MATH 6240, PROBLEM SET 3: CLASSICAL GROUPS

due Friday, December 7

1.(a) Show that if H is a hermitian form on a complex vector space V, then the real part $R = Re(H)$ is a symmetric form on the underlying real vector space V, and the imaginary part $C = Im(H)$ is a skew-symmetric real form on the underlying real space. Observe that R is invariant under multiplication by i: $R(iv, iw) = R(v, w)$. Conversely show that any R which is invariant in this sense is the real part of unique Hermitian form H. For a standard choice of H, compute R and C, and then deduce

$$\mathrm{U}(n) \simeq \mathrm{O}(2n) \cap \mathrm{Sp}(2n, \mathbb{R}).$$

(b) Argue similarly as in (a) to prove

$$\mathrm{Sp}(n) \simeq \mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C}),$$

and, more generally,

$$\mathrm{Sp}(p, q) \simeq U(2p, 2q) \cap \mathrm{Sp}(2n, \mathbb{C}).$$

(c) Show $\mathrm{Sp}(2n, \mathbb{R}) \simeq \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(n, n)$.

(d) Is $\mathrm{O}(p, q)$ naturally a subgroup of $\mathrm{U}(2p, 2q)$?

2. Find a two-to-one homomorphism from $\mathrm{SU}(2)$ to $\mathrm{SO}(3)$.

3. (a) Show directly that $\mathfrak{sp}(2, \mathbb{R})$, $\mathfrak{su}(1, 1)$, and $\mathfrak{so}(2, 1)$ are isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

(b) Show that all of the following groups have isomorphic Lie algebras:

$$\mathrm{SL}(2, \mathbb{R});$$
$$\mathrm{SU}(1, 1);$$
$$\mathrm{SO}(2, 1);$$
$$\mathrm{Sp}(2, \mathbb{R});$$
$$\mathrm{SL}^{\pm}(2, \mathbb{R}) = \{g \in \mathrm{GL}(2, \mathrm{R}) \mid \det(g) = \pm 1\};$$
$$\mathrm{PSp}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})/Z''$$

where Z'' is the center of $\mathrm{Sp}(2, \mathbb{R})$;

$$\mathrm{PGL}'(2, \mathbb{R}) = \mathrm{GL}(2, \mathbb{R})/Z';$$

where Z' is the center of $\mathrm{GL}(2, \mathbb{R})$ and

$$\mathrm{PGL}(2, \mathbb{R}) = [\mathrm{GL}(2, \mathbb{C})/Z]^{\sigma}.$$

In the latter case, Z is the center of $\mathrm{GL}(2, \mathbb{C})$, and σ is the involution which takes each matrix entry to its complex conjugate. In particular, σ preserves Z, and therefore is makes sense to consider $[\mathrm{GL}(2, \mathbb{C})/Z]^{\sigma}$, the fixed points of σ.

(c) Determine all isomorphisms among the groups in (b).

(d) Determine all finite-dimensional representations of the groups in (b).

4. (a) Let $V \simeq \mathbb{C}^2$ be the tautological two-dimensional representation of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Decompose $\mathrm{End}_{\mathbb{C}}(V)$ into irreducible representations of $\mathfrak{g}$.

(b) Decompose $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V)$ into irreducible represenations of $\mathfrak{g}$. (Since $V \simeq V^*$ as representation of $\mathfrak{g}$, we have $\mathrm{End}_{\mathbb{C}}(V) \simeq V \otimes V$; so (a) is the special case of $a = b = 1$ in (b).)