : \mathcal{O}_{k,n-k}^i \to \mathcal{O}_{k,n-k}^{i\pm1}$ *is exact.*

Generalized Verma modules for $\mathfrak{g}_i$ are isomorphic to the quotients

$$M(a_1 \dots a_{i-1} 10 a_{i+2} \dots a_n)/M(a_1 \dots a_{i-1} 10 a_{i+2} \dots a_n).$$

Denote this quotient module by $M_i(a_1 \dots a_{n-2})$. To simplify notations, if $a_1 \dots a_{i-1} a_{i+2} \dots a_n$ is fixed, we will denote $M(a_1 \dots a_{i-1} 10 a_{i+2} \dots a_n)$ by M_{10} and $M(a_1 \dots a_{i-1} 01 a_{i+2} \dots a_n)$ by M_{01}. Then $M_i(a_1 \dots a_{n-2}) = M_{10}/M_{01}$.

Proposition 14.

$$\Gamma_{i\pm1}^1 M_i(a_1 \dots a_{n-2}) = M_{i\pm1}(a_1 \dots a_{n-2}).$$

Proof. We have an exact sequence

$$0 \to M_{01} \to M_{10} \to M_i \to 0$$

which induces a long exact sequence of Zuckerman cohomology functors

$$\cdots \to \Gamma^j_{i-1} M_{01} \to \Gamma^j_{i-1} M_{10} \to \Gamma^j_{i-1} M_i \to \Gamma^{j+1}_{i-1} M_{01} \to \dots$$

Consider the following segment of this sequence

$$\Gamma^1_{i-1} M_{01} \to \Gamma^1_{i-1} M_{10} \to \Gamma^1_{i-1} M_i \to \Gamma^2_{i-1} M_{01} \to \Gamma^2_{i-1} M_{10}. \tag{52}$$

From Proposition 12, we find that

$$\Gamma^1_{i-1} M_{01} = \Gamma^2_{i-1} M_{10} = 0$$

and (52) becomes a short exact sequence

$$0 \to \Gamma^1_{i-1} M_{10} \to \Gamma^1_{i-1} M_i \to \Gamma^2_{i-1} M_{01} \to 0.$$

We proceed by considering two different cases:

(i) If $a_{i-1} = 0$, then by Proposition 12 we have $\Gamma^1_{i-1} M_{10} = M_{i-1}(a_1 \dots a_{n-2})$ and $\Gamma^2_{i-1} M_{01} = 0$.

(ii) If $a_{i-1} = 1$, then $\Gamma^2_{i-1} M_{01} = M_{i-1}(a_1 \dots a_{n-2})$ and $\Gamma^1_{i-1} M_{10} = 0$.

In both cases we get $\Gamma^1_{i-1} M_i = M_{i-1}(a_1 \dots a_{n-2})$. The proof for Γ_{i+1} is the same. □

3.2.2. Equivalences of categories. Let ε_i be the inclusion functor $\mathcal{O}^i_{k,n-k} \to \mathcal{O}_{k,n-k}$. Consider a pair of functors

$$\Gamma^1_{i-1}\varepsilon_i : \mathcal{O}^i_{k,n-k} \to \mathcal{O}^{i-1}_{k,n-k}$$
$$\Gamma^1_i \varepsilon_{i-1} : \mathcal{O}^{i-1}_{k,n-k} \to \mathcal{O}^i_{k,n-k}.$$

These two functors are exact, take generalized Verma modules to generalized Verma modules and the compositions $\Gamma^1_{i-1}\varepsilon_i \Gamma^1_i \varepsilon_{i-1}$ and $\Gamma^1_i \varepsilon_{i-1} \Gamma^1_{i-1}\varepsilon_i$ are identities on generalized Verma modules.

Theorem 5. *Functors $\Gamma^1_{i-1}\varepsilon_i$ and $\Gamma^1_i\varepsilon_{i-1}$ are equivalences of categories $\mathcal{O}^i_{k,n-k}$ and $\mathcal{O}^{i-1}_{k,n-k}$. The composition $\Gamma^1_i\varepsilon_{i-1}\Gamma^1_{i-1}\varepsilon_i$ is isomorphic to the identity functor from the category $\mathcal{O}^i_{k,n-k}$ to itself.*

Proof. By Lemma 2 it remains to show that functors $\Gamma^1_{i-1}\varepsilon_i$ and $\Gamma^1_i\varepsilon_{i-1}$ are two-sided adjoint. We recall that $\Gamma_i : \mathcal{O}_{k,n-k} \to \mathcal{O}^i_{k,n-k}$ is right adjoint to the inclusion functor ε_i. Besides, in the derived category, the derived functor $R\Gamma_i$ is isomorphic to the left adjoint of the shifted inclusion functor $\varepsilon_i[2]$.

Let $M, N \in \mathcal{O}^i_{k,n-k}$. We have natural vector space isomorphisms

$$\begin{aligned}\mathrm{Hom}((\Gamma^1_{i-1}\varepsilon_i)M, N) &\cong \mathrm{Hom}((R\Gamma_{i-1}[1]\circ\varepsilon_i)M, N)\\ &\cong \mathrm{Hom}(\varepsilon_i M, \varepsilon_{i-1}N[1])\\ &\cong \mathrm{Hom}(M, (R\Gamma_i)\varepsilon_{i-1}N[1])\\ &\cong \mathrm{Hom}(M, (R\Gamma_i)[1]\varepsilon_{i-1}N)\\ &\cong \mathrm{Hom}(M, \Gamma^1_i\varepsilon_{i-1}N)\end{aligned}$$

which imply that $\Gamma^1_{i-1}\varepsilon_i$ is left adjoint to $\Gamma^1_i\varepsilon_{i-1}$. A similar computation tells us that $\Gamma^1_{i-1}\varepsilon_i$ is also right adjoint to $\Gamma^1_i\varepsilon_{i-1}$.

Remark. For an abelian category $\mathcal{A}$ the embedding functor $\mathcal{A} \longrightarrow D^b(\mathcal{A})$ is fully faithful and for $M, N \in \mathcal{A}$ we have canonical isomorphisms of hom-spaces

$$\mathrm{Hom}_{\mathcal{A}}(M, N) = \mathrm{Hom}_{D^b(\mathcal{A})}(M, N).$$

That permits us in the chain of identities above to go freely between hom spaces in $\mathcal{O}^i_{k,n-k}$ and $D^b(\mathcal{O}^i_{k,n-k})$. □

Corollary 5. *The categories $\mathcal{O}^i_{k,n-k}$ and $\mathcal{O}^j_{k,n-k}$ are equivalent for all i, j, $1 \leq i, j \leq n-1$.*

Proposition 15. *Let $M \in \mathcal{O}^i_{k,n-k}$. Then*

$$\Gamma^j_i\varepsilon_i M = \begin{cases} M & \text{if } j = 0 \text{ or } j = 2\\ 0 & \text{otherwise.}\end{cases}$$

Proof. $\Gamma^0_i\varepsilon_i M$ is the maximal $\mathfrak{g}_i$-locally finite submodule of M and $\Gamma^2_i\varepsilon_i M$ is the maximal $\mathfrak{g}_i$-locally finite quotient of M. Thus, $\Gamma^0_i\varepsilon_i M = \Gamma^2_i\varepsilon_i M = M$. It remains to check that $\Gamma^1_i\varepsilon_i M = 0$ for any $M \in \mathcal{O}^i_{k,n-k}$.

The derived functor $\mathcal{R}\Gamma_i$ is exact and induces a map of Grothendieck groups

$$[\mathcal{R}\Gamma_i] : K(\mathcal{O}_{k,n-k}) \to K(\mathcal{O}^i_{k,n-k}).$$

Using Proposition 12 we can easily compute $[\mathcal{R}\Gamma_i]$. The bases of $K(\mathcal{O}_{k,n-k})$ and $K(\mathcal{O}^i_{k,n-k})$ consisting of images of Verma modules, respectively generalized Verma modules, are convenient for writing down $[R\Gamma_i]$ explicitly:

$$\begin{aligned}[\mathcal{R}\Gamma_i(M(a_1\dots a_{i-1}00a_i\dots a_{n-2}))] &= 0\\ [\mathcal{R}\Gamma_i(M(a_1\dots a_{i-1}11a_i\dots a_{n-2}))] &= 0\\ [\mathcal{R}\Gamma_i(M(a_1\dots a_{i-1}10a_i\dots a_{n-2}))] &= [M_i(a_1\dots a_{n-2})]\\ [\mathcal{R}\Gamma_i(M(a_1\dots a_{i-1}01a_i\dots a_{n-2}))] &= -[M_i(a_1\dots a_{n-2})]\end{aligned}$$

where, we recall

$$M_i(a_1 \ldots a_{n-2}) = M(a_1 \ldots a_{i-1} 10 a_i \ldots a_{n-2}) / M(a_1 \ldots a_{i-1} 01 a_i \ldots a_{n-2}).$$

Therefore,

$$[\mathcal{R}\Gamma_i \circ \varepsilon_i(M_i(a_1 \ldots a_{n-2}))] = [M_i(a_1 \ldots a_{n-2})] \oplus [M_i(a_1 \ldots a_{n-2})]$$

and

$$[(\mathcal{R}\Gamma_i \circ \varepsilon_i)M] = [M] \oplus [M]$$

for any $M \in \mathcal{O}^i_{k,n-k}$. On the other hand,

$$[(\mathcal{R}\Gamma_i \circ \varepsilon_i)M] = [\Gamma^0_i \varepsilon_i M] - [\Gamma^1_i \varepsilon_i M] + [\Gamma^2_i \varepsilon_i M] = [M] - [\Gamma^1_i \varepsilon_i M] + [M].$$

Thus, $[\Gamma^1_i \varepsilon_i M] = 0$ for any $M \in \mathcal{O}^i_{k,n-k}$ and, hence, $\Gamma^1_i \varepsilon_i M = 0$ for any $M \in \mathcal{O}^i_{k,n-k}$. $\square$

Proposition 16. *Restricting to the subcategory $D^b(\mathcal{O}^i_{k,n-k})$, we have an equivalence of functors*

$$\mathcal{R}\Gamma_i \circ \varepsilon_i \cong \mathrm{Id} \oplus \mathrm{Id}[-2]. \tag{53}$$

Proof. Consider the natural transformation

$$e : \mathrm{Id} \to \mathcal{R}\Gamma_i \circ \varepsilon_i.$$

coming from the adjointness of $\mathcal{R}\Gamma_i$ and ε_i. Let C_i be the functor which is the cone of e. Then by [MP], Lemma 3.2, we have an isomorphism of functors

$$\mathcal{R}\Gamma_i \circ \varepsilon_i = \mathrm{Id} \oplus C_i.$$

From Proposition 15

$$C^j_i M = \begin{cases} M & \text{if } j = 2 \\ 0 & \text{if } j \neq 2. \end{cases}$$

Therefore, $C_i = \mathrm{Id}[-2]$ and we have the isomorphism (53). $\square$

3.2.3. A realization of the Temperley-Lieb algebra by functors. Define functors $\mathcal{V}_i, 1 \leq i \leq n-1$ from $D^b(\mathcal{O}_{k,n-k})$ to $D^b(\mathcal{O}_{k,n-k})$ by

$$\mathcal{V}_i = \varepsilon_i \circ \mathcal{R}\Gamma_i[1]. \tag{54}$$

Theorem 6. *There are natural equivalences of functors*

$$(\mathcal{V}_i)^2 \cong \mathcal{V}_i[-1] \oplus \mathcal{V}_i[1] \tag{55}$$

$$\mathcal{V}_i\mathcal{V}_j \cong \mathcal{V}_j\mathcal{V}_i \quad \textit{for} \quad |i-j| > 1 \tag{56}$$

$$\mathcal{V}_i\mathcal{V}_{i\pm 1}\mathcal{V}_i \cong \mathcal{V}_i. \tag{57}$$

Proof. Isomorphism (55) follows from Proposition 16. Isomorphism (56) is implied by a commutativity isomorphism $\Gamma_i\Gamma_j \cong \Gamma_j\Gamma_i$ for $|i-j| > 1$. The last isomorphism is a corollary of Theorem 5. □

Summing over all k from 0 to n we obtain functors

$$\mathcal{V}_i : D^b(\mathcal{O}_n) \longrightarrow D^b(\mathcal{O}_n)$$

together with isomorphisms (55)–(57). On the Grothendieck group level these functors descend to the action of the Temperley-Lieb algebra $TL_{n,1}$ on $V_1^{\otimes n}$. We thus have a categorification of the action of the Temperley-Lieb algebra on the tensor product $V_1^{\otimes n}$. This action is faithful, so we can loosely say that we have a categorification of the Temperley-Lieb algebra $TL_{n,1}$ itself. In fact if we consider the shift by 1 in the derived category as the analogue of the multiplication by q, then we have categorified the Temperley-Lieb algebra $TL_{n,q}$.

Recall from Section 2.2 that the element U_i of the Temperley-Lieb algebra is the product of morphisms $\cup_{i,n-2}$ and $\cap_{i,n}$ of the Temperley-Lieb category. Morphisms $\cap_{i,n}$ and $\cup_{i,n-2}$ go between objects $\overline{n}$ and $\overline{n-2}$ of the Temperley-Lieb category. On the other hand, the functor $\mathcal{V}_i$, which categorifies U_i, is the composition of functors ε_i and $\mathcal{R}\Gamma_i[1]$. We now explain how to modify the latter functors into functors between derived categories $D^b(\mathcal{O}_n)$ and $D^b(\mathcal{O}_{n-2})$ that cat be viewed as categorifications of morphisms $\cap_{i,n}$ and $\cup_{i,n-2}$.

Let ς_n be the functor from $\mathcal{O}_{k-1,n-k-1}$ to $\mathcal{O}^1_{k,n-k}$ given as follows. First we tensor an $M \in \mathcal{O}_{k-1,n-k-1}$ with the fundamental representation L_2 of $\mathfrak{gl}_2$ to get a $\mathfrak{gl}_2 \oplus \mathfrak{gl}_{n-2}$-module $L_2 \otimes M$. Let Y be the one-dimensional $\mathfrak{gl}_2 \oplus \mathfrak{gl}_{n-2}$-module with weight $\frac{n-3}{2}(e_1+e_2)$ relative to $\mathfrak{gl}_2$ and $-(e_1+\cdots+e_{n-2})$ relative to $\mathfrak{gl}_{n-2}$. Let $\mathfrak{p}$ be the maximal parabolic subalgebra of $\mathfrak{gl}_n$ that contains $\mathfrak{gl}_2 \oplus \mathfrak{gl}_{n-2}$ and the subalgebra of upper triangular matrices. Then $Y \otimes (L_2 \otimes M)$ is naturally a $\mathfrak{p}$-module with the nilradical of $\mathfrak{p}$ acting trivially. Now we parabolically induce from $\mathfrak{p}$ to $\mathfrak{gl}_n$ and define

$$\varsigma_n(M) = U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (Y \otimes L_2 \otimes M).$$

Let ν_n be the functor from $\mathcal{O}^1_{k,n-k}$ to $\mathcal{O}_{n-1,n-k-1}$ defined as follows. For an $M \in \mathcal{O}^1_{k,n-k}$, take the sum of the weight subspaces of M of weights $e_1 + x_3e_3 + \cdots + x_ne_n - \rho_n$, where $x_3, \ldots, x_n \in \mathbb{Z}$ and ρ_n is the half-sum of the positive roots of $\mathfrak{gl}_n$. This direct sum is a $\mathfrak{gl}_{n-2}$-module in a natural way. Define $\nu_n(M)$ as the tensor product of this module with the one-dimensional $\mathfrak{gl}_{n-2}$-module of weight $e_1 + \cdots + e_{n-2}$.

Proposition 17. *Functors ς_n and ν_n are mutually inverse equivalences of categories $\mathcal{O}_{k-1,n-k-1}$ and $\mathcal{O}^1_{k,n-k}$.*

We omit the proof as it is quite standard.

Corollary 6. *The categories $\mathcal{O}^i_{k,n-k}, 1 \le i \le n-1$ and $\mathcal{O}_{k-1,n-k-1}$ are equivalent.*

Denote by $\Xi_{n,i}$ the equivalence of categories

$$\Xi_{n,i} : \mathcal{O}^i_{k,n-k} \longrightarrow \mathcal{O}_{k-1,n-k-1}$$

given by the composition

$$\Xi_{n,i} = \nu_n \circ \Gamma^1_1 \circ \varepsilon_2 \circ \Gamma^1_2 \circ \dots \varepsilon_{i-1} \circ \Gamma^1_{i-1} \circ \varepsilon_i,$$

i.e., $\Xi_{n,i}$ is the composition of equivalences of categories

$$\mathcal{O}^i_{k,n-k} \xrightarrow{\cong} \mathcal{O}^{i-1}_{k,n-k} \xrightarrow{\cong} \dots \xrightarrow{\cong} \mathcal{O}^1_{k,n-k} \xrightarrow{\cong} \mathcal{O}_{k-1,n-k-1}.$$

Denote by $\Pi_{n,i}$ the equivalence of categories

$$\Pi_{n,i} : \mathcal{O}_{k,n-k} \longrightarrow \mathcal{O}^i_{k+1,n+1-k}$$

given by

$$\Pi_{n,i} = \Gamma^1_i \circ \varepsilon_{i-1} \circ \Gamma^1_{i-1} \circ \dots \circ \varepsilon_2 \circ \Gamma^1_2 \circ \varepsilon_1 \circ \varsigma_n.$$

Denote the derived functors of these functors by $\mathcal{R}\Xi_{n,i}$ and $\mathcal{R}\Pi_{n,i}$. Define functors

$$\begin{aligned} \cap_{i,n} :& D^b(\mathcal{O}_{k,n-k}) \longrightarrow D^b(\mathcal{O}_{k-1,n-k-1}) \\ \cup_{i,n} :& D^b(\mathcal{O}_{k,n-k}) \longrightarrow D^b(\mathcal{O}_{k+1,n+1-k}) \end{aligned}$$

by

$$\cap_{i,n} = \mathcal{R}\Xi_{n,i} \circ \mathcal{R}\Gamma_i[1] \tag{58}$$

$$\cup_{i,n} = \varepsilon_i \circ \mathcal{R}\Pi_{n,i}. \tag{59}$$

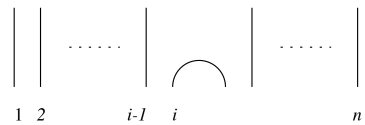
Recall from Section 2.2 defining relations (23)–(29) for the Temperley-Lieb category TL.

Conjecture 1. *There are natural equivalences* (23)–(28) *with functors $\cap_{i,n}$ and $\cup_{i,n}$ defined by* (58) *and* (59).

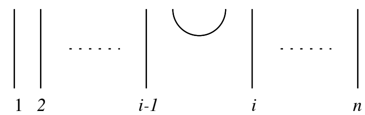
The first two of these equivalences follow from the results of this section. The relation (29) will become $\cap_{i,n+2} \circ \cup_{i,n} \cong \mathrm{Id}[1] \oplus \mathrm{Id}[-1]$.

We next state a conjecture on a functor realization of the category of tangles in $\mathbb{R}^3$. Consider the following 3 elementary tangles

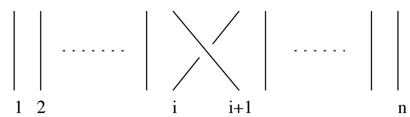
(a)



(b)



(c)



Every tangle in 3-space can be presented as a concatenation of these elementary tangles. We associate to these 3 types of tangles the following functors:

To the tangle (a) associate functor $\cap_{i,n}$ given by the formula (58).

To the tangle (b) associate functor $\cup_{i,n}$ given by the formula (59).

To the tangle (c) associate functor $R_{i,n}$ from $D^b(\mathcal{O}_n)$ to $D^b(\mathcal{O}_n)$ which is the cone of the adjointness morphism of functors

$$\varepsilon_i \circ \mathcal{R}\Gamma_i \longrightarrow \mathrm{Id}\,.$$

Given a presentation α of a tangle t as a composition of elementary tangles of types (a)–(c), to α we associate the functor $f(\alpha)$ which is the corresponding composition of functors $\cap_{i,n}, \cup_{i,n}, R_{i,n}$.

Conjecture 2. *Given two such presentations α, β of a tangle t, functors $f(\alpha)$ and $f(\beta)$ are isomorphic, up to shifts in the derived category.*

This conjecture, if true, will give us functor invariants of tangles. Certain natural transformations between these functors, corresponding to adjointness morphisms between the (shifted) identity functor and compositions of ε_i and $\mathcal{R}\Gamma_i$, are expected to produce invariants of 2-tangles and 2-knots (to be discussed elsewhere). When we have a link rather than a tangle, the associated functor goes between categories of complexes of vector spaces up to homotopies, and the resulting invariants of links will be $\mathbb{Z}$-graded homology groups.

4. Parabolic categories

4.1. Temperley-Lieb algebra and projective functors

4.1.1. On and off the wall translation functors in parabolic categories. Let μ be an integral dominant regular weight and $\mu_i, i = 1, \dots, n-1$ an integral dominant subregular weight on the i-th wall. Let $\mathcal{O}_\mu$ and $\mathcal{O}_{\mu_i}$ be the subcategories of $\mathcal{O}(\mathfrak{gl}_n)$ of modules with generalized central characters $\eta(\mu)$ and $\eta(\mu_i)$. Then $\mathcal{O}_\mu$ is a regular block of $\mathcal{O}(\mathfrak{gl}_n)$ and $\mathcal{O}_{\mu_i}$ is a subregular block of $\mathcal{O}(\mathfrak{gl}_n)$. Verma modules M_μ and M_{μ_i} with highest weights $\mu - \rho$ and $\mu_i - \rho$ are dominant Verma modules in the corresponding categories.

Let T^i, T_i be translation functors on and off the i-th wall

$$\begin{aligned} T^i : &\mathcal{O}_\mu \longrightarrow \mathcal{O}_{\mu_i} \\ T_i : &\mathcal{O}_{\mu_i} \longrightarrow \mathcal{O}_\mu. \end{aligned}$$

These functors are defined up to an isomorphism by the condition that they are projective functors between $\mathcal{O}_\mu$ and $\mathcal{O}_{\mu_i}$ and

1. Functor T^i takes the Verma module M_μ to the Verma module M_{μ_i}.
2. Functor T_i takes the Verma module M_{μ_i} to the projective module $P_{s_i\mu}$ where s_i is the transposition $(i, i+1)$. On the Grothendieck group level,

$$[T_i M_{\mu_i}] = [M_\mu] + [M_{s_i\mu}].$$

Let $\mathfrak{p}_k$ be the maximal parabolic subalgebra of $\mathfrak{gl}_n$ such that $\mathfrak{p}_k \supset \mathfrak{n}_+ \oplus \mathfrak{h}$ and the reductive subalgebra of $\mathfrak{p}_k$ is $\mathfrak{gl}_k \oplus \mathfrak{gl}_{n-k}$. Let $\mathcal{O}^{k,n-k}$, resp $\mathcal{O}^{k,n-k}_i$ be the full subcategory of $\mathcal{O}_\mu$, resp. $\mathcal{O}_{\mu_i}$ consisting of modules that are locally $U(\mathfrak{p}_k)$-finite.

From now on we fix k between 0 and n. Let τ_i^{i+1} be the composition of T_i and T^{i+1}:

$$\tau_i^{i+1} = T^{i+1} \circ T_i.$$

This is a functor from $\mathcal{O}_{\mu_i}$ to $\mathcal{O}_{\mu_{i+1}}$. Similarly, let τ_i^{i-1} be the functor from $\mathcal{O}_{\mu_i}$ to $\mathcal{O}_{\mu_{i-1}}$ given by

$$\tau_i^{i-1} = T^{i-1} \circ T_i.$$

Projective functors preserve subcategories of $U(\mathfrak{p}_k)$-locally finite modules and thus functors $\tau_i^{i\pm1}$ restrict to functors from $\mathcal{O}^{k,n-k}_i$ to $\mathcal{O}^{k,n-k}_{i\pm1}$. Our categorification of the Temperley-Lieb algebra by projective functors is based on the following beautiful result of Enright and Shelton:

Theorem. *Functors $\tau_i^{i\pm1}$ establish equivalences of categories $\mathcal{O}^{k,n-k}_i$ and $\mathcal{O}^{k,n-k}_{i\pm1}$.*

Proof. See [ES], Lemma 10.1 for a proof of a slightly more general statement. An alternative proof that we give below uses

Lemma 4. *The functor $\tau^i_{i+1}\tau^{i+1}_i$ restricted to $\mathcal{O}^{k,n-k}_i$ is isomorphic to the identity functor.*

Proof. We first study this functor as a projective functor from the subregular block $\mathcal{O}_{\mu_i}$ to itself. We will show that this functor is a direct sum of the identity functor and another projective functor that vanishes when restricted to $\mathcal{O}^{k,n-k}_i$.

An isomorphism class of a projective functor is determined by its action on the dominant Verma module. So let us compute the action of $\tau^i_{i+1}\tau^{i+1}_i$ on M_{μ_i} on the Grothendieck group level.

$$\begin{aligned}
[\tau^i_{i+1}\tau^{i+1}_i M_{\mu_i}] &= [T^iT_{i+1}T^{i+1}T_iM_{\mu_i}] \\
&= [T^iT_{i+1}T^{i+1}(M_\mu \oplus M_{s_i\mu})] \\
&= [T^iT_{i+1}(M_{\mu_{i+1}} \oplus M_{s_i\mu_{i+1}})] \\
&= [T^i(M_\mu \oplus M_{s_{i+1}\mu} \oplus M_{s_i\mu} \oplus M_{s_is_{i+1}\mu}] \\
&= [M_{\mu_i} \oplus M_{s_{i+1}\mu_i} \oplus M_{s_i\mu_i} \oplus M_{s_is_{i+1}\mu_i}] \\
&= [M_{\mu_i}] + [M_{s_{i+1}\mu_i}] + [M_{s_i\mu_i}] + [M_{s_is_{i+1}\mu_i}] \\
&= [M_{\mu_i}] + [M_{s_{i+1}\mu_i}] + [M_{\mu_i}] + [M_{s_is_{i+1}\mu_i}].
\end{aligned}$$

The projective module $P_{s_is_{i+1}\mu_i}$ decomposes in the Grothendieck group as the following sum

$$[P_{s_is_{i+1}\mu_i}] = [M_{\mu_i}] + [M_{s_{i+1}\mu_i}] + [M_{s_is_{i+1}\mu_i}].$$

Therefore,

$$[\tau^i_{i+1}\tau^{i+1}_i M_{\mu_i}] = [M_{\mu_i}] + [P_{s_is_{i+1}\mu_i}].$$

From the classification of projective functors (see [BG]), we derive that $\tau^i_{i+1}\tau^{i+1}_i$ is isomorphic to the direct sum of the identity functor and an indecomposable projective functor that takes M_{μ_i} to the indecomposable projective module $P_{s_is_{i+1}\mu_i}$. Denote this functor by $\wp$. Then

$$\tau^i_{i+1}\tau^{i+1}_i \cong \mathrm{Id} \oplus \wp.$$

Let ξ_i be an integral dominant weight on the intersection of the i-th and $(i+1)$-th walls. We require that ξ_i be a generic weight with these conditions, i.e. ξ_i does not lie on any other walls. Let $\mathcal{O}_{\xi_i}$ be the subcategory of $\mathcal{O}(\mathfrak{gl}_n)$ consisting of modules with generalized central character $\eta(\xi_i)$.

Let $T^{\xi_i}_{\mu_i}$ and $T^{\mu_i}_{\xi_i}$ be translation functors from $\mathcal{O}_{\mu_i}$ to $\mathcal{O}_{\xi_i}$ and back. Then $T^{\xi_i}_{\mu_i}$ takes the Verma module M_{μ_i} to the Verma module M_{ξ_i} while $T^{\mu_i}_{\xi_i}$ takes M_{ξ_i} to $P_{s_is_{i+1}\mu_i}$. Therefore, functor $\wp$ is isomorphic to the composition $T^{\mu_i}_{\xi_i}T^{\xi_i}_{\mu_i}$ and

$$\tau^i_{i+1}\tau^{i+1}_i \cong \mathrm{Id} \oplus T^{\mu_i}_{\xi_i}T^{\xi_i}_{\mu_i}.$$

The category $\mathcal{O}_{\xi_i}$ contains no $U(\mathfrak{p}_k)$-locally finite modules other than the zero module. Hence, the functor $T^{\xi_i}_{\mu_i}$, restricted to the subcategory $\mathcal{O}^{k,n-k}_i$ is the zero functor. Therefore, $\tau^i_{i+1}\tau^{i+1}_i$, restricted to $\mathcal{O}^{k,n-k}_i$, is isomorphic to the identity functor. This proves the lemma. □

In exactly the same fashion we establish that $\tau^{i+1}_i\tau^i_{i+1}$ is isomorphic to the identity functor from $\mathcal{O}^{k,n-k}_{i+1}$ to itself. Therefore functors τ^{i+1}_i and τ^i_{i+1} are mutually inverse and provide an equivalence of categories $\mathcal{O}^{k,n-k}_i$ and $\mathcal{O}^{k,n-k}_{i+1}$. □

4.1.2. Projective functor realization of the Temperley-Lieb algebra. Define the functor $\mathcal{U}_i, i = 1, \ldots n-1$ from $\mathcal{O}^{k,n-k}$ to $\mathcal{O}^{k,n-k}$ as the composition of functors T_i and T^i:

$$\mathcal{U}_i = T_i \circ T^i$$

Proposition 18. *There are equivalences of functors*

$$\begin{aligned} \mathcal{U}_i \circ \mathcal{U}_j &\cong \mathcal{U}_j \circ \mathcal{U}_i \quad \textit{for} \quad |i-j|>1 \\ \mathcal{U}_i \circ \mathcal{U}_i &\cong \mathcal{U}_i \oplus \mathcal{U}_i \\ \mathcal{U}_i \circ \mathcal{U}_{i\pm 1} \circ \mathcal{U}_i &\cong \mathcal{U}_i. \end{aligned}$$

Proof. The first two equivalences hold even if we consider $\mathcal{U}_i$ as the functor in the bigger category $\mathcal{O}_\mu$. For example,

$$\mathcal{U}_i \circ \mathcal{U}_i = T_i \circ T^i \circ T_i \circ T^i = (T_i \circ T^i) \oplus (T_i \circ T^i) = \mathcal{U}_i \oplus \mathcal{U}_i$$

where the second equality follows from the result that the composition $T^i \circ T_i$ of projective functors off and on the wall is the direct sum of two copies of the identity functor.

For the third equivalence the restriction to $\mathcal{O}^{k,n-k}$ is absolutely necessary. Then

$$\begin{aligned} \mathcal{U}_i\mathcal{U}_{i+1}\mathcal{U}_i &= T_iT^iT_{i+1}T^{i+1}T_iT^i \\ &= T_i\tau^i_{i+1}\tau^{i+1}_iT^i \\ &= T_iT^i \\ &= \mathcal{U}_i. \end{aligned}$$

□

This proposition gives a functor realization of the Temperley-Lieb algebra by projective functors in parabolic categories $\mathcal{O}^{k,n-k}$. Suitable products of generators U_i produce a basis of the Temperley-Lieb algebra $TL_{n,q}$, which admits a graphical interpretation: elements of this basis correspond to isotopy classes of systems of n simple disjoint arcs in the plane connecting n bottom and n top points. Moreover, this basis is related (see [FG]) to the Kazhdan-Lusztig basis in the Hecke algebra as well as to Lusztig's bases in tensor products of $U_q(\mathfrak{sl}_2)$-representations (see [FK]). Proposition 18 implies that to an element of this basis, we can associate a projective functor from $\mathcal{O}^{k,n-k}$ to $\mathcal{O}^{k,n-k}$, which is defined as a suitable composition of functors $\mathcal{U}_i, 1 \leq i \leq n-1$. We conjecture that these compositions of $\mathcal{U}_i$'s are indecomposable, and, in turn, an indecomposable projective functor from $\mathcal{O}^{k,n-k}$ to $\mathcal{O}^{k,n-k}$ is isomorphic to one of these compositions (compare with Theorem 4).

In [ES] Enright and Shelton, among other things, constructed an equivalence of categories $\mathcal{O}_1^{k,n-k}$ and $\mathcal{O}^{k-1,n-k-1}$ (see [ES], §11). This equivalence allows us to factorize $\mathcal{U}_i$ as a composition of a functor from $\mathcal{O}_{k,n-k}$ to $\mathcal{O}_{k-1,n-k-1}$ and a functor from $\mathcal{O}_{k-1,n-k-1}$ to $\mathcal{O}_{k,n-k}$. This is very much in line with the factorization of the element U_i of the Temperley-Lieb algebra as the composition $\cup_{i,n-2} \circ \cap_{i,n}$ of morphisms $\cup_{i,n-2}$ and $\cap_{i,n}$ of the Temperley-Lieb category.

We now offer the reader a conjecture on realizing the Temperley-Lieb category via functors between parabolic categories $\mathcal{O}^{k,n-k}$. Let

$$\zeta_n : \mathcal{O}_1^{k,n-k} \xrightarrow{\cong} \mathcal{O}^{k-1,n-k-1} \tag{60}$$

be the Enright-Shelton equivalence of categories. Introduce functors

$$\begin{aligned} \cap_{i,n} :& \mathcal{O}^{k,n-k} \longrightarrow \mathcal{O}^{k-1,n-k-1} \\ \cup_{i,n} :& \mathcal{O}^{k,n-k} \longrightarrow \mathcal{O}^{k+1,n+1-k} \end{aligned}$$

given by

$$\cap_{i,n} = \zeta_n \circ \tau_2^1 \circ \tau_3^2 \circ \cdots \circ \tau_{i-1}^{i-2} \circ \tau_i^{i-1} \circ T^i \tag{61}$$

$$\cup_{i,n} = T_i \circ \tau_{i-1}^i \circ \tau_{i-2}^{i-1} \cdots \circ \tau_2^3 \circ \tau_1^2 \circ \zeta_{n+2}^{-1}. \tag{62}$$

Conjecture 3. *There are natural isomorphisms* (23)–(29) (*with q set to -1 in* (29)) *of functors where $\cap_{i,n}$ and $\cup_{i,n}$ are defined by* (61) *and* (62)*.*

Equivalences (23) and (24) are immediate from [ES] and the results of this section. Relation (29) is implied by the fact that the composition of translation functors from and to a wall is equivalent to two copies of the identity functor. To prove the remaining four equivalences, a thorough understanding of the functor ζ_n, defined in [ES] in quite a tricky way, will be required.

To categorify the Temperley-Lieb algebra $TL_{n,q}$ for arbitrary q, rather than just $q=-1$, one needs to work with the mixed version of parabolic categories and projective functors. The conjecture of Irving [Ir] that projective functors admit a mixed structure can probably be approached via a recent work [BGi] of Beilinson and Ginzburg where wall-crossing functors are realized geometrically.

We next state the parabolic analogue of Conjecture 2. It is convenient to suppress parameter k in the definition of $\cap_{i,n}$ and $\cup_{i,n}$ by summing over k and passing to categories $\mathcal{O}^n = \oplus_k \mathcal{O}^{k,n-k}$. We switch to derived categories and extend functors $\cap_{i,n}, \cup_{i,n}$ and $\mathcal{U}_i$ to derived functors

$$\begin{aligned} \cap_{i,n} &: D^b(\mathcal{O}^n) \longrightarrow D^b(\mathcal{O}^{n-2}) \\ \cup_{i,n} &: D^b(\mathcal{O}^n) \longrightarrow D^b(\mathcal{O}^{n+2}) \\ \mathcal{U}_i &: D^b(\mathcal{O}^n) \longrightarrow D^b(\mathcal{O}^n). \end{aligned}$$

Recall elementary tangles (a)–(c) described at the end of Section 3.2.

To the tangle (a) associate functor $\cap_{i,n}$ given by the formula (61).

To the tangle (b) associate functor $\cup_{i,n}$ given by the formula (62).

To the tangle (c) associate functor $R_{i,n}$ from $D^b(\mathcal{O}^n)$ to $D^b(\mathcal{O}^n)$ which is the cone of the adjointness morphism of functors

$$\mathcal{U}_i \longrightarrow \mathrm{Id}\,.$$

Given a presentation α of a tangle t as a composition of elementary tangles of types (a)–(c), to α we associate the functor $g(\alpha)$ which is the corresponding composition of functors $\cap_{i,n}, \cup_{i,n}, R_{i,n}$.

Conjecture 4. *Given two presentations α, β of a tangle t as products of elementary tangles* (a)–(c)*, functors $g(\alpha)$ and $g(\beta)$ are isomorphic, up to shifts in the derived category.*

4.2. $\mathfrak{sl}_2$ and Zuckerman functors

The Grothendieck group of the category $\mathcal{O}^n = \oplus_{k=0}^n \mathcal{O}^{k,n-k}$ has rank 2^n. In the previous section we showed that the projective functors, restricted to $\mathcal{O}^n$, "categorify" the Temperley-Lieb algebra action on $V_1^{\otimes n}$. The Lie algebra $\mathfrak{sl}_2$ action on $V_1^{\otimes n}$ commutes with the Temperley-Lieb algebra action, while Zuckerman functors commute with projective functors. It is an obvious guess now that Zuckerman functors between different blocks of $\mathcal{O}^n$ provide a "categorification" of this $\mathfrak{sl}_2$ action. This fact is well-known and dates back to [BLM] and [GrL], where it is presented in a different language and in the more general case of $\mathfrak{sl}_k$ rather than $\mathfrak{sl}_2$. Beilinson, Lusztig and MacPherson in [BLM] count points over finite fields in certain correspondences between flag varieties. These correspondences define functors between derived categories of sheaves on these flag varieties, smooth along Schubert stratifications. Counting points is equivalent to computing the action of the corresponding functors on Grothendieck groups.

These derived categories are equivalent to derived categories of parabolic subcategories of a regular block of the highest weight category for $\mathfrak{gl}_n$. Pullback and pushforward functors are then isomorphic (up to shifts) to the embedding functor from smaller to bigger parabolic subcategories and its adjoint functors which are Zuckerman functors (see [BGS], Remark (2) on page 504). Combining this observation with the computation of [GrL] for the special case of the Grassmannian rather than an arbitrary partial flag variety, one can check that Zuckerman functors between various pieces of $\mathcal{O}^n$ categorify the action of $E^{(a)}$ and $F^{(a)}$ on $V_1^{\otimes n}$. This fact is stated more accurately below.

Fix $n \in \mathbb{N}$. Recall that we denoted by $\mathfrak{p}_k$ the parabolic subalgebra of $\mathfrak{gl}_n$ consisting of block upper-triangular matrices with the reductive part $\mathfrak{gl}_k \oplus \mathfrak{gl}_{n-k}$. Denote by $\mathfrak{p}_{k,l}$ the parabolic subalgebra of $\mathfrak{gl}_n$ which is the intersection of $\mathfrak{p}_k$ and $\mathfrak{p}_{k+l}$. The maximal reductive subalgebra of $\mathfrak{p}_{k,l}$ is isomorphic to $\mathfrak{gl}_k \oplus \mathfrak{gl}_l \oplus \mathfrak{gl}_{n-k-l}$.

Denote by $\mathcal{O}^{k,l,n-k-l}$ the complete subcategory of $\mathcal{O}_\mu$ consisting of $U(\mathfrak{p}_{k,l})$-locally finite modules. We have embeddings of categories

$$\mathcal{I}_{k,l} : \mathcal{O}^{k,n-k} \longrightarrow \mathcal{O}^{k,l,n-k-l} \tag{63}$$

$$\mathcal{J}_{k,l} : \mathcal{O}^{k+l,n-k-l} \longrightarrow \mathcal{O}^{k,l,n-k-l}. \tag{64}$$

Denote by $\mathcal{K}_{k,l}$ and $\mathcal{L}_{k,l}$ derived functors of the right adjoint functors of $\mathcal{I}_{k,l}$ and $\mathcal{J}_{k,l}$:

$$\mathcal{K}_{k,l} : D^b(\mathcal{O}^{k,l,n-k-l}) \longrightarrow D^b(\mathcal{O}^{k,n-k}) \tag{65}$$

$$\mathcal{L}_{k,l} : D^b(\mathcal{O}^{k,l,n-k-l} \longrightarrow D^b(\mathcal{O}^{k+l,n-k-l}). \tag{66}$$

These are derived functors of Zuckerman functors. Compositions $\mathcal{L}_{k,l} \circ \mathcal{I}_{k,l}$ and $\mathcal{K}_{k,l} \circ \mathcal{J}_{k,l}$ are exact functors between derived categories $D^b(\mathcal{O}^{k,n-k})$ and $D^b(\mathcal{O}^{k+l,n-k-l})$, and in the Grothendieck group of $\mathcal{O}^n = \oplus_{k=0}^n \mathcal{O}^{k,n-k}$ descend to the action of $E^{(l)}$ and $F^{(l)}$ on the module $V_1^{\otimes n}$.

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