Math 6240, Lie Groups and Lie Algebras I
September 3, 2018; to be collected September 10.

We did the computations below in class. The purpose of this exercise is to write the solutions up nicely, in order to make sure you understand all the steps.

Exercise. Let $G = \mathrm{GL}(n, \mathbb{R}) := \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^n)$, the group of linear automorphisms of $\mathbb{R}^n$, i.e. the group of n-by-n invertible matrices. Let 1 denote the identity in G.

(i) Let $\mathfrak{g} = \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n)$, the space of linear endomorphisms of $\mathbb{R}^n$, i.e. all n-by-n matrices. Write down an explicit linear isomorphism between $T_1(G)$ and $\mathfrak{g}$.

(ii) For any $g \in G$, let c_g denote conjugation by g. This is a smooth map from G to G that maps 1 to 1, so we may consider its differential at 1,

$$\mathrm{Ad}(g) := T_1 c_g \;:\; T_1(G) \longrightarrow T_1(G).$$

Using (i), we can interpret $\mathrm{Ad}(g)$ as a map between n-by-n matrices. Compute $\mathrm{Ad}(g)$ explicitly.

(iii) Confirm that Ad is a homomorphism between G and $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$. Then note that the definition of Ad makes sense for any Lie group G, and verify that it is a homomorphism.

(iv) Since Ad is a smooth map between G and $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ that maps the identity to the identity, we can consider its differential:

$$\mathrm{ad} := T_1(\mathrm{Ad}) \;:\; T_1(G) \longrightarrow T_1(\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})).$$

After applying (i) to both the domain and range, we can interpret this as

$$\mathrm{ad} \;:\; \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathfrak{g}).$$

Compute this map explicitly. If we write $[AB] = AB - BA$ for the commutator of two linear endomorphism A and B of the same space, verify that

$$\mathrm{ad}([AB]) = [\mathrm{ad}(A), \mathrm{ad}(B)].$$