Derived functor modules arising as large irreducible constituents of degenerate principal series

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Abstract

For the groups $G = \mathrm{Sp}(p,q)$, $\mathrm{SO}^*(2n)$, and $\mathrm{U}(m,n)$, we consider degenerate principal series whose infinitesimal character coincides with a finite-dimensional representation of G. We prove that each irreducible constituent of maximal Gelfand-Kirillov dimension is a derived functor module. We also show that at an appropriate "most singular" parameter, each irreducible constituent is weakly unipotent and unitarizable. Conversely we show that any weakly unipotent representation associated to a real form of the corresponding Richardson orbit is unique up to isomorphism and can be embedded into a degenerate principal series at the most singular integral parameter (apart from a handful of very even cases in type D). We also discuss edge-of-wedge-type embeddings of derived functor modules into degenerate principal series.
[1]

1 Introduction

In the representation theory of real reductive Lie groups, there are two fundamental constructions of representations: parabolic induction and cohomological induction. For the purposes of this introduction, we call a representation parabolically induced (respectively, cohomologically induced) from a one-dimensional representation a degenerate principal series (respectively, derived functor module). Each construction has a natural geometric interpretation and the geometry involved often suggests a relationship between the two kinds of representations. The situation can become quite complicated (in the case of a split group like $\mathrm{Sp}(2n,\mathbb{R})$, for instance) and it is difficult to extract simple, clean statements. However for groups whose Cartan subgroups are always connected, the situation turns out to be rather more simple. In this article, we study the relation between degenerate principal series and derived functor modules for such groups. Our first main theorem

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is as follows. (We call a weight G-integral if it appears as a weight of a finite-dimensional representation of G.)

Theorem A (Corollary 6.13, Theorem 7.3). *Let G be $\mathrm{Sp}(p,q)$, $\mathrm{SO}^*(2n)$, or $\mathrm{U}(m,n)$. Let P be a parabolic subgroup of G and let X be a representation of G parabolically induced from a one-dimensional representation of P. Assume that X has a G-integral infinitesimal character. Then any irreducible constituent V of X of maximal Gelfand-Kirillov dimension is a derived functor module in the weakly fair range (in the sense of [V4]). Moreover, the multiplicity of V in X is one.*

Among the degenerate principal series representations induced from P, those with the smallest possible integral infinitesimal character (among all degenerate principal series induced from P) are particularly interesting, and so we turn our attention to them. We call such representations *integrally weakly unipotent degenerate principal series*. This is a completely naive definition, but the terminology is potentially dangerous since the more sophisticated notion of weakly unipotent representations has already been defined in [V5] (Definition 5.1 below). According to the results below, the definitions are consonant.

For $G = \mathrm{U}(m,n)$, an integrally weakly unipotent degenerate principal series representation is unitarily induced and its precise structure is known from [Ma3] and [T1]. On the other hand, for $G = \mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$, the situation is a little more complicated. Such representations are not unitarily induced, for instance. Moreover an integrally weakly unipotent degenerate principal series representation for a given parabolic subgroup need not be unique. However, the distribution characters of two such representations coincide, and thus they have the same composition factors. We have the following result.

Theorem B (Theorem 6.10). *Let G be $\mathrm{Sp}(p,q)$ or $\mathrm{SO}^*(2n)$. Let P be a parabolic subgroup of G and let X be an integrally weakly unipotent degenerate principal series representation induced from P. Then each irreducible constituent of X is unitarizable. Moreover, if the Richardson orbit corresponding to P is not very even, then each irreducible constituent of X is weakly unipotent in the sense of [V5].*

Here "the Richardson orbit corresponding to P" means the complex nilpotent orbit induced from the zero orbit of the Levi factor of the complexification of the Lie algebra of P. In particular, the Richardson orbit corresponding to P is very even if and only if $G = \mathrm{SO}^*(4n)$ and the Levi factor of P is isomorphic to a direct product of general linear groups over the quaternionic field $\mathbb{H}$.

For $G = \mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$, we characterize the integrally weakly unipotent representations associated to a Richardson orbit corresponding to a parabolic subgroup as follows.

Theorem C (Theorem 6.12). *Retain the setting of Theorem B. Let $\mathcal{O}$ denote the complex Richardson orbit associated to P. Suppose X is a weakly unipotent representation attached to $\mathcal{O}$ (Definition 5.1). In addition suppose X has integral infinitesimal character. Then X is isomorphic to a derived functor module which arises as an irreducible constituent of an integrally weakly unipotent degenerate principal series representation induced from P. In particular, the associated variety of X is the closure of a unique $K_{\mathbb{C}}$ orbit $\mathcal{O}_K$ on $\mathcal{O} \cap \mathfrak{s}$ (where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ denotes the complexified Cartan decomposition).*

Conversely, for any $K_{\mathbb{C}}$ orbit $\mathcal{O}_K$ on $\mathcal{O} \cap \mathfrak{s}$, there exists a unique (up to isomorphism) weakly unipotent representation with an integral infinitesimal character whose associated *variety is the closure of* $\mathcal{O}_K$*.*

We now turn to the details of the interaction between the geometric interpretations of parabolic and cohomological induction. Let G be a real linear reductive Lie group and let $G_{\mathbb{C}}$ denote its complexification. Write $\mathfrak{g}_0$ (resp. $\mathfrak{g}$) for the Lie algebra of G (resp. $G_{\mathbb{C}}$) and let σ denote the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. We fix a maximal compact subgroup K of G and let θ denote the corresponding Cartan involution. We write $\mathfrak{k}$ for the complexified Lie algebra of K.

We fix a parabolic subgroup P of G with a θ-stable Levi part M and nilradical N. We let $\mathfrak{p}$, $\mathfrak{m}$, and $\mathfrak{n}$ denote the complexified Lie algebras of P, M, and N, respectively, and denote the corresponding analytic subgroups of $G_{\mathbb{C}}$ by $P_{\mathbb{C}}$, $M_{\mathbb{C}}$, and $N_{\mathbb{C}}$. For $Y \in \mathfrak{m}$, we define

$$\delta_P(Y) = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}_{\mathfrak{g}}(Y)|_{\mathfrak{n}})\,.$$

Then δ_P is a one-dimensional representation of $\mathfrak{m}$ and $2\delta_P$ lifts to a holomorphic group homomorphism $\xi_{2\delta} : M_{\mathbb{C}} \to \mathbb{C}^{\times}$. Defining $\xi_{2\delta_P}|_{N_{\mathbb{C}}}$ to be trivial, we may extend $\xi_{2\delta_P}$ to $P_{\mathbb{C}}$. Let $\mathcal{L}_\chi$ be the holomorphic line bundle on $G_{\mathbb{C}}/P_{\mathbb{C}}$ corresponding to a holomorphic character χ of $M_{\mathbb{C}}$. For a character $\eta : P \to \mathbb{C}^{\times}$, we consider the unnormalized parabolically induced representation ${}^u\mathrm{Ind}_P^G(\eta)$. Namely, ${}^u\mathrm{Ind}_P^G(\eta)$ is the K-finite part of the space of the C^∞-sections of the G-homogeneous line bundle on G/P associated to η. ${}^u\mathrm{Ind}_P^G(\eta)$ is a Harish-Chandra $(\mathfrak{g}, K)$-module.

If G/P is orientable, then the trivial representation of G is the unique irreducible quotient of ${}^u\mathrm{Ind}_P^G(\xi_{2\delta})$. If G/P is not orientable, there is a character ω on P such that ω is trivial on the identity component of P and the trivial representation of G is the unique irreducible quotient of ${}^u\mathrm{Ind}_P^G(\xi_{2\delta}\otimes\omega)$. This motivates the first of the following definitions.

Definition 1.1 (Definitions 4.2 and 4.6 below). (a) We call a holomorphic character χ of $M_{\mathbb{C}}$ good if ${}^u\mathrm{Ind}_P^G(\xi_{2\delta}\otimes\chi)$ has a finite-dimensional representation of G as a quotient. In particular, if G/P is orientable, the trivial character is good.

(b) Let $\mathcal{V}$ be an open G-orbit on $G_{\mathbb{C}}/P_{\mathbb{C}}$. We say $\mathcal{V}$ is fine if there is a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{q} \in \mathcal{V}$. For example, if G has a compact Cartan subgroup, any open G-orbit on $G_{\mathbb{C}}/P_{\mathbb{C}}$ is fine.

(c) For each fine open G-orbit $\mathcal{V}$ on $G_{\mathbb{C}}/P_{\mathbb{C}}$, we put

$$A_{\mathcal{V}}(\chi) = \mathrm{H}^{\dim \mathfrak{u}\cap\mathfrak{k}}(\mathcal{V}, \mathcal{L}_\chi \otimes \mathcal{L}_{2\delta_P})_{K\text{-finite}}.$$

From [Wo], $A_{\mathcal{V}}(\chi)$ is a derived functor module [VZ]. If χ is good, $A_{\mathcal{V}}(\chi)$ is in the good range in the sense of [V4]. We call an open G-orbit $\mathcal{V}$ in $G_{\mathbb{C}}/P_{\mathbb{C}}$ good if the Gelfand-Kirillov dimension of $A_{\mathcal{V}}(\mathbb{1})$ equals that of ${}^u\mathrm{Ind}_P^G(\xi_{2\delta})$.

Recall that if G is of Hermitian type and P is a Siegel parabolic subgroup of G, there are two open G-orbits in $G_{\mathbb{C}}/P_{\mathbb{C}}$ each isomorphic to G/K (the Siegel upper and lower half planes). For such orbits, $A_{\mathcal{V}}(\chi)$ are holomorphic or anti-holomorphic discrete series for a good character χ, and it is well-known that we can embed $A_{\mathcal{V}}(\chi)$ into ${}^u\mathrm{Ind}_P^G(\xi_{2\delta}\otimes\chi)$ by taking boundary values at the Shilov boundary.

Our purpose here is to consider other orbits. In this case, the corresponding embedding should be a higher cohomological analog of boundary value maps. We begin with an example.

Example 1.2. Let $G = SO_0(n,2)$ with $n \geq 3$ and let P be a parabolic subgroup whose Levi factor has semisimple part isomorphic to $SO_0(n-1,1)$. For simplicity set $\mathcal{L} = \mathcal{L}_{2\delta_P}$. In this case $X = G_{\mathbb{C}}/P_{\mathbb{C}}$ has three open G-orbits. Two of them (say $\mathcal{O}_+$ and $\mathcal{O}_-$) are Hermitian symmetric spaces (symmetric domains of type IV). The remaining one (say $\mathcal{O}_0$)is non-Stein and isomorphic to $G/(SO(2) \times SO_0(n-2,2))$. Let $\overline{\mathcal{O}}_+$ and $\overline{\mathcal{O}}_-$ be the closures of $\mathcal{O}_+$ and $\mathcal{O}_-$, respectively. In this case, we have $(X - \overline{\mathcal{O}}_+) \cap (X - \overline{\mathcal{O}}_-) = \mathcal{O}_0$ and $(X - \overline{\mathcal{O}}_+) \cup (X - \overline{\mathcal{O}}_-) = X - X_{\mathbb{R}}$ where $X_{\mathbb{R}} = G/P$. Hence we have the following Mayer-Vietoris exact sequence:

$$H^{2n-2}(X - \overline{\mathcal{O}}_+, \mathcal{L}) \oplus H^{2n-2}(X - \overline{\mathcal{O}}_-, \mathcal{L}) \to H^{2n-2}(\mathcal{O}_0, \mathcal{L}) \to H^{2n-1}(X - X_{\mathbb{R}}, \mathcal{L}).$$

We also have the following exact sequences:

$$H^{2n-2}(X, \mathcal{L}) \to H^{2n-2}(X - \overline{\mathcal{O}}_+, \mathcal{L}) \to H^{2n-1}_{\overline{\mathcal{O}}_+}(X, \mathcal{L}),$$
$$H^{2n-2}(X, \mathcal{L}) \to H^{2n-2}(X - \overline{\mathcal{O}}_-, \mathcal{L}) \to H^{2n-1}_{\overline{\mathcal{O}}_-}(X, \mathcal{L}).$$

From [Ko], we have $H^{2n-2}(X, \mathcal{L}) = 0$. We can regard $\overline{\mathcal{O}}_+$ and $\overline{\mathcal{O}}_-$ as closed convex set in an open cell of X. Hence, from the edge-of-wedge theorem (see [KaL] Théorème 1.1.2), we have $H^{2n-1}_{\overline{\mathcal{O}}_+}(X, \mathcal{L}) = H^{2n-1}_{\overline{\mathcal{O}}_-}(X, \mathcal{L}) = 0$. Hence, $H^{2n-2}(X - \overline{\mathcal{O}}_+, \mathcal{L}) = H^{2n-2}(X - \overline{\mathcal{O}}_-, \mathcal{L}) = 0$. We also have the following exact sequence:

$$0 = H^{2n-1}(X, \mathcal{L}) \to H^{2n-1}(X - X_{\mathbb{R}}, \mathcal{L}) \to H^{2n}_{X_{\mathbb{R}}}(x, \mathcal{L}).$$

Hence, we have

$$H^{2n-2}(\mathcal{O}_0, \mathcal{L}) \hookrightarrow H^{2n-1}(X - X_{\mathbb{R}}, \mathcal{L}) \hookrightarrow H^{2n}_{X_{\mathbb{R}}}(x, \mathcal{L}).$$

In this case $X_{\mathbb{R}}$ is orientable if and only if n is even. Hence, the local cohomology $H^{2n}_{X_{\mathbb{R}}}(x, \mathcal{L})$ is the space of hyperfunction sections of the degenerate principal series of G with respect to $\mathcal{L}$ (resp. $\mathcal{L} \otimes \omega$) if n is even (resp. odd). Taking the K-finite part, we have

$$A_{\mathcal{O}_0}(\mathbb{1}) \hookrightarrow {}^u\mathrm{Ind}_P^G(\xi_{2\delta}) \quad \text{(if } n \text{ is even,)}$$
$$\mathcal{A}_{\mathcal{O}_0}(\mathbb{1}) \hookrightarrow {}^u\mathrm{Ind}_P^G(\xi_{2\delta} \otimes \omega) \quad \text{(if } n \text{ is odd.)}$$

□

Such a relatively easy construction of embeddings as in Example 1.2 seems to be difficult to imitate in the general case. However, [Ma1], [SaSt], [Gi], etc. suggests some evidences of the existence of the edge-of-wedge embeddings in greater generality. Therefore, we consider the following problem in the setting of Harish-Chandra modules.

Problem D. Let G be a real linear reductive Lie group and let P be a parabolic subgroup. In the terminology of Definition 1.1, suppose $\mathcal{O}$ is a good open G-orbit in $X = G_{\mathbb{C}}/P_{\mathbb{C}}$ and let χ be a good character of $M_{\mathbb{C}}$. Does there exist a character ω of P which is trivial on the identity component of P such that $\mathcal{A}_{\mathcal{O}}(\chi) \hookrightarrow {}^u\mathrm{Ind}_P^G(\xi_{2\delta} \otimes \chi\omega)$?

If the nilradical of P is commutative, the answer is known by [Sa1], [SaSt], [Sa2], and [Zh]. For instance, suppose G is the rank n symplectic group $\mathrm{Sp}(n, \mathbb{R})$ and P is the Siegel parabolic subgroup. If n is even, then the cited references give an affirmative answer to the question posed in Problem D. However, if n is odd, Problem D admits a negative answer, except of course for the holomorphic and antiholomorphic discrete series as mentioned above.

For quaternionic discrete series, an affirmative answer to Problem D is established in [W].

Here is the affirmative answer to Problem D for an arbitrary parabolic subgroup of $\mathrm{U}(m,n)$.

Theorem E (Theorem 7.5). *Let $G = \mathrm{U}(m,n)$ and let P be a parabolic subgroup. Recall the terminology of Definition 1.1. Let χ be a good holomorphic character of $M_{\mathbb{C}}$ and let $\mathcal{G}$ denote the set of the good orbits in $X_P = G_{\mathbb{C}}/P_{\mathbb{C}}$. Then we have the following description of the socle*

$$\mathrm{Socle}({}^u\mathrm{Ind}_P^G(\xi_{2\delta_P} \otimes \chi)) = \bigoplus_{\mathcal{V}\in\mathcal{G}} A_{\mathcal{V}}(\chi).$$

For $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$, the situation is similar.

Theorem F (Theorem 6.18, Theorem 6.19). *Let $G = \mathrm{Sp}(p,q)$ (respectively $\mathrm{SO}^*(2n)$) and let $P = MN$ be a parabolic subgroup. Suppose M has the following form*

$$M = \mathrm{GL}(k_1,\mathbb{H}) \times \cdots \times \mathrm{GL}(k_s,\mathbb{H}) \times \mathrm{Sp}(p',q')$$

(respectively,

$$M = \mathrm{GL}(k_1,\mathbb{H}) \times \cdots \times \mathrm{GL}(k_s,\mathbb{H}) \times \mathrm{SO}^*(2n')).$$

For each positive integer ℓ, let $m_M(\ell)$ denote the number of i such that $k_i = \ell$. Furthermore assume that for each $\ell > p' + q'$ (respectively $\ell \leqslant p' + q'$), $m_M(\ell)$ is even. Let χ be a good holomorphic character of $M_{\mathbb{C}}$. Again let $\mathcal{G}$ denote the set of the good orbits in $X_P = G_{\mathbb{C}}/P_{\mathbb{C}}$. Then,

$$\mathrm{Socle}({}^u\mathrm{Ind}_P^G(\xi_{2\delta_P} \otimes \chi)) = \bigoplus_{\mathcal{V}\in\mathcal{G}} A_{\mathcal{V}}(\chi).$$

We also consider the analogous problem for $\mathrm{GL}(n,\mathbb{H})$ and for complex semisimple groups in Sections 8 and 9.

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2 Background and notation

2.1 General notation

As usual we denote the quaternionic field, the complex number field, the real number field, the rational number field, the ring of integers, and the set of non-negative integers by $\mathbb{H}$, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. For a ring A and a left A-module M, we let $\mathrm{Ann}_A(M)$ denote the annihilator of M in A.

Let H be a real linear Lie group and let $H_{\mathbb{C}}$ denote its complexification. We write $\mathfrak{h}_0$ for its Lie algebra and $\mathfrak{h}$ for its complexification. Given a subalgebra $\mathfrak{h}_0'$ of $\mathfrak{h}_0$, we let H' denote the corresponding analytic subgroup of H, and adopt analogous conventions for subalgebras of $\mathfrak{h}$ and analytic subgroups of $H_{\mathbb{C}}$. We let $\mathrm{U}(\mathfrak{h})$ denote the universal enveloping algebra of $\mathfrak{h}$.

We let G denote a real reductive linear Lie group with a Cartan involution θ. We write K for the maximal compact subgroup of G corresponding to θ. The corresponding complexified Cartan decomposition is denoted $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. $K_{\mathbb{C}}$ acts via Ad on $\mathfrak{s}$.

For a complex reductive Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$, we let $\Delta(\mathfrak{g}, \mathfrak{h})$ denote the root system for $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, we let $\alpha^\vee$ denote the corresponding coroot in $\mathfrak{h}^*$.

We let $\mathcal{P}$ denote the set of integral weights,

$$\mathcal{P} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \text{for all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})\}.$$

For a connected reductive complex linear group $G_{\mathbb{C}}$ whose Lie algebra is $\mathfrak{g}$, we set

$$\mathcal{P}_{G_{\mathbb{C}}} = \{\lambda \in \mathfrak{h}^* | \lambda \text{ appears as a weight of some finite dimensional representation of } G_{\mathbb{C}}.\}.$$

2.2 Nilpotent orbits

Let G be a linear reductive Lie group. We frequently use an invariant form to identify $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ (or $\mathfrak{g}$ and $\mathfrak{g}^*$) without comment. We let $\mathcal{N}_0$ denote the nilpotent cone in $\mathfrak{g}_0$, and write $\mathcal{N}$ for the nilpotent cone in $\mathfrak{g}$. There are only a finite number of $G_{\mathbb{C}}$ orbits on $\mathcal{N}$ and a finite number of G orbits on $\mathcal{N}_0$. Let $\mathcal{N}(\mathfrak{s}) = \mathcal{N} \cap \mathfrak{s}$. The action of $K_{\mathbb{C}}$ on $\mathfrak{s}$ preserves $\mathcal{N}(\mathfrak{s})$, and there are only a finite number of $K_{\mathbb{C}}$ orbits on $\mathcal{N}(\mathfrak{s})$.

Fix an orbit $\mathcal{O}$ of $G_{\mathbb{C}}$ on $\mathcal{N}$. A K-form of $\mathcal{O}$ is defined to be a $K_{\mathbb{C}}$ orbit on $\mathcal{O} \cap \mathfrak{s}$. We denote the set of K forms of $\mathcal{O}$ by $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})$. If K is connected, this notation causes no confusion since it is indeed the case that the irreducible component of $\mathcal{O} \cap \mathfrak{s}$ are precisely the $K_{\mathbb{C}}$ orbits on $\mathcal{O} \cap \mathfrak{s}$. But if K is disconnected, a typical $K_{\mathbb{C}}$ orbit is a union of such irreducible components. According to [V7], each K form of $\mathcal{O}$ is a Lagrangian subvariety of $\mathcal{O}$ and hence $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})$ is equidimensional.

On the other hand, a real form of $\mathcal{O}$ is defined to be an orbit of G on $\mathcal{O} \cap \mathfrak{g}$. We denote this set by $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{g}_0)$ and a similar caveat applies to this notation.

The Kostant-Sekiguchi correspondence provides a bijection between the set of $K_{\mathbb{C}}$ orbits on $\mathcal{N}(\mathfrak{s})$ and the set of G orbits on $\mathcal{N}_0$ (e.g. [CMc, Chapter 9]). Given a $K_{\mathbb{C}}$ orbit $\mathcal{O}_K$, we write $\mathbf{KS}(\mathcal{O}_K)$ for the corresponding real orbit. In fact, for a fixed complex orbit $\mathcal{O}$, the correspondence restricts to a bijection of $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{g}_0)$ and $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})$, the real and K forms of $\mathcal{O}$.

2.3 Primitive ideals

Let $\mathfrak{g}$ denote a complex reductive Lie algebra. Let $\mathfrak{h}$ denote a Cartan subalgebra in $\mathfrak{g}$ and write W for the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. Each infinitesimal character $\lambda \in \mathfrak{h}^*/W$ parameterizes a maximal ideal Z_λ in the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$.

Recall that a two-sided ideal in $\mathrm{U}(\mathfrak{g})$ is called primitive if it is the annihilator of a simple $\mathrm{U}(\mathfrak{g})$ module. If we write $\mathrm{Prim}(\mathfrak{g})$ for the set of primitive ideals in $\mathrm{U}(\mathfrak{g})$ then we have

$$\mathrm{Prim}(\mathfrak{g}) = \coprod_{\lambda \in \mathfrak{h}^*/W} \mathrm{Prim}_\lambda(\mathfrak{g}),$$

where $\mathrm{Prim}_\lambda(\mathfrak{g})$ consists of the subset of primitive ideals containing Z_λ. If $I \in \mathrm{Prim}_\lambda(\mathfrak{g})$, then I is said to have infinitesimal character λ. For each λ there is a unique maximal ideal $\mathrm{J}^{\max}(\lambda) \in \mathrm{Prim}_\lambda(\mathfrak{g})$.

Given a primitive ideal I in $\mathrm{U}(\mathfrak{g})$, one may consider its associated variety $\mathrm{AV}(I) \subset \mathfrak{g}^*$. $\mathrm{AV}(I)$ is defined to be the closed points of the support of the $\mathrm{S}(\mathfrak{g}) = \mathrm{grU}(\mathfrak{g})$ module $\mathrm{gr}I$ where the gradings are provided by the degree filtration on $\mathrm{U}(\mathfrak{g})$. In fact $\mathrm{AV}(I)$ is the closure of a single nilpotent orbit in $\mathfrak{g}^*$ ([Jo3]).

2.4 Associated varieties, asymptotic supports, and the Barbasch-Vogan conjecture.

Let $\mathfrak{g}$ be a complex reductive Lie algebra and let X be a finitely generated $U(\mathfrak{g})$-module and let (X^j) be a good filtration of X. The corresponding graded object $\mathrm{gr}X$ is a finitely generated $\mathrm{grU}(\mathfrak{g})$-, hence $\mathrm{S}(\mathfrak{g})$-, module. The associated variety $\mathrm{AV}(X)$ of X is defined to be the support of $\mathrm{gr}X$; see [V7].

Hereafter, we assume X is a Harish-Chandra $(\mathfrak{g}, K)$-module. It is not difficult to show that $\mathrm{AV}(X)$ is a union of closures of $K_\mathbb{C}$ orbits on $\mathcal{N}(\mathfrak{s})$, and we may write

$$\mathrm{AV}(X) = \overline{\mathcal{O}_K^1} \cup \cdots \cup \overline{\mathcal{O}_K^j}$$

with each $\mathcal{O}_K^j$ a $K_\mathbb{C}$ orbit on $\mathcal{N}(\mathfrak{s})$. By keeping track of the rank of $\mathrm{gr}X$ along each irreducible component $\mathcal{O}_K^j$, we obtain an integral linear combination

$$\mathcal{AV}(X) = \sum_{\mathcal{O}_K} n_{\mathcal{O}_K}[\mathcal{O}_K],$$

where the sum is over $K_\mathbb{C}$ orbits on $\mathcal{N}(\mathfrak{s})$ and each n is an integer. $\mathcal{AV}(X)$ is called the associated cycle of X. The associated cycle construction is additive on short exact sequences, and hence descends to the Grothendieck group of virtual Harish-Chandra modules. If X is annihilated by a primitive ideal I — for instance, if X is irreducible — then (in the terminology of Section 2.2) each of the orbits $\mathcal{O}_K^j$ appearing in $\mathrm{AV}(X)$ is in fact a K form of the complex orbit $\mathcal{O}$ which is dense in the associated variety of the annihilator of X ([V7]).

Nilpotent orbits also enter the representation theory of X as follows. According to [BaV1], the distribution character of X has an asymptotic expansion whose leading term is a linear combination of Fourier transforms of canonical measures of orbits of G on $\mathcal{N}_0$. This linear combination is called the asymptotic cycle of X and is denoted $\mathcal{AS}(X)$.

Recall the Kostant-Sekiguchi correspondence **KS** of Section 2.1. The Barbasch-Vogan conjecture (now a theorem due to Schmid and Vilonen ([SV])) asserts that

$$\mathcal{AV}(X) = \sum n_{\mathcal{O}_K}[\mathcal{O}_K]$$

if and only if

$$\mathcal{AS}(X) = \sum n_{\mathcal{O}_K}[\mathbf{KS}(\mathcal{O}_K)].$$

For a finitely generated $U(\mathfrak{g})$-module X, we let $\mathrm{Dim}(X)$ denote the Gelfand-Kirillov dimension of X (cf. [V1],[V7]). It is well-known that $\mathrm{Dim}(X) = \dim \mathrm{AV}(X)$.

For a Harish-Chandra $(\mathfrak{g}, K)$-module V, we let $[V]$ denote the distribution character of V. Let V and M be Harish-Chandra $(\mathfrak{g}, K)$-modules. We write $V \approx M$, if a virtual character $[V] - [M]$ is a linear combination of the distribution characters of irreducible Harish-Chandra modules whose Gelfand-Kirillov dimensions are strictly smaller than $\mathrm{Dim}(V)$.

2.5 Induced representations

Let G be a real reductive Lie group. For a parabolic subgroup P of G, we define (unnormalized and normalized) induction as follows. Write $P = MN$ for the Levi decomposition of P (and assume that M is θ-stable). Let Z be a $(\mathfrak{l}, M \cap K)$-module and (π, H)a Hilbert space globalization of Z. Define ${}^u\mathrm{Ind}_P^G(Z)$ to be the K-finite part of

$$\{f \in C^\infty(G) \otimes H \mid f(g\ell n) = \pi(\ell^{-1})f(g) \quad (g \in G, \ell \in M, n \in N)\}.$$

We regard ${}^u\mathrm{Ind}_P^G(Z)$ as a Harish-Chandra $(\mathfrak{g}, K)$-module as usual. In addition, we also consider the normalized induction:

$${}^n\mathrm{Ind}_P^G(Z) = {}^u\mathrm{Ind}_P^G(Z \otimes \mathbb{C}_{\delta_P})$$

where $(\delta_P, \mathbb{C}_{\delta_P})$ is a one-dimensional representation of P defined as follows. Let $M = {}^oMA$ be the Langlands decomposition of M such that ${}^oM$ and A are θ-stable. We let $\mathfrak{a}_0$ denote the Lie algebra of A and $\log : A \to \mathfrak{a}_0$ the inverse of the exponential map. We put

$$\delta_P(man) = e^{\frac{1}{2}\mathrm{tr}(\mathrm{ad}(log a)|_{\mathfrak{n}})} \quad (m \in {}^oM, a \in A, n \in N).$$

2.6 Harish-Chandra cells

For Harish-Chandra $(\mathfrak{g}, K)$-modules X and Y, we write $X \lesssim Y$ if there exists a finite-dimensional irreducible G-representation E such that Y is isomorphic to a subquotient of $X \otimes E$. We write $X \sim Y$ if the both $X \lesssim Y$ and $Y \lesssim X$ hold. It is not hard to show that $X \lesssim Y$ implies that $\mathrm{AV}(Y) \subseteq \mathrm{AV}(X)$, and so $X \sim Y$ implies that $\mathrm{AV}(Y) = \mathrm{AV}(X)$.

We fix a positive system of $\Delta(\mathfrak{g}, \mathfrak{h})$ and let ρ denote the half-sum of all the positive roots; so ρ is the infinitesimal character of the trivial representation. We let $\mathcal{E}$ denote the space of invariant eigendistribution on G of the infinitesimal character ρ. $\mathcal{E}$ has a basis

$$\mathbb{B} = \{[V] \mid V \text{ is an irreducible Harish-Chandra module with infinitesimal character } \rho.\}.$$

We let W denote the Weyl group for $(\mathfrak{g}, \mathfrak{h})$. W acts on $\mathcal{E}$ via the coherent continuation representation ([Zu]).

For each $[V] \in \mathbb{B}$, we write $\mathrm{Cone}(V)$ for the $\mathbb{C}$-subspace of $\mathcal{E}$ spanned by $\{[X] \in \mathbb{B} \mid V \lesssim X\}$. Then $\mathrm{Cone}(V)$ is a W-submodule of $\mathcal{E}$. Moreover, if $V \lesssim X$ then $\mathrm{Cone}(X) \subseteq \mathrm{Cone}(V)$. Set

$$\mathrm{Cell}(V) = \mathrm{Cone}(V) / \sum_{X \in \mathbb{B},\, V \lnsim X} \mathrm{Cone}(X).$$

If we write $[Y]$ again for the image of $[Y] \in \mathrm{Cone}(V)$ in $\mathrm{Cell}(V)$, $\mathrm{Cell}(V)$ has a basis $\{[Y] \in \mathbb{B} \mid V \sim Y\}$. A Harish-Chandra cell is a subquotient of $\mathcal{E}$ which is of the form $\mathrm{Cell}(V)$ for some $[V] \in \mathbb{B}$. Let V be any irreducible Harish-Chandra $(\mathfrak{g}, K)$-module whose infinitesimal character is in $\mathcal{P}_{G_{\mathbb{C}}}$. A standard application of the translation principle implies that there exists some $[X] \in \mathbb{B}$ such that $V \sim X$. We write $\mathrm{Cell}(V)$ for $\mathrm{Cell}(X)$.

3 Associated cycles of degenerate principal series

We begin by recalling how to compute the asymptotic cycle of a parabolically induced representation. A much more general statement is contained in [Ba, Corollary 5.0.10]. We extract only what we need for applications.

Theorem 3.1 (Barbasch). *Suppose G is a real reductive linear group with parabolic subgroup $P = MN$. Let χ denote a character of M, and consider the parabolically induced representation $\mathrm{Ind}_P^G(\chi)$.*

1. *Let $\mathcal{O}_0$ denote a real nilpotent orbit for $\mathfrak{g}_0$. Then $\overline{\mathcal{O}_0}$ appears in the asymptotic support of $\mathrm{Ind}_P^G(\chi)$ if and only if $\mathcal{O}_0 \cap \mathfrak{n}_0$ is non-empty and open in $\mathfrak{n}_0$.*
2. *Let $P_{\mathbb{C}}$ denote the complexification of P in $G_{\mathbb{C}}$. Suppose that the moment map from the cotangent bundle $T^*(G_{\mathbb{C}}/P_{\mathbb{C}})$ is birational onto its image. Then if $\mathcal{O}_0$ appears in the asymptotic support of $\mathrm{Ind}_P^G(\chi)$, its multiplicity is exactly one.*

We immediately obtain a multiplicity one result.

Corollary 3.2. *Retain the setting of Theorem 3.1(2); in particular assume the indicated moment map is birational. If X is an irreducible constituent of $\mathrm{Ind}_P^G(\chi)$ of maximal Gelfand-Kirillov dimension, then X appears with multiplicity one. Moreover, if X and Y are irreducible constituents of $\mathrm{Ind}_P^G(\chi)$ of maximal Gelfand-Kirillov dimension such that $\mathrm{AV}(X) = \mathrm{AV}(Y)$, then we have $X = Y$.*

For certain groups, the above results give a complete description of associated cycles of degenerate principal series. The main result of this section is as follows.

Proposition 3.3. *Recall the notation of Section 2.2.*

1. *Let G be $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$, or $\mathrm{SO}^*(2n)$. Let P be a parabolic subgroup of G and χ be a one-dimensional representation of P. Then*

$$\mathcal{AV}(\mathrm{Ind}_P^G(\chi)) = \sum_{\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})} [\overline{\mathcal{O}_K}]. \tag{3.1}$$

and

$$\mathcal{AS}(\mathrm{Ind}_P^G(\chi)) = \sum_{\mathcal{O}_{\mathbb{R}} \in \mathrm{Irr}(\mathcal{O}\cap\mathfrak{g}_0)} [\overline{\mathcal{O}_{\mathbb{R}}}].$$

2. *Let G be* $\mathrm{GL}(n,\mathbb{R})$ *or* $\mathrm{GL}(n,\mathbb{H})$*. Let P be a parabolic subgroup of G and χ be a one-dimensional representation of P. Then $\mathcal{O}\cap\mathfrak{s}$ consists of a single $K_{\mathbb{C}}$ orbit $\mathcal{O}_K^1$ and likewise $\mathcal{O}\cap\mathfrak{g}_0$ consists of a simple G orbit $\mathcal{O}_{\mathbb{R}}^1$, and*

$$\mathcal{AV}(\mathrm{Ind}_P^G(\chi)) = [\overline{\mathcal{O}_K^1}] \tag{3.2}$$

and

$$\mathcal{AS}(\mathrm{Ind}_P^G(\chi)) = [\overline{\mathcal{O}_{\mathbb{R}}^1}]$$

Proof. The equivalence of the associated cycle and asymptotic cycle statements follows from the Barbasch-Vogan Conjecture (Section 2.4). The statement in (2) asserting that $\mathcal{O}\cap\mathfrak{s}$ and $\mathcal{O}\cap\mathfrak{g}_0$ are single orbits follows essentially from the existence of Jordan canonical form (see also [CMc, Chapter 9]). Finally using [He], it is easy to verify that the birationality hypothesis in Theorem 3.1(2) is satisfied for each of the degenerate principal series considered in the proposition. Thus if an orbit $\mathcal{O}_K$ contributes to the associated cycle in(3.1) and (3.2), it necessarily appears with multiplicity one.

Combining the above statements gives a proof of part (2) of the proposition. To finish the proof of part (1), it only remains to prove that each orbit indicated in (3.1) actually appears.

We suppose $G = \mathrm{U}(p,q)$. Let P be a parabolic subgroup of G. Since $\mathcal{AV}(\mathrm{Ind}_P^G(\chi))$ does not depend on χ, we assume that χ is a one-dimensional unitary representation of P such that ${}^n\mathrm{Ind}_P^G(\chi)$ has an integral infinitesimal character. Then, ${}^n\mathrm{Ind}_P^G(\chi)$ is decomposed into a direct sum of derived functor modules ([Ma5, Theorem 3.3.1]) explicitly. We may calculate the associated variety of each irreducible constituent of ${}^n\mathrm{Ind}_P^G(\chi)$ via an algorithm described in [T1]. It is straightforward to check the desired conclusion in this case.

For $G = \mathrm{Sp}(m,n)$ or $\mathrm{SO}^*(2n)$, using Theorem 3.1(1), we are reduced to the following computation.

Lemma 3.4. *Let G be* $\mathrm{Sp}(m,n)$*, or* $\mathrm{SO}^*(2n)$*. Let $P = MN$ be a parabolic subgroups for the real group G. Write $\mathcal{O}$ for the Richardson orbit with respect to the complexified Lie algebra $\mathfrak{p}$ of P. Let $\mathcal{O}_0$ be a real form of $\mathcal{O}$ with respect to G (Section 2.2). Then, $\mathcal{O}_0\cap\mathfrak{n}_0$ is non-empty and open in $\mathfrak{n}_0$.*

To prove Lemma 3.4, we first develop a general result (Lemma 3.7). So let G be an arbitrary real reductive Lie group, and let $X \in \mathfrak{g}_0$ be a nilpotent element. Extend X to a $\mathfrak{sl}_2$-triple (X,H,Y). Let $\mathfrak{p}_0(X)$ denote the parabolic subalgebra consisting of the nonnegative eigenvalues of $\mathrm{ad}(H)$ on $\mathfrak{g}_0$. Recall that X is even if the eigenvalues of $\mathrm{ad}(H)$ are all even.

Definition 3.5. A parabolic subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}_0$ is called *even* if there exists an even nilpotent element X such that $\mathfrak{p}_0 = \mathfrak{p}_0(X)$. A parabolic subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}_0$ is called *quasieven* if there exists an even parabolic subalgebra $\mathfrak{p}_0'$ such that $\mathfrak{p}_0$ and $\mathfrak{p}_0'$ have a common Levi part. Finally, a parabolic subgroup P of G is called *quasieven* if its Lie algebra is quasieven.

Example 3.6. Consider the parabolic subgroups appearing in Definition 6.6.

1. The parabolic subgroup for $\mathrm{Sp}(p,q)$ is quasieven if $n_i \le p' + q'$ for all i.

2. The parabolic subgroup for $\mathrm{SO}^*(2n)$ is quasieven if $n_i \le n'$ for all i or $n' = 0$.

Lemma 3.7. *Let $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{n}_0$ be any quasieven parabolic subalgebra of $\mathfrak{g}_0$. Let $\mathcal{O}$ denote the complex Richardson orbit induced from $\mathfrak{l}$. Then each real form of $\mathcal{O}$ intersects $\mathfrak{n}_0$ in a dense set.*

Proof. If $\mathfrak{p}_0$ is an even parabolic subalgebra, then the conclusion of the lemma follows immediately from the Dynkin-Kostant theory. (Here it is important to remark that the Jacobson-Morozov Theorem holds for real reductive Lie algebras and real $\mathfrak{sl}(2)$ triples.) Let $\mathfrak{p}_0$ be any quasieven parabolic subalgebra. Let $\mathfrak{p}_0'$ be an even parabolic subalgebra such that $\mathfrak{p}_0$ and $\mathfrak{p}_0'$ have a common Levi part. We write $\mathfrak{n}_0$ (respectively $\mathfrak{n}_0'$) for the nilradical of $\mathfrak{p}_0$ (respectively $\mathfrak{p}_0'$). Arguing as in [CMc, Theorem 7.1.3], it follows that $\mathrm{Ad}(G)\mathfrak{n}_0 = \mathrm{Ad}(G)\mathfrak{n}_0'$. The lemma follows. □

Example 3.8. For $\mathfrak{g}_0 = \mathfrak{sp}(2,\mathbb{R})$ (rank 2), there are two maximal parabolic subalgebras (up to the conjugation), namely the Siegel and Jacobi parabolics. The Siegel parabolic subalgebra is even; the Jacobi is not even quasieven. In this case, the corresponding Richardson orbit is subregular nilpotent orbit and there are three real forms. The nilradical of the Jacobi parabolic subalgebra intersects only one of them. Thus the conclusion of the lemma fails for the Jacobi parabolic subalgebra indicating the necessity of the quasieven hypothesis in general.

Lemma 3.7 together with Example 3.6 provides a proof of Lemma 3.4 many, but not all, cases for $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$. For the remaining cases, we must supply an additional argument.

Suppose first that $G = \mathrm{Sp}(p,q)$. Let $P = MN$ with

$$M = \mathrm{Sp}(p',q') \times \mathrm{GL}(n_1,\mathbb{H}) \times \cdots \times \mathrm{GL}(n_k,\mathbb{H}).$$

Set $n' = p' + q'$ and let

$$n_i^0 = \min(n_i, n')$$
$$n_i^1 = n_i - n_i^1.$$

of M. Set $p^0 = p' + \sum_i n_i^0$, $q^0 = q' + \sum_i n_i^0$, and $p^1 = q^1 = \sum_i n_i^1$. Set $G^j = \mathrm{Sp}(p^j, q^j)$ and consider the parabolic subgroups $P^j = M^j N^j$ of G^j with

$$M^0 = \mathrm{Sp}(p',q') \times \mathrm{GL}(n_1^0,\mathbb{H}) \times \cdots \times \mathrm{GL}(n_k^0,\mathbb{H})$$
$$M^1 = \mathrm{GL}(n_1^1,\mathbb{H}) \times \cdots \times \mathrm{GL}(n_k^1,\mathbb{H}).$$

Let $\mathcal{O}^j$ denote the orbit for $\mathfrak{g}^j$ induced from there zero orbit of $\mathfrak{m}^j$. According to Example 3.6(2), P^0 is quasieven, and hence every real form of $\mathcal{O}^0$ meets $\mathfrak{n}^0$. The key observation from the classification of real forms ([CMc, Chapter 9]) is *the the number of real forms of*

$\mathcal{O}$ *equals the number of real forms of* $\mathcal{O}^0$. So we can deduce the statement of Lemma 3.4 for $\mathcal{O}$ from the corresponding one for $\mathcal{O}^0$. We offer a few more details.

Notice $P^0 \times P^1$ embeds in P in a natural way such that

$$\mathfrak{n}_0^0 \oplus \mathfrak{n}_0^1 \subset \mathfrak{n}_0. \tag{3.3}$$

As we remarked, P^0 is quasieven, so we can find representative $N_1, \ldots, N_r \in \mathfrak{n}_0^0$ for each of the r real forms of $\mathcal{O}^0$. On the other hand, from the classification of real forms, $\mathcal{O}^1$ has a single real form. We may find a representative of it, say N, in $\mathfrak{n}_0^1$. With the decomposition of Equation (3.3) in mind, consider the r elements

$$(N_i, N) \in \mathfrak{n}_0.$$

From the classification of real forms, one may deduce that since the N_i are not conjugate by G^1, none of the r elements (N_i, N) are conjugate by G. The italicized remark of the previous paragraph thus implies that they are a complete set of representative of real forms of $\mathcal{O}$. Lemma 3.4 follows for $\mathrm{Sp}(p,q)$. A very similar argument works for $\mathrm{SO}^*(2n)$. We omit the details.

This completes the proof of Lemma 3.4 and hence Proposition 3.3. □

To conclude, we introduce some notions in the general setting.

Definition 3.9. Let G be an arbitrary real reductive Lie group. An irreducible Harish-Chandra $(\mathfrak{g}, K)$-module V is called quasi P-cofinite if there exists some irreducible finite-dimensional representation F of M such that V is isomorphic to a subquotient of ${}^n\mathrm{Ind}_P^G(F)$.

The following lemma is easily obtained from the fact that parabolic induction commutes with tensoring with a finite-dimensional (the "Mackey isomorphism") together with the exactness of parabolic induction.

Lemma 3.10. *Let V be a quasi P-cofinite irreducible Harish-Chandra $(\mathfrak{g}, K)$-module and let X be an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module such that $V \lesssim X$ (with notation as in Section 2.6). Then X is quasi P-cofinite.*

4 Correspondence of orbits

Let G be a real reductive linear Lie group and retain the notation of Section 2.1. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ and let Q be the corresponding parabolic subgroup of $G_{\mathbb{C}}$. The generalized flag manifold $G_{\mathbb{C}}/Q$ is identified naturally with the set of the parabolic subalgebras of $\mathfrak{g}$ which are $G_{\mathbb{C}}$-conjugate to $\mathfrak{q}$, and we often regard a point of $G_{\mathbb{C}}/Q$ as a parabolic subalgebra.

The following proposition is a special case of the Matsuki duality theorem.

Proposition 4.1 ([M2]). *For each open G-orbit $\mathcal{V}$ in $G_{\mathbb{C}}/Q$, there is a unique closed $K_{\mathbb{C}}$-orbit $\mathcal{C}$ such that $\mathcal{C} \subseteq \mathcal{V}$. This correspondence gives a bijection of the set of the open G-orbits in $G_{\mathbb{C}}/Q$ onto the set of the closed $K_{\mathbb{C}}$-orbits.*

In order to associate a derived functor module to an open G-orbit (or a closed $K_{\mathbb{C}}$-orbit), the following property of the orbit is required.

Definition 4.2. A closed $K_{\mathbb{C}}$-orbit $\mathcal{C}$ in $G_{\mathbb{C}}/Q$ is called fine if there is a θ-stable parabolic subalgebra contained in $\mathcal{C}$. An open G-orbit $\mathcal{V}$ in $G_{\mathbb{C}}/Q$ is called fine if the corresponding $K_{\mathbb{C}}$-orbit is fine. We say that a parabolic subgroup Q of $G_{\mathbb{C}}$ is fine with respective to G if all the open G-orbits in $G_{\mathbb{C}}/Q$ are fine.

Since the $K_{\mathbb{C}}$ action commutes with θ, all points of a fine closed $K_{\mathbb{C}}$-orbit are θ-stable parabolic subalgebras. Also notice that if $\mathfrak{q}$ is a Borel subalgebra, then all open G-orbits are fine ([M1]); in other words, Q is fine. Our aim here is to discuss other sufficient conditions ensuring that a particular open G-orbit is fine. We begin with the following easy fact.

Lemma 4.3. *Let $\mathcal{C}$ be a closed $K_{\mathbb{C}}$-orbit in $G_{\mathbb{C}}/Q$ and let $\mathfrak{q}'$ be a parabolic subalgebra contained in $\mathcal{C}$. Then there exists a θ-stable Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{b} \subseteq \mathfrak{q}'$.*

Proof. Let B be a Borel subgroup of $G_{\mathbb{C}}$ such that $B \subseteq Q$. We consider the natural projection $p : G_{\mathbb{C}}/B \to G_{\mathbb{C}}/Q$. Since $p^{-1}(\mathcal{C})$ is a closed $K_{\mathbb{C}}$-stable subset of $G_{\mathbb{C}}/B$, there is a closed $K_{\mathbb{C}}$-orbit $\mathcal{C}'$ such that $\mathcal{C}' \subseteq p^{-1}(\mathcal{C})$. Since p is $K_{\mathbb{C}}$-equivariant, there is a Borel subalgebra $\mathfrak{b}'$ such that $\mathfrak{b}' \in p^{-1}(\mathfrak{q}') \cap \mathcal{C}'$. But $\mathfrak{b}' \in p^{-1}(\mathfrak{q}')$ means that $\mathfrak{b}' \subseteq \mathfrak{q}$, and since $\mathcal{C}'$ is fine (by the comment in the paragraph preceding the lemma), $\mathfrak{b}'$ is θ-stable. The lemma thus follows. □

Proposition 4.4. *Retain the notation of Definition 4.2. If G has a compact Cartan subgroup, then any parabolic subgroup of $G_{\mathbb{C}}$ is fine with respect to G.*

Proof. Let $\mathcal{C}$ be a closed $K_{\mathbb{C}}$-orbit in $G_{\mathbb{C}}/Q$ and let $\mathfrak{q}'$ be a parabolic subalgebra contained in $\mathcal{C}$. We take a θ-stable Borel subalgebra $\mathfrak{b}' \subseteq \mathfrak{q}'$ as in Lemma 4.3. The we can choose a compact Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{h} \subseteq \mathfrak{b}'$. Thus $\mathfrak{q}'$ corresponds to a subset S of the set of simple roots for $(\mathfrak{b}', \mathfrak{h})$. Since θ acts trivially on $\mathfrak{h}$, S is θ-stable. Hence $\mathfrak{q}'$ is θ-stable. □

We discuss another sufficient condition.

Definition 4.5. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$. We choose a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{b} \subseteq \mathfrak{q}$. We call $\mathfrak{q}$ neat if $\mathfrak{q}$ is stable under any automorphism of $\mathfrak{g}$ preserving $\mathfrak{b}$. It is easy to see that the condition of being neat does not depend on the choice of $\mathfrak{b}$. A parabolic subgroup Q is called neat if its Lie algebra $\mathfrak{q}$ is neat. (Lemma 4.3 implies that any neat parabolic subgroup is fine.)

We next introduce a little more notation. We let μ_Q denote the moment map for the cotangent bundle $T^*G_{\mathbb{C}}/Q$ to $\mathfrak{g}^*$. For a closed $K_{\mathbb{C}}$-orbit $\mathcal{C}$, we consider the conormal bundle $T^*_{\mathcal{C}}G_{\mathbb{C}}/Q$ of $\mathcal{C}$ in $G_{\mathbb{C}}/Q$. We let $\mathcal{C}^\vee$ denote the image of $T^*_{\mathcal{C}}G_{\mathbb{C}}/Q$ under μ_Q. Let $\mathfrak{q}' \in \mathcal{C}$ and write $\mathfrak{u}'$ for the nilradical of $\mathfrak{q}'$. After unwinding the definitions, it is easy to see that $\mathcal{C}^\vee = \mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u} \cap \mathfrak{s})$; here we are implicitly identifying $\mathfrak{g}^*$ and $\mathfrak{g}$.

Definition 4.6. Retain the notation above. Let $\mathcal{O}$ be the Richardson orbit in $\mathfrak{g}$ with respect to $\mathfrak{q}$. A closed and fine G-orbit $\mathcal{C}$ is called good if $\dim \mathcal{C}^\vee = \dim \mathfrak{s} \cap \mathcal{O}$. An open G-orbit is called good if the corresponding closed $K_{\mathbb{C}}$-orbit is good.

For a good closed $K_{\mathbb{C}}$-orbit $\mathcal{C}$, $\mathcal{C}^\vee$ is the closure of an element of $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})$ (with notation as in Section 2.2). If the moment map μ_Q is birational to its image, μ_Q gives bijection of $\mu_Q^{-1}(\mathcal{O})$ onto $\mathcal{O}$, and so we have the following lemma.

Lemma 4.7. *Assume that the moment map μ_Q is birational onto its image. Then the map $\mathcal{C} \mapsto \mathcal{C}^\vee$ of the set of the good closed $K_{\mathbb{C}}$-orbits in $G_{\mathbb{C}}/Q$ to closures of elements of $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})$ is injective.*

For our purposes, it is important to describe the map $\mathcal{C} \rightsquigarrow \mathcal{C}^\vee$. The proof of the following proposition is straightforward.

Proposition 4.8. *Retain the notation above. Let X be an even nilpotent element of $\mathfrak{g}_0$. Choose a $\mathfrak{sl}_2$-triple (X, H, Y) such that $X, H, Y \in \mathfrak{g}_0$. Let $\mathfrak{q}$ denote the parabolic subalgebra consisting of the nonnegative eigenvalues of $\mathrm{ad}(H)$ on $\mathfrak{g}$. Define the Cayley element $C_X \in G_{\mathbb{C}}$ by $C_X = \exp\left(\frac{\pi i}{4}(X+Y)\right)$. Then $\mathrm{Ad}(C_X)\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$. Let $\mathcal{O}_K(X)$ denote the $K_{\mathbb{C}}$-nilpotent orbit in $\mathfrak{s}$ which corresponds to $\mathrm{Ad}(G)X$ via the Kostant-Sekiguchi correspondence. Let $\mathcal{C}(X)$ be the closed $K_{\mathbb{C}}$-orbit in $G_{\mathbb{C}}/Q$ such that $\mathrm{Ad}(C_X)\mathfrak{q} \in \mathcal{C}(X)$. Then we have*

$$\mathcal{C}(X)^\vee = \mathcal{O}_K(X).$$

In general, we do not know a good conceptual description of the correspondence $\mathcal{C} \rightsquigarrow \mathcal{C}^\vee$. However, for classical groups, some combinatorial algorithms computing Richardson orbits in real groups are obtained by [T1] and [T2] (see also [Y]). Examining those algorithms we obtain the following proposition.

Proposition 4.9. *Let G be $\mathrm{U}(m,n)$, $\mathrm{Sp}(p,q)$, or $\mathrm{SO}^*(2n)$. Consider the following Levi subalgebra $\mathfrak{l}$ of the complexified Lie algebra $\mathfrak{g}$ of G,*

$$\begin{aligned}
&\mathfrak{l} \simeq \mathfrak{gl}(n_1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(n_k,\mathbb{C}) \quad \text{if } G = \mathrm{U}(m,n);\\
&\mathfrak{l} \simeq \mathfrak{gl}(2n_1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(2n_k,\mathbb{C}) \oplus \mathfrak{sp}(2n',\mathbb{C}) \quad \text{if } G = \mathrm{Sp}(p,q); \text{ and}\\
&\mathfrak{l} \simeq \mathfrak{gl}(2n_1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(2n_k,\mathbb{C}) \oplus \mathfrak{so}(2n',\mathbb{C}) \quad \text{if } G = \mathrm{SO}^*(2n).
\end{aligned}$$

Let $\mathcal{O}$ denote the complex Richardson orbit induced from $\mathfrak{l}$. Let $\mathfrak{q}$ be an arbitrary parabolic subalgebra of $\mathfrak{g}$ whose Levi part is $\mathfrak{l}$ and let Q denote the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{q}$. In this case, Q is fine by Proposition 4.4.

Then the map $\mathcal{C} \mapsto \mathcal{C}^\vee$ from the set of the good closed $K_{\mathbb{C}}$-orbits in $G_{\mathbb{C}}/Q$ to $\mathrm{Irr}(\overline{\mathcal{O}} \cap \mathfrak{s})$ is bijective. Moreover, the correspondence $\mathcal{C} \rightsquigarrow \mathcal{C}^\vee$ is described explicitly by the combinatorial data described in [T1] and [T2].

5 Weakly unipotent primitive ideals associated to a Richardson orbit

We briefly recall some features of the theory of unipotent primitive ideals. The contents of this section is more or less known (but not collected in one place). Definitions 5.1 and 5.3 provide the notion of weak unipotence and distinguished unipotence for primitive ideals. Propositions 5.5 gives a nice characterizing of distinguished unipotence for primitive ideals whose associated variety is Richardson. Again in the context of Richardson orbits, Proposition 5.7 shows that indeed the notions of weak and distinguished unipotence coincide. (In general, the latter implies the former, but not conversely.) We then turn to specific examples. The main results needed for applications below are Proposition 5.10, 5.12, and 5.14. These may be also deduced from the classification of primitive ideals for classical algebras, but we prefer to avoid invoking the classification.

Let $\mathfrak{g}$ denote a complex reductive Lie algebra and recall the notation of Section 2.3. Set $\mathfrak{h}^{ss} = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ and let $G_{\mathbb{C}}^{ss}$ denote the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $[\mathfrak{g}, \mathfrak{g}]$.

Definition 5.1 ([KnV, Definition 12.3], [V5, Definition 12.10]).

(1) An irreducible $U(\mathfrak{g})$-module X with an infinitesimal character λ is called *weakly unipotent* if

   (a) the restriction of λ to $\mathfrak{h}^{ss}$ is in the $\mathbb{R}$-linear span of the roots;

   (b) for each finite-dimensional $\mathrm{U}(\mathfrak{g})$ module F, $X \otimes_{\mathbb{C}} F$ has finite length (as a $\mathrm{U}(\mathfrak{g})$ module); and

   (c) if there exists a nonzero subquotient of $X \otimes F$ annihilated by an ideal of the form Z_μ, then $\langle \lambda|_{\mathfrak{h}^{ss}}, \lambda|_{\mathfrak{h}^{ss}} \rangle \leqslant \langle \mu|_{\mathfrak{h}^{ss}}, \mu|_{\mathfrak{h}^{ss}} \rangle$.

   If in addition $\mathcal{O}$ is dense in $\mathrm{AV}(\mathrm{Ann}(X))$, then we say that X is *weakly unipotent attached to $\mathcal{O}$.*

(2) A two-sided ideal of I in $\mathrm{U}(\mathfrak{g})$ is called a *weakly unipotent primitive ideal* if it is the annihilator of a weakly unipotent irreducible $\mathrm{U}(\mathfrak{g})$-module.

If we apply results in [V6, Section 7], it is not difficult to deduce the following result.

Corollary 5.2. *If an irreducible* $\mathrm{U}(\mathfrak{g})$*-module X with a weakly unipotent annihilator satisfies condition (b) in Definition 5.1, then X is a weakly unipotent* $\mathrm{U}(\mathfrak{g})$*-module.*

Definition 5.3 (Compare [V5, Definition 12.6]). Let $\lambda \in \mathfrak{h}^*$ be such that $\lambda|_{\mathfrak{h}^{ss}}$ be contained in the real linear span of the roots. Let $I \in \mathrm{Prim}_\lambda(\mathfrak{g})$. Let $\mathrm{AV}(I) = \overline{\mathcal{O}}$ and set

$$\mathrm{Prim}(\mathcal{O}) = \{I' \in \mathrm{Prim}(\mathfrak{g}) \mid \mathrm{AV}(I') = \overline{\mathcal{O}}\},$$

and

$$\mathrm{IC}(\mathcal{O}) = \{\lambda' \mid \mathrm{Prim}_{\lambda'}(\mathfrak{g}) \cap \mathrm{Prim}(\mathcal{O}) \neq \emptyset\}.$$

Finally set

$$\mathrm{IC}_{G_{\mathbb{C}}}(\mathcal{O}, \lambda) = \{\lambda' \in \mathrm{IC}(\mathcal{O}) \mid \lambda' - \lambda \in \mathcal{P}_{G_{\mathbb{C}}}\}.$$

Then I is called $G_{\mathbb{C}}$-distinguished unipotent if $\langle \lambda|_{\mathfrak{h}^{ss}}, \lambda|_{\mathfrak{h}^{ss}} \rangle \leqslant \langle \lambda'|_{\mathfrak{h}^{ss}}, \lambda'|_{\mathfrak{h}^{ss}} \rangle$ for all $\lambda' \in \mathrm{IC}_{G_{\mathbb{C}}}(\mathcal{O}, \lambda)$. In this case, we say that I is *attached to $\mathcal{O}$*. If the infinitesimal character of I is in $\mathcal{P}_{G_{\mathbb{C}}}$, we say that I is *$G_{\mathbb{C}}$-integral distinguished unipotent attached to $\mathcal{O}$*. We let $\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$ denote the set of the $G_{\mathbb{C}}$-integral distinguished unipotent primitive ideals attached to $\mathcal{O}$. Finally, if X is an irreducible Harish-Chandra module whose annihilator is an element of $\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$, we say that X is *distinguished unipotent attached to $\mathcal{O}$.*

We now describe a general method of constructing $G_{\mathbb{C}}$-integral distinguished unipotent primitive ideals attached to Richardson orbits. So suppose that $\mathcal{O}$ is induced from the zero orbit of a Levi factor $\mathfrak{l}$. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{l}$ (and hence in $\mathfrak{g}$) and fix a positive system $\Delta^+(\mathfrak{g},\mathfrak{h})$ for $\Delta(\mathfrak{g},\mathfrak{h})$. Let $\mathfrak{b}$ be the Borel subalgebra containing $\mathfrak{h}$ corresponds to $\Delta^+(\mathfrak{g},\mathfrak{h})$. Put $\Delta^+(\mathfrak{l},\mathfrak{h}) = \Delta^+(\mathfrak{g},\mathfrak{h}) \cap \Delta(\mathfrak{l},\mathfrak{h})$. Write $\delta(\mathfrak{l}) \in \mathfrak{h}^*$ (respectively, $\rho \in \mathfrak{h}^*$) for the half-sum of positive roots of $\mathfrak{h}$ in $\mathfrak{l}$ (respectively, $\mathfrak{g}$). Fix a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that $\mathfrak{b} \subseteq \mathfrak{q}$. Set

$$\mathcal{P}^{++}(\mathfrak{l}) = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \{1,2,3,\ldots\} \quad (\alpha \in \Delta^+(\mathfrak{l},\mathfrak{h}))\}$$

and
(5.1)

$$\mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l}) = \{\lambda \in \mathcal{P}^{++}(\mathfrak{l}) \cap \mathcal{P}_{G_{\mathbb{C}}} \mid \langle \lambda|_{\mathfrak{h}^{ss}}, \lambda|_{\mathfrak{h}^{ss}} \rangle \leqslant \langle \mu|_{\mathfrak{h}^{ss}}, \mu|_{\mathfrak{h}^{ss}} \rangle \text{ for all } \mu \in \mathcal{P}^{++}(\mathfrak{l}) \cap \mathcal{P}_{G_{\mathbb{C}}}\}.$$

For $\mu \in \mathcal{P}^{++}(\mathfrak{l})$, let $E_{\mathfrak{l}}(\mu-\rho)$ denote the irreducible finite-dimensional $\mathrm{U}(\mathfrak{l})$-module with highest weight $\mu - \rho$. Introducing the trivial action of the nilradical of $\mathfrak{q}$, we regard $E_{\mathfrak{l}}(\mu-\rho)$ as a $\mathrm{U}(\mathfrak{q})$-module. We define the generalized Verma module induced from E via

$$M_{\mathfrak{q}}(\mu) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{q})} E_{\mathfrak{l}}(\mu-\rho).$$

For $\mu \in \mathfrak{h}^*$, write $L(\mu)$ for the irreducible $\mathrm{U}(\mathfrak{g})$-module with a highest weight $\lambda - \rho$. A $\mathrm{U}(\mathfrak{g})$-module X is called $\mathfrak{q}$-finite if $\dim \mathrm{U}(\mathfrak{q})v < \infty$ holds for each $v \in X$. For $\mu \in \mathcal{P}^{++}(\mathfrak{l})$, $L(\mu)$ is $\mathfrak{q}$-finite. Conversely, each $\mathfrak{q}$-finite irreducible $\mathrm{U}(\mathfrak{g})$-module is isomorphic to $L(\mu)$ for some $\mu \in \mathcal{P}^{++}(\mathfrak{l})$. In other words, a $\mathfrak{q}$-finite irreducible $\mathrm{U}(\mathfrak{g})$-module is a (unique) quotient of a generalized Verma module.

Lemma 5.4. *Let $\mathcal{O}$ denote the Richardson orbit induced from a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. Let I and J be primitive ideals in $\mathrm{U}(\mathfrak{g})$ with integral infinitesimal character λ such that $I \subseteq J$. Assume that $I \in \mathrm{Prim}(\mathcal{O})$. Then there exists some irreducible $\mathfrak{q}$-finite $\mathrm{U}(\mathfrak{g})$-module E such that J is the annihilator of E in $\mathrm{U}(\mathfrak{g})$.*

Proof. The translation principle easily reduces matters to the case of regular infinitesimal character. Since the $\mathfrak{q}$-finite irreducible $\mathrm{U}(\mathfrak{g})$-modules with a given regular integral infinitesimal character form a single right cone in category **O**, the lemma follows from [LX, Theorem 3.2]. (Actually, Lusztig-Xi's result is stated in the context of affine Weyl groups but the same proof is applicable to the case of finite Weyl groups. We also remark that we may of course interchange "L" and "R" in the statement of their result.) ☐

Lemma 5.4 gives the follows two results.

Proposition 5.5. *Fix a Richardson orbit $\mathcal{O}$ induced from $\mathfrak{l}$ and Retain the above notation (especially that of Definition 5.3 and Equation (5.1)). Then*

$$\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O}) = \mathrm{Prim}(\mathcal{O}) \cap \bigcup_{\lambda \in \mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})} \mathrm{Prim}_{\lambda}(\mathfrak{g})$$

Proposition 5.6. *Let $\lambda \in \mathcal{P}^{++}(\mathfrak{l})$ be such that $\mathcal{P}^{++}(\mathfrak{l}) \cap W\lambda = \{\lambda\}$. Then*

(1) $M_{\mathfrak{q}}(\lambda)$ is irreducible and $J_{\max}(\lambda) = \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{q}}(\lambda))$.

(2) $\mathrm{Prim}_\lambda(\mathfrak{g}) \cap \mathrm{Prim}(\mathcal{O}) = \{J_{\max}(\lambda)\}$.

Next we consider the weakly unipotent primitive ideals attached to a Richardson orbit.

Proposition 5.7. *Retain the above notation. In particular, assume that $\mathcal{O}$ is induced from the zero orbit of a Levi factor $\mathfrak{l}$. Further assume that $G_{\mathbb{C}}^{ss}$ is simply connected. Let $\lambda \in \mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})$ (with notation as in (5.1). Fix $I \in \mathrm{Prim}_\lambda(\mathfrak{g})$ and assume there exists some $I' \in \mathrm{Prim}(\mathcal{O})$ such that $I' \subseteq I$. Then I is weakly unipotent. In particular, each element of $\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$ is weakly unipotent. Conversely, if $I \in \mathrm{Prim}(\mathcal{O})$ is weakly unipotent then $I \in \mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$. Consequently the notions of weakly unipotent and distinguished unipotent coincide for $\mathrm{Prim}(\mathcal{O})$.*

Proof. Fix $I \in \mathrm{Prim}_\lambda(\mathfrak{g})$ and assume there exists some $I' \in \mathrm{Prim}(\mathcal{O})$ such that $I' \subseteq I$. Then it follows from Lemma 5.4 that I is the annihilator of an irreducible $\mathfrak{q}$-finite $\mathrm{U}(\mathfrak{g})$-module, say X. For any finite-dimensional $\mathrm{U}(\mathfrak{g})$-module F, $X \otimes_{\mathbb{C}} F$ is $\mathfrak{q}$-finite. Thus the infinitesimal character of any irreducible subquotient of $X \otimes_{\mathbb{C}} F$ belongs $\mathcal{P}^{++}(\mathfrak{l})$ (up to the action of the Weyl group). Since X is isomorphic to $L(\mu)$ for some $\mu \in \mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})$, it follows from Definition 5.1 and the definition of Equation (5.1) that I is weakly unipotent.

Consider the converse. Let $I \in \mathrm{Prim}(\mathcal{O})$ be weakly unipotent. If $I \notin \mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$, then there exists some $\mu \in \mathcal{P}^{++}(\mathfrak{h})$ such that $\mu \notin \mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})$ and so that I is the annihilator of $L(\mu)$ (by Lemma 5.4). Since the $\mathfrak{q}$-finite irreducible $\mathrm{U}(\mathfrak{g})$-modules with a given regular integral infinitesimal character form a single right cone for category **O**, it follows that an irreducible generalized Verma module with respect to $\mathfrak{q}$ (say V) is an irreducible subquotient of $L(\mu) \otimes_{\mathbb{C}} F_1$ for some finite-dimensional $\mathrm{U}(\mathfrak{g})$-module F_1. Thus any irreducible $\mathfrak{q}$-finite $\mathrm{U}(\mathfrak{g})$-module with an integral infinitesimal character is an irreducible constituent of $V \otimes_{\mathbb{C}} F_2$ for some finite-dimensional $\mathrm{U}(\mathfrak{g})$-module F_2. Since $G_{\mathbb{C}}^{ss}$ is assumed to be simply connected, $\mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})$ is nonempty. So for $\eta \in \mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})$, there is some finite-dimensional $\mathrm{U}(\mathfrak{g})$-module F_3 such that $L(\eta)$ is an irreducible constituent of $L(\mu) \otimes_{\mathbb{C}} F_3$. This contradicts the assumption that I is weakly unipotent. □

We now consider examples. Suppose first that $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $G_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$, and $\mathfrak{l} \cong \mathfrak{gl}(n_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(n_k, \mathbb{C})$; in coordinates

$$\delta(\mathfrak{l}) = \Big(\overbrace{(n_1-1)/2, \cdots, -(n_1-1)/2}^{n_1}, \cdots\cdots \overbrace{(n_k-1)/2, \cdots, -(n_k-1)/2}^{n_k} \Big).$$

Write $\mathcal{O}$ for the Richardson orbit induced from the zero orbit of $\mathfrak{l}$. We put

$$\varepsilon_i = \begin{cases} 0 & \text{if } n_i \text{ is odd,} \\ 1 & \text{if } n_i \text{ is even.} \end{cases}$$

$$\varepsilon = \Big(\overbrace{\varepsilon_1, \cdots, \varepsilon_1}^{n_1}, \cdots\cdots \overbrace{\varepsilon_k, \cdots, \varepsilon_k}^{n_k} \Big).$$

The next two lemmas are simple computational exercises.

Lemma 5.8. *In the above setting, $\mathcal{P}^{++}(\mathfrak{l}) \cap W\lambda = \{\lambda\}$ for all $\lambda \in \mathcal{P}^{++}(\mathfrak{l})$ such that $\lambda|_{\mathfrak{h}^{ss}} = \big(\delta(\mathfrak{l}) \pm \frac{1}{2}\varepsilon\big)\big|_{\mathfrak{h}^{ss}}$.*

Lemma 5.9. *Retain the above setting. Put*

$$s(\mathfrak{l}) = \Big\{ \delta(\mathfrak{l}) + \eta\frac{1}{2}\varepsilon \;\Big|\; \eta = \pm 1 \text{ and the length of } \big(\delta(\mathfrak{l}) + \eta\frac{1}{2}\varepsilon\big)\big|_{\mathfrak{h}^{ss}} \text{ is less than or equal to the length of } \big(\delta(\mathfrak{l}) - \eta\frac{1}{2}\varepsilon\big)\big|_{\mathfrak{h}^{ss}} \Big\};$$

i. e., roughly speaking, $s(\mathfrak{l})$ consists of the shorter of the two elements $\delta(\mathfrak{l}) \pm \frac{1}{2}\varepsilon$ (or both if they have the same length). Then

$$\mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l}) = \{\lambda \in \mathcal{P}^{++}(\mathfrak{l}) \mid \lambda|_{\mathfrak{h}^{ss}} = \mu|_{\mathfrak{h}^{ss}} \text{ for some } \mu \in s(\mathfrak{l})\}.$$

In particular, at least one of $\delta(\mathfrak{l}) \pm \frac{1}{2}\varepsilon$ is in $\mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})$.

Hence, from Propositions 5.6 and 5.7, we obtain the following.

Proposition 5.10. *For $G_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$,*

$$\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O}) = \{J_{\max}(\lambda) \mid \lambda|_{\mathfrak{h}^{ss}} = \mu|_{\mathfrak{h}^{ss}} \text{ for some } \mu \in s(\mathfrak{l})\}$$

where $s(\mathfrak{l})$ is defined in Lemma 5.8.

We now turn our attention to the cases of $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$ or $G_{\mathbb{C}} = \mathrm{SO}(2n, \mathbb{C})$. Fix a Cartan subgroup $\mathfrak{h}$ of $\mathfrak{g}$. Then we can choose an orthonormal basis $e_1, ..., e_n$ of $\mathfrak{h}^*$ such that

$$\Delta(\mathfrak{g}, \mathfrak{h}) = \begin{cases} \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} & \text{if } \mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}), \\ \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm 2e_i \mid 1 \leqslant i \leqslant n\} & \text{if } \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) \end{cases}.$$

Let $E_1, ..., E_n$ be the dual basis of $\mathfrak{h}$ to $e_1, ..., e_n$. We fix a simple system for $\Delta(\mathfrak{g}, \mathfrak{h})$ in each case as follows. If $G_{\mathbb{C}} = \mathrm{SO}(2n, \mathbb{C})$, then let $\Pi = \{e_1 - e_2, ..., e_{n-1} - e_n, e_{n-1} + e_n\}$, and if $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$, set $\Pi = \{e_1 - e_2, ..., e_{n-1} - e_n, 2e_n\}$. We let Δ^+ denote the corresponding positive system of $\Delta(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{b}$ denote the corresponding Borel subalgebra containing $\mathfrak{h}$.

Fix a Levi subalgebra of the following form.

$$\mathfrak{l} \cong \begin{cases} \mathfrak{gl}(2k_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(2k_s, \mathbb{C}) \oplus \mathfrak{sp}(n', \mathbb{C}) & \text{if } \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}); \\ \mathfrak{gl}(2k_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(2k_s, \mathbb{C}) \oplus \mathfrak{so}(2n', \mathbb{C}) & \text{if } \mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}). \end{cases} \tag{5.2}$$

Set $k_i^* = k_1 + \cdots + k_i$ for $1 \leqslant i \leqslant s$ and $k_0^* = 0$. We assume that $\mathfrak{h} \subseteq \mathfrak{l}$ and that $E_{2k_{i-1}^*+1}, E_{2k_{i-1}^*+2}, ..., E_{2k_i^*}$ are contained in the $\mathfrak{gl}(2k_i, \mathbb{C})$-factor in the above direct sum decomposition.

If $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$, in coordinates

$$\delta(\mathfrak{l}) = \Big(\overbrace{k_1 - \frac{1}{2}, k_1 - \frac{3}{2}, \cdots, -k_1 + \frac{1}{2}}^{2k_1}, \cdots\cdots \overbrace{k_s - \frac{1}{2}, k_s - \frac{3}{2}, \cdots, -k_s + \frac{1}{2}}^{2k_s}, n', n' - 1, ..., 1 \Big).$$

If $\mathfrak{g}=\mathfrak{so}(2n,\mathbb{C})$, in coordinates

$$\delta(\mathfrak{l})=\Big(\overbrace{k_1-\frac{1}{2},k_1-\frac{3}{2},\cdots,-k_1+\frac{1}{2}}^{2k_1},\cdots\cdots\overbrace{k_s-\frac{1}{2},k_s-\frac{3}{2},\cdots,-k_s+\frac{1}{2}}^{2k_s},n'-1,n'-2,...,0\Big).$$

Finally let $\mathfrak{q}$ be a parabolic subalgebra with a Levi part $\mathfrak{l}$.

For $\vec{\eta}=(\eta_1,....,\eta_s)\in\{\pm1\}^s$, define

$$[\vec{\eta}]=(\overbrace{\eta_1,...,\eta_1}^{2k_1},\cdots\cdots,\overbrace{\eta_k,....,\eta_k}^{2k_s},\overbrace{0,...,0}^{n'})$$

Let $\mathcal{O}$ denote the Richardson orbit induced from the zero orbit of $\mathfrak{l}$. The next lemma follows easily.

Lemma 5.11. *Retain the above setting. Then*

$$\mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})=\left\{\delta(\mathfrak{l})+\frac{1}{2}[\vec{\eta}]\,\middle|\,\vec{\eta}\in\{\pm1\}^s\right\}.$$

In particular, any two elements in $\mathcal{S}_{G_{\mathbb{C}}}(\mathfrak{l})$ are conjugate under the Weyl group action.

Proposition 5.12. *Retain the above setting. Set $\vec{\eta}_0=(\overbrace{-1,...,-1}^{s})$ and $\lambda_{\mathfrak{q}}=\delta+[\vec{\eta}_0]$. Then*

$$\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})=\{\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{q}}(\lambda_{\mathfrak{q}})\}.$$

Proof. From [V2, Proposition 8.5], $M_{\mathfrak{q}}(\lambda_{\mathfrak{q}})$ is irreducible. Hence, from Lemma 5.11, we have $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{q}}(\lambda_{\mathfrak{q}}))\in\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$. Let $I\in\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$. According to Proposition 5.5 and Lemma 5.11, there is some $\vec{\eta}\in\{\pm1\}^s$ such that $I=\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(L(\delta(\mathfrak{l})+[\vec{\eta}]))$. Since $E_{\mathfrak{l}}(\delta(\mathfrak{l})+[\vec{\eta}])$ is one-dimensional, the Bernstein degree (e.g. [V1]) of $M_{\mathfrak{q}}(\delta(\mathfrak{l})+[\vec{\eta}])$ is one. On the other hand, since $\mathcal{O}$ is dense in the associated variety of both $L_{\mathfrak{q}}(\delta(\mathfrak{l})+[\vec{\eta}])$ and $M_{\mathfrak{q}}(\delta(\mathfrak{l})+[\vec{\eta}])$, it follows that $L_{\mathfrak{q}}(\delta(\mathfrak{l})+[\vec{\eta}])$ is the unique constituent of $M_{\mathfrak{q}}(\delta(\mathfrak{l})+[\vec{\eta}])$ of maximal GK dimension. On the other hand, from [BoJa, 4.10 Corollar], for each $\vec{\eta}\in\{\pm1\}^s$, we have $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{q}}(\delta(\mathfrak{l})+[\vec{\eta}])=\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{q}}(\lambda_{\mathfrak{q}})$. Hence we have $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{q}}(\lambda_{\mathfrak{q}}))\subset I$, both of which have the same associated variety (the closure of $\mathcal{O}$). Then [BoKr, 3.6] implies that indeed $I=\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{q}}(\lambda_{\mathfrak{q}}))$. ∎

Definition 5.13. In the setting of Proposition 5.12, we let $I_{\mathcal{O}}$ denote the unique element of $\mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$. In addition, we let $\lambda(\mathcal{O})$ denote the infinitesimal character of $I_{\mathcal{O}}$ (namely $\lambda_{\mathfrak{q}}$).

Proposition 5.14. *Retain the above setting; in particular recall the notation of Definition 5.1, Proposition 5.12 and Definition 5.13.*

(1) If $\mathfrak{g}=\mathfrak{sp}(n,\mathbb{C})$, then $I_{\mathcal{O}}$ is the unique integral weakly unipotent primitive ideal associated to $\mathcal{O}$.

(2) If $\mathfrak{g}=\mathfrak{so}(2n,\mathbb{C})$ and $k_1+\cdots+k_s<n$, then $I_{\mathcal{O}}$ is the unique integral weakly unipotent primitive ideal associated to $\mathcal{O}$.

(3) If $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and $k_1 + \cdots + k_s = n$, then $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{q}}(\delta(\mathfrak{l})))$ is the unique integral weakly unipotent primitive ideal associated to $\mathcal{O}$.

Proof. (1) follows from Proposition 5.12 and Proposition 5.7. (2) and (3) are obtained by applying Proposition 5.7 to $G_{\mathbb{C}} = \mathrm{Spin}(2n, \mathbb{C})$. We omit the details. □

Example 5.15. It is interesting to note that in general, $I_{\mathcal{O}}$ (of Definition 5.13) is not necessarily maximal. For example, from [Ma5, Theorem 3.2.2 and Lemma 3.6.1], we have the following example. Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and $\mathfrak{l} = \mathfrak{gl}(2k, \mathbb{C}) \oplus \mathfrak{sp}(n-2k, \mathbb{C})$. Then $I_{\mathcal{O}}$ is maximal if and only if $3k > n$.

Remark 5.16. Recall that if $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and $k_1 + \cdots + k_s = n$, then the corresponding Richardson orbit is called very even.

6 $\mathrm{Sp}(p, q)$ and $\mathrm{SO}^*(2n)$

In this section, we assume G is either $\mathrm{Sp}(p, q)$ or $\mathrm{SO}^*(2n)$ and prove the theorems outlined in the introduction relating cohomologically induced representations to certain degenerate principal series. We begin by considering some weakly unipotent derived functor modules. As a starting point, the following result follows immediately from Proposition 4.9.

Corollary 6.1. *Let $G = \mathrm{Sp}(p, q)$ or $\mathrm{SO}^*(2n)$. Suppose $\mathcal{O}$ is a complex orbit induced from a Levi factor of the form appearing in Proposition 4.9. Recall the notation of Section 2.2 and write*

$$\mathrm{Irr}(\mathcal{O} \cap \mathfrak{s}) = \{\mathcal{O}_K^1, \cdots, \mathcal{O}_K^r\}.$$

Let $\mathfrak{p}$ be an arbitrary parabolic subalgebra of $\mathfrak{g}$ whose Levi part is $\mathfrak{l}$. Then, for each j there exists a θ-stable parabolic subalgebra

$$\mathfrak{q}^j = \mathfrak{l}^j \oplus \mathfrak{u}^j$$

such that $\mathfrak{p}$ and $\mathfrak{q}^j$ are $\mathrm{Ad}(G_{\mathbb{C}})$-conjugate and

$$K_{\mathbb{C}} \cdot (\mathfrak{u}^j \cap \mathfrak{s}) = \overline{\mathcal{O}_K^j}. \tag{6.1}$$

Moreover

$$G_{\mathbb{C}} \cdot (\mathfrak{u}^j \cap \mathfrak{s}) = \overline{\mathcal{O}} \tag{6.2}$$

for each j. Such a $\mathfrak{q}^j$ is unique up to $K_{\mathbb{C}}$-conjugacy for each choice of $\mathfrak{p}$.

We now attach a derived functor module to each $\mathcal{O}_K^j$. Retain the setting as above and fix $\mathcal{O}_K^j \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})$. We fix a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ whose Levi part is $\mathfrak{l}$ as in Corollary 6.1. Let $\mathfrak{q}^j = \mathfrak{l}^j \oplus \mathfrak{u}^j$ be the θ-stable parabolic defined by Corollary 6.1. Fix a Cartan $\mathfrak{h}$ in $\mathfrak{l}$, and choose a system of positive roots for $\mathfrak{h}$ in $\mathfrak{g}$ that contains the roots of $\mathfrak{p}$. Since $\mathfrak{p}$ and $\mathfrak{q}^j$ are $\mathrm{Ad}(G_{\mathbb{C}})$-conjugate, we fix $g^j \in G_{\mathbb{C}}$ such that $\mathrm{Ad}(g^j)\mathfrak{p} = \mathfrak{q}^j$. Put $\mathfrak{h}^j = \mathrm{Ad}(g^j)\mathfrak{h}$. By abuse of notation, we identify $(\mathfrak{h}^j)^*$ and $\mathfrak{h}$ via $\mathrm{Ad}(g^j)$. Write δ for the half-sum of the positive roots. Recall the infinitesimal character $\lambda(\mathcal{O}) \in \mathfrak{h}^*/W$ of

Definition 5.13, and choose a representative, also denoted $\lambda(\mathcal{O})$, dominant with respect to our choice of positive roots. Set

$$\lambda^j = \lambda(\mathcal{O}) - \delta. \tag{6.3}$$

Since $\lambda(\mathcal{O})$ differs from $\delta(\mathfrak{l})$ by a shift orthogonal to $\delta(\mathfrak{l}^j)$, it follow that λ^j differs from $\delta(\mathfrak{u}^j)$ by a shift orthogonal to $\delta(\mathfrak{l})$. Moreover, λ^j is integral, and hence it exponentiates to a one-dimensional $(\mathfrak{l}^j, L^j \cap K)$ module. We may thus form the derived functor module $A(\mathcal{O}_K^j) := A_{\mathfrak{q}^j}(\lambda^j)$ as in [KnV, Chapter 5].

Proposition 6.2. *Retain the setting of the previous paragraph and Corollary 6.1. Then $A(\mathcal{O}_K^j)$ is an irreducible unitary representation of G with infinitesimal character $\lambda(\mathcal{O})$ (Definition 5.13). Moreover*

$$\mathrm{AV}(A(\mathcal{O}_K^j)) = \overline{\mathcal{O}_K^j},$$

and

$$\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(A(\mathcal{O}_K^j)) = I_{\mathcal{O}}.$$

In fact, $\mathcal{O}_K^j$ occurs with multiplicity one in the associated cycle of $A(\mathcal{O}_K^j)$.

Proof. All the assertion except the last two are contained in Section 6 of [V4]. (The key hypotheses are that the induction is in the weakly fair range — see Remark 6.3 — and the conclusion of Equation (6.2).) The final assertion follows from the argument of [PT, Proposition 6.2]. $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(A(\mathcal{O}_K^j)) = I_{\mathcal{O}}$ is obtained from the final assertion and Proposition 5.12 (also see [V3, Proposition 16.8]). ☐

Remark 6.3. In the terminology of [KnV, Definition 0.52], λ^j is in the weakly fair range for $\mathfrak{q}^j$. In fact it is clear that $A(\mathcal{O}_K^j)$ is the most singular module possible, given that $\delta(\mathfrak{u}^j)$ does not exponentiate to L^j. Loosely one may say that $A(\mathcal{O}_K^j)$ is on the edge of the weakly fair range.

Remark 6.4. The definition of $A(\mathcal{O}_K^j)$ appears to depend on the choice of $\mathfrak{p}$. Later we see that $A(\mathcal{O}_K^j)$ indeed only depends on $\mathcal{O}_K^j$.

Next we introduce the degenerate principal series of $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$ of particular interest to us. The key is to arrange for them to have the infinitesimal character $\lambda(\mathcal{O})$ of Definition 5.13. The following lemma is a simple exercise.

Lemma 6.5. *Retain the setting of Corollary 6.1. There is a Levi subgroup M of G such that $\mathcal{O}$ is induced from the zero orbit of the complexified Lie algebra $\mathfrak{m}$ of M. Such a subgroup M is unique up to $\mathrm{Ad}(G)$-conjugation. Moreover, we may assume that M is θ-stable.*

If $G = \mathrm{Sp}(p,q)$, then we have

$$M \simeq \mathrm{GL}(k_1, \mathbb{H}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{H}) \times \mathrm{Sp}(p', q')$$

with $p = p' + \sum k_i$ and $q = q' + \sum k_i$.

If $G = \mathrm{SO}^*(2n)$, then we have

$$M \simeq \mathrm{GL}(k_1, \mathbb{H}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{H}) \times \mathrm{SO}^*(2n')$$

with $n = n' + \sum k_i$.

To each Richardson orbit appearing here, we define a corresponding degenerate principal series $\mathcal{I}_P(\mathcal{O};\vec{\eta})$ for each polarization and some additional data $\vec{\eta}$.

Definition 6.6. Retain the setting and notation of Lemma 6.5 and, in particular, fix a θ-stable Levi subgroup M as in the lemma. Let P be any parabolic subgroup of G with a Levi decomposition $P = MN$. For $\mathrm{GL}(n,\mathbb{H})$, we let $\det^{1/2}$ denote the following character. Given $A \in \mathrm{GL}(n,\mathbb{H})$, let $A_{\mathbb{C}}$ denote its image in $\mathrm{GL}(2n,\mathbb{C})$ (under any of the natural embeddings). Then $\det(A_{\mathbb{C}})$ is a well-defined real number, and we let $\det^{1/2}(A)$ denote its positive real fourth root. The result is a character $\det^{1/2}$ of $\mathrm{GL}(n,\mathbb{H})$ with weight $(1/2,\ldots,1/2)$. Fix $\vec{\eta} = (\eta_1,...,\eta_s) \in \{\pm1\}^s$. For $G = \mathrm{Sp}(p,q)$ (respectively $G = \mathrm{SO}^*(2n)$), we let $\det^{\tilde{\eta}/2}$ denote the character of M which is trivial on $\mathrm{Sp}(p',q')$ (respectively $\mathrm{SO}^*(2n')$) but which restricts to $\det^{\eta_i/2}$ on each $\mathrm{GL}(k_i,\mathbb{H})$ factor. Finally we set

$$\mathcal{I}_P(\mathcal{O};\vec{\eta}) = {}^n\mathrm{Ind}_P^G(\det^{\tilde{\eta}/2}),$$

where the induction is normalized (Section 2.5).

The following lemma is immediate.

Lemma 6.7. *Retain the notation and setting of Definition 6.6. In particular, fix P. Fix also $\vec{\eta},\vec{\eta}_1 \in \{\pm1\}^s$. Then there is a parabolic subgroup P' satisfying the following (1)-(3).*

(1) There is a Levi decomposition $P = LN'$.

(2) P' is $\mathrm{Ad}(G)$-conjugate to P.

(3) $\mathcal{I}_P(\mathcal{O};\vec{\eta}_1) \cong \mathcal{I}_{P'}(\mathcal{O};\vec{\eta})$.

The Harish-Chandra module $\mathcal{I}_P(\mathcal{O};\vec{\eta})$ may depend on the choice of the polarization P and the additional data $\vec{\eta} \in \{\pm1\}^s$. However, a result of Harish-Chandra tells us that the distribution character of a parabolically induced representation from an admissible representation of a Levi subgroup does not depend on the choice of polarization. Hence we have the following result.

Proposition 6.8. *We retain the setting and notation of Definition 6.6. Let $\vec{\eta},\vec{\eta}_1 \in \{\pm1\}^s$ and let P and P' be parabolic subgroups with a common Levi subgroup M. Then the distribution character $[\mathcal{I}_P(\mathcal{O};\vec{\eta})]$ coincides with $[\mathcal{I}_{P'}(\mathcal{O};\vec{\eta}_1)]$ and hence depends only on $\mathcal{O}$.*

Proposition 6.9. *Retain the setting and notation of Definition 6.6 and recall Definition 5.13. Then*

$$\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(\mathcal{I}_P(\mathcal{O};\vec{\eta})) = I_{\mathcal{O}}.$$

Proof. The associated variety of the annihilator of a representation induced from π_M is the orbit induced from the associated variety of the annihilator of π_M. Hence the associated variety of the annihilator of $\mathcal{I}_P(\mathcal{O};\vec{\eta})$ is $\overline{\mathcal{O}}$. Since there is a perfect pairing between $\mathcal{I}_P(\mathcal{O};\vec{\eta})$ and $M_{\mathfrak{p}}(\delta(\mathfrak{m})-[\vec{\eta}])$, we see, from that the annihilator of $\mathcal{I}_P(\mathcal{O};\vec{\eta})$ is dual to $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{p}}(\delta(\mathfrak{m}) - [\vec{\eta}]))$. From the argument in the proof of Proposition 5.12 we see $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(M_{\mathfrak{p}}(\delta(\mathfrak{m})-[\vec{\eta}]))$ coincides with the annihilator of an irreducible generalized Verma module. Hence $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(\mathcal{I}_P(\mathcal{O};\vec{\eta}))$ is a primitive ideal. Since $\mathcal{I}_P(\mathcal{O};\vec{\eta})$ has infinitesimal character $\lambda(\mathcal{O})$, the current proposition follows from Proposition 5.12. □

As a consequence, we conclude that the degenerate principal series of Definition 6.6 are a source of unitary weakly unipotent representations.

Theorem 6.10. *Retain the setting and notation of Definition 6.6. Each irreducible constituent of $I_P(\mathcal{O};\vec{\eta})$ is unitarizable. Moreover, if $\mathcal{O}$ is not very even (Remark 5.16), then each irreducible constituent of $I_P(\mathcal{O};\vec{\eta})$ is weakly unipotent (Definition 5.1).*

Proof. We seek to show that ${}^n\mathrm{Ind}_P^G((\det^{\tilde{\eta}/2})^t)$ is irreducible for $0 \leqslant t < 1$. Then from a standard argument using a Jantzen filtration (for example see [V2]), we can deduce that each irreducible constituent of $I_P(\mathcal{O};\vec{\eta})$ is unitarizable. From [Ma4, Corollary 5.1.2], ${}^n\mathrm{Ind}_P^G(\mathbb{1})$ is irreducible; this handles the case of $t = 0$. Using [Ma4, Lemma 4.1.3], we can reduce the irreducibility of ${}^n\mathrm{Ind}_P^G((\det^{\tilde{\eta}/2})^t)$ for $0 < t < 1$ to that of ${}^n\mathrm{Ind}_{\bar{P}}^{\bar{G}}((\det^{1/2})^t A)$ where $\bar{G} = \mathrm{GL}(k_1 + \cdots + k_s, \mathbb{H})$, $\bar{P}$ is a parabolic subgroup of $\bar{G}$ with a Levi part $\bar{M} = \mathrm{GL}(k_1, \mathbb{H}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{H})$, and $\det^{1/2}$ is a one-dimensional representation of $\bar{M}$ which restricts to $\det^{1/2}$ on each $\mathrm{GL}(k_i, \mathbb{H})$ factor. Tensoring the one-dimensional representation $(\det^{1/2})^{-t}$ of $\bar{G}$ with ${}^n\mathrm{Ind}_{\bar{P}}^{\bar{G}}((\det^{1/2})^t)$ gives the unitarily induced representation ${}^n\mathrm{Ind}_{\bar{P}}^{\bar{G}}(\mathbb{1})$, which is irreducible ([V3], for example). We have thus shown that each irreducible constituent of $I_P(\mathcal{O};\vec{\eta})$ is indeed unitarizable.

From Proposition 5.12, 5.14, and 6.9, any irreducible constituent of $I_P(\mathcal{O};\vec{\eta})$ is weakly unipotent for each orbit $\mathcal{O}$ which is not very even. The proof is complete. □

As preparation for Theorem 6.12, we need the following result.

Proposition 6.11. *Let Z be an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module with an infinitesimal character in $\mathcal{P}_{G_{\mathbb{C}}}$ (with notation as in Section 2.1). Fix a Richardson orbit $\mathcal{O}$ as in Corollary 6.1 and assume that $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(Z) \in \mathrm{Prim}(\mathcal{O})$ (Definition 5.3). Then V is quasi P-cofinite (Definition 3.9). In particular, each $A(\mathcal{O}_K^j)$ (defined before Proposition 6.2) is quasi P-cofinite.*

Proof. Let Z be as in the proposition. Recall the notation of Section 2.6. From the translation principle, we see that there is some $[Y] \in \mathbb{B}$ such that $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(Y) \in \mathrm{Prim}(\mathcal{O})$ and $Y \sim Z$. From Lemma 3.10, we have only to show that Y is quasi P-cofinite.

It follows from [Mc] that that there exists a j so that Y and $A(\mathcal{O}_K^j)$ belong to the same cell. (McGovern actually proves that the number of cells consisting of representations annihilated by an element of $\mathrm{Prim}(\mathcal{O})$ is exactly the number r of elements in $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{s}) = \{\mathcal{O}_K^1, \cdots, \mathcal{O}_K^r\}$. The r representations $A(\mathcal{O}_K^j$ each have different associated varieties, and since associated varieties are constant on cells, they must belong to distinct cells. Hence each cell of representations whose annihilator is in $\mathrm{Prim}(\mathcal{O})$ contains a unique element of the form $A(\mathcal{O}_K^j)$.) Let $\mathcal{C}^j$ denote the cell containing $A(\mathcal{O}_K^j)$.

Let X be an irreducible constituent of ${}^n\mathrm{Ind}_P^G(\mathbb{1})$ such that $\mathcal{O}_K^j$ is dense in an irreducible component of $\mathrm{AV}(X)$; such an X exists by the computation of Proposition 3.3. Since $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(X) = \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}({}^n\mathrm{Ind}_P^G(\mathbb{1})) \in \mathrm{Prim}(\mathcal{O})$, the discussion of the previous paragraph implies that X is an element of $\mathcal{C}^j$. From Lemma 3.10, we conclude Y is quasi P-cofinite. The proposition follows. □

Theorem 6.10 showed that the degenerate principal series of Definition 6.6 were a source of weakly unipotent representations attached to $\mathcal{O}$. The following theorem implies that all such unipotent representations are obtained this way (apart from the very even case).

Theorem 6.12. *Retain the notation of Definition 6.6 and Proposition 6.2.*

(1) In the notation introduced at the end of Section 2.4,

$$\mathcal{I}_P(\mathcal{O};\vec{\eta}) \approx \bigoplus_{j=1}^{r} A(\mathcal{O}_K^j).$$

That is, $\mathcal{I}_P(\mathcal{O};\vec{\eta})$ has a composition series in which each $A(\mathcal{O}_K^j)$ appears exactly once. Moreover, $A(\mathcal{O}_K^1), ..., A(\mathcal{O}_K^r)$ exhaust the irreducible constituent of maximal Gelfand-Kirillov dimension.

(2) Suppose X is an integral weakly unipotent representation attached to $\mathcal{O}$ (Definition 5.1). Assume that $\mathcal{O}$ is not very even (Remark 5.16). Then there exists j such that

$$X \cong A(\mathcal{O}_K^j).$$

(3) Suppose X is a distinguished unipotent representation attached to $\mathcal{O}$ (Definition 5.1). Then there exists j such that

$$X \cong A(\mathcal{O}_K^j).$$

Proof. Suppose X is any representation whose infinitesimal character lies in $\mathcal{P}_{G_\mathbb{C}}$ and whose annihilator lies in $\mathrm{Prim}(\mathcal{O})$. Then Proposition 6.11 implies that there is some finite-dimensional representation F such that X is isomorphic to an irreducible constituent of ${}^n\mathrm{Ind}_P^G(F)$. But at the infinitesimal character $\lambda(\mathcal{O})$, ${}^n\mathrm{Ind}_P^G(F)$ must be of the form $\mathcal{I}_P(\mathcal{O};\vec{\eta})$ given in Definition 6.6, and we know from Proposition 6.8 that all such representations have the same distribution character.

Take $X = A(\mathcal{O}_K^j)$ in the preceding paragraph. We conclude that each $A(\mathcal{O}_K^j)$ appears as a composition factor of each $\mathcal{I}_P(\mathcal{O};\vec{\eta})$. Corollary 3.2 says each $A(\mathcal{O}_K^j)$ appears exactly once. Propositions 3.3(1) and 6.2 and the additivity of the associated cycle on modules of the same maximal GK dimension shows there can be no other constituents of maximal GK dimension. We thus obtain the first assertion of the current proposition. Now parts (2) and (3) follow from (1) and the previous paragraph. $\square$

Using the translation principle, it turns out that we deduce from the "most singular" case of Theorem 6.12 results about other less singular infinitesimal characters. This is the content of Proposition 6.13. To formulate it, we need further notation.

For each $1 \leqslant j \leqslant r$, fix a θ-stable parabolic subalgebra $\mathfrak{q}^j = \mathfrak{l}^j \oplus \mathfrak{u}^j$ as in Proposition 4.9. Since $\mathfrak{p}$ and $\mathfrak{q}^j$ are $\mathrm{Ad}(G_\mathbb{C})$-conjugate, we fix $g^j \in G_\mathbb{C}$ such that $\mathrm{Ad}(g^j)\mathfrak{p} = \mathfrak{q}^j$. Set $\mathfrak{h}^j = \mathrm{Ad}(g^j)\mathfrak{h}$. Let $\chi : M_\mathbb{C} \to \mathbb{C}^\times$ be a holomorphic one-dimensional representation. We define $\chi^j = \chi \circ \mathrm{Ad}(g^j)^{-1}$, a holomorphic character of the analytic subgroup $L_\mathbb{C}^j$ of $G_\mathbb{C}$ corresponding to $\mathfrak{l}^j$.

Write X_P for the complexified generalized flag variety $G_\mathbb{C}/P_\mathbb{C}$. For each $1 \leqslant j \leqslant r$, let $\mathcal{V}^j$ denote the (open) G-orbit in $G_\mathbb{C}/P_\mathbb{C}$ containing $\mathfrak{q}^j$. Proposition 4.9 implies that $\mathcal{V}^1, ..., \mathcal{V}^r$ exhaust all the good open G-orbits in X_P.

For $X \in \mathfrak{m}$, define

$$\delta_P(X) = \frac{1}{2}\mathrm{tr}\left(\mathrm{ad}_{\mathfrak{g}}(X)|_{\mathfrak{n}}\right).$$

Then δ is a one-dimensional representation of $\mathfrak{m}$, and $2n\delta$ lifts to a holomorphic group homomorphism $\xi_{2n\delta_P} : M_{\mathbb{C}} \to \mathbb{C}^\times$ for any $n \in \mathbb{Z}$. As usual, we may regard $\xi_{2\delta}$ as a one-dimensional representation of P. With this notation $\xi_0 = \xi_{0\delta_P}$ is the trivial representation of P.

Let ω_P be the canonical line bundle on X_P and let $\mathcal{L}_\chi$ denote the homogeneous holomorphic line bundle on X_P associated to the character χ. So, for instance, $\omega_P = \mathcal{L}_{\xi_{2\delta_P}}$. For a character χ and $1 \leqslant j \leqslant r$, we may consider the derived functor module

$$A_{\mathcal{V}^j}(\chi) = A_{\mathfrak{q}^j}(\chi^j) = H^{\dim(\mathfrak{k}\cap\mathfrak{u}^j)}(\mathcal{V}, \mathcal{L}_\chi \otimes \omega_P)_{K\text{-finite}}. \tag{6.4}$$

Fix a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{p}$ and $\mathfrak{h} \subseteq \mathfrak{m}$. Let $\Delta^+(\mathfrak{g},\mathfrak{h})$ denote the positive system of $\Delta(\mathfrak{g},\mathfrak{h})$ corresponding to $\mathfrak{b}$.

Let χ be a holomorphic character of $M_{\mathbb{C}}$ and write $d\chi$ for the differential of χ. In addition, write $d\chi$ for the restriction of $d\chi : \mathfrak{m} \to \mathbb{C}$ to $\mathfrak{h}$. We call χ P-good (respectively P-weakly fair) if $\langle d\chi + \rho, \alpha^\vee\rangle > 0$ (respectively $\langle d\chi + \delta_P, \alpha^\vee\rangle \geqslant 0$) for all $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})$. If χ is P-good (respectively P-weakly fair), then $A_{\mathcal{V}^j}(\chi)$ is in the good range (respectively the weakly fair range) in the sense of [V4].

Proposition 6.13. *Retain the notation introduced just before the proposition (especially that of (6.4).) In particular, let χ be a P-weakly fair holomorphic character of $M_{\mathbb{C}}$. Let $\mathcal{I}_P(\chi)$ denote the degenerate principal series representation ${}^n\mathrm{Ind}_P^G(\mathbb{C}_{\delta_P} \otimes \chi)$. Then (in the notation introduced at the end of Section 2.4),*

$$\mathcal{I}_P(\chi) \approx \bigoplus_{j=1}^{r} A_{\mathcal{V}^j}(\chi).$$

Proof. First, we remark that for any $\vec{\eta} \in \{\pm 1\}^s$, $\det^{\tilde{\eta}/2} \otimes \xi_{-\delta_P}$ can be extended to a holomorphic character of $M_{\mathbb{C}}$ (say $\chi(\vec{\eta})$). Then we easily see $\mathcal{I}_P(\mathcal{O},\vec{\eta}) = \mathcal{I}_P(\chi(\vec{\eta}))$. Moreover, if we appropriately choose $\vec{\eta} \in \{\pm 1\}^s$, then $\det^{\tilde{\eta}/2} \otimes \mathbb{C}_{-\delta_P}$ is P-weakly fair. Hereafter we fix such a choice for $\vec{\eta}$. We need the following lemma, which is well-known ([V4], [V6], etc.).

Lemma 6.14. *For each P-weakly fair χ, we have $T_{d\chi+\rho}^{d\chi(\vec{\eta})+\rho}(A_{\mathcal{V}^j}(\chi)) = A_{\mathcal{V}^j}(\chi(\vec{\eta})) = A(\mathcal{O}_K^j)$. Here, $T_{d\chi'+\rho}^{d\chi+\rho}$ means the translation functor from $d\chi+\rho$ to $d\chi(\vec{\eta})+\rho$ defined by tensoring with the irreducible finite-dimensional representation with lowest weight $d\chi(\vec{\eta}) - d\chi$.*

Continuation of the proof of Proposition 6.13. From Corollary 3.2 and arguing as in the proof of Theorem 6.12, we have only to show $A_{\mathcal{V}^j}(\chi)$ appears in $\mathcal{I}_P(\chi)$ as an irreducible constituent. Let Z be an irreducible constituent of $\mathcal{I}_P(\chi)$. From the proof of Proposition 6.11, we know that Z lies in the same cell as some $A_{\mathcal{V}^j}(\chi)$. Thus $\mathrm{AV}(Z) = \overline{\mathcal{O}_K^j}$ for some j. Put $Z' = T_{d\chi+\rho}^{d\chi(\vec{\eta})+\rho}(Z)$. Since the differential of $\det^{\tilde{\eta}/2}$ is $[\vec{\eta}]$, we have $d\chi(\vec{\eta}) + \rho = \delta(\mathfrak{m}) + [\vec{\eta}]$. Since $\mathrm{AV}(Z') = \overline{\mathcal{O}_K^j}$, we conclude $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(Z') \in \mathrm{Dist}_{G_{\mathbb{C}}}(\mathcal{O})$. Theorem 6.12 (2) implies that $Z' \cong A(\mathcal{O}_K^j)$. On the other hand, from Lemma 6.14, we have $A(\mathcal{O}_K^j) = T_{d\chi+\rho}^{d\chi(\vec{\eta})+\rho}(A_{\mathcal{V}^j}(\chi))$. Therefore, from [V6, Proposition 7.7], we have $Z \cong A_{\mathcal{V}^j}(\chi)$. □

As a corollary of the above theorem, we obtain a nice characterization of constituents of maximal GK dimension in integral degenerate principal series.

Corollary 6.15. *Let P be a parabolic subgroup of G and let X be a representation of G parabolically induced from a one-dimensional representation of P. Assume that the infinitesimal character of X lies in $\mathcal{P}_{G_{\mathbb{C}}}$. Then any irreducible constituent of X of the maximal Gelfand-Kirillov dimension is a derived functor module in the weakly fair range.*

Proof. We may write X as $\mathcal{I}_P(\chi)$ for some holomorphic character. We easily see that there is some parabolic subgroup P' whose Levi part coincides with that of P such that χ is P'-weakly fair. Since $\mathcal{I}_P(\chi)$ and $\mathcal{I}_{P'}(\chi)$ have the same distribution character (Proposition 6.8), the corollary follows from Proposition 6.13. □

Remark 6.16. In the proof of Corollary 6.15, P' is not necessarily $\mathrm{Ad}(G)$-conjugate to P. So, although an irreducible constituent of X in Corollary 6.15 can be written in the form $A_{\mathfrak{q}}(\lambda)$, $\mathfrak{q}$ is not necessarily $\mathrm{Ad}(G_{\mathbb{C}})$-conjugate to $\mathfrak{p}$. This means that $A_{\mathfrak{q}}(\lambda)$ is not necessarily attached to an open G-orbit in the complexified generalized flag manifold X_P.

To conclude this section, we turn our attention to edge-of-wedge type embeddings. We begin with the following lemma in the general setting. In its statement, we let M^h denote the Hermitian dual of a Harish-Chandra module M (e.g. [KnV, Section VI.2]).

Lemma 6.17. *Let G be a real reductive group. Let I be a Harish-Chandra $(\mathfrak{g}, K)$-module and let $\Psi : I^h \to I$ be a $(\mathfrak{g}, K)$-homomorphism. We invoke the following three assumptions:*

(a1) $\mathrm{Dim}(\mathrm{Kernel}(\Psi)) < \mathrm{Dim}(I^h)$.

(a2) Let V be an arbitrary irreducible constituent of I such that $\mathrm{Dim}(V) = \mathrm{Dim}(I)$. *Then, $V^h \cong V$ and the multiplicity of V in I is one.*

(a3) Let W be any submodule of I, then $\mathrm{Dim}(W) = \mathrm{Dim}(I)$.

Let $\mathrm{Socle}(I)$ *denote the largest semisimple submodule of I. Then*

(1) $\mathrm{Socle}(I) = \Psi(I^h)$.

(2) $\mathrm{Dim}(I/\mathrm{Socle}(I)) < \mathrm{Dim}(I)$.

In particular, any irreducible constituent Y of I such that $\mathrm{Dim}(Y) = \mathrm{Dim}(I)$ *is a submodule of I.*

Proof. Let Y be any irreducible submodule of I. From (a3), we have $\mathrm{Dim}(Y) = \mathrm{Dim}(I)$. From (a1), we see $\Psi(I^h)$ contains an irreducible constituent isomorphic to Y. But since the multiplicity of Y in I is one (by (a2)), it follows that indeed Y is in the image $\Psi(I^h)$. Hence $\mathrm{Socle}(I) \subseteq \Psi(I^h)$.

In order to show (1), we thus have only to show that $\Psi(I^h) \cong I^h/\mathrm{Kernel}(\Psi)$ is semisimple. Let W be any irreducible quotient of $I^h/\mathrm{Kernel}(\Psi)$. Then W is also a quotient of I^h and so W^h is realized as a submodule of I. According to (a2), $W \simeq W^h$, and thus from the previous paragraph, we conclude that the image $\Psi(I^h)$ contains a submodule isomorphic to W. By (a2), W has multiplicity one in $\Psi(I^h)$. We conclude that any irreducible quotient W of $\Psi(I^h)$ is also a submodule. Thus $\Psi(I^h)$ is semisimple and (1) follows. Assertion (2) follows from (a1) and (1). □

We consider the following special case. Fix a parabolic subgroup P of G with a θ-stable Levi part M. Consider the unnormalized generalized Verma modules as above,

$$^u M_{\mathfrak{p}}(\mu) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathbb{C}_\mu.$$

Let χ be a holomorphic character of $M_{\mathbb{C}}$. Recall that there is a perfect pairing between $^u M_{\mathfrak{p}}(-2\delta_P - d\chi)$ and $\mathcal{I}_P(\chi)$ (where $\mathcal{I}(\chi)$ is defined in Proposition 6.13).

Here is an edge-of-wedge embedding result for $\mathrm{Sp}(p,q)$.

Theorem 6.18. *Let $G = \mathrm{Sp}(p,q)$ and fix a parabolic subgroup $P = MN$ with*

$$M = \mathrm{GL}(k_1, \mathbb{H}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{H}) \times \mathrm{Sp}(p', q').$$

For each positive integer ℓ, let $m_M(\ell)$ be the number of i between 1 and s such that $k_i = \ell$. Assume

(C1) *For each $\ell > p' + q'$, $m_M(\ell)$ is even.*

Let χ be a P-weakly fair holomorphic character of $M_{\mathbb{C}}$. Then,

(a) There is an embedding of generalized Verma modules

$$\psi_\chi : {^u M_{\mathfrak{p}}}(-2\delta_P - d\chi) \hookrightarrow {^u M_{\mathfrak{p}}}(d\chi).$$

This induces an intertwining operator

$$\Psi_\chi : \mathcal{I}_P(\chi^{-1} \otimes \xi_{-2\delta_P}) \to \mathcal{I}_P(\chi).$$

(cf. [CS, 2]).

(b) We have $\mathrm{Socle}(\mathcal{I}_P(\chi)) = \Psi_\chi(\mathcal{I}_P(\chi^{-1} \otimes \xi_{-2\delta_P}))$. *Moreover,* $\mathrm{Socle}(\mathcal{I}_P(\chi)) \approx \mathcal{I}_P(\chi)$ *(with notation as in the end of Section 2.4).*

(c) We have that the decomposition of Proposition 6.13 is indeed the socle,

$$\mathrm{Socle}(\mathcal{I}_P(\chi)) = \bigoplus_{j=1}^{r} A_{\mathcal{V}^j}(\chi).$$

Proof. Set $I = \mathcal{I}_P(\chi)$. Then $I^h = \mathcal{I}(\chi^{-1} \otimes \xi_{-2\delta_P})$. Part (a) is obtained in [Ma5, Theorem 5.1.2]. (This is where assumption (C1) is needed.) Thus to establish the theorem, we need only verify that the assumptions (a1)-(a3) of Lemma 6.17 are satisfied. Since Kernel(Ψ) has a perfect pairing with ${}^uM_{\mathfrak{p}}(d\chi)/{}^uM_{\mathfrak{p}}(-2\delta_P - d\chi)$ and $\mathrm{Dim}({}^uM_{\mathfrak{p}}(d\chi)/M_{\mathfrak{p}}(-2\delta_P - d\chi)) < \mathrm{Dim}(M_{\mathfrak{p}}(-2\delta_P - d\chi)) = \mathrm{Dim}(I)$, we have $\mathrm{Dim}(\mathrm{Kernel}(\Psi)) < \mathrm{Dim}(I)$. This is (a1).

Condition (a2) follows from Proposition 6.13. Moreover, since ${}^uM_{\mathfrak{p}}(-2\delta_P - d\chi)$ is irreducible, any submodule of I also has a perfect pairing with $M_{\mathfrak{p}}(-2\delta_P - d\chi)$. This implies that any submodule of I has the same Gelfand-Kirillov dimension as I. This is (a3) and the proof is complete. □

The following result for $\mathrm{SO}^*(2n)$ is obtained in a similar way.

Theorem 6.19. *Let $G = \mathrm{SO}^*(2n)$ and let $P = MN$ be a parabolic subgroup with*

$$M = \mathrm{GL}(k_1, \mathbb{H}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{H}) \times \mathrm{SO}^*(2n').$$

For each positive integer ℓ, let $m_M(\ell)$ be the number of i between 1 and s such that $k_i = \ell$. Assume that

(C2) *For each $\ell \leq n'$, $m_M(\ell)$ is even.*

Let χ be a P-weakly fair holomorphic character of $M_{\mathbb{C}}$. Then, the conclusions (a)-(c) of Theorem 6.18 hold in the present context.

7 U(m, n)

For $\mathrm{U}(m,n)$, Theorems 7.3 and 7.4 below establish results analogous to those proved for $\mathrm{Sp}(p,q)$ in Corollary 6.15 and Theorem 6.18. It is possible to imitate the approach of Section 6 in the context of $\mathrm{U}(p,q)$, and in particular it is possible to define a version of the representation $\mathcal{I}_P(\mathcal{O})$ considered in Definition 6.6. However, such a $\mathcal{I}_P(\mathcal{O})$ is not necessarily a class one degenerate principal series. Moreover, the argument leading to the proof of Corollary 6.15 cannot be imitated for $\mathrm{U}(m,n)$. In this section we instead prove the main results of Theorems 7.3 and 7.4 in a different way.

Henceforth in this section we fix $G = \mathrm{U}(m,n)$. We fix a maximal compact subgroup $K \simeq \mathrm{U}(m) \times \mathrm{U}(n)$ of G and let θ denote the corresponding Cartan involution. As usual, we let $\mathfrak{g}$ (resp. $\mathfrak{k}$) denote the complexified Lie algebra of G (resp. K); so $\mathfrak{g} \cong \mathfrak{gl}(m+n, \mathbb{C})$ and $\mathfrak{k} \cong \mathfrak{gl}(m, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$.

Let P be a parabolic subgroup of G and choose a θ-stable maximally split Cartan subgroup ${}^sH$ of G such that ${}^sH \subseteq P$. (We remark that all Cartan subgroups of G are connected.) Let M denote the θ-stable Levi part of P (and so ${}^sH \subseteq M$), and write N for the nilradical of P. We let ${}^s\mathfrak{h}$ the complexified Lie algebra of ${}^sH$.

Fix a Borel subalgebra $\mathfrak{b}$ such that ${}^s\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{p}$. For simplicity, write Δ for $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$. We let Δ^+ denote the system of positive roots corresponding to $\mathfrak{b}$ and let Π denote the corresponding basis of Δ. We choose an orthonormal basis $e_1, ..., e_{m+n}$ of ${}^s\mathfrak{h}^*$ so that

$$\Delta = \{e_i - e_j \mid 1 \leqslant i, j \leqslant m+n, i \neq j\},$$

$$\Pi = \{e_1 - e_2, ..., e_{m+n-1} - e_{m+n}\}.$$

We modify our original choice of $\mathfrak{b}$ appropriately so that $\theta(e_i) = e_{m+n-i+1}$ for all $1 \leqslant i \leqslant \min\{m,n\}$ and $\theta(e_i) = e_i$ for all $\min\{m,n\} < i < m+n-\min\{m,n\}$.

Let $\kappa = (k_1, ..., k_s)$ be a finite sequence of positive integers such that

$$k = k_1 + \cdots + k_s \leqslant \min\{m,n\}.$$

Put $k_i^* = k_1 + \cdots + k_i$ for $1 \leqslant i \leqslant s$ and $k_0^* = 0$. Hence $k_s^* = k$. Set $m' = m-k$ and $n' = n-k$. Define a subset $S(\kappa)$ of Π as follows,

$$S(\kappa) = \Pi - \left\{e_{k_i^*} - e_{k_i^*+1}, e_{m+n-k_i^*} - e_{m+n-k_i^*+1} \;\middle|\; 1 \leqslant i \leqslant s\right\}.$$

We easily see that there exists some $\kappa = (k_1, ..., k_s)$ as above such that $\mathfrak{p}$ is the standard parabolic subalgebra containing the Borel subalgebra $\mathfrak{b}$ corresponding to $S(\kappa)$, i.e. such that $S(\kappa)$ is a basis of $\Delta(\mathfrak{m}, {}^s\mathfrak{h}) \cap \Delta^+$.

Formally, we let $\mathrm{U}(0,0)$ denote the trivial group $\{1\}$ and we let $\mathrm{GL}(\kappa, \mathbb{C})$ denote the product $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{C})$. Then,

$$M_\kappa \cong \mathrm{GL}(\kappa, \mathbb{C}) \times \mathrm{U}(m', n'),$$

and $\mathrm{GL}(\kappa, \mathbb{C})$ and $\mathrm{U}(m', n')$ can be identified with subgroups of M. The Cartan involution θ induces Cartan involutions on M, $\mathrm{GL}(\kappa, \mathbb{C})$, $\mathrm{U}(m', n')$ and we denote them by the same letter θ.

Next we consider induced representations from characters of parabolic subgroups. Let μ and ν be complex numbers such that $\mu + \nu \in \mathbb{Z}$. We define a one-dimensional representation $(\eta^k_{\mu,\nu}, \mathbb{C}^k_{\mu,\nu})$ of $\mathrm{GL}(k, \mathbb{C})$ as follows.

$$\eta^k_{\mu,\nu}(g) = \det(g)^\mu \overline{\det(g)}^{-\nu} \quad (g \in \mathrm{GL}(k, \mathbb{C})).$$

For $h \in \mathbb{Z}$, we define a one-dimensional representation $(\eta^{p,q}_h, \mathbb{C}^{p,q}_h)$ of $\mathrm{U}(p,q)$ as follows.

$$\eta^{p,q}_h(g) = \det(g)^h \quad (g \in \mathrm{U}(p,q)).$$

Let $\mathbf{u} = (u_1, ..., u_s)$ and $\mathbf{v} = (v_1, ..., v_s)$ be sequences of complex numbers such that $u_i + v_i \in \mathbb{Z}$ for all $1 \leqslant i \leqslant s$. Let $h \in \mathbb{Z}$. We define a one-dimensional representation $\mathbb{C}^\kappa_{\mathbf{u};h;\mathbf{v}}$ of $\mathrm{GL}(k, \mathbb{C}) \times \mathrm{U}(m-k, n-k)$ by

$$\mathbb{C}^\kappa_{\mathbf{u};h;\mathbf{v}} = \mathbb{C}^{k_1}_{u_1,v_1} \boxtimes \cdots \boxtimes \mathbb{C}^{k_s}_{u_s,v_s} \boxtimes \mathbb{C}^{m-k,n-k}_h.$$

When we identify M with $\mathrm{GL}(\kappa, \mathbb{C}) \times \mathrm{U}(m-k, n-k)$, there are ambiguities arising from automorphisms of $GL(\kappa, \mathbb{C})$; i.e. any identification we choose may be twisted by complex conjugation. We choose the identification so that the differential of the restriction of $\mathbb{C}^\kappa_{\mathbf{u},h,\mathbf{v}}$ to ${}^sH$ is

$$\sum_{j=1}^{s} \sum_{i=1}^{k_j} [u_j e_{k^*_{j-1}+i} + v_j e_{m+n-k^*_{j-1}-i+1} \in {}^s\mathfrak{h}^*] + h \sum_{\ell=1}^{m+n-2k} e_{k+\ell};$$

i.e. we assign $e_1, ..., e_k$ (respectively $e_{m+n-k+1}, ..., e_{m+n}$) to the holomorphic (respectively anti-holomorphic) part. We consider the following degenerate principal series representation of G,

$$\mathcal{I}_P[\mathbf{u}; h; \mathbf{v}] = {}^n\mathrm{Ind}_P^G(\mathbb{C}^{\kappa}_{\mathbf{u};h;\mathbf{v}}). \tag{7.1}$$

We let $\delta^{\kappa}_{m,n}$ denote the element of $\mathbb{C}^s$ whose i-th entry is $\frac{m+n-k_i^*+k_i}{2}$. If $m = n = k$, we regard $\mathbb{C}^{0,0}_h$ as the trivial representation of the trivial group. In this case $\mathcal{I}_P[\mathbf{u}; h; \mathbf{v}]$ does not depend on h.

We let $J(\mathbf{u}, h, \mathbf{v})$ denote the annihilator of $\mathcal{I}_P[\mathbf{u}; h; \mathbf{v}]$ in the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$.

We define a normalized generalized Verma module as follows. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g} = \mathfrak{gl}(m+n)$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$. We define a one-dimensional representation $-\rho_{\mathfrak{p}}$ of $\mathfrak{l}$ by $-\rho_{\mathfrak{p}}(X) = \frac{1}{2}(\mathrm{ad}(X)|_{\mathfrak{n}})$ $(X \in \mathfrak{l})$. For a one-dimensional representation ξ of $\mathfrak{l}$ we extend $\xi \otimes -\rho_{\mathfrak{p}}$ to a one-dimensional representation of $\mathfrak{p}$ as usual and define ${}^nM_{\mathfrak{p}}(\xi) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} (\xi \otimes -\rho_{\mathfrak{p}})$.

We define a weight on ${}^s\mathfrak{h}$ by

$$\xi(\mathbf{u}, h, \mathbf{v}) = \sum_{i=1}^{s} u_i \left(\sum_{j=1}^{k_i} e_{k_{i-1}^*+j} \right) + h \sum_{j=1}^{m+n-2k} e_{k+j} + \sum_{i=1}^{s} v_{s-i} \left(\sum_{j=1}^{k_i} e_{m+n-k+k_{i-1}^*+j} \right).$$

Obviously, $\xi(\mathbf{u}, h, \mathbf{v})$ can be extend to a one-dimensional representation of $\mathfrak{l}$. We let $\bar{\mathfrak{p}}_{\kappa}$ denote the opposite parabolic subalgebra to $\mathfrak{p}_{\kappa}$.

Recall that ${}^n\mathcal{I}_P[\mathbf{u}; h; \mathbf{v}]$ and ${}^nM_{\mathfrak{p}_\kappa}(-\xi(\mathbf{u}, h, \mathbf{v}))$ admit perfect pairing. ${}^nM_{\bar{\mathfrak{p}}_\kappa}(\xi(\mathbf{u}, h, \mathbf{v}))$ and ${}^nM_{\mathfrak{p}_\kappa}(-\xi(\mathbf{u}, h, \mathbf{v}))$ also admit a perfect pairing. Hence we have:

Lemma 7.1. *$J(\mathbf{u}, h, \mathbf{v})$ coincides with the annihilator of ${}^nM_{\bar{\mathfrak{p}}_\kappa}(\xi(\mathbf{u}, h, \mathbf{v}))$.*

Let $\mathbf{c} = (c_1, ..., c_\ell)$ be a sequence of non-negative integers such that $c_1 + \cdots + c_\ell = m+n$. Put $c_i^* = c_1 + \cdots + c_i$ and $c_0^* = 0$. We define a subset $S[\mathbf{c}]$ of Π as follows,

$$S[\mathbf{c}] = \Pi - \{e_{c_i^*} - e_{c_i^*+1} | 1 \leqslant i \leqslant \ell\}.$$

We let $\mathfrak{p}(\mathbf{c})$ denote the standard parabolic subalgebra of $\mathfrak{g}$ associated to $S[\mathbf{c}]$. Under an appropriate realization of $\mathfrak{g}$ as $\mathfrak{gl}(m+n, \mathbb{C})$, $\mathfrak{p}(\mathbf{c})$ is the block-upper-triangular parabolic subalgebra of $\mathfrak{gl}(m+n, \mathbb{C})$ with blocks of sizes $c_1, ..., c_\ell$ along the diagonal. If $c_i = 0$ for some i, "a block of size 0" means nothing: we simply neglect it. We let $\bar{\mathfrak{p}}(\mathbf{c})$ denote the parabolic subalgebra opposite to $\mathfrak{p}(\mathbf{c})$. For a sequence $\mathbf{h} = (h_1, ..., h_\ell)$ of complex numbers, we define a weight on ${}^s\mathfrak{h}$ as follows.

$$\xi(\mathbf{h}) = \sum_{i=1}^{\ell} h_i \left(\sum_{j=1}^{c_i} e_{c_{i-1}^*+j} \right).$$

Fix a degenerate principal series representation ${}^n\mathcal{I}_P[\mathbf{u}; h; \mathbf{v}]$ with an integral infinitesimal character. In particular, $\mathbf{u}$ and $\mathbf{v}$ are fixed so that $\mathbf{u} - \delta^{\kappa}_{m,n}, \mathbf{v} + \delta^{\kappa}_{m,n} \in \mathbb{Z}^s$. We define

$\mathbf{h} = (h_1, ..., h_{2s+1})$ as follows.

$$h_1 = h,$$
$$h_{2i} = u_{s-i+1} \quad (1 \leqslant i \leqslant s),$$
$$h_{2i+1} = v_i \quad (1 \leqslant i \leqslant s).$$

We also define a sequence of positive integer $\mathbf{c}$ by

$$c_1 = m + n - 2k,$$
$$c_{2i} = c_{2i+1} = k_{s-i+1} \quad (1 \leqslant i \leqslant s).$$

Finally, we define

$$\Xi(\mathbf{u}, h, \mathbf{v}) = \{\tau \in \mathfrak{S}_{2s+1} \mid h_{\tau(1)} \geqslant \cdots \geqslant h_{\tau(2s+1)}\}.$$

Obviously $\Xi(\mathbf{u}, h, \mathbf{v})$ is non-empty. For $\tau \in \mathfrak{S}_{2s+1}$, put $\mathbf{c}^\tau = (c_{\tau(1)}, ..., c_{\tau(2s+1)})$ and $\mathbf{h}^\tau = (h_{\tau(1)}, ..., h_{\tau(2s+1)})$ From [BoJa, 4.10 Corollar] and Lemma 7.1, we have:

Proposition 7.2. *If $\tau \in \Xi(\mathbf{u}, h, \mathbf{v})$, then $J(\mathbf{u}, h, \mathbf{v})$ coincides with the annihilator of ${}^n M_{\bar{\mathfrak{p}}(\mathbf{c}^\tau)}(\xi(\mathbf{h}^\tau))$.*

From [V2], it follows ${}^n M_{\bar{\mathfrak{p}}(\mathbf{c}^\tau)}(\xi(\mathbf{h}^\tau))$ is irreducible for $\tau \in \Xi(\mathbf{u}, h, \mathbf{v})$. Hence we see that $J(\mathbf{u}, h, \mathbf{v})$ is a primitive ideal.

We now discuss θ-stable parabolic subalgebras with respect to G (cf. [V8, Example 4.5]). Let ℓ be a positive integer. Let $\mathbb{P}_\ell(m, n)$ denote the set

$$\left\{((m_1, ..., m_\ell), (n_1, ..., n_\ell)) \in \mathbb{N}^\ell \times \mathbb{N}^\ell \;\middle|\; \sum_{i=1}^{\ell} m_i = m, \sum_{i=1}^{\ell} n_i = n, \right.$$
$$\left. \text{and } m_j + n_j > 0 \text{ for all } 1 \leq j \leq \ell \right\}.$$

We also put $\mathbb{P}(m, n) = \bigcup_{\ell > 0} \mathbb{P}_\ell(m, n)$ and $\mathbb{P}(0, 0) = \mathbb{P}_0(0, 0) = \{(\emptyset, \emptyset)\}$. If $(\mathbf{m}, \mathbf{n}) \in \mathbb{P}(m, n)$ satisfies $(\mathbf{m}, \mathbf{n}) \in \mathbb{P}_\ell(m, n)$, we call ℓ the length of $(\mathbf{m}, \mathbf{n})$. For $(\mathbf{m}, \mathbf{n}) \in \mathbb{P}(m, n)$, we define

$$I_{(\mathbf{m}, \mathbf{n})} = \mathrm{diag}(I_{m_1}, -I_{n_1}, ..., I_{m_\ell}, -I_{n_\ell})$$

Here, for a positive integer I_k means the $k \times k$ identity matrix. (If $k = 0$, we simply ignore I_0.) Then we can consider the following realization of G,

$$G = \left\{ g \in \mathrm{GL}(m + n, \mathbb{C}) |{}^t\bar{g} I_{(\mathbf{m}, \mathbf{n})} g = I_{(\mathbf{m}, \mathbf{n})} \right\}.$$

Let θ be the Cartan involution given by conjugation by $I_{(\mathbf{m}, \mathbf{n})}$. In this realization, we let $\mathfrak{q}(\mathbf{m}, \mathbf{n})$ denote the block-upper-triangular parabolic subalgebra of $\mathfrak{g} = \mathfrak{gl}(m + n, \mathbb{C})$ with blocks of sizes $p_1 + q_1, ..., p_\ell + q_\ell$ along the diagonal. Then, $\mathfrak{q}(\mathbf{m}, \mathbf{n})$ is a θ-stable parabolic subalgebra. The corresponding Levi subgroup $\mathrm{U}(\mathbf{m}, \mathbf{n})$ consists of diagonal blocks,

$$\mathrm{U}(\mathbf{m}, \mathbf{n}) \cong \mathrm{U}(m_1, n_1) \times \cdots \times \mathrm{U}(m_\ell, n_\ell).$$

We let $\mathfrak{g}(\mathbf{m},\mathbf{n})$ denote the complexified Lie algebra of $\mathrm{U}(\mathbf{m},\mathbf{n})$, and write $\mathfrak{v}(\mathbf{m},\mathbf{n})$ the nilradical of $\mathfrak{q}(\mathbf{m},\mathbf{n})$.

Via the above construction of $\mathfrak{q}(\mathbf{m},\mathbf{n})$, $K_{\mathbb{C}}$-conjugacy classes of θ-stable parabolic subalgebras of G are parametrized by $\mathbb{P}(m,n)$.

Let $(\mathbf{m},\mathbf{n}) = ((m_1,...,m_\ell),(n_1,...,n_\ell)) \in \mathbb{P}_\ell(m,n)$ and $\mathbf{h} = (h_1,....,h_\ell) \in \mathbb{Z}^\ell$. We define $(\eta_{\mathbf{h}}, \mathbb{C}_{\mathbf{h}})$ to be the one-dimensional representation of $\mathrm{U}(\mathbf{m},\mathbf{n})$ which restricts to $\mathbb{C}_{h_i}^{m_i,n_i}$ on each $\mathrm{U}(m_i,n_i)$ factor. We let $\mathcal{A}_{(\mathbf{m},\mathbf{n})}[\mathbf{h}]$ denote the derived functor module

$$A_{\mathfrak{q}(\mathbf{m},\mathbf{n})}(\eta_{\mathbf{h}}).$$

Recall the terminology of the good, weakly fair, and mediocre ranges. (The first two are standard (e.g. [KnV, Introduction]), the third is introduced in [T1].) A simple check shows that $\mathcal{A}_{(\mathbf{m},\mathbf{n})}[\mathbf{h}]$ is in the good range for $\mathfrak{q}(\mathbf{m},\mathbf{n})$ if and only if $h_i \geqslant h_{i+1}$ for all $1 \leqslant i < \ell$, and $\mathcal{A}_{(\mathbf{m},\mathbf{n})}[\mathbf{h}]$ is a derived functor module in the weakly fair range for $\mathfrak{q}(\mathbf{m},\mathbf{n})$ if and only if

$$h_i - h_{i+1} \geqslant -\frac{m_i + n_i + m_{i+1} + n_{i+1}}{2} \quad (1 \leqslant i < \ell).$$

For $(\mathbf{m},\mathbf{n}) \in \mathbb{P}(m,n)$,

$$\mathrm{Dim}(\mathcal{A}_{\mathbf{m},\mathbf{n}}[\mathbf{h}]) \leqslant \frac{1}{2} \dim \mathcal{O}(\mathbf{m},\mathbf{n}).$$

We call $(\mathbf{m},\mathbf{n}) \in \mathbb{P}(m,n)$ normal if

$$\mathrm{Dim}(\mathcal{A}_{\mathbf{m},\mathbf{n}}[\mathbf{h}]) = \frac{1}{2} \dim \mathcal{O}(\mathbf{m},\mathbf{n})$$

holds in the good range. It is known that for a normal $(\mathbf{m},\mathbf{n}) \in \mathbb{P}_\ell(m,n)$ and any mediocre $\mathbf{h} \in \mathbb{Z}^\ell$, we have

$$\mathrm{Dim}(\mathcal{A}_{\mathbf{m},\mathbf{n}}[\mathbf{h}]) = \frac{1}{2} \dim \mathcal{O}(\mathbf{m},\mathbf{n}) \text{ and } \mathrm{AV}(\mathcal{A}_{\mathbf{m},\mathbf{n}}[\mathbf{h}]) = \mathrm{AV}(\mathcal{A}_{\mathbf{m},\mathbf{n}}[\mathbf{h}']),$$

where $\mathbf{h}'$ is any parameter in the good range (cf. [T1]).

Fix a sequence of positive integers $\mathbf{c} = (c_1,...,c_\ell)$ such that $c_1 + \cdots + c_\ell = m+n$. Put

$$\mathbb{O}(\mathbf{c}) = \{(\mathbf{m},\mathbf{n}) \in \mathbb{P}_\ell(m,n) \mid (\mathbf{m},\mathbf{n}) \text{ is normal and } m_i + n_i = c_i \text{ for all } 1 \leqslant i \leqslant \ell\}$$

This set is of importance when determining the irreducible constituents of degenerate principal series of maximal Gelfand-Kirillov dimension. We take that up now.

Theorem 7.3. *In the notation of this section, consider a degenerate principal series representation of the form $\mathcal{I}_P[\mathbf{u};h;\mathbf{v}]$ whose infinitesimal character is a weight-lattice translate of the infinitesimal character of the trivial representation. Define $\mathbf{c}$ and $\mathbf{h}$ from $(\mathbf{u},h,\mathbf{v})$ as above. For $\tau \in \Xi(\mathbf{u},h,\mathbf{v})$ (and using the notation at the end of Section 2.4), we have*

$$\mathcal{I}_P[\mathbf{u};h;\mathbf{v}] \approx \bigoplus_{(\mathbf{m},\mathbf{n}) \in \mathbb{O}(\mathbf{c}^\tau)} \mathcal{A}_{(\mathbf{m},\mathbf{n})}[\check{\mathbf{h}}^\tau];$$

here $\check{\mathbf{h}}^\tau = (\check{h}_1^\tau,...,\check{h}_{2s+1}^\tau)$ is defined by

$$\check{h}_i^\tau = h_{\tau(i)} - \frac{m+n-c_{\tau(i)}}{2} + c^{\tau *}_{i-1} \quad \textit{for } 1 \leqslant i \leqslant 2s+1,$$

where

$$c^{\tau *}_{i-1} = c_{\tau(1)} + \cdots + c_{\tau(i-1)} \qquad \textit{for } 2 \leqslant i \leqslant 2s+1,$$

and $c^{\tau *}_0 = 0$.

Proof. Applying the translation principle into the weakly fair range ([V2], [V4]), we may assume that $h_{\tau(1)} \gg \cdots \gg h_{\tau(2s+1)}$.

For simplicity, we say that an infinitesimal character is ρ-integral if it is a weight lattice translate of the infinitesimal character of the trivial representation. From [BaV2] (see also [T1, Section 6]) it follows that

(∗) If J is a primitive ideal with ρ-integral infinitesimal character, and $\mathcal{O}_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})$ where $\mathcal{O}$ is dense in the associated variety of J, then there is a unique irreducible Harish-Chandra module $A(J, \mathcal{O}_K)$ whose annihilator is J and whose associated variety is the closure of $\mathcal{O}_K$.

Now let $\mathcal{O}$ denote the Richardson orbit associated to $\mathfrak{p}$. From the discussion, $J(\mathbf{u}, h, \mathbf{v})$ is primitive and its associated variety is the closure of $\mathcal{O}$. According to the definitions it follows that every constituent of maximal GK dimension in $\mathcal{I}_P[\mathbf{u}; h; \mathbf{v}]$ is annihilated by $J(\mathbf{u}, h, \mathbf{v})$. Thus if we set $J = J(\mathbf{u}, h, \mathbf{v})$ and write the associated cycle of $\mathcal{I}_P[\mathbf{u}; h; \mathbf{v}]$ according to Proposition 3.3(1) as

$$\sum_{\mathcal{O}^j_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})} [\mathcal{O}^j_K],$$

item (∗) and the additivity of the associated cycle on constituents of maximal GK dimension immediately implies that

$$\mathcal{I}_P[\mathbf{u}; h; \mathbf{v}] \approx \bigoplus_{\mathcal{O}^j_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})} A(J, \mathcal{O}^j_K). \tag{7.2}$$

From [V3, Proposition 16.8], we conclude that the annihilator of $\mathcal{A}_{(\mathbf{m},\mathbf{n})}[\check{\mathbf{h}}^\tau]$ contains the annihilator of ${}^n M_{\bar{\mathfrak{p}}(\mathbf{c}^\tau)}(\xi(\mathbf{h}^\tau))$. Since $\mathrm{Dim}(\mathcal{A}_{(\mathbf{m},\mathbf{n})}[\check{\mathbf{h}}^\tau]) = \frac{1}{2} \dim \mathcal{O}$, Proposition 7.2 implies that the annihilator of $\mathcal{A}_{(\mathbf{m},\mathbf{n})}[\check{\mathbf{h}}^\tau]$ coincides with J. On the other hand, it is easy to check (using the algorithm of [T1, Section 5], for instance) that

$$\left\{ \mathrm{AV}(\mathcal{A}_{(\mathbf{m},\mathbf{n})}[\check{\mathbf{h}}^\tau]) \;\middle|\; (\mathbf{m}, \mathbf{n}) \in \mathbb{O}(\mathbf{c}^\tau) \right\} = \mathrm{Irr}(\mathcal{O} \cap \mathfrak{s}).$$

Thus

$$\bigoplus_{\mathcal{O}^j_K \in \mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})} A(J, \mathcal{O}^j_K) = \bigoplus_{(\mathbf{m},\mathbf{n}) \in \mathbb{O}(\mathbf{c}^\tau)} \mathcal{A}_{(\mathbf{m},\mathbf{n})}[\check{\mathbf{h}}^\tau],$$

and the theorem now follows from (7.2). □

Next, we consider the case of the P-weakly fair range. (The terminology "P-weakly fair" is introduced just before Proposition 6.13.) In that case, we have a result analogous to Theorem 6.18: the socle of the degenerate principal series is just the direct sum of the irreducible constituents of maximal Gelfand-Kirillov dimension and, moreover, each constituent corresponds to an open orbit in the complexified generalized flag manifold. The following result is proved in the same way as Theorem 6.18.

Theorem 7.4. *Assume* $\mathbf{u} = (u_1, ..., u_s), \mathbf{v} = (v_1, ..., v_s) \in \mathbb{Z}^s$ *and* $h \in \mathbb{Z}$ *satisfy*

$$u_1 \geqslant \cdots \geqslant u_s \geqslant h \geqslant v_s \geqslant \cdots \geqslant v_1.$$

Set

$$\mathbf{c}(m, n, \kappa) = (k_1, ..., k_s, m + n - 2k_s^*, k_s, ..., k_1) \in \mathbb{N}^{2s+1}.$$

Let $\delta_{m,n}^{\kappa}$ *denote the element of* $\mathbb{C}^s$ *whose the i-th entry is* $\frac{m+n-k_{i-1}^*}{2}$, *where* $k_i^* = k_1 + \cdots + k_i$.

(1) We have

$$\operatorname{Socle}(\mathcal{I}_P[\mathbf{u} + \delta_{m,n}^{\kappa}; h; \mathbf{v} - \delta_{m,n}^{\kappa}]) \cong \bigoplus_{(\mathbf{m},\mathbf{n}) \in \mathbb{O}(\mathbf{c}(m,n;\kappa))} \mathcal{A}_{(\mathbf{m},\mathbf{n})}[u_1, ..., u_s, h, v_s, ..., v_1].$$

(2) There exists an embedding of a generalized Verma modules

$${}^{n}M_{\mathfrak{p}_\kappa}(-\xi(\mathbf{u} + \delta_{m,n}^{\kappa}, h, \mathbf{v} - \delta_{m,n}^{\kappa})) \hookrightarrow {}^{n}M_{\mathfrak{p}_\kappa}(-\xi(\mathbf{v} - \delta_{m,n}^{\kappa}, h, \mathbf{u} + \delta_{m,n}^{\kappa})).$$

which induces an intertwining operator

$$\varphi : \mathcal{I}_P[\mathbf{v} - \delta_{m,n}^{\kappa}; h; \mathbf{u} + \delta_{m,n}^{\kappa}]) \to \mathcal{I}_P[\mathbf{u} + \delta_{m,n}^{\kappa}; h; \mathbf{v} - \delta_{m,n}^{\kappa}]).$$

Moreover,

$$\operatorname{Image}(\varphi) = \operatorname{Socle}(\mathcal{I}_P[\mathbf{u} + \delta_{m,n}^{\kappa}; h; \mathbf{v} - \delta_{m,n}^{\kappa}]).$$

Since $\mathrm{U}(m, n))$ has a compact Cartan subgroup, Proposition 4.4 implies that $K_{\mathbb{C}}$-conjugacy classes of θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ parametrize the open G-orbits on the complexified generalized flag variety $X_P = G_{\mathbb{C}}/P_{\mathbb{C}}$. Let χ be a holomorphic character of $M_{\mathbb{C}}$. We let $\mathcal{L}_\chi$ (respectively ω_P) denote the holomorphic line bundle on X_P associated to χ (respectively the canonical line bundle on X_P). For an open G-orbit $\mathcal{V}$ in X_P, we consider the associated derived functor module

$$A_{\mathcal{V}}(\chi) = H^{\dim(\mathfrak{k} \cap \mathfrak{u})}(\mathcal{V}, \mathcal{L}_\chi \otimes \omega_P)_{K\text{-finite}}.$$

We may thus rewrite Theorem 7.4 as follows.

Corollary 7.5. *Recall the definition of the* P*-weakly fair range given before Proposition 6.13. For each* P*-weakly fair holomorphic character* χ *of* $M_{\mathbb{C}}$, *we have*

$$\operatorname{Socle}({}^{n}\operatorname{Ind}_P^G(\mathbb{C}_{\delta_P} \otimes \chi)) \cong \bigoplus \mathcal{A}_{\mathcal{V}}(\chi),$$

where the sum is taken over all good open G*-orbits in* X_P *(Definition 4.6).*

8 $\mathrm{GL}(n, \mathbb{H})$

We turn to $G = \mathrm{GL}(n, \mathbb{H})$ and consider a parabolic subgroup $P = MN$ of G with

$$M \simeq \mathrm{GL}(k_1, \mathbb{H}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{H}).$$

Here, $k_1 + \cdots + k_s = n$. Write $\mathcal{O}$ for the Richardson orbit induced from the zero orbit of $\mathfrak{m}$. In this case $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{s})$ consists of a since $K_{\mathbb{C}}$ orbit $\mathcal{O}_K$. Define

$$\mathcal{I}(\mathcal{O}) = {}^n\mathrm{Ind}_P^G(\mathbb{1}). \tag{8.1}$$

This is unitarily induced and it follows from [V3] that $\mathcal{I}(\mathcal{O})$ is irreducible. The associated variety of $\mathcal{I}(\mathcal{O})$ is $\overline{\mathcal{O}_K^1}$ (as can be deduced from Theorem 3.1, for instance) and it has the special unipotent infinitesimal character associated to $\mathcal{O}$ (cf. [BaV3]). Thus the results of [BaV3] imply that $\mathcal{I}(\mathcal{O})$ is annihilated by $\mathrm{J}^{\max}(\lambda(\mathcal{O}))$.

The structure of the Harish-Chandra cells for $\mathrm{GL}(n, \mathbb{H})$ is easier than that for $\mathrm{Sp}(p, q)$ and $\mathrm{SO}^*(2n)$ (e.g. [Mc]). From an argument similar to the one leading to Theorem 6.12, we obtain the following result.

Theorem 8.1. *$\mathcal{I}(\mathcal{O})$ is the unique weakly unipotent representation attached to $\mathcal{O}$ with integral infinitesimal character.*

On the other hand for $G = \mathrm{GL}(n, \mathbb{H})$ the unique open G-orbit in the complexified flag manifold $X_P = G_{\mathbb{C}}/P_{\mathbb{C}}$ is not necessarily good. In fact, we have the following result.

Proposition 8.2. *The unique open G-orbit in X_P (say $\mathcal{V}$) is good (Definition 4.6) if and only if P is G-conjugate to its opposite parabolic subgroup.*

Proof. First, we remark it is obvious that $\mathfrak{p}$ is neat in the sense of Definition 4.5 if and only if P is G-conjugate to its opposite parabolic subgroup. So the if-part of the current proposition follows from Lemma 4.3. The only-if part is proved as follows. Let $\mathcal{C}$ be the unique $K_{\mathbb{C}}$-orbit contained in $\mathcal{V}$. Let $\mathfrak{q}$ be any parabolic subalgebra contained in $\mathcal{C}$. Then, from Lemma 4.3, there is θ-stable Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{b} \subseteq \mathfrak{q}$. We choose a θ-stable Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{h} \subseteq \mathfrak{b}$. If we consider the basis of the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ corresponding to $\mathfrak{b}$, θ induces the non-trivial automorphism of the Dynkin diagram. Hence, we see that $\theta(\mathfrak{q})$ is $\mathrm{Ad}(G_{\mathbb{C}})$-conjugate to the opposite parabolic subalgebra. So, $\mathfrak{p}$ is $\mathrm{Ad}(G_{\mathbb{C}})$-conjugate to the opposite parabolic subalgebra. The theorem follows. □

We have the following analog of Theorem 6.12.

Theorem 8.3. *Let P be a parabolic subgroup of $G = \mathrm{GL}(n, \mathbb{H})$ such that P is G-conjugate to its opposite parabolic subgroup. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra in the unique closed $K_{\mathbb{C}}$-orbit in $G_{\mathbb{C}}/P_{\mathbb{C}}$. Define $\delta_{\mathfrak{q}}$ by $\delta_{\mathfrak{q}}(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{u}})$. Then, we have*

$$\mathcal{I}(\mathcal{O}) \cong A_{\mathfrak{q}}(-\delta_{\mathfrak{q}}).$$

(This is a derived functor module on the "edge" of the weakly fair range in the sense of Remark 6.3.)

Proof. One may verify that $A_{\mathfrak{q}}(-\delta_{\mathfrak{q}})$ is irreducible and integrally weakly unipotent attached to $\mathcal{O}$. So the result follows from Theorem 8.1.

As in the case of $\mathrm{Sp}(p, q)$ and $\mathrm{SO}^*(2n)$, we have the following result which is deduced in the same way as Corollary 6.15.

Corollary 8.4. *Let P be a parabolic subgroup of $G = \mathrm{GL}(n, \mathbb{H})$ and let X be a representation of G parabolically induced from a one-dimensional representation of P. Assume that P is G-conjugate to its opposite parabolic subgroup. Moreover, assume that X has integral infinitesimal character. Then any irreducible constituent of X of the maximal Gelfand-Kirillov dimension is a derived functor module in the weakly fair range.*

We define $\mathcal{I}_P(\chi)$ and $A_{\mathcal{V}}(\chi)$ in the same way as the case of $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$. Since every involution in the Weyl group of the type A is a Duflo involution, [Ma2, Proposition 2.1.2] implies the following result.

Theorem 8.5. *Let $G = \mathrm{GL}(n, \mathbb{H})$ and assume that the parabolic subgroup $P = MN$ is G-conjugate to its opposite. Let χ be a P-weakly fair holomorphic character of $M_{\mathbb{C}}$. Then,*

(a) There is an embedding of generalized Verma module

$$\psi_\chi : {}^u M_{\mathfrak{p}}(-2\delta_P - d\chi) \hookrightarrow {}^u M_{\mathfrak{p}}(d\chi)$$

which induces an intertwining operator

$$\Psi_\chi : \mathcal{I}_P(\chi^{-1} \otimes \xi_{-2\delta_P}) \to \mathcal{I}_P(\chi).$$

(b) We have $\mathrm{Socle}(\mathcal{I}_P(\chi)) = \Psi_\chi(\mathcal{I}_P(\chi^{-1} \otimes \xi_{-2\delta_P}))$ *and, moreover,*

$$\mathrm{Socle}(\mathcal{I}_P(\chi)) \approx \mathcal{I}_P(\chi).$$

(c) We have

$$\mathrm{Socle}(\mathcal{I}_P(\chi)) \cong A_{\mathcal{V}}(\chi).$$

9 Complex groups

Let G be a connected complex reductive Lie group. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathrm{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and Δ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let w_0 denote the longest element of W. Fix a positive system Δ^+ of Δ and let Π denote the corresponding basis of Δ. Fix a triangular decomposition $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ such that $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\bar{n} = \sum_{-\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Set $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$.

Let $\mathfrak{g}_0$ denote the normal real form of $\mathfrak{g}$ which is compatible with the above decomposition and let $X \rightsquigarrow \bar{X}$ denote the corresponding complex conjugation. Then there is an anti-automorphism of $\mathrm{U}(\mathfrak{g})$, the so-called Chevalley anti-automorphism, denoted $u \mapsto {}^t u$ which satisfies the following when restricted to $\mathfrak{g}$,

(1) ${}^t\mathfrak{g}_0 = \mathfrak{g}_0$.

(2) ${}^t\mathfrak{n} = \bar{\mathfrak{n}}$, ${}^t\bar{\mathfrak{n}} = \mathfrak{n}$.

(3) ${}^tX = X \quad (X \in \mathfrak{h})$.

Define a homomorphism of real Lie algebras $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ via $X \rightsquigarrow (X, \bar{X})$ for $X \in \mathfrak{g}$. Then the image of this homomorphism is a real form of $\mathfrak{g} \times \mathfrak{g}$, and so we can regard $\mathfrak{g} \times \mathfrak{g}$ as the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Then $\mathfrak{k}_{\mathbb{C}} := \{(X, -{}^tX) \mid X \in \mathfrak{g}\}$ is identified with the complexification of a compact form of $\mathfrak{g}$, and of course $\mathfrak{k}_{\mathbb{C}}$ is also identified (as a complex Lie algebra) with $\mathfrak{g}$ via $X \rightsquigarrow (X, -{}^tX)$. We may regard $G \times G$ as the complexification $G_{\mathbb{C}}$ of G.

Let P be a standard parabolic subgroup of G; namely assume that the Lie algebra $\mathfrak{p}$ of P contains $\mathfrak{b}$. Let M (resp. $\mathfrak{m}$) denote the Levi part of P (resp. $\mathfrak{p}$) stable under the Chevalley anti-automorphism. Write $\mathfrak{n}_{\mathfrak{p}}$ for the nilradical of $\mathfrak{p}$. Let S be the subset of Π corresponding to $\mathfrak{p}$; i.e. S is the basis of the root system of $\mathfrak{h}$ in $\mathfrak{m}$. Put $S' = \{-w_0\alpha | \alpha \in S\}$. Let $\mathfrak{p}'$ denote the standard parabolic subalgebra corresponding to S'. Let $w_{\mathfrak{p}}$ denote the longest element of the Weyl group with respect to $(\mathfrak{m}, \mathfrak{h})$.

Under the above identification $G_{\mathbb{C}} \cong G \times G$, the complexification $P_{\mathbb{C}}$ of P is identified with a subgroup $P \times P$ of $G \times G$, and the complex generalized flag variety $X = G_{\mathbb{C}}/P_{\mathbb{C}}$ is identified with $G/P \times G/P$. X can be regarded as the set of parabolic subalgebras of $\mathfrak{g}$ which are $\mathrm{Ad}(G_{\mathbb{C}})$-conjugate to $Lie(P_{\mathbb{C}}) = \mathfrak{p}_{\mathbb{C}} \cong \mathfrak{p} \times \mathfrak{p}$.

The following is well-known.

Proposition 9.1. *X has a unique G-orbit (say $\mathcal{O}_0$). The following five conditions are equivalent to each other.*

(1) $\mathcal{O}_0$ contains a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

(2) $\mathfrak{p}$ and the parabolic subalgebra opposite to $\mathfrak{p}$ are G-conjugate.

(3) $\mathfrak{p} = \mathfrak{p}'$.

(4) S is stable under the action of $-w_0$.

(5) $w_0 w_{\mathfrak{p}} = w_{\mathfrak{p}} w_0$. (Namely $w_0 w_{\mathfrak{p}}$ is an involution.)

For $X \in \mathfrak{m}$,define $\delta_{\mathfrak{p}}(X) = \frac{1}{2}\mathrm{tr}\left(\mathrm{ad}_{\mathfrak{g}}(X)|_{\mathfrak{n}_{\mathfrak{p}}}\right)$. Then, $\delta_{\mathfrak{p}}$ is a one-dimensional representation of $\mathfrak{m}$, and $2n\delta_{\mathfrak{p}}$ lifts to a holomorphic group homomorphism $\xi_{2n\delta_{\mathfrak{p}}} : M \to \mathbb{C}^\times$ for all $n \in \mathbb{Z}$. Identifying $M_{\mathbb{C}} \cong M \times M$, for $n, m \in 2\mathbb{Z}$ we can regard $\chi_{n,m} = \xi_{n\delta_{\mathfrak{p}}} \boxtimes \xi_{m\delta_{\mathfrak{p}}}$ as a real analytic group homomorphism $M \to \mathbb{C}^\times$. Taking account of the natural projection $P \to M$, we regard $\chi_{n,m}$ as a real analytic one-dimensional representation of P. We may thus consider an unnormalized degenerate principal series representation,

$$ {}^u\mathrm{Ind}_P^G(\chi_{n,m}) = \{f \in C^\infty(G) | f(gp) = \chi_{n,m}(p)^{-1} f(g) \quad (g \in G, p \in P)\}_{\mathfrak{k}_{\mathbb{C}}\text{-finite}}. $$

If $\mathfrak{p} = \mathfrak{b}$, then $\delta_{\mathfrak{b}} = \rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$. For $t \in \mathbb{R}$, we define an unnormalized generalized Verma module,

$$ {}^uM_{\mathfrak{p}}(t) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathbb{C}_{t\delta_{\mathfrak{p}}}. $$

Here, $\mathbb{C}_{t\delta_{\mathfrak{p}}}$ means the one-dimensional module of $\mathfrak{p}$ such that $\mathfrak{m}$ acts on it by $t\delta_{\mathfrak{p}}$. If $\mathfrak{p} = \mathfrak{b}$, we simply write ${}^uM(t) = {}^uM_{\mathfrak{b}}(t)$. We let $I_{\mathfrak{p}}(t)$ denote the annihilator of $M_{\mathfrak{p}}(t)$ in $\mathrm{U}(\mathfrak{g})$. From [V2], $M_{\mathfrak{p}}(t)$ is irreducible for $t \leqslant -1$. Therefore $I_{\mathfrak{p}}(t)$ is a primitive ideal for $t \leqslant -1$.

Under the identification $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \times \mathfrak{g}$, we identify $\mathrm{U}(\mathfrak{g}_{\mathbb{C}}) \cong \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$. We write $u \rightsquigarrow \check{u}$ ($u \in \mathrm{U}(\mathfrak{g})$) for the anti-automorphism of $\mathrm{U}(\mathfrak{g})$ generated by $X \rightsquigarrow -X$ ($X \in \mathfrak{g}$). If V is a $\mathrm{U}(\mathfrak{g})$-bimodule, we can regard V as a $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$-module as follows,

$$(u_1 \otimes u_2)v = {}^t\check{u}_1 v \check{u}_2 \quad (u_1, u_2 \in \mathrm{U}(\mathfrak{g}), v \in V).$$

In particular, we may regard $\mathrm{U}(\mathfrak{g})/I_{\mathfrak{p}}(t)$ as a $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$-module.

We quote:

Theorem 9.2 ([ConD, 2.12 and 6.3]). *If n is an even integer such that $n \geqslant 2$, then ${}^u\mathrm{Ind}_P^G(\chi_{n,n})$ is isomorphic to $\mathrm{U}(\mathfrak{g})/I_{\mathfrak{p}}(-n)$ as $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$-modules.*

Together with [BoKr, 3.6] and the primitivity of $I_{\mathfrak{p}}(-n)$ $(n \geqslant 0)$, we have:

Corollary 9.3. *If $n \geqslant 2$, then ${}^u\mathrm{Ind}_P^G(\chi_{n,n})$ has a unique irreducible submodule (say Y_n). The Gelfand-Kirillov dimension of Y_n equals that of ${}^u\mathrm{Ind}_P^G(\chi_{n,n})$. The Gelfand-Kirillov dimension of ${}^u\mathrm{Ind}_P^G(\chi_{n,n})/Y_n$ is strictly smaller than that of Y_n.*

Since ${}^u\mathrm{Ind}_P^G(\chi_{2,2})$ is the induced module associated with the canonical bundle, we only consider the case of $n = 2$. For simplicity, we put $I = I_{\mathfrak{p}}(-2) = I_{\mathfrak{p}'}(0)$. (The latter equality follows from [BoJa, 4.10 Corollar] where is it proved that $I_{\mathfrak{p}}(-n) = I_{\mathfrak{p}'}(n-2)$.) Let M and N be $\mathrm{U}(\mathfrak{g})$-modules. $\mathrm{Hom}_{\mathbb{C}}(M, N)$ has a natural structure of a $\mathrm{U}(\mathfrak{g})$-bimodule (using the Chevalley anti-automorphism), and so we we may consider $\mathrm{Hom}_{\mathbb{C}}(M, N)$ as a $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$ module. We denote the $\mathfrak{k}_{\mathbb{C}}$-finite part of $\mathrm{Hom}(M, N)$ by $L(M, N)$. The functor $V \rightsquigarrow L({}^uM(0), V)$ defines an equivalence of categories between a category of highest weight modules and the category of Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$-modules (Bernstein-Gelfand-Joseph-Enright cf. [BeG], [Jo1]). Via this equivalence of categories, the 2-sided ideals of $\mathrm{U}(\mathfrak{g})/I$ correspond to the $\mathrm{U}(\mathfrak{g})$-submodules of ${}^uM(0)/I^uM(0)$. In particular Y_2 corresponds to the irreducible highest weight module with the highest weight $\tau\rho - \rho$ ([Jo1], also see [Jo2, page 43]). Here, $\tau \in W$ is the Duflo involution associated to the primitive ideal I. For integral weights $\mu, \nu \in \mathfrak{h}^*$, we let $V(\nu, \mu)$ denote the Langlands (Zhelobenko) subquotient of ${}^u\mathrm{Ind}_B^G(\xi_{\nu+\rho} \boxtimes \xi_{\mu+\rho})$. From [Jo1, 4.5], we have:

Theorem 9.4. *The unique irreducible submodule Y_2 of ${}^u\mathrm{Ind}_P^G(\chi_{2,2})$ is isomorphic to $V(-\tau\rho, -\rho) = V(w_0\tau w_0\rho, \rho)$.*

Using the equivalent conditions in Proposition 9.1, we may compare Y_2 with $\mathcal{A}_{\mathcal{O}_0} = H^{\dim \mathfrak{n}_{\mathfrak{p}}}(\mathcal{O}_0, \mathcal{L})_{\mathfrak{k}_{\mathbb{C}}\text{-finite}}$. For orientation, we include the following result.

Theorem 9.5 ([E], [VZ]). *Assume that $\mathfrak{p}$ satisfies the equivalent conditions in Proposition 9.1. Then we have*

$$\mathcal{A}_{\mathcal{O}_0} \cong {}^u\mathrm{Ind}_P^G(\chi_{2,0}) \cong {}^u\mathrm{Ind}_P^G(\chi_{0,2}).$$

Since $2\delta_{\mathfrak{p}} = \rho - w_0 w_{\mathfrak{p}}\rho$, we see $\mathcal{A}_{\mathcal{O}_0}$ is the Langlands subquotient $V(-w_0w_{\mathfrak{p}}\rho, -\rho)$ of ${}^u\mathrm{Ind}_B^G(\xi_{-w_0w_{\mathfrak{p}}\rho+\rho} \boxtimes \xi_{-\rho+\rho})$. Since ${}^uM_{\mathfrak{p}}(-2)$ is the irreducible highest weight module of the highest weight $-2\delta_{\mathfrak{p}} = w_0w_{\mathfrak{p}}\rho - \rho$, we have:

Theorem 9.6. *Assume that $\mathfrak{p}$ satisfies the equivalent conditions in Proposition 1.2.1. Then $\mathcal{A}_{\mathcal{O}_0}$ is isomorphic to the unique submodule of ${}^u\mathrm{Ind}_P^G(\chi_{2,2})$ if and only if $w_0 w_{\mathfrak{p}}$ is a Duflo involution in W.*

In case that $w_0 w_{\mathfrak{p}}$ is a Duflo involution, the embedding of $\mathcal{A}_{\mathcal{O}_0}$ into ${}^u\mathrm{Ind}_P^G(\chi_{2,2})$ can be regarded as giving rise to intertwining operators between the following degenerate principal series representation,

$$ {}^u\mathrm{Ind}_P^G(\chi_{2,0}) \hookrightarrow {}^u\mathrm{Ind}_P^G(\chi_{2,2}), \tag{9.1} $$

$$ {}^u\mathrm{Ind}_P^G(\chi_{0,2}) \hookrightarrow {}^u\mathrm{Ind}_P^G(\chi_{2,2}). \tag{9.2} $$

In fact, the following result holds:

Proposition 9.7 ([Ma2]). *Let t be a non-negative even integer. Then we have*

$$ {}^uM_{\mathfrak{p}}(-t-2) \hookrightarrow {}^uM_{\mathfrak{p}}(t) $$

if and only if $w_0 w_{\mathfrak{p}}$ is a Duflo involution in W.

Taking account of the isomorphism ${}^u\mathrm{Ind}_P^G(\chi_{m,n}) \cong ({}^uM_{\mathfrak{p}}(-m) \otimes {}^uM_{\mathfrak{p}}(-n))^*_{\mathfrak{k}_{\mathbb{C}}\text{-finite}}$, the intertwining operators of (9.1) and (9.2) are induced from the embeddings the generalized Verma modules in the proposition.

Finally, if W is a Weyl group of the type A, each involution in W is a Duflo involution ([Du]). Hence, we obtain:

Corollary 9.8. *If $G = \mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{p}$ satisfies the equivalent conditions in Proposition 1.1, the socle of ${}^u\mathrm{Ind}_P^G(\chi_{2,2})$ is isomorphic to $\mathcal{A}_{\mathcal{O}_0}$.*

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