DIRAC COHOMOLOGY FOR GRADED AFFINE HECKE ALGEBRAS

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Abstract. We define an analogue of the Casimir element for a graded affine Hecke algebra $\mathbb{H}$, and then introduce an approximate square-root called the Dirac element. Using it, we define the Dirac cohomology $H^D(X)$ of an $\mathbb{H}$ module X, and show that $H^D(X)$ carries a representation of a canonical double cover of the Weyl group $\widetilde{W}$. Our main result shows that the $\widetilde{W}$ structure on the Dirac cohomology of an irreducible $\mathbb{H}$ module X determines the central character of X in a precise way. This can be interpreted as p-adic analogue of a conjecture of Vogan for Harish-Chandra modules. We also apply our results to the study of unitary representations of $\mathbb{H}$.

1. Introduction

This paper develops the theory of Dirac cohomology for modules over a graded affine Hecke algebra. The cohomology of such a module X is a representation of canonical double cover $\widetilde{W}$ of a relevant Weyl group. Our main result shows that when X is irreducible, the $\widetilde{W}$ representation (when nonzero) determines the central character of X. This can be interpreted as a p-adic analogue of Vogan's Conjecture (proved by Huang and Pandžić [HP]) for Harish-Chandra modules.

In more detail, fix a root system R (not necessarily crystallographic), let V_0 (resp. V) denote its real (resp. complex) span, write $V_0^\vee$ (resp. $V^\vee$) for the real (resp. complex) span of the coroots, W for the Weyl group, and fix a W-invariant inner product $\langle\ ,\ \rangle$ on $V_0^\vee$ (and hence V_0 as well). Extend $\langle\ ,\ \rangle$ to an inner product on $V^\vee$ (and V). Let $\mathbb{H}$ denote the associated graded affine Hecke algebra with parameters defined by Lusztig [L1] (Definition 2.1). As a complex vector space, $\mathbb{H} \simeq \mathbb{C}[W] \otimes S(V^\vee)$. Lusztig proved that maximal ideals in the center of $\mathbb{H}$, and hence central characters of irreducible $\mathbb{H}$ modules, are parametrized by (W orbits of) elements of V. In particular, using $\langle\ ,\ \rangle$ it makes sense to speak of the length of the central character of an irreducible $\mathbb{H}$ module.

After introducing certain Casimir-type elements in Section 2, we then turn to the Dirac element in Section 3. Let $C(V_0^\vee)$ denote the corresponding Clifford algebra for the inner product $\langle\ ,\ \rangle$. For a fixed orthonormal basis $\{\omega_i\}$ of $V_0^\vee$, the Dirac element is defined (Definition 3.1) as

$$\mathcal{D} = \sum_i \widetilde{\omega}_i \otimes \omega_i \in \mathbb{H} \otimes C(V_0^\vee),$$

where $\widetilde{\omega}_i \in \mathbb{H}$ is given by (2.14). In Theorem 3.1, we prove $\mathcal{D}$ is roughly the square root of the Casimir element $\sum_i \omega_i^2 \in \mathbb{H}$ (Definition 2.4).

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For a fixed space of spinors S for $C(V_0^\vee)$ and a fixed $\mathbb{H}$ module X, $\mathcal{D}$ acts as an operator D on $X \otimes S$. Since W acts by orthogonal transformation on $V_0^\vee$, we can consider its preimage $\widetilde{W}$ in $\mathsf{Pin}(V_0^\vee)$. By restriction, X is a representation of W, and so $X \otimes S$ is a representation of $\widetilde{W}$. Lemma 3.4 shows that $\mathcal{D}$ (and hence D) are approximately $\widetilde{W}$ invariant. Thus $\ker(D)$ is also a representation of $\widetilde{W}$. Corollary 3.6 shows that if X is irreducible, unitary, and $\ker(D)$ is nonzero, then any irreducible representation of $\widetilde{W}$ occurring in $\ker(D)$ determines the length of the central character of X. This is an analogue of Parthasarathy's fundamental calculation [P] (cf. [SV, Section 7]) for Harish-Chandra modules.

We then define the Dirac cohomology of X as $H^D(X) = \ker(D)/(\ker(D) \cap \mathrm{im}(D))$ in Definition 4.1. (For unitary representations $H^D(X) = \ker(D)$.) Once again $H^D(X)$ is a representation of $\widetilde{W}$. One is naturally led to the following version of Vogan's Conjecture: if X is irreducible and $H^D(X)$ is nonzero, then any irreducible representation of $\widetilde{W}$ occurring in $H^D(X)$ determines the central character of X, not just its length. This is the content of Theorem 4.4 below. For algebras $\mathbb{H}$ attached to crystallographic root systems and equal parameters, a much more precise statement is given in Theorem 5.8. As explained in Remark 5.9, the proof of Theorem 5.8 also applies for the special kinds of unequal parameters arising in Lusztig's geometric theory [L2, L3].

To make Theorem 5.8 precise, we need a way of passing from an irreducible $\widetilde{W}$ representation to a central character, i.e. an element of V. This is a fascinating problem in its own right. The irreducible representations of $\widetilde{W}$ — the so-called spin representations of W — have been known for a long time from the work of Schur, Morris, Reade, and others. But only recently has a uniform parametrization of them in terms of nilpotent orbits emerged [C]. This parametrization (partly recalled in Theorem 5.1) provides exactly what is needed for the statement of Theorem 5.8.

One of the main reasons for introducing the Dirac operator (as in the real case) is to study unitary representations. We give applications in Section 5.2. Corollary 5.4 and Remarks 5.5–5.6 contain powerful general statements about unitary representations, in particular the "spectral gap" measuring the degree to which the trivial representation is isolated in the unitary dual of $\mathbb{H}$. Given the machinery of the Dirac operator, the proofs of these results are very simple. As an example, we include the following remarkable result. It follows immediately from Theorem 5.8.

Theorem 1.1. *Suppose $\mathbb{H}$ is attached to crystallographic root system with equal parameters (Definition 2.1), and let $\mathfrak{g}$ denote the corresponding complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h} \simeq V$. Let X be an irreducible $\mathbb{H}$ module with central character χ_ν for $\nu \in V \simeq \mathfrak{h}$ (as in Definition 2.2). Suppose further that X is unitary with respect to the $*$-operation of Section 2.5, and that the kernel of the Dirac operator for X is nonzero (Definition 3.1). Then there exists an $\mathfrak{sl}_2$ triple $\{e, h, f\} \subset \mathfrak{g}$ such that the centralizer in $\mathfrak{g}$ of e is solvable and such that*

$$\nu = \frac{1}{2}h.$$

We remark that while this paper is inspired by the ideas of Huang-Pandžić, Kostant, Parthasarathy, Schmid, and Vogan, it is essentially self-contained. There are two exceptions, and both arise during the proof of the sharpened Theorem 5.8. We have already mentioned that we use the main results of [C] in Section 5. The other nontrivial result we need is the classification and W-module structure of certain tempered $\mathbb{H}$-modules ([KL, L1, L3]). These results (in the form we use them) are not available at arbitrary parameters. This explains the crystallographic condition and restrictions on parameters in the statement of Theorem 5.8 and in Remark 5.9. For applications to unitary representations of p-adic groups, these hypotheses are natural. Nonetheless we expect a version of Theorem 5.8 to hold for arbitrary parameters and noncrystallographic roots systems.

The results of this paper suggest generalizations to other types of related Hecke algebras. They also suggest possible generalizations along the lines of [K] for a version of Kostant's cubic Dirac operator. Finally, in [EFM] and [CT] (and also in unpublished work of Hiroshi Oda) functors between Harish-Chandra modules and modules for associated graded affine Hecke algebras are introduced. It would be interesting to understand how these functors relate Dirac cohomology in the two categories.

2. Casimir operators

2.1. **Root systems.** Fix a root system $\Phi = (V_0, R, V_0^\vee, R^\vee)$ over the real numbers. In particular: $R \subset V_0 \setminus \{0\}$ spans the real vector space V_0; $R^\vee \subset V^\vee \setminus \{0\}$ spans the real vector space $V_0^\vee$; there is a perfect bilinear pairing

$$(\cdot,\cdot) : V_0 \times V_0^\vee \to \mathbb{R};$$

and there is a bijection between R and $R^\vee$ denoted $\alpha \mapsto \alpha^\vee$ such that $(\alpha, \alpha^\vee) = 2$ for all α. Moreover, for $\alpha \in R$, the reflections

$$\begin{aligned} s_\alpha &: V_0 \to V_0, \quad s_\alpha(v) = v - (v, \alpha^\vee)\alpha, \\ s_\alpha^\vee &: V_0^\vee \to V_0^\vee, \ s_\alpha^\vee(v') = v' - (\alpha, v')\alpha^\vee \end{aligned}$$

leave R and $R^\vee$ invariant, respectively. Let W be the subgroup of $GL(V_0)$ generated by $\{s_\alpha \mid \alpha \in R\}$. The map $s_\alpha \mapsto s_\alpha^\vee$ given an embedding of W into $GL(V_0^\vee)$ so that

$$(v, wv') = (wv, v') \tag{2.1}$$

for all $v \in V_0$ and $v' \in V_0^\vee$.

We will assume that the root system Φ is reduced, meaning that $\alpha \in R$ implies $2\alpha \notin R$. However, initially we do not need to assume that Φ is crystallographic, meaning that for us $(\alpha, \beta^\vee)$ need not always be an integer. We will fix a choice of positive roots $R^+ \subset R$, let Π denote the corresponding simple roots in R^+, and let $R^{\vee,+}$ denote the corresponding positive coroots in $R^\vee$. Often we will write $\alpha > 0$ or $\alpha < 0$ in place of $\alpha \in R^+$ or $\alpha \in (-R^+)$, respectively.

We fix, as we may, a W-invariant inner product $\langle \cdot,\cdot \rangle$ on $V_0^\vee$. The constructions in this paper of the Casimir and Dirac operators depend, up to a positive scalar, on the choice of this inner product. Using the bilinear pairing $(\cdot,\cdot)$, we define a dual inner product on V_0 as follows. Let $\{\omega_i \mid i = 1, \cdots, n\}$ and $\{\omega^i \mid i = 1, \dots, n\}$ be $\mathbb{R}$-bases of $V_0^\vee$ which are in duality; i.e. such that $\langle \omega_i, \omega^j \rangle = \delta_{i,j}$, the Kronecker delta. Then for $v_1, v_2 \in V_0$, set

$$\langle v_1, v_2 \rangle = \sum_{i=1}^{n} (v_1, \omega_i)(v_2, \omega^i). \tag{2.2}$$

(Since the inner product on $V_0^\vee$ is also denoted $\langle \cdot,\cdot \rangle$, this is an abuse of notation. But it causes no confusion in practice.) Then (2.2) defines an inner product on V_0

which once again is W-invariant. It does not depend on the choice of bases $\{\omega_i\}$ and $\{\omega^i\}$. If v is a vector in V or in $V^\vee$, we set $|v| := \langle v, v\rangle^{1/2}$.

Finally, we extend $\langle\ ,\ \rangle$ to an inner product on the complexification $V^\vee := V_0^\vee \otimes_{\mathbb{R}} \otimes\mathbb{C}$ (and $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$).

2.2. **The graded affine Hecke algebra.** Fix a root system Φ as in the previous section. Set $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, and $V^\vee = V_0^\vee \otimes_{\mathbb{R}} \mathbb{C}$. Fix a W-invariant "parameter function" $c : R \to \mathbb{R}$, and set $c_\alpha = c(\alpha)$.

Definition 2.1 ([L1] §4)**.** The graded affine Hecke algebra $\mathbb{H} = \mathbb{H}(\Phi, c)$ attached to the root system Φ and with parameter function c is the complex associative algebra with unit generated by the symbols $\{t_w \mid w \in W\}$ and $\{t_f \mid f \in S(V^\vee)\}$, subject to the relations:

(1) The linear map from the group algebra $\mathbb{C}[W] = \bigoplus_{w\in W} \mathbb{C}w$ to $\mathbb{H}$ taking w to t_w is an injective map of algebras.
(2) The linear map from the symmetric algebra $S(V^\vee)$ to $\mathbb{H}$ taking an element f to t_f is an injective map of algebras.

We will often implicitly invoke these inclusions and view $\mathbb{C}[W]$ and $S(V^\vee)$ as subalgebras of $\mathbb{H}$. As is customary, we also write f instead of t_f in $\mathbb{H}$. The final relation is

(3)
$$\omega t_{s_\alpha} - t_{s_\alpha} s_\alpha(\omega) = c_\alpha(\alpha, \omega), \quad \alpha \in \Pi,\ \omega \in V^\vee; \tag{2.3}$$
here $s_\alpha(\omega)$ is the element of $V^\vee$ obtained by s_α acting on ω.

Proposition 4.5 in [L1] says that the center $Z(\mathbb{H})$ of $\mathbb{H}$ is $S(V^\vee)^W$. Therefore maximal ideals in $Z(\mathbb{H})$ are parametrized by W orbits in V.

Definition 2.2. For $\nu \in V$, we write χ_ν for the homomorphism from $Z(\mathbb{H})$ to $\mathbb{C}$ whose kernel is the maximal ideal parametrized by the W orbit of ν. By a version of Schur's lemma, $Z(\mathbb{H})$ acts in any irreducible $\mathbb{H}$ module X by a scalar $\chi : Z(\mathbb{H}) \to \mathbb{C}$. We call χ the central character of (π, X). In particular, there exists $\nu \in V$ such that $\chi = \chi_\nu$.

2.3. **The Casimir element of $\mathbb{H}$.**

Definition 2.3. Let $\{\omega_i : i = 1, n\}$ and $\{\omega^i : i = 1, n\}$ be dual bases of $V_0^\vee$ with respect to $\langle\ ,\ \rangle$. Define
$$\Omega = \sum_{i=1}^{n} \omega_i \omega^i \in \mathbb{H}. \tag{2.4}$$
It follows from a simple calculation that Ω is well-defined independent of the choice of bases.

Lemma 2.4. *The element Ω is central in $\mathbb{H}$.*

Proof. To see that Ω is central, in light of Definition 2.1, it is sufficient to check that $t_{s_\alpha}\Omega = \Omega t_{s_\alpha}$ for every $\alpha \in \Pi$. Using (2.3) twice and the fact that $(\alpha, s_\alpha(\omega)) = -(\alpha, \omega)$ (as follows from (2.1)), we find
$$t_{s_\alpha}(\omega_i\omega^i) = (s_\alpha(\omega_i)s_\alpha(\omega^i))t_{s_\alpha} + c_\alpha(\alpha, \omega^i)s_\alpha(\omega_i) + c_\alpha(\alpha, \omega_i)\omega^i. \tag{2.5}$$

Therefore, we have

$$
\begin{aligned}
t_{s_\alpha}\Omega &= \sum_{i=1}^{n} s_\alpha(\omega_i)s_\alpha(\omega^i)t_{s_\alpha} + c_\alpha \sum_{i=1}^{n}(\alpha,\omega^i)s_\alpha(\omega_i) + c_\alpha\sum_{i=1}^{n}(\alpha,\omega_i)\omega^i \\
&= \Omega t_{s_\alpha} + c_\alpha \sum_{i=1}^{n}(\alpha, s_\alpha(\omega^i))\omega_i + c_\alpha \sum_{i=1}^{n}(\alpha,\omega_i)\omega^i, \\
&= \Omega t_{s_\alpha} - c_\alpha \sum_{i=1}^{n}(\alpha,\omega^i)\omega_i + c_\alpha\sum_{i=1}^{n}(\alpha,\omega_i)\omega^i.
\end{aligned}
\tag{2.6}
$$

But the last two terms cancel (which can be seen by taking $\{\omega_i\}$ to be a self-dual basis, for example). So indeed $t_{s_\alpha}\Omega = \Omega t_{s_\alpha}$. □

Lemma 2.5. *Let (π, X) is an irreducible $\mathbb{H}$-module with central character χ_ν for $\nu \in V$ (as in Definition 2.2). Then*

$$\pi(\Omega) = \langle \nu, \nu\rangle \,\mathrm{Id}_X.$$

Proof. Since Ω is central (by Lemma 2.4), it acts by a multiple of the identity on X. We use the weight decomposition of (π, X) with respect to the abelian subalgebra $S(V^\vee)$. Let $x \neq 0$ be an eigenvector for the weight $w\nu \in V$, $w \in W$. Then we have:

$$\pi(\omega_i\omega^i)x = (w\nu,\omega_i)(w\nu,\omega^i)x,$$

and when we sum over the dual bases $\{\omega_i\}, \{\omega^i\}$, we find

$$\pi(\Omega)x = \sum_{i=1}^{n}(w\nu,\omega_i)(w\nu,\omega^i)x = \langle w\nu, w\nu\rangle x = \langle \nu,\nu\rangle x, \tag{2.7}$$

by (2.2) and the W-invariance of $\langle\ ,\ \rangle$. □

2.4. We will need the following formula. To simplify notation, we define

$$t_{w\beta} := t_w t_{s_\beta} t_{w^{-1}}, \text{ for } w \in W,\ \beta \in R. \tag{2.8}$$

Lemma 2.6. *For $w \in W$ and $\omega \in V^\vee$,*

$$t_w \omega t_w^{-1} = w(\omega) + \sum_{\beta>0 \text{ s.t. } w\beta<0} c_\beta(\beta,\omega)t_{w\beta}. \tag{2.9}$$

Proof. The formula holds if $w = s_\alpha$, for $\alpha \in \Pi$, by (2.3):

$$t_{s_\alpha}\omega t_{s_\alpha} = s_\alpha(\omega) + c_\alpha(\alpha,\omega)t_{s_\alpha}. \tag{2.10}$$

We now do an induction on the length of w. Suppose the formula holds for w, and let α be a simple root such that $s_\alpha w$ has strictly greater length. Then

$$
\begin{aligned}
t_{s_\alpha}t_w\omega t_{w^{-1}}t_{s_\alpha} = t_{s_\alpha}&\left[w(\omega) + \sum_{\beta\,:\,w\beta<0} c_\beta(\beta,\omega)t_{w\beta}\right]t_{s_\alpha} = \\
&= s_\alpha w(\omega) + c_\alpha(\alpha,\omega)t_{s_\alpha} + \sum_{\beta\,:\,w\beta<0} c_\beta(\beta,\omega)t_{s_\alpha}t_{w\beta}t_{s_\alpha} \\
&= s_\alpha w(\omega) + c_\alpha(\alpha,\omega)t_{s_\alpha} + \sum_{\beta\,:\,w\beta<0} c_\beta(\beta,\omega)t_{s_\alpha w\beta}.
\end{aligned}
\tag{2.11}
$$

The claim follows. □

2.5. **The $*$-operation, Hermitian and unitary representations.** The algebra $\mathbb{H}$ has a natural conjugate linear anti-involution defined on generators as follows ([BM2, Section 5]):

$$\begin{aligned} t_w^* &= t_{w^{-1}}, \quad w \in W, \\ \omega^* &= -\omega + \sum_{\beta>0} c_\beta(\beta,\omega) t_{s_\beta}, \quad \omega \in V_0^\vee. \end{aligned} \tag{2.12}$$

In general there are other conjugate linear anti-involutions on $\mathbb{H}$, but this one is distinguished by its relation to the canonical notion of unitarity for p-adic group representations [BM1]-[BM2].

An $\mathbb{H}$-module (π, X) is said to be $*$-Hermitian (or just Hermitian) if there exists a Hermitian form $(\ ,\ )_X$ on X which is invariant in the sense that:

$$(\pi(h)x, y)_X = (x, \pi(h^*)y)_X, \quad \text{for all } h \in \mathbb{H},\ x, y \in X. \tag{2.13}$$

If such a form exists which is also positive definite, then X is said to be $*$-unitary (or just unitary).

Because the second formula in (2.12) is complicated, we need other elements which behave more simply under $*$. For every $\omega \in V^\vee$, define

$$\widetilde{\omega} = \omega - \frac{1}{2}\sum_{\beta>0} c_\beta(\beta,\omega) t_{s_\beta} \ \in \mathbb{H}. \tag{2.14}$$

Then it follows directly from the definitions that $\omega^* = -\omega$. Thus if (π, X) is Hermitian $\mathbb{H}$-module

$$(\pi(\widetilde{\omega})x, \pi(\widetilde{\omega})x)_X = (\pi(\widetilde{\omega}^*)\pi(\widetilde{\omega})x, x)_X = -(\pi(\widetilde{\omega}^2)x, x)_X. \tag{2.15}$$

If we further assume that X is unitary, then

$$(\pi(\widetilde{\omega}^2)x, x)_X \leq 0, \quad \text{for all } x \in X,\ \omega \in V_0^\vee. \tag{2.16}$$

For each ω and x, this is a necessary condition for a Hermitian representation X to be unitary. It is difficult to apply because the operators $\pi(\widetilde{\omega}^2)$ are intractable in general. Instead we introduce a variation on the Casimir element of Definition 2.3 whose action in an $\mathbb{H}$-module will be seen to be tractable.

Definition 2.7. Let $\{\omega_i\}, \{\omega^i\}$ be dual bases of $V_0^\vee$ with respect to $\langle\ ,\ \rangle$. Define

$$\widetilde{\Omega} = \sum_{i=1}^{n} \widetilde{\omega}_i \widetilde{\omega}^i \ \in \mathbb{H}. \tag{2.17}$$

It will follow from Theorem 2.11 below that $\widetilde{\Omega}$ is independent of the bases chosen.

If we sum (2.16) over a self-dual orthonormal basis of $V_0^\vee$, we immediately obtain the following necessary condition for unitarity.

Proposition 2.8. *A Hermitian $\mathbb{H}$-module (π, X) with invariant form $(\ ,\ )_X$ is unitary only if*

$$(\pi(\widetilde{\Omega})x, x)_X \leq 0, \quad \textit{for all } x \in X. \tag{2.18}$$

The remainder of this section will be aimed at computing the action of $\widetilde{\Omega}$ in an irreducible $\mathbb{H}$ module as explicitly as possible (so that the necessary condition of 2.8 becomes as effective as possible). Since $\widetilde{\Omega}$ is no longer central, nothing as simple as Lemma 2.5 is available. But Proposition 2.10(2) below immediately implies that $\widetilde{\Omega}$ invariant under conjugation by t_w for $w \in W$. It therefore acts by a scalar on each W isotypic component of $\mathbb{H}$ module. We compute these scalars next, the main results being Theorem 2.11 and Corollary 2.12.

To get started, set

$$T_\omega = \omega - \widetilde{\omega} = \frac{1}{2}\sum_{\beta>0} c_\beta(\beta,\omega)t_{s_\beta} \in \mathbb{H} \tag{2.19}$$

and

$$\Omega_W = \frac{1}{4}\sum_{\substack{\alpha>0,\beta>0\\ s.t.\ s_\alpha(\beta)<0}} c_\alpha c_\beta\langle\alpha,\beta\rangle t_{s_\alpha}t_{s_\beta} \in \mathbb{C}[W]. \tag{2.20}$$

Note that Ω_W is invariant under the conjugation action of W. As we shall see, Ω_W plays the role of a Casimir element for $\mathbb{C}[W]$.

Lemma 2.9. *If $\omega_1,\omega_2 \in V^\vee$, we have*

$$[T_{\omega_1},T_{\omega_2}] = \frac{1}{4}\sum_{\substack{\alpha>0,\beta>0\\ s.t.\ s_\alpha(\beta)<0}} c_\alpha c_\beta((\alpha,\omega_1)(\beta,\omega_2)-(\beta,\omega_1)(\alpha,\omega_2))t_{s_\alpha}t_{s_\beta}.$$

Proof. From the definition (2.19), we see that

$$[T_{\omega_1},T_{\omega_2}] = \frac{1}{4}\sum_{\alpha>0,\beta>0} c_\alpha c_\beta((\alpha,\omega_1)(\beta,\omega_2)-(\beta,\omega_1)(\alpha,\omega_2))t_{s_\alpha}t_{s_\beta}.$$

Assume $\alpha>0,\beta>0$ are such that $s_\alpha(\beta)>0$. Notice that if $\gamma = s_\alpha(\beta)$, then $t_{s_\gamma}t_{s_\alpha} = t_{s_\alpha}t_{s_\beta}$. Also, it is elementary to verify (by a rank 2 reduction to the span of $\alpha^\vee$ and $\beta^\vee$, for instance) that

$$(s_\alpha(\beta),\omega_1)(\alpha,\omega_2)-(\alpha,\omega_1)(s_\alpha(\beta),\omega_2) = -((\alpha,\omega_1)(\beta,\omega_2)-(\beta,\omega_1)(\alpha,\omega_2)).$$

Since c is W-invariant, this implies that the contributions of the pairs of roots $\{\alpha,\beta\}$ and $\{s_\alpha(\beta),\alpha\}$ (when $s_\alpha(\beta)>0$) cancel out in the above sum. The claim follows. $\square$

Proposition 2.10. *Fix $w\in W$ and $\omega,\omega_1,\omega_2\in V^\vee$. The elements defined in (2.14) have the following properties:*

1. *$\widetilde{\omega}^* = -\widetilde{\omega}$;*
2. *$t_w\widetilde{\omega}t_{w^{-1}} = \widetilde{w(\omega)}$;*
3. *$[\widetilde{\omega}_1,\widetilde{\omega}_2] = -[T_{\omega_1},T_{\omega_2}]$.*

Proof. As remarked above, property (1) is obvious from (2.12). For (2), using Lemma 2.6, we have:

$$\begin{aligned} t_w\widetilde{\omega}t_{w^{-1}} &= t_w\omega t_{w^{-1}} - \frac{1}{2}\sum_{\beta>0} c_\beta(\beta,\omega)t_w t_{s_\beta}t_{w^{-1}} \\ &= w(\omega) + \sum_{\beta>0:w\beta<0} c_\beta(\beta,\omega)t_{w\beta} - \frac{1}{2}\sum_{\beta>0} c_\beta(\beta,\omega)t_{w\beta} \\ &= w(\omega) + \frac{1}{2}\sum_{\beta>0:w\beta<0} c_\beta(\beta,\omega)t_{w\beta} - \frac{1}{2}\sum_{\beta>0:w\beta>0} c_\beta(\beta,\omega)t_{w\beta} \\ &= w(\omega) - \frac{1}{2}\sum_{\beta'>0} c_{\beta'}(w^{-1}\beta',\omega)t_{s_{\beta'}} = \widetilde{w(\omega)}. \end{aligned} \tag{2.21}$$

For the last step, we set $\beta' = -w\beta$ in the first sum and $\beta' = w\beta$ in the second sum, and also used that $c_{\beta'} = c_\beta$ since c is W-invariant.

Finally, we verify (3). We have

$$\begin{aligned}[\widetilde{\omega}_1, \widetilde{\omega}_2] &= [\omega_1 - T_{\omega_1}, \omega_2 - T_{\omega_2}] \\ &= [T_{\omega_1}, T_{\omega_2}] - ([T_{\omega_1}, \omega_2] + [\omega_1, T_{\omega_2}]).\end{aligned}$$

We do a direct calculation:

$$[T_{\omega_1}, \omega_2] = \frac{1}{2}\sum_{\alpha>0} c_\alpha(\alpha, \omega_1)(t_{s_\alpha}\omega_2 t_{s_\alpha} - \omega_2)t_{s_\alpha}.$$

Applying Lemma 2.6, we get

$$\begin{aligned}[T_{\omega_1}, \omega_2] &= \frac{1}{2}\sum_{\alpha>0} c_\alpha(\alpha, \omega_1)(s_\alpha(\omega_2) - \omega_2)t_{s_\alpha} + \frac{1}{2}\sum_{\substack{\alpha>0,\beta>0\\ s.t.\ s_\alpha(\beta)<0}} c_\alpha c_\beta(\alpha, \omega_1)(\beta, \omega_2)t_{s_{\alpha}(\beta)}t_{s_\alpha} \\ &= -\frac{1}{2}\sum_{\alpha>0} c_\alpha(\alpha, \omega_1)(\alpha, \omega_2)\alpha^\vee t_{s_\alpha} + \frac{1}{2}\sum_{\substack{\alpha>0,\beta>0\\ s.t.\ s_\alpha(\beta)<0}} c_\alpha c_\beta(\alpha, \omega_1)(\beta, \omega_2)t_{s_\alpha}t_{s_\beta}.\end{aligned}$$

From this and Lemma 2.9, it follows immediately that $[T_{\omega_1}, \omega_2] + [\omega_1, T_{\omega_2}] = 2[T_{\omega_1}, T_{\omega_2}]$. This completes the proof of (3). □

Theorem 2.11. *Let $\widetilde{\Omega}$ be the W-invariant element of $\mathbb{H}$ from Definition 2.7. Recall the notation of* (2.19) *and* (2.20)*. Then*

$$\widetilde{\Omega} = \Omega - \Omega_W. \tag{2.22}$$

Proof. From Definition 2.7, we have

$$\widetilde{\Omega} = \sum_{i=1}^n \omega_i\omega^i - \sum_{i=1}^n(\omega_i T_{\omega^i} + T_{\omega_i}\omega^i) + \sum_{i=1}^n T_{\omega_i}T_{\omega^i}. \tag{2.23}$$

On the other hand, we have $\widetilde{\omega} = (\omega - \omega^*)/2$, and so

$$\widetilde{\omega}_i\widetilde{\omega}^i = (\omega_i\omega^i + \omega_i^*\omega^{i^*})/4 - (\omega_i\omega^{i^*} + \omega_i^*\omega^i)/4. \tag{2.24}$$

Summing (2.24) over i from 1 to n, we find:

$$\begin{aligned}\widetilde{\Omega} &= \sum_{i=1}^n \frac{\omega_i\omega^i + \omega_i^*\omega^{i^*}}{4} - \sum_{i=1}^n \frac{\omega_i\omega^{i^*} + \omega_i^*\omega^i}{4} \\ &= \frac{1}{2}\sum_{i=1}^n \omega_i\omega^i - \frac{1}{4}\sum_{i=1}^n[\omega_i(-\omega^i + 2T_{\omega^i}) + (-\omega_i + 2T_{\omega_i})\omega^i] \\ &= \sum_{i=1}^n \omega_i\omega^i - \frac{1}{2}\sum_{i=1}^n(\omega_i T_{\omega^i} + T_{\omega_i}\omega^i).\end{aligned} \tag{2.25}$$

We conclude from (2.23) and (2.25) that

$$\sum_{i=1}^n T_{\omega_i}T_{\omega^i} = \frac{1}{2}\sum_{i=1}^n(\omega_i T_{\omega^i} + T_{\omega_i}\omega^i), \tag{2.26}$$

and

$$\widetilde{\Omega} = \Omega - \frac{1}{2}\sum_{i=1}^n(\omega_i T_{\omega^i} + T_{\omega_i}\omega^i) = \Omega - \sum_{i=1}^n T_{\omega_i}T_{\omega^i}. \tag{2.27}$$

This is the first assertion of the theorem. For the remainder, write out the definition of T_{ω_i} and T_{ω^i}, and use (2.2):

$$\sum_{i=1}^{n} T_{\omega_i} T_{\omega^i} = \frac{1}{4} \sum_{\alpha,\beta>0} c_\alpha c_\beta \langle \alpha, \beta \rangle t_{s_\alpha} t_{s_\beta} = \frac{1}{4} \sum_{\substack{\alpha>0,\beta>0 \\ s_\alpha(\beta)<0}} c_\alpha c_\beta \langle \alpha, \beta \rangle t_{s_\alpha} t_{s_\beta}, \tag{2.28}$$

with the last equality following as in the proof of Lemma 2.9. □

Corollary 2.12. *Retain the setting of Proposition 2.8 but further assume* (π, X) *is irreducible and unitary with central character* χ_ν *with* $\nu \in V$ *(as in Definition 2.2). Let* (σ, U) *be an irreducible representation of* W *such that* $\mathrm{Hom}_W(U, X) \neq 0$. *Then*

$$\langle \nu, \nu \rangle \leq c(\sigma) \tag{2.29}$$

where

$$c(\sigma) = \frac{1}{4} \sum_{\alpha>0} c_\alpha^2 \langle \alpha, \alpha \rangle + \frac{1}{4} \sum_{\substack{\alpha>0,\beta>0 \\ \alpha\neq\beta, s_\alpha(\beta)<0}} c_\alpha c_\beta \langle \alpha, \beta \rangle \frac{\mathrm{tr}_\sigma(s_\alpha s_\beta)}{\mathrm{tr}_\sigma(1)} \tag{2.30}$$

is the scalar by which Ω_W *acts in* U *and* tr_σ *denotes the character of* σ.

Proof. The result follow from the formula in Theorem 2.11 by applying Proposition 2.8 to a vector x in the σ isotypic component of X. □

Theorem 2.11 will play an important role in the proof of Theorem 3.5 below.

3. The Dirac operator

Throughout this section we fix the setting of Section 2.1.

3.1. The Clifford algebra.

Denote by $C(V_0^\vee)$ the Clifford algebra defined by $V_0^\vee$ and $\langle\ ,\ \rangle$. More precisely, $C(V_0^\vee)$ is the quotient of the tensor algebra of $V_0^\vee$ by the ideal generated by

$$\omega \otimes \omega' + \omega' \otimes \omega + 2\langle \omega, \omega' \rangle, \quad \omega, \omega' \in V_0^\vee.$$

Equivalently, $C(V_0^\vee)$ is the associative algebra with unit generated by $V_0^\vee$ with relations:

$$\omega^2 = -\langle \omega, \omega \rangle, \quad \omega\omega' + \omega'\omega = -2\langle \omega, \omega' \rangle. \tag{3.1}$$

Let $\mathsf{O}(V_0^\vee)$ denote the group of orthogonal transformation of $V_0^\vee$ with respect to $\langle\ ,\ \rangle$. This acts by algebra automorphisms on $C(V_0^\vee)$, and the action of $-1 \in \mathsf{O}(V_0^\vee)$ induces a grading

$$C(V_0^\vee) = C(V_0^\vee)_{\mathsf{even}} + C(V_0^\vee)_{\mathsf{odd}}. \tag{3.2}$$

Let ϵ be the automorphism of $C(V_0^\vee)$ which is $+1$ on $C(V_0^\vee)_{\mathsf{even}}$ and -1 on $C(V_0^\vee)_{\mathsf{odd}}$. Let ^{t} be the transpose antiautomorphism of $C(V_0^\vee)$ characterized by

$$\omega^t = -\omega,\ \omega \in V^\vee, \quad (ab)^t = b^t a^t,\ a, b \in C(V_0^\vee). \tag{3.3}$$

The Pin group is

$$\mathsf{Pin}(V_0^\vee) = \{a \in C(V_0^\vee) \mid \epsilon(a) V_0^\vee a^{-1} \subset V_0^\vee,\ a^t = a^{-1}\}. \tag{3.4}$$

It sits in a short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathsf{Pin}(V_0^\vee) \xrightarrow{\ p\ } \mathsf{O}(V_0^\vee) \longrightarrow 1, \tag{3.5}$$

where the projection p is given by $p(a)(\omega) = \epsilon(a)\omega a^{-1}$. (Note the appearance of ϵ in the definition of $\mathsf{Pin}(V_0^\vee)$. This insures that p is surjective.)

We call acomplex simple $C(V_0^\vee)$ module (γ, S) of dimension $2^{[\dim V/2]}$ a spin module for $C(V_0^\vee)$. When $\dim V$ is even, there is only one such module (up to equivalence), but if $\dim V$ is odd, there are two inequivalent spin modules. We may endow such a module with a positive definite Hermitian form $\langle\ ,\ \rangle_S$ such that

$$\langle\gamma(a)s, s'\rangle_S = \langle s, \gamma(a^t)\rangle_S, \quad \text{for all } a \in C(V_0^\vee) \text{ and } s, s' \in S. \tag{3.6}$$

In all cases, (γ, S) restricts to an irreducible unitary representation of $\mathsf{Pin}(V_0^\vee)$.

3.2. **The Dirac operator D.**

Definition 3.1. Let $\{\omega_i\}$, $\{\omega^i\}$ be dual bases of $V_0^\vee$, and recall the elements $\widetilde{\omega}_i \in \mathbb{H}$ from (2.14). The Dirac element is defined as

$$\mathcal{D} = \sum_i \widetilde{\omega}_i \otimes \omega_i \in \mathbb{H} \otimes C(V_0^\vee).$$

It is elementary to verify that $\mathcal{D}$ does not depend on the choice of dual bases.

Frequently we will work with a fixed spin module (γ, S) for $C(V_0^\vee)$ and a fixed $\mathbb{H}$-module (π, X). In this setting, it will be convenient to define the Dirac operator for X (and S) as $D = (\pi \otimes \gamma)(\mathcal{D})$. Explicitly,

$$D = \sum_{i=1}^n \pi(\widetilde{\omega}_i) \otimes \gamma(\omega^i) \in \mathrm{End}_{\mathbb{H}\otimes C(V_0^\vee)}(X \otimes S). \tag{3.7}$$

Lemma 3.2. *Suppose X is a Hermitian $\mathbb{H}$-module with invariant form $(\ ,\ )_X$. With notation as in* (3.6)*, endow $X \otimes S$ with the Hermitian form $(x \otimes s, x' \otimes s')_{X\otimes S} = (x, x')_X \otimes (s, s')_S$. Then the operator D is self adjoint with respect to $(\ ,\ )_{X\otimes S}$,*

$$(D(x \otimes s), x' \otimes s')_{X\otimes S} = (x \otimes s, D(x' \otimes s'))_{X\otimes S} \tag{3.8}$$

Proof. This follows from a straight-forward verification. □

We immediately deduce the following analogue of Proposition 2.8.

Proposition 3.3. *In the setting of Lemma 3.2, a Hermitian $\mathbb{H}$-module is unitary only if*

$$(D^2(x \otimes s), x \otimes s)_{X\otimes S} \geq 0, \qquad \text{for all } x \otimes s \in X \otimes S. \tag{3.9}$$

To be a useful criterion for unitarity, we need to establish a formula for D^2 (Theorem 3.5 below).

3.3. **The spin cover $\widetilde{W}$.**

The Weyl group W acts by orthogonal transformations on $V_0^\vee$, and thus is a subgroup of $\mathsf{O}(V_0^\vee)$. We define the group $\widetilde{W}$ in $\mathsf{Pin}(V_0^\vee)$:

$$\widetilde{W} := p^{-1}(\mathsf{O}(V_0^\vee)) \subset \mathsf{Pin}(V_0^\vee), \text{ where } p \text{ is as in (3.5).} \tag{3.10}$$

Therefore, $\widetilde{W}$ is a central extension of W,

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{W} \xrightarrow{\ p\ } W \longrightarrow 1. \tag{3.11}$$

We will need a few details about the structure of $\widetilde{W}$. For each $\alpha \in R$, define elements $f_\alpha \in C(V_0^\vee)$ via

$$f_\alpha = \alpha^\vee / |\alpha^\vee| \in V^\vee \subset C(V_0^\vee). \tag{3.12}$$

It follows easily that $p(f_\alpha) = s_\alpha$, the reflection in W through α. Thus $\{f_\alpha \mid \alpha \in R\}$ (or just $\{f_\alpha \mid \alpha \in \Pi\}$) generate $\widetilde{W}$. Obviously $f_\alpha^2 = -1$. Slightly more delicate considerations (e.g. [M, Theorem 3.2]) show that if $\alpha, \beta \in R$, $\gamma = s_\alpha(\beta)$, then

$$f_\beta f_\alpha = -f_\alpha f_\gamma. \tag{3.13}$$

A representation of $\widetilde{W}$ is called genuine if it does not factor to W, i.e. if -1 acts nontrivially. Otherwise it is called nongenuine. (Similar terminology applies to $\mathbb{C}[\widetilde{W}]$ modules.) Via restriction, we can regard a spin module (γ, S) for $C(V_0^\vee)$ as a unitary $\widetilde{W}$ representation. Clearly it is genuine. Since $R^\vee$ spans $V^\vee$, it is also irreducible (e.g. [M] Theorem 3.3). For notational convenience, we lift the sgn representation of W to a nongenuine representation of $\widetilde{W}$ which we also denote by sgn.

We write ρ for the diagonal embedding of $\mathbb{C}[\widetilde{W}]$ into $\mathbb{H} \otimes C(V_0^\vee)$ defined by extending

$$\rho(\widetilde{w}) = t_{p(\widetilde{w})} \otimes \widetilde{w} \tag{3.14}$$

linearly.

Lemma 3.4. *Recall the notation of Definition 3.1 and* (3.14)*. For* $\widetilde{w} \in \widetilde{W}$,

$$\rho(\widetilde{w})\mathcal{D} = \mathsf{sgn}(\widetilde{w})\mathcal{D}\rho(\widetilde{w})$$

as elements of $\mathbb{H} \otimes C(V^\vee)$.

Proof. From the definitions and Proposition 2.10(2), we have

$$\begin{aligned}\rho(\widetilde{w})\mathcal{D}\rho(\widetilde{w}^{-1}) &= \sum_i t_{p(\widetilde{w})}\widetilde{\omega}_i t_{p(\widetilde{w}^{-1})} \otimes \widetilde{w}\omega^i\widetilde{w}^{-1} \\ &= \sum_i \widetilde{p(\widetilde{w}) \cdot \omega_i} \otimes \widetilde{w}\omega^i\widetilde{w}^{-1}\end{aligned}$$

where we have used the $\cdot$ to emphasize the usual action of W on $S(V_0^\vee)$. We argue that in $C(V_0^\vee)$

$$\widetilde{w}\omega^i\widetilde{w}^{-1} = \mathsf{sgn}(\widetilde{w})(p(\widetilde{w}) \cdot \omega^i). \tag{3.15}$$

Then the lemma follows from the fact that the definition of $\mathcal{D}$ is independent of the choice of dual bases.

Since $\widetilde{W}$ is generated by the various f_α for α simple, it is sufficient to verify (3.15) for $\widetilde{w} = f_\alpha$. This follows from direct calculation: $f_\alpha\omega^i f_\alpha^{-1} = -\frac{1}{\langle\alpha^\vee,\alpha^\vee\rangle}\alpha^\vee\omega^i\alpha^\vee = -\frac{1}{\langle\alpha^\vee,\alpha^\vee\rangle}\alpha^\vee(-\alpha^\vee\omega^i - 2\langle\omega,\alpha^\vee\rangle) = -\omega^i + (\omega,\alpha)\alpha^\vee = -s_\alpha \cdot \omega^i$. □

3.4. **A formula for $\mathcal{D}^2$.** Set

$$\Omega_{\widetilde{W}} = \frac{1}{4}\sum_{\substack{\alpha>0,\beta>0\\ s_\alpha(\beta)<0}} c_\alpha c_\beta |\alpha||\beta| f_\alpha f_\beta. \tag{3.16}$$

This is a complex linear combination of elements of $\widetilde{W}$, i.e. an element of $\mathbb{C}[\widetilde{W}]$. Using (3.13), it is easy to see $\Omega_{\widetilde{W}}$ is invariant under the conjugation action of $\widetilde{W}$. Once again $\Omega_{\widetilde{W}}$ will play the role of a Casimir element for $\mathbb{C}[\widetilde{W}]$. The following result should be compared with [P, Proposition 3.1].

Theorem 3.5. *With notation as in* (2.4), (3.7), (3.14), *and* (3.16),

$$\mathcal{D}^2 = -\Omega \otimes 1 + \rho(\Omega_{\widetilde{W}}), \tag{3.17}$$

as elements of $\mathbb{H} \otimes C(V^\vee)$.

Proof. It will be useful below to set

$$R_\circ^2 := \{(\alpha, \beta) \in R \times R : \alpha > 0, \beta > 0, \alpha \neq \beta, s_\alpha(\beta) < 0\}. \tag{3.18}$$

To simplify notation, we fix a self-dual (orthonormal) basis $\{\omega_1 : \ i = 1, \dots, n\}$ of $V^\vee$. From Definition 3.1, we have

$$\mathcal{D}^2 = \sum_{i=1}^n \widetilde{\omega}_i^2 \otimes \omega_i^2 + \sum_{i \neq j} \widetilde{\omega}_i \widetilde{\omega}_j \otimes \omega_i \omega_j$$

Using $\omega_i^2 = -1$ and $\omega_i\omega_j = -\omega_i\omega_j$ in $C(V^\vee)$ and the notation of Definition 2.7, we get

$$\mathcal{D}^2 = -\widetilde{\Omega} \otimes 1 + \sum_{i<j} [\widetilde{\omega}_i, \widetilde{\omega}_j] \otimes \omega_i \omega_j.$$

Applying Theorem 2.11 to the first term and Proposition 2.10(3) to the second, we have

$$\begin{aligned}\mathcal{D}^2 = &- \Omega \otimes 1 + \frac{1}{4}\sum_{\alpha>0} \langle \alpha, \alpha \rangle + \frac{1}{4} \sum_{(\alpha,\beta) \in R_\circ^2} c_\alpha c_\beta \langle \alpha, \beta \rangle t_{s_\alpha} t_{s_\beta} \otimes 1 \\ &- \sum_{i<j} [T_{\omega_i}, T_{\omega_j}] \otimes \omega_i \omega_j.\end{aligned}$$

Rewriting $[T_{\omega_i}, T_{\omega_j}]$ using Proposition 2.10(3) , this becomes

$$\begin{aligned}\mathcal{D}^2 = &- \Omega \otimes 1 + \frac{1}{4}\sum_{\alpha>0} \langle \alpha, \alpha \rangle + \frac{1}{4} \sum_{(\alpha,\beta) \in R_\circ^2} c_\alpha c_\beta \langle \alpha, \beta \rangle t_{s_\alpha} t_{s_\beta} \otimes 1 \\ &- \frac{1}{4} \sum_{i<j} \sum_{(\alpha,\beta) \in R_\circ^2} c_\alpha c_\beta ((\omega_i, \alpha)(\omega_j, \beta) - (\omega_i, \beta)(\omega_j, \alpha)) t_{s_\alpha} t_{s_\beta} \otimes \omega_i \omega_j,\end{aligned}$$

and since $\omega_i\omega_j = -\omega_j\omega_i$ in $C(V^\vee)$,

$$\begin{aligned}\mathcal{D}^2 = &-\Omega \otimes 1 + \frac{1}{4}\sum_{\alpha>0} \langle \alpha, \alpha \rangle + \frac{1}{4} \sum_{(\alpha,\beta) \in R_\circ^2} c_\alpha c_\beta \langle \alpha, \beta \rangle t_{s_\alpha} t_{s_\beta} \otimes 1 \\ &- \frac{1}{4} \sum_{(\alpha,\beta) \in R_\circ^2} c_\alpha c_\beta t_{s_\alpha} t_{s_\beta} \otimes \sum_{i \neq j} ((\omega_i, \alpha)(\omega_j, \beta)) \omega_i \omega_j.\end{aligned}$$

Using (2.2) and the definition of f_α in (3.12), we get

$$\begin{aligned}\mathcal{D}^2 &= -\Omega \otimes 1 + \frac{1}{4}\sum_{\alpha>0} \langle \alpha, \alpha \rangle + \frac{1}{4} \sum_{(\alpha,\beta) \in R_\circ^2} c_\alpha c_\beta \langle \alpha, \beta \rangle t_{s_\alpha} t_{s_\beta} \otimes 1 \\ &\qquad - \frac{1}{4} \sum_{(\alpha,\beta) \in R_\circ^2} c_\alpha c_\beta t_{s_\alpha} t_{s_\beta} \otimes (|\alpha||\beta| f_\alpha f_\beta + \langle \alpha, \beta \rangle) \\ &= -\Omega \otimes 1 + \frac{1}{4}\sum_{\alpha>0} \langle \alpha, \alpha \rangle + \frac{1}{4} \sum_{(\alpha,\beta) \in R_\circ^2} c_\alpha c_\beta |\alpha||\beta| t_{p(f_\alpha)} t_{p(f_\beta)} \otimes f_\alpha f_\beta.\end{aligned}$$

The theorem follows. ∎

Corollary 3.6. *In the setting of Proposition 3.3, assume further that X is irreducible and unitary with central character χ_ν with $\nu \in V$ (as in Definition 2.2). Let $(\widetilde{\sigma}, \widetilde{U})$ be an irreducible representation of $\widetilde{W}$ such that $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, X \otimes S) \neq 0$. Then*

$$\langle \nu, \nu \rangle \leq c(\widetilde{\sigma}) \tag{3.19}$$

where

$$c(\widetilde{\sigma}) = \frac{1}{4} \sum_{\alpha > 0} c_\alpha^2 \langle \alpha, \alpha \rangle + \frac{1}{4} \sum_{\substack{\alpha>0, \beta>0, \alpha \neq \beta \\ s_\alpha(\beta)<0}} c_\alpha c_\beta |\alpha||\beta| \frac{\mathrm{tr}_{\widetilde{\sigma}}(f_\alpha f_\beta)}{\mathrm{tr}_{\widetilde{\sigma}}(1)}, \tag{3.20}$$

is the scalar by which $\Omega_{\widetilde{W}}$ acts in $\widetilde{U}$ and $\mathrm{tr}_{\widetilde{\sigma}}$ denotes the character of $\widetilde{\sigma}$.

Proof. The corollary follows by applying Proposition 3.3 to a vector $x \otimes s$ in the $\widetilde{\sigma}$ isotypic component of $X \otimes S$, and then using the formula for $D^2 = (\pi \otimes \gamma)(\mathcal{D}^2)$ from Theorem 3.5 and the formula for $\pi(\Omega)$ from Lemma 2.5. □

4. Dirac cohomology and Vogan's conjecture

Suppose (π, X) is an irreducible $\mathbb{H}$ module with central character χ_ν. By Lemma 3.4, the kernel of the Dirac operator on $X \otimes S$ is invariant under $\widetilde{W}$. Suppose $\ker(D)$ is nonzero and that $\widetilde{\sigma}$ is an irreducible representation of $\widetilde{W}$ appearing in $\ker(D)$. Then in the notation of Corollary 3.6, Theorem 3.5 and Lemma 2.5 imply that

$$\langle \nu, \nu \rangle = c(\widetilde{\sigma}).$$

In particular, the length of ν is determined by the $\widetilde{W}$ structure of $\ker(D)$. Theorem 4.4 below says that χ_ν itself is determined by this information.

4.1. **Dirac cohomology.**

As discussed above, Vogan's Conjecture suggests that for an irreducible unitary representation X, the $\widetilde{W}$ structure of the kernel of the Dirac operator D should determine the central character of X. This is certainly false for nonunitary representations. But since it is difficult to imagine a proof of an algebraic statement which applies only to unitary representations, we use an idea of Vogan and enlarge the class of irreducible unitary representations for which $\ker(D)$ is nonzero to the class of representations with nonzero Dirac cohomology in the following sense.

Definition 4.1. In the setting of Definition 3.1, define

$$H^D(X) := \ker D \big/ (\ker D \cap \mathrm{im} D) \tag{4.1}$$

and call it the Dirac cohomology of X. (For example, if X is unitary, Lemma 3.2 implies $\ker(D) \cap \mathrm{im}(D) = 0$, and so $H^D(X) = \ker(D)$.)

We roughly follow the outline proposed for real groups by Vogan in [V, Lecture 3] and completed in [HP]. The main technical result of this section is the following algebraic statement. (See Theorem 2.5 and Corollary 3.5 in [HP] for the analogue result for real groups.)

Theorem 4.2. *Let $\mathbb{H}$ be the graded affine Hecke algebra attached to a root system Φ and parameter function c (Definition 2.1). Let $z \in Z(\mathbb{H})$ be given. Then there exists $a \in \mathbb{H} \otimes C(V_0^\vee)$ and a unique element $\zeta(z)$ in the center of $\mathbb{C}[\widetilde{W}]$ such that*

$$z \otimes 1 = \rho(\zeta(z)) + \mathcal{D} a + a \mathcal{D}$$

as elements in $\mathbb{H} \otimes C(V_0^\vee)$*. Moreover, the map* $\zeta : Z(\mathbb{H}) \to \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ *is a homomorphism of algebras.*

The proof of Theorem 4.2 is given after Remark 4.6.

Definition 4.3. Theorem 4.2 allows one to attach canonically to every irreducible $\widetilde{W}$-representation $(\widetilde{\sigma}, \widetilde{U})$ a homomorphism $\chi^{\widetilde{\sigma}} : Z(\mathbb{H}) \to \mathbb{C}$, i.e. a central character of $\mathbb{H}$. More precisely, for every $z \in Z(\mathbb{H})$, $\widetilde{\sigma}(\zeta(z))$ acts by a scalar, and we denote this scalar by $\chi^{\widetilde{\sigma}}(z)$. Since ζ is an algebra homomorphism by Theorem 4.2, the map $\chi^{\widetilde{\sigma}}$ is in fact a homomorphism.

With this definition, Vogan's conjecture takes the following form.

Theorem 4.4. *Let* $\mathbb{H}$ *be the graded affine Hecke algebra attached to a root system* Φ *and parameter function* c *(Definition 2.1). Suppose* (π, X) *is an* $\mathbb{H}$ *module with central character* χ_ν *with* $\nu \in V$ *(as in Definition 2.2). In the setting of Definition 4.1, suppose that* $H^D(X) \neq 0$*. Let* $(\widetilde{\sigma}, \widetilde{U})$ *be an irreducible representation of* $\widetilde{W}$ *such that* $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X)) \neq 0$*. Then*

$$\chi_\nu = \chi^{\widetilde{\sigma}},$$

where $\chi^{\widetilde{\sigma}}$ *is as in Definition 4.3.*

Theorem 4.5. *Theorem 4.2 implies Theorem 4.4.*

Proof. In the setting of Theorem 4.4, suppose $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X)) \neq 0$. Then there exists $\widetilde{x} = x \otimes s \neq 0$ in the $\widetilde{\sigma}$ isotypic component of $X \otimes S$ such that $\widetilde{x} \in \ker D \backslash \mathrm{im} D$. Then for every $z \in Z(\mathbb{H})$, we have

$$(\pi(z) \otimes 1)\widetilde{x} = \chi_\nu(z)\widetilde{x}$$

and

$$(\pi \otimes \gamma)(\rho(\zeta(z)))\widetilde{x} = \widetilde{\sigma}(\zeta(z))\widetilde{x}.$$

Note that the right-hand sides of the previous two displayed equations are scalar multiples of $\widetilde{x}$. Assuming Theorem 4.2, we have

$$(\pi(z) \otimes 1 - (\pi \otimes \gamma)\rho(\zeta(z)))\,\widetilde{x} = (Da + aD)\widetilde{x} = Da\widetilde{x}, \tag{4.2}$$

which would imply that $\widetilde{x} \in \mathrm{im}D$, unless $Da\widetilde{x} = 0$. So we must have $Da\widetilde{x} = 0$, and therefore

$$\widetilde{\sigma}(\zeta(z)) \text{ acts by the scalar } \chi_\nu(z) \text{ for all } z \in Z(\mathbb{H}). \tag{4.3}$$

Now the claim follows by comparison with Definition 4.3. □

Remark 4.6. In Section 5, we shall describe explicitly the central characters $\chi^{\widetilde{\sigma}}$, when the graded Hecke algebra $\mathbb{H}$ has a geometric realization ([L2, L3]).

4.2. **Proof of Theorem 4.2.** Motivated by [HP, Section 3], we define

$$d \;:\; \mathbb{H} \otimes C(V_0^\vee) \longrightarrow \mathbb{H} \otimes C(V_0^\vee). \tag{4.4}$$

on a simple tensor of the form $a = h \otimes v_1 \cdots v_k$ (with $h \in H$ and $v_i \in V^\vee$) via

$$d(a) = \mathcal{D}a - (-1)^k a\mathcal{D},$$

and extend linearly to all of $\mathbb{H} \otimes C(V_0^\vee)$.

Then Lemma 3.4 implies that d interchanges the spaces

$$(\mathbb{H} \otimes C(V_0^\vee))^{\mathsf{triv}} = \{a \in \mathbb{H} \otimes C(V_0^\vee) \mid \rho(\widetilde{w})a = a\rho(\widetilde{w})\} \tag{4.5}$$

and

$$(\mathbb{H} \otimes C(V_0^\vee))^{\mathsf{sgn}} = \{a \in \mathbb{H} \otimes C(V_0^\vee) \mid \rho(\widetilde{w})a = \mathsf{sgn}(\widetilde{w})a\rho(\widetilde{w})\}. \tag{4.6}$$

(Such complications are not encountered in [HP] since the underlying real group is assumed to be connected.) Let d^{triv} (resp. d^{sgn}) denote the restriction of d to the space in (4.5) (resp. (4.6)). We will deduce Theorem 4.2 from the following result.

Theorem 4.7. *With notation as in the previous paragraph,*

$$\ker(d^{\mathsf{triv}}) = \operatorname{im}(d^{\mathsf{sgn}}) \oplus \rho(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}).$$

To see that Theorem 4.7 implies the first assertion of Theorem 4.2, take $z \in Z(\mathbb{H})$. Since $z \otimes 1$ is in $(\mathbb{H} \otimes C_{\mathsf{even}}(V_0^\vee))^{\mathsf{triv}}$ and clearly commutes with $\mathcal{D}$, $z \otimes 1$ is in the kernel of d^{triv}. So the conclusion of Theorem 4.7 implies $z \otimes 1 = d^{\mathsf{sgn}}(a) + \rho(\zeta(z))$ for a unique $\zeta(z) \in \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ and an element a of $(\mathbb{H} \otimes C_{\mathsf{odd}}(V_0^\vee))^{\mathsf{sgn}}$. In particular $d^{\mathsf{sgn}}(a) = \mathcal{D}a + a\mathcal{D}$. Thus

$$z \otimes 1 = \rho(\zeta(z)) + \mathcal{D}a + a\mathcal{D},$$

which is the main conclusion of the Theorem 4.2. (The assertion that ζ is an algebra homomorphism is treated in Lemma 4.16 below.)

Thus everything comes down to proving Theorem 4.7. We begin with some preliminaries.

Lemma 4.8. *We have*

$$\rho(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}) \subset \ker(d^{\mathsf{triv}}).$$

Proof. Fix $\widetilde{w} \in \widetilde{W}$ and let $s_{\alpha_1} \cdots s_{\alpha_k}$ be a reduced expression of $p(w)$ with α_i simple. Then (after possibly replacing α_1 with $-\alpha_1$), $\widetilde{w} = f_{\alpha_1} \cdots f_{\alpha_k}$. Set $a = \rho(\widetilde{w})$. Then the definition of d and Lemma 3.4 imply

$$\begin{aligned} d(a) = \mathcal{D}a - (1)^k a\mathcal{D} &= (1 - (-1)^k \mathsf{sgn}(\widetilde{w}))\mathcal{D}a \\ &= (1 - (-1)^k(-1)^k)\mathcal{D}a = 0, \end{aligned}$$

as claimed. □

Lemma 4.9. *We have* $(d^{\mathsf{triv}})^2 = (d^{\mathsf{sgn}})^2 = 0$.

Proof. For any $a \in \mathbb{H} \otimes C(V_0^\vee)$, one computes directly from the definition of d to find

$$d^2(a) = \mathcal{D}^2 a - a\mathcal{D}^2.$$

By Theorem 3.5, $\mathcal{D}^2 = -\Omega \otimes 1 + \rho(\Omega_{\widetilde{W}})$. By Lemma 2.4, $-\Omega \otimes 1$ automatically commutes with a. If we further assume that a is in $(\mathbb{H} \otimes C(V_0^\vee))^{\mathsf{triv}}$, then a commutes with $\rho(\Omega_{\widetilde{W}})$ as well. Since each term in the definition $\Omega_{\widetilde{W}}$ is in the kernel of sgn, the same conclusion holds if a is in $(\mathbb{H} \otimes C(V_0^\vee))^{\mathsf{sgn}}$. The lemma follows. □

We next introduce certain graded objects. Let $S^j(V^\vee)$ denote the subspace of elements of degree j in $S(V^\vee)$. Let $\mathbb{H}^j$ denote the subspace of $\mathbb{H}$ consisting of products elements in the image of $\mathbb{C}[W]$ and $S^j(V^\vee)$ under the maps described in (1) and (2) in Definition 2.1. Then it is easy to check (using (2.3)) that $\mathbb{H}^0 \subset \mathbb{H}^1 \subset \cdots$ is an algebra filtration. Set $\overline{\mathbb{H}}^j = \mathbb{H}^j/\mathbb{H}^{j-1}$ and let $\overline{\mathbb{H}} = \bigoplus_j \overline{\mathbb{H}}^j$ denote the associated graded algebra. Then $\overline{\mathbb{H}}$ identifies with $\mathbb{C}[W] \ltimes S(V^\vee)$ with $\mathbb{C}[W]$ acting in natural way:

$$t_w \omega t_{w^{-1}} = w(\omega).$$

We will invoke these identification often without comment. Note that $\overline{\mathbb{H}}$ does not depend on the parameter function c used to define $\mathbb{H}$.

The map d of (4.4) induces a map

$$\overline{d} \,:\, \overline{\mathbb{H}} \otimes C(V_0^\vee) \longrightarrow \overline{\mathbb{H}} \otimes C(V_0^\vee). \tag{4.7}$$

Explicitly, if we fix a self-dual basis $\{\omega_1, \dots, \omega_n\}$ of $V^\vee$, then the value of $\overline{d}$ on a simple tensor of the form $a = t_w f \otimes v_1 \cdots v_k$ (with $t_w f \in \mathbb{C}[W] \ltimes S(V^\vee)$ and $v_i \in V^\vee$) is given by

$$\begin{aligned}\overline{d}(a) &= \sum_i \omega_i t_w f \otimes \omega_i v_1 \cdots v_k - (-1)^k \sum_i t_w f \omega_i \otimes v_1 \cdots v_k \omega_i \\ &= \sum_i t_w w^{-1}(\omega_i) f \otimes \omega_i v_1 \cdots v_k - (-1)^k \sum_i t_w f \omega_i \otimes v_1 \cdots v_k \omega_i.\end{aligned} \tag{4.8}$$

We will deduce Theorem 4.7 from the computation of the cohomology of $\overline{d}$. We need some final preliminaries.

Lemma 4.10. *The map $\overline{d}$ of* (4.8) *is an odd derivation in the sense that if $a = t_w f \otimes v_1 \cdots v_k \in \overline{\mathbb{H}}$ and $b \in \overline{\mathbb{H}}$ is arbitrary, then*

$$\overline{d}(ab) = \overline{d}(a)b + (-1)^k a\overline{d}(b).$$

Proof. Fix a as in the statement of the lemma. A simple induction reduces the general case of the lemma to the following three special cases: (i) $b = \omega \otimes 1$ for $\omega \in V^\vee$; (ii) $b = 1 \otimes \omega$ for $\omega \in V^\vee$; and (iii) $b = t_s \otimes 1$ for $s = s_\alpha$ a simple reflection in W. (The point is that these three types of elements generate $\overline{\mathbb{H}}$.) Each of these cases follows from a straight-forward verification. For example, consider the first case, $b = \omega \otimes 1$. Then from the definition of $\overline{d}$, we have

$$\overline{d}(ab) = \sum_i \omega_i t_w f \omega \otimes \omega_1 v_1 \cdots v_k - (-1)^k \sum_i t_w f \omega \omega_i \otimes v_1 \cdots v_k \omega_i. \tag{4.9}$$

On the other hand, since it is easy to see that $d(b) = 0$ in this case, we have

$$\begin{aligned}\overline{d}(a)b + (-1)^k a\overline{d}(b) &= \overline{d}(a)b \\ &= \sum_i \omega_i t_w f \omega \otimes \omega_1 v_1 \cdots v_k - (-1)^k \sum_i t_w f \omega_i \omega \otimes v_1 \cdots v_k \omega_i.\end{aligned} \tag{4.10}$$

Since $S(V^\vee)$ is commutative, (4.9) and (4.10) coincide, and the lemma holds in this case. The other two remaining cases hold by similar direct calculation. We omit the details. □

Lemma 4.11. *The map $\bar{d}$ satisfies $\bar{d}^2 = 0$.*

Proof. Fix $a = t_w f \otimes v_1 \cdots v_k \in \overline{\mathbb{H}}$ and let b be arbitrary. Using Lemma 4.10, one computes directly from the definitions to find

$$\overline{d}^2(ab) = \overline{d}^2(a)b + a\overline{d}^2(b).$$

It follows that to establish the current lemma in general, it suffices to check that $d^2(b) = 0$ for each of the three kinds of generators b appearing in the proof of Lemma 4.10. Once again this is a straight-forward verification whose details we omit. (Only case (iii) is nontrivial.) □

Lemma 4.12. *Let $\overline{\rho}$ denote the diagonal embedding of $\mathbb{C}[\widetilde{W}]$ in $\overline{\mathbb{H}} \otimes C(V_0^\vee)$ defined by linearly extending*

$$\overline{\rho}(\widetilde{w}) = t_{p(f_\alpha)} \otimes \widetilde{w}$$

for $\widetilde{w} \in \widetilde{W}$. Then

$$\overline{\rho}(\mathbb{C}[\widetilde{W}]) \subset \ker(\overline{d}).$$

Proof. As noted in Section 3.3, the various $f_\alpha = \alpha^\vee/|\alpha^\vee|$ (for α simple) generate $\widetilde{W}$. Furthermore $p(f_\alpha) = s_\alpha$. So Lemma 4.10 implies that the current lemma will follow if we can prove

$$\overline{d}(t_{s_\alpha} \otimes \alpha^\vee) = 0$$

for each simple α. For this we compute directly,

$$\begin{aligned}
\overline{d}(t_{s_\alpha} \otimes \alpha^\vee) &= \sum_i \omega_i t_{s_\alpha} \otimes \omega_i \alpha^\vee + \sum_i t_{s_\alpha} \omega_i \otimes \alpha^\vee \omega_i \\
&= \sum_i t_{s_\alpha} s_\alpha(\omega_i) \otimes \omega_i \alpha^\vee + \sum_i t_{s_\alpha} \omega_i \otimes \alpha^\vee \omega_i \\
&= \sum_i t_{s_\alpha} (\omega_i - (\alpha, \omega_i))\alpha^\vee \otimes \omega_i \alpha^\vee + \sum_i t_{s_\alpha} \omega_i \otimes \alpha^\vee \omega_i \\
&= \sum_i t_{s_\alpha} \omega_i \otimes (\omega_i \alpha^\vee + \alpha^\vee \omega_i) - \sum_i t_{s_\alpha} (\alpha, \omega_i) \alpha^\vee \otimes \omega_i \alpha^\vee \\
&= -2 \sum_i t_{s_\alpha} \omega_i \otimes \langle \alpha^\vee, \omega_i \rangle - \sum_i t_{s_\alpha} \alpha^\vee \otimes (\alpha, \omega_i) \omega_i \alpha^\vee \\
&= -2 \sum_i t_{s_\alpha} \langle \alpha^\vee, \omega_i \rangle \omega_i \otimes 1 - \sum_i t_{s_\alpha} \alpha^\vee \otimes \frac{\langle \omega_i, \alpha \rangle}{\langle \alpha^\vee, \alpha^\vee \rangle} \omega_i \alpha^\vee \\
&= -2 t_{s_\alpha} \alpha^\vee \otimes 1 \quad - \quad t_{s_\alpha} \alpha^\vee \otimes \frac{(\alpha^\vee)^2}{\langle \alpha^\vee, \alpha^\vee \rangle} \\
&= -2 t_{s_\alpha} \alpha^\vee \otimes 1 \quad + \quad 2 t_{s_\alpha} \alpha^\vee \otimes 1 = 0.
\end{aligned}$$

□

Note from (4.8), it follows that $\overline{d}$ preserves the subspace $S(V^\vee) \otimes C(V_0^\vee) \subset \overline{\mathbb{H}} \otimes C(V_0^\vee)$. Write $\overline{d}'$ for the restriction of $\overline{d}$ to $S(V^\vee) \otimes C(V_0^\vee)$.

Lemma 4.13. *With notation as in the previous paragraph,*

$$\ker(\overline{d}') = \operatorname{im}(\overline{d}') \oplus \mathbb{C}(1 \otimes 1).$$

Proof. An elementary calculation shows that $\overline{d}'$ is a multiple of the differential in the Koszul complex whose cohomology is well-known. (See [HP, Lemma 4.1], for instance.) □

We can now assemble these lemmas into the computation of the cohomology of $\overline{d}$.

Proposition 4.14. *We have*

$$\ker(\overline{d}) = \operatorname{im}(\overline{d}) \oplus \overline{\rho}(\mathbb{C}[\widetilde{W}]).$$

Proof. By Lemmas 4.11 and 4.12, $\mathrm{im}(\overline{d}) + \overline{\rho}(\mathbb{C}[\widetilde{W}]) \subset \ker(\overline{d})$. Since it follows from the definition of $\overline{d}$ that $\mathrm{im}(\overline{d})$ and $\overline{\rho}(\mathbb{C}[\widetilde{W}])$ intersect trivially, we need only establish the reverse inclusion. Fix $a \in \ker(\overline{d})$ and write it as a sum of simple tensors of the form $t_w f \otimes v_1 \cdots v_k$. For each $w_j \in W$, let a_j denote the sum of the simple tensors appearing in this expression for a which have t_{w_j} in them. Thus $a = a_1 + \cdots a_l$, and we can arrange the indexing so that each a_i is nonzero. Since $\overline{d}(a) = 0$,

$$\overline{d}(a_1) + \cdots + \overline{d}(a_l) = 0. \tag{4.11}$$

Each term $\overline{d}(a_i)$ is a sum of simple tensors of the form $t_{w_i} f \otimes v_1 \cdots v_k$. Since the w_i are distinct, the only way (4.11) can hold is if each $\overline{d}(a_i) = 0$. Choose $\widetilde{w}_i \in \widetilde{W}$ such that $p(\widetilde{w}_i) = w_i$. Set

$$a_i' = \overline{\rho}(\widetilde{w}_1^{-1}) a_i \in S(V) \otimes C(V_0^\vee).$$

Using Lemmas 4.10 and 4.12, we have

$$\overline{\rho}(\widetilde{w}_i)\overline{d}(a_i') = \overline{d}(\overline{\rho}(\widetilde{w}_i)a_i) = \overline{d}(a_i) = 0.$$

Thus for each i,

$$\overline{d}(a_i') = 0.$$

Since each $a_i' \in S(V^\vee) \otimes C(V_0^\vee)$, Lemma 4.13 implies $a_i' = \overline{d}(b_i') \oplus c_i(1 \otimes 1)$ with $b_i' \in S(V^\vee) \otimes C(V_0^\vee)$ and $c_i \in \mathbb{C}$. Using Lemmas 4.10 and 4.12 once again, we have

$$\begin{aligned} a_i &= \overline{\rho}(\widetilde{w}_i) a_i' \\ &= \overline{\rho}(\widetilde{w}_i) \left(\overline{d}'(b_i') + c_i(1 \otimes 1) \right) \\ &= \overline{d}(\rho(\widetilde{w}_i) b_i') + c_i \overline{\rho}(\widetilde{w}_i) \\ &\in \mathrm{im}(\overline{d}) + \rho(\mathbb{C}[\widetilde{W}]). \end{aligned}$$

Hence $a = a_1 + \cdots + a_l \in \mathrm{im}(\overline{d}) + \rho(\mathbb{C}[\widetilde{W}])$ and the proof is complete. □

The considerations around (4.5) also apply in the graded setting. In particular, using an argument as in the proof of Lemma 3.4, we conclude $\overline{d}$ interchanges the spaces

$$(\overline{\mathbb{H}} \otimes C(V_0^\vee))^{\mathsf{triv}} = \{a \in \overline{\mathbb{H}} \otimes C(V_0^\vee) \mid \rho(\widetilde{w})a = a\rho(\widetilde{w})\} \tag{4.12}$$

and

$$(\overline{\mathbb{H}} \otimes C(V_0^\vee))^{\mathsf{sgn}} = \{a \in \overline{\mathbb{H}} \otimes C(V_0^\vee) \mid \rho(\widetilde{w})a = \mathsf{sgn}(\widetilde{w})a\rho(\widetilde{w})\}. \tag{4.13}$$

As before, let $\overline{d}^{\mathsf{triv}}$ (resp. $\overline{d}^{\mathsf{sgn}}$) denote the restriction of d to the space in (4.5) (resp. (4.6)). Passing to the subspace

$$(\overline{\mathbb{H}} \otimes C(V_0^\vee))^{\mathsf{triv}} \oplus (\overline{\mathbb{H}} \otimes C(V_0^\vee))^{\mathsf{sgn}}$$

in Proposition 4.14 we obtain the following corollary.

Corollary 4.15. *With notation as in the previous paragraph,*

$$\ker(\overline{d}^{\mathsf{triv}}) = \mathrm{im}(\overline{d}^{\mathsf{sgn}}) \oplus \overline{\rho}(\mathbb{C}[\widetilde{W}]^{\widetilde{W}}).$$

Theorem 4.7, and hence the first part of Theorem 4.2, now follow from Corollary 4.15 by an easy induction based on the degree of the filtration. All that remains is to prove that the map $\zeta : Z(\mathbb{H}) \to \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ is an algebra homomorphism.

Lemma 4.16. *The map* $\zeta : Z(\mathbb{H}) \to \mathbb{C}[\widetilde{W}]^{\widetilde{W}}$ *defined by Theorem 4.2 is a homomorphism of algebras.*

Proof. The nontrivial part of the claim is that $\zeta(z_1z_2) = \zeta(z_1)\zeta(z_2)$, for all $z_1, z_2 \in Z(\mathbb{H})$. From Theorem 4.7, there exist elements $a_1, a_2 \in (\mathbb{H} \otimes C_{\mathsf{odd}}(V_0^\vee))^{\mathsf{sgn}}$ such that $z_i \otimes 1 = \rho(\zeta(z_i)) + d^{\mathsf{sgn}}(a_i)$, $i = 1, 2$. Therefore,

$$\begin{aligned} z_1z_2 \otimes 1 &= \rho(\zeta(z_1)\zeta(z_2)) + \rho(\zeta(z_1))d^{\mathsf{sgn}}(a_2) + d^{\mathsf{sgn}}(a_1)\rho(\zeta(z_2)) + d^{\mathsf{sgn}}(a_1)d^{\mathsf{sgn}}(a_2) \\ &= \rho(\zeta(z_1)\zeta(z_2)) + d^{\mathsf{sgn}}(\rho(\zeta(z_1))a_2 + a_1\rho(\zeta(z_2)) + a_1d^{\mathsf{sgn}}(a_2)), \end{aligned}$$

by Lemma 4.8 and Lemma 4.10. The claim now follows by applying Theorem 4.7 to $z_1z_2 \otimes 1$. □

This completes the proof of Theorem 4.2 and hence (by Theorem 4.5) Theorem 4.4 as well.

5. Vogan's conjecture and geometry of nilpotent orbits

In this section (for the reasons mentioned in the introduction), we fix a crystallographic root system Φ and set the parameter function c in Definition 2.1 to be identically 1, i.e. $c_\alpha = 1$ for all $\alpha \in R$.

5.1. **Geometry of irreducible representations of $\widetilde{W}$.** Let $\mathfrak{g}$ denote the complex semisimple Lie algebra corresponding to Φ. In particular, $\mathfrak{g}$ has a Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{h} \simeq V$ canonically. Write $\mathcal{N}$ for the nilpotent cone in $\mathfrak{g}$. Let G denote the adjoint group $\mathrm{Ad}(\mathfrak{g})$ acting by the adjoint action on $\mathcal{N}$.

Given $e \in \mathcal{N}$, let $\{e, h, f\} \subset \mathfrak{g}$ denote an $\mathfrak{sl}_2$ triple with $h \in \mathfrak{h}$ semisimple. Set

$$\nu_e = \frac{1}{2}h \in \mathfrak{h} \simeq V. \tag{5.1}$$

The element ν_e depends on the choices involved. But its W-orbit (and in particular $\langle \nu_e, \nu_e \rangle$ and the central character χ_{ν_e} of Definition 2.2) are well-defined independent of the G orbit of e.

Let

$$\mathcal{N}_{\mathsf{sol}} = \{e \in \mathcal{N} \mid \text{ the centralizer of } e \text{ in } \mathfrak{g} \text{ is a solvable Lie algebra}\}. \tag{5.2}$$

Then G also acts on $\mathcal{N}_{\mathsf{sol}}$.

Next let $A(e)$ denote the component group of the centralizer of $e \in \mathcal{N}$ in G. To each $e \in \mathcal{N}$, Springer has defined a graded representation of $W \times A(e)$ (depending only on the G orbit of e) on the total cohomology $H^\bullet(\mathcal{B}^e)$ of the Springer fiber over e. Set $d(e) = 2\dim(\mathcal{B}^e)$, and define

$$\sigma_{e,\phi} = \left(H^{d(e)}(\mathcal{B}^e)\right)^\phi \in \mathsf{Irr}(W) \cup \{0\}, \tag{5.3}$$

the ϕ invariants in the top degree. (In general, given a finite group H, we write $\mathsf{Irr}(H)$ for the set of equivalence classes of its irreducible representations.) Let $\mathsf{Irr}_0(A(e)) \subset \mathsf{Irr}(A(e))$ denote the subset of representations of "Springer type", i.e. those ϕ such that $\sigma_{e,\phi} \neq 0$.

Finally, let $\mathsf{Irr}_{\mathsf{gen}}(\widetilde{W}) \subset \mathsf{Irr}(\widetilde{W})$ denote the subset of genuine representations.

Theorem 5.1 ([C])**.** *Recall the notation of* (5.1), (5.2), *and* (5.3). *Then there is a surjective map*

$$\Psi : \ \mathsf{Irr}_{\mathsf{gen}}(\widetilde{W}) \longrightarrow G\backslash\mathcal{N}_{\mathsf{sol}} \tag{5.4}$$

with the following properties:

(1) *If* $\Psi(\widetilde{\sigma}) = G \cdot e$, *then*
$$c(\widetilde{\sigma}) = \langle \nu_e, \nu_e \rangle, \tag{5.5}$$
where $c(\widetilde{\sigma})$ *is defined in* (3.20).

(2) (a) *For a fixed spin module* (γ, S) *for* $C(V_0^\vee)$, *if* $e \in \mathcal{N}_{\mathsf{sol}}$ *and* $\phi \in \mathsf{Irr}_0(A(e))$, *then there exists* $\widetilde{\sigma} \in \Psi^{-1}(G \cdot e)$ *so that*
$$\mathrm{Hom}_W\,(\sigma_{e,\phi}, \widetilde{\sigma} \otimes S) \neq 0.$$
(b) *If* $\Psi(\widetilde{\sigma}) = G \cdot e$, *then there exists* $\phi \in \mathsf{Irr}_0(A(e))$ *and a spin module* (γ, S) *for* $C(V_0^\vee)$, *such that*
$$\mathrm{Hom}_W\,(\sigma_{e,\phi}, \widetilde{\sigma} \otimes S) \neq 0.$$

Together with Corollary 3.6, we immediately obtain the following.

Corollary 5.2. *Suppose* (π, X) *is an irreducible unitary* $\mathbb{H}$*-module with central character* χ_ν *with* $\nu \in V$ *(as in Definition 2.2). Fix a spin module* (γ, S) *for* $C(V^\vee)$.

(a) *Let* $(\widetilde{\sigma}, \widetilde{U})$ *be a representation of* $\widetilde{W}$ *such that* $\mathrm{Hom}_{\widetilde{W}}\left(\widetilde{U}, X \otimes S\right) \neq 0$. *In the notation of Theorem 5.1, write* $\Psi(\widetilde{\sigma}) = G \cdot e$. *Then*
$$\langle \nu, \nu \rangle \leq \langle \nu_e, \nu_e \rangle. \tag{5.6}$$
(b) *Suppose* $e \in \mathcal{N}_{\mathsf{sol}}$ *and* $\phi \in \mathsf{Irr}_0(A(e))$ *such that* $\mathrm{Hom}_W(\sigma_{(e,\phi)}, X) \neq 0$. *Then*
$$\langle \nu, \nu \rangle \leq \langle \nu_e, \nu_e \rangle. \tag{5.7}$$

Remark 5.3. The bounds in Corollary 5.2 represents the best possible in the sense that there exist X such that the inequalities are actually equalities. For example, consider part (b) of the corollary, fix $\phi \in \mathsf{Irr}_0(A(e))$, and let $X_t(e, \phi)$ be the unique tempered representation of $\mathbb{H}$ parametrized by (e, ϕ) in the Kazhdan-Lusztig classification ([KL]). Thus $X_t(e, \phi)$ is an irreducible unitary representation with central character χ_{ν_e} and, as a representation of W,
$$X_t(e, \phi) \simeq H^\bullet(\mathcal{B}_e)^\phi.$$
In particular, $\sigma_{e,\phi}$ occurs with multiplicity one in $X_t(e, \phi)$ (in the top degree). Thus the inequality in Corollary 5.2(b) applied to $X_t(e, \phi)$ is an equality.

The representations $X_t(e, \phi)$ will play an important role in our proof of Theorem 5.8.

5.2. **Applications to unitary representations.** Recall that there exists a unique open dense G-orbit in $\mathcal{N}$, the regular orbit; let $\{e_{\mathsf{r}}, h_{\mathsf{r}}, f_{\mathsf{r}}\}$ be a corresponding $\mathfrak{sl}_2$ with $h_{\mathsf{r}} \in \mathfrak{h}$, and set $\nu_{\mathsf{r}} = \frac{1}{2} h_{\mathsf{r}}$. If $\mathfrak{g}$ is simple, then there exists a unique open dense G-orbit in the complement of $G \cdot e_{\mathsf{r}}$ in $\mathcal{N}$ called the subregular orbit. Let $\{e_{\mathsf{sr}}, h_{\mathsf{sr}}, f_{\mathsf{sr}}\}$ be an $\mathfrak{sl}_2$ triple for the subregular orbit with $h_{\mathsf{sr}} \in \mathfrak{h}$, and set $\nu_{\mathsf{sr}} = \frac{1}{2} h_{\mathsf{sr}}$.

The tempered module $X_t(e_{\mathsf{r}}, \mathsf{triv})$ is the Steinberg discrete series, and we have $X_t(e_{\mathsf{r}}, \mathsf{triv})|_W = \mathsf{sgn}$. When $\mathfrak{g}$ is simple, the tempered module $X_t(e_{\mathsf{sr}}, \mathsf{triv})$ has dimension $\dim V + 1$, and $X_t(e_{\mathsf{sr}}, \mathsf{triv})|_W = \mathsf{sgn} \oplus \mathsf{refl}$, where refl is the reflection W-type.

Now we can state certain bounds for unitary $\mathbb{H}$-modules.

Corollary 5.4. *Let* (π, X) *be an irreducible unitary* $\mathbb{H}$*-module with central character* χ_ν *with* $\nu \in V$ *(as in Definition 2.2). Then, we have:*

(1) $\langle \nu, \nu \rangle \leq \langle \nu_{\mathsf{r}}, \nu_{\mathsf{r}} \rangle$;
(2) *if* $\mathfrak{g}$ *is simple of rank at least* 2*, and* X *is* not *the trivial or the Steinberg* $\mathbb{H}$*-module, then* $\langle \nu, \nu \rangle \leq \langle \nu_{\mathsf{sr}}, \nu_{\mathsf{sr}} \rangle$.

Proof. The first claim follows from Corollary 5.2(1) since $\langle \nu_e, \nu_e \rangle \leq \langle \nu_{\mathsf{r}}, \nu_{\mathsf{r}} \rangle$, for every $e \in \mathcal{N}$.

For the second claim, assume X is not the trivial or the Steinberg $\mathbb{H}$-module. Then X contains a W-type σ such that $\sigma \neq \mathsf{triv}, \mathsf{sgn}$. We claim that $\sigma \otimes S$, where S is a fixed irreducible spin module, contains a $\widetilde{W}$-type $\widetilde{\sigma}$ which is not a spin module. If this were not the case, assuming for simplicity that $\dim V$ is even, we would find that $\sigma \otimes S = S \oplus \cdots \oplus S$, where there are $\mathrm{tr}_\sigma(1)$ copies of S in the right hand side. In particular, we would get $\mathrm{tr}_\sigma(s_\alpha s_\beta)\,\mathrm{tr}_S(f_\alpha f_\beta) = \mathrm{tr}_\sigma(1)\,\mathrm{tr}_S(f_\alpha f_\beta)$. Notice that this formula is true when $\dim V$ is odd too, since the two inequivalent spin modules in this case have characters which have the same value on $f_\alpha f_\beta$. If $\langle \alpha, \beta \rangle \neq 0$, then we know that $\mathrm{tr}_S(f_\alpha f_\beta) \neq 0$ ([M]). This means that $\mathrm{tr}_\sigma(s_\alpha s_\beta) = \mathrm{tr}_\sigma(1)$, for all non-orthogonal roots α, β. One verifies directly that, when Φ is simple of rank two, this relation does not hold. Thus we obtain a contradiction.

Returning to the second claim in the corollary, let $\widetilde{\sigma}$ be a $\widetilde{W}$-type appearing in $X \otimes S$ which is not a spin module. Let e be a nonregular nilpotent element such that $\Psi(\widetilde{\sigma}) = G \cdot e$. Corollary 5.2 says that $\langle \nu, \nu \rangle \leq \langle \nu_e, \nu_e \rangle$. To complete the proof, recall that if $\mathfrak{g}$ is simple, the largest value for $\langle \nu_e, \nu_e \rangle$, when e is not a regular element, is obtained when e is a subregular nilpotent element. □

Remark 5.5. Standard considerations for reducibility of principal series allow one to deduce a strengthened version of Corollary 5.4(1) assuming $\nu \in V_0$, namely that ν is contained in the convex hull of the Weyl group orbit of ν_{r}. This corresponds to a classical results of Howe and Moore [HM]. We also note that Corollary 5.4(2) implies, in particular, that, when the roots system is simple, the only unitary irreducible $\mathbb{H}$-modules with central character ν_{r} are the trivial and the Steinberg module. This is a version of a well-known result of Casselman [Ca]. Moreover, Corollary 5.4(2) shows that the trivial $\mathbb{H}$-module is isolated in the unitary dual of $\mathbb{H}$ for all simple root systems of rank at least two, and it gives the best possible spectral gap for the trivial module.

Remark 5.6. There is another, subtler application of Corollary 5.2(2). Assume that $X(s, e, \psi)$ is an irreducible $\mathbb{H}$-module parametrized in the Kazhdan-Lusztig classification by the G-conjugacy class of $\{s, e, \psi\}$, where $s \in V_0$, $[s, e] = e$, $\psi \in \mathsf{Irr}_0 A(s, e)$. The group $A(s, e)$ embeds canonically in $A(e)$. Let $\mathsf{Irr}_0 A(s, e)$ denote the subset of elements in $\mathsf{Irr} A(s, e)$ which appear in the restriction of an element of $\mathsf{Irr}_0 A(e)$. The module $X(s, e, \psi)$ is characterized by the property that it contains every W-type $\sigma_{(e,\phi)}$, $\phi \in \mathsf{Irr}_0 A(e)$ such that $\mathrm{Hom}_{A(s,e)}(\psi, \phi) \neq 0$.

Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$ triple containing e. One may choose s such that $s = \frac{1}{2}h + s_z$, where $s_z \in V_0$ centralizes $\{e, h, f\}$ and s_z is orthogonal to h with respect to $\langle\ ,\ \rangle$. When $s_z = 0$, we have $A(s, e) = A(h, e) = A(e)$, and $X(\frac{1}{2}h, e, \psi)$ is the tempered module $X_t(e, \phi)$ ($\phi = \psi$) from before.

Corollary 5.2(2) implies that if $e \in \mathcal{N}_{\mathsf{sol}}$, then $X(s, e, \psi)$ is unitary *if and only if* $X(s, e, \psi)$ is tempered.

5.3. **Dirac cohomology and nilpotent orbits.** In this setting, we can sharpen the results from Section 4, in particular Theorem 4.4, by making use of Theorem 5.1.

(For comments related to dropping the equal-parameter crystallographic condition, see Remark 5.9.)

Proposition 5.7. *Let (π, X) be an irreducible $\mathbb{H}$ module with central character χ_ν with $\nu \in V$ (as in Definition 2.2). In the setting of Definition 3.1, suppose $(\widetilde{\sigma}, \widetilde{U})$ is an irreducible representation of $\widetilde{W}$ such that $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, X \otimes S) \neq 0$. Write $\Psi(\widetilde{\sigma}) = G \cdot e$ as in Theorem 5.1. Assume further that $\langle \nu, \nu \rangle = \langle \nu_e, \nu_e \rangle$. Then*

$$\mathrm{Hom}_{\widetilde{W}}\left(\widetilde{U}, H^D(X)\right) \neq 0.$$

Proof. Let $x \otimes s$ be an element of the $\widetilde{\sigma}$ isotypic component of $X \otimes S$. By Theorem 3.5 and Theorem 5.1, have

$$D^2(x \otimes s) = (-\langle \nu, \nu \rangle + \langle \nu_e, \nu_e \rangle)\,(x \otimes s) = 0. \tag{5.8}$$

Since X is unitary, $\ker D \cap \mathrm{im} D = 0$, and so (5.8) implies $x \otimes s \in \ker(D) = H^D(X)$. □

Theorem 5.8. *Let $\mathbb{H}$ be the graded affine Hecke algebra attached to a crystallographic root system Φ and constant parameter function $c \equiv 1$ (Definition 2.1). Suppose (π, X) is an $\mathbb{H}$ module with central character χ_ν with $\nu \in V$ (as in Definition 2.2). In the setting of Definition 4.1, suppose that $H^D(X) \neq 0$. Let $(\widetilde{\sigma}, \widetilde{U})$ be a representation of $\widetilde{W}$ such that $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X)) \neq 0$. Using Theorem 5.1, write $\Psi(\widetilde{\sigma}) = G \cdot e$. Then*

$$\chi_\nu = \chi_{\nu_e}.$$

Proof. The statement of Theorem 5.8 will follow from Theorem 4.4, if we can show $\chi^{\widetilde{\sigma}}(z) = \chi_{\nu_e}(z)$ for all $z \in Z(\mathbb{H})$ where $\Psi(\widetilde{\sigma}) = G \cdot e$ as in Theorem 5.1.

Using Theorem 5.1(2b), choose $\phi \in \mathsf{Irr}_0(A(e))$ such that

$$\mathrm{Hom}_{\widetilde{W}}(\widetilde{\sigma}, \sigma_{e,\phi} \otimes S) \neq 0, \tag{5.9}$$

and consider the unitary $\mathbb{H}$ module $X(e, \phi)$ of Remark 5.3 with central character χ_{ν_e}. Then since $X_t(e, \phi)$ contains the W type $\sigma_{e,\phi}$, (5.9) implies

$$\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, X \otimes S) \neq 0.$$

So Proposition 5.7 implies that

$$\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X)) \neq 0.$$

Since $X(e, \phi)$ has central character χ_{ν_e}, (4.3) applies to give that $\widetilde{\sigma}(\zeta(z))$ acts by the scalar $\chi_{\nu_e}(z)$ for all $z \in \mathbb{H}$. This completes the proof. □

Remark 5.9. Note that the proof of Theorem 5.8 depended on two key ingredients beyond Theorems 4.2 and 4.4: Theorem 5.1 and the classification (and W-structure) of tempered modules. Both results are available for the algebras considered by Lusztig in [L2], the former by [C, Theorem 3.10.1] and the latter by [L3]. Thus our proof establishes a version of Theorem 5.8 for cases of the unequal parameters as in [L2].

It would be interesting to consider the problem of identifying the central characters $\chi^{\widetilde{\sigma}}$ (Definition 4.3), for every irreducible $\widetilde{W}$-representation $\widetilde{\sigma}$, in the setting of a graded affine Hecke algebra $\mathbb{H}$ attached to an arbitrary root system Φ and an arbitrary parameter function c. We expect that the set $\{\chi^{\widetilde{\sigma}} : \widetilde{\sigma} \in \mathsf{Irr}\widetilde{W}\}$ is always a subset of the set of central characters of irreducible tempered $\mathbb{H}$-modules, and at least when c is non-constant, we expect it is, in fact, precisely the set of central characters for elliptic tempered $\mathbb{H}$-modules (in the sense of [OS]).

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